Universal Norms of p-Units in Some Non-Commutative Galois Extensions

dedicated to Professor John Coates on the occasion of his 60th birthday

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1 Introduction.

Fix a prime number p. Let F be a finite extension of $\mathbb{Q}$ and let F_∞ be an algebraic extension of F. We will consider the $\mathbb{Z}_p$-submodule $U(F_\infty/F)$ of $O_F[1/p]^\times \otimes \mathbb{Z}_p$ defined by

$$U(F_\infty/F) = \mathrm{Image}(\varprojlim_L (O_L[1/p]^\times \otimes \mathbb{Z}_p) \to O_F[1/p]^\times \otimes \mathbb{Z}_p),$$

where L ranges over all finite extensions of F contained in F_∞ and where the inverse limit is taken with respect to the norm maps.
In the case F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F, the understanding of $U(F_\infty/F)$ is related to profound aspects in Iwasawa theory studied by Coates and other people, as we will shortly recall in §3. Concerning bigger Galois extensions F_∞/F, the following result is (essentially) contained in Corollary 3.23 of Coates and Sujatha [4] (see §3 of this paper).

Assume F_∞/F is a Galois extension and $\mathrm{Gal}(F_\infty/F)$ *is a commutative p-adic Lie group. Assume also that there is only one place of F lying over p. Then $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.*

We ask what happens in the case of non-commutative Lie extensions.
The purpose of this paper is to prove the following theorem, which was conjectured by Coates.

Theorem 1.1. *Let $a_1, \cdots, a_r \in F$, and let*

$$F_n = F(\zeta_{p^n}, a_1^{1/p^n}, \cdots, a_r^{1/p^n}), \quad F_\infty = \cup_{n\geq 1} F_n,$$

where ζ_{p^n} denotes a primitive p^n-th root of 1. Let F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F. Then:

(1) The quotient group $U(F^{\mathrm{cyc}}/F)/U(F_\infty/F)$ is finite.

(2) If there is only one place of F lying over p, then $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.

An interesting point in the proof is that we use the finiteness of the higher K-groups $K_{2n}(O_F)$ for $n \geq 1$, for this result on the muliplicative group K_1. The author does not have any result on $\varprojlim_L O_F[1/S]^\times$ without $\otimes \mathbb{Z}_p$.
The plan of this paper is as follows. In §2, we review basic facts. In §3, we review some known results in the case F_∞/F is an abelian extension. In §4 and §5, we prove Theorem 1.1 (we will prove a slightly stronger result Theorem 5.1).

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2 Basic facts.

We prepare basic facts related to $U(F_\infty/F)$. Most materials appear in Coates and Sujatha [4]. We principally follow their notation.

2.1. Let p be a prime number, and let F be a finite extension of $\mathbb{Q}$. In the case $p = 2$, we assume F is totally imaginary, for simplicity.
Let F_∞ be a Galois extension of F such that the Galois group $G = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group and such that only finitely many finite places of F ramify in F_∞.
Let $\mathbb{Z}_p[[G]]$ be the completed group ring of G, that is, the inverse limit of the group rings $\mathbb{Z}_p[G/U]$ where U ranges over all open subgroups of G.

2.2. We define $\mathbb{Z}_p[[G]]$-modules

$$\mathcal{Z}^i(F_\infty) \quad \text{and} \quad \mathcal{Z}^i_S(F_\infty) \quad (i \geq 0)$$

where S is a finite set of finite places of F contaning all places of F lying over p. Let

$$\mathcal{Z}^i_S(F_\infty) = \varprojlim_L H^i(O_L[1/S], \mathbb{Z}_p(1))$$

where L ranges over all finite extensions of F contained in F_∞, $O_L[1/S]$ denotes the subring of L consisting of all elements which are integral at any finite place of L not lying over S, and H^i is the étale cohomology. In the case S is the set of all places of F lying over p, we denote $\mathcal{Z}^i_S(F_\infty)$ simply by $\mathcal{Z}^i(F_\infty)$.

Since

$$(1) \quad H^1(O_L[1/S], \mathbb{Z}_p(1)) \simeq O_L[1/S]^\times \otimes \mathbb{Z}_p$$

by Kummer theory,

$$(2) \quad \mathcal{Z}^1_S(F_\infty) \simeq \varprojlim_L (O_L[1/S]^\times \otimes \mathbb{Z}_p).$$

Note that $H^i(O_L[1/S], \mathbb{Z}_p(1))$ are finitely generated $\mathbb{Z}_p$-modules and $\mathcal{Z}^i(F_\infty)$ are finitely generated $\mathbb{Z}_p[[G]]$-modules. These modules are zero if $i \geq 3$ for the reason of cohomological dimension (here in the case $p = 2$, we use our assumption F is totally imaginary).

2.3. Let $U_S(F_\infty/F)$ be the image of $\varprojlim_L (O_L[1/S]^\times \otimes \mathbb{Z}_p)$ in $O_F[1/S]^\times \otimes \mathbb{Z}_p$. Here L ranges over all finite extensions of F contained in F_∞.
The main points of the preparation in this section are the isomorphisms (1b) and (2b) below.

(1) Assume S contains all finite places of F which ramify in F_∞. Then there are canonical isomorphisms

$$(1a) \qquad H_0(G, \mathcal{Z}^2_S(F_\infty)) \simeq H^2(O_F[1/S], \mathbb{Z}_p(1)),$$

$$(1b) \quad H_1(G, \mathcal{Z}^2_S(F_\infty)) \simeq (O_F[1/S]^\times \otimes \mathbb{Z}_p)/U_S(F_\infty/F).$$

(2) Assume F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc}. Then we have canonical isomorphisms

$$(2a) \qquad H_0(G, \mathcal{Z}^2(F_\infty/F)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^2(O_F[1/p], \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$(2b) \quad H_1(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Here $H_m(G, ?) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, ?)$ denotes the G-homology. Note that $H_m(G, M)$ are finitely generated $\mathbb{Z}_p$-modules for any finitely generated $\mathbb{Z}_p[[G]]$-module M.

