On the Leading Terms of Zeta Isomorphisms and p-Adic L-functions in Non-Commutative Iwasawa Theory

Dedicated to John Coates

David Burns and Otmar Venjakob

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Abstract. We discuss the formalism of Iwasawa theory descent in the setting of the localized K_1-groups of Fukaya and Kato. We then prove interpolation formulas for the 'leading terms' of the global Zeta isomorphisms that are associated to certain Tate motives and of the p-adic L-functions that are associated to certain critical motives.

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1. Introduction

In the last few years there have been several significant developments in non-commutative Iwasawa theory.
Firstly, in [11], Coates, Fukaya, Kato, Sujatha and the second named author formulated a main conjecture for elliptic curves without complex multiplication. More precisely, if F_∞ is any Galois extension of a number field F which contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F and is such that $\mathrm{Gal}(F_\infty/F)$ is a compact p-adic Lie group with no non-trivial p-torsion, then Coates et al. formulated a $\mathrm{Gal}(F_\infty/F)$-equivariant main conjecture for any elliptic curve which is defined over F, has good ordinary reduction at all places above p and whose Selmer group (over F_∞) satisfies a certain natural torsion condition.
Then, in [16], Fukaya and Kato formulated a natural main conjecture for any compact p-adic Lie extension of F and any critical motive M which is defined over F and has good ordinary reduction at all places above p.
The key feature of [11] is the use of the localization sequence of algebraic K-theory with respect to a canonical Ore set. However, the more general approach of [16] is rather more involved and uses a notion of 'localized K_1-groups' together with Nekovář's theory of Selmer complexes and the (conjectural) existence of certain canonical p-adic L-functions. See [39] for a survey.

The p-adic L-functions of Fukaya and Kato satisfy an interpolation formula which involves both the 'non-commutative Tamagawa number conjecture' (this is a natural refinement of the 'equivariant Tamagawa number conjecture' formulated by Flach and the first named author in [7] and hence also implies the 'main conjecture of non-abelian Iwasawa theory' discussed by Huber and Kings in [19]) as well as a local analogue of the non-commutative Tamagawa number conjecture. Indeed, by these means, at each continuous finite dimensional p-adic representation ρ of $\mathrm{Gal}(F_\infty/F)$, the 'value at ρ' of the p-adic L-function is explicitly related to the value at the central critical point of the complex L-function associated to the 'ρ^*-twist' $M(\rho^*)$ of M, where ρ^* denotes the contragredient of the representation ρ. However, if the Selmer module of $M(\rho^*)$ has strictly positive rank (and by a recent result of Mazur and Rubin [21], which is itself equivalent to a special case of an earlier result of Nekovář [24, Th. 10.7.17], this should often be the case), then both sides of the Fukaya-Kato interpolation formula are equal to zero.

The main aim of the present article is therefore to extend the formalism of Fukaya and Kato in order to obtain an interesting interpolation formula for all representations ρ as above. To this end we shall introduce a notion of 'the leading term at ρ' for elements of suitable localized K_1-groups. This notion is defined in terms of the Bockstein homomorphisms that have already played significant roles (either implicitly or explicitly) in work of Perrin-Riou [27, 29], of Schneider [34, 33, 32, 31] and of Greither and the first named author [9, 4] and have been systematically incorporated into Nekovář's theory of Selmer complexes [24]. We then give two explicit applications of this approach in the setting of extensions F_∞/F with $F_{\mathrm{cyc}} \subseteq F_\infty$. We show first that the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre [35] and interpreted by Tate in [37], can be reinterpreted as providing interpolation formulas for the leading terms of the global Zeta isomorphisms associated to certain Tate motives in terms of the leading terms at $s = 1$ (in the classical sense) of the p-adic Artin L-functions that are constructed by combining Brauer induction with the fundamental results of Deligne and Ribet and of Cassou-Nogués. We then also prove an interpolation formula for the leading terms of the Fukaya-Kato p-adic L-functions which involves the leading term at the central critical point of the associated complex L-function, the Neron-Tate pairing and Nekovář's p-adic height pairing.

In a subsequent article we shall apply the approach developed here to describe the leading terms of the 'algebraic p-adic L-functions' that are introduced by the first named author in [5], and we shall use the resulting description to prove that the main conjecture of Coates et al. for an extension F_∞/F and an elliptic curve E implies the equivariant Tamagawa number conjecture for the motive $h^1(E)(1)$ at each finite degree subextension of F_∞/F. We note that this result provides a partial converse to the theorem of Fukaya and Kato which shows that, under a natural torsion hypothesis on Selmer groups, the main conjecture of Fukaya and Kato specialises to recover the main conjecture of Coates et al.

The main contents of this article are as follows. In §2 we recall some basic facts regarding (non-commutative) determinant functors and the localized K_1-groups of Fukaya and Kato. In §3 we discuss the formalism of Iwasawa theory descent in the setting of localized K_1-groups and we introduce a notion of the leading terms at p-adic representations for the elements of such groups. We explain how this formalism applies in the setting of the canonical Ore sets introduced by Coates et al., we show that it can be interpreted as taking values after 'partial derivation in the cyclotomic direction', and we use it to extend several well known results concerning Generalized Euler-Poincaré characteristics. In §4 we recall the 'global Zeta isomorphisms' that are conjectured to exist by Fukaya and Kato, and in §5 we prove an interpolation formula for the leading terms of the global Zeta isomorphisms that are associated to certain Tate motives. Finally, in §6, we prove an interpolation formula for the leading terms of the p-adic L-functions that are associated to certain critical motives. We shall use the same notation as in [39].

It is clear that the recent developments in non-commutative Iwasawa theory are due in large part to the energy, encouragement and inspiration of John Coates. It is therefore a particular pleasure for us to dedicate this paper to him on the occasion of his sixtieth birthday.

This collaboration was initiated during the conference held in Boston in June 2005 in recognition of the sixtieth birthday of Ralph Greenberg. The authors are very grateful to the organizers of this conference for the opportunity to attend such a stimulating meeting.

2. Preliminaries

2.1. Determinant functors. For any associative unital ring R we write $B(R)$ for the category of bounded (cohomological) complexes of (left) R-modules, $C(R)$ for the category of bounded (cohomological) complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules and $C^p(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. We also write $D^p(R)$ for the category of perfect complexes as full triangulated subcategory of the bounded derived category $D^b(R)$ of (left) R-modules. We write $(P(R), \mathrm{is})$, $(C^p(R), \mathrm{quasi})$ and $(D^p(R), \mathrm{is})$ for the subcategories of isomorphisms in $P(R)$, quasi-isomorphisms in $C^p(R)$ and isomorphisms in $D^p(R)$ respectively.

For each complex $C = (C^\bullet, d_C^\bullet)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d^i_{C[r]} = (-1)^r d_C^{i+r}$ for each integer i.

We recall that in [16, §1.2] Fukaya and Kato construct an explicit alternative to the category of virtual objects that is used in [7]. Indeed, they construct explicitly a category $\mathcal{C}_R$ and a 'determinant functor'

$$\mathbf{d}_R : (P(R), \mathrm{is}) \to \mathcal{C}_R$$

which possess the following properties:

a) $\mathcal{C}_R$ has an associative and commutative product structure $(M,N) \mapsto M \cdot N$ (which we often write more simply as MN) with canonical unit object $\mathbf{1}_R = \mathbf{d}_R(0)$. If P is any object of $P(R)$, then in $\mathcal{C}_R$ the object $\mathbf{d}_R(P)$ has a canonical inverse $\mathbf{d}_R(P)^{-1}$. Every object of $\mathcal{C}_R$ is of the form $\mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)^{-1}$ for suitable objects P and Q of $P(R)$;
b) All morphisms in $\mathcal{C}_R$ are isomorphisms and elements of the form $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic in $\mathcal{C}_R$ if and only if P and Q correspond to the same element of the Grothendieck group $K_0(R)$. There is a natural identification $\mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R) \cong K_1(R)$ and if $\mathrm{Mor}_{\mathcal{C}_R}(M,N)$ is non-empty, then it is a $K_1(R)$-torsor where each element α of $K_1(R) \cong \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R)$ acts on $\phi \in \mathrm{Mor}_{\mathcal{C}_R}(M,N)$ to give $\alpha\phi : M = \mathbf{1}_R \cdot M \xrightarrow{\alpha\cdot\phi} \mathbf{1}_R \cdot N = N$;
c) $\mathbf{d}_R$ preserves the product structure: specifically, for each P and Q in $P(R)$ one has $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

The functor $\mathbf{d}_R$ can be extended to give a functor

$$\mathbf{d}_R : (C^p(R), \text{quasi}) \to \mathcal{C}_R$$

in the following way: for each $C \in C^p(R)$ one sets

$$\mathbf{d}_R(C) := \mathbf{d}_R(\bigoplus_{i\in\mathbb{Z}} C^{2i})\mathbf{d}_R(\bigoplus_{i\in\mathbb{Z}} C^{2i+1})^{-1}.$$

This extended functor then has the following properties for all objects C, C' and C'' of $C^p(R)$:

d) If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence in $C^p(R)$, then there exists a canonical morphism in $\mathcal{C}_R$ of the form

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C''),$$

which we take to be an identification;
e) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical morphism in $\mathcal{C}_R$ of the form

$$\mathbf{1}_R = \mathbf{d}_R(0) \to \mathbf{d}_R(C);$$

f) For any integer r there exists a canonical morphism $\mathbf{d}_R(C[r]) \cong \mathbf{d}_R(C)^{(-1)^r}$ in $\mathcal{C}_R$ which we take to be an identification;
g) The functor $\mathbf{d}_R$ factorizes through the image of $C^p(R)$ in $D^p(R)$ and extends (uniquely up to unique isomorphism) to give a functor

$$\mathbf{d}_R : (D^p(R), \text{is}) \to \mathcal{C}_R.$$

h) For each $C \in D^b(R)$ we write $H(C)$ for the complex with $H(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $H(C)$ belongs to $D^p(R)$ (in which case we shall say that C is *cohomologically perfect*), then there are canonical morphisms in $\mathcal{C}_R$ of the form

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(H(C)) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i};$$

i) If R' is any other (associative unital) ring and Y is an (R', R)-bimodule that is both finitely generated and projective as a left R'-module, then the functor $Y \otimes_R - : P(R) \to P(R')$ extends to give a diagram

$$\begin{array}{ccc} (D^p(R), \mathrm{is}) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ {\scriptstyle Y\otimes^{\mathbb{L}}_R -}\downarrow & & \downarrow{\scriptstyle Y\otimes_R -} \\ (D^p(R'), \mathrm{is}) & \xrightarrow{\mathbf{d}_{R'}} & \mathcal{C}_{R'} \end{array}$$

which commutes (up to canonical isomorphism). In particular, if $R \to R'$ is any ring homomorphism and $C \in \mathrm{D}^p(R)$, then we often write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

REMARK 2.1. Unless R is a regular ring, property d) does not extend to arbitrary exact triangles in $D^p(R)$. In general therefore all constructions in the sequel which involve complexes must be made in such a way to avoid this problem (nevertheless, we suppress any explicit discussion of this issue in the present manuscript and simply refer the reader to [7] for details as to how this problem can be overcome). The second displayed morphism in h) is induced by the properties d) and f). However, whilst a precise description of the first morphism in h) is important for the purposes of explicit computations, it is actually rather difficult to find in the literature. Here we use the description given by Knudsen in [20, §3].

REMARK 2.2. In the sequel we will have to distinguish between two inverses of a morphism $\phi : C \to D$ with $C, D \in \mathcal{C}_R$. The inverse with respect to composition will be denoted by $\overline{\phi} : D \to C$ while

$$\phi^{-1} := \overline{\mathrm{id}_{D^{-1}} \cdot \phi \cdot \mathrm{id}_{C^{-1}}} : C^{-1} \to D^{-1}$$

is the unique isomorphism such that $\phi \cdot \phi^{-1} = \mathrm{id}_{\mathbf{1}_R}$ under the identification $X \cdot X^{-1} = \mathbf{1}_R$ for both $X = C$ and $X = D$. If $D = C$, then $\phi : C \to C$ corresponds uniquely to an element of $K_1(R) \cong \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R)$ by the rule $\phi \cdot \mathrm{id}_{C^{-1}} : \mathbf{1}_R \to \mathbf{1}_R$. Under this identification $\overline{\phi}$ and ϕ^{-1} agree in $K_1(R)$ and are inverse to ϕ. Furthermore, the following relation between $\circ$ and $\cdot$ is easily verified: if $\phi : A \to B$ and $\psi : B \to C$ are morphisms in $\mathcal{C}_R$, then one has $\psi \circ \phi = \psi \cdot \phi \cdot \mathrm{id}_{B^{-1}}$.

We shall use the following

CONVENTION: If $\phi : \mathbf{1} \to A$ is a morphism and B an object in $\mathcal{C}_R$, then we write $B \xrightarrow{\cdot\,\phi} B \cdot A$ for the morphism $\mathrm{id}_B \cdot \phi$. In particular, any morphism $B \xrightarrow{\phi} A$ can be written as $B \xrightarrow{\cdot\,(\mathrm{id}_{B^{-1}} \cdot \phi)} A$.

REMARK 2.3. In this remark we let C denote the complex $P_0 \xrightarrow{\phi} P_1$, in which the first term is placed in degree 0 and $P_0 = P_1 = P$. Then, by definition, one has $\mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$. However, if ϕ is an isomorphism (so C is acyclic), then by property e) there is also a canonical morphism $\mathbf{1}_R \xrightarrow{acyc} \mathbf{d}_R(C)$. This latter morphism coincides with the composite

$$\mathbf{1}_R = \mathbf{d}_R(P_1)\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}_R(\phi)^{-1}\cdot \mathrm{id}_{\mathbf{d}_R(P_1)^{-1}}} \mathbf{d}_R(P_0)\mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

and thus depends on ϕ. Indeed, Remark 2.2 shows that the composite morphism

$$\mathbf{1}_R \xrightarrow{acyc} \mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$$

corresponds to the element $\mathbf{d}_R(\phi)^{-1}$ of $K_1(R)$. Thus, in order to distinguish between the above identifications of $\mathbf{1}_R$ with $\mathbf{d}_R(C)$, we shall say that C is *trivialized by the identity* when using either $\mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$ or its inverse with respect to composition.

Remark 2.4. Let $\mathcal{O} = \mathcal{O}_L$ be the valuation ring of a finite extension L of $\mathbb{Q}_p$ and A a finite $\mathcal{O}$-module. Then for any morphism in $\mathcal{C}_{\mathcal{O}}$ of the form $a : \mathbf{1}_{\mathcal{O}} \to \mathbf{d}_{\mathcal{O}}(A)$, and in particular therefore for that induced by any exact sequence of $\mathcal{O}$-modules of the form $0 \longrightarrow \mathcal{O}^n \longrightarrow \mathcal{O}^n \longrightarrow A \longrightarrow 0$, we obtain a canonical element $c = c(a) \in L^\times \cong \mathrm{Aut}_{\mathcal{C}_L}(\mathbf{1}_L)$ by means of the composite

$$\mathbf{1}_L \xrightarrow{a_L} L \otimes_{\mathcal{O}} \mathbf{d}_{\mathcal{O}}(A) = \mathbf{d}_L(L \otimes_{\mathcal{O}} A) \xrightarrow{acyc} \mathbf{1}_L$$

where the map 'acyc' is induced by property e). As an immediate consequence of the elementary divisor theorem one checks that $\mathrm{ord}_L(c) = \mathrm{length}_{\mathcal{O}}(A)$.

2.2. The localized K_1-group. In [16, §1.3] a *localized* K_1-group is defined for any full subcategory Σ of $C^p(R)$ which satisfies the following four conditions:

(i) $0 \in \Sigma$,

(ii) if C, C' are in $C^p(R)$ and C is quasi-isomorphic to C', then $C \in \Sigma \Leftrightarrow C' \in \Sigma$,

(iii) if $C \in \Sigma$, then $C[n] \in \Sigma$ for all $n \in \mathbb{Z}$,

(iv) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in $C^p(R)$ with both $C' \in \Sigma$ and $C'' \in \Sigma$, then $C \in \Sigma$.

Since we want to apply the same construction to a subcategory which is not necessarily closed under extensions, we weaken the last condition to

(iv′) if C' and C'' belong to Σ, then $C' \oplus C''$ belongs to Σ.

Definition 2.5. (Fukaya and Kato) Let Σ be any full subcategory of $C^p(R)$ which satisfies the conditions (i), (ii), (iii) and (iv′). Then the *localized K_1-group* $K_1(R, \Sigma)$ is defined to be the (multiplicatively written) abelian group which has as generators all symbols of the form $[C, a]$ where $C \in \Sigma$ and a is a morphism $\mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$, and as relations

(0) $[0, \mathrm{id}_{\mathbf{1}_R}] = 1$,

(1) $[C', \mathbf{d}_R(f) \circ a] = [C, a]$ if $f : C \to C'$ is a quasi-isomorphism with C (and thus also C') in Σ,

(2) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in Σ, then
$$[C,a] = [C',a'] \cdot [C'',a'']$$
where a is the composite of $a' \cdot a''$ with the isomorphism induced by property d),

(3) $[C[1], a^{-1}] = [C,a]^{-1}$.

Remark 2.6. Relation (3) is a simple consequence of the relations (0), (1) and (2). Note also that this definition of $K_1(R,\Sigma)$ makes no use of the conditions (iii) and (iv$'$) that the category Σ is assumed to satisfy. In particular, if Σ satisfies (iv) (rather than only (iv$'$)), then the above definition coincides with that given by Fukaya and Kato. We shall often refer to a morphism in $\mathcal{C}_R$ of the form $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ or $a : \mathbf{d}_R(C) \to \mathbf{1}_R$ as a *trivialization* (of C).

We now assume to be given a left denominator set S of R and we let $R_S := S^{-1}R$ denote the corresponding localization and Σ_S the full subcategory of $C^p(R)$ consisting of all complexes C such that $R_S \otimes_R C$ is acyclic. For any $C \in \Sigma_S$ and any morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ we write $\theta_{C,a}$ for the element of $K_1(R_S)$ which corresponds under the canonical isomorphism $K_1(R_S) \cong \mathrm{Aut}_{\mathcal{C}_{R_S}}(\mathbf{1}_{R_S})$ to the composite

$$\mathbf{1}_{R_S} \to \mathbf{d}_{R_S}(R_S \otimes_R C) \to \mathbf{1}_{R_S} \tag{1}$$

where the first arrow is induced by a and the second by the fact that $R_S \otimes_R C$ is acyclic. Then it can be shown that the assignment $[C,a] \mapsto \theta_{C,a}$ induces an isomorphism of groups
$$\mathrm{ch}_{R,\Sigma_S} : K_1(R,\Sigma_S) \cong K_1(R_S)$$
(cf. [16, Prop. 1.3.7]). Hence, if Σ is any subcategory of Σ_S we also obtain a composite homomorphism
$$\mathrm{ch}_{R,\Sigma} : K_1(R,\Sigma) \to K_1(R,\Sigma_S) \cong K_1(R_S).$$
In particular, we shall often use this construction in the following case: C is a fixed object of $D^p(R)$ which is such that $R_S \otimes_R C$ is acyclic and Σ denotes the smallest full subcategory Σ_C of $C^p(R)$ which contains all objects of $C^p(R)$ that are isomorphic in $D^p(R)$ to C and also satisfies the conditions (i), (ii), (iii) and (iv) that are described above. (With this definition, it is easily seen that $\Sigma_C \subset \Sigma_S$).

3. Leading terms

In this section we define a notion of the leading term at a continuous finite dimensional p-adic representation of elements of suitable localized K_1-groups. To do this we introduce an appropriate 'semisimplicity' hypothesis and use a natural construction of Bockstein homomorphisms. We also discuss several alternative characterizations of this notion. We explain how this formalism applies in the context of the canonical localizations introduced in [11] and we use it to extend several well known results concerning Generalized Euler-Poincaré characteristics.

