Λ-Adic Euler Characteristics of Elliptic Curves

To John Coates on the occasion of his sixtieth birthday

Daniel Delbourgo

Received: July 11, 2005
Revised: February 17, 2006

Abstract. Let $E_{/\mathbb{Q}}$ be a modular elliptic curve, and $p > 3$ a good ordinary or semistable prime.

Under mild hypotheses, we prove an exact formula for the μ-invariant associated to the weight-deformation of the Tate module of E. For example, at ordinary primes in the range $3 < p < 100$, the result implies the triviality of the μ-invariant of $X_0(11)$.

2000 Mathematics Subject Classification: 11G40; also 11F33, 11R23, 11G05

0. Introduction

A central aim in arithmetic geometry is to relate global invariants of a variety, with the behaviour of its L-function. For elliptic curves defined over a number field, these are the numerical predictions made by Birch and Swinnerton-Dyer in the 1960's. A decade or so later, John Coates pioneered the techniques of Iwasawa's new theory, to tackle their conjecture prime by prime. Together with Andrew Wiles, he obtained the first concrete results for elliptic curves admitting complex multiplication.

Let p be a prime number, and F_∞ a p-adic Lie extension of a number field F. From the standpoint of Galois representations, one views the Iwasawa theory of an elliptic curve E defined over F, as being the study of the p^∞-Selmer group

$$\mathrm{Sel}_{F_\infty}(E) \quad \subset \quad H^1\Big(\mathrm{Gal}(\overline{F}/F),\ \mathbb{A}_{F_\infty}\Big)\ .$$

Here $\mathbb{A}_{F_\infty} = \mathrm{Hom}_{\mathrm{cont}}\Big(\mathrm{Ta}_p(E)[[\mathrm{Gal}(F_\infty/F)]],\ \mathbb{Q}/\mathbb{Z}\Big)$ denotes the Pontrjagin dual to the $\mathrm{Gal}(F_\infty/F)$-deformation of the Tate module. The field F_∞ is often taken to be the cyclotomic $\mathbb{Z}_p$-extension of F, or sometimes the anti-cyclotomic extension. Hopefully a more complete picture becomes available over $F_\infty = F\big(E[p^\infty]\big)$, the field obtained by adjoining all p-power division points on E. If E has no complex multiplication, then $\mathrm{Gal}\big(F_\infty/F\big)$ is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ by a theorem of Serre, which means the underlying Iwasawa algebras are no longer commutative.

In this article we study a special kind of Selmer group, namely the one which is associated to a Hida deformation of $\mathrm{Ta}_p(E)$. This object is defined by imposing the local condition that every 1-cocycle lies within a compatible family of points, living on the pro-jacobian of $\hat{X} = \varprojlim_r X_1(Np^r)$. There is a natural action of the diamond operators on the universal nearly-ordinary representation, which extends to a continuous action of $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ on our big Selmer group. By the structure theory of Λ-modules, we can define an analogue of the μ-invariant for a weight deformation, μ^{wt} say. One can also deform both the Tate-Shafarevich group and the Tamagawa factors $[E(F_\nu) : E_0(F_\nu)]$, as sheaves over weight-space. Conjecturally the deformation of Ш should be mirrored by the behaviour of the *improved* p-adic L-function in [GS, Prop 5.8], which interpolates the L-values of the Hida family at the point $s = 1$. The Λ-adic Tamagawa factors $\mathrm{Tam}_{\Lambda,l}$ are related to the arithmetic of $F_\infty = F\big(E[p^\infty]\big)$, as follows.

For simplicity suppose that E is defined over $F = \mathbb{Q}$, and is without complex multiplication. Let $p \geq 5$ be a prime where E has good ordinary reduction, and assume there are no rational cyclic p-isogenies between E and any other elliptic curve. Both Howson and Venjakob have proposed a definition for a μ-invariant associated to the full GL_2-extension. Presumably, this invariant should represent the power of p occurring in the leading term of a hypothetical p-adic L-function, interpolating critical L-values of E at twists by Artin representations factoring through $\mathrm{Gal}(F_\infty/\mathbb{Q})$.

Recall that for a discrete p-primary $\mathrm{Gal}(F_\infty/\mathbb{Q})$-module M, its $\mathrm{Gal}(F_\infty/\mathbb{Q})$-Euler characteristic is the product

$$\chi\Big(\mathrm{Gal}(F_\infty/\mathbb{Q}),\ M\Big) \ := \ \prod_{j=0}^{\infty} \Big(\#H^j\big(F_\infty/\mathbb{Q}, M\big)\Big)^{(-1)^j}.$$

Under the twin assumptions that $L(E,1) \neq 0$ and $\mathrm{Sel}_{F_\infty}(E)$ is cotorsion over the non-abelian Iwasawa algebra, Coates and Howson [CH, Th 1.1] proved that

$$\chi\Big(\mathrm{Gal}(F_\infty/\mathbb{Q}),\ \mathrm{Sel}_{F_\infty}(E)\Big) \ = \prod_{\text{bad primes } l} \big|L_l(E,1)\big|_p \ \times \ \Big(\#\widetilde{E}(\mathbb{F}_p)[p^\infty]\Big)^2 \times \Big(\text{the } p\text{-part of the BS,D formula}\Big).$$

Let μ^{GL_2} denote the power of p occurring above. It's straightforward to combine the main result of this paper (Theorem 1.4) with their Euler characteristic calculation, yielding the upper bound

$$\mu^{\mathrm{wt}} \quad \leq \quad \mu^{\mathrm{GL}_2} \;+ \sum_{\text{bad primes } l} \Big\{ \mathrm{ord}_p\big(L_l(E,1)\big) - \mathrm{ord}_p\big(\mathrm{Tam}_{\Lambda,l}\big) \Big\} \,.$$

In other words, the arithmetic of the weight-deformation is controlled in the p-adic Lie extension. This is certainly consistent with the commonly held belief, that the Greenberg-Stevens p-adic L divides the projection (to the Iwasawa algebra of the maximal torus) of some 'non-abelian L-function' living in $\mathbb{Z}_p\Big[\Big[\mathrm{Gal}\big(\mathbb{Q}\big(E[p^\infty]\big)/\mathbb{Q}\big)\Big]\Big]$. The non-commutative aspects currently remain shrouded in mystery, however.

Finally, we point out that many elliptic curves E possess Λ-adic Tamagawa factors, which differ from the p-primary component of the standard factor $\mathrm{Tam}(E)$. P. Smith has estimated this phenomenon occurs infrequently – a list of such curves up to conductor $< 10,000$ has been tabulated in [Sm, App'x A].

Acknowledgement: We dedicate this paper to John Coates on his sixtieth birthday. The author thanks him heartily for much friendly advice, and greatly appreciates his constant support over the last decade.

1. Statement of the Results

Let E be an elliptic curve defined over the rationals. We lose nothing at all by supposing that E be a strong Weil curve of conductor N_E, and denote by $\pm\phi$ the non-constant morphism of curves $\phi : X_0(N_E) \twoheadrightarrow E$ minimal amongst all $X_0(N_E)$-parametrisations. In particular, there exists a normalised eigenform $f_E \in \mathcal{S}_2^{\mathrm{new}}\big(\Gamma_0(N_E)\big)$ satisfying $\phi^*\omega_E = c_E^{\mathrm{Man}} f_E(q)dq/q$, where ω_E denotes a Néron differential on E and c_E^{Man} is the Manin constant for ϕ.

Fix a prime number $p \geq 5$, and let's write $N = p^{-\mathrm{ord}_p N_E} N_E$ for the tame level. We shall assume E has either good ordinary or multiplicative reduction over $\mathbb{Q}_p$,

hence $\mathbf{f}_2 := \begin{cases} f_E(q) - \beta_p f_E(q^p) & \text{if } p \nmid N_E \\ f_E(q) & \text{if } p \| N_E \end{cases}$ will be the p-stabilisation of f_E at p.

Hypothesis($\mathcal{R}_E$). $\quad$ *$\mathbf{f}_2$ is the unique p-stabilised newform in $\mathcal{S}_2^{\mathrm{ord}}\big(\Gamma_0(Np)\big)$.*

Throughout $\Lambda = \mathbb{Z}_p[[\Gamma]]$ denotes the completed group algebra of $\Gamma = 1 + p\mathbb{Z}_p$, and $\mathcal{L} = \mathrm{Frac}(\Lambda)$ its field of fractions. There are non-canonical isomorphisms $\Lambda \cong \mathbb{Z}_p[[X]]$ given by sending a topological generator $u_0 \in \Gamma$ to the element $1 + X$. In fact the $\mathbb{Z}_p$-linear extension of the map $\sigma_k : u_0 \mapsto u_0^{k-2}$ transforms Λ into the Iwasawa functions $\mathcal{A}_{\mathbb{Z}_p} = \mathbb{Z}_p\langle\langle k \rangle\rangle$, convergent everywhere on the closed unit disk.

Under the above hypothesis, there exists a unique Λ-adic eigenform $\mathbf{f} \in \Lambda[[q]]$ lifting the cusp form $\mathbf{f}_2$ at weight two; furthermore

$$\mathbf{f}_k \ := \ \sum_{n=1}^{\infty} \sigma_k\big(a_n(\mathbf{f})\big) q^n \ \in \ \mathcal{S}_k^{\mathrm{ord}}\big(\Gamma_1(Np^r)\big)$$

is a p-stabilised eigenform of weight k and character ω^{2-k}, for all integers $k \geq 2$. Hida and Mazur-Wiles [H1,H2,MW] attached a continuous Galois representation

$$\rho_\infty : G_{\mathbb{Q}} \ \longrightarrow \ \mathrm{GL}_2(\Lambda) \ = \ \mathrm{Aut}_\Lambda(\mathbb{T}_\infty)$$

interpolating Deligne's p-adic representations for every eigenform in the family. The rank two lattice $\mathbb{T}_\infty$ is always free over Λ, unramified outside of Np, and the characteristic polynomial of $\rho_\infty\big(\mathrm{Frob}_l\big)$ will be $1 - a_l(\mathbf{f})x + l\big\langle l\big\rangle x^2$ for primes $l \nmid Np$. If we restrict to a decomposition group above p,

$$\rho_\infty \otimes_\Lambda \mathcal{A}_{\mathbb{Z}_p}\Big|_{G_{\mathbb{Q}_p}} \ \sim \ \begin{pmatrix} \chi_{\mathrm{cy}} < \chi_{\mathrm{cy}} >^{k-2} \phi_k^{-1} & * \\ 0 & \phi_k \end{pmatrix} \quad \text{where } \phi_k : G_{\mathbb{Q}_p}/I_p \to \mathbb{Z}_p^\times$$

is the unramified character sending Frob_p to the eigenvalue of U_p at weight k.