(1a) and (1b) follow from the spectral sequence

$$E_2^{i,j} = H_{-i}(G, \mathcal{Z}^j_S(F_\infty)) \Rightarrow E^i_\infty = H^i(O_F[1/S], \mathbb{Z}_p(1)),$$

the isomorphisms 2.2 (1) (2), and the fact $\mathcal{Z}^j_S(F_\infty) = 0$ for $j \geq 3$. The above spectral sequence is given in [9] Proposition 8.4.8.3 in the case G is commutative. In general, we have the above spectral sequence by [6] 1.6.5 (3).
The proofs of (2a) and (2b) are given in 2.6 later.

2.4. By Kummer theory and by the well known structure theorem of the Brauer group of a global field, we have an exact sequence

$$(1) \quad 0 \to \mathrm{Pic}(O_F[1/S])\{p\} \to H^2(O_F[1/S], \mathbb{Z}_p(1)) \to \oplus_{v \in S} \mathbb{Z}_p \xrightarrow{\mathrm{sum}} \mathbb{Z}_p \to 0,$$

where $\{p\}$ denotes the p-primary part. Let

$$Y_S(F_\infty) = \varprojlim_L \mathrm{Pic}(O_L[1/S])\{p\},$$

where L ranges over all finite extensions of F contained in F_∞. In the case S is the set of all places of F lying over p, we denote $Y_S(F_\infty)$ simply by $Y(F_\infty)$. Then the exact sequences (1) with F replaced by L give an exact sequence of $\mathbb{Z}_p[[G]]$-modules

$$(2) \quad 0 \to Y_S(F_\infty) \to \mathcal{Z}_S^2(F_\infty) \to \oplus_{v\in S}\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

where for each $v \in S$, $G_v \subset G$ is the decomposition group of a place of F_∞ lying over v.

If S contains all finite place of F which ramify in F_∞, the composite homomorphism

$$(3) \quad (O_F[1/S]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F) \simeq H_1(G, \mathcal{Z}_S^2(F_\infty))$$
$$\to \oplus_{v\in S} H_1(G, \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p) = \oplus_{v\in S} H_1(G_v, \mathbb{Z}_p)$$

induced by (1b) and (2) coincides with the homomorphism induced by the reciprocity maps

$$F_v^\times \to G_v^{\mathrm{ab}}(p) \simeq H_1(G_v, \mathbb{Z}_p)$$

of local class field theory, where G_v^{ab} denotes the abelian quotient of G_v and (p) means the pro-p part.

2.5. Assume $F_\infty \supset F^{\mathrm{cyc}}$. Then we have isomorphisms

$$\mathcal{Z}^1(F_\infty) \xrightarrow{\simeq} \mathcal{Z}_S^1(F_\infty), \quad Y(F_\infty) \xrightarrow{\simeq} Y_S(F_\infty).$$

The first isomorphism shows $U(F_\infty/F) = U_S(F_\infty/F)$.

In fact, for each finite extension L of F contained in F_∞, we have an exact sequence

$$\begin{aligned} 0 \to O_L[1/p]^\times \otimes \mathbb{Z}_p \to O_L[1/S]^\times \otimes \mathbb{Z}_p \to \\ \to \oplus_w \mathbb{Z}_p \to \mathrm{Pic}(O_L[1/p])\{p\} \to \mathrm{Pic}(O_L[1/S])\{p\} \to 0 \end{aligned}$$

where w ranges over all places of L lying over S but not lying over p. If L' is a finite extension of F such that $L \subset L' \subset F_\infty$, and if w' is a place of L' lying over w, the transition map from $\mathbb{Z}_p$ at w' to $\mathbb{Z}_p$ at w is the multiplication by the degree of the residue extension of w'/w. Since the residue extension of v in F^{cyc}/F for v not lying over p is a $\mathbb{Z}_p$-extension, this shows that the inverse limit of $\oplus_w \mathbb{Z}_p$ for varying L is zero. Hence we have the above isomorphisms.

2.6. We prove (2a) (2b) of 2.3. Take S containing all finite places of F which ramify in F_∞. Let T be the set of all elements of S which do not lie over p.

By 2.4 (2) and by $Y(F_\infty) \xrightarrow{\simeq} Y_S(F_\infty)$ in 2.5, we have an exact sequence of $\mathbb{Z}_p[[G]]$-modules

$$0 \to \mathcal{Z}^2(F_\infty) \to \mathcal{Z}^2_S(F_\infty) \to \oplus_{v\in T}\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_v]]} \mathbb{Z}_p \to 0.$$

This gives a long exact sequence

$$\begin{aligned}\cdots \to H_m(G, \mathcal{Z}^2(F_\infty)) \to H_m(G, \mathcal{Z}^2_S(F_\infty)) \to \\ \to \oplus_{v\in T} H_m(G_v, \mathbb{Z}_p) \to H_{m-1}(G, \mathcal{Z}^2(F_\infty)) \to \cdots .\end{aligned}$$

Let $G^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ and for $v \in T$, let G_v^{cyc} be the image of G_v in G^{cyc}. Then v is unramified in F^{cyc}/F, and we have a canonical isomorphism $G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$ which sends the Frobenius of v in G_v^{cyc} to $1 \in \mathbb{Z}_p$. Let H_v ($v \in T$) be the kernel of $G_v \to G_v^{\mathrm{cyc}}$. Since G is a p-adic Lie group and since the characteristic of the residue field of v is different from p, H_v is of dimension ≤ 1 as a p-adic Lie group. Furthermore, if H_v is infinite, for an element σ_v of G_v whose image in G_v^{cyc} is the Frobenius of v, the inner automorphism on H_v by σ_v is of infinite order as is seen from the usual description of the tame quotient of the absolute Galois group of F_v. These prove

(1) For $v \in T$, the kernel and the cokernel of the canonical map $H_m(G_v, \mathbb{Z}_p) \to H_m(G_v^{\mathrm{cyc}}, \mathbb{Z}_p)$ are finite for any m.