3.1. Bockstein homomorphisms. Let G be a compact p-adic Lie group which contains a closed normal subgroup H such that the quotient group $\Gamma := G/H$ is topologically isomorphic to $\mathbb{Z}_p$. We fix a topological generator γ of Γ and denote by

$$\theta \in H^1(G, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Z}_p)$$

the unique homomorphism $G \twoheadrightarrow \Gamma \to \mathbb{Z}_p$ which sends γ to 1. We write $\Lambda(G)$ for the Iwasawa algebra of G. Then, since $H^1(G, \mathbb{Z}_p) \cong \mathrm{Ext}^1_{\Lambda(G)}(\mathbb{Z}_p, \mathbb{Z}_p)$ by [25, Prop. 5.2.14], the element θ corresponds to a canonical extension of $\Lambda(G)$-modules of the form

$$0 \to \mathbb{Z}_p \to E_\theta \to \mathbb{Z}_p \to 0. \tag{2}$$

Indeed, one has $E_\theta = \mathbb{Z}_p^2$ upon which G acts via the matrix $\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$.
For any $A^\bullet$ in $B(\Lambda(G))$ we endow the complex $A^\bullet \otimes_{\mathbb{Z}_p} E_\theta$ with the natural diagonal G-action. Then (2) induces an exact sequence in $B(\Lambda(G))$ of the form

$$0 \to A^\bullet \to A^\bullet \otimes_{\mathbb{Z}_p} E_\theta \to A^\bullet \to 0.$$

This sequence in turn induces a 'cup-product' morphism in $D^b(\Lambda(G))$ of the form

$$A^\bullet \xrightarrow{\theta} A^\bullet[1]. \tag{3}$$

It is clear that this morphism depends upon the choice of γ, but nevertheless we continue to denote it simply by θ.
We now let $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ be a (continuous) representation of G on $T_\rho := \mathcal{O}^n$, where $\mathcal{O} = \mathcal{O}_L$ denotes the valuation ring of a finite extension L of $\mathbb{Q}_p$. Then in the sequel we are mainly interested in the morphism

$$\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^\bullet \xrightarrow{\theta_*} \mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^\bullet[1]$$

that is induced by (3), where we consider $\mathcal{O}^n$ as a right $\Lambda(G)$-module via the transpose ρ^t of ρ. In particular, in each degree i we shall refer to the induced homomorphism

$$\mathfrak{B}_i : \mathbb{T}\mathrm{or}_i^{\Lambda(G)}(T_\rho, A^\bullet) \to \mathbb{T}\mathrm{or}_{i-1}^{\Lambda(G)}(T_\rho, A^\bullet)$$

of hyper-tor groups

$$\mathbb{T}\mathrm{or}_i^{\Lambda(G)}(T_\rho, A^\bullet) := H^{-i}(\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^\bullet)$$

as the *Bockstein homomorphism (in degree i) of* $(A^\bullet, T_\rho, \gamma)$.

3.2. The case $G = \Gamma$. In this section we consider the case $G = \Gamma$ and take the trivial Γ-module $\mathbb{Z}_p$ for ρ. We set $T := \gamma - 1 \in \Lambda(\Gamma)$.

3.2.1. *Bockstein homomorphisms.* For any complex $A^\bullet \in B(\Lambda(\Gamma))$ it is clear that the canonical short exact sequence

$$0 \to \Lambda(\Gamma) \xrightarrow{\times T} \Lambda(\Gamma) \to \mathbb{Z}_p \to 0$$

induces an exact triangle in $D^b(\Lambda(\Gamma))$ of the form

$$A^\bullet \xrightarrow{\times T} A^\bullet \to \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\bullet \to A^\bullet[1]. \tag{4}$$

However, in order to be as concrete as possible, we choose to describe this result on the level of complexes. To this end we fix the following definition of the mapping cone of a morphism $f : A^\bullet \to B^\bullet$ of complexes:

$$\mathrm{cone}(f) := B^\bullet \oplus A^\bullet[1],$$

with differential in degree i equal to

$$d^i_{\mathrm{cone}(f)} := \begin{pmatrix} d^i_{B^\bullet} & f^i \\ 0 & -d^{i+1}_{A^\bullet} \end{pmatrix} : B^i \oplus A^{i+1} \to B^{i+1} \oplus A^{i+2}.$$

If $A^\bullet$ is a bounded complex of projective $\Lambda(\Gamma)$-modules, then we set

$$\mathrm{cone}(A^\bullet) := \mathrm{cone}(A^\bullet \xrightarrow{T} A^\bullet)$$

and

$$A^\bullet_0 := \mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A^\bullet.$$

In any such case there exists a morphism of complexes $\pi : \mathrm{cone}(A^\bullet) \to A^\bullet_0$ of the form

$$\begin{tikzcd}
{} \arrow[r] & A^{i-1} \oplus A^i \arrow[r, "d^{i-1}_{\mathrm{cone}}"] \arrow[d, "\pi^{i-1}"'] & A^i \oplus A^{i+1} \arrow[r, "d^i_{\mathrm{cone}}"] \arrow[d, "\pi^i"'] & A^{i+1} \oplus A^{i+2} \arrow[r, "d^{i+1}_{\mathrm{cone}}"] \arrow[d, "\pi^{i+1}"'] & {} \\
{} \arrow[r] & A_0^{i-1} \arrow[r, "d^{i-1}_{A_0^\bullet}"] & A_0^i \arrow[r, "d^i_{A_0^\bullet}"] & {A_0}^{i+1} \arrow[r, "d^{i+1}_{A_0^\bullet}"] & {}
\end{tikzcd}$$

where, in each degree i, π^i sends $(a,b) \in A^i \oplus A^{i+1}$ to the image of a in $\mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A^i = A^i_0$. It is easy to check that π is a quasi-isomorphism.
Now from (4) we obtain short exact sequences

$$0 \to H^i(A^\bullet)_\Gamma \to \mathbb{H}_{-i}(\Gamma, A^\bullet) \to H^{i+1}(A^\bullet)^\Gamma \to 0 \tag{5}$$

where

$$\mathbb{H}_i(\Gamma, A^\bullet) := \mathbb{T}\mathrm{or}_i^{\Lambda(\Gamma)}(\mathbb{Z}_p, A^\bullet)$$

denotes the hyper-homology of $A^\bullet$ (with respect to Γ) and for any $\Lambda(\Gamma)$-module M we write $M_\Gamma = M/TM$ and $M^\Gamma = {}_TM$ (= kernel of multiplication by T) for the maximal quotient module, resp. submodule, of M upon which Γ acts trivially.

Lemma 3.1. *Let $A^\bullet$ be a bounded complex of projective $\Lambda(\Gamma)$-modules. Then in each degree i the Bockstein homomorphism of the triple $(A^\bullet, \mathbb{Z}_p, \gamma)$ coincides with the composite*

$$\mathbb{H}_i(\Gamma, A^\bullet) \to H^{-i+1}(A^\bullet)^\Gamma \xrightarrow{\kappa^{-i+1}(A^\bullet)} H^{-i+1}(A^\bullet)_\Gamma \to \mathbb{H}_{i-1}(\Gamma, A^\bullet)$$

where the first and third arrows are as in (5) and $\kappa^{-i+1}(A^\bullet)$ denotes the tautological homomorphism

$$H^{-i+1}(A^\bullet)^\Gamma \hookrightarrow H^{-i+1}(A^\bullet) \twoheadrightarrow H^{-i+1}(A^\bullet)_\Gamma.$$

Proof. As is shown by Rapoport and Zink in [30, Lem. 1.2], on the level of complexes the cup product morphism of the triple $(A^\bullet, \mathbb{Z}_p, \gamma)$ is described by the morphism

$$\theta : \mathrm{cone}(A^\bullet) \to \mathrm{cone}(A^\bullet)[1]$$

which sends $(a,b) \in A^i \oplus A^{i+1}$ to $(b,0) \in A^{i+1} \oplus A^{i+2}$. Now let $\bar{a}$ be in $\ker(d^{-i}_{A^\bullet_0})$ representing a class in $\mathbb{H}_i(\Gamma, A^\bullet)$. Then there exists $(a,b) \in \ker(d^{-i}_{\mathrm{cone}})$ with $\pi^{-i}((a,b)) = \bar{a}$. Since $(a,b) \in \ker(d^{-i}_{\mathrm{cone}})$ one has $b \in \ker(d^{i+1}_{A^\bullet})$ and $Tb = -d^i_{A^\bullet}(a)$. This implies that $d^i_{A^\bullet}(a)$ is divisible by T (in A^{i+1}) and also that $b = -T^{-1}d^i_{A^\bullet}(a) \in A^{i+1}$. Thus θ maps (a,b) to $(-T^{-1}d^i_{A^\bullet}(a), 0)$ and the class in $\mathbb{H}_{i-1}(\Gamma, A^\bullet)$ is represented by $-\overline{T^{-1}d^i_{A^\bullet}(a)} \in \ker(d^{-i+1}_{A^\bullet_0})$. By using the canonical short exact sequence

$$0 \to A^\bullet \to \mathrm{cone}(A^\bullet) \to A^\bullet[1] \to 0$$

one immediately verifies that $\mathfrak{B}_i$ coincides with the composite homomorphism described in the lemma. $\square$

From this description it is clear that for any bounded complex of projective $\Lambda(\Gamma)$-modules $A^\bullet$ the pair

$$(\mathbb{H}_i(\Gamma, A^\bullet), \mathfrak{B}_i) \tag{6}$$

forms a homological complex (which, by re-indexing, we shall consider as cohomological complex whenever convenient). It is also clear that this construction extends in a well-defined fashion to objects $A^\bullet$ of $D^p(\Lambda(\Gamma))$.

3.2.2. *Semisimplicity.*

Definition 3.2. *(Semisimplicity)* For any $A^\bullet \in D^p(\Lambda(\Gamma))$ we set

$$r_\Gamma(A^\bullet) := \sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}_p}(H^i(A^\bullet)^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \in \mathbb{Z}.$$

We say that a complex $A^\bullet \in D^p(\Lambda(\Gamma))$ is *semisimple* if the cohomology of the associated complex (6) is $\mathbb{Z}_p$-torsion (and hence finite) in all degrees. We let Σ_{ss} denote the full subcategory of $C^p(\Lambda(\Gamma))$ consisting of those complexes that are semisimple.

Remark 3.3. (i) If $A^\bullet \in D^p(\Lambda(\Gamma))$ is semisimple, then the cohomology of $A^\bullet$ is a torsion $\Lambda(\Gamma)$-module in all degrees.
(ii) In each degree i Lemma 3.1 gives rise to a canonical exact sequence

$$0 \to \mathrm{cok}(\kappa^{-i}(A^\bullet)) \to \ker(\mathfrak{B}_i)/\mathrm{im}(\mathfrak{B}_{i+1}) \to \ker(\kappa^{-i+1}(A^\bullet)) \to 0.$$

This implies that a complex $A^\bullet \in D^p(\Lambda(\Gamma))$ is semisimple if and only if the homomorphism $\kappa^i(A^\bullet)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is bijective in each degree i, and hence also that in any such case one has

$$r_\Gamma(A^\bullet)=\sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{Q}_p}(H^i(A^\bullet)_\Gamma\otimes_{\mathbb{Z}_p}\mathbb{Q}_p).$$

DEFINITION 3.4. *(The canonical trivialization)* For each $A^\bullet\in D^b(\Lambda(\Gamma))$ we write $(\mathbb{H}_\bullet(\Gamma,A^\bullet),0)$ for the complex with $(\mathbb{H}_\bullet(\Gamma,A^\bullet),0)^i=\mathbb{H}_i(\Gamma,A^\bullet)$ in each degree i and in which all differentials are the zero map. In particular, if $A^\bullet\in\Sigma_{\mathrm{ss}}$, then we obtain a canonical composite morphism

$$\begin{aligned}(7)\quad t(A^\bullet):\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p\otimes_{\Lambda(\Gamma)}A^\bullet)_{\mathbb{Q}_p}&\cong\mathbf{d}_{\mathbb{Z}_p}((\mathbb{H}_\bullet(\Gamma,A^\bullet),0))_{\mathbb{Q}_p}\\&=\mathbf{d}_{\mathbb{Z}_p}((\mathbb{H}_\bullet(\Gamma,A^\bullet),\mathfrak{B}_\bullet))_{\mathbb{Q}_p}\cong\mathbf{1}_{\mathbb{Q}_p}\end{aligned}$$

where the first, resp. last, morphism uses property h) (in §2.1) for the functor $\mathbf{d}_{\mathbb{Z}_p}$, resp. property i) for the natural homomorphism $\mathbb{Z}_p\to\mathbb{Q}_p$ and then property e) for the functor $\mathbf{d}_{\mathbb{Q}_p}$.

REMARK 3.5. If the complex $\mathbb{Q}_p\otimes_{\Lambda(\Gamma)}A^\bullet$ is acyclic, then $t(A^\bullet)$ coincides with the trivialization obtained by directly applying property e) to $\mathbb{Q}_p\otimes_{\Lambda(\Gamma)}A^\bullet$.

The category Σ_{ss} satisfies the conditions (i), (ii), (iii) and (iv$'$) that are described in §2 (but does not satisfy condition (iv)). In addition, as the following result shows, the above constructions behave well on short exact sequences of semisimple complexes.

LEMMA 3.6. *Let $A^\bullet,B^\bullet$ and $C^\bullet$ be objects of Σ_{ss} which together lie in a short exact sequence in $C^p(\Lambda(\Gamma))$ of the form*

$$0\to A^\bullet\to B^\bullet\to C^\bullet\to 0.$$

Then one has

$$r_\Gamma(B^\bullet)=r_\Gamma(A^\bullet)+r_\Gamma(C^\bullet)$$

and, with respect to the canonical morphism

$$\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p\otimes_{\Lambda(\Gamma)}B^\bullet)_{\mathbb{Q}_p}\cong\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p\otimes_{\Lambda(\Gamma)}A^\bullet)_{\mathbb{Q}_p}\cdot\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p\otimes_{\Lambda(\Gamma)}C^\bullet)_{\mathbb{Q}_p}$$

that is induced by the given short exact sequence, one has

$$t(B^\bullet)=t(A^\bullet)\cdot t(C^\bullet).$$

Proof. We let $\mathfrak{p}$ denote the kernel of the augmentation map $\Lambda(\Gamma)\to\mathbb{Z}_p$ and R the localization $\Lambda(\Gamma)_\mathfrak{p}$ of $\Lambda(\Gamma)$ at $\mathfrak{p}$. Then R is a discrete valuation ring with uniformizer T and residue class field $R/(T)$ naturally isomorphic to $\mathbb{Q}_p$. Further, if a complex $K^\bullet\in D^p(\Lambda(\Gamma))$ is semisimple, then the structure theory of finitely generated $\Lambda(\Gamma)$-modules implies that in each degree i the R-module $H^i(K^\bullet_\mathfrak{p})$ is isomorphic to a direct sum of (finitely many) copies of $R/(T)$ and hence also to $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^i(K^\bullet)_\Gamma\cong\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^i(K^\bullet)^\Gamma$.

To prove the claimed equality $r_\Gamma(B^\bullet) = r_\Gamma(A^\bullet) + r_\Gamma(C^\bullet)$ it is therefore enough to take dimensions over $\mathbb{Q}_p \cong R/(T)$ in the long exact cohomology sequence of the following short exact sequence in $C^p(R)$

$$0 \to A^\bullet_{\mathfrak{p}} \to B^\bullet_{\mathfrak{p}} \to C^\bullet_{\mathfrak{p}} \to 0. \tag{8}$$

To prove the second claim we note that if $K^\bullet \in C^p(\Lambda(\Gamma))$, then the complex $K^\bullet_{0,\mathfrak{p}} := \mathbb{Q}_p \otimes_{\Lambda(\Gamma)} K^\bullet$ is isomorphic in $D^p(\mathbb{Q}_p)$ to $\mathbb{Q}_p \otimes_R K^\bullet_{\mathfrak{p}}$. Hence, since each term of $C^\bullet_{\mathfrak{p}}$ is a projective R-module, the short exact sequence (8) gives rise to a short exact sequence in $C^p(\mathbb{Q}_p)$ of the form

$$0 \to A^\bullet_{0,\mathfrak{p}} \to B^\bullet_{0,\mathfrak{p}} \to C^\bullet_{0,\mathfrak{p}} \to 0. \tag{9}$$

Now one has a commutative diagram in $\mathcal{C}_{\mathbb{Q}_p}$

$$\begin{array}{ccc}
\mathbf{d}_{\mathbb{Q}_p}(B^\bullet_{0,\mathfrak{p}}) & \longrightarrow & \mathbf{d}_{\mathbb{Q}_p}(A^\bullet_{0,\mathfrak{p}})\mathbf{d}_{\mathbb{Q}_p}(C^\bullet_{0,\mathfrak{p}}) \\
\downarrow & & \downarrow \\
\mathbf{d}_{\mathbb{Q}_p}(H(B^\bullet_{0,\mathfrak{p}})) & \longrightarrow & \mathbf{d}_{\mathbb{Q}_p}(H(A^\bullet_{0,\mathfrak{p}}))\mathbf{d}_{\mathbb{Q}_p}(H(C^\bullet_{0,\mathfrak{p}}))
\end{array}$$

in which the upper, resp. lower, horizontal morphism is induced by (9), resp, by the long exact cohomology sequence of (9), and both vertical arrows are induced by applying property h) of $\mathbf{d}_{\mathbb{Q}_p}$ in §2.1. (For a proof of the commutativity of the above diagram see [2, Thm. 3.3].) Further, in this situation the exact sequences (5) induce short exact sequences $0 \to H^i(A^\bullet_{\mathfrak{p}}) \to H^i(A^\bullet_{0,\mathfrak{p}}) \to H^{i+1}(A^\bullet_{\mathfrak{p}}) \to 0$ (and similarly for $B^\bullet$ and $C^\bullet$) which together lie in a short exact sequence of long exact sequences

$$\begin{array}{cccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H^i(A^\bullet_{\mathfrak{p}}) & \to & H^i(B^\bullet_{\mathfrak{p}}) & \to & H^i(C^\bullet_{\mathfrak{p}}) & \to & H^{i+1}(A^\bullet_{\mathfrak{p}}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H^i(A^\bullet_{0,\mathfrak{p}}) & \to & H^i(B^\bullet_{0,\mathfrak{p}}) & \to & H^i(C^\bullet_{0,\mathfrak{p}}) & \to & H^{i+1}(A^\bullet_{0,\mathfrak{p}}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H^{i+1}(A^\bullet_{\mathfrak{p}}) & \to & H^{i+1}(B^\bullet_{\mathfrak{p}}) & \to & H^{i+1}(C^\bullet_{\mathfrak{p}}) & \to & H^{i+2}(A^\bullet_{\mathfrak{p}}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

where the upper and lower, resp. central, row is the exact cohomology sequence of (8), resp. (9). It is now a straightforward exercise to derive the required equality $t(B^\bullet) = t(A^\bullet) \cdot t(C^\bullet)$ from the commutativity of both of the above diagrams. $\square$

3.2.3. *Leading terms.* We write ρ_{triv} for the trivial representation of Γ.

