# Admissible $p$-adic Measures Attached to Triple Products of Elliptic Cusp Forms

To dear John Coates for his sixtieth birthday

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Abstract. We use the Siegel-Eisenstein distributions of degree three, and their higher twists with Dirichlet characters, in order to construct admissible $p$-adic measures attached to the triple products of elliptic cusp forms. We use an integral representation of Garrett's type for triple products of three cusp eigenforms. For a prime $p$ and for three primitive cusp eigenforms $f_1, f_2, f_3$ of equal weights $k_1 = k_2 = k_3 = k$, we study the critical values of Garrett's triple product $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$ twisted with Dirichlet characters $\chi$. The result is stated in framework of a general program by John Coates, see [Co], [Co-PeRi].

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## 0 Introduction

The purpose of this paper is to give a construction of $p$-adic admissible measures (in the sense of Amice-Vélu) attached to Garrett's triple $L$-function attached to three primitive cusp eigenforms of equal weight $k$, where $p$ is a prime. For this purpose we use the theory of $p$-adic integration with values in spaces of *nearly-holomorphic modular forms* (in the sense of Shimura, see [ShiAr]) over a normed $\mathcal{O}$-algebra $A$ where $\mathcal{O}$ is the ring of integers in a finite extension $K$ of $\mathbb{Q}_p$. Often we simply assume that $A = \mathbb{C}_p$.

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## 0.1 Generalities on triple products

Consider three primitive cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3) \tag{0.1}$$

of weights $k_1, k_2, k_3$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \bmod N_j$ $(j = 1, 2, 3)$, and let $\chi$ denote a Dirichlet character.
The triple product twisted with Dirichlet characters $\chi$ is defined as the following complex $L$-function (an Euler product of degree eight):

$$L^S(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \notin S} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}), \text{ where} \tag{0.2}$$

$$L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \tag{0.3}$$

$$\det\left(1_8 - X \begin{pmatrix} \alpha^{(1)}_{p,1} & 0 \\ 0 & \alpha^{(2)}_{p,1} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,2} & 0 \\ 0 & \alpha^{(2)}_{p,2} \end{pmatrix} \otimes \begin{pmatrix} \alpha^{(1)}_{p,3} & 0 \\ 0 & \alpha^{(2)}_{p,3} \end{pmatrix}\right)$$
$$= \prod_{\eta} (1 - \alpha^{(\eta(1))}_{p,1} \alpha^{(\eta(2))}_{p,2} \alpha^{(\eta(3))}_{p,3} X), \quad \eta : \{1, 2, 3\} \to \{1, 2\}, \text{ and}$$

$$1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha^{(1)}_{p,j}(p)X)(1 - \alpha^{(2)}_{p,j}(p)X), \quad j = 1, 2, 3,$$

are the Hecke $p$–polynomials of forms $f_j$ and the product is extended over all primes $p \notin S$, and $S = \mathrm{Supp}(N_1N_2N_3)$ denotes the set of all prime divisors of the product $N_1N_2N_3$. We always assume that

$$k_1 \geq k_2 \geq k_3, \tag{0.4}$$

including the case of equal weights $k_1 = k_2 = k_3 = k$.
We use the corresponding normalized motivic $L$ function (see [De79], [Co], [Co-PeRi]), which in the case of "balanced" weights (i.e. $k_1 \leq k_2 + k_3 - 2$) has the form:

$$\Lambda^S(f_1 \otimes f_2 \otimes f_3, s, \chi) = \tag{0.5}$$
$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - k_3 + 1)\Gamma_{\mathbb{C}}(s - k_2 + 1)\Gamma_{\mathbb{C}}(s - k_1 + 1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The motivic Gamma-factor

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - k_3 + 1)\Gamma_{\mathbb{C}}(s - k_2 + 1)\Gamma_{\mathbb{C}}(s - k_1 + 1)$$

determines the critical values $s = k_1, \cdots, k_2 + k_3 - 2$ and a (conjectural) functional equation of the form: $s \mapsto k_1 + k_2 + k_3 - 2 - s$.
Throughout the paper we fix an embedding

$$i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p, \text{ and define} \tag{0.6}$$

$$\lambda(p) = \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3}, \text{ where we assume that } |i_p(\alpha^{(1)}_{p,j})| \leq |i_p(\alpha^{(2)}_{p,j})|, j = 1, 2, 3. \tag{0.7}$$

### 0.2 Statement of main results

For a fixed positive integer $N \in \mathbb{N}$ consider the profinite group

$$Y = Y_{N,p} = \varprojlim_{v} Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times .$$

There is a natural projection $y_p : Y \to \mathbb{Z}_p^\times$. Let us fix a normed $\mathcal{O}$-algebra $A$ where $\mathcal{O}$ is the ring of integers in a finite extension $K$ of $\mathbb{Q}_p$.

Definition 0.1 (a) *For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, A)$ denote the $A$-module of locally polynomial functions of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow A^\times$; in particular,*

$$\mathcal{P}^1(Y, A) = \mathcal{C}^{loc-const}(Y, A)$$

*(the $A$-submodule of locally constant functions). We adopt the notation $\Phi(\mathcal{U}) := \Phi(\chi_\mathcal{U})$ for the characteristic function $\chi_\mathcal{U}$ of an open subset $\mathcal{U} \subset Y$. Let also denote $\mathcal{C}^{loc-an}(Y, A)$ the $A$-module of locally analytic functions and $\mathcal{C}(Y, A)$ the $A$-module of continuous functions so that*

$$\mathcal{P}^1(Y, A) \subset \mathcal{P}^h(Y, A) \subset \mathcal{C}^{loc-an}(Y, A) \subset \mathcal{C}(Y, A).$$

(b) *For a given positive integer $h$ we define an $h$-admissible measure on $Y$ with values in an $A$-module $M$ as a homomorphism of $A$-modules:*

$$\tilde{\Phi} : \mathcal{P}^h(Y, A) \to M,$$

*such that for all $a \in Y$ and for $v \to \infty$*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^j d\tilde{\Phi} \right|_{p,M} = o(p^{-v(j-h)}) \quad \text{for all} \quad j = 0, 1, \cdots, h-1,$$

*where $a_p = y_p(a)$.*

We adopt the notation $(a)_v = a + (Np^v)$ for both an element of $Y_v$ and the corresponding open compact subset of $Y$.

### $U_p$–Operator and Method of Canonical projection.

In Section 2.2, we construct an $h$-admissible measure $\widetilde{\Phi}^\lambda : \mathcal{P}^h(Y, A) \to \mathcal{M}(A)$ out of a sequence of distributions

$$\Phi_r : \mathcal{P}^1(Y, A) \to \mathcal{M}(A)$$

with values in an $A$-module $M = \mathcal{M}(A)$ of nearly-holomorphic triple modular forms over $A$ (for all $r \in \mathbb{N}$ with $r \leq h-1$), where $\lambda \in A^\times$ is a fixed non-zero eigenvalue of triple Atkin's operator $U_T = U_{T,p}$, acting on $\mathcal{M}(A)$, and

$h = [2\mathrm{ord}_p\lambda(p)] + 1$. In our case $\mathcal{M}(A) \subset A[\![q_1, q_2, q_3]\!][R_1, R_2, R_3]$, and such modular forms are formal series

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1, n_2, n_3; R_1, R_2, R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \in A[\![q_1, q_2, q_3]\!][R_1, R_2, R_3]$$

such that for $A = \mathbb{C}$, for all $z_j = x_j + iy_j \in \mathbb{H}$ and for $R_j = (4\pi y_j)^{-1}$ the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight $(k, k, k)$ and character $(\psi_1, \psi_2, \psi_3)$, $j = 1, 2, 3$. The usual action of $U = U_p$ on elliptic modular forms of one variable extends to triple Atkin's operator $U_T = U_{T,p} = (U_p)^{\otimes 3}$ acting on triple modular forms by

$$U_T(g) = \sum_{n_1,n_2,n_3=0}^{\infty} a(pn_1, pn_2, pn_3; pR_1, pR_2, pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}. \tag{0.8}$$

We consider the canonical projection operator $\pi_\lambda : \mathcal{M}(A) \to \mathcal{M}(A)^\lambda$ onto the maximal $A$-submodule $\mathcal{M}(A)^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent, and such that $\mathrm{Ker}\,\pi_\lambda = \bigcap_{n \geq 1} \mathrm{Im}\,(U_T - \lambda I)^n$. We define an $A$-linear map

$$\widetilde{\Phi}^\lambda : \mathcal{P}^h(Y, A) \to \mathcal{M}(A)$$

on local monomials $y_p^j$ by

$$\int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\lambda = \pi_\lambda(\Phi_j((a)_v))$$

where $\Phi_j : \mathcal{P}^1(Y, A) \to \mathcal{M}(A)$ is a sequence of $\mathcal{M}(A)$-valued distributions on $Y$ (for $j = 0, 1, \ldots, h-1$). Recall that for a primitive cusp eigenform $f_j = \sum_{n=1}^\infty a_n(f) q^n$ of conductor $C = C_{f_j}$, the function $f_{j,0} = \sum_{n=1}^\infty a_n(f_{j,0}) q^n \in \overline{\mathbb{Q}}[\![q]\!]$ is defined as an eigenfunction of $U = U_p$ with the eigenvalue $\alpha_{p,j}^{(1)} \in \overline{\mathbb{Q}}$ $(U(f_0) = \alpha f_0)$ which satisfies the identity

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j | V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \tag{0.9}$$

$$\sum_{n=1}^{\infty} a_n(f_{j,0}) n^{-s} = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} a_n(f_j) n^{-s} (1 - \alpha_{p,j}^{(1)} p^{-s})^{-1}.$$

For any fixed $n_0 = n \cdot p^m$ with $p \nmid n$ we have $a_{n_0}(f_{j,0}) = a_n(f_j) \cdot (\alpha_{p,j}^{(1)})^m \in \overline{\mathbb{Q}}$ and $a_n(f_j)$ are eigenvalues of Hecke operators $T_n$. Therefore, $U_T(f_{1,0} \otimes f_{2,0} \otimes f_{3,0}) = \lambda(f_{1,0} \otimes f_{2,0} \otimes f_{3,0})$. Moreover,

$$f_j^0 = f_{j,0}^\rho \Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \text{ where } f_{j,0}^\rho = \sum_{n=1}^\infty \overline{a(n, f_0)} q^n. \tag{0.10}$$

Consider the triple product defined by (0.2) as an Euler product of degree eight: $\mathcal{D}(f_1 \otimes f_2 \otimes f_3, s, \chi) = L^{(N)}(f_1 \otimes f_2 \otimes f_3, s, \chi)$, attached to three cusp eigenforms $f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j)$, $(j = 1, 2, 3)$ of weight $k$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \bmod N_j$ $(j = 1, 2, 3)$, where $\chi \bmod Np^v$ is an arbitrary Dirichlet character, and the notation $L^{(N)}$ means that the local factors at primes dividing $N = \mathrm{LCM}(N_1, N_2, N_2)$ are removed from an Euler product. Before giving the precise statements of our results on $p$-adic triple $L$-functions, we describe in more detail critical values of the $L$ function $\mathcal{D}(f_1 \otimes f_2 \otimes f_3, s, \chi)$.

Let us introduce the following normalized $L$-function

$$\begin{aligned}
&\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, s + 2k - 2, \psi_1\psi_2\chi) = \qquad (0.11)\\
&\Gamma_\mathbb{C}(s + 2k - 2)\Gamma_\mathbb{C}(s + k - 1)^3 L^{(N)}(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, s + 2k - 2, \psi_1\psi_2\chi),
\end{aligned}$$

where $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$, and $\Gamma_\mathbb{C}(s + 2k - 2)\Gamma_\mathbb{C}(s + k - 1)^3$ is the motivic Gamma-factor (compare with (0.5), and see [Co], [Co-PeRi], [Pa94]). For an arbitrary Dirichlet character $\chi \bmod Np^v$ consider the following Dirichlet characters:

$$\begin{aligned}
&\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi, \qquad (0.12)\\
&\chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3;
\end{aligned}$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are $Np$-complete (i.e., have the same prime divisors as those of $Np$).

THEOREM A (ALGEBRAIC PROPERTIES OF THE TRIPLE PRODUCT) *Assume that $k \geq 2$. Then for all pairs $(\chi, r)$ such that the corresonding Dirichlet characters $\chi_j$ are $Np$-complete, and $0 \leq r \leq k-2$, we have that*

$$\frac{\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k - 2 - r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T} \in \overline{\mathbb{Q}}$$

*where*

$$\begin{aligned}
\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T &:= \langle f_1^\rho, f_1^\rho\rangle_N \langle f_2^\rho, f_2^\rho\rangle_N \langle f_3^\rho, f_3^\rho\rangle_N\\
&= \langle f_1, f_1\rangle_N \langle f_2, f_2\rangle_N \langle f_3, f_3\rangle_N.
\end{aligned}$$

For the $p$-adic construction, let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. Fix a positive integer $N$, a Dirichlet character $\psi \bmod N$ and consider the commutative profinite group $Y = Y_{N,p} = \varprojlim_m (\mathbb{Z}/Np^m\mathbb{Z})^*$ and its group $X_{N,p} = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continuous) $p$-adic characters (this is a $\mathbb{C}_p$-analytic Lie group). The group $X_{N,p}$ is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A $p$-adic $L$-function $L_{(p)} : X_{N,p} \to \mathbb{C}_p$ is a certain meromorphic function on $X_{N,p}$. Such a function often come from a $p$-adic measure $\mu_{(p)}$ on $Y$ (*bounded* or *admissible* in the sense of Amice-Vélu, see [Am-V]). In this case we write for all $x \in X_{N,p}$

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y) \mathrm{d}\mu_{(p)}(y)$$

In order to establish $p$-adic properties, let us use the product (0.7) $\lambda = \lambda(p) = \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3}$, where we assume that $|i_p(\alpha^{(1)}_{p,j})| \le |i_p(\alpha^{(2)}_{p,j})|, j = 1,2,3.$

Theorem B (on admissible measures attached to the triple product). *Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde\mu^{\lambda}_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function*

$$\mathcal{D}_{(p)}(x, f_1\otimes f_2\otimes f_3) : X_p \to \mathbb{C}_p,$$

*given for all $x \in X_{N,p}$ by the integral*

$$\mathcal{D}_{(p)}(x, f_1\otimes f_2\otimes f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde\mu^{\lambda}_{f_1\otimes f_2\otimes f_3}(y),$$

*and having the following properties:*
(i) *for all pairs $(r,\chi)$ such that $\chi \bmod C_\chi$ is a primitive Dirichlet character modulo $C_\chi$, $\chi \in X^{\mathrm{tors}}_{N,p}$, assuming that all three corresonding Dirichlet characters $\chi_j$ given by (0.12) have $Np$-complete conductor $(j = 1,2,3)$, and $r \in \mathbb{Z}$ is an integer with $0 \le r \le k-2$, the following equality holds:*

$$\begin{aligned}\mathcal{D}_{(p)}(\chi x_p^r, f_1\otimes f_2\otimes f_3) = & \\ i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(2k-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda(p)^{2v}} & \\ \frac{\mathcal{D}^\star(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi)}{\langle f_1^0\otimes f_2^0\otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{T,Np}}\Big) &\end{aligned} \tag{0.13}$$

*where $v = \mathrm{ord}_p(C_\chi)$, $\chi_1 \bmod Np^v = \chi$, $\chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi, \chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi$, $G(\chi)$ denotes the Gauß sum of a primitive Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$).*
(ii) *if $\mathrm{ord}_p\lambda(p) = 0$ then the holomorphic function in (i) is a bounded $\mathbb{C}_p$-analytic function;*
(iii) *in the general case (but assuming that $\lambda(p) \ne 0$) the holomorphic function in (i) belongs to the type $o(\log(x_p^h))$ with $h = [2\mathrm{ord}_p\lambda(p)] + 1$ and it can be represented as the Mellin transform of the $h$-admissible $\mathbb{C}_p$-valued measure $\tilde\mu^{\lambda}_{f_1\otimes f_2\otimes f_3}$ (in the sense of Amice-Vélu) on $Y$*
(iv) *if $h \le k-2$ then the function $\mathcal{D}_{(p)}$ is uniquely determined by the above conditions (i).*