Since the composition $O_F[1/S]^\times \to H_1(G, \mathcal{Z}^2_S(F_\infty)) \to H_1(G_v^{\mathrm{cyc}}, \mathbb{Z}_p) = G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$ for $v \in T$ coincides with the v-adic valuation $O_F[1/S]^\times \to \mathbb{Z}$, (1) shows that the cokernel of $H_1(G, \mathcal{Z}^2_S(F_\infty)) \to \oplus_{v\in T} H_1(G_v, \mathbb{Z}_p)$ is finite. Hence by the above long exact sequence, we have the following commutative diagram with exact rows in which the kernel of the first arrow of each row is finite.

$$\begin{array}{ccccccc} H_0(G, \mathcal{Z}^2(F_\infty)) & \to & H_0(G, \mathcal{Z}^2_S(F_\infty)) & \to & \oplus_{v\in T}\mathbb{Z}_p & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(O_F[1/p], \mathbb{Z}_p(1)) & \to & H^2(O_F[1/S], \mathbb{Z}_p(1)) & \to & \oplus_{v\in T}\mathbb{Z}_p & \to & 0 \end{array}$$

By this diagram and by 2.3 (1a), we have 2.3 (2a).

We next prove 2.3 (2b). By the above (1), $H_2(G_v, \mathbb{Z}_p)$ is finite for $v \in T$. By this and by the case $m = 1$ of the above (1), we see that the complex $0 \to H_1(G, \mathcal{Z}^2(F_\infty)) \to H_1(G, \mathcal{Z}^2_S(F_\infty)) \to \oplus_{v\in T} H_1(G_v^{\mathrm{cyc}}, \mathbb{Z}_p)$ has finite homology groups. By 2.3 (1b) and by $U(F_\infty/F) = U_S(F_\infty/F)$ (2.5), the kernel of the last arrow of this complex is isomorphic to $(O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F_\infty/F)$. This proves 2.3 (2b).

3 Abelian extensions (Review).

In this section, we review the proof of the following result of Coates and Sujatha ([4] Cor. 3.23), and then recall some known facts on $U(F^{\mathrm{cyc}}/F)$.

Proposition 3.1. *Assume F_∞/F is Galois and* $\mathrm{Gal}(F_\infty/F)$ *is a commutative p-adic Lie group. Assume further that there is only one place of F lying over p. Then:*

(1) $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$.

(2) $H_m(G, Y(F_\infty))$ and $H_m(G, \mathcal{Z}^2(F_\infty))$ are finite for any m.

In fact, this result was written in [4] in the situation $\mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^2$. This was because this result appeared in [4] in the study of the arithmetic of a $\mathbb{Z}_p^2$-extension generated by p-power division points of an elliptic curve with complex multiplication. We just check here that the method of their proof works in this generality.

Proof. We may (and do) assume $F_\infty \supset F^{\mathrm{cyc}}$. In the case $p = 2$, to apply our preparation in §2, we assume F is totally imaginary without a loss of generality (we may replace F by a finite extension of F having only one place lying over p for the proof of 3.1).
(1) follows from the finiteness of $H_1(G, \mathcal{Z}^2(F_\infty))$ in (2) by 2.3 (2b). We prove (2).
We have $H_0(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^2(O_F[1/p], \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by 2.3 (2a), and $H^2(O_F[1/p], \mathbb{Z}_p(1))$ is finite by the exact sequence 2.4 (1) and by the assumption that there is only one place of F lying oer p. Hence $H_0(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$. This shows that $H_m(G, \mathcal{Z}^2(F_\infty)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ for any m (Serre [11]). (Here the assumption G is commutative is essential. See 5.6.) This proves $H_m(G, \mathcal{Z}^2(F_\infty))$ is finite for any m.
Let v be the unique place of F lying over p. Then by class field theory, the decomposition group G_v of v in G is of finite index in G. By the exact sequence

$$H_2(G_v, \mathbb{Z}_p) \to H_2(G, \mathbb{Z}_p) \to H_1(G, \mathcal{Z}^2(F_\infty)/Y(F_\infty)) \to H_1(G_v, \mathbb{Z}_p) \to H_1(G, \mathbb{Z}_p)$$

obtained from 2.4 (2), this shows that $H_1(G, \mathcal{Z}^2(F_\infty)/Y(F_\infty))$ and hence the kernel of $H_0(G, Y(F_\infty)) \to H_0(G, \mathcal{Z}^2(F_\infty))$ are finite. Hence $H_0(G, Y(F_\infty))$ is finite, and by Serre [11], $H_m(G, Y(F_\infty))$ is finite for any m. □

3.2. In the rest of this section, we recall some known facts about $U(F^{\mathrm{cyc}}/F)$. Let $G^{\mathrm{cyc}} = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. For a place v of F lying over p, let $G_v^{\mathrm{cyc}} \subset G^{\mathrm{cyc}}$ be the decomposition group of v (so $G_v^{\mathrm{cyc}} \simeq \mathbb{Z}_p$). Let $(\oplus_{v|p} G_v^{\mathrm{cyc}})^0$ be the kernel of the canoncial map $\oplus_{v|p} G_v^{\mathrm{cyc}} \to G^{\mathrm{cyc}}$.
Let

$$\alpha_F \ : \ (O_F[1/p]^\times \otimes \mathbb{Z}_p)/U(F^{\mathrm{cyc}}/F) \to (\oplus_{v|p} G_v^{\mathrm{cyc}})^0$$

be the homomorphism induced by the reciprocity maps of local fields F_v, which appeared in 2.4 (3).
It is known that the following conditions (1) - (3) are equivalent.

(1) $\mathrm{Ker}\,(\alpha_F)$ is finite. (That is, $U(F^{\mathrm{cyc}}/F)$ is of finite index in the kernel of $O_F[1/p]^\times \otimes \mathbb{Z}_p \to (\oplus_{v|p} G_v^{\mathrm{cyc}})^0$.)