DEFINITION 3.7. *(The leading term)* For each $A^\bullet \in \Sigma_{\mathrm{ss}}$ and each morphism $a : \mathbf{1}_{\Lambda(\Gamma)} \to \mathbf{d}_{\Lambda(\Gamma)}(A^\bullet)$ in $\mathcal{C}_{\Lambda(\Gamma)}$ we define the *leading term* $(A^\bullet, a)^*(\rho_{\mathrm{triv}})$ *of the pair* $(A^\bullet, a)$ *at* ρ_{triv} to be equal to $(-1)^{r_\Gamma(A^\bullet)}$ times the element of $\mathbb{Q}_p\backslash\{0\}$ which corresponds via the canonical isomorphisms $\mathbb{Q}_p^\times \cong K_1(\mathbb{Q}_p) \cong \mathrm{Aut}_{\mathcal{C}_{\mathbb{Q}_p}}(\mathbf{1}_{\mathbb{Q}_p})$ to the composite morphism

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\mathbb{Q}_p \otimes_{\Lambda(\Gamma)} a} \mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A^\bullet)_{\mathbb{Q}_p} \xrightarrow{t(A^\bullet)} \mathbf{1}_{\mathbb{Q}_p}.$$

After taking Lemma 3.6 into account, it can be shown that this construction induces a well defined homomorphism of groups

$$\begin{aligned}(-)^*(\rho_{\mathrm{triv}}) : K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}}) &\to \mathbb{Q}_p^\times \\ [A^\bullet, a] &\mapsto [A^\bullet, a]^*(\rho_{\mathrm{triv}}) := (A^\bullet, a)^*(\rho_{\mathrm{triv}}).\end{aligned}$$

In particular therefore, (property g) of the functor $\mathbf{d}_{\Lambda(\Gamma)}$ combines with relation (1) in the definition of $K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}})$ to imply that) the notation $[A^\bullet, a]^*(\rho_{\mathrm{triv}})$ extends in a well-defined fashion to pairs of the form $(A^\bullet, a)$ where $A^\bullet \in D^p(\Lambda(\Gamma))$ is semisimple and a is a morphism in $\mathcal{C}_{\Lambda(\Gamma)}$ of the form $\mathbf{1}_{\Lambda(\Gamma)} \to \mathbf{d}_{\Lambda(\Gamma)}(A^\bullet)$.

The reason for the occurrence of ρ_{triv} in the above definition will become clear in the next subsection. In the remainder of the current section we justify the name 'leading term' by explaining the connection between $(A^\bullet, a)^*(\rho_{\mathrm{triv}})$ and the leading term (in the usual sense) of an appropriate characteristic power series.

To this end we note that Remark 3.3(i) implies that Σ_{ss} is a subcategory of the full subcategory of $C^p(\Lambda(\Gamma))$ consisting of those complexes C for which $Q(\Gamma) \otimes_{\Lambda(\Gamma)} C$ is acyclic, where we write $Q(\Gamma)$ for the quotient field of $\Lambda(\Gamma)$. Hence there exists a homomorphism

$$\mathrm{ch}_\Gamma := \mathrm{ch}_{\Lambda(\Gamma), \Sigma_{\mathrm{ss}}} : K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}}) \to K_1(Q(\Gamma)) \cong Q(\Gamma)^\times.$$

Now the identification between $\Lambda(\Gamma)$ and the power series ring $\mathbb{Z}_p[[T]]$ (which, of course, depends on the choice of $T = \gamma - 1$) allows any element F of $Q(\Gamma)^\times$ to be written uniquely as

$$F(T) = T^r G(T) \tag{10}$$

with $r = r(F) \in \mathbb{Z}$ and $G(T) \in Q(\Gamma)$ such that $G(0) \in \mathbb{Q}_p^\times$. The leading coefficient of F with respect to its expansion in the Laurent series ring $\mathbb{Q}_p\{\{T\}\}$ is therefore equal to $F^*(0) := G(0)$.

PROPOSITION 3.8. *Let $A^\bullet$ be any object of $D^p(\Lambda(\Gamma))$ which is semisimple and a any morphism in $\mathcal{C}_{\Lambda(\Gamma)}$ of the form $\mathbf{1}_{\Lambda(\Gamma)} \to \mathbf{d}_{\Lambda(\Gamma)}(A^\bullet)$.*

(i) (Order of vanishing) *For $\mathcal{L} := [A^\bullet, a]$ one has $r(\mathrm{ch}_\Gamma(\mathcal{L})) = r_\Gamma(A^\bullet)$.*

(ii) (Leading terms) *One has a commutative diagram of abelian groups*

$$\begin{array}{ccc} K_1(\Lambda(\Gamma),\Sigma_{\mathrm{ss}}) & \xrightarrow{\mathrm{ch}_\Gamma} & K_1(Q(\Gamma)) \\ {\scriptstyle (-)^*(\rho_{\mathrm{triv}})}\big\downarrow & & \big\downarrow{\scriptstyle (-)^*(0)} \\ \mathbb{Q}_p^\times & = & \mathbb{Q}_p^\times. \end{array}$$

Proof. We use the localization R of $\Lambda(\Gamma)$ that was introduced in the proof of Lemma 3.6.
It is easy to see that both of the homomorphisms $(-)^*(\rho_{\mathrm{triv}})$ and ch_Γ factor via the flat base change $R\otimes_{\Lambda(\Gamma)} -$ through $K_1(R,\Xi)$, where Ξ denotes the full subcategory of $C^p(R)$ consisting of those complexes $K^\bullet$ with the property that in each degree i the R-module $H^i(K^\bullet)$ is isomorphic to a direct sum of (finitely many) copies of $R/(T)$. Thus it suffices to show the commutativity of the above diagram with $K_1(\Lambda(\Gamma),\Sigma_{\mathrm{ss}})$ replaced by $K_1(R,\Xi)$. Moreover, by Lemma 3.9 below this is reduced to the case where $A^\bullet$ is a complex of the form $R\xrightarrow{d} R$ where R occurs in degrees -1 and 0 and d denotes multiplication by either T or 1. Further, since the complex $R\xrightarrow{\times 1} R$ is acyclic we shall therefore assume that d denotes multiplication by T.
Now $\mathrm{Mor}_{\mathcal{C}_R}(\mathbf{1}_R,\mathbf{d}_R(A^\bullet))$ is a $K_1(R)$-torsor and so all possible trivializations arise in the following way: if ϵ is any fixed element of $R^\times$, then the R-module homomorphism $R\to A^{-1}$, resp. $R\to A^0$, that sends $1\in R$ to $1\in R$, resp. to $\epsilon\in R$, induces a morphism $can_1:\mathbf{d}_R(R)\to\mathbf{d}_R(A^{-1})$, resp. $can_\epsilon:\mathbf{d}_R(R)\to\mathbf{d}_R(A^0)$, in $\mathcal{C}_R$, and hence also a morphism $a_\epsilon:=(can_1)^{-1}\cdot can_\epsilon:\mathbf{1}_R\to\mathbf{d}_R(A^\bullet)$. Setting $\mathcal{L}_\epsilon:=[A^\bullet,a_\epsilon]\in K_1(R,\Xi)$, one checks easily that $\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon)=T^{-1}\epsilon$ and thus $\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon)^*(0)=\epsilon(0)$. On the other hand, the Bockstein homomorphism $\mathfrak{B}_1$ of the triple $(A^\bullet,R/(T),\gamma)$ is equal to $\mathbb{Q}_p\xrightarrow{-1}\mathbb{Q}_p$ as one checks by using the description given in the proof of Lemma 3.1. Thus $\mathcal{L}_\epsilon^*(\rho_{\mathrm{triv}})$ is, by definition, equal to $(-1)^{r_\Gamma(A^\bullet)}$ times the determinant of

$$\mathbb{Q}_p\xrightarrow{\epsilon(0)}\mathbb{Q}_p\xrightarrow{(\mathfrak{B}_1)^{-1}=-1}\mathbb{Q}_p\xrightarrow{1}\mathbb{Q}_p.$$

Hence, observing that $r_\Gamma(A^\bullet)=-1=r(\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon))$, we have $\mathcal{L}_\epsilon^*(\rho_{\mathrm{triv}})=\epsilon(0)=\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon)^*(0)$. This proves both claims of the Proposition. □

Lemma 3.9. *Let R be a discrete valuation ring with uniformizer T and assume that $A^\bullet\in C^p(R)$ is such that in each degree i the R-module $H^i(A^\bullet)$ is annihilated by T. Then $A^\bullet$ is isomorphic in $C^p(R)$ to the direct sum of finitely many complexes of the form $R\to R$ where the differential is equal to multiplication by either 1 or T.*

Proof. Assume that m is the maximal degree such that $A^m\neq 0$ and fix an isomorphism $D:R^d\cong A^m$. Let $(e_1,\dots,e_d)$ be the standard basis of R^d. Then, by assumption, for each integer i with $1\le i\le d$, one has $Te_i\in\mathrm{im}(D^{-1}\circ d^{m-1})$. For each such i we set $h_i:=1$ if $e_i\in\mathrm{im}(D^{-1}\circ d^{m-1})$ and, otherwise, we set $h_i:=T$. We write H for the diagonal $d\times d$-matrix with entries $h_1,\dots,h_d$.

Then, since the image of the map $R^d \xrightarrow{H} R^d$ is equal to $\mathrm{im}(D^{-1}\circ d^{m-1})$, there exists a retraction $E:\ R^d \to A^{m-1}$ (i.e. with left inverse '$H^{-1}\circ D^{-1}\circ d^{m-1}$') that makes the following diagram commutative

$$\begin{tikzcd}
\arrow[r] & 0 \arrow[r] \arrow[d] & R^d \arrow[r, "H"] \arrow[d, "E"'] & R^d \arrow[r] \arrow[d, "D"'] & 0 \arrow[r] \arrow[d] & {} \\
\arrow[r] & A^{m-2} \arrow[r, "d^{m-2}"] & A^{m-1} \arrow[r, "d^{m-1}"] & A^m \arrow[r, "d^m"] & 0 \arrow[r] & {} .
\end{tikzcd}$$

Now if $B^\bullet$ denotes the upper row of this diagram and $C^\bullet := A^\bullet/B^\bullet$ the associated quotient complex (not the mapping cone!), then one checks readily that there exists a *split* exact sequence $0 \to B^\bullet \to A^\bullet \to C^\bullet \to 0$. This implies that $C^\bullet$ belongs to $C^p(R)$ and has cohomology annihilated by T (in all degrees). Thus, since the length of $C^\bullet$ is strictly shorter than the length of $A^\bullet$, the proof can be completed by induction. $\square$

Remark 3.10. It will be clear to the reader that analogous statements hold for all results of this subsection if we replace $\mathbb{Z}_p$ by $\mathcal{O}$, $\mathbb{Q}_p$ by L, $\Lambda(\Gamma)$ by $\Lambda_{\mathcal{O}}(\Gamma) := \mathcal{O}[[\Gamma]]$ and $Q(\Gamma)$ by the quotient field $Q_{\mathcal{O}}(\Gamma)$ of $\Lambda_{\mathcal{O}}(\Gamma)$.

3.3. The general case. We extend the constructions of §3.2 to the setting of the Bockstein homomorphisms that are discussed at the end of §3.1.

If $A^\bullet \in C^p(\Lambda(G))$, then for any continuous representation of G of the form $\rho: G \to \mathrm{GL}_n(\mathcal{O})$ we regard the complex

$$A^\bullet(\rho^*) := \mathcal{O}^n \otimes_{\mathbb{Z}_p} A^\bullet$$

as a complex of (left) $\Lambda_{\mathcal{O}}(G)$-modules by means of the following G-action: $g(x\otimes_{\mathbb{Z}_p} a) := \rho^*(g)(x)\otimes_{\mathbb{Z}_p} g(a)$ for each $g\in G$, $x\in\mathcal{O}^n$ and $a\in A^i$. With this action, there exists a natural isomorphism in $C^p(\mathbb{Z}_p)$ between $\mathbb{Z}_p\otimes_{\Lambda(G)} A^\bullet(\rho^*)$ and the complex $\mathcal{O}^n\otimes_{\Lambda(G)} A^\bullet$ that occurs in §3.1. Further, it can be shown that the Bockstein homomorphisms $\mathfrak{B}_\bullet$ of the triple $(A^\bullet, T_\rho, \gamma)$ give rise to a complex of the form $(\mathbb{H}_\bullet(G, A^\bullet(\rho^*)), \mathfrak{B}_\bullet)$ where for each integer i and each normal closed subgroup J of G we set

$$\mathbb{H}_i(J, A^\bullet(\rho^*))) := H^{-i}(\mathbb{Z}_p\otimes_{\Lambda(J)} A^\bullet(\rho^*)) \cong \mathbb{T}\mathrm{or}_i^{\Lambda(J)}(T_\rho, A^\bullet)$$

(see, for example, the proof of Lemma 3.13 below).

Definition 3.11. *(Semisimplicity at ρ)* For each $A^\bullet\in D^p(\Lambda(G))$ we set

$$r_G(A^\bullet)(\rho) := \sum_{i\in\mathbb{Z}} (-1)^{i+1}\dim_L\left(\mathbb{H}_i(H, A^\bullet(\rho^*))^\Gamma \otimes_{\mathcal{O}} L\right) \in \mathbb{Z},$$

where L is the fraction field of $\mathcal{O}$. We say that a complex $A^\bullet\in D^p(\Lambda(G))$ is *semisimple at* ρ if the cohomology of the associated complex $(\mathbb{H}_\bullet(G, A^\bullet(\rho^*)), \mathfrak{B}_\bullet)$ is $\mathbb{Z}_p$-torsion in each degree. We let $\Sigma_{\mathrm{ss}-\rho}$ denote the full subcategory of $C^p(\Lambda(G))$ consisting of those complexes that are semisimple at ρ, and we note that $\Sigma_{\mathrm{ss}-\rho}$ satisfies the conditions (i), (ii), (iii) and (iv′) that are described in §2.

Definition 3.12. *(Finiteness at ρ)* We say that a complex $A^\bullet \in D^p(\Lambda(G))$ is *finite at* ρ if the groups $\mathbb{H}_i(G, A^\bullet(\rho^*))$ are $\mathbb{Z}_p$-torsion in all degrees i. We let $\Sigma_{\mathrm{fin}-\rho}$ denote the full subcategory of $C^p(\Lambda(G))$ consisting of those complexes that are finite at ρ, and we note that $\Sigma_{\mathrm{fin}-\rho}$ satisfies the conditions (i), (ii), (iii) and (iv) that are described in §2. In particular we have $\Sigma_{\mathrm{fin}-\rho} \subseteq \Sigma_{\mathrm{ss}-\rho}$.

In the next result we consider the tensor product $\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n$ as an $(\Lambda_{\mathcal{O}}(\Gamma), \Lambda(G))$-bimodule where $\Lambda_{\mathcal{O}}(\Gamma)$ acts by multiplication on the left and $\Lambda(G)$ acts on the right via the rule $(\tau \otimes_{\mathcal{O}} x)g := \tau\bar{g} \otimes_{\mathcal{O}} \rho(g)^t(x)$ for each $g \in G$ (with image $\bar{g}$ in Γ), $x \in \mathcal{O}^n$ and $\tau \in \Lambda_{\mathcal{O}}(\Gamma)$. For each complex $A^\bullet \in \Sigma_{\mathrm{ss}-\rho}$ we then set

$$A^\bullet_\rho := (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes_{\Lambda(G)} A^\bullet \in C^p(\Lambda_{\mathcal{O}}(\Gamma)).$$

Lemma 3.13. *Fix $A^\bullet \in C^p(\Lambda(G))$.*

(i) *There are natural quasi-isomorphisms in $C^p(\Lambda_{\mathcal{O}}(\Gamma))$ of the form*

$$A^\bullet_\rho \cong \Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*) \cong \mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(H)} A^\bullet(\rho^*).$$

(ii) *One has $r_G(A^\bullet)(\rho) = r_\Gamma(A^\bullet_\rho)$.*

(iii) *The Bockstein homomorphism in any given degree of $(A^\bullet, T_\rho, \gamma)$ (as defined in §3.1) coincides with the Bockstein homomorphism in the same degree of $(A^\bullet_\rho, \mathbb{Z}_p, \gamma)$.*

(iv) *One has $A^\bullet \in \Sigma_{\mathrm{ss}-\rho}$ if and only if $A^\bullet_\rho \in \Sigma_{\mathrm{ss}}$ (when considered as an object of $C^p(\Lambda_{\mathcal{O}}(\Gamma))$). Further, if this is the case, then the trivialization*

$$t(A^\bullet_\rho) : \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(\Gamma)} A^\bullet_\rho)_L \to \mathbf{1}_L$$

that is defined as in (7) *coincides with the composite morphism*

$$\begin{aligned} t(A^\bullet(\rho^*)) : \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*))_L &\cong \mathbf{d}_{\mathcal{O}}((\mathbb{H}_\bullet(G, A^\bullet(\rho^*)), 0))_L \\ &= \mathbf{d}_{\mathcal{O}}((\mathbb{H}_\bullet(G, A^\bullet(\rho^*)), \mathfrak{B}_\bullet))_L \cong \mathbf{1}_L \end{aligned} \tag{11}$$

where the first, resp. last, morphism uses property h) (in §2.1) for the functor $\mathbf{d}_{\mathcal{O}}$, resp. property i) for the homomorphism $\mathcal{O} \to L$ and then property e) for the functor $\mathbf{d}_L$.

(v) *If $A^\bullet, B^\bullet$ and $C^\bullet$ are objects of $\Sigma_{\mathrm{ss}-\rho}$ which together lie in a short exact sequence in $C^p(\Lambda(G))$ of the form*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0,$$

then one has

$$r_G(B^\bullet)(\rho) = r_G(A^\bullet)(\rho) + r_G(C^\bullet)(\rho)$$

and, with respect to the canonical morphism

$$\mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} B^\bullet(\rho^*))_L = \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*))_L \cdot \mathbf{d}_O(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} C^\bullet(\rho^*))_L$$

that is induced by the given short exact sequence, one has

$$t(B^\bullet(\rho^*)) = t(A^\bullet(\rho^*)) \cdot t(C^\bullet(\rho^*)).$$

Proof. Claim (i) is clear (given the specified actions). Claim (ii) then follows by using the isomorphisms of claim (i) to directly compare the definitions of $r_G(A^\bullet)(\rho)$ and $r_\Gamma(A^\bullet_\rho)$. In a similar way, claims (iii) and (iv) follow from the functorial construction of Bockstein homomorphisms and the fact that there are natural isomorphisms in $C^p(\mathcal{O})$ of the form

$$\begin{aligned} \mathcal{O}^n \otimes_{\Lambda(G)} A^\bullet &\cong \mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*) \\ &\cong \mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(\Gamma)} \big(\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*)\big) \\ &\cong \mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A^\bullet_\rho. \end{aligned}$$

Finally, to prove claim (v) we observe that, by claim (i), the given short exact sequence gives rise to a short exact sequence of semisimple complexes in $C^p(\Lambda_{\mathcal{O}}(\Gamma))$ of the form

$$0 \to A^\bullet_\rho \to B^\bullet_\rho \to C^\bullet_\rho \to 0.$$

The equalities of claim (v) thus follow from claims (ii), (iii) and (iv) and the results of Lemma 3.6 as applied to the last displayed short exact sequence. □

Definition 3.14. *(The leading term at ρ)* For each complex $A^\bullet \in \Sigma_{\mathrm{ss}-\rho}$ and each morphism $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^\bullet)$ in $\mathcal{C}_{\Lambda(G)}$ we define the *leading term* $(A^\bullet, a)^*(\rho)$ *of the pair* $(A^\bullet, a)$ *at* ρ to be equal to $(-1)^{r_G(A^\bullet)(\rho)}$ times the element of $L \setminus \{0\}$ which corresponds via the canonical isomorphisms $L^\times \cong K_1(L) \cong \mathrm{Aut}_{\mathcal{C}_L}(\mathbf{1}_L)$ to the composite morphism

$$\mathbf{1}_L \xrightarrow{L^n \otimes_{\Lambda(G)} a} \mathbf{d}_L(L^n \otimes_{\Lambda(G)} A^\bullet) \xrightarrow{t(A^\bullet(\rho^*))} \mathbf{1}_L.$$

Then, since $\Sigma_{A^\bullet} \subset \Sigma_{\mathrm{ss}-\rho}$, Lemma 3.13(v) can be used to show that this construction induces a well-defined homomorphism of groups

$$\begin{aligned} (-)^*(\rho) : K_1(\Lambda(G), \Sigma_{A^\bullet}) &\to L^\times \\ [A^\bullet, a] &\mapsto [A^\bullet, a]^*(\rho) := (A^\bullet, a)^*(\rho). \end{aligned}$$

In particular, (property g) of the functor $\mathbf{d}_{\Lambda(G)}$ combines with relation (1) in the definition of $K_1(\Lambda(G), \Sigma_{A^\bullet})$ to imply that) the notation $[A^\bullet, a]^*(\rho)$ extends in a well-defined fashion to pairs of the form $(A^\bullet, a)$ where $A^\bullet \in D^p(\Lambda(G))$ is semisimple at ρ and a is a morphism in $\mathcal{C}_{\Lambda(G)}$ of the form $\mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^\bullet)$.