QUESTION. *Can one make a Tamagawa number conjecture for the Λ-adic form* $\mathbf{f}$*, which specialises at arithmetic primes to each Bloch-Kato conjecture?*

The answer turns out to be a cautious 'Yes', provided one is willing to work with p-primary components of the usual suspects. In this article, we shall explain the specialisation to weight two (i.e. elliptic curves) subject to a couple of simplifying assumptions. The general case will be treated in a forthcoming work, and includes the situation where the nearly-ordinary deformation ring $\mathcal{R}_E$ is a non-trivial finite, flat extension of Λ. Let's begin by associating local points to ρ_∞...

For each pair of integers $m, r \in \mathbb{N}$, the multiplication by p^m endomorphism on the p-divisible group $J_r = \mathrm{jac}\ X_1(Np^r)$ induces a tautological exact sequence $0 \to J_r[p^m] \to J_r \stackrel{\times p^m}{\to} J_r \to 0$. Upon taking Galois invariants, we obtain a long exact sequence in $G_{\mathbb{Q}_p}$-cohomology

$$\begin{aligned} 0 \to J_r(\mathbb{Q}_p)[p^m] \to J_r(\mathbb{Q}_p) &\stackrel{\times p^m}{\to} J_r(\mathbb{Q}_p) \\ &\stackrel{\partial_{r,m}}{\to} H^1(\mathbb{Q}_p, J_r[p^m]) \to H^1(\mathbb{Q}_p, J_r)[p^m] \to 0. \end{aligned}$$

The boundary map $\partial_{r,m}$ injects $J_r(\mathbb{Q}_p)\big/p^m$ into $H^1(\mathbb{Q}_p, J_r[p^m])$, so applying the functors $\varprojlim_m$ and $\varprojlim_r$ yields a level-compatible Kummer map

$$\varprojlim_{r,m} \partial_{r,m} : J_\infty(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Z}_p \ \hookrightarrow \ H^1\big(\mathbb{Q}_p, \mathrm{Ta}_p(J_\infty)\big) \quad \text{which is Hecke-equivariant;}$$

here J_∞ denotes the limit $\varprojlim_r$ jac $X_1(Np^r)$ induced from $X_1(Np^{r+1}) \overset{\pi_p}{\twoheadrightarrow} X_1(Np^r)$.

For a compact Λ-module M, we define its twisted dual $A_M := \mathrm{Hom}_{\mathrm{cont}}\big(M, \mu_{p^\infty}\big)$. Recall that Hida [H1] cuts $\mathbb{T}_\infty$ out of the massive Galois representation $\mathrm{Ta}_p(J_\infty)$ using idempotents $\mathbf{e}_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ and $\mathbf{e}_{\mathrm{prim}}$ living in the abstract Hecke algebra (the latter is the projector to the p-normalised primitive part, and in general exists only after extending scalars to $\mathcal{L}$).

Definition 1.1. *(a) We define $X(\mathbb{Q}_p)$ to be the pre-image of the local points*

$$\mathbf{e}_{\mathrm{prim}}\cdot\left(\left(\mathbf{e}_{\mathrm{ord}}\cdot\varprojlim_{r,m}\partial_{r,m}\Big(J_\infty(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Z}_p\Big)\right)\otimes_\Lambda\mathcal{L}\right)$$

under the canonical homomorphism $H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\xrightarrow{-\otimes 1}H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\otimes_\Lambda\mathcal{L}$.

(b) We define the dual group $X^D(\mathbb{Q}_p)$ to be the orthogonal complement

$$\Big\{\mathbf{x}\in H^1\big(\mathbb{Q}_p,A_{\mathbb{T}_\infty}\big)\quad\textit{such that}\;\; inv_{\mathbb{Q}_p}\big(X(\mathbb{Q}_p)\cup\mathbf{x}\big)\;=\;0\Big\}$$

under Pontrjagin duality $H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\times H^1\big(\mathbb{Q}_p,A_{\mathbb{T}_\infty}\big)\;\to\;H^2\big(\mathbb{Q}_p,\mu_{p^\infty}\big)\;\cong\;\mathbb{Q}_p/\mathbb{Z}_p$.

The local condition $X(\mathbb{Q}_p)$ will be Λ-saturated inside its ambient cohomology group. These groups were studied by the author and Smith in [DS], and are intimately connected to the behaviour of big dual exponential maps for the family.

Let Σ denote a finite set containing p and primes dividing the conductor N_E. Write $\mathbb{Q}_\Sigma$ for the maximal algebraic extension of the rationals, unramified outside the set of bad places $\Sigma\cup\{\infty\}$. Our primary object of study is the big Selmer group

$$\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty):=\mathrm{Ker}\left(H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},A_{\mathbb{T}_\infty}\big)\overset{\oplus\mathrm{res}_l}{\longrightarrow}\bigoplus_{l\neq p}H^1\big(\mathbb{Q}_l,A_{\mathbb{T}_\infty}\big)\oplus\frac{H^1\big(\mathbb{Q}_p,A_{\mathbb{T}_\infty}\big)}{X^D(\mathbb{Q}_p)}\right)$$

which is a discrete module over the local ring Λ.
For each arithmetic point in $\mathrm{Spec}(\Lambda)^{\mathrm{alg}}$, the Λ-adic object $\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)$ interpolates the Bloch-Kato Selmer groups associated to the p-stabilisations $\mathbf{f}_k$ of weight $k\geq 2$. At $k=2$ it should encode the Birch and Swinnerton-Dyer formulae, up to some easily computable fudge-factors.

Proposition 1.2. *(a) The Pontrjagin dual*

$$\widehat{Sel_{\mathbb{Q}}(\rho_\infty)} = Hom_{\text{cont}}\big(Sel_{\mathbb{Q}}(\rho_\infty),\ \mathbb{Q}/\mathbb{Z}\big)$$

is a finitely-generated Λ-module;

(b) If $L(E,1) \neq 0$ then $\widehat{Sel_{\mathbb{Q}}(\rho_\infty)}$ is Λ-torsion, i.e. $Sel_{\mathbb{Q}}(\rho_\infty)$ is Λ-cotorsion.

In general, one can associate a characteristic element to $\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)$ via

$$Ш_{\mathbb{Q}}(\rho_\infty) \ := \ \mathrm{char}_\Lambda\Big(\mathrm{Hom}_{\text{cont}}\Big(\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\big/_{\Lambda\text{-div}}\ ,\ \mathbb{Q}/\mathbb{Z}\Big)\Big)$$

where $\big/_{\Lambda\text{-div}}$ indicates we have quotiented by the maximal $\mathfrak{m}_\Lambda$-divisible submodule; equivalently $Ш_{\mathbb{Q}}(\rho_\infty)$ is a generator of the characteristic ideal of $\mathrm{Tors}_\Lambda\Big(\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}\Big)$. If the L-function doesn't vanish at $s=1$ then by 1.2(b), the Pontrjagin dual $\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}$ is already pseudo-isomorphic to a compact Λ-module of the form

$$\bigoplus_{i=1}^{t} \mathbb{Z}/p^{\mu_i}\mathbb{Z} \ \oplus \ \bigoplus_{j=1}^{s} \Lambda/F_j^{e_j}\Lambda$$

where the F_j's are irreducible distinguished polynomials, and all of the $\mu_i, e_j \geq 0$. In this particular case $Ш_{\mathbb{Q}}(\rho_\infty)$ will equal $p^{\mu_1+\cdots\mu_t} \times \prod_{j=1}^s F_j^{e_j}$ modulo $\Lambda^\times$, and so annihilates the whole of $\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}$.

Definition/Lemma 1.3. *For each prime $l \neq p$ and integer weight $k \geq 2$, we set*

$$Tam_l(\rho_\infty; k) \ := \ \#Tors_\Lambda\Big(H^1\big(I_l, \mathbb{T}_\infty\big)\Big)^{\mathrm{Frob}_l=1} \otimes_{\Lambda,\sigma_k} \mathbb{Z}_p \ \in \ p^{\mathbb{N}\cup\{0\}}\ .$$

Then at weight two,

$$\prod_{l\neq p} Tam_l(\rho_\infty; 2) \quad \textit{divides the p-part of} \quad \prod_{l\neq p}\big[\mathcal{C}^{\min}(\mathbb{Q}_l) : \mathcal{C}_0^{\min}(\mathbb{Q}_l)\big]$$

where $\mathcal{C}^{\min}_{/\mathbb{Q}}$ refers to the $\mathbb{Q}$-isogenous elliptic curve of Stevens, for which every optimal parametrisation $X_1(Np) \twoheadrightarrow E$ admits a factorisation $X_1(Np) \to \mathcal{C}^{\min} \to E$.

These mysterious Λ-adic Tamagawa numbers control the specialisation of our big Tate-Shafarevich group Ш at arithmetic points. In particular, for the weight $k=2$ they occur in the leading term of $Ш_{\mathbb{Q}}(\rho_\infty)$ viewed as an element of $\Lambda \cong \mathbb{Z}_p[[X]]$. It was conjectured in [St] that $\mathcal{C}^{\min}$ is the same elliptic curve for which the Manin constant associated to $X_1(Np) \twoheadrightarrow \mathcal{C}^{\min}$ is ± 1. Cremona pointed out the Tamagawa factors $[\mathcal{C}^{\min}(\mathbb{Q}_l) : \mathcal{C}_0^{\min}(\mathbb{Q}_l)]$ tend to be smaller than the $[E(\mathbb{Q}_l) : E_0(\mathbb{Q}_l)]$'s.

To state the simplest version of our result, we shall assume the following:

Hypothesis(Frb). *Either (i)* $p \nmid N_E$ *and* $a_p(E) \neq +1$,
or (ii) $p \| N_E$ *and* $a_p(E) = -1$
or (iii) $p \| N_E$ *and* $a_p(E) = +1$, $p \nmid \mathrm{ord}_p\Big(\mathbf{q}_{\mathrm{Tate}}(\mathcal{C}^{\min})\Big)$.

Note that in case (iii), the condition that p does not divide the valuation of the Tate period $\mathbf{q}_{\mathrm{Tate}}(\mathcal{C}^{\min})$ ensures the p-part of $[\mathcal{C}^{\min}(\mathbb{Q}_p) : \mathcal{C}_0^{\min}(\mathbb{Q}_p)]$ is trivial.

Theorem 1.4. *Assume both* ($\mathcal{R}_E$) *and (Frb) hold. If* $L(E,1) \neq 0$, *then*

$$\sigma_2\Big(Ш_{\mathbb{Q}}(\rho_\infty)\Big) \equiv \mathcal{L}_p^{\mathrm{wt}}(E) \times [E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)] \prod_{l \neq p} \frac{[E(\mathbb{Q}_l) : E_0(\mathbb{Q}_l)]}{\mathrm{Tam}_l(\rho_\infty; 2)} \times \frac{\#Ш_{\mathbb{Q}}(E)}{\#E(\mathbb{Q})^2}$$

modulo $\mathbb{Z}_p^\times$, *where the* $\mathcal{L}^{\mathrm{wt}}$*-invariant at weight two is defined to be*

$$\mathcal{L}_p^{\mathrm{wt}}(E) \quad := \quad \frac{\int_{E(\mathbb{R})} \omega_E}{\int_{\mathcal{C}^{\min}(\mathbb{R})} \omega_{\mathcal{C}^{\min}}} \times \frac{\#\mathcal{C}^{\min}(\mathbb{Q})}{\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma} .$$

In particular, the Γ*-coinvariants of* $A_{\mathbb{T}_\infty}(\mathbb{Q}) = H^0\big(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{\mathbb{T}_\infty}\big)$ *are always finite, and the denominator* $\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma$ *divides into* $\#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]$.