(2) $\mathrm{Coker}\,(\alpha_F)$ is finite.

(3) $H_0(G^{\mathrm{cyc}}, Y(F^{\mathrm{cyc}}))$ is finite.

The equivalence of (1)-(3) is proved as follows. Though this is not at all an essential point, in the case $p = 2$, to apply our preparation in §2, we assume F is totally imaginary without a loss of generality (we can replace F by a finite extension of F for the proof of the equivalence). Let σ be a topological generator of G^{cyc}. Then $H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}}))$ is isomorphic to the cokernel of $\sigma - 1 : \mathcal{Z}^2(F^{\text{cyc}}) \to \mathcal{Z}^2(F^{\text{cyc}})$ and $H_1(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}}))$ is isomorphic to the kernel of it. Since $\mathcal{Z}^2(F^{\text{cyc}})$ is a torsion $\mathbb{Z}_p[[G^{\text{cyc}}]]$-module, this shows that the $\mathbb{Z}_p$-rank of $H_1(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) \simeq (O_F[1/p] \otimes \mathbb{Z}_p)/U(F^{\text{cyc}}/F)$ is equal to the $\mathbb{Z}_p$-rank of $H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) \simeq H^2(O_F[1/p], \mathbb{Z}_p(1))$ which is equal to the $\mathbb{Z}_p$-rank of $(\oplus_{v|p} G_v^{\text{cyc}})^0$ by 2.4 (1). Hence (1) and (2) are equivalent. The exact sequence 2.4 (2) (take $F_\infty = F^{\text{cyc}}$ and S to be the set of all places of F lying over p) shows that $\mathrm{Coker}\,(\alpha_F)$ is isomorphic to the kernel of $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}})) \to H_0(G^{\text{cyc}}, \mathcal{Z}^2(F^{\text{cyc}})) = H^2(O_F[1/p], \mathbb{Z}_p(1))$. The image of the last map is $\mathrm{Pic}(O_F[1/p])\{p\}$ by 2.4 (1) (2), and hence is finite. Hence $\mathrm{Coker}\,(\alpha_F)$ is finite if and only if $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite.

3.3. Greenberg [7] proved that $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite if F is an abelian extension of $\mathbb{Q}$ (hence all (1) - (3) in 3.2 are satisfied in this case).

3.4. In the case F is totally real, by Coates [2] Theorem 1.13, $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ is finite if Leopoldt conjecture for F is true.

3.5. Let F be a CM field. Let F^+ be the real part of F, and let $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^{\pm} \subset H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))$ be the $\pm$-part with respect to the action of the complex conjugation in $\mathrm{Gal}(F/F^+)$. Then by the above result of Coates, $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^+$ is finite if Leopoldt conjecture for F^+ is true. On the other hand, Conjecture 2.2 in Coates and Lichtenbaum [3] says that $H_0(G^{\text{cyc}}, Y(F^{\text{cyc}}))^-$ is finite. In [8], Gross conjectured that the kernel and the cokernel of the (-)-part α_F of α_F is finite (this finiteness is also a consequence of Conjecture 2.2 of [3]), and formulated a conjecture which relates α_F^- to the leading terms of the Taylor expansions at $s = 0$ of p-adic Artin L-functions.
Thus known conjectures support that the equivalent conditions (1) - (3) in 3.2 are satisfied by any CM field F.
A natural question arises: Are (1) - (3) in 3.2 true for any number field F?

4 A result on Tor modules.

The purpose of this section is to prove Proposition 4.2 below.

4.1. For a compact p-adic Lie group G, for a $\mathbb{Z}_p[[G]]$-module T, and for a continuous homomorphism $G \to \mathbb{Z}_p^\times$, let $T(\chi)$ be the $\mathbb{Z}_p[[G]]$-module whose underlying abelian group is that of T and on which $\mathbb{Z}_p[[G]]$ acts by $\mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]] \to \mathrm{End}(T)$, where the first arrow is the automorphism $\sigma \mapsto \chi(\sigma)\sigma$ ($\sigma \in G$) of the topological ring $\mathbb{Z}_p[[G]]$ and the second arrow is the original action of $\mathbb{Z}_p[[G]]$ on T. We call $T(\chi)$ the twist of T by χ.

Proposition 4.2. *Let G be a compact p-adic Lie group, let H be a closed normal subgroup of G, and assume that we are given a finite family of closed normal subgroups H_i $(0 \le i \le r)$ of G such that $\{1\} = H_0 \subset H_1 \subset \cdots \subset H_r = H$, $H_i/H_{i-1} \simeq \mathbb{Z}_p$ for $1 \le i \le r$ and such that the the action of G on H_i/H_{i-1} by inner automorphisms is given by a homomorphism $\chi_i : G/H \to \mathbb{Z}_p^\times$.*
Let M be a finitely generated $\mathbb{Z}_p[[G]]$-module, and let M' be a subquotient of the $\mathbb{Z}_p[[G]]$-module M. Let $m \ge 0$. Then there is a finite family $(S_i)_{1 \le i \le k}$ of $\mathbb{Z}_p[[G/H]]$-submodules of $H_m(H, M')$ satisfying the following (i) *and* (ii)*.*

(i) $0 = S_0 \subset S_1 \subset \cdots \subset S_k = H_m(H, M')$.

(ii) *For each i $(1 \le i \le k)$, there are a subquotient T of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H, M)$ and a family $(s(j))_{1 \le j \le r}$ of non-negative integers $s(j)$ such that $\sharp\{j | s(j) > 0\} \ge m$ and such that S_i/S_{i-1} is isomorphic to the twist $T(\prod_{1 \le j \le k} \chi_j^{s(j)})$ of T.*

Note
$$H_m(H, M) = \mathrm{Tor}_m^{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, M) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G/H]], M)$$
for $\mathbb{Z}_p[[G]]$-modules M.
A key point in the proof of Proposition 3.1 was that for commutative rings, Tor_m vanishes if Tor_0 vanishes. This is not true for non-commutative rings. In the next section, we will use the above relation of Tor_0 and Tor_m in a non-commutative situation for the proof of Theorem 1.1.