If $A^\bullet$ is clear from the context, then we often write $a^*(\rho)$ in place of $[A^\bullet, a]^*(\rho)$. It is easily checked that (in the case $G = \Gamma$ and $\rho = \rho_{\mathrm{triv}}$) these definitions are compatible with those given in §3.2. Further, in §3.4.3 we shall reinterpret the expression $[A^\bullet, a]^*(\rho)$ defined above as the leading term at $s = 0$ of a natural p-adic meromorphic function.

Remark 3.15. If $A^\bullet \in D^p(\Lambda(G))$ is both semisimple at ρ and such that $r_G(A^\bullet)(\rho) = 0$ (which is the case, for example, if $A^\bullet$ is finite at ρ), then we set $[A^\bullet, a](\rho) := [A^\bullet, a]^*(\rho)$ and refer to this as the *value of* $[A^\bullet, a]$ *at* ρ. In particular, after taking account of Remark 3.5, it is clear that this definition coincides with that given in [16, 4.1.5].

3.4. Canonical localizations. We apply the constructions of §3.3 in the setting of the canonical localizations of $\Lambda(G)$ that were introduced in [11].

3.4.1. *The canonical Ore sets.* We recall from [11, §2-§3] that there are canonical left and right denominator sets S and S^* of $\Lambda(G)$ where

$$S := \{\lambda \in \Lambda(G) : \Lambda(G)/\Lambda(G)\lambda \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and

$$S^* := \bigcup_{i\geq 0} p^i S.$$

We write S^*-tor for the category of finitely generated $\Lambda(G)$-modules M which satisfy $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} M = 0$. We further recall from loc. cit. that a finitely generated $\Lambda(G)$-module M belongs to S^*-tor, if and only if $M/M(p)$ is finitely generated when considered as a $\Lambda(H)$-module (by restriction) where $M(p)$ denotes the submodule of M consisting of those elements that are annihilated by some power of p.

3.4.2. *Leading terms.* In this subsection we use the notation of Definition 3.14 and the isomorphism $K_1(\Lambda(G), \Sigma_{S^*}) \cong K_1(\Lambda(G)_{S^*})$ described at the end of §2.2.

If $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ is any continuous representation and $A^\bullet$ any object of Σ_{S^*}, then $\Sigma_{A^\bullet} \subset \Sigma_{S^*}$ and so there exists a canonical homomorphism

$$\mathrm{ch}_{G,A^\bullet} := \mathrm{ch}_{\Lambda(G),\Sigma_{A^\bullet}} : K_1(\Lambda(G), \Sigma_{A^\bullet}) \to K_1(\Lambda(G), \Sigma_{S^*}) \cong K_1(\Lambda(G)_{S^*}).$$

In addition, the ring homomorphism $\Lambda(G)_{S^*} \to M_n(Q(\Gamma))$ which sends each element $g \in G$ to $\rho(g)\bar{g}$ where $\bar{g}$ denotes the image of g in Γ, induces a homomorphism of groups

$$\rho_* : K_1(\Lambda(G)_{S^*}) \to K_1(M_n(Q_{\mathcal{O}}(\Gamma))) \cong K_1(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^\times.$$

Proposition 3.16. *Let $A^\bullet$ be a complex which belongs to both Σ_{S^*} and $\Sigma_{\mathrm{ss}-\rho}$.*

(i) (Order of vanishing) *One has $r_G(A^\bullet)(\rho) = r_\Gamma(A^\bullet_\rho) = r(\rho_* \circ \mathrm{ch}_{G,A^\bullet}(A^\bullet))$.*

(ii) (Leading terms) *The following diagram of abelian groups commutes*

$$\begin{tikzcd} K_1(\Lambda(G), \Sigma_{A^\bullet}) \arrow[r, "\mathrm{ch}_{G,A^\bullet}"] \arrow[d, "(-)^*(\rho)"'] & K_1(\Lambda(G)_{S^*}) \arrow[d, "(\rho_*(-))^*(0)"] \\ L^\times \arrow[r, equal] & L^\times, \end{tikzcd}$$

where $(-)^*(0)$ denotes the 'leading term' homomorphism $K_1(Q_{\mathcal{O}}(\Gamma)) \to L^\times$ which occurs in Proposition 3.8 (and Remark 3.10).

Proof. By Lemma 3.13(i) one has $\mathbb{H}_i(H, A^\bullet(\rho^*)) = H^{-i}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(H)} A^\bullet(\rho^*)) = H^{-i}(A^\bullet_\rho)$ in each degree i. Thus, after taking account of Proposition 3.8 (and Remark 3.10), claim (i) follows directly from Definitions 3.2 and 3.11.

Claim (ii) is proved by the same argument as used in [16, Lem. 4.3.10]. Indeed, one need only observe that the above diagram arises as the following composite commutative diagram

$$\begin{tikzcd}
K_1(\Lambda(G),\Sigma_{A^\bullet}) \arrow[r, "\mathrm{ch}_{\Lambda(G),\Sigma_{A^\bullet}}"] \arrow[d, "(\Lambda_{\mathcal{O}}(\Gamma)\otimes_{\mathcal{O}}\mathcal{O}^n)\otimes_{\Lambda(G)}-"'] & K_1(\Lambda(G)_{S^*}) \arrow[d, "\rho_*"] \\
K_1(\Lambda_{\mathcal{O}}(\Gamma),\Sigma_{\mathrm{ss}}) \arrow[r, "\mathrm{ch}_{\Lambda_{\mathcal{O}}(\Gamma),\Sigma_{\mathrm{ss}}}"] \arrow[d, "(-)^*(\rho_{\mathrm{triv}})"'] & K_1(Q_{\mathcal{O}}(\Gamma)) \arrow[d, "(-)^*(0)"] \\
L^\times \arrow[r, equal] & L^\times
\end{tikzcd}$$

where the lower square is as in Proposition 3.8. □

For any element F of $K_1(\Lambda(G)_{S^*})$ we write $F^*(\rho)$ for the *leading term* $(\rho_*(F))^*(0)$ *of* F *at* ρ. By Proposition 3.16, this notation is consistent with that of Definition 3.14 in the case that F belongs to the image of $\mathrm{ch}_{G,A^\bullet}$. In a similar way, if $r(\rho_*(F)) = 0$, then we shall use the notation $F(\rho) := F^*(\rho)$.

3.4.3. *Partial derivatives.* We now observe that the constructions of the previous section allow an interpretation of the expression $(A^\bullet, a)^*(\rho)$ defined in §3.3 as the leading term (in the usual sense) at $s = 0$ of a natural p-adic meromorphic function.

At the outset we fix a representation of G of the form $\chi : G \twoheadrightarrow \Gamma \to \mathbb{Z}_p^\times$ which has infinite order and set

$$c_{\chi,\gamma} := \log_p(\chi(\gamma)) \in \mathbb{Q}_p^\times.$$

We also fix an object $A^\bullet$ of Σ_{S^*} and a morphism $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^\bullet)$ in $\mathcal{C}_{\Lambda(G)}$, we set $\mathcal{L} := [A^\bullet, a] \in K_1(\Lambda(G), \Sigma_{A^\bullet})$ and for any continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ we define

$$f_\rho(T) := \rho_*(\mathrm{ch}_{G,A^\bullet}(\mathcal{L})) \in K_1(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^\times.$$

Then, since the zeros and poles of elements of $Q_{\mathcal{O}}(\Gamma)$ are discrete, the function

$$s \mapsto f_{\mathcal{L}}(\rho\chi^s) := f_\rho(\chi(\gamma)^s - 1)$$

is a p-adic meromorphic function on $\mathbb{Z}_p$.

Lemma 3.17. *Let $A^\bullet$ and a be as above and set $r := r_G(A^\bullet)(\rho)$. Then,*

(*i*) *in any sufficiently small neighbourhood U of 0 in $\mathbb{Z}_p$ one has*

$$\mathcal{L}^*(\rho\chi^s) = \mathcal{L}(\rho\chi^s) = f_{\mathcal{L}}(\rho\chi^s)$$

for all $s \in U \setminus \{0\}$,

(*ii*) *$c_{\chi,\gamma}^r \mathcal{L}^*(\rho)$ is the (usual) leading coefficient at $s = 0$ of $f_{\mathcal{L}}(\rho\chi^s)$, and*

(*iii*) *if $r \geq 0$, then one has*

$$c_{\chi,\gamma}^r \mathcal{L}^*(\rho) = \frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}s^r} f_{\mathcal{L}}(\rho\chi^s)\big|_{s=0}.$$

Proof. If U is any sufficiently small neighbourhood of 0 in $\mathbb{Z}_p$, then one has $f_{\rho\chi^s}(0) \in L^\times$ for all $s \in U \setminus \{0\}$. Since $f_{\rho\chi^s}(T) = f_\rho(\chi(\gamma)^s(T+1)-1)$ we may therefore deduce from Proposition 3.16 that $\mathcal{L}^*(\rho\chi^s) = \mathcal{L}(\rho\chi^s) = f_{\rho\chi^s}(0) = f_\rho(\chi(\gamma)^s - 1) = f_{\mathcal{L}}(\rho\chi^s)$ for any $s \in U \setminus \{0\}$. This proves claim (i).
In addition, if $r \geq 0$ and we factorize $f_\rho(T)$ as $T^r G_\rho(T)$ with $G_\rho(T) \in Q_{\mathcal{O}}(\Gamma)$, then $G_\rho(0) = f_\rho^*(0)$ and

$$\begin{aligned} \frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}s^r} f_{\mathcal{L}}(\rho\chi^s)\big|_{s=0} &= \lim_{0\neq s\to 0} \frac{f_\rho(\chi(\gamma)^s-1)}{s^r} \\ &= \lim_{0\neq s\to 0} \big(\frac{(\chi(\gamma)^s-1)^r}{s^r}\ G_\rho(\chi(\gamma)^s-1)\big) \\ &= \big(\lim_{0\neq s\to 0} \frac{\chi(\gamma)^s-1}{s}\big)^r G_\rho(0) \\ &= (\log_p(\chi(\gamma)))^r f_\rho^*(0) \\ &= c^r_{\chi,\gamma}\mathcal{L}^*(\rho), \end{aligned}$$

where the last equality follows from Proposition 3.16. This proves claim (iii). Also, if $r < 0$, then (whilst we no longer have the interpretation of the limit as a partial derivative) the same arguments prove the statement concerning the leading coefficient at $s = 0$ that is made in claim (ii). □

Remark 3.18. Lemma 3.17 is of particular interest in the case that χ is equal to the cyclotomic character of G when the above calculus can be interpreted as partial derivation in the 'cyclotomic' direction (cf. Remark 5.6).

3.5. Generalized Euler-Poincaré characteristics. In this subsection we show that the constructions made in §3.3 give rise to a natural extension of certain results from [11, 16, 38].
To do this we fix a continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ and a complex $A^\bullet \in \Sigma_{\mathrm{ss}-\rho}$ and in each degree i we set

$$H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*)) := H^i\big((\mathbb{H}_{-\bullet}(G, A^\bullet(\rho^*)), \mathfrak{B}_{-\bullet})\big).$$

We then define the (generalized) additive, respectively multiplicative, Euler-Poincaré characteristic of the complex $A^\bullet(\rho^*)$ by setting

$$\chi_{\mathrm{add}}(G, A^\bullet(\rho^*)) := \sum_{i\in\mathbb{Z}}(-1)^i \mathrm{length}_{\mathcal{O}}\big(H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*))\big),$$

respectively

$$\chi_{\mathrm{mult}}(G, A^\bullet(\rho^*)) := (\#\kappa_L)^{\chi_{\mathrm{add}}(G, A^\bullet(\rho^*))}$$

where κ_L denotes the residue class field of L. We recall that for a single $\Lambda(G)$-module M, or rather its Pontryagin-dual D, similar Euler characteristics have already been studied by several other authors (cf. [12, 42, 18]). Indeed, they use the Hochschild-Serre spectral sequence to construct differentials

$$d^i : H^i(G, D) \to H^i(H, D)^\Gamma \to H^i(H, D)_\Gamma \to H^{i+1}(G, D)$$

where the second arrow is induced by the identity map on $H^i(H, D)$; then the generalized Euler characteristics studied in loc. cit. are defined just as above but by using the complex $(H^\bullet(G,D),d^\bullet)$ in place of $(\mathbb{H}_{-\bullet}(G,-),\mathfrak{B}_{-\bullet})$. However, Lemma 3.13(i) implies that the Pontryagin dual of d^i is equal to the Bockstein homomorphism $\mathfrak{B}_{i+1} : \mathbb{H}_{i+1}(G,P^\bullet) \to \mathbb{H}_i(G,P^\bullet)$ where $P^\bullet$ is a projective resolution of M.

Proposition 3.19. *Let* ord_L *denote the valuation of* L *which takes the value* 1 *on any uniformizing parameter and* $|-|_p$ *the p-adic absolute value, normalized so that* $|p|_p = p^{-1}$.
If $A^\bullet \in \Sigma_{\mathrm{ss}-\rho}$ *and* $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^\bullet)$ *is any morphism in* $\mathcal{C}_{\Lambda(G)}$*, then for* $\mathcal{L} := [A^\bullet, a]$ *one has*

$$\chi_{\mathrm{add}}(G, A^\bullet(\rho^*)) = \mathrm{ord}_L(\mathcal{L}^*(\rho))$$

and

$$\chi_{\mathrm{mult}}(G, A^\bullet(\rho^*)) = |\mathcal{L}^*(\rho)|_p^{-[L:\mathbb{Q}_p]}.$$

Proof. We observe first that by combining Lemma 3.13 with property h) in §2.1 (with $R = \mathcal{O}$) we obtain canonical morphisms

$$\begin{aligned}\mathbf{1}_{\mathcal{O}} \xrightarrow{\mathcal{O}^n \otimes_{\Lambda(G)} a} \mathbf{d}_{\mathcal{O}}(\mathcal{O}^n \otimes_{\Lambda(G)} A^\bullet) \quad &\cong \quad \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)} A^\bullet(\rho^*)))\\ &\cong \quad \mathbf{d}_{\mathcal{O}}((\mathbb{H}_{-\bullet}(G, A^\bullet(\rho^*)), \mathfrak{B}_{-\bullet}))\\ &\cong \quad \prod_{i\in\mathbb{Z}} \mathbf{d}_{\mathcal{O}}\big(H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*))\big)^{(-1)^i}.\end{aligned}$$

After applying $L\otimes_{\mathcal{O}}-$ to this composite morphism and then identifying all factors in the product expression with $\mathbf{1}_L$ by acyclicity we recover the definition of the leading term $\mathcal{L}^*(\rho) := (A^\bullet, a)^*(\rho)$. On the other hand, if we take the product over all i of any arbitrarily chosen maps $f_i : \mathbf{1}_{\mathcal{O}} \to \mathbf{d}_{\mathcal{O}}\big(H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*))\big)^{(-1)^i}$, this will coincide with the above map modulo $\mathcal{O}^\times$. Thus the product over all i of the maps

$$(\mathbf{1}_{\mathcal{O}})_L \xrightarrow{(f_i)_L} \mathbf{d}_{\mathcal{O}}\big(H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*))\big)_L^{(-1)^i} \xrightarrow{acyc} \mathbf{1}_L\ ,$$

which calculate the length of $H^i_{\mathfrak{B}}(G, A^\bullet(\rho^*))$ by Remark 2.4, differs from $\mathcal{L}^*(\rho)$ only by a unit in $\mathcal{O}$ and hence the claimed result follows. ☐

Remark 3.20. If the complex $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} A^\bullet(\rho^*)$ is acyclic, then the leading term $\mathcal{L}^*(\rho)$ is equal to the value of $\mathcal{L}$ at ρ (in the sense of Remark 3.15). This implies that Proposition 3.19 recovers the results of [11, Thm. 3.6], [38, Prop. 6.3] and [16, Rem. 4.1.13].

4. Global Zeta isomorphisms

In this section we recall the non-commutative Tamagawa Number Conjecture that has been formulated by Fukaya and Kato.

4.1. Galois cohomology. The main reference for this section is [16, §1.6], but see also [7]; here we use the same notation as in the survey article [39]. For simplicity we assume throughout this section that p is odd.
We fix a finite set S of places of $\mathbb{Q}$ which contains both $S_p := \{p\}$ and $S_\infty := \{\infty\}$ and let U denote the corresponding dense open subset $\mathrm{Spec}(\mathbb{Z}[\frac{1}{S}])$ of $\mathrm{Spec}(\mathbb{Z})$. We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and, for each place v of $\mathbb{Q}$, an algebraic closure $\bar{\mathbb{Q}}_v$ of $\mathbb{Q}_v$. We then set $G_\mathbb{Q} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $G_{\mathbb{Q}_v} := \mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$ and write G_S for the Galois group of the maximal extension of $\mathbb{Q}$ inside $\bar{\mathbb{Q}}$ which is unramified outside S. If X is any topological abelian group which is endowed with a continuous action of G_S, then we write $\mathrm{R}\Gamma(U, X)$ ($\mathrm{R}\Gamma_c(U, X)$) for global Galois cohomology with restricted ramification (and compact support) and for any place v of $\mathbb{Q}$ we denote by $\mathrm{R}\Gamma(\mathbb{Q}_v, X)$ the corresponding local Galois cohomology complex.
We let L denote a finite extension of $\mathbb{Q}_p$, we write $\mathcal{O}$ for the valuation ring of L and we let V denote a finite dimensional L-vector space which is endowed with a continuous action of $G_\mathbb{Q}$. Then the 'finite parts' of global and local Galois cohomology are written as $\mathrm{R}\Gamma_f(\mathbb{Q}, V)$ and $\mathrm{R}\Gamma_f(\mathbb{Q}_v, V)$ respectively, and there exists a canonical exact triangle of the form
(12)

$$\mathrm{R}\Gamma_c(U, V) \longrightarrow \mathrm{R}\Gamma_f(\mathbb{Q}, V) \longrightarrow \bigoplus_{v\in S} \mathrm{R}\Gamma_f(\mathbb{Q}_v, V) \longrightarrow \mathrm{R}\Gamma_c(U, V)[1].$$

We set $t_p(V) := D_{dR}(V)/D^0_{dR}(V)$ and also $t_\ell(V) := 0$ for each prime number $\ell \neq p$. Then, for each prime ℓ, Fukaya and Kato define a canonical morphism in $\mathcal{C}_L$ of the form

$$\eta_\ell(V): \quad \mathbf{1}_L \quad \to \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_\ell, V))\mathbf{d}_L(t_\ell(V)). \tag{13}$$

For the explicit definition of this morphism we refer the reader either to the original reference [16, §2.4.4] or to the survey article [39, Appendix].

4.2. K-Motives over $\mathbb{Q}$. For further background on this (standard) material we refer the reader to either [16, §2.2, 2.4], [7, §3] or [39, §2].
We fix a finite extension K of $\mathbb{Q}$ and a motive M that is defined over $\mathbb{Q}$ and has coefficients K. As usual we write M_B, M_{dR}, M_ℓ and M_λ for the Betti, de Rham, ℓ-adic and λ-adic realizations of M, where ℓ ranges over rational primes and λ over non-archimedean places of K. We also let t_M denote the tangent space M_{dR}/M^0_{dR} of M. For any ring R and $R[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module X we denote by X^+ and X^- the R-submodule of X upon which complex conjugation acts as multiplication by $+1$ and -1 respectively.
In our later calculations we will use each of the following isomorphisms:

- The comparison isomorphisms between the Betti and λ-adic realizations of M induce canonical isomorphisms of K_λ-modules, respectively K_ℓ-modules, of the form

$$g^+_\lambda : K_\lambda \otimes_K M^+_B \cong M^+_\lambda, \text{ respectively } g^+_\ell : K_\ell \otimes_K M^+_B \cong M^+_\ell. \tag{14}$$

- We set $K_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} K$. Then the comparison isomorphism between the de Rham and Betti realizations of M induces a canonical $K_{\mathbb{R}}$-equivariant period map

$$\mathbb{R} \otimes_{\mathbb{Q}} M_B^+ \xrightarrow{\alpha_M} \mathbb{R} \otimes_{\mathbb{Q}} t_M. \tag{15}$$

- For each p-adic place λ of K, the comparison isomorphism between the p-adic and de Rham realizations of M induces a canonical isomorphism of K_λ-modules of the form

$$t_p(M_\lambda) = D_{dR}(M_\lambda)/D_{dR}^0(M_\lambda) \xrightarrow[\cong]{g_{dR}^t} K_\lambda \otimes_K t_M. \tag{16}$$

We further recall that the 'motivic cohomology groups' $H_f^0(M) := H^0(M)$ and $H_f^1(M)$ of M are K-modules that can be defined either in terms of algebraic K-theory or motivic cohomology in the sense of Voevodsky (cf. [7]). They are both conjectured to be finite dimensional.