This equation is a special case of a more general Tamagawa number formalism. Whilst none of the assumptions ($\mathcal{R}_E$), (Frb) and $L(E,1) \neq 0$ are actually necessary, the full result requires a weight-regulator term, the relative covolume of $X(\mathbb{Q}_p)$ and various other additional factors – we won't consider these complications here.

Example 1.5. Consider the modular curve $E = X_0(11)$ given by the equation

$$E \ : \ y^2 \ + \ y \ = \ x^3 \ - \ x^2 \ - \ 10x \ - \ 20 \, .$$

The Tamagawa number of E at the bad prime 11 equals 5, whereas elsewhere the curve has good reduction. Let's break up the calculation into three parts: (a) Avoiding the supersingular prime numbers 19 and 29, one checks for every good ordinary prime $7 \leq p \leq 97$ that both of the hypotheses ($\mathcal{R}_E$) and (Frb) hold true (to check the former, we verified that there are no congruences modulo p between f_E and any newform at level $11p$). Now by Theorem 1.4,

$$\sigma_2\Big(Ш_{\mathbb{Q}}(\rho_\infty)\Big) \quad \equiv \quad \frac{\mathcal{L}_p^{\mathrm{wt}}(E) \times 5 \times \#Ш_{\mathbb{Q}}(E)}{\mathrm{Tam}_{11}(\rho_\infty; 2) \times 5^2} \quad \equiv \quad 1 \quad \text{modulo } \mathbb{Z}_p^\times$$

since the $\mathcal{L}_p^{\mathrm{wt}}$-invariant is a p-adic unit, and the size of $Ш_{\mathbb{Q}}(E)$ is equal to one.

(b) At the prime $p = 11$ the elliptic curve E has split multiplicative reduction. The optimal curve $\mathcal{C}^{\min}$ is $X_1(11)$ whose Tamagawa number is trivial, hence so is $\mathrm{Tam}_{11}(\rho_\infty;2)$. Our theorem implies $\sigma_2\Big(Ш_{\mathbb{Q}}(\rho_\infty)\Big)$ must then be an 11-adic unit.

(c) When $p = 5$ the curve E fails to satisfy (Frb) as the Hecke eigenvalue $a_5(E) = 1$. Nevertheless the deformation ring $\mathcal{R}_E \cong \Lambda$, and E has good ordinary reduction. Applying similar arguments to the proof of 1.4, one can show that

$$\left|\sigma_2\Big(Ш_{\mathbb{Q}}(\rho_\infty)\Big)\right|_5^{-1} \quad \text{divides} \quad \frac{\#\widetilde{X_1(11)}(\mathbb{F}_5)[5^\infty] \;\times\; \#Ш_{\mathbb{Q}}\big(X_1(11)\big)[5^\infty]}{\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma \;\times\; \#X_1(11)(\mathbb{Q})[5^\infty]}\,.$$

The right-hand side equals one, since $X_1(11)(\mathbb{Q})$ and the reduced curve $\widetilde{X_1(11)}(\mathbb{F}_5)$ possess a non-trivial 5-torsion point. As the left-hand side is 5-integral, clearly $\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma = 1$ and it follows that $\sigma_2\Big(Ш_{\mathbb{Q}}(\rho_\infty)\Big)$ is a 5-adic unit.

COROLLARY 1.6. *For all prime numbers p such that $5 \le p \le 97$ and $a_p\big(X_0(11)\big) \neq 0$,*

the μ^{wt}-invariant associated to the Hida deformation of $\mathrm{Sel}_{\mathbb{Q}}\big(X_0(11)\big)[p^\infty]$ is zero.

In fact the μ^{wt}-invariant is probably zero at all primes p for which $X_0(11)$ has good ordinary reduction, but we need a more general formula than 1.4 to prove this.

2. Outline of the Proof of Theorem 1.4

We begin with some general comments.

The rank two module $\mathbb{T}_\infty \otimes_{\Lambda,\sigma_2} \mathbb{Z}_p$ is isomorphic to the dual of $H^1_{\text{ét}}\big(\overline{E},\mathbb{Z}_p\big)$, in general only after tensoring by $\mathbb{Q}_p$. Consider instead the arithmetic pro-variety $\hat{X} = \varprojlim_{r\ge1} X_1(Np^r)$ endowed with its canonical $\mathbb{Q}$-structure. The specialisation $(\sigma_2)_* : \mathbb{T}_\infty \twoheadrightarrow \big(\mathbb{T}_\infty\big)_\Gamma \hookrightarrow \mathrm{Ta}_p\Big(\mathrm{jac}\ X_1(Np)\Big)$ is clearly induced from $\hat{X} \overset{\mathrm{proj}}{\longrightarrow} X_1(Np)$. It follows from [St, Th 1.9] that $\mathbb{T}_\infty \otimes_{\Lambda,\sigma_2} \mathbb{Z}_p \cong \mathrm{Ta}_p(\mathcal{C}^{\min})$ on an integral level, where $\mathcal{C}^{\min}$ denotes the same elliptic curve occurring as a subvariety of jac $X_1(Np)$, alluded to earlier in 1.3.

Taking twisted duals of $0 \to \mathbb{T}_\infty \overset{u_0-1}{\to} \mathbb{T}_\infty \to \mathrm{Ta}_p(\mathcal{C}^{\min}) \to 0$, we obtain a corresponding short exact sequence

$$0 \;\to\; \mathrm{Hom}_{\mathrm{cont}}\Big(\mathrm{Ta}_p(\mathcal{C}^{\min}), \mu_{p^\infty}\Big) \;\to\; A_{\mathbb{T}_\infty} \overset{u_0-1}{\to} A_{\mathbb{T}_\infty} \;\to\; 0$$

of discrete Λ-modules. The Weil pairing on the optimal curve $\mathcal{C}^{\min}$ implies that $\mathrm{Hom}_{\mathrm{cont}}\Big(\mathrm{Ta}_p(\mathcal{C}^{\min}), \mu_{p^\infty}\Big) \cong \mathcal{C}^{\min}[p^\infty]$. We thus deduce that $\mathrm{Ta}_p(\mathcal{C}^{\min}) \ncong \mathrm{Ta}_p(E)$ if and only if there exists a cyclic p^n-isogeny defined over $\mathbb{Q}$, between the two elliptic curves E and $\mathcal{C}^{\min}$ (note this can only happen when the prime p is very small).

Let G denote either $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$, or a decomposition group $\mathrm{Gal}(\overline{\mathbb{Q}_l}/\mathbb{Q}_l)$ at some prime number l. For indices $j = 0, 1, 2$ there are induced exact sequences

$$0 \to H^j\big(G, A_{\mathbb{T}_\infty}\big) \otimes_{\Lambda,\sigma_2} \mathbb{Z}_p \to H^{j+1}\big(G, \mathcal{C}^{\min}[p^\infty]\big) \to H^{j+1}\big(G, A_{\mathbb{T}_\infty}\big)^\Gamma \to 0$$

and in continuous cohomology,

$$0 \to H^j\big(G, \mathbb{T}_\infty\big) \otimes_{\Lambda,\sigma_2} \mathbb{Z}_p \to H^j\big(G, \mathrm{Ta}_p(\mathcal{C}^{\min})\big) \to H^{j+1}\big(G, \mathbb{T}_\infty\big)^\Gamma \to 0\,.$$

From now on, we'll just drop the '$\ _{\sigma_2}$' from the tensor product notation altogether.

Remark: Our strategy is to compare $\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)$ with the p-primary Selmer group for $\mathcal{C}^{\min}$ over the rationals. We can then use the Isogeny Theorem to exchange the optimal curve $\mathcal{C}^{\min}$ with the strong Weil curve E.

For each prime $l \neq p$, we claim there is a natural map

$$\delta_l \ :\ \frac{H^1\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)}{H^1_{\mathrm{nr}}\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)} \longrightarrow H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)^\Gamma\ ;$$

here $H^1_{\mathrm{nr}}\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)$ denotes the orthogonal complement to the p-saturation of $H^1\left(\mathrm{Frob}_l, \mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}\right)$ inside $H^1\big(\mathbb{Q}_l, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)$. To see why this map exists, note that $H^1\big(\mathbb{Q}_l, \mathbb{T}_\infty\big)$ is Λ-torsion, hence $H^1\big(\mathbb{Q}_l, \mathbb{T}_\infty\big) \otimes_\Lambda \mathbb{Z}_p$ is p^∞-torsion and must lie in any p-saturated subgroup of $H^1\big(\mathbb{Q}_l, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)$. Consequently the Γ-coinvariants

$$H^1\big(\mathbb{Q}_l, \mathbb{T}_\infty\big)_\Gamma \hookrightarrow \text{ the } p\text{-saturation of } H^1\left(\mathrm{Frob}_l, \mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}\right)\ ,$$

and then dualising we obtain δ_l.

Let's now consider what happens when $l = p$. In [DS, Th 2.1] we identified the family of local points $X(\mathbb{Q}_p)$ with the cohomology subgroup

$$H^1_{\mathcal{G}}\big(\mathbb{Q}_p, \mathbb{T}_\infty\big) \ :=\ \mathrm{Ker}\left(H^1_{\mathrm{cont}}\big(\mathbb{Q}_p, \mathbb{T}_\infty\big) \overset{(-\otimes 1)\otimes 1}{\longrightarrow} H^1_{\mathrm{cont}}\Big(\mathbb{Q}_p, \mathbb{T}_\infty \otimes \mathbb{B}_{\mathrm{dR}}\Big) \otimes_\Lambda \mathcal{L}\right)$$

where $\mathbb{B}_{\mathrm{dR}}$ denotes Iovita and Stevens' period ring. In particular, we showed that

$$X(\mathbb{Q}_p)_\Gamma \ =\ H^1_{\mathcal{G}}\big(\mathbb{Q}_p, \mathbb{T}_\infty\big) \otimes_\Lambda \mathbb{Z}_p \hookrightarrow H^1_g\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big) \cong \mathcal{C}^{\min}(\mathbb{Q}_p)\widehat{\otimes}\mathbb{Z}_p$$

the latter isomorphism arising from [BK, Section 3]. Dualising the above yields

$$\delta_p \ :\ \frac{H^1\big(\mathbb{Q}_p, \mathcal{C}^{\min}[p^\infty]\big)}{\mathcal{C}^{\min}(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \left(\frac{H^1\big(\mathbb{Q}_p, A_{\mathbb{T}_\infty}\big)}{X^D(\mathbb{Q}_p)}\right)^\Gamma$$

because $H^1_g\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)^\perp \cong \mathcal{C}^{\min}(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $X(\mathbb{Q}_p)^\perp = X^D(\mathbb{Q}_p)$.