4.3. We denote this proposition with fixed r by (A_r). Let (B_r) be the case $M = M'$ of (A_r).

Since (B_r) is a special case of (A_r), (B_r) follows from (A_r).
In 4.4, we show that conversely, (A_r) follows from (B_r). In 4.5, we prove (B_1). In 4.6, for $r \ge 1$, we prove (B_r) assuming (A_{r-1}) and (B_1). These give a proof of Prop.4.2.

4.4. We can deduce (A_r) from (B_r) as follows. Let M'' be the quotient of the $\mathbb{Z}_p[[G]]$-module M such that M' is a $\mathbb{Z}_p[[G]]$-submodule of M''. We have an exact sequence of $\mathbb{Z}_p[[G/H]]$-modules
$$H_{m+1}(H, M''/M') \to H_m(H, M') \to H_m(H, M'').$$
Then (A_r) for the pair (M, M') is obtained from (B_r) applied to M''/M' and to M'' since $H_0(H, M''/M')$ and $H_0(H, M'')$ are quotients of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H, M)$.

4.5. We prove (B_1). Assume $r = 1$. Let $\chi = \chi_1$.
Note that $H \simeq \mathbb{Z}_p$. Let α be a topological generator of H, and let $N = \alpha - 1 \in \mathbb{Z}_p[[G]]$. Let $I = \mathrm{Ker}\,(\mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G/H]]) = \mathbb{Z}_p[[G]]N = N\mathbb{Z}_p[[G]]$.
We have

(1) For $\sigma \in G$, $\sigma N \sigma^{-1}$ is expressed as a power series in N with coefficients in $\mathbb{Z}_p$ which is congruent to $\chi(\sigma)N \mod N^2$. In particular, $\sigma N \sigma^{-1} \equiv \chi(\sigma)N$ mod I^2.

In fact, $\sigma N \sigma^{-1} = \alpha^{\chi(\sigma)} - 1 = (1+N)^{\chi(\sigma)} - 1 = \chi(\sigma)N + \sum_{n \geq 2} c_i N^i$ for some $c_i \in \mathbb{Z}_p$.
Concerning $H_m(H, M)$ $(m \geq 0)$, we have:

(2) $N(M)$ is a $\mathbb{Z}_p[[G]]$-submodule of M, I kills $M/N(M)$, and there is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules

$$H_0(H, M) \simeq M/N(M).$$

(3) $\mathrm{Ker}\,(N : M \to M)$ is a $\mathbb{Z}_p[[G]]$-submodule of M, I kills $\mathrm{Ker}\,(N : M \to M)$, and there is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules

$$H_1(H, M) \simeq \mathrm{Ker}\,(N : M \to M)(\chi).$$

(4) $H_m(H, M) = 0$ fo $m \geq 2$.

We prove (2)–(4). We have a projective resolution

$$0 \to I \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G/H]] \to 0$$

of the right $\mathbb{Z}_p[[G]]$-module $Z_p[[G/H]]$. Since $H_m(H, ?) = \mathrm{Tor}_m^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G/H]], ?)$, $H_0(H, M)$ (resp. $H_1(H, M)$) is isomorphic to the cokernel (resp. kernel) of $I \otimes_{\mathbb{Z}_p[[G]]} M \to M$, and $H_m(H, M) = 0$ for all $m \geq 2$. This proves (2) and (4). Furthermore,

$$H_1(H, M) \simeq \mathrm{Ker}\,(I \otimes_{\mathbb{Z}_p[[G]]} M \to M) \simeq I \otimes_{\mathbb{Z}_p[[G]]} \mathrm{Ker}\,(N : M \to M)$$

$$\simeq I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M).$$

Consider the bijection

$$\mathrm{Ker}\,(N : M \to M) \to I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M)\ ;\ x \mapsto N \otimes x.$$

By the above (1), for $\sigma \in G$, we have $\sigma N \otimes x = \chi(\sigma)N\sigma \otimes x = \chi(\sigma)N \otimes \sigma x$ in $I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M)$. Hence

$$I/I^2 \otimes_{\mathbb{Z}_p[[G/H]]} \mathrm{Ker}\,(N : M \to M) \simeq \mathrm{Ker}\,(N : M \to M)(\chi)$$

as $\mathbb{Z}_p[[G/H]]$-modules. This proves (3).
Let

$$V_n = \mathrm{Ker}\,(N^n : M \to M) \quad (n \geq 0), \quad V = \cup_n V_n.$$

Then, since $\mathbb{Z}_p[[G]]N^n = N^n\mathbb{Z}_p[[G]]$, V_n is a $\mathbb{Z}_p[[G]]$-submodule of M. Since $\mathbb{Z}_p[[G]]$ is Noetherian and M is a finitely generated $\mathbb{Z}_p[[G]]$-module, $V = V_n$ for some n. That is, N is nilpotent on V. Since $\mathrm{Ker}\,(N : M/V \to M/V) = 0$, we have $H_1(H, M/V) = 0$ by (3). Hence

(5) $H_1(H,V) = H_1(H,M)$,

(6) $H_0(H,V) \to H_0(H,M)$ is injective.

Consider the monodromy filtration $(W_i)_i$ on the abelian group V given by the nilpotent endomorphism N in the sense of Deligne [5] 1.6. It is an increasing filtration characterized by the properties $N(W_i) \subset W_{i-2}$ for all i, and $N^i : \mathrm{gr}_i^W \overset{\cong}{\to} \mathrm{gr}_{-i}^W$ for all $i \geq 0$.

(7) W_i are $\mathbb{Z}_p[[G]]$-submodules of V.