Remark 0.2 *It was checked by B.Gorsse and G.Robert that*

$$\langle f_1^{0,\rho}\otimes f_2^{0,\rho}\otimes f_3^{0,\rho}, f_{1,0}^\rho\otimes f_{2,0}^\rho\otimes f_{3,0}^\rho\rangle_{T,Np} = \beta\cdot\langle f_1,f_1\rangle_N\langle f_2,f_2\rangle_N\langle f_3,f_3\rangle_N$$

*for some $\beta \in \overline{\mathbb{Q}}^*$ (see [Go-Ro]).*

### 0.3 Scheme of the Proof

We construct $\overline{\mathbb{Q}}$-valued distributions denoted by $\mu_{f_1\otimes f_2\otimes f_3,r}$ on the profinite group $Y_{N,p}$, and attached to the special values at $s = 2k-2-r$ with $0 \le r \le k-2$ of the triple product $L(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho, s, \psi_1\psi_2\chi)$ twisted with a Dirichlet character $\psi_1\psi_2\chi \bmod Np^v$. We use an integral representation of this special value in terms of a $\mathcal{C}^\infty$-Siegel-Eisenstein series $F_{\chi,r}$ of degree 3 and of weight $k$ (to be specified later), where $0 \le r \le k-2$. Such a series $F_{\chi,r}$ depends on the character $\chi$, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, and it is defined by $F_{\chi,r} = G^\star(\mathcal{Z}, -r; k, (Np^v)^2, \boldsymbol{\psi})$, where $\mathcal{Z}$ denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(\mathcal{Z}, s; k, (Np^v)^2, \boldsymbol{\psi})$ is given by (A.12). This series depends on $s = -r$, and for the critical values at integral points $s \in \mathbb{Z}$ such that $2-k \le s \le 0$, it represents a (*nearly-*) *holomorphic* Siegel modular form in the sense of Shimura [ShiAr].
Our construction consists of the following steps:
1) We consider the profinite ring $A_{N,p} = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})$. Starting from any sequence $F_r$ of nearly-holomorphic Siegel modular forms we construct first a sequence $\Psi_{F_r}$ of modular distributions on the additive profinite group

$$S = S_{N,p} := \left\{ \boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix} \middle| \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23} \in A_{N,p} \right\};$$

such distributions take values in $\mathcal{C}^\infty$-(nearly-holomorphic) modular forms on the Siegel half plane $\mathbb{H}_3$. This construction, given in Section 1, generalizes the *higher twist* of $F_r$, already utilized in the work [Boe-Schm], in a simpler situation.
2) Next we consider the (real analytic) Siegel-Eisenstein series $F_{\chi,r}$ as a formal (nearly-holomorphic) Fourier series, whose coefficients admit explicit polynomial expressions (see Section 1 and Appendix A), and we use the fact that they may be written in terms of $p$-adic integrals of $\chi$ over $Y$ (see [PaSE] and [PaIAS]).
A crucial point of our construction is the *higher twist* in Section 1. We define the higher twist of the series $F_{\chi,r}$ by the characters (0.12) as the following formal nearly-holomorphic Fourier expansion:

$$F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \sum_{\mathcal{T}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23}) Q(R, \mathcal{T}; k-2r, r) a_{\chi,r}(\mathcal{T}) q^{\mathcal{T}}. \tag{0.14}$$

The series (0.14) can be naturally interpreted as an integral of the Dirichlet character $\chi$ on the group $Y$ with respect to a *modular distribution* $\Psi_r$:

$$F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \int_Y \chi(y)\,\mathrm{d}\Psi_r(y) =: \Psi_r(\chi). \tag{0.15}$$

These modular distributions take values in the ring of formal Fourier expansion whose coefficients are polynomials in $R = (4\pi \operatorname{Im}(\mathcal{Z}))^{-1}$ over the field $\overline{\mathbb{Q}}$ (which

is imbedded into $\mathbb{C}_p$ via (0.6). The distributions $\Psi_r$ are uniformly bounded (coefficient-by-coefficient).
3) If we consider the diagonal embedding

$$\mathrm{diag} : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}_3,$$

then the restriction produces a sequence $\Phi_r = 2^r \,\mathrm{diag}^* \,\Psi_r$ of distributions on $Y$ with values in the tensor product $\mathcal{M}_{k,r}(\overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r}(\overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r}(\overline{\mathbb{Q}})$ of three spaces of elliptic nearly-holomorphic modular forms on the Poincaré upper half plane $\mathbb{H}$ (the normalizing factor $2^r$ is neeeded in order to prove certain congruences between $\Phi_r$ in Section 3).
The important property of these distributions, established in Section 1, is that the nebentypus character of the triple modular form $\Phi_r(\chi)$ is fixed and is equal to $(\psi_1, \psi_2, \psi_3)$, see Proposition 1.5. Using this property, and applying the canonical projector $\pi_\lambda$ of Section 2 to $\Phi_r(\chi)$, we prove in Section 3 that the sequence of modular distributions $\Phi_r$ on $Y$ produces a $p$-adic admissible measure $\tilde{\Phi}^\lambda$ (in the sense of Amice-Vélu, [Am-V]) with values in a finite dimensional subspace

$$\mathcal{M}^\lambda(\mathbb{C}_p) \subset \mathcal{M}(\mathbb{C}_p),\ \mathcal{M}(\mathbb{C}_p) = \mathcal{M}_{k,r}(\mathbb{C}_p) \otimes \mathcal{M}_{k,r}(\mathbb{C}_p) \otimes \mathcal{M}_{k,r}(\mathbb{C}_p)$$

of the $\mathbb{C}_p$-vector space $\mathcal{M}(\mathbb{C}_p) = \bigcup_{v\geq 0} \mathcal{M}_{k,r}(Np^v, \psi_1, \psi_2, \psi_3; \mathbb{C}_p)$ of formal nearly-holomorphic triple modular forms of levels $Np^v$ and the fixed nebentypus characters $(\psi_1, \psi_2, \psi_3)$. We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}(\overline{\mathbb{Q}})$ (they have cyclotomic formal Fourier coefficients), and a general admissibility criterion (see Theorem 2.4). Proof of the Main Congruence is contained in Section 3.
4) Application of a $\overline{\mathbb{Q}}$-valued linear form of type

$$\mathcal{L}\ : h \longmapsto \frac{\left\langle \tilde{f}_1 \otimes \tilde{f}_2 \otimes \tilde{f}_3, h \right\rangle}{\left\langle \tilde{f}_1, \tilde{f}_1 \right\rangle \left\langle \tilde{f}_2, \tilde{f}_2 \right\rangle \left\langle \tilde{f}_3, \tilde{f}_3 \right\rangle}$$

for $h \in \mathcal{M}_{k,r}(\overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r}(\overline{\mathbb{Q}}) \otimes \mathcal{M}_{k,r}(\overline{\mathbb{Q}})$, produces a sequence of $\overline{\mathbb{Q}}$-valued distributions given by $\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r)(\chi))$, $\lambda \in \overline{\mathbb{Q}}^\times$. More precisely, we consider three auxilliary modular forms

$$\tilde{f}_j(z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j} e(nz) \in S_k(\Gamma_0(N_j p^{\nu_j}), \psi_j) \quad (1 \leq j \leq 3, \nu_j \geq 1), \qquad (0.16)$$

with the same eigenvalues as those of (0.1), for all Hecke operators $T_q$, with $q$ prime to $Np$. In our construction we use as $\tilde{f}_j$ certain "easy transforms" of primitive cusp forms in (0.1). In particular, we choose as $\tilde{f}_j$ eigenfunctions $\tilde{f}_j = f_j^0$ of the adjoint Atkin's operator $U_p^*$, in this case we denote by $f_{j,0}$ the corresponding eigenfunctions of $U_p$. The $\overline{\mathbb{Q}}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued

admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ starting from the modular $p$-adic admissible measure $\tilde{\Phi}^\lambda$ of stage 3), where $\ell : \mathcal{M}(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$. See Section 4 for the construction of $\tilde{\mu}^\lambda$.
5) We show in Section 5 that for any suitable Dirichlet character $\chi$ mod $Np^v$ the integral

$$\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi)))$$

coincides (up to a normalisation) with the special $L$-value

$\mathcal{D}^*(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi)$ (under the above assumptions on $\chi$ and $r$).

We use a general integral representation of Section B. The basic idea how a Dirichlet character $\chi$ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out. Note however that the existence of a $\mathbb{C}_p$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ established at stage 4), does not depend on this technical computation, and details will appear elsewhere.

Remark 0.3 *Similar techniques can be applied in the case of three arbitrary "balanced" weights (0.4) $k_1 \geq k_2 \geq k_3$, i.e. when $k_1 \leq k_2+k_3-2$, using various differential operators acting on modular forms (the Maaß-Shimura differential operators (see [ShiAr], [Or]), and Ibukiyama's differential operators (see [Ibu], [BSY]). More precisely, one applies these operators to a twisted Eisenstein series. In this case the critical values of the L function $\mathcal{D}(f_1 \otimes f_2 \otimes f_3, s, \chi)$ correspond to $s = k_1, \cdots, k_2 + k_3 - 2$. The equality of weights in the present paper is made to avoid (for lack of space) the calculus of differential operators.*

### 0.4 Conclusion: some advantages of our $p$-adic method

The whole construction works in various situations and it can be split into several independent steps:

1) Construction of modular distributions $\Phi_r$ (on a profinite or even adelic space $Y$ of type $Y = \mathbb{A}_K^*/K^*$ for a number field $K$) with values in an infinite dimensional modular tower $\mathcal{M}(\mathcal{A})$ over complex numbers (or in an $\mathcal{A}$-module of infinite rank over some $p$-adic algebra $\mathcal{A}$).

2) Application of a canonical projector of type $\pi_\lambda$ onto a finite dimensional subspace $\mathcal{M}^\lambda(\mathcal{A})$ of $\mathcal{M}(\mathcal{A})$ (or over a locally free $\mathcal{A}$-module of finite rank over some $\mathcal{A}$) in the form: $\pi_\lambda(g) = (U^\lambda)^{-v}\pi_{\lambda,1}(U^v(g)) \in \mathcal{M}^\lambda(Np, \mathcal{A})$ as in (2.3) of Section 2 (this method works only for $\lambda \in \mathcal{A}^\times$, and gives the $\lambda$-characteristic projector of $g \in \mathcal{M}(Np^v, \mathcal{A})$ (independently of a sufficiently large $v$)).

3) One proves the admissibility criterium of Theorem 2.4 saying that the sequence $\pi_\lambda(\Phi_r)$ of distributions with values in $\mathcal{M}^\lambda(\mathcal{A})$ determines an $h$-admissible measure $\tilde{\Phi}^\lambda$ with values in this finite dimensional space for a suitable $h$ (determined by the slope $\mathrm{ord}_p(\lambda)$).

4) Application of a linear form $\ell$ of type $g \mapsto \langle f^0, \pi_\lambda(g)\rangle / \langle f, f\rangle$ to the modular distributions $\Phi_r$ produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda = \ell(\pi_\lambda(\Phi_r))$, and an $\mathcal{A}$-valued admissible measure. The growth condition can be verified starting from congruences between modular forms $\Phi_j(\chi)$, generalizing our Main Congruence of Section 3.

5) One shows that certain integrals $\mu_j^\lambda(\chi)$ of the constructed distributions $\mu_j^\lambda$ coincide with normalized $L$-values; however, computing these integrals is not needed for the construction of $p$-adic admissible measures $\tilde{\mu}^\lambda$ (which is already done at stage 4)).

6) Under some assumptions, one can show a result on uniqueness for the constructed $h$-admissibles measures: they are determined by the integrals $\mu_j^\lambda(\chi)$ over almost all Dirichlet characters and sufficiently many $j = 0, 1, \cdots, h-1$ (this stage is not necessary, but it is nice to have uniqueness of the construction), see [JoH05].

7) If we are lucky, we can prove a functional equation for the constructed measure $\tilde{\mu}^\lambda$ (using the uniqueness in 6)), and using a functional equation for the $L$-values (over complex numbers), computed at stage 5), for almost all Dirichlet characters (again, this stage is not necessary, but it is nice to have a functional equation).

This strategy is applicable in various cases (described above), cf. [PaJTNB], [Puy], [Go02]. An interesting discussion in the Bourbaki talk [Colm03] of P.Colmez indicates the use of this method for constructing Euler systems.

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## Contents

# 1 Modular distributions attached to the higher twist of Eisenstein series

## 1.1 Higher twists of the Siegel-Eisenstein series

In this Section we study a $\mathcal{C}^\infty$-Siegel-Eisenstein series $F_{\chi,r}$ of degree 3 and of weight $k$, where $0 \le r \le k-2$. As in the Introduction, consider the Dirichlet characters (0.12) $\chi_1 \bmod Np^v = \chi$, $\chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi$, $\chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi$.

The series $F_{\chi,r} = G^\star(\mathcal{Z}, -r; k, (Np^v)^2, \boldsymbol{\psi})$, depends on the character $\chi$, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$. Here $\mathcal{Z}$ denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(\mathcal{Z}, s; k, (Np^v)^2, \boldsymbol{\psi})$ is given by (A.12). This series depends on $s = -r$, and for the critical values at integral points $s \in \mathbb{Z}$ such that $2-k \le s \le 0$, it represents a (*nearly-*) *holomorphic* function in the sense of Shimura [ShiAr] viewed as formal (nearly-holomorphic) Fourier series, whose coefficients admit explicit polynomial expressions in terms of simple $p$-adic integrals for $p \nmid \det(\mathcal{T})$:

$$F_{\chi,r} = \sum_{\mathcal{T}\in B_3} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k-2r, r) a_{\chi,r}(\mathcal{T}) q^{\mathcal{T}},$$

where $B_3 = \{\mathcal{T} = (\mathcal{T}_{ij}) \in \mathrm{M}_3(\mathbb{R}) \mid \mathcal{T} = {}^t\mathcal{T}, \mathcal{T} \ge 0, \mathcal{T}_{ij}, 2\mathcal{T}_{ii} \in \mathbb{Z}\}$, and $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}\mathcal{Z}))$, $R = (4\pi \mathrm{Im}(\mathcal{Z}))^{-1}$. More precisely, for any $\mathcal{T}$ with $p \nmid \det(\mathcal{T})$ there exists a bounded measure $\mathcal{F}_{\mathcal{T}}$ on $Y$ with values in $\overline{\mathbb{Q}}$ such that

$$a_{\chi,r}(\mathcal{T}) = \int_Y y_p^r \chi(y) d\mathcal{F}_{\mathcal{T}} = \prod_{\ell \mid \det(2\mathcal{T})} M_\ell(\mathcal{T}, \boldsymbol{\psi}(\ell)\ell^{-k+2r}), \tag{1.1}$$

where $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$ (see (A.17), Theorem A.2 in Appendix A, also in [PaSE], [PaIAS]). Here we use arithmetical nearly-holomorphic Siegel modular forms (see [ShiAr] and Appendix A.2 for more details) viewed as formal power series $g = \sum_{\mathcal{T}\in B_m} a(\mathcal{T}, R_{i,j}) q^{\mathcal{T}} \in \overline{\mathbb{Q}}[\![q^{B_m}]\!][R_{i,j}]$ such that for all $\mathcal{Z} \in \mathbb{H}_m$ the series converges to a $\mathcal{C}^\infty$-Siegel modular form of a given weight $k$ and character $\psi$. As in the introduction, (0.14), we define the higher twist of the series $F_{\chi,r}$

by the characters (0.12) as the following formal nearly-holomorphic Fourier expansion:

$$F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \sum_{\mathcal{T}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23})Q(R,\mathcal{T};k-2r,r)a_{\chi,r}(\mathcal{T})q^{\mathcal{T}} = \Psi_r(\chi).$$

We construct in this section a sequence of distributions $\Phi_r$ on $Y$ using the restriction to the diagonal

$$\Phi_r(\chi) := 2^r \operatorname{diag}^* \Psi_r(\chi) = 2^r F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3} \circ \operatorname{diag} \tag{1.2}$$

$$\begin{aligned} &= 2^r \sum_{t_1,t_2,t_3\geq 0} \sum_{\substack{\mathcal{T}: t_{11}=t_1,\\ t_{22}=t_2, t_{23}=t_3}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23}) \det(\mathcal{T})^{k-2r-\kappa}\times \\ &\times Q(\operatorname{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)a_{\chi,r}(\mathcal{T})q_1^{t_1}q_2^{t_2}q_3^{t_3}, \\ &\text{where } \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23}) = \bar\chi(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23}), \end{aligned}$$

taking values in the tensor product of three spaces of nearly-holomorphic elliptic modular forms on the Poincaré upper half plane $\mathbb{H}$ (recall that the normalizing factor $2^r$ is neeeded in order to prove congruences between $\Phi_r$ in Section 3). We show in Proposition 1.5 that the (diagonal) nebentypus character of $F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3}$ is $(\psi_1,\psi_2,\psi_3)$, thus it *does not depend* on $\chi$.