LEMMA 2.1. *For all prime numbers $l \in \Sigma$, the kernel of δ_l is a finite p-group.*

We defer the proof until the next section, but for $l \neq p$ it's straightforward. This discussion may be neatly summarised in the following commutative diagram, with left-exact rows:

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Sel}_{\mathbb{Q}}(\mathcal{C}^{\min})[p^\infty] & \to & H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big) & \xrightarrow{\lambda_0} & \bigoplus_{l\in\Sigma} \dfrac{H^1\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)}{H^1_\star\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)} \\
 & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\oplus\delta_l} \\
0 & \to & \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)^\Gamma & \to & H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\Big)^\Gamma & \xrightarrow{\lambda_\infty} & \bigoplus_{l\in\Sigma} \left(\dfrac{H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)}{H^1_\star\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)}\right)^\Gamma .
\end{array}$$

FIGURE 1.

At primes $l \neq p$ the notation $H^1_\star$ represents H^1_{nr}. When $l = p$ we have written $H^1_\star\big(\mathbb{Q}_l, \mathcal{C}^{\min}[p^\infty]\big)$ for the points $\mathcal{C}^{\min}(\mathbb{Q}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p$, and analogously $H^1_\star\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)$ in place of our family of local points $X^D(\mathbb{Q}_p)$.

Applying the Snake Lemma to the above, we obtain a long exact sequence

$$0 \ \to\ \mathrm{Ker}(\alpha)\ \to\ \mathrm{Ker}(\beta)\ \to\ \mathrm{Im}(\lambda_0) \cap \left(\bigoplus_{l\in\Sigma} \mathrm{Ker}(\delta_l)\right)\ \to\ \mathrm{Coker}(\alpha)\ \to\ 0$$

as the map β is surjective. The kernel of β equals $H^0\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\big) \otimes_\Lambda \mathbb{Z}_p$ i.e., the Γ-coinvariants $H^1\Big(\Gamma,\ H^0\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\big)\Big)$. As Γ is pro-cyclic and $A_{\mathbb{T}_\infty}$ is discrete,

$$\begin{aligned}
\#H^1\Big(\Gamma,\ H^0\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\big)\Big) &\leq\ \#H^0\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ H^0(\Gamma, A_{\mathbb{T}_\infty})\Big) \\
&=\ \#H^0\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{T}_\infty \otimes_\Lambda \mathbb{Z}_p\ ,\ \mu_{p^\infty}\big)\Big) \\
&=\ \#H^0\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big)\ \ =\ \#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]\ .
\end{aligned}$$

In other words, the size of $\mathrm{Ker}(\beta)$ is bounded by $\#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]$. By a well-known theorem of Mazur on torsion points, the latter quantity is at most 16.

REMARKS: (i) Let's recall that for any elliptic curve A over the rational numbers, its Tate-Shafarevich group can be defined by the exactness of

$$0\ \to\ A(\mathbb{Q}) \otimes \mathbb{Q}/\mathbb{Z}\ \to\ H^1(\mathbb{Q}, A)\ \to\ Ш_{\mathbb{Q}}(A)\ \to\ 0\ .$$

(ii) Lemma 2.1 implies every term occurring in our Snake Lemma sequence is finite, and as a direct consequence $\mathrm{Sel}_{\mathbb{Q}}(\mathcal{C}^{\min})[p^\infty] \xrightarrow{\alpha} \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)^\Gamma$ is a quasi-isomorphism. The coinvariants $\Big(\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}\Big)_\Gamma$ must then be of finite type over

$\mathbb{Z}_p$, Nakayama's lemma forces $\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}$ to be of finite type over Λ, and Proposition 1.2(a) follows.

(iii) Assume further that $L(E,1) \neq 0$. By work of Kolyvagin and later Kato [Ka], both $E(\mathbb{Q})$ and $\text{Ш}_{\mathbb{Q}}(E)$ are finite. Since $\mathcal{C}^{\min}$ is $\mathbb{Q}$-isogenous to E, clearly the Mordell-Weil and Tate-Shafarevich groups of the optimal curve must also be finite. Equivalently $\#\mathrm{Sel}_{\mathbb{Q}}(\mathcal{C}^{\min}) < \infty$, whence

$$\mathrm{rank}_\Lambda\Big(\widehat{\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)}\Big) \ \leq$$
$$\leq \ \mathrm{corank}_{\mathbb{Z}_p}\Big(\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)^\Gamma\Big) \ = \ \mathrm{corank}_{\mathbb{Z}_p}\Big(\mathrm{Sel}_{\mathbb{Q}}(\mathcal{C}^{\min})[p^\infty]\Big) \ = \ 0.$$

It follows that $\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)$ is Λ-cotorsion, and Proposition 1.2(b) is established. The special value of $\text{Ш}_{\mathbb{Q}}(\rho_\infty)$ at σ_2 is determined (modulo p-adic units) by the Γ-Euler characteristic of $\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)$, namely

$$\chi\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \ := \ \prod_{j=0}^{\infty}\Big(\#H^j\big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\big)\Big)^{(-1)^j} \ = \ \frac{\#H^0\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big)}{\#H^1\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big)}$$

as Γ has cohomological dimension one.

After a brisk diagram chase around Figure 1, we discover that

$$\chi\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \ = \ \frac{\#\mathrm{Sel}_{\mathbb{Q}}(\mathcal{C}^{\min})[p^\infty] \ \times \ \#\Big(\mathrm{Im}(\lambda_0) \cap \big(\bigoplus_{l\in\Sigma}\mathrm{Ker}(\delta_l)\big)\Big)}{\#\mathrm{Ker}(\beta) \ \times \ \#H^1\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big)}$$
$$= \ \frac{\#\text{Ш}_{\mathbb{Q}}(\mathcal{C}^{\min})[p^\infty] \quad \times \quad \prod_{l\in\Sigma}\#\mathrm{Ker}(\delta_l)}{\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma \ \times \ \#H^1\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \ \times \ \prod_{l\in\Sigma}\Big[\mathrm{Ker}(\delta_l) : \mathrm{Im}(\lambda_0)\cap\mathrm{Ker}(\delta_l)\Big]}.$$

Proposition 2.2.
(a) $\#\mathit{Ker}(\delta_l) \ = \ \Big|\big[\mathcal{C}^{\min}(\mathbb{Q}_l) : \mathcal{C}_0^{\min}(\mathbb{Q}_l)\big]\Big|_p^{-1} \times \Big|\mathit{Tam}_l(\rho_\infty;2)\Big|_p$ *if* $l \neq p$;

(b) $\#\mathit{Ker}(\delta_p) = 1$ *and* $\Big|\big[\mathcal{C}^{\min}(\mathbb{Q}_p) : \mathcal{C}_0^{\min}(\mathbb{Q}_p)\big]\Big|_p = 1$ *if Hypothesis(Frb) holds for* E.

Proposition 2.3. *If* $L(E,1) \neq 0$, *then*

$$\#H^1\Big(\Gamma, \mathit{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \ \times \ \prod_{l\in\Sigma}\Big[\mathit{Ker}(\delta_l) : \mathit{Im}(\lambda_0)\cap \mathit{Ker}(\delta_l)\Big] \quad = \quad \#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]\ .$$

The former result is proved in the next section, and the latter assertion in §4. Substituting them back into our computation of the Γ-Euler characteristic,

$$\chi\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \ \approx \ \frac{\#\text{Ш}_{\mathbb{Q}}(\mathcal{C}^{\min}) \ \times \ \prod_{l\in\Sigma}\big[\mathcal{C}^{\min}(\mathbb{Q}_l) : \mathcal{C}_0^{\min}(\mathbb{Q}_l)\big]}{\#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma \ \times \ \#\mathcal{C}^{\min}(\mathbb{Q}) \ \times \ \prod_{l\in\Sigma-\{p\}}\mathrm{Tam}_l(\rho_\infty;2)}$$

where the notation $x \approx y$ is employed whenever $x = uy$ for some unit $u \in \mathbb{Z}_p^\times$. Setting $\mathcal{L}_p^{\mathrm{wt},\dagger}(E) := \#\mathcal{C}^{\min}(\mathbb{Q}) \Big/ \#A_{\mathbb{T}_\infty}(\mathbb{Q})_\Gamma$, the above can be rewritten as

$$\frac{\mathcal{L}_p^{\mathrm{wt},\dagger}(E)}{\prod_{l\in\Sigma-\{p\}} \mathrm{Tam}_l(\rho_\infty;2)} \times \frac{\#\text{Ш}_{\mathbb{Q}}(\mathcal{C}^{\min}) \times \prod_{l\in\Sigma}\left[\mathcal{C}^{\min}(\mathbb{Q}_l):\mathcal{C}_0^{\min}(\mathbb{Q}_l)\right]}{\#\mathcal{C}^{\min}(\mathbb{Q})^2}.$$

Cassels' Isogeny Theorem allows us to switch $\mathcal{C}^{\min}$ with the isogenous curve E, although this scales the formula by the ratio of periods $\int_{E(\mathbb{R})}\omega_E \Big/ \int_{\mathcal{C}^{\min}(\mathbb{R})}\omega_{\mathcal{C}^{\min}}$. Observing that $\sigma_2\Big(\text{Ш}_{\mathbb{Q}}(\rho_\infty)\Big) \approx \chi\Big(\Gamma,\mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big)$, Theorem 1.4 is finally proved.

