Euler and Infinite Speed

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The famous Swiss mathematician Leonhard Euler was born in Basel on the 15th of April 1707. Already in 1720 when he was still a thirteen-year-old boy, he enrolled at the University of Basel. One year later, he obtained the Bachelor's degree. In 1723 when he was sixteen years old, he obtained his Master's degree (A. L. M. = Master of Liberal Arts).

In 1727 without ever having obtained the Ph. D. degree he submitted a short habilitation thesis (consisting of fifteen pages); that is, a thesis in application for the vacant professorship of physics at the University of Basel. At that time he had published two papers, one of them being partially faulty. No wonder that the commission which looked for a suitable candidate for the professorship did not elect him. Yet Euler was very much infuriated by this decision. Still in 1727, he went to St. Petersburg in order to work at the newly founded Academy of Sciences. He never came back to Switzerland. Between 1741 and 1766 he lived and worked in Berlin at the reformed Academy of Sciences and Literature of Berlin. In 1766 he returned to St. Petersburg where he died on the 18th of September 1783.

The complete title of his habilitation thesis reads:

> *May it bring you happiness and good fortune – Physical dissertation on sound which Leonhard Euler, Master of the liberal arts submits to the public examination of the learned in the juridical lecture-room on February 18, 1727 at 9 o'clock looking at the free professorship of physics by order of the magnificent and wisest class of philosophers whereby the divine will is nodding assent. The most eminent young man Ernst Ludwig Burchard, candidate of philosophy, is responding.*

As we know, all imploring was in vain: Euler did not get the position. The thesis is all the more interesting because Euler had added a supplement in which he formulated six statements regarding utterly different subjects. For example he maintained that Leibniz's theory of preestablished harmony between body

Figure 1: Leonhard Euler (1707–1783) (L. Euler, *Opera omnia*, series I, vol. 1, Leipzig – Berlin 1911, Engraving after p. I)

and soul is false, without mentioning the name of his illustrious predecessor. Another statement prescribed the construction of a ship mast.

The third statement considered a thought experiment: What would happen at the centre of the earth if a stone were dropped into a straight tunnel drilled to the centre of the earth and beyond to the other side of the planet?

Euler distinguished between exactly three possiblities: Either the stone will rest at the centre or will at once proceed beyond it or it will immediately return from the centre to us. There is no mention of speed. Euler just stated that the last case will take place. No justification or explanation is given, though none of these three possibilities had the slightest evidence. What is worse, a far better answer had been already given by Galileo in 1632.

In the second day of his *Dialogue about the two main world systems* Galileo discussed this thought experiment in order to refute Aristotle's distinction between natural and unnatural motions. The natural motion of heavy bodies is the straight fall to the centre of the earth. But what about a cannon ball

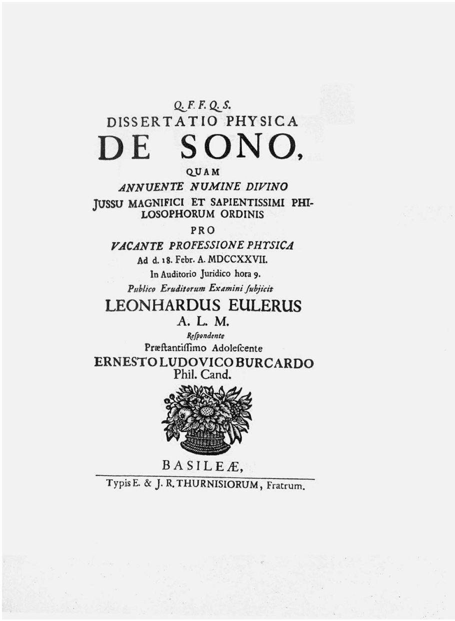
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Figure 2: Title page of Euler's *Physical dissertation on sound* (L. Euler, *Opera omnia*, series III, vol. 1, Leipzig – Berlin 1926, p. 181)

that has dropped into such an earth tunnel? Even the Aristotelian Simplicio avowed that the cannon ball would reach the same height from which it had dropped into the tunnel in the other half of the tunnel. The natural motion would change into an unnatural motion.