## 1.2 The higher twist as a distribution

Let us fix a Dirichlet character $\chi$ mod $Np^v$ as above with $v \geq 1$, and an arbitrary $\mathcal{C}^\infty$-modular function

$$F \in \mathcal{M}_k^{(3)}(\Gamma_0(Np^v),\boldsymbol{\psi})^\infty,$$

with a Dirichlet character $\boldsymbol{\psi}$ mod $Np^v$ which depends on $\chi$ mod $Np^v$, for example, the series $F_{\chi,r}$ with the nebentypus character $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\bar\psi_3$. Then the higher twist of $F$ with $\chi_1,\chi_2,\chi_3$ was initially defined by the formula

$$\tilde F = \sum_{\varepsilon_{12},\varepsilon_{13},\varepsilon_{23} \bmod Np^v} \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23})F|_k t_{\boldsymbol{\varepsilon},Np^v} \tag{1.3}$$

where we use the translation $t_{\boldsymbol{\varepsilon},Np^v} = \begin{pmatrix} 1_3 & \frac{1}{Np^v}\boldsymbol{\varepsilon} \\ 0_3 & 1_3 \end{pmatrix}$ on $\mathbb{H}_3$ with $\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix}$. The idea of the construction. We wish to interpret the series (1.3) in terms of a distribution on a profinite group, using the following model example: consider the profinite ring $A_{N,p} = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})$, and a compact open subset $\alpha + (Np^v) \subset A_{N,p}$ with $\alpha$ an integer mod $Np^v$, and $N$ is prime to $p$. For any formal series $f = \sum_{n\geq 1} a_n q^n \in \mathbb{C}[\![q]\!]$ and for any open subset $\alpha + (Np^v) \subset A_{N,p}$ consider the following partial series:

$$\mu_f(\alpha + (Np^v)) = \sum_{\substack{n\geq 1 \\ n\equiv\alpha \bmod Np^v}} a_n q^n \in \mathbb{C}[\![q]\!].$$

If $q = \exp(2\pi i z)$ with $z \in \mathbb{H}$, it follows from the orthogonality relations that

$$\mu_f(\alpha + (Np^v)) = (Np^v)^{-1} \sum_{\beta \bmod Np^v} \exp(-2\pi i\alpha\beta/Np^v) f\left(z + \frac{\beta}{Np^v}\right),$$

and that for any Dirichlet character $\chi$ mod $Np^v$ one has

$$\int_{A_{N,p}} \chi(\alpha) d\mu_f(\alpha) = \sum_{n\geq 1} \chi(n) a_n q^n = f(\chi) \in \mathbb{C}[\![q]\!].$$

(the series $f$ twisted by the character $\chi$).
In the same fashion, consider the additive profinite group

$$S = S_{N,p} := \left\{ \boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix} \middle| \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23} \in A_{N,p} \right\};$$

equipped with the scalar product $\langle \cdot, \cdot \rangle : S_{N,p} \times S_{N,p} \longrightarrow A_{N,p}$:

$$\left\langle \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)} \right\rangle = \mathrm{tr}(\boldsymbol{\varepsilon}^{(1)}\boldsymbol{\varepsilon}^{(2)}) = 2\varepsilon_{12}^{(1)}\varepsilon_{12}^{(2)} + 2\varepsilon_{13}^{(1)}\varepsilon_{13}^{(2)} + 2\varepsilon_{23}^{(1)}\varepsilon_{23}^{(2)}, \text{ where}$$

$$\boldsymbol{\varepsilon}^{(1)} = \begin{pmatrix} 0 & \varepsilon_{12}^{(1)} & \varepsilon_{13}^{(1)} \\ \varepsilon_{12}^{(1)} & 0 & \varepsilon_{23}^{(1)} \\ \varepsilon_{13}^{(1)} & \varepsilon_{23}^{(1)} & 0 \end{pmatrix}, \boldsymbol{\varepsilon}^{(2)} = \begin{pmatrix} 0 & \varepsilon_{12}^{(2)} & \varepsilon_{13}^{(2)} \\ \varepsilon_{12}^{(2)} & 0 & \varepsilon_{23}^{(2)} \\ \varepsilon_{13}^{(2)} & \varepsilon_{23}^{(2)} & 0 \end{pmatrix}.$$

PROPOSITION 1.1 *Suppose that the function $F$ is invariant with respect to any integer translation of type $t_{\boldsymbol{\varepsilon},1}$ : $F|t_{\varepsilon,1} = F$. Then*
1) *The action $F|t_{\boldsymbol{\varepsilon},Np^v}$ depends only on the class of $\boldsymbol{\varepsilon} \in S/Np^vS$, and the additive character $e_{\boldsymbol{\varepsilon}^{(0)}} : \boldsymbol{\varepsilon} \mapsto \exp(\langle \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(0)} \rangle / Np^v)$ on $S$ is trivial iff $\boldsymbol{\varepsilon}^{(0)} \in Np^vS$.*
2) *The formula*

$$\Psi_F(\boldsymbol{\varepsilon}^{(0)} + (Np^v)) = (Np^v)^{-3} \sum_{\boldsymbol{\varepsilon}\in S \bmod Np^vS} \exp(-2\pi i\left\langle \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(0)} \right\rangle/Np^v) F|t_{\boldsymbol{\varepsilon},Np^v}$$
$$= (Np^v)^{-3} \sum_{\boldsymbol{\varepsilon}\in S \bmod Np^vS} e(-\left\langle \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(0)} \right\rangle/Np^v) F|t_{\boldsymbol{\varepsilon},Np^v} \tag{1.4}$$

*defines a distribution with values in $\mathcal{C}^\infty$-functions on $\mathbb{H}_3$, where $e(\alpha/Np^v) := \exp(2\pi i\alpha/Np^v)$ is well-defined for all $\alpha \in A_N$.*

*Proof*: 1) Follows directly from the invariance: $F|t_{\varepsilon,1} = F$.
2) It suffices to check the finite-additivity condition:

$$\Psi_F(\varepsilon^{(0)} + (Np^v)) = \sum_{\varepsilon^{(1)} \in S \bmod p} \Psi_F(\varepsilon^{(0)} + Np^v\varepsilon^{(1)} + (Np^{v+1})), \tag{1.5}$$

i.e.,

$$(Np^v)^{-3} \sum_{\varepsilon \in S/Np^vS} e(-\left\langle \varepsilon, \varepsilon^{(0)} \right\rangle / Np^v) F|t_{\varepsilon, Np^v} \tag{1.6}$$

$$= (Np^{v+1})^{-3} \times$$

$$\sum_{\varepsilon^{(1)} \in S/pS} \sum_{\varepsilon^{(2)} \in S/Np^{v+1}S} e(-\left\langle \varepsilon^{(2)}, (\varepsilon^{(0)} + Np^v\varepsilon^{(1)}) \right\rangle / Np^{v+1}) F|t_{\varepsilon^{(2)}, Np^{v+1}}. \tag{1.7}$$

For all $\varepsilon^{(2)}$ the sum on the right on $\varepsilon^{(1)} \in S/pS$ in (1.6) becomes

$$(Np^{v+1})^{-3} \sum_{\varepsilon^{(1)} \in S/pS} e(-\left\langle \varepsilon^{(2)}, (\varepsilon^{(0)} + Np^v\varepsilon^{(1)}) \right\rangle / Np^{v+1}) F|t_{\varepsilon^{(2)}, Np^{v+1}} \tag{1.8}$$

$$= (Np^{v+1})^{-3} e(-\frac{\left\langle \varepsilon^{(2)}, \varepsilon^{(0)} \right\rangle}{Np^{v+1}}) F|t_{\varepsilon^{(2)}, Np^{v+1}} \sum_{\varepsilon^{(1)} \in S/pS} e(-\frac{\left\langle \varepsilon^{(2)}, Np^v\varepsilon^{(1)} \right\rangle}{Np^{v+1}})$$

$$= (Np^{v+1})^{-3} e(-\left\langle \varepsilon^{(2)}, \varepsilon^{(0)} \right\rangle / Np^{v+1}) F|t_{\varepsilon^{(2)}, Np^{v+1}} \sum_{\varepsilon^{(1)} \in S/pS} e\left(-\left\langle \varepsilon^{(2)}, \varepsilon^{(1)} \right\rangle\right).$$

It remains to notice that

$$\sum_{\varepsilon^{(1)} \in S/pS} e(-\left\langle \varepsilon^{(2)}, \varepsilon^{(1)} \right\rangle / p) = \begin{cases} p^3, & \text{if } \varepsilon^{(2)} = p\varepsilon^{(3)}, \varepsilon^{(3)} \in S \\ 0, & \text{otherwise,} \end{cases} \tag{1.9}$$

because $\varepsilon^{(1)} \mapsto e(-\left\langle \varepsilon^{(2)}, \varepsilon^{(1)} \right\rangle / p)$ is a non trivial character of $S/pS$ iff $\varepsilon^{(2)} \in pS$. The right hand side of (1.6) becomes

$$(Np^{v+1})^{-3} \sum_{\varepsilon^{(1)} \in S/pS} \sum_{\varepsilon^{(2)} \in S/Np^{v+1}S} e(-\left\langle \varepsilon^{(2)}, (\varepsilon^{(0)} + Np^v\varepsilon^{(1)}) \right\rangle / Np^{v+1}) F|t_{\varepsilon^{(2)}, Np^{v+1}} \tag{1.10}$$

$$= (Np^{v+1})^{-3} p^3 \sum_{\varepsilon^{(3)} \in S/Np^vS} e(-\left\langle \varepsilon^{(3)}, \varepsilon^{(0)} \right\rangle / Np^v) F|t_{\varepsilon^{(3)}, Np^v}.$$

Remark 1.2 *The Fourier expansions of the nearly-holomorphic Siegel modular form*

$$F_{\varepsilon,v} :=$$

$$\Psi_F(\varepsilon + (Np^v)) = (Np^v)^{-3} \sum_{\varepsilon' \in S \bmod Np^vS} \exp(-2\pi i \langle \varepsilon', \varepsilon \rangle / Np^v) F|t_{\varepsilon', Np^v}.$$

*is given as the following partial Fourier series*

$$F_{\boldsymbol{\varepsilon},v}(\mathcal{Z}) = \sum_{\substack{\mathcal{T},t_{12}\equiv\varepsilon_{12} \bmod Np^v \\ t_{13}\equiv\varepsilon_{13},t_{23}\equiv\varepsilon_{23} \bmod Np^v}} a(\mathcal{T},R)q^{\mathcal{T}}, \tag{1.11}$$

*where $F$ is a nearly-holomorphic Siegel modular form, which is a periodic function on $\mathbb{H}_3$: $F = \sum_{\mathcal{T}} a(\mathcal{T},R)q^{\mathcal{T}}$, and $\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33}\end{pmatrix}$ runs over half integral symmetric non negative matrices.*

Indeed,
$F|t_{\boldsymbol{\varepsilon}',Np^v} = \sum_{\mathcal{T}} a(\mathcal{T},R)q^{\mathcal{T}}|t_{\boldsymbol{\varepsilon}',Np^v} = \sum_{\mathcal{T}} \exp(2\pi i \mathrm{tr}(\boldsymbol{\varepsilon}'\mathcal{T})/Np^v)a(\mathcal{T},R)q^{\mathcal{T}}$,
hence

$$F_{\boldsymbol{\varepsilon},v} = (Np^v)^{-3}\sum_{\boldsymbol{\varepsilon}'\in S \bmod Np^vS} \exp(-2\pi i\langle\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}\rangle/Np^v)\sum_{\mathcal{T}}\exp(2\pi i\mathrm{tr}(\boldsymbol{\varepsilon}'\mathcal{T})/Np^v)a(\mathcal{T},R)q^{\mathcal{T}}.$$

It suffices to notice that

$$\mathrm{tr}(\boldsymbol{\varepsilon}'\mathcal{T}) = \mathrm{tr}\left(\begin{pmatrix} 0 & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{12} & 0 & \varepsilon'_{23} \\ \varepsilon'_{13} & \varepsilon'_{23} & 0\end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33}\end{pmatrix}\right) = 2(\varepsilon'_{12}t_{12}+\varepsilon'_{13}t_{13}+\varepsilon'_{23}t_{23}).$$

∎

Let us consider now three Dirichlet characters $\chi_1$, $\chi_2$, $\chi_3$ mod $Np^v$, and let us compute the corresponding integrals against the constructed modular distribution (1.4) of the locally constant function $\boldsymbol{\varepsilon} \mapsto \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23})$ on the profinite additive group

$$S = S_N := \left\{\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0\end{pmatrix}\middle|\, \varepsilon_{12},\varepsilon_{13},\varepsilon_{23}\in A_N\right\}.$$

Proposition 1.3 *Let $F$ be a function invariant with respect to any translation of type $t_{\boldsymbol{\varepsilon},1}$ : $F|t_{\boldsymbol{\varepsilon},1} = F$. Let us write $F_{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \int_S \bar\chi_1(\varepsilon_{12})\bar\chi_2(\varepsilon_{13})\bar\chi_3(\varepsilon_{23})d\Psi_F(\boldsymbol{\varepsilon})$. Then*

$$F_{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \tag{1.12}$$
$$(Np^v)^{-3}\sum_{\boldsymbol{\varepsilon}\in S/Np^vS} G_{Np^v}(\bar\chi_1,-\varepsilon_{12})G_{Np^v}(\bar\chi_2,-\varepsilon_{13})G_{Np^v}(\bar\chi_3,-\varepsilon_{23})F|t_{\boldsymbol{\varepsilon},Np^v}.$$

*Here $G_{Np^v}(\chi,\boldsymbol{\varepsilon}) := \sum_{\alpha'} e(\boldsymbol{\varepsilon}\alpha'/Np^v)\chi(\alpha')$ denotes the Gauß sum (of a non necessarily primitive Dirichlet character $\chi$).*

Remarks 1.4 *1) The advantage of the expression* (1.12) *in compare with* (1.3) *is that it does not depend on a choice of* $v$*.*
*2)It follows from (1.11, that the Fourier expansion of the series (1.12) is given by*

$$F_{\bar\chi_1,\bar\chi_2,\bar\chi_3} = \sum_{\mathcal{T}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23})a(\mathcal{T},R)q^{\mathcal{T}}. \tag{1.13}$$

*Proof* is similar to that of Proposition 1.1, and it follows from the definitions. ∎

## 1.3 The level of the higher twist

Let us consider the symplectic inclusion:

$$i \ : \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Sp}_3(\mathbb{Z}) \tag{1.14}$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & & & b_1 & & \\ & a_2 & & & b_2 & \\ & & a_3 & & & b_3 \\ c_1 & & & d_1 & & \\ & c_2 & & & d_2 & \\ & & c_3 & & & d_3 \end{pmatrix}$$

We study the behaviour of the modular form $F_{\bar\chi_1,\bar\chi_2,\bar\chi_3}$ with respect to the subgroup

$$i(\Gamma_0(N^2p^{2v})^3) \subset \Gamma_0^{(3)}(N^2p^{2v}),$$

where $(\chi_1 \otimes \chi_2 \otimes \chi_3)(\boldsymbol{\varepsilon}) = \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23})$.
We will have to study two different types of twist; we can treat them simultaneously if we consider a function

$$\phi : \mathbb{Z}/N\mathbb{Z} \longmapsto \mathbb{C}$$

which is "$\varphi$-spherical" i.e.