3. Computing the Local Kernels

We now examine the kernels of the homorphisms δ_l for all prime numbers $l \in \Sigma$. Let's start by considering $l \neq p$. By its very definition, δ_l is the dual of

$$\widehat{\delta_l} \;:\; H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)\otimes_\Lambda \mathbb{Z}_p \;\hookrightarrow\; H^1_{\mathrm{nr}}\big(\mathbb{Q}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\big)$$

where $H^1_{\mathrm{nr}}(\cdots)$ denotes the p-saturation of $H^1\left(\mathrm{Frob}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}\right) \cong \frac{\mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}}{(\mathrm{Frob}_l-1)}$.
The key term we need to calculate is

$$\#\mathrm{Ker}(\delta_l) \;=\; \#\mathrm{Coker}\big(\widehat{\delta_l}\big) \;=\; \Big[H^1_{\mathrm{nr}}\big(\mathbb{Q}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\big) : H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)\otimes_\Lambda \mathbb{Z}_p\Big].$$

Firstly, the sequence $0 \to \mathbb{T}_\infty^{I_l}\otimes_\Lambda \mathbb{Z}_p \to \mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l} \to H^1\big(I_l,\mathbb{T}_\infty\big)^\Gamma \to 0$ is exact, and $\mathbb{T}_\infty^{I_l}\otimes_\Lambda \mathbb{Z}_p$ coincides with $\big(\mathbb{T}_\infty\otimes_\Lambda \mathbb{Z}_p\big)^{I_l} = \mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}$ since the Galois action and diamond operators commute on $\mathbb{T}_\infty$. As a corollary $H^1\big(I_l,\mathbb{T}_\infty\big)^\Gamma$ must be zero.
The group $\mathrm{Gal}\big(\mathbb{Q}_l^{\mathrm{unr}}/\mathbb{Q}_l\big)$ is topologically generated by Frobenius, hence

$$\begin{aligned} H^1\Big(\mathrm{Frob}_l,\mathbb{T}_\infty^{I_l}\Big)_\Gamma &\cong \left(\frac{\mathbb{T}_\infty^{I_l}}{(\mathrm{Frob}_l-1).\mathbb{T}_\infty^{I_l}}\right)\otimes_\Lambda \mathbb{Z}_p \\ &= \left(\frac{\big(\mathbb{T}_\infty\otimes_\Lambda \mathbb{Z}_p\big)^{I_l}}{(\mathrm{Frob}_l-1).\big(\mathbb{T}_\infty\otimes_\Lambda \mathbb{Z}_p\big)^{I_l}}\right) \cong H^1\Big(\mathrm{Frob}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}\Big). \end{aligned}$$

Since the local cohomology $H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)$ is always Λ-torsion when the prime $l \neq p$, inflation-restriction provides us with a short exact sequence

$$0 \to H^1\Big(\mathrm{Frob}_l,\mathbb{T}_\infty^{I_l}\Big) \overset{\mathrm{infl}}{\to} H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big) \overset{\mathrm{rest}}{\to} \mathrm{Tors}_\Lambda\Big(H^1\big(I_l,\mathbb{T}_\infty\big)^{\mathrm{Frob}_l}\Big) \to 0\,.$$

The boundary map $\mathrm{Tors}_\Lambda\Big(H^1\big(I_l,\mathbb{T}_\infty\big)^{\mathrm{Frob}_l}\Big)^\Gamma \to H^1\Big(\mathrm{Frob}_l,\mathbb{T}_\infty^{I_l}\Big)_\Gamma$ trivialises because $H^1\big(I_l,\mathbb{T}_\infty\big)^\Gamma = 0$, so the Γ-coinvariants $H^1\Big(\mathrm{Frob}_l,\mathbb{T}_\infty^{I_l}\Big)_\Gamma$ inject into $H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)_\Gamma$ under inflation.

We deduce that there is a commutative diagram, with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
H^1\Big(\mathrm{Frob}_l,\mathbb{T}_\infty^{I_l}\Big)_\Gamma \arrow[r, hook, "\mathrm{infl}"] \arrow[d, equal] & H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)_\Gamma \arrow[r, two heads, "\mathrm{rest}"] \arrow[d] & \mathrm{Tors}_\Lambda\Big(H^1\big(I_l,\mathbb{T}_\infty\big)^{\mathrm{Frob}_l}\Big)_\Gamma \arrow[d, "\theta"'] \\
H^1\Big(\mathrm{Frob}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})^{I_l}\Big) \arrow[r, hook, "\mathrm{infl}"] & H^1_{\mathrm{nr}}\big(\mathbb{Q}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\big) \arrow[r, two heads, "\mathrm{rest}"] \arrow[d] & H^1\big(I_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\big)^{\mathrm{Frob}_l}[p^\infty] \arrow[d] \\
 & H^2\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)^\Gamma \arrow[r, phantom, "\cong"] \arrow[d] & \mathrm{Coker}(\theta) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Figure 2.

Remark: Using Figure 2 to compute indices, general nonsense informs us that

$$\begin{aligned}
\#\mathrm{Ker}(\delta_l) \quad &= \quad \Big[H^1_{\mathrm{nr}}\big(\mathbb{Q}_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\big) : H^1\big(\mathbb{Q}_l,\mathbb{T}_\infty\big)_\Gamma\Big] \quad = \quad \#\mathrm{Coker}(\theta) \\
&= \quad \frac{\#H^1\Big(I_l,\mathrm{Ta}_p(\mathcal{C}^{\min})\Big)^{\mathrm{Frob}_l}[p^\infty]}{\#\mathrm{Tors}_\Lambda\Big(H^1\big(I_l,\mathbb{T}_\infty\big)^{\mathrm{Frob}_l}\Big)_\Gamma} \quad \approx \quad \frac{\Big[\mathcal{C}^{\min}(\mathbb{Q}_l) : \mathcal{C}_0^{\min}(\mathbb{Q}_l)\Big]}{\mathrm{Tam}_l(\rho_\infty;2)} .
\end{aligned}$$

In one fell swoop this proves Proposition 2.2(a), Lemma 1.3 and half of Lemma 2.1.

Let's concentrate instead on $l=p$. The kernel of δ_p is dual to the cokernel of

$$\widehat{\delta_p} \ : \ H^1_{\mathcal{G}}\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\otimes_\Lambda \mathbb{Z}_p \quad\hookrightarrow\quad H^1_g\big(\mathbb{Q}_p,\mathrm{Ta}_p(\mathcal{C}^{\min})\big) .$$

Clearly the $\mathbb{Z}_p$-rank of $H^1_{\mathcal{G}}\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\otimes_\Lambda\mathbb{Z}_p$ is bounded below by the Λ-rank of $H^1_{\mathcal{G}}\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)$ which equals one, thanks to a specialisation argument in [DS, Th 3]. On the other hand

$$\mathrm{rank}_{\mathbb{Z}_p}\Big(H^1_g\big(\mathbb{Q}_p,\mathrm{Ta}_p(\mathcal{C}^{\min})\big)\Big) \quad = \quad \dim_{\mathbb{Q}_p}\Big(\mathcal{C}^{\min}(\mathbb{Q}_p)\,\widehat{\otimes}\,\mathbb{Q}_p\Big) \quad = \quad 1$$

because the formal group of $\mathcal{C}^{\min}_{/\mathbb{Z}_p}$ has semistable height one. We conclude that

$$\#\mathrm{Ker}(\delta_p) \quad = \quad \#\mathrm{Coker}\big(\widehat{\delta_p}\big) \quad = \quad \Big[H^1_g\big(\mathbb{Q}_p,\mathrm{Ta}_p(\mathcal{C}^{\min})\big) : H^1_{\mathcal{G}}\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)_\Gamma\Big]$$

must be finite, which completes the demonstration of Lemma 2.1.

Remarks: (i) For any de Rham $G_{\mathbb{Q}_p}$-representation V, Bloch and Kato [BK] define a dual exponential map

$$\exp_V^* \ : \ H^1\big(\mathbb{Q}_p, V\big) \ \longrightarrow \ \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V) \ := \ \Big(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+\Big)^{G_{\mathbb{Q}_p}}$$

whose kernel is $H^1_g\big(\mathbb{Q}_p, V\big)$. If V equals the p-adic representation $\mathrm{Ta}_p(\mathcal{C}^{\min})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, then the cotangent space $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V) \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} H^1_{\mathrm{dR}}\big(\mathcal{C}^{\min}/\mathbb{Q}\big)$ is a $\mathbb{Q}_p$-line, generated by a Néron differential $\omega_{\mathcal{C}^{\min}}$ on the optimal elliptic curve.

(ii) Applying $\exp_V^*$ above and then cupping with the dual basis $\omega^*_{\mathcal{C}^{\min}}$, we obtain a homomorphism

$$\exp_\omega^* \ : \ \frac{H^1\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)}{H^1_g\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)} \ \longrightarrow \ \Big(\mathrm{Ta}_p(\mathcal{C}^{\min}) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+\Big)^{G_{\mathbb{Q}_p}} \ \overset{-\cup\,\omega^*_{\mathcal{C}^{\min}}}{\longrightarrow} \ \mathbb{Q}_p$$

which sends Kato's zeta element [Ka, Th 13.1] to a non-zero multiple of $\frac{L_{Np}(\mathcal{C}^{\min},1)}{\Omega^+_{\mathcal{C}^{\min}}}$. In particular $L_{Np}(\mathcal{C}^{\min},1) = L_{Np}(E,1) \neq 0$, so the image of the composition $\exp_\omega^*$ must be a lattice $p^{n_1}\mathbb{Z}_p \subset \mathbb{Q}_p$ say. Let's abbreviate the quotient H^1/H^1_g by $H^1_{/g}$. Notice also that the $\mathbb{Z}_p$-rank of $H^1_{/g}\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big)$ equals one and the module is p^∞-torsion free, hence $\exp_\omega^*$ is injective.

In [De, Th 3.3] we showed the existence of a big dual exponential map

$$\mathrm{EXP}^*_{\mathbb{T}_\infty} \ : \ H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big) \ \longrightarrow \ \Lambda[1/p]\,, \qquad \mathrm{Ker}\big(\mathrm{EXP}^*_{\mathbb{T}_\infty}\big) \ = \ H^1_{\mathcal{G}}\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)$$

interpolating the standard $\exp^*$'s at the arithmetic points (we skip over the details). At weight two, $\mathrm{EXP}^*_{\mathbb{T}_\infty}$ modulo $u_0 - 1$ coincides with $\exp^*_\omega$ up to a non-zero scalar. The weight-deformation of Kato's zeta-element lives in $\mathrm{loc}_p\Big(H^1\big(\mathbb{Q}, \mathbb{T}_\infty\big)\Big)$, and via

$$H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big) \ \overset{\mathrm{mod}\ u_0-1}{\longrightarrow} \ H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma \ \overset{\mathrm{proj}}{\twoheadrightarrow} \ \frac{H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma}{H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma \cap H^1_g} \ \overset{\exp^*_\omega}{\hookrightarrow} \ \mathbb{Q}_p$$

is sent to the L-value $\frac{L_{Np}(\mathcal{C}^{\min},1)}{\Omega^+_{\mathcal{C}^{\min}}} \times$ (a Λ-adic period). In this case, the image of $H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma$ under $\exp^*_\omega$ will be a lattice $p^{n_2}\mathbb{Z}_p \subset \mathbb{Q}_p$ for some $n_2 \geq n_1$.
Key Claim: There is a commutative diagram, with exact rows

$$\begin{array}{ccccccc}
0 \to & H^1_{\mathcal{G}}\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma & \overset{\varepsilon}{\longrightarrow} & H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)_\Gamma & \overset{\exp^*(-)\,\cup\,\omega^*_{\mathcal{C}^{\min}}}{\longrightarrow} & p^{n_2}\mathbb{Z}_p \to 0 \\
 & \downarrow & & \mathrm{nat}\downarrow & & \mathrm{id}\downarrow \\
0 \to & H^1_g\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big) & \longrightarrow & H^1\big(\mathbb{Q}_p, \mathrm{Ta}_p(\mathcal{C}^{\min})\big) & \overset{\exp^*(-)\,\cup\,\omega^*_{\mathcal{C}^{\min}}}{\longrightarrow} & p^{n_1}\mathbb{Z}_p \to 0.
\end{array}$$