In fact, for $\sigma \in G$, the filtration $(\sigma W_i)_i$ also has the characterizing property of $(W_i)_i$ by (1).
Now we define an increasing filtration $(W'_i)_i$ of the $\mathbb{Z}_p[[G/H]]$-module $H_0(H,V)$ and an increasing filtration $(W''_i)_i$ on the $\mathbb{Z}_p[[G/H]]$-module $H_1(H,V) = H_1(H,M)$ as follows. By identifying $H_0(H,V)$ with $\mathrm{Coker}\,(N : V \to V)$, let $W'_i = W_i(\mathrm{Coker}\,(N : V \to V))$ (i.e. the image of W_i in $\mathrm{Coker}\,(N : V \to V)$). By identifying $H_1(H,V)$ with $\mathrm{Ker}\,(N : V \to V)(\chi)$, let $W''_i = W_i(\mathrm{Ker}\,(N : V \to V))(\chi)$ (i.e. $(W_i \cap \mathrm{Ker}\,(N : V \to V))(\chi)$). Then $W''_0 = H_1(H,M)$, and $W''_i = 0$ if i is sufficiently small. We prove:

(8) For any $i \geq 0$,
$$\mathrm{gr}_{-i}^{W''} \simeq \mathrm{gr}_i^{W'}(\chi^{i+1})$$
as $\mathbb{Z}_p[[G/H]]$-modules.

By the injectivity of $H_0(H,V) \to H_0(H,M)$ (6), this proves (B_1).

We prove (8). By (1), we have

(9) The map $N : \mathrm{gr}_i^W \to \mathrm{gr}_{i-2}^W$ satisfies $\sigma N \sigma^{-1} = \chi(\sigma) N$ for $\sigma \in G$.

Let $P_i \subset \mathrm{gr}_i^W$ ($i \leq 0$) be the primitive part $\mathrm{Ker}\,(N : \mathrm{gr}_i^W \to \mathrm{gr}_{i-2}^W)$ ([5] 1.6.3). Then for $i \geq 0$, the canonical map $\mathrm{gr}_{-i}^W(\mathrm{Ker}\,(N : V \to V)) \to P_{-i}$ is an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules ([5] 1.6.6). Furthermore, we have a bijection $P_{-i} \overset{\cong}{\to} \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))$ as the composition

$$P_{-i} \to \mathrm{gr}_{-i}^W \overset{N^i}{\leftarrow} \mathrm{gr}_i^W \to \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))$$

([5] 1.6.4, 1.6.6, and the dual statement of 1.6.6 for $\mathrm{Coker}\,(N)$). By (9), this gives an isomorphism of $\mathbb{Z}_p[[G/H]]$-modules $P_{-i} \simeq \mathrm{gr}_i^W(\mathrm{Coker}\,(N : V \to V))(\chi^i)$. Hence we have (8).

4.6. Let $r \geq 1$. We prove (B_r) assuming (A_{r-1}) and (B_1). Let $J = H_1$. By the spectral sequence

$$E_2^{-i,-j} = H_i(H/J, H_j(J,M)) \Rightarrow E_\infty^{-m} = H_m(H,M)$$

in which $H_j(J,M)=0$ for $j\geq 2$, we have an exact sequence of $\mathbb{Z}_p[[G/H]]$-modules

$$(1)\quad H_{m-1}(H/J,H_1(J,M))\to H_m(H,M)\to H_m(H/J,H_0(J,M)).$$

We consider $H_{m-1}(H/J,H_1(J,M))$ first. By (B_1) applied to the triple (G,J,M), $H_1(J,M)$ is a successive extension of twists of subquotients of $H_0(J,M)$ by χ_1^i $(i\geq 1)$. By (A_{r-1}) applied the triple $(G/J,H/J,H_0(J,M))$, $H_{m-1}(H/J,?)$ of these subquotients of $H_0(J,M)$ are successive extensions of twists of subquotients of $H_0(H/J,H_0(J,M))=H_0(H,M)$ by $\prod_{2\leq j\leq r}\chi_j^{s(j)}$ such that $s(j)\geq 0$ for all j and such that $\sharp(\{j\,|\,s(j)>0\}\geq m-1$. Hence $H_{m-1}(H/J,H_1(J,M))$ is a successive extension of twists of subquotients of $H_0(H,M)$ by $\prod_{1\leq j\leq r}\chi_j^{s(j)}$ such that $s(j)\geq 0$ for all j and such that $\sharp(\{j\,|\,s(j)>0\}\geq m$.

We consider $H_m(H/J,H_0(J,M))$ next. By (B_{r-1}) (which is assumed since we assume (A_{r-1})) applied to the triple $(G/J,H/J,H_0(J,M))$, $H_m(H/J,H_0(J,M))$ is a successive extension of twists of subquotients of $H_0(H/J,H_0(J,M))=H_0(H,M)$ by $\prod_{2\leq j\leq r}\chi_i^{s(j)}$ such that $s(j)\geq 0$ for all j and such that $\sharp(\{j\,|\,s(j)>0\}\geq m$.

By these properties of $H_{m-1}(H/J,H_1(J,M))$ and $H_m(H/J,H_0(J,M))$, the exact sequence (1) proves (B_r) (assuming (A_{r-1}) and (B_1)).

5 Some non-commutative Galois extensions.

Theorem 1.1 in Introduction is contained in Corollary 5.2 of the following Theorem 5.1, for the extension F_∞/F in Theorem 1.1 satisfies the assumption of Theorem 5.1 with $n(i)=1$ for all i.