$$\phi(gXh) = \varphi(g)\varphi(h)\phi(X)$$

for all $g, h \in (\mathbb{Z}/N\mathbb{Z})^\times, X \in \mathbb{Z}/N\mathbb{Z}$, where $\varphi$ is a Dirichlet character mod $N$.
Let us use Proposition 1.12 and the spherical function

$$\phi : (\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}) \mapsto G_{Np^v}(\bar\chi_1, -\varepsilon_{12})G_{Np^v}(\bar\chi_2, -\varepsilon_{13})G_{Np^v}(\bar\chi_3, -\varepsilon_{23}),$$

with respect to three variables $(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$, and the Dirichlet characters

$$(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}) \mapsto \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23}).$$

Proposition 1.5 *Consider a (nearly-holomorphic) Siegel modular form $F$ for the group $\Gamma_0^{(3)}(Np^v)$ and the Dirichlet character $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$).*
*Then for all $M = i\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) \in \Gamma_0^{(3)}(N^2p^{2v})$ one has:*
*1) $\tilde{F}|M = \underbrace{\boldsymbol{\psi}\bar{\chi}_1\bar{\chi}_2(d_1)}_{\psi_1}\underbrace{\boldsymbol{\psi}\bar{\chi}_1\bar{\chi}_3(d_2)}_{\psi_2}\underbrace{\boldsymbol{\psi}\bar{\chi}_2\bar{\chi}_3(d_3)}_{\psi_3}\tilde{F}$, where $\tilde{F}$ is defined by (1.3),*
*2) $F_{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3}|M = \underbrace{\boldsymbol{\psi}\bar{\chi}_1\bar{\chi}_2(d_1)}_{\psi_1}\underbrace{\boldsymbol{\psi}\bar{\chi}_1\bar{\chi}_3(d_2)}_{\psi_2}\underbrace{\boldsymbol{\psi}\bar{\chi}_2\bar{\chi}_3(d_3)}_{\psi_3}F_{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3}$, where $F_{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3}$ is defined by (1.12).*

*Proof.* We study modular forms on $\mathbb{H}_3$. Let us consider a more general situation and write $N$ instead of $Np^v$. We use the (somewhat unconventional) congruence subgroup (with $N \mid M$):

$$\Gamma_1^{(3)}(M,N) := \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(3)}(M) \,\middle|\, D \equiv \operatorname{diag}(D_1,D_2,D_3) \bmod N\right\}.$$

Here the $D_i$ denote integers along the diagonal of $D$. It is easy to see that this defines a subgroup of $\mathrm{Sp}(3,\mathbb{Z})$ and that a similar congruence also holds for $A$. The appropriate space of modular forms, denoted by $\mathcal{M}_k^{(3)}(M,N;\chi;\psi_1,\psi_2,\psi_3)$, with Dirichlet characters $\psi_j$ mod $N$ and a Dirichlet character $\chi$ mod $M$ is then the set of holomorphic functions on $\mathbb{H}_3$ satisfying

$$f\mid_k \gamma = \chi(\det D)\left(\prod_{j=1}^{3}\psi_j(D_j)\right) f$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_1^{(3)}(M,N)$. For any $\alpha \in \mathbb{R}$ and any $1 \le i < j \le 3$ we define a symmetric matrix of size 3 by

$$S_{ij}^{(3)}(\alpha) := \begin{pmatrix} & & \alpha \\ & & \\ \alpha & & \end{pmatrix}$$

(the number $\alpha$ sits in the $(i,j)$th and $(j,i)$positions). Then, for a function $F \in \mathcal{M}_k^{(3)}(M,N;\chi;\psi_1,\psi_2,\psi_3)$ we define a new function $F_{ij}^{\phi}$ on $\mathbb{H}_3$ by

$$F_{ij}^{\phi}(\mathcal{Z}) = \sum_{\alpha \bmod N} \phi(\alpha)\cdot F(\mathcal{Z} + S_{ij}^{(3)}(\frac{\alpha}{N}))$$

Proposition 1.6 *Assume that $N^2 \mid M$, $\chi$ is a character* mod $\frac{M}{N}$, *and $F \in \mathcal{M}_k^{(3)}(M,N;\chi;\psi_1,\psi_2,\psi_3)$. Then*

$$F_{ij}^{\phi} \in \mathcal{M}_k^{(3)}(M,N;\chi;\psi_1',\psi_2',\psi_3')$$

*with*

$$\psi'_r = \begin{cases} \psi_r & \text{if} \quad r \notin \{i,j\} \\ \psi_r \cdot \overline{\varphi} & \text{if} \quad r \in \{i,j\} \end{cases}$$

Remarks 1.7 *1) We mention here two basic types of $\varphi$-spherical functions $\phi : \mathbb{Z}/N\mathbb{Z}$:*
Type I: *"Dirichlet character" $\phi(X) := \varphi(X)$*
Type II: *"Gauß sum" $\phi(X) = G(\overline{\varphi}, -X)$ where $G(\varphi, X)$ denotes a Gauß sum (a version of such spherical functions of matrix argument was studied in [Boe-Schm]):*

$$G(\varphi, X) := \sum_{\alpha \bmod N} \varphi(\alpha) \exp(2\pi i \frac{1}{N} \alpha X)$$

*2) Our* basic example *is as follows: let $\varphi_1, \varphi_2, \varphi_3$ be three Dirichlet characters mod $N$ and let $\phi_i$ be $\varphi_i$-spherical functions on $\mathbb{Z}/N\mathbb{Z}$. Furthermore let $F \in \mathcal{M}_k^{(3)}(\Gamma_0(M), \chi)$ with $N^2 \mid M$ and $\chi$ a Dirichlet character* $\bmod \frac{M}{N}$*. Then*

$$h(z_1, z_2, z_3) := \sum_{\alpha,\beta,\gamma \bmod N} \phi_1(\alpha)\phi_2(\beta)\phi_3(\gamma) F(\begin{pmatrix} z_1 & \frac{\alpha}{N} & \frac{\beta}{N} \\ \frac{\alpha}{N} & z_2 & \frac{\gamma}{N} \\ \frac{\beta}{N} & \frac{\gamma}{N} & z_3 \end{pmatrix})$$

*is an element of*

$$\mathcal{M}_k(\Gamma_0(M), \chi\bar{\varphi}_1\bar{\varphi}_2) \otimes \mathcal{M}_k(\Gamma_0(M), \chi\bar{\varphi}_1\bar{\varphi}_3) \otimes \mathcal{M}_k(\Gamma_0(M), \chi\bar{\varphi}_2\bar{\varphi}_3)$$

*(note that the definition of $h$ depends on $N$)*
*3) Other important cases are treated in [Boe-Schm] it can also (by iteration) be applied to cases of block matrices of different size which e.g. occur in the work [Boe-Ha] on the $L$-function for* $\mathrm{GSp}(2) \times \mathrm{GL}(2)$.

*Proof.* We first try to find $X \in \mathrm{Sym}_3(\frac{1}{N}\mathbb{Z})$ such that

$$\begin{pmatrix} 1_3 & S(\frac{\alpha}{N}) \\ 0_3 & 1_3 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} 1_3 & -X \\ 0_3 & 1_3 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{A} + S(\frac{\alpha}{N})\mathcal{C} & -\mathcal{A}X + \mathcal{B} - S(\frac{\alpha}{N})\mathcal{C}X + S(\frac{\alpha}{N})\mathcal{D} \\ \mathcal{C} & -\mathcal{C}X + \mathcal{D} \end{pmatrix}$$

is in $\Gamma_0^{(3)}(M)$ (for the moment we only assume here that $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ is integral. The conditions $N^2 \mid M$ and the congruences mod $M$ and $N$ will then be forced to hold). The first (evident) condition is that $\mathcal{C} \equiv 0 \bmod M$. It is easy to see that the two numbers on the diagonal

$$-\mathcal{C}X + \mathcal{D} \qquad \text{and} \qquad \mathcal{A} + S(\frac{\alpha}{N})\mathcal{C}$$

are integers, if $\mathcal{C}$ is congruent to 0 modulo $N$.

The remaining condition is that

$$-\mathcal{A}X + \mathcal{B} - S(\frac{\alpha}{N})\mathcal{C}X + S(\frac{\alpha}{N})\mathcal{D}$$

is integral, which is satisfied if $\mathcal{C} \equiv 0 \bmod N^2$ and $-\mathcal{A} \cdot X + S(\frac{\alpha}{N})\mathcal{D}$ is integral. Therefore we should choose any $X$ satisfying

$$(NX) \equiv \overline{\mathcal{A}}S(\alpha)\mathcal{D} \quad \bmod N$$

where $\overline{\mathcal{A}}$ is a (multiplicative) inverse of the matrix $\mathcal{A}$ mod $N$. Now we use the fact that $\mathcal{A} \equiv \mathrm{diag}(A_1, A_2, A_3) \bmod N$ and $\mathcal{D} \equiv \mathrm{diag}(D_1, D_2, D_3) \bmod N$ are matrices which are diagonal modulo $N$, we may therefore choose the integral symmetric matrix $NX$ to be modulo $N$ equal to

$$NX := S^{(3)}_{ij}\left(\overline{A_i} \cdot \alpha \cdot D_j\right) \Rightarrow X = X(\alpha) = S^{(3)}_{ij}\left(\frac{\overline{A_i} \cdot \alpha \cdot D_j}{N}\right).$$

By the above,

$$\begin{aligned} F^{\phi}_{ij} |_k \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} &= \sum_{\alpha \bmod N} \phi(\alpha) F |_k \begin{pmatrix} 1 & S(\frac{\alpha}{N}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \\ &= \sum_{\alpha \bmod N} \phi(\alpha) F |_k \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \begin{pmatrix} 1 & X(\alpha) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix} \in \Gamma^{(3)}_1(M, N)$ with

$$\tilde{\mathcal{A}} \equiv \mathcal{A} \bmod \frac{M}{N} \quad \text{and} \quad \tilde{\mathcal{D}} \equiv \mathcal{D} \quad \bmod \frac{M}{N}$$

(in particular, these congruences hold mod $N$). Therefore

$$F^{\phi}_{ij} |_k \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \chi(\det(\mathcal{D}))\psi_1(D_1) \dots \psi_n(D_n) \sum_{\alpha \bmod N} \phi(\alpha) F |_k \begin{pmatrix} 1_3 & X(\alpha) \\ 0_3 & 1_3 \end{pmatrix}.$$

Instead of summing over $\alpha$ we may as well sum over $\beta := D_i \cdot \alpha \cdot D_j \bmod N$. Then we obtain

$$\chi(\det(\mathcal{D}))\psi_1(D_1) \dots \psi_n(D_n)\overline{\varphi}(D_i)\overline{\varphi}(D_j) \sum_{\beta \bmod N} \phi(\beta) F |_k \begin{pmatrix} 1_3 & S^{(3)}_{ij}(\frac{\beta}{N}) \\ 0_3 & 1_3 \end{pmatrix}$$

$$= \chi(\det(\mathcal{D})\psi_1(D_1) \dots \psi_n(D_n)\overline{\varphi}(D_i)\overline{\varphi}(D_j) F^{\phi}_{ij}. \quad \blacksquare$$

Notice that the properties of Propositions 1.6 hold for the *iterated twists*, and Propositions 1.5 follows from Propositions 1.6 by three iterated twists with $N$ equal to $Np^v$. $\blacksquare$

## 2 Computation of the canonical projection

### 2.1 A general construction: the canonical $\lambda$-characteristic projection

We explain now a general method which associates a $p$-adic measure $\mu_{\lambda,\Phi}$ on a profinite group $Y$, to a sequence of distributions $\Phi_r$ on $Y$ with values in a suitable (infinite dimensional) vector space $\mathcal{M}$ of modular forms, and to a nonzero eigenvalue $\lambda$ of the Atkin operator $U = U_p$ acting on $\mathcal{M}$. We consider holomorphic (or nearly-holomorphic) modular forms in a space of the type

$$\mathcal{M} = \mathcal{M}_k(\psi, \overline{\mathbb{Q}}) = \bigcup_{v\geq 0} \mathcal{M}_k(Np^v, \psi, \overline{\mathbb{Q}}) \subset \mathcal{M}(\mathbb{C}_p) = \bigcup_{v\geq 0} \mathcal{M}_k(Np^v, \psi, \mathbb{C}_p),$$

with finite dimensional vector spaces $\mathcal{M}_k(Np^v, \psi, \overline{\mathbb{Q}})$ at each fixed level, endowed with a natural $\overline{\mathbb{Q}}$-rational structure (for example, given by the Fourier coefficients). The parameters here are triples $k = (k_1, k_2, k_3)$, $\psi = (\psi_1, \psi_2, \psi_3)$ of weights and characters. The important property of our construction is that does not use *passage to a $p$-adic limit.* We put

$$\mathcal{M}_k(Np^v, \psi, A) = \mathcal{M}_k(Np^v, \psi, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} A.$$

for any $\overline{\mathbb{Q}}$-algebra $A$.

Definition 2.1 *Let $A = \mathbb{C}_p$ , $A = \overline{\mathbb{Q}}$, or $A = \mathbb{C}$, and $\mathcal{M} = \mathcal{M}(A)$.*
(a) *For a $\lambda \in A$ let us define $\mathcal{M}^{(\lambda)} = \mathrm{Ker}\,(U - \lambda I)$ the subspace of eigenvectors with eigenvalue $\lambda$).*
(b) *Let us define the $\lambda$-characteristic subspace of $U$ on $\mathcal{M}$ by*

$$\mathcal{M}^{\lambda} = \bigcup_{n\geq 1} \mathrm{Ker}\,(U - \lambda I)^n$$

(c) *Let us define for any $v \geq 0$*

$$\mathcal{M}^{\lambda}(Np^v) = \mathcal{M}^{\lambda} \cap \mathcal{M}(Np^v), \quad \mathcal{M}^{(\lambda)}(Np^v) = \mathcal{M}^{(\lambda)} \cap \mathcal{M}(Np^v).$$

Proposition 2.2 *Let $\psi \bmod N$ be a fixed Dirichlet character, then $U^v(\mathcal{M}(Np^{v+1}, \psi)) \subset \mathcal{M}(Np, \psi)$.*

*Proof* follows from a known formula of J.-P. Serre: for $g \in \mathcal{M}_k(Np^{v+1}, \psi)$,

$$g|_k U^v = p^{3v(k/2-1)} g|_k W_{Np^{v+1}} \mathrm{Tr}_{Np}^{Np^{v+1}} W_{Np}, \tag{2.1}$$

where $W_N : \mathcal{M}(N, \psi) \to \mathcal{M}(N, \bar{\psi})$ is the involution (over $\mathbb{C}$) of level $N$ (see [Se73] for the elliptic modular case, which extends to the triple modular case).