To verify this assertion, we need to prove the injectivity of the top-left map ε. Recall that $H^1_{\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)=X(\mathbb{Q}_p)$ is Λ-saturated inside the local H^1, thus the quotient $H^1_{/\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)$ is Λ-free. In particular, both $H^1_{\mathcal{G}}$ and H^1 share the same Λ-torsion submodules, so at weight two $H^1_{\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma$ and $H^1(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma$ must have identical $\mathbb{Z}_p$-torsion. It follows from the invariants/coinvariants sequence

$$\begin{array}{llllll} 0 & \rightarrow & H^1_{\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)^\Gamma & \rightarrow H^1(\mathbb{Q}_p,\mathbb{T}_\infty)^\Gamma & \rightarrow & H^1_{/\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)^\Gamma \\ & \stackrel{\partial}{\rightarrow} & H^1_{\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma & \stackrel{\varepsilon}{\rightarrow} H^1(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma & \rightarrow & H^1_{/\mathcal{G}}(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma \ \rightarrow \ 0 \end{array}$$

that ε fails to be injective, if and only if the image of ∂ has $\mathbb{Z}_p$-rank at least one. However,

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}_p}\mathrm{Im}(\partial) \ &= \\ &= \ \mathrm{rank}_{\mathbb{Z}_p}\Big(H^1_{\mathcal{G}}(\cdots)_\Gamma\Big)-\mathrm{rank}_{\mathbb{Z}_p}\Big(H^1(\cdots)_\Gamma\Big)+\mathrm{rank}_{\mathbb{Z}_p}\Big(H^1_{/\mathcal{G}}(\cdots)_\Gamma\Big) \\ &\leq \ \mathrm{rank}_{\mathbb{Z}_p}\Big(H^1_{\mathcal{G}}(\cdots)_\Gamma\Big)-\mathrm{rank}_{\mathbb{Z}_p}\Big(H^1(\cdots)_\Gamma\Big)+\mathrm{rank}_{\mathbb{Z}_p}\big(p^{n_2}\mathbb{Z}_p\big) \end{aligned}$$

as the rank of $H^1_{/\mathcal{G}}(\cdots)_\Gamma$ is bounded by the rank of $H^1(\cdots)_\Gamma\Big/\big(H^1(\cdots)_\Gamma\cap H^1_g\big)$. The right-hand side above is equal to zero, hence $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Im}(\partial)$ is forced to be zero. The non-triviality of the boundary map ∂ can therefore never happen, and the injectivity of ε follows as well.

Remark: Using our Key Claim to calculate $\Big[H^1_g(\cdots):H^1_{\mathcal{G}}(\cdots)_\Gamma\Big]$, we find that

$$\begin{aligned} \#\mathrm{Ker}(\delta_p) \ &= \ p^{-(n_2-n_1)}\times\Big[H^1\big(\mathbb{Q}_p,\mathrm{Ta}_p(\mathcal{C}^{\min})\big):H^1(\mathbb{Q}_p,\mathbb{T}_\infty)_\Gamma\Big] \\ &= \ p^{-(n_2-n_1)}\times\#H^2(\mathbb{Q}_p,\mathbb{T}_\infty)^\Gamma \ = \ p^{-(n_2-n_1)}\times\#H^0(\mathbb{Q}_p,A_{\mathbb{T}_\infty})_\Gamma \end{aligned}$$

where the very last equality arises from the non-degeneracy of the local pairing $H^2(\mathbb{Q}_p,\mathbb{T}_\infty)\times H^0(\mathbb{Q}_p,A_{\mathbb{T}_\infty})\rightarrow\mathbb{Q}_p/\mathbb{Z}_p$.

By an argument familiar from §2,

$$\begin{aligned} &\#H^0(\mathbb{Q}_p,A_{\mathbb{T}_\infty})_\Gamma \\ &\qquad\leq \ \#H^0\big(\mathbb{Q}_p,A_{\mathbb{T}_\infty^\Gamma}\big) \ = \ \#H^0\Big(\mathbb{Q}_p,\mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{T}_\infty\otimes_\Lambda\mathbb{Z}_p,\mu_{p^\infty}\big)\Big) \\ &\qquad= \ \#H^0\Big(\mathbb{Q}_p,\mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Ta}_p(\mathcal{C}^{\min}),\mu_{p^\infty}\big)\Big) \ = \ \#\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty] \end{aligned}$$

again due to the pro-cyclicity of Γ. Because $n_2-n_1\geq 0$, we get an upper bound

$$\#\mathrm{Ker}(\delta_p) \ \leq \ p^{-(n_2-n_1)}\#\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty] \quad \leq \quad \#\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty]\ ;$$

we proceed by showing that the right-hand side is trivial under Hypothesis(Frb).

Case (i): $p \nmid N_E$ and $a_p(E) \neq +1$.

Here E and the isogenous curve $\mathcal{C}^{\min}$ have good ordinary reduction at the prime p; in particular, the formal group of $\mathcal{C}^{\min}_{/\mathbb{Z}_p}$ possesses no points of order p since $p \neq 2$. It follows that $\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty]$ injects into the subgroup of $\mathbb{F}_p$-rational points on $\widetilde{\mathcal{C}^{\min}}$, the reduced elliptic curve. Moreover

$$\#\widetilde{\mathcal{C}^{\min}}(\mathbb{F}_p) \quad = \quad p+1-a_p(E) \quad \not\equiv \quad 0 \pmod{p} \qquad \text{as} \quad a_p(E) \not\equiv +1,$$

meaning $\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty] \cong \widetilde{\mathcal{C}^{\min}}(\mathbb{F}_p)[p^\infty]$ is the trivial group.

Case (ii): $p \| N_E$ and $a_p(E) = -1$.

Both E and $\mathcal{C}^{\min}$ have non-split multiplicative reduction at p. The Tamagawa factor $[\mathcal{C}^{\min}(\mathbb{Q}_p) : \mathcal{C}_0^{\min}(\mathbb{Q}_p)]$ is either 1, 2, 3 or 4, all of which are coprime to $p \geq 5$. We thus have an isomorphism $\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty] \cong \mathcal{C}_0^{\min}(\mathbb{Q}_p)[p^\infty]$. Again the formal group is p-torsion free, so $\mathcal{C}_0^{\min}(\mathbb{Q}_p)[p^\infty]$ coincides with the p^∞-torsion in the group of non-singular points $\widetilde{\mathcal{C}^{\min}}(\mathbb{F}_p) - \{\text{node}\}$. But these non-singular points look like $\mathbb{F}_p^\times$ which has no points of order p, so neither does $\mathcal{C}^{\min}(\mathbb{Q}_p)$.

Case (iii): $p \| N_E$ and $a_p(E) = +1$, $p \nmid \mathrm{ord}_p\Big(\mathbf{q}_{\mathrm{Tate}}(\mathcal{C}^{\min})\Big)$.

This last situation corresponds to our elliptic curves being split multiplicative at p. The group of connected components $\mathcal{C}^{\min}(\mathbb{Q}_p)/\mathcal{C}_0^{\min}(\mathbb{Q}_p) \cong \mathbb{Z}/\mathrm{ord}_p\Big(\mathbf{q}_{\mathrm{Tate}}(\mathcal{C}^{\min})\Big)\mathbb{Z}$ has order coprime to p, by assumption. Again $\mathcal{C}^{\min}(\mathbb{Q}_p)[p^\infty] \cong \mathcal{C}_0^{\min}(\mathbb{Q}_p)[p^\infty]$, and an identical argument to case (ii) establishes that the p-part of $\mathcal{C}^{\min}(\mathbb{Q}_p)$ is trivial.

4. Global Euler-Poincaré Characteristics

It remains to give the proof of Proposition 2.3, i.e. to demonstrate why

$$\#H^1\Big(\Gamma, \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \times \prod_{l \in \Sigma} \Big[\mathrm{Ker}(\delta_l) : \mathrm{Im}(\lambda_0) \cap \mathrm{Ker}(\delta_l)\Big] \quad = \quad \#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]$$

whenever the analytic rank of E is zero.

Let's start by writing down the Poitou-Tate sequence for the optimal curve. It is an easy exercise to verify that $H^1\big(\mathbb{Q}_l,\ \mathcal{C}^{\min}[p^\infty]\big)\Big/H^1_\star\big(\mathbb{Q}_l,\ \mathcal{C}^{\min}[p^\infty]\big)$ is isomorphic to $H^1\big(\mathbb{Q}_l, \mathcal{C}^{\min}\big)[p^\infty]$ where '$\star = \mathrm{nr}$' if $l \neq p$, and '$\star = g$' if $l = p$. The exactness of the sequence

$$0 \to \mathrm{Sel}_{\mathbb{Q}}\big(\mathcal{C}^{\min}\big)[p^\infty] \to H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big) \xrightarrow{\lambda_0} \bigoplus_{l \in \Sigma} H^1\Big(\mathbb{Q}_l,\ \mathcal{C}^{\min}\Big)[p^\infty]$$
$$\to \mathrm{Hom}_{\mathrm{cont}}\Big(\mathcal{C}^{\min}(\mathbb{Q})\widehat{\otimes}\mathbb{Z}_p\ ,\ \mathbb{Q}/\mathbb{Z}\Big) \to H^2\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big) \to \cdots$$

is then an old result of Cassels.

Lemma 4.1. *If* $Sel_{\mathbb{Q}}\big(\mathcal{C}^{\min}\big)[p^\infty]$ *is finite, then* $H^2\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\big)=0.$

The proof is well-known to the experts. It's a basic consequence of the cyclotomic Iwasawa theory of elliptic curves, e.g. see Coates' textbook on the subject.

If we mimic the same approach Λ-adically, the Poitou-Tate exact sequence reads as

$$0 \to \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty) \to$$
$$\to H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{\mathbb{T}_\infty}\Big) \xrightarrow{\lambda_\infty^\dagger} \bigoplus_{l\in\Sigma} \frac{H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)}{H^1_\star\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)} \to \widehat{\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)} \to \cdots$$

where the compact Selmer group is defined to be

$$\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty) :=$$
$$:= \mathrm{Ker}\left(H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathbb{T}_\infty\big) \xrightarrow{\oplus \mathrm{res}_l} \bigoplus_{l\neq p} \frac{H^1\big(\mathbb{Q}_l, \mathbb{T}_\infty\big)}{H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)^\perp} \oplus \frac{H^1\big(\mathbb{Q}_p, \mathbb{T}_\infty\big)}{X(\mathbb{Q}_p)}\right).$$

In fact $H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)$ is orthogonal to all of $H^1\big(\mathbb{Q}_l, \mathbb{T}_\infty\big)$ under Pontrjagin duality, so the local conditions at $l \neq p$ are completely redundant.

Proposition 4.2. *If* $L(E,1)\neq 0$, *then the compact version* $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ *is zero.*

The proof is rather lengthy – we postpone it till the end of this section.