Theorem 5.1. *Assume that F_∞ is a Galois extension of F, $F_\infty\supset\cup_n F(\zeta_{p^n})$, and that there is a finite family of closed normal subgroups H_i $(1\leq i\leq r)$ of $G=\mathrm{Gal}(F_\infty/F)$ satisfying the following condition. Let F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F and let H be the kernel of $G\to G^{\mathrm{cyc}}=\mathrm{Gal}(F^{\mathrm{cyc}}/F)$. Then $\{1\}=H_0\subset H_1\subset\cdots\subset H_r$, H_r is an open subgroup of H, and for $1\leq i\leq r$, $H_i/H_{i-1}\simeq\mathbb{Z}_p$ and the action of G on it by inner automorphism is the $n(i)$-th power of the cyclotomic character $G\to\mathbb{Z}_p^\times$ for some positive integer $n(i)>0$. Let S be any finite set of finite places of F containing all places lying over p. Then the kernel and the cokernel of the canonical maps*

$$H_m(G,\mathcal{Z}_S^2(F_\infty))\to H_m(G^{\mathrm{cyc}},\mathcal{Z}_S^2(F^{\mathrm{cyc}})),$$
$$H_m(G,Y(F_\infty))\to H_m(G^{\mathrm{cyc}},Y(F^{cyc}))$$

are finite for any m.

In particular (since $H_m(G^{\mathrm{cyc}},?)=0$ for $m\geq 2$), $H_m(G,\mathcal{Z}_S^2(F_\infty))$ and $H_m(G,Y(F_\infty))$ are finite for any $m\geq 2$.

Corollary 5.2. *Let the assumption be as in Theorem 5.1. Then:*

(1) The quotient group $U(F^{cyc}/F)/U(F_\infty/F)$ is finite.
(2) If there is only one place of F lying over p, then $U(F_\infty/F)$ is of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$, and $H_m(G, Y(F_\infty))$ and $H_m(G, \mathcal{Z}^2(F_\infty))$ are finite for any m.

(3) If F is an abelian extension over $\mathbb{Q}$, then $H_m(G, Y(F_\infty))$ is finite for any m.

In fact, by 2.3 (2b), (1) of Corollary 5.2 follows from the finiteness of the kernel and the cokernel of $H_1(G, \mathcal{Z}^2(F_\infty)) \to H_1(G^{\mathrm{cyc}}, \mathcal{Z}^2(F^{\mathrm{cyc}}))$ which is a special case of Theorem 5.1. (2) follows from (1) and the case $F_\infty = F^{\mathrm{cyc}}$ of Proposition 3.1. (3) follows from (1) and the result of Greenberg introduced in 3.3.

Corollary 5.3. *Let the assumption be as in Theorem 5.1. Then $H_m(G, \mathcal{Z}^1(F_\infty))$ for $m \geq 1$ and the kernel of the canonical map $H_0(G, \mathcal{Z}^1(F_\infty)) \to O_F[1/p]^\times \otimes \mathbb{Z}_p$ are finite.*

In fact, for S containing all finite places which ramify in F_∞, since $\mathcal{Z}^1(F_\infty) \xrightarrow{\simeq} \mathcal{Z}^1_S(F_\infty)$ (2.5), the spectral sequence in 2.3 shows that $H_m(G, \mathcal{Z}^1(F_\infty))$ for $m \geq 1$ is isomorphic to $H_{m+2}(G, \mathcal{Z}^2_S(F_\infty))$, and the kernel of $H_0(G, \mathcal{Z}^1(F_\infty)) \to O_F[1/p]^\times \otimes \mathbb{Z}_p$ is isomorphic to $H_2(G, \mathcal{Z}^2_S(F_\infty))$. Hence this corollary follows from the finiteness of $H_m(G, \mathcal{Z}^2_S(F_\infty))$ for $m \geq 2$ in Theorem 5.1.

5.4. We prove Theorem 5.1. First in this 5.4, we show that the kernel and the cokernel of $H_m(G, \mathcal{Z}^2_S(F_\infty)) \to H_m(G^{\mathrm{cyc}}, \mathcal{Z}^2_S(F^{\mathrm{cyc}}))$ are finite for any m assuming that S contains all finite places of F which ramify in F_∞,.
We may replace F by a finite extension of F. Hence we may assume that $H_r = H$, $\cup_{n\geq 1} F(\zeta_{p^n}) = F^{\mathrm{cyc}}$, and that in the case $p = 2$, F is totally imaginary. Let $\mathfrak{p}$ be the augmentation ideal of $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$. It is a prime ideal of $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$. By the spectral sequence $E_2^{-i,-j} = H_i(G^{\mathrm{cyc}}, H_j(H, ?)) \Rightarrow E_\infty^{-m} = H_m(G, ?)$, it is sufficient to prove that $H_i(G^{\mathrm{cyc}}, H_m(H, \mathcal{Z}^2_S(F_\infty)))$ is finite for any i and for any $m \geq 1$. For a finitely generated $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-module M, $H_i(G^{\mathrm{cyc}}, M)$ is isomorphic to $M/\mathfrak{p}M$ if $i = 0$, to the part of M annihilated by $\mathfrak{p}$ if $i = 1$, and is zero if $i \geq 2$. Applying this taking $M = H_m(H, \mathcal{Z}^2_S(F_\infty))$, we see that it is sufficient to prove

$$(1) \quad H_m(H, \mathcal{Z}^2_S(F_\infty))_{\mathfrak{p}} = 0 \quad \text{for any} \quad m \geq 1,$$

where $(?)_{\mathfrak{p}}$ denotes the localization at the prime ideal $\mathfrak{p}$.
We apply Proposition 4.2 to the case $M = M' = \mathcal{Z}^2_S(F_\infty)$. By this proposition, to prove (1), it is sufficient to show that for any subquotient T of the $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-module $H_0(H, M) = \mathcal{Z}^2_S(F^{\mathrm{cyc}})$ and for any integer $k \geq 1$, we have $T(k)_{\mathfrak{p}} = 0$. Here $T(k)$ is the k-th Tate twist. It is sufficient to prove that $H_0(G^{\mathrm{cyc}}, T(k))$ is finite. Since $\mathcal{Z}^2_S(F^{\mathrm{cyc}})$ is a finitely generated torsion $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-module, the $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-module T is a successive extension of $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-modules which are

either finite or isomorphic to $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ of $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$ of height one. We may assume $T \simeq \mathbb{Z}_p[[G^{\mathrm{cyc}}]]/\mathfrak{q}$. Then there is a $\mathbb{Z}_p[[G^{\mathrm{cyc}}]]$-homomorphism $\mathcal{Z}^2_S(F^{\mathrm{cyc}}) \to T$ with finite cokernel. Hence it is sufficient to prove that $H_0(G^{\mathrm{cyc}}, \mathcal{Z}^2(F^{\mathrm{cyc}})(k)))$ is finite for any $k \geq 1$. But

$$H_0(G^{\mathrm{cyc}}, \mathcal{Z}^2(F^{\mathrm{cyc}})(k))) \simeq H^2(O_F[1/S], \mathbb{Z}_p(k+1)).$$