Proposition 2.3 *Let $A = \mathbb{C}_p$ or $A = \overline{\mathbb{Q}}$, $\mathcal{M} = \mathcal{M}(A)$, $\lambda \in A^*$, and let $U^\lambda$ be the restriction of $U$ on $\mathcal{M}^\lambda$, then*
(a) $(U^\lambda)^v : \mathcal{M}^\lambda(Np^{v+1}) \xrightarrow{\sim} \mathcal{M}^\lambda(Np)$ *is an $A$-linear invertible operator, where* $U^\lambda = U|_{\mathcal{M}^\lambda(Np^{v+1})}$.
(b) *The vector subspace $\mathcal{M}^\lambda(Np^{v+1}) = \mathcal{M}^\lambda(Np)$ does not depend on $v$.*
(c) *Let $\pi_{\lambda,v+1} : \mathcal{M}(Np^{v+1}) \to \mathcal{M}^\lambda(Np^{v+1})$ be the projector on the $\lambda$-characteristic subspace of $U$ with the kernel* $\mathrm{Ker}\,(\pi_{\lambda,v}) = \bigcap_{n\geq 1} \mathrm{Im}\,(U-\lambda I)^n = \bigoplus_{\beta\neq\lambda} \mathcal{M}^\beta(N_0p^v))$, *then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{M}(Np^{v+1}) & \xrightarrow[\pi_{\lambda,v+1}]{} & \mathcal{M}^\lambda(Np^{v+1}) \\ U^v \big\downarrow & & \big\downarrow \wr\, U^v \\ \mathcal{M}(Np) & \xrightarrow[\pi_{\lambda,1}]{} & \mathcal{M}^\lambda(Np) \end{array} \tag{2.2}$$

Let us use the notation

$$\pi_\lambda(g) = (U^\lambda)^{-v}\pi_{\lambda,1}(U^v(g)) \in \mathcal{M}^\lambda(\Gamma_0(Np), \psi, \mathbb{C}) \tag{2.3}$$

for the canonical $\lambda$-characteristic projection of $g \in \mathcal{M}(\Gamma_0(Np^{v+1}), \psi, \mathbb{C})$.
*Proof* of (a). The linear operator $(U^\lambda)^v$ acts on the $A$-linear vector space $\mathcal{M}^\lambda(Np^{v+1})$ of finite dimension, and its determinant is in $A^*$, hence the $A$-linear operator $(U^\lambda)^v$ is *invertible*.
*Proof* of (b). We have the obvious inclusion of vector spaces: $\mathcal{M}^\lambda(Np) \subset \mathcal{M}^\lambda(Np^{v+1})$. On the other hand the $A$-vector spaces $\mathcal{M}^\lambda(Np^{v+1})$ and $\mathcal{M}^\lambda(Np)$ are isomorphic by (a), hence they coincide:

$$\mathcal{M}^\lambda(Np) \subset \mathcal{M}^\lambda(Np^{v+1}) = U^v(\mathcal{M}^\lambda(Np^{v+1})) \subset \mathcal{M}^\lambda(Np).$$

*Proof* of (c). Following the theory of reduction of endomorphisms in finite dimensional vector spaces over a field $K$, the canonical projector $\pi_{\lambda,v}$ onto the $\lambda$-characteristic subspace $\bigcup_{n\geq 1} \mathrm{Ker}\,(U-\lambda I)^n$ with the kernel $\bigcap_{n\geq 1} \mathrm{Im}\,(U-\lambda I)^n$ can be expressed, on one hand, as a polynomial of $U$ over $K$, hence $\pi_{\lambda,v}$ commutes with $U$. On the other hand, the restriction of $\pi_{\lambda,v+1}$ on $\mathcal{M}(Np)$ coincides with $\pi_{\lambda,1} : \mathcal{M}(Np) \to \mathcal{M}^\lambda(Np)$, because its image is

$$\bigcup_{n\geq 1} \mathrm{Ker}\,(U-\lambda I)^n \cap \mathcal{M}(Np) = \bigcup_{n\geq 1} \mathrm{Ker}\,(U|_{\mathcal{M}(Np)} - \lambda I)^n,$$

and its kernel is

$$\bigcap_{n\geq 1} \mathrm{Im}\,(U-\lambda I)^n \cap \mathcal{M}(Np) = \bigcap_{n\geq 1} \mathrm{Im}\,(U|_{\mathcal{M}(Np)} - \lambda I)^n. \quad \blacksquare$$

## 2.2 A general result on admissible measures with values in modular forms (a criterion for admissibility)

Consider the profinite group $Y = \varprojlim_v Y_v$ where $Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times$. There is a natural projection $y_p : Y \to \mathbb{Z}_p^\times$. Let $A$ be a normed ring over $\mathbb{Z}_p$, and $M$ be a normed $A$-module with the norm $|\cdot|_{p,M}$.
Let us recall Definition 0.1, c): for a given positive integer $h$ an *$h$-admissible measure* on $Y$ with values in $M$ is an $A$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y, A) \to M$$

such that for fixed $a \in Y$ and for $v \to \infty$

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,M} = o(p^{-v(h'-h)}) \text{ for all } h' = 0, 1, \dots, h-1,$$

where $a_p = y_p(a)$, $\mathcal{P}^h(Y, A)$ denotes the $A$-module of *locally polynomial functions* of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow A^\times$. We adopt the notation $(a)_v = a + (Np^v)$ for both an element of $Y_v$ and the corresponding open compact subset of $Y$.
We wish now to construct an $h$-admissible measure $\tilde{\Phi}^\lambda : \mathcal{P}^h(Y, A) \to \mathcal{M}(A)$ out of a sequence of distributions

$$\Phi_r^\lambda : \mathcal{P}^1(Y, A) \to \mathcal{M}(A)$$

with values in an $A$-module $M = \mathcal{M}(A)$ of modular forms over $A$ as in Section 2.1).
For this purpose we recall first Proposition 2.3, (c). Suppose that $\lambda \in A^\times$ is an invertible element of the algebra $A$. Recall that the *$\lambda$-characteristic projection operator*

$$\pi_{\lambda,v} : \mathcal{M}(Np^v; A) \to \mathcal{M}(Np^v; A)^\lambda \subset \mathcal{M}(Np^v; A) \ (v \geq 1)$$

is determined by the kernel $\bigcap_{n\geq 1} \mathrm{Im}(U - \lambda I)^n$; this projector is given as a polynomial of $U$ over $A$ whose degree is bounded by the *rank* of $\mathcal{M}(Np^v; A)$.
Using Proposition 2.3(c), the sequence of projectors $\pi_{\lambda,v}$ can be glued to the *canonical projection operator*

$$\pi_\lambda : \mathcal{M}(A) \to \mathcal{M}(A)^\lambda \subset \mathcal{M}(A) \tag{2.4}$$

given for all $g \in \mathcal{M}(A)$ by

$$\pi_\lambda(g) = g^\lambda = U^{-v} \left[\pi_{\lambda,1} U^v(g)\right]$$

($g^\lambda$ is well defined if $v$ is sufficiently large so that $g \in \mathcal{M}(Np^{v+1})$).

Next we construct an admissible measure

$$\tilde{\Phi}^{\lambda} : \mathcal{P}^h(Y, A) \to \mathcal{M}(Np; A)$$

such that

$$\int_{(a)_v} y_p^r \, d\tilde{\Phi}^{\lambda} = \Phi_r^{\lambda}((a)_v) = \pi_{\lambda}(\Phi_r((a)_v))$$

where $\Phi_r : \mathcal{P}^1(Y, A) \to \mathcal{M}(A)$ are $\mathcal{M}(A)$-valued distributions on $Y$ for $r = 0, 1, \ldots, h-1$ , and $\Phi_r^{\lambda}((a)_v)$ are their $\lambda$-characteristic projections given by

$$\Phi_r^{\lambda}((a)_v) = U^{-v'} \left[ \pi_{\lambda,1} U^{v'} \Phi_r((a)_v) \right]$$

for any sufficiently large $v'$. Note first of all that the definition

$$\int_{(a)_v} y_p^r \, d\tilde{\Phi}^{\lambda} = \Phi_r^{\lambda}((a)_v) = U^{-\kappa v} \left[ \pi_{\lambda,1} U^{\kappa v} \Phi_r((a)_v) \right].$$

of the linear form $\tilde{\Phi}^{\lambda} : \mathcal{P}^h(Y, A) \to \mathcal{M}(A)$ is independent on the choice of the level: for any sufficiently large $v'$, we have by Proposition 2.3 the following comutative diagram

$$\begin{array}{ccc}
\mathcal{M}(Np^{v'+1}; A) & \stackrel{\pi_{\lambda,v'+1}}{\longrightarrow} & \mathcal{M}(Np^{v'+1}; A)^{\lambda} \\
U^{v'} \big\downarrow & & \big\downarrow \wr \; U^{v'} \\
\mathcal{M}(Np; A) & \stackrel{\pi_{\lambda,1}}{\longrightarrow} & \mathcal{M}(Np; A)^{\lambda}
\end{array}$$

in which the right vertical arrow is an $A$-isomorphism by Proposition 2.3 (b), and the $A$-linear endomorphism $U$ commutes with the characteristic projectors $\pi_{\lambda,v'+1}$, $\pi_{\lambda,1}$. Hence the following sequence stabilizes: for some $v_0'$ and for all $v' \geq v_0'$ we have that

$$U^{-v'} \left[ \pi_{\lambda,1} U^{v'} \Phi_r((a)_v) \right] = U^{-v_0'} \left[ \pi_{\lambda,1} U^{v_0'} \Phi_r((a)_v) \right].$$

Theorem 2.4 *Let $\lambda \in A$ be an element whose absolute value is a positive constant with $0 < |\lambda|_p < 1$. Suppose that there exists a positive integer $\varkappa$ such that for any $(a)_v \subset Y$ the following two conditions are satisfied:*

$$\Phi_r\big((a)_v\big) \in \mathcal{M}(N'p^{\varkappa v}), \text{ with } N' \text{ independent of } v, \qquad (level)$$

$$\left| U^{\varkappa v} \Big( \sum_{r'=0}^{r} \binom{r}{r'} (-y_p^0)^{r-r'} \Phi_{r'}\big((a)_v\big) \Big) \right|_p \leq C p^{-vr} \qquad (growth)$$

*for all $r = 0, 1, \ldots, h-1$ with $h = [\varkappa \mathrm{ord}_p(\lambda)] + 1$.*
*Then there exists an $h$-admissible measure $\tilde{\Phi}^{\lambda} : \mathcal{P}^h(Y, A) \to \mathcal{M}$ such that for all $((a)_v) \subset Y$ and for all $r = 0, 1, \ldots, h-1$ one has*

$$\int_{(a)_v} y_p^r \, d\tilde{\Phi}^{\lambda} = \Phi_r^{\lambda}((a)_v)$$

*where*

$$\Phi_r^\lambda((a)_v) = \pi_\lambda(\Phi_r((a)_v)) := U^{-\varkappa v}\left[\pi_{\lambda,1}U^{\varkappa v}\Phi_r((a)_v)\right]$$

*is the canonical projection of* $\pi_\lambda$ *of the modular form* $\Phi_r((a)_v)$ *(note that* $U^{\varkappa v}\Phi_r((a)_v) \in \mathcal{M}(Np^{\varkappa v}; A)^\lambda = \mathcal{M}(Np; A)^\lambda$ *because of the inclusion* $U^{\varkappa v-1}(\mathcal{M}(Np^{\varkappa v}; A)) \subset \mathcal{M}(Np; A)$ for all $v \geq 1$, *see Proposition 2.3 (a))*

*Proof.* We need to check the $h$-growth condition of Definition 0.1, c) for the linear form

$$\tilde{\Phi}^\lambda : \mathcal{P}^h(Y, A) \to \mathcal{M}(A)^\lambda$$

(given by the condition of Theorem 2.4). This growth condition says that for all $a \in Y$ and for $v \to \infty$

$$\left|\int_{(a)_v} (y_p - y_p^0)^r \, d\tilde{\Phi}^\lambda\right|_{p,\mathcal{M}} = o(p^{-v(r-h)})$$

for all $r = 0, 1, \ldots, h-1$, where $h = [\varkappa \mathrm{ord}_p(\lambda)] + 1$ and $y_p^0 = y_p(a)$.
Let us develop the definition of $\tilde{\Phi}^\lambda$ using the binomial formula:

$$\begin{aligned}\int_{(a)_v} (y_p - y_p^0)^r \, d\tilde{\Phi}^\lambda &= \sum_{r'=0}^{r} \binom{r}{r'} (-y_p^0)^{r-r'} \Phi_{r'}^\lambda((a)_v) = \lambda^{-v\varkappa} \cdot \\ &\lambda^{v\varkappa} \cdot U^{-v\varkappa}\left[\pi_{\lambda,1}U^{\varkappa v}\Big(\sum_{r'=0}^{r} \binom{r}{r'} (-y_p^0)^{r-r'} \Phi_{r'}\big((a)_v\big)\Big)\right]. \qquad (2.5)\end{aligned}$$

First we notice that all the operators

$$\lambda^{v\varkappa} \cdot U^{-v\varkappa} = \big(\lambda^{-1}U\big)^{-v\varkappa} = \big(I + \lambda^{-1}Z\big)^{-v\varkappa} = \sum_{j=0}^{n-1} \binom{-v\varkappa}{j} \big(\lambda^{-1}Z\big)^j$$

are uniformly bounded for $v \to \infty$ by a positive constant $C_1$ (where $U = \lambda I + Z$ and $Z^n = 0$ where $n$ is the rank of $\mathcal{M}(Np; A)$). Note that the binomial coefficients $\binom{-v\varkappa}{j}$ are all $\mathbb{Z}_p$-integral.

On the other hand by the condition (*growth*) of the theorem (for the distributions $\Phi_r$) we have the following inequality:

$$\left|U^{\varkappa v}\Big(\sum_{r'=0}^{r} \binom{r}{r'} (-y_p^0)^{r-r'} \Phi_{r'}((a)_v)\Big)\right|_{p,\mathcal{M}} \leq Cp^{-vr}$$

for all $r = 0, 1, \ldots, \varkappa h - 1$. If we apply to this estimate the previous bounded operators we get

$$\left|\int_{(a)_v} (y_p - y_p^0)^r \, d\tilde{\Phi}^\lambda\right|_{p,\mathcal{M}} \leq C \cdot C_1 |\lambda^{-v\varkappa}|_p \cdot p^{-vr} = o(p^{-v(r-h)})$$

because of the estimate

$$|\lambda^{-v\varkappa}|_p = \left(p^{\mathrm{ord}_p(\lambda)}\right)^{v\varkappa} = o(p^{vh}), \text{ and } \varkappa\mathrm{ord}_p(\lambda) < h = [\varkappa\mathrm{ord}_p(\lambda)] + 1. \quad \blacksquare \tag{2.6}$$

We apply Theorem 2.4 in Section 5.1 in order to obtain a $p$-adic measure in the form $\mu_{\lambda,\Phi} = \ell(\pi_\lambda(\Phi))$. Here $\lambda$ is a non-zero eigenvalue of Atkin's operator $U = U_p$ acting on $\mathcal{M}$, $\ell : \mathcal{M}^\lambda(Np; A) \to A$ is an $A$-linear form, applied to the projection $\pi_\lambda\ : \mathcal{M} \to \mathcal{M}^\lambda \subset \mathcal{M}^\lambda(Np; A)$ of a modular distribution $\Phi$, where $A = \mathbb{C}_p$.

## 3 Main Congruence for the higher twists of the Siegel-Eisenstein series

The purpose of this section is to show that the admissibility criterion of Theorem 2.4 with $h^* = 2$ is satisfied by a sequence of modular distributions (1.2), constructed in Section 1.

### 3.1 Construction of a sequence of modular distributions

As in the Introduction, consider the series $F_{\chi,r} = G^\star(\mathcal{Z}, -r; k, (Np^v)^2, \boldsymbol{\psi})$, given by (A.12), viewed as formal (nearly-holomorphic) Fourier series, whose coefficients admit explicit polynomial expressions. The only property that we use in this section is the fact that they can be written in terms of simple $p$-adic integrals:

$$F_{\chi,r} = \sum_{\mathcal{T}} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k-2r, r) a_{\chi,r}(\mathcal{T}) q^{\mathcal{T}},$$

[PaSE], [PaIAS] and (1.1)). Here we use a universal polynomial, described in [CourPa], Theorem 3.14 as follows:

$$\begin{aligned} Q(R, \mathcal{T}) &= Q(R, \mathcal{T}; k-2r, r) \qquad (3.1) \\ &= \sum_{t=0}^{r} \binom{r}{t} \det(\mathcal{T})^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k + r) Q_L(R, \mathcal{T}), \\ Q_L(R, \mathcal{T}) &= \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\mathcal{T})\right) \cdot \ldots \cdot \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\mathcal{T})\right)), \end{aligned}$$

where we use the natural representation $\rho_r : \mathrm{GL}_m(\mathbb{C}) \longrightarrow \mathrm{GL}(\wedge^r\mathbb{C}^m)$ $(0 \le r \le m)$ of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$. Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of $z$ of degree $r$. Put $\rho^\star_r(z) = \det(z)\rho_{m-r}({}^tz)^{-1}$. Then the representations $\rho_r$ and $\rho^\star_r$ turn out to be polynomial representations so that for each $z \in \mathrm{M}_m(\mathbb{C})$ the linear operators $\rho_r(z)$, $\rho^\star_r(z)$ are well defined. In (3.1), $L$ runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$. The coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ in (3.1) are polynomials in $\beta$ of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$.