4.3. The Tamagawa Number Conjecture. For each embedding $K \to \mathbb{C}$ the complex L-function that is associated to a K-motive M is defined (for the real part of s large enough) as an Euler product

$$L_K(M,s) = \prod_\ell P_\ell(M, p^{-s})^{-1}$$

over all rational primes ℓ. We assume meromorphic continuation of this function and write $L_K^*(M) \in \mathbb{C}^\times$ and $r(M) \in \mathbb{Z}$ for its leading coefficient and order of vanishing at $s = 0$ respectively.
To establish a link between $L_K^*(M)$ and Galois cohomology one uses the 'fundamental line'

$$\begin{aligned}\Delta_K(M): \quad = \quad & \mathbf{d}_K(H_f^0(M))^{-1}\mathbf{d}_K(H_f^1(M))\mathbf{d}_K(H_f^0(M^*(1))^*)\mathbf{d}_K(H_f^1(M^*(1))^*)^{-1} \\ & \mathbf{d}_K(M_B^+)\mathbf{d}_K(t_M)^{-1}.\end{aligned}$$

Indeed, as described in [16, §2.2.7], it is conjectured that archimedean regulators and height pairings combine with the period map α_M to induce a canonical morphism in $\mathcal{C}_{K_{\mathbb{R}}}$ (the 'period-regulator isomorphism') of the form

$$\vartheta_\infty(N) : K_{\mathbb{R}} \otimes_K \Delta_K(M) \cong \mathbf{1}_{K_{\mathbb{R}}}. \tag{17}$$

In addition, a standard conjecture on cycle class maps and Chern class maps induces, for each non-archimedean place λ of K, a canonical 'λ-adic period-regulator isomorphism' in C_{K_λ} (which involves the morphism in (13))

$$\vartheta_\lambda(N) : \Delta_K(M)_{K_\lambda} \cong \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M_\lambda))^{-1}. \tag{18}$$

We now fix a compact p-adic Lie extension F_∞ of $\mathbb{Q}$ which is unramified outside S. We set $G := \mathrm{Gal}(F_\infty/\mathbb{Q})$ and write $\Lambda(G)$ for the associated Iwasawa algebra. For any motive M over $\mathbb{Q}$ we fix a $G_{\mathbb{Q}}$-stable full $\mathbb{Z}_p$-sublattice T_p of M_p and define a (left) Λ-module by setting

$$\mathbb{T} := \Lambda(G) \otimes_{\mathbb{Z}_p} T_p$$

on which $\Lambda(G)$ acts via left multiplication (on the left hand factor) and each element g of $G_{\mathbb{Q}}$ acts diagonally via $g(x\otimes_{\mathbb{Z}_p} y) = x\bar{g}^{-1}\otimes_{\mathbb{Z}_p} g(y)$, where $\bar{g}$ denotes the image of g in $G \subset \Lambda(G)$.

For any non-archimedean place λ of K we write $\mathcal{O}_\lambda$ for the valuation ring of K_λ. We consider a continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O}_\lambda)$ of G which, with respect to a suitable choice of basis, is the λ-adic realization N_λ of a K-motive N. We continue to denote by ρ the induced ring homomorphism $\Lambda(G) \to \mathrm{M}_n(\mathcal{O}_\lambda)$ and we consider $\mathcal{O}_\lambda^n$ as a right $\Lambda(G)$-module via action by the transpose ρ^t on the left, viewing $\mathcal{O}_\lambda^n$ as set of column vectors (contained in K_λ^n). Note that, setting $M(\rho^*) := N^* \otimes M$, we obtain an isomorphism of Galois representations

$$\mathcal{O}_\lambda^n \otimes_{\Lambda(G)} \mathbb{T} \cong T_\lambda(M(\rho^*)),$$

where $T_\lambda(M(\rho^*))$ is the $\mathcal{O}_\lambda$-lattice $\mathcal{O}_\lambda^n \otimes T_p$ of $M(\rho^*)_\lambda$, on which $g \in G_{\mathbb{Q}}$ acts diagonally: $g(x \otimes t) = \rho^*(g)x \otimes g \cdot t$ denoting by ρ^* the contragredient representation of ρ.

Conjecture 4.1 (Fukaya and Kato, [16, Conj. 2.3.2]). *Set $\Lambda := \Lambda(G)$. Then there exists a canonical morphism in $\mathcal{C}_\Lambda$*

$$\zeta_\Lambda(M) := \zeta_\Lambda(\mathbb{T}) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{R}\Gamma_c(U, \mathbb{T}))^{-1}$$

with the following property: for all K, λ and ρ as above the (generalized) base change $K_\lambda^n \otimes_\Lambda -$ sends $\zeta_\Lambda(M)$ to the composite morphism

$$\mathbf{1}_{K_\lambda} \xrightarrow{\zeta_K(M(\rho^*))_{K_\lambda}} \Delta_K(M(\rho^*))_{K_\lambda} \xrightarrow{\vartheta_\lambda(N)} \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M(\rho^*)_\lambda))^{-1},$$

where

$$\zeta_K(M(\rho^*)) : \mathbf{1}_K \to \Delta_K(M(\rho^*))$$

denotes the unique morphism which is such that, for every embedding $K \to \mathbb{C}$, the leading coefficient $L_K^*(M(\rho^*))$ is equal to the composite

$$\mathbf{1}_{\mathbb{C}} \xrightarrow{\zeta_K(M(\rho^*))_{\mathbb{C}}} \Delta_K(M(\rho^*))_{\mathbb{C}} \xrightarrow{(\vartheta_\infty(N))_{\mathbb{C}}} \mathbf{1}_{\mathbb{C}}.$$

Fukaya and Kato refer to the (conjectural) morphism '$\zeta_\Lambda(M)$' in Conjecture 4.1 as a *global Zeta isomorphism*. We note also that it is straightforward to show that Conjecture 4.1 implies the 'p-primary component' of the Equivariant Tamagawa Number Conjecture that is formulated by Flach and the first named author in [7, Conj. 4(iv)] and hence also implies the 'main conjecture of non-abelian Iwasawa theory' that is discussed by Huber and Kings in [19]. For a further discussion of Conjecture 4.1 see [39, §4].

5. The interpolation formula for Tate motives

In this section we give a first explicit application of the formalism developed in §3. More precisely, we show that the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre in [35] and discussed by Tate in [37, Chap. VI, §5], can be naturally interpreted as an interpolation formula for the leading term (in the sense of Definition 3.14) of certain global Zeta isomorphisms that are predicted to exist by Conjecture 4.1 in terms of the leading terms (in the classical sense) of suitable p-adic Artin L-functions. Interested readers can find further explicit results concerning Conjecture 4.1 in the special case that we consider here in, for example, both [3] and [8].

Throughout this section we set $\mathrm{G}(F/E) := \mathrm{Gal}(F/E)$ for any Galois extension of fields F/E. We also fix an odd prime p and a totally real Galois extension F_∞ of $\mathbb{Q}$ which contains the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}$ of $\mathbb{Q}$ and is such that $G := \mathrm{G}(F_\infty/\mathbb{Q})$ is a compact p-adic Lie group. We assume further that $F_\infty/\mathbb{Q}$ is unramified outside a finite set of prime numbers S (which therefore contains p). We set $H := \mathrm{G}(F_\infty/\mathbb{Q}_{\mathrm{cyc}})$ and $\Gamma := \mathrm{G}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \cong G/H$. We fix a subfield E of F_∞ which is both Galois and of finite degree over $\mathbb{Q}$, we set $\bar{G} := \mathrm{G}(E/\mathbb{Q})$ and we write $S_p(E)$ for the set of p-adic places of E and E_{cyc}, $E_{w,\mathrm{cyc}}$ for each $w \in S_p(E)$ and $\mathbb{Q}_{p,\mathrm{cyc}}$ for the cyclotomic $\mathbb{Z}_p$-extensions of E, E_w and $\mathbb{Q}_p$ respectively. For simplicity, we always assume that the following condition is satisfied

$$E \cap \mathbb{Q}_{\mathrm{cyc}} = \mathbb{Q} \text{ and } E_w \cap \mathbb{Q}_{p,\mathrm{cyc}} = \mathbb{Q}_p \text{ for all } w \in S_p(E). \tag{19}$$

We note that this condition implies that there is a direct product decomposition $\mathrm{G}(E_{\mathrm{cyc}}/\mathbb{Q}) \cong \Gamma \times \bar{G}$ and hence allows us to regard γ as a topological generator of each of the groups Γ, $\mathrm{G}(E_{\mathrm{cyc}}/E)$, $\mathrm{G}(E_{w,\mathrm{cyc}}/E_w)$ for $w \in S_p(E)$ and $\mathrm{G}(\mathbb{Q}_{p,\mathrm{cyc}}/\mathbb{Q}_p)$.

We let $\mathbb{T}$ denote the (left) $\Lambda(G)$-module $\Lambda(G)$ endowed with the following (left) action of $G_{\mathbb{Q}}$: each $\sigma \in G_{\mathbb{Q}}$ acts on $\mathbb{T}$ as right multiplication by the element $\chi_{\mathrm{cyc}}(\bar{\sigma})\bar{\sigma}^{-1}$ where $\bar{\sigma}$ denotes the image of σ in G and χ_{cyc} is the cyclotomic character $G \to \Gamma \to \mathbb{Z}_p^\times$. For each subfield F of F_∞ which is Galois over $\mathbb{Q}$ we let $\mathbb{T}_F$ denote the (left) $\Lambda(\mathrm{G}(F/\mathbb{Q}))$-module $\Lambda(\mathrm{G}(F/\mathbb{Q})) \otimes_{\Lambda(G)} \mathbb{T}$. We also set $U := \mathrm{Spec}(\mathbb{Z}[\frac{1}{S}])$ and note that for each such field F there is a natural isomorphism in $D^p(\Lambda(\mathrm{G}(F/\mathbb{Q})))$ of the form

$$\Lambda(\mathrm{G}(F/\mathbb{Q})) \otimes^{\mathbb{L}}_{\Lambda(G)} \mathrm{R}\Gamma_c(U, \mathbb{T}) \cong \mathrm{R}\Gamma_c(U, \mathbb{T}_F). \tag{20}$$

We regard each character of $\bar{G}$ as a character of G via the natural projection $G \twoheadrightarrow \bar{G}$. For any field C we write $R_C^+(\overline{G})$ and $R_C(\overline{G})$ for the set of finite dimensional C-valued characters of $\overline{G}$ and for the ring of finite dimensional C-valued virtual characters of $\overline{G}$, respectively. For each $\rho \in R_C^+(\overline{G})$ we fix a representation space V_ρ of character ρ and for any $\mathbb{Q}_p[\overline{G}]$-module N, respectively endomorphism α of a $\mathbb{Q}_p[\bar{G}]$-module N, we write N^ρ for the $\mathbb{C}_p$-module

$$\mathrm{Hom}_{\bar{G}}(V_\rho, \mathbb{C}_p \otimes_{\mathbb{Q}_p} N) \cong ((V_{\rho^*})_{\mathbb{C}_p} \otimes_{\mathbb{Q}_p} N)_{\bar{G}},$$

respectively α^ρ for the induced endomorphism of N^ρ. We use similar notation for complex characters ρ and $\mathbb{Q}[\bar{G}]$-modules N.

For any abelian group A we write $A\hat{\otimes}\mathbb{Z}_p$ for its p-adic completion $\varprojlim_n A/p^nA$.

5.1. LEOPOLDT'S CONJECTURE. We recall that Leopoldt's Conjecture (for the field E at the prime p) is equivalent to the injectivity of the natural localisation map

$$\lambda_p : \mathcal{O}_E\left[\frac{1}{p}\right]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{w\in S_p(E)} E_w^\times \hat{\otimes}\mathbb{Z}_p.$$

If $\rho \in R^+_{\mathbb{C}_p}(\bar{G})$, then in the sequel we say that Leopoldt's Conjecture 'is valid at ρ' if one has $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(\lambda_p))^\rho = 0$.
We set $c_\gamma := c_{\chi_{\mathrm{cyc}},\gamma} \in \mathbb{Q}_p^\times$ (see §3.4.3) and for each $\rho \in R^+_{\mathbb{C}_p}(\bar{G})$ we define

$$\langle \rho, 1\rangle := \dim_{\mathbb{C}_p}(H^0(\bar{G}, V_\rho)) = \dim_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho).$$

Lemma 5.1. *We fix $\rho \in R^+_{\mathbb{C}_p}(\bar{G})$ and assume that Leopoldt's Conjecture is valid at ρ.*

(i) *There are canonical isomorphisms*

$$(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_c(U, \mathbb{T}_E))^\rho \cong \begin{cases} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_p))^\rho, & \text{if } i = 2 \\ (\mathbb{Q}_p)^\rho, & \text{if } i = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $\mathrm{R}\Gamma_c(U, \mathbb{T})$ *is semisimple at ρ and one has $r_G(\mathrm{R}\Gamma_c(U, \mathbb{T}))(\rho) = \langle \rho, 1\rangle$.*

(iii) *For each $w \in S_p(E)$ we write $\mathrm{N}_{E_w/\mathbb{Q}_p}$ for the homomorphism $E_w^\times \hat{\otimes}\mathbb{Z}_p \to \mathbb{Q}_p^\times \hat{\otimes}\mathbb{Z}_p$ that is induced by the field theoretic norm map. Then, with respect to the identifications given in claim (i), the Bockstein homomorphism in degree -2 of $(\mathrm{R}\Gamma_c(U, \mathbb{T}), T_\rho, \gamma)$ is equal to $-c_\gamma^{-1}$ times the homomorphism*

$$(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2_c(U, \mathbb{T}_E))^\rho \to (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^3_c(U, \mathbb{T}_E))^\rho$$

that is induced by the homomorphism

$$\log_{p,E} : \prod_{w\in S_p(E)} E_w^\times \hat{\otimes}\mathbb{Z}_p \to \mathbb{Z}_p$$

which sends each element $(e_w)_w$ to $\sum_w \log_p(\mathrm{N}_{E_w/\mathbb{Q}_p}(e_w))$.

Proof. Claim (i) can be verified by combining the exact cohomology sequence of the tautological exact triangle

$$\mathrm{R}\Gamma_c(U, \mathbb{T}_E) \to \mathrm{R}\Gamma(U, \mathbb{T}_E) \to \bigoplus_{\ell\in S} \mathrm{R}\Gamma(\mathbb{Q}_\ell, \mathbb{T}_E) \to \mathrm{R}\Gamma_c(U, \mathbb{T}_E)[1] \tag{21}$$

together with the canonical identifications $H^i(U, \mathbb{T}_E) \cong H^i(\mathcal{O}_E[\frac{1}{S}], \mathbb{Z}_p(1))$ and $H^i(\mathbb{Q}_\ell, \mathbb{T}_E) \cong \bigoplus_{w\in S_\ell(E)} H^i(E_w, \mathbb{Z}_p(1))$ and an explicit computation of each of the groups $H^i(\mathcal{O}_E[\frac{1}{S}], \mathbb{Z}_p(1))$ and $H^i(E_w, \mathbb{Z}_p(1))$. As this is routine we leave explicit details to the reader except to note that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2_c(U, \mathbb{T}_E)$ is canonically isomorphic to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_p)$ (independently of Leopoldt's Conjecture), whilst the fact that E is totally real implies that the vanishing of $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1_c(U, \mathbb{T}_E))^\rho$ is equivalent to that of $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(\lambda_p))^\rho$.

To prove claims (ii) and (iii) we note first that, in terms of the notation used in §3.3, the isomorphism (20) (with $F = E_{\mathrm{cyc}}$) induces a canonical isomorphism in $D^p(\Lambda_{\mathcal{O}}(\Gamma))$ of the form

$$\mathrm{R}\Gamma_c(U, \mathbb{T})_\rho \cong \mathcal{O}^n \otimes_{\mathbb{Z}_p[\bar{G}]} \mathrm{R}\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}), \tag{22}$$

where Γ acts naturally on the right hand factor in the tensor product.
From Lemma 3.13(iv) we may therefore deduce that $\mathrm{R}\Gamma_c(U, \mathbb{T})$ is semisimple at ρ if and only if the complex $\mathcal{O}^n \otimes_{\mathbb{Z}_p[\bar{G}]} \mathrm{R}\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}) \in D^p(\Lambda_{\mathcal{O}}(\Gamma))$ is semisimple. But the latter condition is easy to check by using the criterion of Remark 3.3(ii): indeed, one need only note that $H^i_c(U, \mathbb{T}_{E_{\mathrm{cyc}}})$ is finite if $i \notin \{2, 3\}$, that $H^3_c(U, \mathbb{T}_{E_{\mathrm{cyc}}})$ identifies with $\mathbb{Z}_p$ (as a Γ-module) and that the exact sequences of (5) combine with the descriptions of claim (i) to imply that $((\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}))^\rho)^\Gamma$ and $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}))^\rho_\Gamma$ both vanish. In addition, the same observations combine with Lemma 3.13(ii) to imply that $r_G(\mathrm{R}\Gamma_c(U, \mathbb{T}))(\rho) = \dim_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho)$.
Regarding claim (iii), the isomorphism (22) combines with Lemma 3.13(iii) to imply that $(\mathfrak{B}_{-2})^\rho = (\hat{\mathfrak{B}}_{-2})^\rho$ where $\hat{\mathfrak{B}}_{-2}$ is the Bockstein homomorphism in degree -2 of $(\mathrm{R}\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}), \mathbb{Z}_p, \gamma)$, with γ regarded as a topological generator of $\mathrm{G}(E_{\mathrm{cyc}}/E)$. Also, by comparing (21) to the corresponding exact triangle with E_{cyc} in place of E, we obtain a morphism of exact triangles of the form

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}_{E_{\mathrm{cyc}}}) \arrow[r, "\gamma-1"] \arrow[d] & \mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}_{E_{\mathrm{cyc}}}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}_E) \arrow[r] \arrow[d] & {} \\
\mathrm{R}\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}})[1] \arrow[r, "\gamma-1"] & \mathrm{R}\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}})[1] \arrow[r] & \mathrm{R}\Gamma_c(U, \mathbb{T}_E)[1] \arrow[r] & {} .
\end{tikzcd}$$

Thus, by combining the description of Lemma 3.1 with consideration of the long exact cohomology sequences of this diagram we obtain a commutative diagram

$$\begin{tikzcd}
\bigoplus_{w \in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(E_w, \mathbb{Z}_p(1)) \arrow[r] \arrow[d, "(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{B}_{-1,w})_w"'] & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2_c(U, \mathbb{T}_E) \arrow[d, "(-1)\times(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{\mathfrak{B}}_{-2})"] \\
\bigoplus_{w \in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(E_w, \mathbb{Z}_p(1)) \arrow[r] & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^3_c(U, \mathbb{T}_E).
\end{tikzcd}$$