As a corollary, the restriction map $\lambda_\infty^\dagger$ must be surjective at the Λ-adic level. Taking Γ-cohomology, we obtain a long exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)^\Gamma \longrightarrow H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\Big)^\Gamma \xrightarrow{\lambda_\infty} \bigoplus_{l\in\Sigma}\left(\frac{H^1\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)}{H^1_\star\big(\mathbb{Q}_l, A_{\mathbb{T}_\infty}\big)}\right)^\Gamma$$
$$\longrightarrow H^1\Big(\Gamma,\ \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \longrightarrow H^1\Big(\Gamma,\ H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\big)\Big).$$

The right-most term is zero, since it is contained inside $H^2\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big)$ which vanishes by Lemma 4.1. We can then compare the cokernels of λ_0 and

λ_∞ via the commutative diagram, with exact columns:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\Big\downarrow & & \Big\downarrow \\
H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathcal{C}^{\min}[p^\infty]\Big) & \xrightarrow{\beta} & H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\ A_{\mathbb{T}_\infty}\Big)^\Gamma \\
{\scriptstyle\lambda_0}\Big\downarrow & & {\scriptstyle\lambda_\infty}\Big\downarrow \\
\bigoplus_{l\in\Sigma} H^1\Big(\mathbb{Q}_l,\ \mathcal{C}^{\min}\Big)[p^\infty] & \xrightarrow{\oplus\delta_l} & \bigoplus_{l\in\Sigma}\left(\dfrac{H^1(\mathbb{Q}_l, A_{\mathbb{T}_\infty})}{H^1_\star(\mathbb{Q}_l, A_{\mathbb{T}_\infty})}\right)^\Gamma \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Hom}_{\mathrm{cont}}\Big(\mathcal{C}^{\min}(\mathbb{Q})\widehat{\otimes}\mathbb{Z}_p\ ,\ \mathbb{Q}/\mathbb{Z}\Big) & \dashrightarrow & H^1\Big(\Gamma,\ \mathrm{Sel}_{\mathbb{Q}}(\rho_\infty)\Big) \\
\Big\downarrow & & \Big\downarrow \\
0 & & 0
\end{array}.$$

Figure 3.

Remark: Focussing momentarily on the homomorphisms δ_l and λ_0, one deduces

$$\Big[\mathrm{Ker}\big(\oplus\,\delta_l\big) : \mathrm{Ker}\big(\oplus\,\delta_l\big)\cap\mathrm{Im}(\lambda_0)\Big] \quad = \quad \frac{\Big[\bigoplus_{l\in\Sigma} H^1\big(\mathbb{Q}_l,\ \mathcal{C}^{\min}\big)[p^\infty] : \mathrm{Im}(\lambda_0)\Big]}{\Big[\mathrm{Im}\big(\oplus\,\delta_l\big) : \oplus\delta_l\big(\mathrm{Im}(\lambda_0)\big)\Big]}$$

upon applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\big(\oplus\,\delta_l\big) & \longrightarrow & H^1\big(\mathbb{Q}_l,\ \mathcal{C}^{\min}\big)[p^\infty] & \xrightarrow{\oplus\delta_l} & \mathrm{Im}\big(\oplus\,\delta_l\big) & \longrightarrow & 0 \\
 & & \bigcup & & \bigcup & & \bigcup & & \\
0 & \longrightarrow & \mathrm{Ker}\big(\oplus\,\delta_l\big)\cap\mathrm{Im}(\lambda_0) & \longrightarrow & \mathrm{Im}(\lambda_0) & \xrightarrow{\oplus\delta_l} & \oplus\delta_l\big(\mathrm{Im}(\lambda_0)\big) & \longrightarrow & 0\,.
\end{array}$$

The numerator above equals $\#\mathrm{Hom}_{\mathrm{cont}}\Big(\mathcal{C}^{\min}(\mathbb{Q})\widehat{\otimes}\mathbb{Z}_p, \mathbb{Q}/\mathbb{Z}\Big)$, which has the same size as the p-primary subgroup of $\mathcal{C}^{\min}(\mathbb{Q})$. Casting a cold eye over Figure 3, one exploits the surjectivity of $\oplus\delta_l$ to conclude the denominator term is $\#\mathrm{Coker}(\lambda_\infty)$. Equivalently,

$$\prod_{l\in\Sigma}\Big[\mathrm{Ker}(\delta_l) : \mathrm{Im}(\lambda_0)\cap\mathrm{Ker}(\delta_l)\Big] \quad = \quad \frac{\#\mathcal{C}^{\min}(\mathbb{Q})[p^\infty]}{\#\mathrm{Coker}(\lambda_\infty)}$$

which finishes off the demonstration of 2.3.

The proof of Proposition 4.2:
There are three stages. We first show that the compact Selmer group is Λ-torsion. Using a version of Nekovář's control theory along the critical line $(s,k) \in \{1\} \times \mathbb{Z}_p$, we next establish its finiteness. Lastly, we embed $\mathfrak{Sel}$ inside a tower of rational points, whose structure is narrow enough to imply the Selmer group is zero.

Examining the behaviour of our big dual exponential $\mathrm{EXP}^*_{\mathbb{T}_\infty}$ from [De, Th 3.3], there is a tautological sequence of Λ-homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty) & \longrightarrow & H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathbb{T}_\infty\big) & \xrightarrow{\mathrm{loc}_p(-)\ \mathrm{mod}\ X(\mathbb{Q}_p)} & \dfrac{H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)}{X(\mathbb{Q}_p)} \\ & & & & & & \Big\downarrow{\scriptstyle \mathrm{EXP}^*_{\mathbb{T}_\infty}} \\ & & & & & & \Lambda[1/p] \end{array}$$

which is exact along the row. A global Euler characteristic calculation shows that

$$\begin{aligned} \mathrm{rank}_\Lambda\Big(H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}_\infty\big)\Big) \ &= \ \mathrm{rank}_\Lambda\Big(H^2\big(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}_\infty\big)\Big) + 1 \\ &\le \ \mathrm{rank}_{\mathbb{Z}_p}\Big(H^2\big(\mathbb{Q}_\Sigma/\mathbb{Q},\mathrm{Ta}_p(\mathcal{C}^{\min})\big)\Big) + 1 \\ &\overset{\text{by Kato}}{=} \ 0\ +\ 1 \end{aligned}$$

– the final equality lies very deep, and follows from [Ka, Th 14.5(1)].
On the other hand, the weight-deformation of Kato's zeta-element will generate rank one Λ-submodules inside both of $H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathbb{T}_\infty\big)$ and $H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\Big/X(\mathbb{Q}_p)$. To verify this claim, observe that $\mathrm{EXP}^*_{\mathbb{T}_\infty}$ modulo u_0-1 sends the zeta-element to a multiple of $\frac{L_{Np}(\mathcal{C}^{\min},1)}{\Omega^+_{\mathcal{C}^{\min}}}$, which is non-zero. This means the image of $\mathrm{EXP}^*_{\mathbb{T}_\infty} \circ \mathrm{loc}_p$ is not contained in the augmentation ideal, and so is abstractly isomorphic to Λ.

Remark: In summary, we have just shown that the global H^1 has Λ-rank one. Because the quotient $H^1\big(\mathbb{Q}_p,\mathbb{T}_\infty\big)\Big/X(\mathbb{Q}_p)$ is Λ-torsion free and also has rank one, we may identify $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ with the Λ-torsion submodule of $H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\ \mathbb{T}_\infty\big)$.

Question. *Does $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ contain any pseudo-summands of the form $\Lambda\big/F_j^{e_j}\Lambda$*

for some irreducible distinguished polynomial F_j and for $e_j \in \mathbb{N}$?

To provide an answer, we will need to specialise at arithmetic points of $\mathrm{Spec}(\Lambda)^{\mathrm{alg}}$. For any de Rham $\mathrm{Gal}\big(\mathbb{Q}_\Sigma/\mathbb{Q}\big)$-lattice $\mathbf{T}$, the Selmer group $H^1_{g,\mathrm{Spec}\mathbb{Z}}$

is defined by

$$H^1_{g,\mathrm{Spec}\mathbb{Z}}\big(\mathbb{Q},\mathbf{T}\big) := \mathrm{Ker}\left(H^1\big(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbf{T}\big) \overset{\oplus \mathrm{res}_l}{\longrightarrow} \bigoplus_{l\neq p} H^1\big(I_l,\mathbf{T}\big) \oplus \frac{H^1\big(\mathbb{Q}_p,\mathbf{T}\big)}{H^1_g\big(\mathbb{Q}_p,\mathbf{T}\big)}\right) .$$

Control Theorem. [Sm, Th 5.1] *For all bar finitely many integral weights $k \geq 2$, the induced specialisation*

$$\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p \quad\longrightarrow\quad H^1_{g,\mathrm{Spec}\mathbb{Z}}\Big(\mathbb{Q}\ ,\ \mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\Big)$$

has finite kernel and cokernel, bounded independently of the choice of $\sigma_k : \Lambda \twoheadrightarrow \mathbb{Z}_p$.

Kato's Theorem. [Ka, Th 14.2] *For all integral weights $k \geq 3$, the Bloch-Kato compact Selmer group $H^1_{f,\mathrm{Spec}\mathbb{Z}}\Big(\mathbb{Q}\ ,\ \mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\Big)$ is finite.*

Actually Kato proves this result for discrete Selmer groups, but they are equivalent statements. Note that $\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p$ is a lattice inside $V^*_{\mathbf{f}_k}$, the contragredient of Deligne's $G_{\mathbb{Q}}$-representation attached to the eigenform $\mathbf{f}_k \in \mathcal{S}^{\mathrm{ord}}_k\big(\Gamma_0(Np^r),\omega^{2-k}\big)$. The non-vanishing of the L-value $L(\mathbf{f}_k,1)$ forces these Selmer groups to be finite.

Corollary 4.3. *For almost all $k \geq 2$, the order of $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p$ is bounded.*

Proof: We first observe that $H^1_{g,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q},V^*_{\mathbf{f}_k})$ coincides with $H^1_{f,\mathrm{Spec}\mathbb{Z}}(\mathbb{Q},V^*_{\mathbf{f}_k})$ unless the local condition $H^1_g(\mathbb{Q}_p,V^*_{\mathbf{f}_k})$ is strictly larger than $H^1_f(\mathbb{Q}_p,V^*_{\mathbf{f}_k})$. However,

$$\dim_{\mathbb{Q}_p}\Big(H^1_{g/f}(\mathbb{Q}_p,V^*_{\mathbf{f}_k})\Big) \quad=\quad \dim_{\mathbb{Q}_p}\Big(\mathbf{D}_{\mathrm{cris}}\big(V_{\mathbf{f}_k}(1)\big)/(\varphi-1)\Big) \text{ by [BK, Cor 3.8.4]}$$

and an argument involving slopes of the Frobenius φ shows this dimension is zero.