### 3.2 Utilizing the admissibility criterion

Recall an important property of the sequence of distributions $\Phi_r$ defined by (1.2), Section 1: the nebentypus character of $\Phi_r(\chi)$ is $(\psi_1, \psi_2, \psi_3)$, so that it *does not depend* on $\chi$. Now let us prove that the sequence of distributions $\Phi_r$ on $Y$ produces a certain *admissible measure* $\tilde{\Phi}$ with values in a finite dimensional $\mathbb{C}_p$-vector subspace

$$\mathcal{M}^\lambda \subset \mathcal{M},\ \mathcal{M} = \mathcal{M}_{k,r}(\mathbb{C}_p) \otimes \mathcal{M}_{k,r}(\mathbb{C}_p) \otimes \mathcal{M}_{k,r}(\mathbb{C}_p),$$

(of nearly-holomorphic triple modular forms over $\mathbb{C}_p$) using a general admissibility criterion (see Theorem 2.4).

### 3.3 Sufficient conditions for admissibility of measures with values in nearly-holomorphic modular forms

In order to construct the admissible measures of Theorem B we use the admissible measures $\tilde{\mu}^\lambda(f_1 \otimes f_2 \otimes f_3, y)$ constructed in Section 5 out of the modular distributions $\Phi_r$ in the form

$$\tilde{\mu}^\lambda(f_1 \otimes f_2 \otimes f_3)(\chi y_p^r) = \ell(\pi_\lambda(\Phi_r)(\chi)).$$

The growth condition for $\tilde{\mu}^\lambda$ follows then from a growth condition for $\Phi_r$:

$$\sup_{a \in Y} \left| \int_{a+(Np^v)} (y_p - a_p)^r d\widetilde{\Phi}^\lambda \right|_p = o\left( |Np^v|_p^{r-2\mathrm{ord}_p\lambda} \right), \tag{3.2}$$

where

$$\tilde{\Phi}^\lambda(\chi y_p^r) = \pi_\lambda(\Phi_r(\chi)).$$

Let us use a general result giving a sufficient condition for the admissibility of measures with values in nearly-holomorphic Siegel modular forms (given in Theorem 2.4) with $\varkappa = 2$, $h = [2\mathrm{ord}_p\lambda] + 1$. Then we need to check that the nearly-holomorphic triple modular forms $\Phi_r(\chi)$ are of level $N^2\chi^{2v}$, nebentypus $(\psi_1, \psi_2, \psi_3)$, and satisfy the congruences

$$\left| U_T^{2v} \Big( \sum_{r'=0}^{r} \binom{r}{r'} (-a_p^0)^{r-r'} \Phi_{r'}\big((a)_v\big) \Big) \right|_p \le Cp^{-vr} \tag{3.3}$$

and for all $r = 0, 1, \cdots, k-2$.

### 3.4 Special Fourier coefficients of the higher twist of the Siegel-Eisenstein distributions

Let us use the Fourier expansions (1.13) for $\Psi_r(\chi)$. These formulas directly imply the Fourier expansion of $\Phi_r(\chi)|U_p^{2v}$ as follows

$$\Phi_r(\chi)|U_p^{2v} = \sum_{t_1, t_2 t_3 \ge 0} a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r) q_1^{t_1} q_2^{t_2} q_3^{t_3} \tag{3.4}$$

with

$$a(p^{2v}t_1, p^{2v}t_2, p^{2v}t_3; p^{2v}R_1, p^{2v}R_2, p^{2v}R_3, r)$$

$$= \sum_{\mathcal{T}:\mathrm{diag}(\mathcal{T})=(p^{2v}t_1,p^{2v}t_2,p^{2v}t_3)} \bar\chi(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times$$

$$\times \det(\mathcal{T})^{k-2r-\kappa}Q(p^{2v}\,\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T})$$

$$= \sum_{\mathcal{T}:\mathrm{diag}(\mathcal{T})=(p^{2v}t_1,p^{2v}t_2,p^{2v}t_3)} v_{\chi,r}(\mathcal{T},\mathrm{diag}(R_1,R_2,R_3)),$$

where

$$v_{\chi,r}(\mathcal{T},\mathrm{diag}(R_1,R_2,R_3)) = \bar\chi(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times \tag{3.5}$$

$$\times \det(\mathcal{T})^{k-2r-\kappa}Q(p^{2v}\,\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T})$$

$$= \chi^{(p)}(2)\bar\chi^{(p)}(\mathcal{T})\chi^\circ(t_{12}t_{13}t_{23})\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\times$$

$$\times \det(\mathcal{T})^{k-2r-\kappa}Q(p^{2v}\,\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T}).$$

Let us notice that, for any $\mathcal{T}$ with $\mathrm{diag}(\mathcal{T}) = (p^{2v}t_1, p^{2v}t_2, p^{2v}t_3)$ one has

$$\det(\mathcal{T}) \equiv 2t_{12}t_{13}t_{23} \bmod p^{2v},$$
$$\chi^{(p)}(2t_{12}t_{13}t_{23}) = \chi^{(p)}(\det(\mathcal{T})) = \chi(\det(\mathcal{T})\overline{\chi^\circ(\det(\mathcal{T})},$$
$$2^r a_{\chi,r}(\mathcal{T}) = \int_Y y_p^r\chi(y)\mathrm{d}\mathcal{F}_{\mathcal{T}},$$
$$\text{with } \chi = \chi^{(p)}\chi^\circ, \chi^{(p)} \bmod p^v, \chi^\circ \bmod N, \text{ and } p\nmid N,$$

for a bounded measure $\mathcal{F}_{\mathcal{T}}$ on $Y$ with values in $\overline{\mathbb{Q}}$. It follows that

$$v_{\chi,r}(\mathcal{T},\mathrm{diag}(R_1,R_2,R_3))$$
$$= \chi^{(p)}(2)\bar\chi(\det(\mathcal{T}))\det(\mathcal{T})^{-r}\overline{\chi^\circ(\det(\mathcal{T})}\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})\cdot \tag{3.6}$$

$$\cdot \det(\mathcal{T})^{k-r-\kappa}Q(p^{2v}\,\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)2^r a_{\chi,r}(\mathcal{T}) \tag{3.7}$$

$$= \det(\mathcal{T})^{k-r-\kappa}Q(p^{2v}\,\mathrm{diag}(R_1,R_2,R_3),\mathcal{T};k-2r,r)\overline{\chi^\circ}(2)\int_Y \chi y_p^r \mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3},$$

where $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$ denotes the bounded measure defined by the equality:

$$\int_Y \chi y_p^r \mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3} \tag{3.8}$$
$$= \chi^{(p)}(2)\chi^\circ(2)2^r\bar\chi(\det(\mathcal{T}))\det(\mathcal{T})^{-r}\overline{\chi^\circ(\det(\mathcal{T})}\bar\psi_2\psi_3(t_{13})\bar\psi_1\psi_3(t_{23})a_{\chi,r}(\mathcal{T}).$$

## 3.5 Main Congruence for the Fourier expansions

Let us use the orthogonality relations for Dirichlet characters in order to prove the admissibility of the distributions given by the sequence $\pi_\lambda(\Phi_r(\chi))$ using the Fourier expansions (3.4). According to the admissibility criterion of Theorem 2.4 we need to check the following *Main Congruence*:

$$\Big|\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\frac{1}{\varphi(Np^v)}\sum_{\chi \mod Np^v}\chi^{-1}(a)v_{\chi,r'}(\mathcal{T},p^{2v}\operatorname{diag}(R_1,R_2,R_3))\Big|_p$$
$$\le Cp^{-vr}, \tag{3.9}$$

where we use the notation (3.6) for $v_{\chi,r'}(\mathcal{T},\operatorname{diag}(R_1,R_2,R_3))$, implying that the coefficients

$$i_p(v_{\chi,r'}(\mathcal{T},\operatorname{diag}(R_1,R_2,R_3)))$$

in (3.5) are given as sums of the following expressions:

$$B_r(\chi,\mathcal{T}) = \overline{\chi^\circ}(2)\det(\mathcal{T})^{k-r-\kappa}\int_Y \chi y_p^r\mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}\cdot \tag{3.10}$$
$$\cdot\sum_{t=0}^{r}\binom{r}{t}\det(\mathcal{T})^{r-t}\sum_{|L|\le mt-t}R_L(\kappa-k+r)Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T}),$$

where $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$ denotes the bounded measure defined by (3.8). Using the expressions (3.10), the main congruence (3.9) is reduced to proving the congruence for the numbers $B_r(\chi,\mathcal{T})$: there exists a non-zero integer $C_k$ such that

$$C_k\cdot\sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\frac{1}{\varphi(Np^v)}\sum_{\chi \mod Np^v}\chi^{-1}(a)B_{r'}(\chi,\mathcal{T})\equiv 0 \mod p^{vr}$$
$$\Longleftrightarrow C_k\cdot A\equiv 0 \mod Np^{vr}, \tag{3.11}$$

where we use the notation

$$A = A_r(\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3) = \sum_{r'=0}^{r}\binom{r}{r'}(-a_p^0)^{r-r'}\frac{1}{\varphi(Np^v)}\sum_{\chi \bmod Np^v}\chi^{-1}(a)\cdot \tag{3.12}$$

$$\cdot\,\overline{\chi^\circ}(2)\det(\mathcal{T})^{k-r'-\kappa}\int_Y \chi y_p^{r'}\,\mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}\sum_{t=0}^{r'}\binom{r'}{t}\det(\mathcal{T})^{r'-t}$$

$$\sum_{|L|\le mt-t} R_L(\kappa-k+r')Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T}).$$

Note that $R_L(\kappa-k+r')$ is a polynomial of degree $mt-|L| = 3t-|L|$ in $\kappa-k+r'$ (see (3.1)), hence in $r'$, and $\binom{r'}{t}$ is a polynomial of degree $t$ in $r'$. One can therefore write

$$\binom{r'}{t}R_L(\kappa-k+r) = \sum_{n=0}^{4t-|L|}\mu_n\frac{(r'+n+1)!}{(r'+1)!}.$$

Here the coefficients $\mu_n$ are fixed rational numbers (independent of $r'$).
Using the orthogonality relations for Dirichlet characters $\bmod Np^v$, we see that the sum over $r'$ in (3.12), denoted by $C = C_r(t,L,\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3)$, takes the form

$$C_r(t,L,\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3) = \overline{\chi^\circ}(2)\det(\mathcal{T})^{k-t-\kappa}$$

$$\int_{y\equiv a \bmod p^v}\sum_{n=0}^{4t-|L|}\mu_n\underbrace{\sum_{r'=0}^{r}\binom{r}{r'}(-a)^{r-r'}\frac{(r'+n+1)!}{(r'+1)!}y^{r'}}_{y^{-n}\frac{\partial^n}{\partial y^n}\left(y^{n+1}(y-a)^r\right)}\mathrm{d}\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}(y)$$

Note that we write $\chi = \chi^\circ\chi^{(p)}$, fix $\chi^\circ$, and sum over all characters $\chi^{(p)} \bmod p^v$. We have therefore $(y-a)^r \equiv 0 \bmod (p^v)^r$ in the integration domain $y \equiv a \bmod p^v$, implying the congruence

$$c_kC_r(t,L,\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3) \equiv 0 \pmod{(p^v)^{r-n}} \equiv 0 \pmod{(p^v)^{r-4t+|L|}}, \tag{3.13}$$

where $c_k \in \mathbb{Q}^*$ is a nonzero constant coming from the denominators of the fixed rational numbers $\mu_n$, and of the bounded distributions $\mathcal{F}_{\mathcal{T};\chi^\circ,\psi_1,\psi_2,\psi_3}$.

## 3.6 Proof of the Main Congruence

Now the expression (3.12) transforms to

$$A_r(\mathcal{T}) = \sum_{t=0}^{r}\sum_{|L|\le 2t}\det(\mathcal{T})^t\cdot C(t,L,\mathcal{T})\det(\mathcal{T})^{k-2r-\kappa}Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T}), \tag{3.14}$$

where $Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T})$ is a homogeneuos polynomial of degree $3t-|L|$ in the variables $R_{ij}$ implying the congruence

$$Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T}) \equiv 0 \pmod{(p^{2v})^{(3t-|L|)}}. \tag{3.15}$$

On the other hand we know from the description (3.1) of the polynomial

$$Q(R,\mathcal{T}) = Q(R,\mathcal{T};k-2r,r) = \sum_{t=0}^{r}\binom{r}{t}\det(\mathcal{T})^{r-t}\sum_{|L|\le 2t} R_L(\kappa-k+r)Q_L(R,\mathcal{T}),$$

$$Q_L(R,\mathcal{T}) = \operatorname{tr}\left({}^t\rho_{3-l_1}(R)\rho^\star_{l_1}(\mathcal{T})\right)\cdot\ldots\cdot\operatorname{tr}\left({}^t\rho_{3-l_t}(R)\rho^\star_{l_t}(\mathcal{T})\right),$$

that $2t-|L|\ge 0$ so we obtain the desired congruence as follows

$$\begin{cases} c_kC_r(t,L,\mathcal{T}) \equiv 0 \pmod{(p^v)^{r-4t+|L|}} \\ Q_L(p^{2v}\operatorname{diag}(R_1,R_2,R_3),\mathcal{T}) \equiv 0 \pmod{(p^{2v})^{(3t-|L|)}} \end{cases} \tag{3.16}$$
$$\Rightarrow c_kA_r(\mathcal{T}) \equiv 0 \pmod{p^{vr}},$$

since $v(r-4t+|L|)+2v(3t-|L|) = vr+2vt-v|L| \ge vr$, proving (3.9). ∎

### 3.7 Construction of admissible measures with values in nearly-holomorphic modular forms

We wish now to construct an $h$-admissible measure $\tilde\Phi^\lambda : \mathcal{P}^h(Y,A)\to\mathcal{M}_T(A)$ on $Y$ out of the following sequence of the higher twists of Siegel-Eisenstein distributions given by the equality (1.2):

$$\Phi_r := 2^r\operatorname{diag}^*\Psi_r = 2^rF^{\bar\chi_1,\bar\chi_2,\bar\chi_3}_{\chi,r},\ \Phi_r : \mathcal{P}^1(Y,A)\to\mathcal{M}_T(A)$$

(they take values in the $A$-module

$$M = \mathcal{M}_T(\psi_1,\psi_2,\psi_3;A)\subset\mathcal{M}_{k,r}(\psi_1;A)\otimes\mathcal{M}_{k,r}(\psi_2;A)\otimes\mathcal{M}_{k,r}(\psi_3;A)$$

of triple modular forms over $A=\mathbb{C}_p$ or $A=\overline{\mathbb{Q}}$).