The last group is finite by Soulé [12]. In fact, by Quillen [10] and Borel [1], $K_{2k}(O_F[1/S])$ is finite, and by Soulé [12], we have a surjective Chern class map from $K_{2k}(O_F[1/S])$ to $H^2(O_F[1/S], \mathbb{Z}_p(k+1))$.

5.5. We complete the proof of Theorem 5.1. Let S be a finite set of finite places of F which contains all places of F lying over p. Take a finite set S' of finite places of F such that $S \subset S'$ and such that S' contains all finite places of F which ramify in F_∞.
By comparing the exact sequence 2.4 (2) for F_∞/F and that for F^{cyc}/F, we see that the finiteness of the kernel and the cokernel of $H_m(G, \mathcal{Z}^2_S(F_\infty)) \to H_m(G^{\mathrm{cyc}}, \mathcal{Z}^2_S(F^{\mathrm{cyc}}))$ for all m and that of $H_m(G, Y(F_\infty)) \to H_m(G^{\mathrm{cyc}}, Y(F^{\mathrm{cyc}}))$ for all m are consequences of the following (1) - (3).

(1) The kernel and the cokernel of $H_m(G, \mathcal{Z}^2_{S'}(F_\infty)) \to H_m(G^{\mathrm{cyc}}, \mathcal{Z}^2_{S'}(F^{\mathrm{cyc}}))$ are finite for all m.

(2) The kernel and the cokernel of $H_m(G, \mathbb{Z}_p) \to H_m(G^{\mathrm{cyc}}, \mathbb{Z}_p)$ are finite for all m.

(3) The kernel and the cokernel of $H_m(G_v, \mathbb{Z}_p) \to H_m(G_v^{\mathrm{cyc}}, \mathbb{Z}_p)$ are finite for all m and for all finite places v of F. Here $G_v \subset G$ denotes a decomposition group of a place of F_∞ lying over v, and G_v^{cyc} denotes the image of G_v in G^{cyc}.

We proved (1) already in 5.4. (2) and (3) follow from the case $M = M' = \mathbb{Z}_p$ of Proposition 4.2.

REMARK 5.6. There is an example of a p-adic Lie extension F_∞/F for which there is only one place of F lying over p but $U(F_\infty/F)$ is not of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p$. For example, let $F = \mathbb{Q}$, let E be an elliptic curve over F with good ordinary reduction at p, and let F_∞ be the field generated over F by p^n-division points of E for all n. Then $U(F_\infty/F) = \{1\}$ and is not of finite index in $O_F[1/p]^\times \otimes \mathbb{Z}_p = \mathbb{Z}[1/p]^\times \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p$. In fact $U(F_\infty/F)$ must be killed by the reciprocity map of local class field theory of $\mathbb{Q}_p$ into $G_p^{\mathrm{ab}}(p) \simeq \mathbb{Z}_p^2$, where $G_p \subset G = \mathrm{Gal}(F_\infty/F)$ denotes the decomposition group at p, and $G_p^{\mathrm{ab}}(p)$ denotes the pro-p part of the abelian quotient of G_p. The image of $p \in \mathbb{Z}[1/p]^\times$ in $G_p^{\mathrm{ab}}(p)$ is of infinite order. This proves $U(F_\infty/F) = \{1\}$. In this case, $H_0(G, \mathcal{Z}^2(F_\infty))$ is finite, but $H_1(G, \mathcal{Z}^2(F_\infty))$ is not finite.

REMARK 5.7. There is an example of a p-adic Lie extension F_∞/F for which $G = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^2$ and $H_0(G, Y(F_\infty/F))$ is not finite. Let K be an imaginary quadratic field in which p splits, let K_∞ be the unique Galois extension of K such that $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2$, let F be a finite extension of K in which p splits completely, and let $F_\infty = FK_\infty$. Then the $\mathbb{Z}_p$-rank of $H_1(G, Y(F_\infty))$ is $\geq [F : K] - 1$ as is shown below. Hence it is not zero if $F \neq K$. In fact, from the exact sequence 2.4 (2) with S the set of all places of F lying over p, we can obtain

$$\begin{aligned} &\mathrm{rank}_{\mathbb{Z}_p} H_1(G, Y(F_\infty)) \geq \\ &\quad \geq (\sum_{v \in S} \mathrm{rank}_{\mathbb{Z}_p} H_1(G_v, \mathbb{Z}_p)) - \mathrm{rank}_{\mathbb{Z}_p} H_1(G, \mathbb{Z}_p) - \mathrm{rank}_{\mathbb{Z}}\, O_F[1/p]^\times. \end{aligned}$$

But $\mathrm{rank}_{\mathbb{Z}_p} H_1(G_v, \mathbb{Z}_p) = 2$ for any $v \in S$, $\mathrm{rank}_{\mathbb{Z}_p} H_1(G, \mathbb{Z}_p) = 2$, $\mathrm{rank}_{\mathbb{Z}} O_F[1/p]^\times = 3[F : K] - 1$ by Dirichlet's unit theorem, and hence the right hand side of the above inequality is $2[F : \mathbb{Q}] - 2 - (3[F : K] - 1) = [F : K] - 1$.

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