Here the upper row is the (tautological) surjection that is induced by the canonical identifications $H^1(E_w, \mathbb{Z}_p(1)) \cong E_w^\times \hat{\otimes} \mathbb{Z}_p$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2_c(U, \mathbb{T}_E) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_p)$, the lower row is the surjection induced by the canonical identifications $H^2(E_w, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ and $H^3_c(U, \mathbb{T}_E) \cong \mathbb{Z}_p$ together with the identity map on $\mathbb{Z}_p$, $\mathfrak{B}_{-1,w}$ is the Bockstein homomorphism in degree -1 of $(\mathrm{R}\Gamma(E_{w,\mathrm{cyc}}, \mathbb{Z}_p(1)), \mathbb{Z}_p, \gamma)$ where γ is considered as a topological generator of $\mathrm{G}(E_{w,\mathrm{cyc}}/E_w)$, and the factor -1 occurs on the right hand vertical arrow because of the 1-shift in the lower row of the previous diagram.
Further, for each $w \in S_p(E)$ the natural isomorphism (in $D^p(\mathbb{Z}_p)$)

$$\mathbb{Z}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathrm{G}(E_w/\mathbb{Q}_p)]} \mathrm{R}\Gamma(E_w, \mathbb{Z}_p(1)) \cong \mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{Z}_p(1))$$

induces a commutative diagram

$$\begin{tikzcd} H^1(E_w, \mathbb{Z}_p(1)) \arrow[r] \arrow[d, "\mathfrak{B}_{-1,w}"'] & H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \arrow[d, "\mathfrak{B}_{-1,p}"] \\ H^2(E_w, \mathbb{Z}_p(1)) \arrow[r] & H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \end{tikzcd}$$

where the upper horizontal arrow is induced by the canonical identifications $H^1(E_w, \mathbb{Z}_p(1)) \cong E_w^\times \hat{\otimes} \mathbb{Z}_p$ and $H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Q}_p^\times \hat{\otimes} \mathbb{Z}_p$ together with the map $\mathrm{N}_{E_w/\mathbb{Q}_p}$, the lower horizontal arrow is induced by the canonical identifications $H^2(E_w, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ and $H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ together with the identity map on $\mathbb{Z}_p$, and $\mathfrak{B}_{-1,p}$ is the Bockstein homomorphism in degree -1 of $(\mathrm{R}\Gamma(\mathbb{Q}_{p,\mathrm{cyc}}, \mathbb{Z}_p(1)), \mathbb{Z}_p, \gamma)$. To prove claim (iii) it thus suffices to recall that, with respect to the natural identifications $H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Q}_p^\times \hat{\otimes} \mathbb{Z}_p$ and $H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$, the map $\mathfrak{B}_{-1,p}$ is equal to $c_\gamma^{-1} \cdot \log_p$ (see, for example, [9, p. 352]). $\square$

5.2. The p-adic Stark conjecture at $s = 1$. For each character $\chi \in R_{\mathbb{C}}(\bar{G})$ we write $L_S(s, \chi)$ for the Artin L-function of χ that is truncated by removing the Euler factors attached to primes in S (cf. [37, Chap. 0, §4]). Then, for each character $\rho \in R_{\mathbb{C}_p}(G)$ there exists a unique p-adic meromorphic function $L_{p,S}(\cdot, \rho) : \mathbb{Z}_p \to \mathbb{C}_p$ such that for each strictly negative integer n and each isomorphism $\iota : \mathbb{C}_p \cong \mathbb{C}$ one has

$$L_{p,S}(n, \rho)^\iota = L_S(n, (\rho \cdot \omega^{n-1})^\iota)$$

where $\omega : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is the Teichmüller character (cf. [37, Chap. V., Thm. 2.2]). Indeed, this function is the 'S-truncated p-adic Artin L-function' of ρ that is constructed by Greenberg in [17] by combining techniques of Brauer induction with the fundamental results of Deligne and Ribet [15] and Cassou-Noguès [10]. For typographical simplicity in the sequel, we fix an isomorphism $\iota : \mathbb{C}_p \cong \mathbb{C}$ as above and hence often omit it from the notation.

In this section we recall a conjecture of Serre regarding the 'leading term at $s = 1$' of $L_{p,S}(s, \rho)$. To this end we set $E_\infty := \mathbb{R} \otimes_{\mathbb{Q}} E \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and write $\log_\infty(\mathcal{O}_E^\times)$ for the inverse image of $\mathcal{O}_E^\times \hookrightarrow E_\infty^\times$ under the (componentwise) exponential map $\exp_\infty : E_\infty \to E_\infty^\times$. We set $E_0 := \{x \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x) = 0\}$. Then $\log_\infty(\mathcal{O}_E^\times)$ is a lattice in $\mathbb{R} \otimes_{\mathbb{Q}} E_0$ and so there is a canonical isomorphism of $\mathbb{C}[\bar{G}]$-modules

$$\mu_\infty : \mathbb{C} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C} \otimes_{\mathbb{Q}} E_0.$$

By a standard argument (cf. [14, §6, Exer. 6]) this implies that the $\mathbb{Q}[\bar{G}]$-modules E_0 and $\mathbb{Q} \otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times)$ are (non-canonically) isomorphic. We also note that the composite homomorphism

$$\text{(23)}\qquad \log_\infty(\mathcal{O}_E^\times) \xrightarrow{\exp_\infty} \mathcal{O}_E^\times \xrightarrow{\lambda_p} \prod_{w\in S_p(E)} U^1_{E_w}$$
$$\xrightarrow{(u_w)_w\mapsto(\log_p(u_w))_w} \prod_{w\in S_p(E)} E_w \cong \mathbb{Q}_p\otimes_{\mathbb{Q}} E,$$

factors through the inclusion $\mathbb{Q}_p\otimes_{\mathbb{Q}} E_0 \subset \mathbb{Q}_p\otimes_{\mathbb{Q}} E$ and hence induces an isomorphism of $\mathbb{Q}_p[\bar{G}]$-modules

$$\mu_p : \mathbb{Q}_p\otimes_{\mathbb{Z}} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{Q}_p\otimes_{\mathbb{Q}} E_0.$$

CONJECTURE 5.2 (Serre). *For each $\rho\in R^+_{\mathbb{C}_p}(\bar{G})$ we set*

$$L^*_{p,S}(1,\rho) := \lim_{s\to 1}(s-1)^{\langle\rho,1\rangle}\cdot L_{p,S}(s,\rho).$$

Then $L^*_{p,S}(1,\rho)$ is equal to the leading term of $L_{p,S}(s,\rho)$ at $s=1$, and for each choice of isomorphism of $\mathbb{Q}[\bar{G}]$-modules $g: E_0\to\mathbb{Q}\otimes_{\mathbb{Z}}\log_\infty(\mathcal{O}_E^\times)$ one has

$$\frac{L^*_{p,S}(1,\rho)}{\det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{Q}_p}\mu_p)\circ(\mathbb{C}_p\otimes_{\mathbb{Q}} g))^\rho} = \frac{L^*_S(1,\rho)}{\det_{\mathbb{C}}(\mu_\infty\circ(\mathbb{C}\otimes_{\mathbb{Q}} g))^\rho}.$$

REMARK 5.3. This conjecture is the 'p-adic Stark conjecture at $s=1$' as discussed by Tate in [37, Chap. VI, §5], where it is attributed to Serre [35]. More precisely, there are some slight imprecisions in the discussion of [37, Chap. VI, §5] (for example, and as already noted by Solomon in [36, §3.3], the intended meaning of the symbols '$\log U$' and 'μ_p' in [37, p. 137] is unclear) and Conjecture 5.2 represents a natural clarification of the presentation given in loc. cit..

REMARK 5.4. We fix a subgroup J of $\bar{G}$ and write 1_J for the trivial character of J. If $\rho = \mathrm{Ind}_J^{\bar{G}}\,1_J$, then the inductive behaviour of L-functions combines with the analytic class number formula for E^J to show that Conjecture 5.2 is valid for ρ if and only if the p-adic zeta function of the field E^J has a simple pole at $s=1$ with residue equal to $2^{[E^J:\mathbb{Q}]-1}hR_pe_p/\sqrt{|d|}$ where h, R_p and d are the class number, p-adic regulator and absolute discriminant of E^J respectively and $e_p := \prod_{v\in S_p(E^J)}(1-\mathrm{N}v^{-1})$ (cf. [37, Rem., p. 138]). From the main result (§5, Thm.) of Colmez in [13] one may thus deduce that Conjecture 5.2 is valid for $\rho = \mathrm{Ind}_J^{\bar{G}}\,1_J$ if and only if Leopoldt's Conjecture is valid for E^J. We note also that if Leopoldt's Conjecture is valid for E, then it is valid for all such intermediate fields E^J.

5.3. THE INTERPOLATION FORMULA. We now reinterpret the equality of Conjecture 5.2 as an interpolation formula for the Zeta isomorphism $\zeta_{\Lambda(G)}(\mathbb{T})$ that is predicted to exist by Conjecture 4.1.

THEOREM 5.5. *If Conjecture 5.2 is valid, then for each $\rho\in R^+_{\mathbb{C}_p}(\bar{G})$ the complex $\mathrm{R}\Gamma_c(U,\mathbb{T})$ is semisimple at ρ and one has both $r_G(\mathrm{R}\Gamma_c(U,\mathbb{T}))(\rho) = \langle\rho,1\rangle$ and*

$$\text{(24)}\qquad c_\gamma^{\langle\rho,1\rangle}\cdot\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho) = L^*_{p,S}(1,\rho).$$

Remark 5.6. One can naturally interpret (24) as an equality of leading terms of p-adic meromorphic functions. Indeed, whilst Conjecture 5.2 predicts that $L^*_{p,S}(1,\rho)$ is the leading term at $s=1$ of $L_{p,S}(s,\rho)$, Lemma 3.17 interprets the left hand side of (24) as the leading term at $s=0$ of the function $f_{\mathcal{L}}(\rho\chi^s_{\mathrm{cyc}})$ with $\mathcal{L} := [\mathrm{R}\Gamma_c(U,\mathbb{T}), \zeta_{\Lambda(G)}(\mathbb{T})] \in K_1(\Lambda(G), \Sigma_{\mathrm{ss}-\rho})$.

Proof. We note first that if Conjecture 5.2 is valid, then Remark 5.4 implies that Leopoldt's Conjecture is valid for E and so Lemma 5.1(ii) implies that $r_G(\mathrm{R}\Gamma_c(U,\mathbb{T}))(\rho) = \langle\rho, 1\rangle$ for each $\rho \in R^+_{\mathbb{C}_p}(\bar{G})$ and also that $\mathrm{R}\Gamma_c(U,\mathbb{T})$ is semisimple at each such ρ.
We now fix $\rho \in R^+_{\mathbb{C}_p}(\bar{G})$ and a number field K over which the character ρ can be realised. We fix an embedding $K \hookrightarrow \mathbb{C}$ and write λ for the place of K which is induced by the fixed isomorphism $\iota : \mathbb{C}_p \cong \mathbb{C}$. We set $M := h^0(\mathrm{Spec}\, E)(1)$ and note that $M([\rho]^*) := M \otimes [\rho]^*$ is a K-motive, where $[\rho]^*$ denotes the dual of the Artin motive corresponding to ρ.
To evaluate $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ we need to make Definition 3.14 explicit. To do this we use the observations of [6, §1.1, §1.3] to explicate the isomorphism $\zeta_K(M([\rho]^*))_{K_\lambda}$ which occurs in Conjecture 4.1. Indeed one has $H^1_f(M) = \mathcal{O}^\times_E \otimes_{\mathbb{Z}} \mathbb{Q}$, $H^0_f(M^*(1)) = \mathbb{Q}$, $t_M = E$ and $H^0_f(M) = H^1_f(M^*(1)) = M^+_B = 0$ (the latter since E is totally real). This implies that

$$\mathbb{C} \otimes_K \Delta_K(M([\rho]^*)) = \mathbf{d}_{\mathbb{C}}((\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}^\times_E)_\rho)\mathbf{d}_{\mathbb{C}}((\mathbb{Q})_\rho)\mathbf{d}_{\mathbb{C}}((E)_\rho)^{-1}$$

and that $\zeta_K(M([\rho]^*))_{K_\lambda}$ is equal to the composite morphism

$$\begin{aligned} \mathbf{1}_{\mathbb{C}_p} \to & \mathbf{1}_{\mathbb{C}_p} \\ \to & \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}^\times_E)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E)^\rho)^{-1} \\ \to & \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} H^2_c(U, M([\rho]^*)_\lambda))^{-1}\mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} H^3_c(U, M([\rho]^*)_\lambda) \\ \to & \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} \mathrm{R}\Gamma_c(U, M([\rho]^*)_\lambda))^{-1}. \end{aligned} \tag{25}$$

In this displayed formula we have used the following notation: the first map corresponds to multiplication by $L^*_S(1,\rho)$; the second map is induced by applying $(\mathbb{C}_p \otimes_{\mathbb{R},\iota^{-1}} -)^\rho$ to both the natural isomorphism $\mathbb{R} \otimes_{\mathbb{Q}} E \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and also the exact sequence

$$0 \to \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}^\times_E \xrightarrow{(\log\circ\sigma)_\sigma} \prod_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{R} \xrightarrow{(x_\sigma)_\sigma \mapsto \sum_\sigma x_\sigma} \mathbb{R} \to 0; \tag{26}$$

the third map is induced by Lemma 5.1(i) and the inverse of the isomorphism

$$\prod_{w \in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U^1_{E_w} \xrightarrow{(u_w)_w \mapsto (\log_p(u_w))_w} \prod_{w \in S_p(E)} E_w \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E; \tag{27}$$

the last map is induced by property h) as described in §2.1 (with $R = \mathbb{C}_p$).

Also, from Lemma 5.1(iii) we know that $\mathbb{C}_p \otimes_{K_\lambda} t(\mathrm{R}\Gamma_c(U,\mathbb{T})(\rho^*))$ is equal to the composite

$$\begin{aligned}(28)\qquad & \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} \mathrm{R}\Gamma_c(U, M([\rho]^*)_\lambda)^{-1} \\ &\to \mathbf{d}_{\mathbb{C}_p}(H^2_c(U, M([\rho]^*)_\lambda)^{-1}\mathbf{d}_{\mathbb{C}_p}(H^3_c(U, M([\rho]^*)_\lambda) \\ &\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho)^{-1}\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho) = \mathbf{1}_{\mathbb{C}_p}\end{aligned}$$

where the first arrow is induced by property h) in §2.1 (with $R = \mathbb{C}_p$) and the second by Lemma 5.1(i) and the homomorphism $-c_\gamma^{-1}\log_{p,E}$ described in Lemma 5.1(iii).

Now, after taking account of Lemma 5.1(ii), the leading term $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ is defined (in Definition 3.14) to be equal to $(-1)^{\langle\rho,1\rangle}$ times the element of $\mathbb{C}_p^\times$ which corresponds to the composite of (25) and (28). Thus, after noting that there is a commutative diagram of the form

$$\begin{array}{ccc} \prod_{w\in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U^1_{E_w} & \longrightarrow & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_p) \\ {\scriptstyle (27)}\downarrow & & \downarrow{\scriptstyle \log_{p,E}} \\ \mathbb{Q}_p \otimes_{\mathbb{Q}} E & \xrightarrow{\mathrm{Tr}_{E/\mathbb{Q}}} & \mathbb{Q}_p \end{array}$$

where the upper horizontal arrow is the tautological projection, the observations made above imply that

$$(29)\qquad c_\gamma^{\langle\rho,1\rangle}\cdot\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho) = L^*_S(1,\rho)\cdot\xi$$

where ξ is the element of $\mathbb{C}_p^\times$ that corresponds to the composite morphism

$$\begin{aligned}(30)\qquad \mathbf{1}_{\mathbb{C}_p} &= \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)^{-1} \\ &\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)^{-1} \\ &\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)^{-1} = \mathbf{1}_{\mathbb{C}_p}.\end{aligned}$$

Here the first arrow is induced by applying $\mathbb{C}_p \otimes_{\mathbb{R},\iota^{-1}} -$ to the isomorphism $\mathbb{R}\otimes_{\mathbb{Z}}\mathcal{O}_E^\times \cong \mathbb{R}\otimes_{\mathbb{Q}} E_0$ coming from the map $(\log\circ\sigma)_\sigma$ in (26) and the second by the isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E_0$ coming from the second and third arrows in (23). (Note also that the factor $(-1)^{\langle\rho,1\rangle}$ in the definition of $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ cancels against the factor -1 in the term $-c_\gamma^{-1}$ which occurs in the morphism (28) and hence does not occur in the formula (29)).

But, upon comparing the definitions of μ_∞ and μ_p in §5.2 with the maps involved in (30), one finds that ξ is equal to

$$\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p)\circ(\mathbb{C}_p \otimes_{\mathbb{C},\iota^{-1}} \mu_\infty)^{-1})^\rho = \frac{\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p)\circ(\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho}{\det_{\mathbb{C}}(\mu_\infty\circ(\mathbb{C}\otimes_{\mathbb{Q}} g))^\rho}$$

and hence (29) implies that

$$\frac{c_\gamma^{\langle\rho,1\rangle}\cdot\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)}{\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p)\circ(\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho} = \frac{L^*_S(1,\rho)}{\det_{\mathbb{C}}(\mu_\infty\circ(\mathbb{C}\otimes_{\mathbb{Q}} g))^\rho}.$$

The claimed equality (24) now follows immediately upon comparing this equality to that of Conjecture 5.2. □

Corollary 5.7. *If Leopoldt's Conjecture is valid for E at p, then for every finite dimensional $\mathbb{Q}$-rational character ρ of $\bar{G}$ there exists a natural number n_ρ such that*

$$(c_\gamma^{\langle\rho,1\rangle}\cdot\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho))^{n_\rho} = L^*_{p,S}(1,\rho)^{n_\rho}.$$

Further, if ρ is a permutation character, then one can take $n_\rho = 1$.

Proof. If ρ is $\mathbb{Q}$-rational, then Artin's Induction Theorem implies the existence of a natural number n_ρ such that in $R_{\mathbb{C}_p}(G)$ one has $n_\rho\cdot\rho = \sum_H n_H\cdot\mathrm{Ind}_H^G 1_H$ where H runs over the set of subgroups of $\bar{G}$ and each n_H is an integer (cf. [37, Chap. II, Thm. 1.2]). Further, ρ is said to be a permutation character if and only if there exists such a formula with $n_\rho = 1$. The stated result thus follows by combining Theorem 5.5 with Remark 5.4 and the fact that each side of (24) is both additive and inductive in ρ. □

6. The interpolation formula for critical motives

As a second application of the formalism introduced in §3, in this section we prove an interpolation formula for the leading terms (in the sense of Definition 3.14) of the p-adic L-functions that Fukaya and Kato conjecture to exist for any critical motive which has good ordinary reduction at all places above p. (We recall that a motive M is said to be 'critical' if the map (15) is bijective). To study these p-adic L-functions we must combine Conjecture 4.1 together with a local analogue of this conjecture (which is also due to Fukaya and Kato, and is recalled as Conjecture 6.1 below) and aspects of Nekovář's theory of Selmer complexes and of the theory of p-adic height pairings.

6.1. Local epsilon isomorphisms. At the outset we fix a 'p-adic period' t (that is, a topological generator of $\mathbb{Z}_p(1)$). Let L be any finite extension of $\mathbb{Q}_p$ and V any finite-dimensional L-vector space with continuous $G_{\mathbb{Q}_p}$-action. Then we write $\epsilon_p(V) := \epsilon(D_{pst}(V))$ for Deligne's epsilon-factor at p, where $D_{pst}(V)$ is endowed with the linearized action of the Weil group and thereby considered as a representation of the Weil-Deligne group, see [16, §3.2] or [29, App. C]. (Note that this notation hides dependence on the choice of a Haar measure and p-adic period. Note also that the choice of $t = (t_n) \in \mathbb{Z}_p(1)$ determines a homomorphism $\psi_p : \mathbb{Q}_p \to \overline{\mathbb{Q}_p}^\times$ with $\ker(\psi_p) = \mathbb{Z}_p$ by sending $\frac{1}{p^n}$ to $t_n \in \mu_{p^n}$). The subfield of inertial invariants $(B_{dR})^{I_p}$ of B_{dR} identifies with the completion $\widehat{\mathbb{Q}_p^{nr}}$ of the maximal unramified extension $\mathbb{Q}_p^{nr}$ of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$. For L and V as above we set $\widetilde{L} := \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L$ and

$$\Gamma^*(-j) := \begin{cases}\Gamma(j) = (j-1)!, & \text{if } j > 0,\\ \lim_{s\to j}(s-j)\Gamma(s) = (-1)^j((-j)!)^{-1}, & \text{if } j \le 0,\end{cases}$$

and

$$\Gamma_L(V) := \prod_{j\in\mathbb{Z}} \Gamma^*(j)^{-h(-j)},$$

where $h(j) := \dim_L gr^j D_{dR}(V)$.