Galileo erroneously presupposed a constant gravitation. But he rightly deduced an oscillating motion of the cannon ball. Euler did not mention the Italian mathematician. Presumably he did not know his solution of the thought experiment. Nine years later he came back to this question in his *Mechanics or the science of motion set forth analytically.* Now he explicitly concluded that the speed of the falling stone will become infinitely large in the centre of the earth. Nevertheless it will immediately return to the starting-point.

Euler admitted:

> This seems to differ from truth because hardly any reason is obvious why a body, having infinitely large speed that it has acquired in C, should proceed to any other region than to CB, especially because the direction of the infinite speed turns to this region. However that may be, here we have to be confídent more in the calculation than in our judgement and to confess that we do not understand at all the jump if it is done from the infinite into the finite.

SCHOLION 2

272. Hoc quidem veritati minus videtur consentaneum; vix enim apparet ratio, cur corpus celeritate sua infinite magna, quam in C acquisivit, in aliam potius plagam quam in CB sit progressurum, praesertim cum huius celeritatis infinitae directio sit secundum hanc plagam. Quicquid autem sit, hic calculo potius quam nostro iudicio est fidendum atque statuendum, nos saltum, si fit ex infinito in finitum, penitus non comprehendere. Eo autem magis in hac sententia confirmamur simili exemplo, quod infra plene explanatum occurret (§ 655), si est $n = -2$; hoc enim casu corporis in C pervenientis celeritas quoque est infinita et secundum CB directa; nihilo vero minus corpus non ultra C progreditur, sed subito ex C versus A revertitur pariter ac accesserat. Ex quo perspicitur, quoties celeritas in C existat infinita, iudicium de ulteriori corporis motu esse suspendendum. Tam diu autem hoc tantum fiat, quoad ad motus curvilineos perveniamus; ex iisque enim, qui sint rectilinei, evidentius colligetur (§ 762). Neque enim calculus, qui tum instituetur, obnoxius est huic incommodo, ut a hypothesi dissentiat; sed quaqua versus vis centripeta aequalis ponetur non refragante calculo.

Figure 3: L. Euler, *Mechanics*, vol. 1, 1736, § 272 (Explanation 2) (L. Euler, *Opera omnia*, series II, vol. 1, Leipzig – Berlin 1912, p. 88)

Euler's result was the consequence of his mathematical modelling of the situation (an impermissible commutation of limits). When in 1739 Benjamin Robbins wrote his review of Euler's Mechanics he put as follows:

> When y, the distance of the body from the center, is made negative, the terms of the distance expressed by $\mathrm{y^n}$, where n may be any number affirmative, or negative, whole number or fraction, are sometimes changed with it. The centripetal force being as some power of the fraction; if, when y is supposed negative, $\mathrm{y^n}$ be still affirmative, the solution gives the velocity of the body in its subsequent ascent from the center; but if $\mathrm{y^n}$ by this supposition becomes also negative, the solution exhibits the velocity, after the body has passed the center, upon condition, that the centripetal force becomes centrifugal; and when on this supposition $\mathrm{y^n}$ becomes impossible, the determination of the velocity beyond the center is impossible, the condition being so.

The French physicist Pierre-Louis Moreau de Maupertuis was the president of the Academy of Sciences and of Literature in Berlin at the beginning of Euler's sojourn in Berlin. He unfortunately proposed to construct such an earth tunnel. His proposal was ridiculed by Voltaire on the occasion of the famous quarrel about the principle of least action between Maupertuis, Euler, and the Prussian king Frederick the Great on the one side and the Swiss mathematician Samuel König on the other side. Thus Euler's curious statement about the dropping stone had a satirical aftermath. In 1753 Voltaire published his *Lampoon of*

Doctor Akakia. Therein he made Euler regret that he had more confídence in his calculation than in human judgement. In truth Euler never recanted his solution.

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