By Kato's theorem $H^1_{f,\mathrm{Spec}\mathbb{Z}}$ is finite, so it lies in $H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\Big)[p^\infty]$; the latter torsion is identified with $H^0\Big(\mathbb{Q}_\Sigma/\mathbb{Q},\big(\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\big)\otimes\mathbb{Q}/\mathbb{Z}\Big)$ via a standard technique in continuous cohomology. It follows from the Control Theorem, that

$$\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p \overset{\mathrm{nat}}{\to} H^1_{f,\mathrm{Spec}\mathbb{Z}}\Big(\mathbb{Q},\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\Big) \hookrightarrow \Big(\big(\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\big)\otimes\mathbb{Q}/\mathbb{Z}\Big)^{G_{\mathbb{Q}}}$$

has kernel killed by a universal power p^{ν_1} say, independent of the weight k. Let us choose a prime $l \nmid Np$. By definition $1-a_l(\mathbf{f}_k).\mathrm{Frob}_l + l < l >^{k-2} .\mathrm{Frob}_l^2$ is zero on $V^*_{\mathbf{f}_k}$, and $1-a_l(\mathbf{f}_k)+l < l >^{k-2}$ must kill off $\Big(\big(\mathbb{T}_\infty\otimes_{\Lambda,\sigma_k}\mathbb{Z}_p\big)\otimes\mathbb{Q}/\mathbb{Z}\Big)^{G_{\mathbb{Q}}}$

because Frobenius acts trivially on the $G_{\mathbb{Q}}$-invariants. We claim that there are infinitely many choices of l for which $1 - a_l(\mathbf{f}_k) + l < l >^{k-2} \neq 0$. If not,

$$1 - a_l(\mathbf{f}_k) l^{-s} + l < l >^{k-2} l^{-2s} \quad = \quad \big(1 - l^{-s}\big)\big(1 - \omega^{2-k}(l) l^{k-1-s}\big) \quad \text{for all } l \notin S$$

where S is some finite set containing Σ. Proceeding further down this cul-de-sac, we obtain an equality of incomplete L-functions $L_S(\mathbf{f}_k, s) = \zeta_S(s) L_S(\omega^{2-k}, s+1-k)$ which is patently ridiculous, as $\mathbf{f}_k$ is not an Eisenstein series!

If $k \equiv k' \bmod (p-1)p^c$, then

$$1 - a_l(\mathbf{f}_k) + l < l >^{k-2} \;\equiv\; 1 - a_l(\mathbf{f}_{k'}) + l < l >^{k'-2} \quad \text{modulo } p^{c+1} .$$

For each class τ modulo $p-1$, we can cover weight-space by a finite collection of open disks $D_1^\tau, \dots, D_{n(\tau)}^\tau$ upon which $\mathrm{ord}_p\big(1 - a_l(\mathbf{f}_k) + l < l >^{k-2}\big)$ is constant for every $k \in D_j^\tau$, $k \equiv \tau (\bmod\ p-1)$. Setting ν_2 equal to the non-negative integer

$$\max_{\tau \bmod p-1} \left\{ \max_{1 \le j \le n(\tau)} \left\{ \mathrm{ord}_p\big(1 - a_l(\mathbf{f}_k) + l < l >^{k-2}\big) \text{ with } k \in D_j^\tau,\, k \equiv \tau \right\} \right\},$$

clearly p^{ν_2} annihilates all the $\Big(\big(\mathbb{T}_\infty \otimes_{\Lambda, \sigma_k} \mathbb{Z}_p\big) \otimes \mathbb{Q}/\mathbb{Z}\Big)^{G_{\mathbb{Q}}}$'s. We deduce that $p^{\nu_1+\nu_2}$ kills off $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty) \otimes_{\Lambda, \sigma_k} \mathbb{Z}_p$ for almost all $k \geq 3$, and the corollary is proved.

Remark: The answer to the question posed above is therefore negative, i.e. there can exist no pseudo-summands of the shape $\Lambda / F_j^{e_j} \Lambda$ lying inside of $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ (otherwise the specialisations $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty) \otimes_{\Lambda, \sigma_k} \mathbb{Z}_p$ would have unbounded order for varying weights $k \geq 2$, which violates Corollary 4.3). The compact Selmer group is of finite-type over the local ring Λ, and it follows from the structure theory that $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ must be a finite abelian p-group, of order dividing $p^{\nu_1+\nu_2}$.

Let us recall the definition of the degeneration maps between modular curves. For integers $d \geq 1$ and $m, n \geq 5$ with $dm \big| n$, the finite map $\pi_d : X_1(n) \to X_1(m)$ operates on the affine curves $Y_1(-)$ by the rule

$$\pi_d \left(A,\ \mu_n \overset{\theta}{\hookrightarrow} A[n] \right) \quad = \quad \left(A',\ \mu_m \overset{\theta'}{\hookrightarrow} A'[m] \right)$$

where $A' = A / \theta(\mu_d)$, and the injection $\theta' : \mu_m \hookrightarrow \mu_{n/d} \overset{d}{\overset{\sim}{\leftarrow}} \mu_n / \mu_d \overset{\theta \bmod \mu_d}{\hookrightarrow} A / \theta(\mu_d)$.

Hida [H1] identified the $\Gamma^{p^{r-1}}$-coinvariants of $\mathbb{T}_\infty$, with the Tate module of a p-divisible subgroup of jac $X_1(n)$ at level $n = Np^r$. The natural composition

$$H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbb{T}_\infty\Big) \cong \varprojlim_{r\geq 1} H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \big(\mathbb{T}_\infty\big)_{\Gamma^{p^{r-1}}}\Big) \hookrightarrow \varprojlim_{\pi_{p\,*}} H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathrm{Ta}_p(J_r)^{\mathrm{ord}}\Big)$$

injects $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ into the projective limit $\varprojlim_{\pi_{p\,*}}\Big(H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathrm{Ta}_p(J_r)^{\mathrm{ord}}\Big)[p^\infty]\Big)$. Again it's continuous cohomology, so the $\mathbb{Z}_p$-torsion in $H^1\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathrm{Ta}_p(J_r)^{\mathrm{ord}}\Big)$ is then isomorphic to $H^0\Big(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathrm{Ta}_p(J_r)^{\mathrm{ord}} \otimes \mathbb{Q}/\mathbb{Z}\Big) = J_r^{\mathrm{ord}}(\mathbb{Q})[p^\infty]$ as finite groups.

LEMMA. *(Nekovář)* [NP, 1.6.6] *(i)* $\pi_{1\,*}\big(\mathbf{e}_{\mathrm{ord}}.Ta_p(J_{r+1})\big) \subset p\,\big(\mathbf{e}_{\mathrm{ord}}.Ta_p(J_r)\big)$; *Let* $\frac{1}{p}\pi_{1\,*} : Ta_p(J_{r+1})^{\mathrm{ord}} \to Ta_p(J_r)^{\mathrm{ord}}$ *denote the map satisfying* $p\left(\frac{1}{p}\pi_{1\,*}\right) = \pi_{1\,*}$.

(ii) $\left(\frac{1}{p}\pi_{1\,*}\right) \circ \pi_1^* =$ *multiplication by* p *on* $\mathbf{e}_{\mathrm{ord}}.Ta_p(J_r)$;

(iii) $\pi_1^* \circ \left(\frac{1}{p}\pi_{1\,*}\right) = \sum_{\gamma\in\Gamma_r/\Gamma_{r+1}}\langle\gamma\rangle$ *on* $\mathbf{e}_{\mathrm{ord}}.Ta_p(J_{r+1})$ *where* $\Gamma_r = \Gamma^{p^{r-1}}$;

(iv) $\pi_{p\,*} = U_p \circ \left(\frac{1}{p}\pi_{1\,*}\right)$ *on* $\mathbf{e}_{\mathrm{ord}}.Ta_p(J_{r+1})$.

We shall use these facts directly, to show the triviality of the compact Selmer group. Because it is finite of order dividing $p^{\nu_1+\nu_2}$, for large enough $r \gg 1$ we can realise $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty)$ as a subgroup $\mathbb{S}_r$ of jac $X_1(Np^r)^{\mathrm{ord}}(\mathbb{Q})[p^{\nu_1+\nu_2}]$.

The sequence of $\mathbb{S}_r$'s is compatible with respect to the degeneration maps $\pi_{p\,*}$ and π_1^* : jac $X_1(Np^r)(\overline{\mathbb{Q}})[p^\infty] \longrightarrow$ jac $X_1(Np^{r+1})(\overline{\mathbb{Q}})[p^\infty]$, so for any $e \geq 0$

$$\mathbb{S}_r \;=\; \big(\pi_{p\,*}\big)^e\Big(\mathbb{S}_{r+e}\Big) \;\cong\; \big(\pi_{p\,*}\big)^e \circ \big(\pi_1^*\big)^e\Big(\mathbb{S}_r\Big)\,.$$

By part (iv) of this lemma $\big(\pi_{p\,*}\big)^e$ coincides with $\Big(U_p \circ \left(\frac{1}{p}\pi_{1\,*}\right)\Big)^e$, and the covariant action of the U_p-operator is invertible on the ordinary locus. Consequently

$$\mathbb{S}_r \;\cong\; a_p(\mathbf{f})^e \times \left(\frac{1}{p}\pi_{1\,*}\right)^e \circ \big(\pi_1^*\big)^e\Big(\mathbb{S}_r\Big) \;\stackrel{\text{by (ii)}}{=}\; a_p(\mathbf{f})^e \times p^e\,\big(\mathbb{S}_r\big)$$

and picking $e \geq \nu_1+\nu_2$, we see that $\mathfrak{Sel}_{\mathbb{Q}}(\mathbb{T}_\infty) \cong \mathbb{S}_r \subset J_r[p^{\nu_1+\nu_2}]$ must be zero. The proof of Proposition 4.2 is thankfully over.

REFERENCES

[BK] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in the Grothendieck Festchrift I, Progress in Math. 86, Birkhäuser (1990), 333-400.

[CH] J. Coates and S. Howson, *Euler characteristics and elliptic curves II*, Journal Math. Soc. Japan 53 (2001), 175-235.

[De] D. Delbourgo, *Super Euler systems and ordinary deformations of modular symbols*, preprint (2004).

[DS] D. Delbourgo and P. Smith, *Kummer theory for big Galois representations*, to appear in Math. Proc. of the Camb. Phil. Soc.

[GS] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. 111 (1993), 401-447.

[H1] H. Hida, *Galois representations into* $GL_2(\mathbb{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. 85 (1986), 545-613.

[H2] H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) 19 (1986), 231-273.

[Ka] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, preprint (2002).

[MW] B. Mazur and A. Wiles, *On p-adic analytic families of Galois representations*, Compositio Math. 59 (1986), 231-264.

[NP] J. Nekovář and A. Plater, *On the parity ranks of Selmer groups*, Asian Journal Math. (2) 4 (2000), 437-498.

[Sm] P. Smith, PhD Thesis, University of Nottingham (2006).

[St] G. Stevens, *Stickelberger elements and modular parametrizations of elliptic curves*, Invent. Math. 98 (1989), 75-106.

Daniel Delbourgo
Department of Mathematics
University Park
Nottingham
England NG7 2RD
dd@maths.nott.ac.uk