We let

$$\epsilon_{p,L}(V) : \mathbf{1}_{\widetilde{L}} \to \big(\mathbf{d}_L(\mathrm{R}\Gamma(\mathbb{Q}_p, V))\mathbf{d}_L(V)\big)_{\widetilde{L}}$$

denote the morphism that is obtained by taking the product of $\Gamma_L(V)$ with the morphisms $\eta_\ell(V)$ and $\overline{(\eta_\ell(V^*(1))^*)}$ from (13) and the morphism

$$\epsilon_{dR}(V) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_{\widetilde{L}}(V)\mathbf{d}_{\widetilde{L}}(D_{dR}(V))^{-1}$$

that is constructed by Fukaya and Kato in [16, Prop. 3.3.5].

We set $\Lambda := \Lambda(G)$ and define

$$\widetilde{\Lambda} := W(\overline{\mathbb{F}_p})[\![G]\!] = \varprojlim_U \big(W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G/U]\big),$$

where U runs over all open normal subgroups of G and $W(\overline{\mathbb{F}_p})$ denotes the Witt ring of $\overline{\mathbb{F}_p}$. Now we fix a finite-dimensional $\mathbb{Q}_p$-linear representation V of $G_{\mathbb{Q}_p}$, a full Galois stable $\mathbb{Z}_p$-sublattice T of V, set $\mathbb{T} := \Lambda \otimes_{\mathbb{Z}_p} T$ and we write $\mathcal{O}$ for the valuation ring of L. For any continuous representations $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ we denote by $V(\rho^*)$ the Galois representation $\rho^* \otimes V := \mathcal{O}^n \otimes_{\mathbb{Z}_p} V$, on which $G_{\mathbb{Q}_p}$ acts diagonally, via ρ^* on the first factor.

The following conjecture will play a key role in the sequel (for further discussion of this conjecture see [39, Conj. 5.9]).

CONJECTURE 6.1 (Fukaya and Kato, [16, Conj. 3.4.3]). *There exists a canonical morphism in $\mathcal{C}_{\widetilde{\Lambda}}$ of the form*

$$\epsilon_{p,\Lambda}(\mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda}} \to \big(\mathbf{d}_\Lambda(\mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T})) \cdot \mathbf{d}_\Lambda(\mathbb{T})\big)_{\widetilde{\Lambda}}$$

which is such that for all finite degree extensions L of $\mathbb{Q}_p$, with valuation ring $\mathcal{O}$, and all continuous representations $\rho : G \to \mathrm{GL}_n(\mathcal{O}) \subseteq \mathrm{GL}_n(L)$ such that $V(\rho^*)$ is de Rham one has

$$L^n \otimes_\Lambda \epsilon_{p,\Lambda}(\mathbb{T}) = \epsilon_{p,L}(V(\rho^*)).$$

6.2. SELMER COMPLEXES. We fix a continuous finite-dimensional L-linear representation W of $G_{\mathbb{Q}}$ which satisfies the following 'condition of Dąbrowski-Panchishkin':

(DP) W is de Rham and there exists a $G_{\mathbb{Q}_p}$-subrepresentation $\hat{W}$ of W (restricted to $G_{\mathbb{Q}_p}$) such that $D^0_{dR}(\hat{W}) = t_p(W) := D_{dR}(W)/D^0_{dR}(W)$.

Thus we have an exact sequence of $G_{\mathbb{Q}_p}$-representations

$$0 \to \hat{W} \to W \to \tilde{W} \to 0$$

such that $D^0_{dR}(\hat{W}) = t_p(\tilde{W}) = 0$ (cf. [23, Prop. 1.28]). Setting $Z := W^*(1)$, $\hat{Z} := \tilde{W}^*(1)$ and $\tilde{Z} := \hat{W}^*(1)$ we obtain by Kummer duality the analogous exact sequence

$$0 \to \hat{Z} \to Z \to \tilde{Z} \to 0$$

and we note that Z also satisfies the condition (DP).
We now fix a finite set S of places of $\mathbb{Q}$ which contains both $S_\infty := \{\infty\}$ and $S_p := \{p\}$ and is such that W (and hence also Z) is a representation of G_S, and we set $U := \mathrm{Spec}(\mathbb{Z}[\frac{1}{S}])$.
Then the *Selmer complex* $SC_U(\hat{W}, W)$ is defined to be the natural mapping fibre
(31)

$$SC_U(\hat{W}, W) \longrightarrow \mathrm{R}\Gamma(U, W) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_p, W/\hat{W}) \oplus \bigoplus_{\ell \neq p} \mathrm{R}\Gamma(\mathbb{Q}_\ell, W) \longrightarrow$$

while the *modified Selmer complex* $SC(\hat{W}, W)$ is defined to be the natural mapping fibre
(32)

$$SC(\hat{W}, W) \longrightarrow \mathrm{R}\Gamma(U, W) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_p, W/\hat{W}) \oplus \bigoplus_{\ell \neq p} \mathrm{R}\Gamma_{/f}(\mathbb{Q}_\ell, W) \longrightarrow$$

where in both cases ℓ runs over all prime numbers that are distinct from p. Also, for each such ℓ, the complex $\mathrm{R}\Gamma_{/f}(\mathbb{Q}_\ell, W)$ is defined as the natural mapping cone

$$\mathrm{R}\Gamma_f(\mathbb{Q}_\ell, W) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_\ell, W) \longrightarrow \mathrm{R}\Gamma_{/f}(\mathbb{Q}_\ell, W) \longrightarrow \tag{33}$$

For any $G_{\mathbb{Q}_p}$-representation V and prime number ℓ we define an element of the polynomial ring $L[u]$ by setting

$$P_\ell(V, u) := P_{L,\ell}(V, u) := \begin{cases} \det_L(1 - \varphi_\ell u | V^{I_\ell}), & \text{if } \ell \neq p, \\ \det_L(1 - \varphi_p u | D_{cris}(V)), & \text{if } \ell = p, \end{cases}$$

where φ_ℓ denotes the geometric Frobenius automorphism of ℓ.
Then the following three conditions are easily seen to be equivalent:

(A_1) $P_\ell(W, 1) P_\ell(Z, 1) \neq 0$ for all primes $\ell \neq p$,
(A_2) $H^0(\mathbb{Q}_\ell, W) = H^0(\mathbb{Q}_\ell, Z) = 0$ for all primes $\ell \neq p$,
(A_3) $\mathrm{R}\Gamma_f(\mathbb{Q}_\ell, W)$ is quasi-null for all primes $\ell \neq p$.

We also consider the following conditions:

(B_1) $P_p(W, 1) P_p(Z, 1) \neq 0$,
(B_2) $D_{cris}(W)^{\varphi_p - 1} = D_{cris}(Z)^{\varphi_p - 1} = 0$,
(B_3) $H^0(\mathbb{Q}_p, W) = H^0(\mathbb{Q}_p, Z) = 0$.

We note that (B_1) is equivalent to (B_2) and that [23, Thm. 1.15] shows that (B_3) implies (B_2).
Finally we consider the following mutually equivalent conditions (to see that (C_2) is equivalent to (C_3) one uses loc. cit. and the fact that $t_p(\tilde{W}) = t_p(\tilde{Z}) = 0$) :

(C_1) $P_p(\tilde{W}, 1) P_p(\tilde{Z}, 1) \neq 0$,
(C_2) $D_{cris}(\tilde{W})^{\varphi_p - 1} = D_{cris}(\tilde{Z})^{\varphi_p - 1} = 0$,
(C_3) $H^0(\mathbb{Q}_p, \tilde{W}) = H^0(\mathbb{Q}_p, \tilde{Z}) = 0$.

Lemma 6.2. *Let X denote either W or Z.*

(i) *If condition (A_1) is satisfied, then for every prime $\ell \neq p$ all of the following complexes are quasi-null*

$$\mathrm{R}\Gamma(\mathbb{Q}_\ell, X) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_\ell, X) \cong \mathrm{R}\Gamma_{/f}(\mathbb{Q}_\ell, X) \cong 0.$$

(ii) *If condition (C_1) is satisfied, then there are isomorphisms in $D^p(L)$ of the form*

$$\mathrm{R}\Gamma_{/f}(\mathbb{Q}_p, X) \cong \mathrm{R}\Gamma(\mathbb{Q}_p, \tilde{X})$$

and

$$\mathrm{R}\Gamma_f(\mathbb{Q}_p, X) \cong \mathrm{R}\Gamma(\mathbb{Q}_p, \hat{X}) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_p, \hat{X}).$$

(iii) *If conditions (A_1) and (C_1) are both satisfied, then there exists an isomorphism in $D^p(L)$ of the form*

$$SC_U(\hat{W}, W) \cong \mathrm{R}\Gamma_f(\mathbb{Q}, W).$$

Proof. We assume (A_1). Then by local duality and the local Euler characteristic formula it follows immediately that $\mathrm{R}\Gamma(\mathbb{Q}_\ell, X)$ is quasi-null. The other statements in claim (i) are then obvious. To prove claim (ii) we assume (C_1). Then, since every bounded complex of finitely generated L-modules is canonically isomorphic in $D^b(L)$ to its cohomology, considered as a complex with zero differentials, we have $\mathrm{R}\Gamma(\mathbb{Q}_p, \hat{X}) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_p, \hat{X}) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_p, X)$ by [16, Lem. 4.1.7]. Thus the exact triangles

$$\mathrm{R}\Gamma(\mathbb{Q}_p, \hat{X}) \to \mathrm{R}\Gamma(\mathbb{Q}_p, X) \to \mathrm{R}\Gamma(\mathbb{Q}_p, \tilde{X}) \to$$

and

$$\mathrm{R}\Gamma_f(\mathbb{Q}_p, X) \to \mathrm{R}\Gamma(\mathbb{Q}_p, X) \to \mathrm{R}\Gamma_{/f}(\mathbb{Q}_p, X) \to$$

are naturally isomorphic in $D^p(L)$. Finally, we note that claim (iii) follows immediately from claims (i) and (ii) and the respective definitions of $SC_U(\hat{W}, W)$ and $\mathrm{R}\Gamma_f(\mathbb{Q}, W)$. □

6.3. p-adic height pairings. To prepare for our derivation of the interpolation formula in §6.4 we now discuss certain preliminaries regarding p-adic height pairings.

We let M be any motive over $\mathbb{Q}$, $V = M_p$ its p-adic realization, ρ an Artin representation defined over the number field K and $[\rho]$ the corresponding Artin motive. We fix a p-adic place λ of K, set $L := K_\lambda$ and write $\mathcal{O}$ for the valuation ring of L. Then the λ-adic realisation

$$W := N_\lambda = V \otimes_{\mathbb{Q}_p} [\rho]^*_\lambda \tag{34}$$

of the motive $N := M(\rho^*) := M \otimes [\rho]^*$ is an L-adic representation. We assume that V (and hence, since $[\rho]^*$ is pure of weight zero, also W) satisfies the condition (DP). We fix a full Galois stable $\mathbb{Z}_p$-sublattice T of V and set $T_\rho := T \otimes_{\mathbb{Z}_p} \mathcal{O}^n$, a Galois stable lattice in W (where we assume that without loss of generality $[\rho]^*_\lambda$ is given as $\rho^* : G_\mathbb{Q} \to \mathrm{GL}_n(\mathcal{O})$). Similarly we fix a full $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-sublattice $\hat{T}$ of $\hat{V}$ and we define $\tilde{T}$ to be the lattice in $\tilde{V}$ that is induced from T. Finally we set $\hat{T}_\rho := \hat{T} \otimes_{\mathbb{Z}_p} \mathcal{O}^n$ and $\tilde{T}_\rho := \tilde{T} \otimes_{\mathbb{Z}_p} \mathcal{O}^n$ (which are Galois stable $\mathcal{O}$-sublattices of $\hat{W}$ and $\tilde{W}$ respectively).

Example 6.3. Let A be an abelian variety that is defined over $\mathbb{Q}$ and set $M := h^1(A)(1)$. If A has good ordinary reduction at p, then $W := N_\lambda$ satisfies the conditions (DP), (A_1), (B_1) and (C_1). Indeed, the last three conditions are valid for weight reasons, and more generally, condition (DP) is known to be valid for any motive which has good ordinary reduction at p (see [28]). More precisely, for A (still in the good ordinary case) we have $\hat{W} = \hat{V} \otimes [\rho]^*_\lambda$ where $\hat{V} = V_p(\widehat{A^\vee})$ denotes the p-adic Tate-module of the formal group of the dual abelian variety $A^\vee$ of A. However, if, for example, A is an elliptic curve with (split) multiplicative reduction at p, then M does not satisfy the condition (B_1).

Now we define a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-sublattice of $\hat{V}$ by setting

$$\hat{T} := T \cap \hat{V}.$$

As before we let $\mathbb{T}$ denote the Galois representation $\Lambda \otimes_{\mathbb{Z}_p} T$ and set $\hat{\mathbb{T}} := \Lambda \otimes_{\mathbb{Z}_p} \hat{T}$ similarly. Then $\hat{\mathbb{T}}$ is a $G_{\mathbb{Q}_p}$-stable Λ-submodule of $\mathbb{T}$. It is in fact a direct summand of $\mathbb{T}$ and there exists a morphism in $\mathcal{C}_{\tilde{\Lambda}}$ of the form

$$\beta : \mathbf{d}_\Lambda(\mathbb{T}^+)_{\tilde{\Lambda}} \cong \mathbf{d}_\Lambda(\hat{\mathbb{T}})_{\tilde{\Lambda}}. \tag{35}$$

Now the Selmer complexes $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ and $SC(\hat{\mathbb{T}}, \mathbb{T})$ are defined analogously as for W above.

Then $SC_U(\hat{X}, X)$ coincides with the Selmer complex $\widetilde{\mathrm{R}\Gamma_f}(X)$ that occurs in [24, (11.3.1.5)] for $X \in \{W, Z\}$. More generally, we set $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q})$ and define

$$\mathbb{T}_{\mathrm{cyc},\rho} := \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T_\rho$$

and similarly also $\hat{\mathbb{T}}_{\mathrm{cyc},\rho}$ and $\tilde{\mathbb{T}}_{\mathrm{cyc},\rho}$. Then $SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho}, \mathbb{T}_{\mathrm{cyc},\rho})$ identifies with the Selmer complex $\widetilde{\mathrm{R}\Gamma_{f,\mathrm{Iw}}}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}, T_\rho)$ that is defined in [24, (8.8.5)] (with Nekovář's local conditions induced by setting $T_\ell^+ := \hat{\mathbb{T}}_{\mathrm{cyc}}(\rho)$ if $\ell = p$ and $T_\ell^+ := 0$ otherwise, and with Nekovář's set Σ taken to be the set of all rational primes). Thus we obtain a pairing

$$h_p(W) : H^1_f(\mathbb{Q}, W) \times H^1_f(\mathbb{Q}, Z) \to L$$

from [24, §11] where $h_p(W)$ is denoted $\tilde{h}_{\pi,1,1}$. Now, by [24, Thm. 11.3.9], the pairing $h_p(W)$ coincides up to sign with the height pairings constructed by Schneider [32] (in the case of abelian varieties) and Perrin-Riou [26] (for semi-stable representations) and also those constructed earlier by Nekovář [23]: see also [loc. cit., §8.1] and the papers of Mazur and Tate [22] and Zarhin [41] for alternative definitions of related height pairings.

It follows from the construction of Nekovář's height pairing (cf. [24, the sentence after (11.1.3.2)]) that the induced map

$$\mathrm{ad}(h_p(W)) : H^1_f(\mathbb{Q}, W) \to H^1_f(\mathbb{Q}, Z)^* \tag{36}$$

is equal to the composite

$$(37)\quad H^1_f(\mathbb{Q},W)\cong H^1(SC_U(\hat{W},W))\xrightarrow{\mathfrak{B}} H^2(SC_U(\hat{W},W)) \cong H^2_f(\mathbb{Q},W)\cong H^1_f(\mathbb{Q},Z)^*$$

where the first and third maps are by Lemma 6.2(iii), $\mathfrak{B}$ denotes the Bockstein homomorphism for $SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho},\mathbb{T}_{\mathrm{cyc},\rho})$ and the last map comes from global duality.

6.4. THE INTERPOLATION FORMULA. In this section we assume that the motive $N := M(\rho^*)$ is critical. Then, assuming the conjecture [39, Conj. 3.3] of Fontaine and Perrin-Riou to be valid, the motivic cohomology groups

(D_1) $H^0_f(N)=H^0_f(N^*(1))=0$

both vanish. In fact, if we also assume the validity of a well-known conjecture [39, Conj. 3.6] on p-adic regulator maps, this last condition is equivalent to the condition

(D_2) $H^0_f(\mathbb{Q},W)=H^0_f(\mathbb{Q},Z)=0$

where W is defined in (34) and $Z := W^*(1)$.
We also consider the condition

(F) The pairing $h_p(W)$ is non-degenerate.

EXAMPLE 6.4. If A is an abelian variety over $\mathbb{Q}$, then the motive $M = h^1(A)(1)$ satisfies the conditions (D_1) and (D_2). However, very little is known about the non-degeneracy of the p-adic height pairing in the ordinary case. Indeed, as far as we are aware, the only theoretical evidence for non-degeneracy is a result of Bertrand [1] that for an elliptic curve with complex multiplication, the height of a point of infinite order is non-zero (but even this is unknown in the non CM case). Computationally, however, there has been a lot of work done recently by Stein and Wuthrich [40]. We are grateful to J. Coates, P. Schneider and C. Wuthrich for providing us with these examples.

We now fix a compact p-adic Lie extension F_∞ of $\mathbb{Q}$ which contains $\mathbb{Q}_{\mathrm{cyc}}$ and is unramified outside S. We let G denote the group $\mathrm{Gal}(F_\infty/\mathbb{Q})$, with quotient $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q})$, and we set $\Lambda := \Lambda(G)$.
In [16] Fukaya and Kato use the morphisms $\zeta_\Lambda(M)$ and $\epsilon_{p,\Lambda}(\hat{\mathbb{T}})$ that are predicted to exist by Conjecture 4.1 and Conjecture 6.1 to construct canonical 'p-adic L-function' morphisms in $\mathcal{C}_\Lambda$ of the form

$$(38)\qquad \mathcal{L}_{U,\beta} := \mathcal{L}_{U,\beta}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(SC_U(\hat{\mathbb{T}},\mathbb{T}))$$

and

$$(39)\qquad \mathcal{L}_{\beta} := \mathcal{L}_{\beta}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(SC(\hat{\mathbb{T}},\mathbb{T}))$$

both depending on the isomorphism β in (35). We set $SC_U := SC_U(\hat{\mathbb{T}},\mathbb{T})$ and $SC := SC(\hat{\mathbb{T}},\mathbb{T})$. Then the morphisms $\mathcal{L}_{U,\beta}$ and $\mathcal{L}_\beta$ give rise to elements $[SC_U,\mathcal{L}_{U,\beta}]$ and $[SC,\mathcal{L}_\beta]$ of $K_1(\Lambda(G),\Sigma_{SC_U})$ and $K_1(\Lambda(G),\Sigma_{SC})$ respectively

(where we use the notation Σ_C introduced at the end of §2.2), and for simplicity we continue to denote these elements by $\mathcal{L}_{U,\beta}$ and $\mathcal{L}_\beta$ respectively.
We write Υ for the set of all primes $\ell \neq p$ with the property that the ramification index of ℓ in $F_\infty/\mathbb{Q}$ is infinite. We note that Υ is empty if G has a commutative open subgroup.