Theorem 3.1 *Let $\lambda\in A$ be an element whose absolute value is a positive constant with $0<|\lambda|_p<1$, and define $h=[2\mathrm{ord}_p(\lambda)]+1$. Then the sequence (1.2) satisfies for any $(a)_v\subset Y$ the following two conditions:*

$$\Phi_r\big((a)_v\big)\in\mathcal{M}(N'p^{2v}),\ \text{with } N' \text{ independent of } v, \tag{level}$$

$$\left|U_T^{2v}\Big(\sum_{r'=0}^{r}\binom{r}{r'}(-y_p^0)^{r-r'}\Phi_{r'}\big((a)_v\big)\Big)\right|_p \le Cp^{-vr} \tag{growth}$$

*for all $r=0,1,\ldots,h-1$.*
*Moreover, there exists an $h$-admissible measure $\tilde\Phi^\lambda:\mathcal{P}^h(Y,A)\to\mathcal{M}_T$ such that for all $((a)_v)\subset Y$ and for all $r=0,1,\ldots,h-1$ one has*

$$\int_{(a)_v} y_p^r\,d\tilde\Phi^\lambda = \Phi_r^\lambda((a)_v)$$

*where*

$$\Phi_r^\lambda((a)_v) = \pi_{\lambda,T}(\Phi_r((a)_v)) := U_T^{-2v}\left[\pi_{\lambda,1}U_T^{2v}\Phi_r((a)_v)\right]$$

*is the canonical projection of* $\pi_\lambda$ *of the triple modular form* $\Phi_r((a)_v)$ *(note that* $U_T^{2v}\Phi_r((a)_v) \in \mathcal{M}_T(Np^{2v};A)^\lambda = \mathcal{M}_T(Np;A)^\lambda$ *because of the inclusion* $U_T^{2v-1}(\mathcal{M}_T(Np^{2v};A)) \subset \mathcal{M}_T(Np;A)$ for all $v \geq 1$*, see Proposition 2.3 (a)).*

*Proof.* We use Theorem 2.4 with $\varkappa = 2$, and we to check the $h$-growth condition for the $A$-linear map

$$\tilde{\Phi}^\lambda : \mathcal{P}^h(Y,A) \to \mathcal{M}_T(A)$$

defined in Theorem 3.1. We have to check that for any $((a)_v) \in Y$ the following two conditions are satisfied: for all $r = 0,1,\dots,h-1$,

$$\Phi_r((a)_v) \in \mathcal{M}(N^2p^{2v}), \qquad (level)$$

$$\left| U_T^{2v}\left(\sum_{r'=0}^{r}\binom{r}{r'}(-y_p^0)^{r-r'}\Phi_{r'}((a)_v)\right)\right|_p \leq Cp^{-vr}. \qquad (growth)$$

The (*level*) condition is implied by the definition (1.2)

$$\Phi_r(\chi) = 2^r \operatorname{diag}^* F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3},$$

and Proposition 1.5.
The (*growth*) is deduced from the Main Congruence (3.9) (proved in Section 3.6) for the Fourier coefficients of the functions (1.2). ∎

## 4 A TRILINEAR FORM ON THE CHARACTERISTIC SUBSPACE OF THE $U$-OPERATOR

### 4.1 THE ADJOINT OPERATOR $U^*$

Let $f = \sum_{n=1}^\infty a_n q^n$ denote a primitive cusp eigenform of conductor dividing $Np$, with coefficients $i_p(a_n)$ in a finite extension $K$ of $\mathbb{Q}_p$ and of Dirichlet character $\psi$ modulo $N$. Let $\alpha \in K$ be a root of the Hecke polynomial $x^2 - a_p(f)x + \psi(p)p^{k-1}$ as above, and let $\alpha'$ denote the other root.
Recall that the function $f_0 = \sum_{n=1}^\infty a_n(f_0)q^n \in \overline{\mathbb{Q}}[[q]]$ is defined by (0.9) as an eigenfunction of $U = U_p$ with the eigenvalue $\alpha \in \overline{\mathbb{Q}}$. In the following proposition, let $U^*$ denote the operator adjoint to

$$U = U_p : \mathcal{M}_{r,k}(\Gamma_1(Np),\mathbb{C}) \to \mathcal{M}_{r,k}(\Gamma_1(Np),\mathbb{C})$$

in the complex vector space $\mathcal{M}_{r,k}(\Gamma_1(Np),\mathbb{C})$ with respect to the Petersson inner product.

PROPOSITION 4.1 (a) *The following operator identity holds:* $U^* = W_{Np}^{-1}UW_{Np}$ *(in the complex vector space* $\mathcal{M}_{r,k}(\Gamma_1(Np),\mathbb{C})$*).*

(b) *There are the following identities in* $\mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C})$*:*

$$f^0|U^* = \overline{\alpha} f^0 \text{ and } T_l(f^0) = a_l(f) f^0$$

*for all "good primes"* $l \nmid Np$.
(c) *The linear form* $g \mapsto \langle f^0, g \rangle_{Np}$ *on* $\mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C})$ *vanishes on the complex vector subspace* $\mathrm{Ker}\pi_{\alpha,1} = \mathrm{Im}(U - \alpha I)^{n_1}$ *where* $n_1 = \dim \mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C})$*, and we use the same notation as above*

$$\pi_{\alpha,1} : \mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C}) \to \mathcal{M}^{\alpha}_{r,k}(\Gamma_1(Np), \mathbb{C})$$

*for the complex characteristic projection onto the* $\alpha$*-primary subspace of the operator* $U$ *(acting on the finite-dimensional complex vector space* $\mathcal{M}_{r,k}(\Gamma_1(Np), \mathbb{C})$*) hence*

$$\langle f^0, g \rangle_{Np} = \langle f^0, \pi_{\alpha,1}(g) \rangle_{Np}$$

(d) *If* $g \in \mathcal{M}(Np^{v+1}; \overline{\mathbb{Q}})$ *and* $\alpha \neq 0$*, then we have the equality*

$$\langle f^0, \pi_\alpha(g) \rangle_{Np} = \alpha^{-v} \langle f^0, U^v g \rangle_{Np}$$

*where*

$$\pi_\alpha(g) = g^\alpha = U^{-v}\left[\pi_{\alpha,1} U^v g\right] \in \mathcal{M}^\alpha(Np)$$

*is the* $\alpha$*-part of* $g$.
(e) *The linear form*

$$\mathcal{L}_{f,\alpha} : \mathcal{M}(Np^v; \mathbb{C}) \to \mathbb{C}, \quad g \mapsto \frac{\langle f^0, \alpha^{-v} U^v(g) \rangle_{Np}}{\langle f^0, f_0 \rangle_{Np}}$$

*is defined over* $\overline{\mathbb{Q}}$*:*

$$\mathcal{L}_{f,\alpha} : \mathcal{M}(Np^v; \overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}$$

*and there exists a unique* $\mathbb{C}_p$*-linear form* $\ell_{f,\alpha}$ *on* $\mathcal{M}(Np^v; \mathbb{C}_p) = \mathcal{M}(Np^v; \overline{\mathbb{Q}}) \otimes_{i_p} \mathbb{C}_p$ *such that* $\ell_{f,\alpha}(g) = i_p(\mathcal{L}_{f,\alpha}(g))$ *for all* $g \in i_p(\mathcal{M}(Np^v; \overline{\mathbb{Q}}))$.

*Proof* (a) See [Miy], Theorem 4.5.5 (see also [Ran90]).
(b) Let us use directly the statement a):

$$f^0|U^* = f_0^\rho | W_{Np} W_{Np}^{-1} U W_{Np} = \bar{\alpha} f^\rho | W_{Np} = \bar{\alpha} f^0.$$

(c) If $g \in \mathrm{Ker}\pi_{\alpha,1} = \mathrm{Im}(U - \alpha I)^{n_1}$ then $g = (U - \alpha I)^{n_1} g_1$ and

$$\langle f^0, (U - \alpha I)^{n_1} g_1 \rangle_{Np} = \langle (U^* - \bar{\alpha} I) f^0, (U - \alpha I)^{n_1 - 1} g_1 \rangle_{Np} = 0$$

hence $\langle f^0, g \rangle_{Np} = 0$; moreover

$$\langle f^0, g \rangle_{Np} = \langle f^0, \pi_{\alpha,1}(g) + (g - \pi_{\alpha,1}(g)) \rangle_{Np} = \langle f^0, \pi_{\alpha,1}(g) \rangle_{Np}.$$

(d) Let us use the definitions and write the following product:

$$\alpha^v\langle f^0, \pi_\alpha g\rangle_{Np} = \langle U^{*v}(f^0), U^{-v}\left[\pi_{\alpha,1}U^v g\right]\rangle_{Np}$$
$$= \langle f^0, \pi_{\alpha,1}(U^v g)\rangle_{Np} = \langle f^0, U^v g\rangle_{Np}$$

by (c) as $U^v g \in \mathcal{M}(Np)$.
(e) Note that $\mathcal{L}_{f,\alpha}(f_0) = 1$, $f_0 \in \mathcal{M}(Np; \overline{\mathbb{Q}})$. Consider the complex vector space

$$\mathrm{Ker}\mathcal{L}_{f,\alpha} = \langle f^0\rangle^\perp = \{g \in \mathcal{M}(Np^v; \mathbb{C}) \mid \langle f^0, g\rangle_{Np^v} = 0\}.$$

It admits a $\overline{\mathbb{Q}}$-rational basis (as it is stable under all "good" Hecke operators $T_l$ $(l \nmid Np)$:

$$\langle f^0, g\rangle_{Np^v} = 0 \Rightarrow \langle f^0, T_l g\rangle_{Np^v} = \langle T_l^* f^0, g\rangle_{Np^v} = 0$$

and diagonalizing the action of $T_l$ (over $\overline{\mathbb{Q}}$) we get such a basis establishing e). We obtain then the $\mathbb{C}_p$-linear form $\ell_{f,\alpha}$ on $\mathcal{M}(Np^v; \mathbb{C}_p) = \mathcal{M}(Np^v; \overline{\mathbb{Q}}) \otimes_{i_p} \mathbb{C}_p$ such that $\ell_{f,\alpha}(g) = i_p(\mathcal{L}_{f,\alpha}(g))$ by extending scalars from $\overline{\mathbb{Q}}$ to $\mathbb{C}_p$ via the imbedding $i_p$.
Note that we use here only the $\alpha$-part $\mathcal{M}(Np^v; A)^\alpha$ because the constructed linear form $\ell_{f,\alpha}$ passes through the $\pi_\alpha$ (for $A = \mathbb{C}_p$ , $A = \overline{\mathbb{Q}}$, or $A = \mathbb{C}$). Moreover, $f_0$ can be included to a basis $\{f_0, g_i\}_{i=2,\cdots,n}$ of $\mathcal{M}(Np^v; A)^\alpha$, where $g_i$ are eigenfunctions of all Hecke operators $T_l$ for primes $l \nmid Np$; they are algebraically orthogonal to $f_0$ (in the sense of the algebraic Petersson product studied by Hida [Hi90]) so that projection to the $f_0$ part of this basis gives such an $A$-linear form.

## 4.2 The triple $U$-operator

In the following proposition, we consider the triple $U$-operator

$$U_T = U_{1,p} \otimes U_{2,p} \otimes U_{3,p} : \mathcal{M}_T(\Gamma_1(Np), \mathbb{C}) \to \mathcal{M}_T(\Gamma_1(Np), \mathbb{C}), \text{ where} \tag{4.1}$$

$$\mathcal{M}_T(\Gamma_1(Np), \mathbb{C}) = \mathcal{M}_{k_1}(\Gamma_1(Np), \mathbb{C}) \otimes \mathcal{M}_{k_2}(\Gamma_1(Np), \mathbb{C}) \otimes \mathcal{M}_{k_3}(\Gamma_1(Np), \mathbb{C}),$$

acting on the complex vector space $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$ endowed with the triple Petersson inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle g_1 \otimes g_2 \otimes g_3, h_1 \otimes h_2 \otimes h_3\rangle_T = \langle g_1, h_1\rangle_{Np}\langle g_2, h_2\rangle_{Np}\langle g_3, h_3\rangle_{Np}.$$

Let

$$U_T^* = U_{1,p}^* \otimes U_{2,p}^* \otimes U_{3,p}^*$$

denote the adjoint operator on $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$ for the triple Petersson inner product. Recall the notation (0.9) and (0.10):

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j|V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j|\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}$$
$$f_{j,0}^\rho = \sum_{n=1}^\infty \overline{a(n, f_0)} q^n,\ f_j^0 = f_{j,0}^\rho|_k\ W_{Np} = f_{j,0}^\rho\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0\end{pmatrix}.$$

Proposition 4.2 (a) *The following operator identity holds:*

$$U_T^* = W_{Np}^{-1}U_{p,1}W_{Np} \otimes W_{Np}^{-1}U_{p,2}W_{Np} \otimes W_{Np}^{-1}U_{p,3}W_{Np}$$

*(in the complex vector space $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$).*
(b) *There are the following identities in $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$:*

$$U_T^*(f_1^0 \otimes f_2^0 \otimes f_3^0) = \overline{\lambda}(f_1^0 \otimes f_2^0 \otimes f_3^0).$$

(c) *The linear form on $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$ defined by*

$$g_1 \otimes g_2 \otimes g_3 \otimes \mapsto \left\langle f_1^0 \otimes f_2^0 \otimes f_3^0, g_1 \otimes g_2 \otimes g_3 \right\rangle_T = \langle f_1^0, g_1\rangle_{Np} \langle f_2^0, g_2\rangle_{Np} \langle f_3^0, g_3\rangle_{Np}$$

*vanishes on the complex vector subspace $\operatorname{Ker} \pi_{\lambda,T,1} = \operatorname{Im}(U_T - \lambda I)^{n_T}$ where we write $n_T = \dim \mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$, and we use the notation*

$$\pi_{\lambda,T,1} : \mathcal{M}_T(\Gamma_1(Np), \mathbb{C}) \to \mathcal{M}_T^\lambda(\Gamma_1(Np), \mathbb{C})$$

*for the complex characteristic projection onto the $\lambda$-primary subspace of the operator $U_T$ acting on the finite-dimensional complex vector space $\mathcal{M}_T(\Gamma_1(Np), \mathbb{C})$. Moreover, the following equality holds*

$$\left\langle f_1^0 \otimes f_2^0 \otimes f_3^0, g_1 \otimes g_2 \otimes g_3 \right\rangle_T = \left\langle f_1^0 \otimes f_2^0 \otimes f_3^0, \pi_{\lambda,T,1}(g_1 \otimes g_2 \otimes g_3) \right\rangle_T.$$

(d) *If $g \in \mathcal{M}_T(Np^{v+1}; \overline{\mathbb{Q}})$ and $\lambda \neq 0$, then we have the equality*

$$\langle f_1^0 \otimes f_2^0 \otimes f_3^0, \pi_{\lambda,T}(g)\rangle_{T,Np} = \lambda^{-v} \langle f_1^0 \otimes f_2^0 \otimes f_3^0, U_T^v g\rangle_{T,Np}$$

*where*

$$\pi_{\lambda,T}(g) = g^\lambda = U_T^{-v}\left[\pi_{\lambda,T,1} U_T^v g\right] \in \mathcal{M}_T^\lambda(Np)$$

*is the $\lambda$-part of $g$.*
(e) *The linear form*

$$\mathcal{L}_{T,\lambda} : \mathcal{M}_T(Np^v; \mathbb{C}) \to \mathbb{C}, \quad g \mapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, \lambda^{-v} U_T^v g\rangle_{T,Np}}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,Np}}$$

*is defined over $\overline{\mathbb{Q}}$:*

$$\mathcal{L}_{T,\lambda} : \mathcal{M}_T(Np^v; \overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}$$

*and there exists a unique $\mathbb{C}_p$-linear form $\ell_{T,\lambda}$ on $\mathcal{M}_T(Np^v; \mathbb{C}_p) = \mathcal{M}_T(Np^v; \overline{\mathbb{Q}}) \otimes_{i_p} \mathbb{C}_p$ such that $\ell_{T,\lambda}(g) = i_p(\mathcal{L}_{f,\alpha}(g))$ for all $g \in i_p(\mathcal{M}_T(Np^v; \overline{\mathbb{Q}}))$.*

Remark 4.3 *We may view the trilinear form*

$$(g_1, g_2, g_3) \mapsto \ell_{T,\lambda}(g_1 \otimes g_2 \otimes g_3)$$

*as a $p$-adic version of the triple Petersson product following Hida [Hi90].*

*Proof* of Proposition 4.2, a), b) follows directly from that of Proposition 4.1. In order to prove c) we need to show that the linear form on $\mathcal{M}_T(\Gamma_1(Np),\mathbb{C})$ defined by

$$\begin{aligned} g_1\otimes g_2\otimes g_3\otimes &\mapsto \left\langle f_1^0\otimes f_2^0\otimes f_3^0, g_1\otimes g_2\otimes g_3\right\rangle_{T,Np} \\ &= \langle f_1^0, g_1\rangle_{Np}\langle f_2^0, g_2\rangle_{Np}\langle f_3^0, g_3\rangle_{Np} \end{aligned}$$

vanishes on the complex vector subspace

$$\operatorname{Ker}\pi_{\lambda,T,1} = \operatorname{Im}(U_T-\lambda I)^{n_T} = (\operatorname{Ker}(U_T^*-\overline{\lambda}I)^{n_T})^\perp.$$

It suffices to notice that

$$f_1^0\otimes f_2^0\otimes f_3^0\in \operatorname{Ker}(U_T^*-\overline{\lambda}I)\subset \operatorname{Ker}(U_T^*-\overline{\lambda}I)^{n_T},$$

because of the equality

$$U_T^*(f_1^0\otimes f_2^0\otimes f_3^0) = U_{1,p}^*(f_1^0)\otimes U_{2,p}^*(f_2^0)\otimes U_{3,p}^*(f_3^0) = \overline{\lambda}(f_1^0\otimes f_2^0\otimes f_3^0).$$