Theorem 6.5. *We assume that the motive $M(\rho^*)$ is critical, that the representation W defined in (34) satisfies the conditions (DP), (A_1), (B_1), (C_1), (D_2) and (F) and that the morphisms $\zeta_\Lambda(M)$ and $\epsilon_{p,\Lambda}(\hat{\mathbb{T}})$ that are described in Conjecture 4.1 and Conjecture 6.1 exist.*
Then both $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ and $SC(\hat{\mathbb{T}}, \mathbb{T})$ are semisimple at ρ, one has $r := r_G(SC_U(\hat{\mathbb{T}}, \mathbb{T}))(\rho) = r_G(SC(\hat{\mathbb{T}}, \mathbb{T}))(\rho) = \dim_L H^1_f(\mathbb{Q}, W)$ and the leading term $\mathcal{L}^*_\beta(\rho)$ (respectively $\mathcal{L}^*_{U,\beta}(\rho)$) is equal to the product

$$(40)\quad (-1)^r \frac{L^*_{K,B}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_{p,\beta}(M(\rho^*))R_p(M(\rho^*)) \cdot \Gamma_{\mathbb{Q}_p}(\hat{V})^{-1} \cdot \frac{P_{L,p}(\hat{W}^*(1), 1)}{P_{L,p}(\hat{W}, 1)},$$

where $L^*_{K,B}(M(\rho^*))$ denotes the leading term at $s = 0$ of the B-truncated complex L-function of $M(\rho^*)$ with $B := \Upsilon \cup S_p$ (respectively $B := S \setminus S_\infty$). Further, the regulator terms $R_\infty(M(\rho^*))$ and $R_p(M(\rho^*))$ and period terms $\Omega_\infty(M(\rho^*))$ and $\Omega_{p,\beta}(M(\rho^*))$ that occur in the above formula are as defined in the course of the proof given below.

Remark 6.6. The formulas of Theorem 6.5 represent a natural generalization of the formulas obtained by Perrin-Riou in [29, 4.2.2 and 4.3.6]. Further, by slightly altering the definition of the complex L-function an analogous formula can be proved even in the case that the condition (B_1) is not satisfied. Indeed, if condition (B_1) fails, then one can have $P_{L,p}(W, 0) = 0$ and so the order of vanishing at $s = 0$ of the functions $L_{K,B}(M(\rho^*), s)$ and $L_K(M(\rho^*), s)$ may differ. However, to avoid this problem, in formula (40) one need only replace $P_{L,p}(\hat{W}, 1)$ by the leading coefficient of $P_{L,p}(\hat{W}, p^s)$ at $s = 0$, or equivalently one can replace the term $\frac{L^*_{K,B}(M(\rho^*))}{P_{L,p}(\hat{W},1)}$ by $\frac{L^*_{K,B\setminus\{p\}}(M(\rho^*))}{\{P_{L,p}(W,u)^{-1}P_{L,p}(\hat{W},u)\}_{u=1}}$.

Proof. We first prove all of the assertions concerning $SC_U(\hat{\mathbb{T}}, \mathbb{T})$.
By [16, 4.1.4(2)] there exists a canonical isomorphism

$$(41)\qquad (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes^{\mathbb{L}}_{\Lambda(G)} SC_U(\hat{\mathbb{T}}, \mathbb{T}) \cong SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho}, \mathbb{T}_{\mathrm{cyc},\rho}).$$

Lemma 3.13 therefore combines with the following result to imply that, under the stated conditions, $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ is semisimple at ρ and one has $r_G(SC_U(\hat{\mathbb{T}}, \mathbb{T}))(\rho) = \dim_L H^1_f(\mathbb{Q}, W)$.

Lemma 6.7. *We assume that the conditions (A_1), (C_1) and (D_2) are satisfied.*

(i) *Then $SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho}, \mathbb{T}_{\mathrm{cyc},\rho})$ is semisimple if and only if the condition (F) holds.*

(ii) *Further, if condition (F) is satisfied, then* $r_\Gamma(SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho}, \mathbb{T}_{\mathrm{cyc},\rho})) = \dim_L H^1_f(\mathbb{Q}, W)$.

Proof. By assumption, the condition (D_2) can be combined with the isomorphism of Lemma 6.2 (iii) and the global duality isomorphism $H^3_f(\mathbb{Q}, W) \cong H^0_f(\mathbb{Q}, Z)^*$ to imply that $SC_U(\hat{W}, W)$ is acyclic outside degrees 1 and 2. Both claims therefore follow from the fact that the homomorphisms (36) and (37) are known to coincide and that there are canonical isomorphisms $L \otimes_{\Lambda(\Gamma)} \mathbb{T}_{\mathrm{cyc},\rho} \cong W$, $L \otimes_{\Lambda(\Gamma)} \hat{\mathbb{T}}_{\mathrm{cyc},\rho} \cong \hat{W}$ and thus $L \otimes_{\Lambda(\Gamma)} SC_U(\hat{\mathbb{T}}_{\mathrm{cyc},\rho}, \mathbb{T}_{\mathrm{cyc},\rho}) \cong SC_U(\hat{W}, W)$. $\square$

We next prove the explicit formula (40) for the leading term $\mathcal{L}^*_{U,\beta}(\rho)$. Our proof of this result is closely modeled on that of [16, Thm. 4.2.26] (as amplified in [39, proof of Thm. 6.4]).

At the outset we set $N := M(\rho^*)$, let $\gamma = (\gamma_i)_i$ and $\delta = (\delta_i)_i$ denote a choice of 'good bases' (in the sense of [16, 4.2.24(3)]) of M^+_B and t_M for $\hat{\mathbb{T}}$ and write γ' and δ' for the induced K-bases of N^+_B and t_N respectively. Then these choices induce a morphism

$$can_{\gamma',\delta'} : \mathbf{1}_K \to \mathbf{d}_K(N^+_B)\mathbf{d}_K(t_N)^{-1}. \tag{42}$$

Furthermore, we let $P^\vee = (P^\vee_1, \dots, P^\vee_{d(N)})$ and $P = (P_1, \dots, P_{d(N)})$ be K-bases of $H^1_f(N)$ and $H^1_f(N^*(1))$ respectively. Then, letting $P^d := (P^d_1, \dots, P^d_{d(N)})$ denote the dual basis of P, we obtain a similar morphism

$$can_{P^\vee,P^d} : \mathbf{1}_K \to \mathbf{d}_K(H^1_f(N))\mathbf{d}_K(H^1_f(N^*(1))^*)^{-1}. \tag{43}$$

Then $can := can_{\gamma',\delta'} \cdot can_{P^\vee,P^d}$ is a morphism

(44)

$$can : \mathbf{1}_K \to \Delta_K(N) = \mathbf{d}_K(N^+_B)\mathbf{d}_K(t_N)^{-1}\mathbf{d}_K(H^1_f(N))\mathbf{d}_K(H^1_f(N^*(1))^*)^{-1}.$$

We fix an embedding of K into $\mathbb{C}$. We let $\Omega_\infty(N)$ denote the determinant of the canonical isomorphism

$$\alpha_N : (N^+_B)_\mathbb{C} \to (t_N)_\mathbb{C} \tag{45}$$

with respect to the bases γ' and δ', and $R_\infty(N)$ the determinant of the inverse of the canonical isomorphism

$$h_\infty(N) : \left(H^1_f(N^*(1))^*\right)_\mathbb{C} \to H^1_f(N)_\mathbb{C} \tag{46}$$

with respect to the bases P^d and $P^\vee$ respectively. Thus we have morphisms

$$\Omega_\infty(N) : \ \mathbf{1}_\mathbb{C} \xrightarrow{(can_{\gamma',\delta'})_\mathbb{C}} \mathbf{d}_K(N^+_B)_\mathbb{C}\mathbf{d}_K(t_N)^{-1}_\mathbb{C} \xrightarrow{\mathbf{d}(\alpha_N)\cdot \mathrm{id}} \mathbf{1}_\mathbb{C}$$

and

$$R_\infty(N) : \ \mathbf{1}_\mathbb{C} \xrightarrow{(can_{P^\vee,P^d})_\mathbb{C}} \mathbf{d}_K(H^1_f(N))_\mathbb{C}\mathbf{d}_K(H^1_f(N^*(1))^*)^{-1}_\mathbb{C} \xrightarrow{\mathrm{id}\cdot\mathbf{d}(h_\infty(N))^{-1}} \mathbf{1}_\mathbb{C}$$

whose product gives

$$\Omega_\infty(N)R_\infty(N) : \ \mathbf{1}_\mathbb{C} \xrightarrow{can} \Delta_K(N)_\mathbb{C} \xrightarrow{(\vartheta_\infty(N))_\mathbb{C}} \mathbf{1}_\mathbb{C}.$$

Upon comparing this with the leading term

$$L_K^*(M): \quad \mathbf{1}_{\mathbb{C}} \xrightarrow{\zeta_K(N)_{\mathbb{C}}} \Delta_K(N)_{\mathbb{C}} \xrightarrow{(\vartheta_\infty(N))_{\mathbb{C}}} \mathbf{1}_{\mathbb{C}}$$

we deduce that $\zeta_K(N): \mathbf{1}_K \to \Delta_K(N)$ is equal to the morphism

$$\frac{L_K^*(M)}{\Omega_\infty(N)R_\infty(N)} \cdot can: \mathbf{1}_K \to \Delta_K(N).$$

Before proceeding we recall the relevant descent properties of Selmer complexes.

Lemma 6.8. *We use the notation of §6.3.*

(i) *There exist canonical isomorphisms of the form*

$$L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} \mathrm{R}\Gamma_c(U,\mathbb{T}) \cong \mathrm{R}\Gamma_c(U,W), \quad L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} SC_U(\hat{\mathbb{T}},\mathbb{T}) \cong SC_U(\hat{W},W).$$

(ii) *There exists an exact triangle of the form*

$$L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} SC(\hat{\mathbb{T}},\mathbb{T}) \longrightarrow SC(\hat{W},W) \longrightarrow \bigoplus_{\ell\in\Upsilon} \mathrm{R}\Gamma_f(\mathbb{Q}_\ell,W) \longrightarrow .$$

Proof. See [16, Prop. 1.6.5 and Prop. 4.2.17]. □

Now, after taking account of Lemma 6.8(i), the leading term $\mathcal{L}^*_{U,\beta}(\rho)$ is defined (in Definition 3.14) to be equal to $(-1)^r$ times the morphism

$$\mathbf{1}_{\tilde{L}} \xrightarrow{\zeta_\Lambda(M)(\rho)_{\tilde{L}}} \mathbf{d}_L(\mathrm{R}\Gamma_c(U,W))^{-1}_{\tilde{L}} \xrightarrow{\beta(\rho)\epsilon(\hat{\mathbb{T}})^{-1}(\rho)} \mathbf{d}_L(SC_U(\hat{W},W))^{-1}_{\tilde{L}} \xrightarrow{t(SC_U(\rho^*))_{\tilde{L}}} \mathbf{1}_{\tilde{L}}$$

where $\zeta_\Lambda(M)(\rho) := L^n \otimes_\Lambda \zeta_\Lambda(N)$, $\beta(\rho) := L^n \otimes_\Lambda \beta$ and $\epsilon(\hat{\mathbb{T}})(\rho) := L^n \otimes_\Lambda \epsilon_{p,\Lambda}(\hat{\mathbb{T}})$. But Conjecture 4.1 implies that $\zeta_\Lambda(M)(\rho)$ is equal to

$$\mathbf{1}_{\tilde{L}} \xrightarrow{\zeta_K(N)_{\tilde{L}}} \Delta_K(N)_{\tilde{L}} \xrightarrow{\vartheta_\lambda(N)} \mathbf{d}_L(\mathrm{R}\Gamma_c(U,W))^{-1}_{\tilde{L}},$$

while Conjecture 6.1 implies that

$$\epsilon(\hat{\mathbb{T}})(\rho) = \epsilon_{p,L}(\hat{W}),$$

and hence it follows that $\mathcal{L}^*_{U,\beta}(\rho)$ is equal to the product of the following seven terms (47)-(53):

$$(-1)^r \frac{L_K^*(N)}{\Omega_\infty(M(\rho^*))R_\infty(N)}; \tag{47}$$

$$\Gamma_L(\hat{W})^{-1} = \Gamma_{\mathbb{Q}_p}(\hat{V})^{-1}; \tag{48}$$

(49)

$$\Omega_{p,\beta}(M(\rho^*)): \ \mathbf{d}_L(\hat{W})_{\tilde{L}} \xrightarrow{\cdot\,\epsilon_{dR}(\hat{W})^{-1}} \mathbf{d}_L(D_{dR}(\hat{W}))_{\tilde{L}} \xrightarrow{\mathbf{d}(g^t_{dR})} \mathbf{d}_K(t_{M(\rho^*)})_{\tilde{L}} \xrightarrow{\cdot\, can_{\gamma,\delta}} \mathbf{d}_K\big(M(\rho^*)^+_B\big)_{\tilde{L}} \xrightarrow{\mathbf{d}(g^+_\lambda)} \mathbf{d}_L(W^+)_{\tilde{L}} \xrightarrow{\beta(\rho)} \mathbf{d}_L(\hat{W})_{\tilde{L}},$$

where we use $D^0_{dR}(\hat{W}) = 0$ for the second isomorphism and where we apply Remark 2.2 to regard this as an automorphism of $\mathbf{1}_{\tilde{L}}$;

$$(50) \qquad \prod_{\ell \in S \setminus \{p,\infty\}} P_{L,\ell}(W,1):$$

$$\mathbf{1}_L \xrightarrow{\prod \eta_\ell(W)} \prod_{\ell \in S\setminus\{p,\infty\}} \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_\ell, W)) \xrightarrow{acyc} \mathbf{1}_L,$$

where the first map comes from the trivialization by the identity and the second from the acyclicity;

$$(51) \quad \{P_{L,p}(W,u)P_{L,p}(\hat{W},u)^{-1}\}_{u=1}:$$

$$\mathbf{1}_L \xrightarrow{\eta_p(W)\cdot\eta_p(\hat{W})^{-1}} \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_p, W))\mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_p,\hat{W}))^{-1} \xrightarrow{quasi} \mathbf{1}_L,$$

where we use that $t_p(W) = D_{dR}(\hat{W}) = t_p(\hat{W})$ and the quasi-isomorphism described in Lemma 6.2(ii);

$$(52) \qquad P_{L,p}(\hat{W}^*(1),1): \ \mathbf{1}_L \xrightarrow{\overline{(\eta_p(\hat{W}^*(1)))^*}} \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_p, \hat{W}^*(1))) \xrightarrow{acyc} \mathbf{1}_L,$$

where we use the fact that $t_p(\hat{W}^*(1)) = D^0_{dR}(\hat{W}) = 0$;

$$(53) \quad R_p(N): \ \ \mathbf{1}_L \xrightarrow{(can_{P^\vee,P^d})_L} \mathbf{d}_K(H^1_f(N))_L \mathbf{d}_K(H^1_f(N^*(1))^*)_L^{-1} \xrightarrow{\cong}$$

$$\mathbf{d}_K(H^1_f(\mathbb{Q},W))_L\mathbf{d}_K(H^1_f(\mathbb{Q},Z)^*)_L^{-1} \xrightarrow{h_p(W)} \mathbf{1}_L$$

which is equal to the determinant over L of the isomorphism $\mathrm{ad}(h_p(W))$ with respect to the chosen bases $P^\vee$ and P.

Indeed, in order to compare $\mathcal{L}^*_{U,\beta}(\rho)$ with the product of the above terms (47)-(53) one just has to verify that after revealing all definitions and identifications, in particular all comparison isomorphisms, the same constituents show up in both expressions (here we rely on Remark 2.2 which implies that all compositions of maps in $\mathcal{C}_{\tilde{L}}$ can be interpreted as products and hence are independent of any ordering). Thus we shortly indicate how the constituents of $\mathcal{L}^*_{U,\beta}(\rho)$ give rise to precisely those in the product: As we remarked earlier, $\zeta_\Lambda(M)(\rho)$ decomposes up to the comparison isomorphism $\mathbf{d}(g^+_\lambda)$, which contributes to factor (49), into $\zeta_K(N)_L$ and $\vartheta_\lambda(N)$. While $\zeta_K(N)_L$ gives the full factor (47) and contributes with $can_{\gamma,\delta}$ and $can_{P^\vee,P^d}$ to the factors (49) and (53), respectively, the second part $\vartheta_\lambda(N)$ gives the full factor (50), the half factor (51) in the form of $\eta_p(W)$ and contributes $\mathbf{d}(g^+_{dR})$ to factor (49). Further, $\beta(\rho)$ contributes to factor (49), while according to [16, §3.3] $\epsilon(\hat{\mathbb{T}})^{-1}(\rho) = \epsilon_{p,L}(\hat{W})^{-1}$ gives the full factors (48) and (52), the other half of (51) in the form of $\eta_p(\hat{W})^{-1}$ and adds $\epsilon_{dR}(\hat{W})$ to factor (49). Finally, we had observed at the end of §6.3 that $t(SC_U(\rho^*))$ is equal to $h_p(W)$.

Since $\mathcal{L}^*_{U,\beta}(\rho)$ is equal to the product of the terms (47)-(53), it is therefore enough to show that the product of these terms is also equal to the explicit product expression in (40). But this follows immediately by a direct comparison of the maps involved and then using the fact that

$$L^*_{K,B}(N) = L^*_K(N) \cdot \prod_{\ell \in S \setminus S_\infty} P_{L,\ell}(W,1) \cdot P_{L,p}(\hat{W},1)^{-1} \cdot P_{L,p}(\hat{W}^*(1),1).$$

At this stage we have proved all of the claims in Theorem 6.5 concerning $SC_U(\hat{\mathbb{T}},\mathbb{T})$ and so it only remains to prove the analogous claims for the complex $SC(\hat{\mathbb{T}},\mathbb{T})$. But these claims can be proved easily by combining the above argument with consideration of the exact triangle

$$SC_U(\hat{W},W) \to L^n \otimes^{\mathbb{L}}_\Lambda SC(\hat{\mathbb{T}},\mathbb{T}) \to \bigoplus_{\ell \notin (S_p \cup \Upsilon)} \mathrm{R}\Gamma_f(\mathbb{Q}_\ell, W) \to$$

(which itself results from comparing the defining exact triangles (31) and (32) firstly with each other and then with the exact triangle in Lemma 6.8(ii)) and the equality

$$L^*_{K,\Upsilon'}(N) = L^*_{K,B'}(N) \prod_{\ell \in B \setminus \Upsilon} P_{L,\ell}(W,1)^{-1}$$

with $\Upsilon' = \Upsilon \cup \{p\}$ and $B' = S \setminus S_\infty$. □

Example 6.9. Let E be an elliptic curve defined over $\mathbb{Q}$. Set $M := h^1(E)(1)$ and $F_\infty := \mathbb{Q}(E(p))$ where $E(p)$ denotes the p-power torsion subgroup of $E(\overline{\mathbb{Q}})$. Then it is conjectured that $SC_U(\hat{\mathbb{T}},\mathbb{T})$ always belongs to Σ_{S^*} (cf. [11, Conj. 5.1] and [16, 4.3.5 and Prop. 4.3.7]). Further, as was shown in [16], the existence of a morphism $\mathcal{L}_\beta(M)$ as in (39) implies the existence of the element $\mathcal{L}_E$ of $K_1(\Lambda(G)_{S^*})$ that [11, Conj. 5.7] predicts to exist with a precise interpolation property for all Artin representations ρ such that $r_G(SC(\hat{\mathbb{T}},\mathbb{T}))(\rho) = 0$. More generally, the formula (40) now gives a precise interpolation property for (the leading term of) the element $\mathcal{L}_E$ at all Artin representations at which the underlying archimedean and p-adic height pairings are non-degenerate.

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David Burns
King's College London
Dept. of Mathematics
London WC2R 2LS
United Kingdom

Otmar Venjakob
Universität Bonn
Mathematisches Institut
Beringstraße 1
D-53115 Bonn
Germany