More precisely, if $g\in\operatorname{Ker}\pi_{\lambda,T,1} = \operatorname{Im}(U_T-\lambda I)^{n_T}$ then $g = (U_T-\lambda I)^{n_T}g_1$ and

$$\begin{aligned} &\langle f_1^0\otimes f_2^0\otimes f_3^0, (U_T-\lambda I)^n g_1\rangle_{T,Np} \\ &= \langle (U_T^*-\overline{\lambda}I)(f_1^0\otimes f_2^0\otimes f_3^0, (U_T-\lambda I)^{n-1}g_1)\rangle_{T,Np} = 0 \end{aligned}$$

hence $\langle f_1^0\otimes f_2^0\otimes f_3^0, g\rangle_{T,Np} = 0$. Moreover, the following equality holds

$$\left\langle f_1^0\otimes f_2^0\otimes f_3^0, g_1\otimes g_2\otimes g_3\right\rangle_T = \left\langle f_1^0\otimes f_2^0\otimes f_3^0, \pi_{\lambda,T,1}(g_1\otimes g_2\otimes g_3)\right\rangle_T,$$

by the definition of the projection $\pi_{\lambda,T,1}$:

$$g_1\otimes g_2\otimes g_3 - \pi_{\lambda,T,1}(g_1\otimes g_2\otimes g_3)\in\operatorname{Ker}\pi_{\lambda,T,1}.$$

d) Let us use the definitions and write the following product:

$$\begin{aligned} &\lambda^v\langle f_1^0\otimes f_2^0\otimes f_3^0, \pi_{\lambda,T}g\rangle_{T,Np} = \langle U_T^{*v}(f_1^0\otimes f_2^0\otimes f_3^0), U_T^{-v}\left[\pi_{\lambda,T,1}U_T^v g\right]\rangle_{T,Np} = \\ &\langle f_1^0\otimes f_2^0\otimes f_3^0, \pi_{\lambda,T,1}(U_T^v g)\rangle_{T,Np} = \langle f_1^0\otimes f_2^0\otimes f_3^0, U_T^v g\rangle_{T,Np} \end{aligned}$$

by c) as $U_T^v g\in\mathcal{M}_T(Np)$.
e) Note that $\mathcal{L}_{T,\lambda}(f_1^0\otimes f_2^0\otimes f_3^0) = 1$, $f_1^0\otimes f_2^0\otimes f_3^0\in\mathcal{M}_T(Np;\overline{\mathbb{Q}})$. Consider the complex vector space

$$\operatorname{Ker}\mathcal{L}_{T,\lambda} = \langle f_1^0\otimes f_2^0\otimes f_3^0\rangle^\perp = \{g\in\mathcal{M}_T(Np^v;\mathbb{C}) \mid \langle f^0, g\rangle_{T,Np^v} = 0\}.$$

It admits a $\overline{\mathbb{Q}}$-rational basis (as in Proposition 4.1) establishing e).
We obtain then the $\mathbb{C}_p$-linear form $\ell_{T,\lambda}$ on $\mathcal{M}_T(Np^v;\mathbb{C}_p) = \mathcal{M}_T(Np^v;\overline{\mathbb{Q}})\otimes_{i_p}\mathbb{C}_p$ such that $\ell_{T,\lambda}(g) = i_p(\mathcal{L}_{T,\lambda}(g))$ by extending scalars from $\overline{\mathbb{Q}}$ to $\mathbb{C}_p$ via the imbedding $i_p$. ∎

## 5 Computation of $p$-adic integrals and $L$-values

### 5.1 Construction of $p$-adic measures

Let $\mathcal{M} = \mathcal{M}_T(A) = \bigcup_{v\geq 0}\mathcal{M}_{k,r}(Np^v,\psi_1;A) \otimes_A \mathcal{M}_{k,r}(Np^v,\psi_2;A) \otimes_A \mathcal{M}_{k,r}(Np^v,\psi_3;A)$ be the $A$-module of nearly-holomorphic triple modular forms with formal Fourier coefficients in $A$, where $A=\mathbb{C}_p$. Let us define an $A$-valued measure

$$\tilde{\mu}^\lambda(y;f_1\otimes f_2\otimes f_3) : \mathcal{C}^{loc-an}(Y,A)\to A$$

by applying the trilinear form $\ell_{T,\lambda}:\mathcal{M}(Np^v;A)\to A$ of Proposition 4.2

$$\tilde{\mu}^\lambda(y;f_1\otimes f_2\otimes f_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda) \tag{5.1}$$

to the $h$-admissible measure $\tilde{\Phi}^\lambda$ of Theorem 2.4 on $Y$ with values in $\mathcal{M}(A)^\lambda\subset \mathcal{M}(Np;A)$. That $h$-admissible measure was defined as an $A$-linear map $\tilde{\Phi}^\lambda$ : $\mathcal{P}^h(Y,A)\to\mathcal{M}(A)^\lambda$ satisfying for any $(a)_\nu\subset Y$ and for all $r=0,1,\dots,h-1$ the following equality:

$$\int_{(a)_\nu} y_p^r\, d\tilde{\Phi}^\lambda = \pi_\lambda(\Phi_r((a)_\nu))\in\mathcal{M}(Np),$$

where $h=[2\mathrm{ord}_p\lambda(p)]+1$, hence

$$\int_{(a)_\nu} y_p^r\,\mathrm{d}\tilde{\mu}^\lambda(y;f_1\otimes f_2\otimes f_3) = \ell_{T,\lambda}\left(\int_{(a)_\nu} y_p^r\,\mathrm{d}\tilde{\Phi}^\lambda(y)\right). \tag{5.2}$$

### 5.2 Evaluation of the integral

$$\int_Y \chi(y)\, y_p^r\,\mathrm{d}\tilde{\mu}^\lambda(y;f_1\otimes f_2\otimes f_3) \tag{5.3}$$

for $r\in\mathbb{N}$, $0\leq r\leq k-2$. The result is given in terms of Garrett's triple $L$ function $\mathcal{D}^\star(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho, 2k-2-r,\psi_1\psi_2\chi)$. Let us use the action of the involution $W_{N_j}=\begin{pmatrix}0 & -1\\ N_j & 0\end{pmatrix}$ of the exact level $N_j$ of $f_j$:

$$f_j\big|_k W_{N_j} = \begin{pmatrix}0 & -1\\ N_j & 0\end{pmatrix} = \gamma_j\cdot f_j^\rho, \qquad f_j^\rho\big|_k W_{N_j} = \begin{pmatrix}0 & -1\\ N_j & 0\end{pmatrix} = \bar{\gamma}_j\cdot f_j,$$

$$\text{where } f_j^\rho(z) = \sum_{n=1}^\infty \bar{a}_{n,j}e(nz)\in\mathcal{S}_k(N_j,\bar{\psi}_j), \tag{5.4}$$

$$(j=1,2,3) \text{ and } \gamma_j \text{ is the corresponding root number.} \tag{5.5}$$

Recall the notation (0.9) and (0.10):

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j|V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

$$f_{j,0}^{\rho} = \sum_{n=1}^{\infty} \overline{a(n, f_0)} q^n, \ f_j^0 = f_{j,0}^{\rho}|_k \ W_{Np} = f_{j,0}^{\rho}\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}.$$

Proposition 5.1 *Under the notations and assumptions as in Theorem B.2, the value of the integral (5.3) is given for $0 \le r \le k-2$ by the image under $i_p$ of the following algebraic number*

$$T \cdot \lambda^{-2v} \mathfrak{L}_{Np}(-r) \frac{\mathcal{D}^{\star}(f_1^{\rho} \otimes f_2^{\rho} \otimes f_3^{\rho}, 2k-2-r, \psi_1\psi_2\chi)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0} \otimes \rangle_{T,N^2p^{2v}}},$$

*where*

$$T = 2^{-r} \frac{((Np)^3/N_1N_2N_3)^{k/2} \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 (\chi_1\chi_2\chi_3)(2) p^{3\cdot v(k-2)}}{N_{1,1}N_{1,2}N_{1,3} G(\chi_{1,0}) G(\chi_{2,0}) G(\chi_{3,0})} \times$$
$$\times (Np^{2v})^{k-2r} \frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}) : \Gamma(N^2p^{2v})]^3}.$$

*$\gamma_j$ is the corresponding root number, given by (5.4), and the factor $\mathfrak{L}_{Np}(-r)$, given by (5.13).*

Remark. In particular, Propostion 5.1 implies Theorem A, using a computation by B.Gorsse and G.Robert (see [Go-Ro]) that for some $\beta \in \overline{\mathbb{Q}}^*$

$$\langle f_1^{0,\rho} \otimes f_2^{0,\rho} \otimes f_3^{0,\rho}, f_{1,0}^{\rho} \otimes f_{2,0}^{\rho} \otimes f_{3,0}^{\rho} \rangle_{T,Np} = \beta \cdot \langle f_1, f_1 \rangle_N \langle f_2, f_2 \rangle_N \langle f_3, f_3 \rangle_N.$$

## 5.3 Evaluation of the trilinear form

In order to compute the $p$-adic integral, the next step of the proof uses computations similar to those in [Hi85], §4 and §7. More precisely let us write the integral in the form

$$\int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}_\lambda(y; f_1 \otimes f_2 \otimes f_3) = \sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, \mathrm{d}\ell_{T,\lambda}(\tilde{\Phi}^\lambda)(y)) =$$
$$= \ell_{T,\lambda} \left( \sum_{a \in Y_v} \chi(a) \int_{(a)_v} y_p^r \, \mathrm{d}\tilde{\Phi}^\lambda(y) \right) = \ell_{T,\lambda} \left( \sum_{a \in Y_v} \chi(a) \Phi_r^\lambda((a)_v) \right), \tag{5.6}$$

where $(a)_v = (a + (Np^v)) \subset Y$, and by definition (5.1)

$$\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3) = \ell_{T,\lambda}(\tilde{\Phi}^\lambda)(y), \tag{5.7}$$

$$\int_{(a)_v} y_p^r \, \mathrm{d}(\tilde{\Phi}^\lambda) = \Phi_r^\lambda((a)_v) \in \mathcal{M}_T^\lambda(Np) \tag{5.8}$$

for $r = 0, 1, \dots, h-1$. Moreover $\Phi_r\big((a)_v\big)$ is a triple modular form given by (1.2) of level $N^2p^{2v}$ as a value of a higher twist of a Siegel-Eisenstein distributions, hence

$$\Phi_r^\lambda(\chi) = U_T^{-2v}\left[\pi_{\lambda,T,1}U_T^{2v}\left(2^r F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3}\circ \mathrm{diag}\right)\right]. \tag{5.9}$$

Taking into account the equalities (5.9), the integral (5.6) transforms to the following

$$\int_Y \chi(y)\, y_p^r \,\mathrm{d}\tilde\mu^\lambda(y; f_1\otimes f_2\otimes f_3) = \ell_{T,\lambda}\left(\sum_{a\in Y_v}\chi(a)\Phi_r^\lambda((a)_v)\right) \tag{5.10}$$
$$= \ell_{T,\lambda}\Big(U_T^{-2v}\Big[\pi_{\lambda,T,1}U_T^{2v}\big(2^r F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3}\circ\mathrm{diag}\big)\Big]\Big)$$

Notice that then it follows that the sum in the right hand side of the equality (5.10) can be expressed through the functions (1.2):

$$\int_Y \chi(y)\, y_p^r\,\mathrm{d}\tilde\mu^\lambda(y; f_1\otimes f_2\otimes f_3)(y)$$
$$\ell_{T,\lambda}\left(U_T^{-2v}\Big[\pi_{\lambda,T,1}U_T^{2v}\big(2^r F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3}\circ\mathrm{diag}\big)\Big]\right) \tag{5.11}$$

where we use the functions (1.2). The function

$$g = \Phi_r(\chi) = 2^r F_{\chi,r}^{\bar\chi_1,\bar\chi_2,\bar\chi_3}\circ\mathrm{diag}$$

is computed in (B.5), Appendix B as follows:

$$\mathcal{E}(z_1,z_2,z_3;-r,k,Np^v,\boldsymbol{\psi},\chi_1,\chi_2,\chi_3)$$
$$= N_{1,1}N_{1,2}N_{1,3}(\bar\chi_1\bar\chi_2\bar\chi_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})2^{-r}\Phi_r(\chi),$$

thus it is a *nearly-holomorphic* triple modular form in in the $\mathbb{Q}^{\mathrm{ab}}$-module

$$M(\mathbb{Q}^{\mathrm{ab}}) = \mathcal{M}_T(N^2p^{2v},\psi_1\otimes\psi_2\otimes\psi_3;\mathbb{Q}^{\mathrm{ab}})$$
$$\subset \mathcal{M}_{k,r}(N^2p^{2v},\psi_1;\mathbb{Q}^{\mathrm{ab}})\otimes\mathcal{M}_{k,r}(N^2p^{2v},\psi_2;\mathbb{Q}^{\mathrm{ab}})\otimes\mathcal{M}_{k,r}(N^2p^{2v},\psi_3;\mathbb{Q}^{\mathrm{ab}}).$$

Then by the general formula of Proposition 4.2 e) we have:

$$\mathcal{L}_{T,\lambda}:\mathcal{M}_T(N^2p^{2v};\mathbb{C})\to\mathbb{C},\quad g\mapsto \frac{\langle f_1^0\otimes f_2^0\otimes f_3^0, \lambda^{-2v}U_T^{2v}g\rangle_{T,N^2p}}{\langle f_1^0\otimes f_2^0\otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{T,N^2p}}, \tag{5.12}$$

$$\ell_{T,\lambda}\left(U_T^{-2v}\big[\pi_{\lambda,T,1}U_T^{2v}(g)\big]\right) = i_p\left(\frac{\langle f_1^0\otimes f_2^0\otimes f_3^0, \lambda^{-2v}U_T^{2v}(g)\rangle_{T,N^2p}}{\langle f_1^0\otimes f_2^0\otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{N^2p}}\right)$$
$$= i_p\left(\lambda^{-2v}p^{3\cdot 2v(k-1)}\cdot\frac{\langle V^{2v}(f_1^0\otimes f_2^0\otimes f_3^0), g\rangle_{T,N^2p^{2v+1}}}{\langle f_1^0\otimes f_2^0\otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{T,N^2p}}\right).$$

The scalar products in 5.12 can be computed using Theorem B.2, but we omit here the details. This implies Proposition 5.1 using the integral representation of Theorem B.2 for modular forms $\tilde{f}_{j,2v}(z) = \sum_{n=1}^{\infty} a_{j,n,2v} e(nz)$ as above:

$$\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi_1) \tag{5.13}$$

$$(Np^{2v})^{k-2r} \frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}):\Gamma(N^2p^{2v})]^3} \times \mathfrak{L}_{Np}(-r)$$

$$= \left\langle \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v}, \mathcal{E}(z_1, z_2, z_3; -r, k, N^2p^{2v}, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3) \right\rangle_{T, N^2p^{2v}},$$

where

$$\mathfrak{L}_{Np}(s) = \mathfrak{L}_{Np}(s; \tilde{f}_{1,2v} \otimes \tilde{f}_{2,2v} \otimes \tilde{f}_{3,2v}) := \sum_{n|N^\infty} G_N(\overline{\psi_1\psi_2\chi_1}, 2n) \frac{a_{n,1,2v}a_{n,2,2v}a_{n,3,2v}}{n^{2s+2k-2}}.$$

## 5.4 Proof of Theorem B

Let us use Propostion 5.1 and (5.13):

$$2^{-r}\int_Y \chi(y)\, y_p^r \, \mathrm{d}\tilde{\mu}^\lambda(y; f_1 \otimes f_2 \otimes f_3)(y) = 2^{-r}\ell_{T,\lambda}\left(U_T^{-2v}\left[\pi_{\lambda,T,1}U_T^{2v}(g)\right]\right) \tag{5.14}$$

$$= \frac{((Np)^3/N_1N_2N_3)^{k/2}\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3(\chi_1\chi_2\chi_3)(2)p^{3\cdot v(k-2)}}{\lambda^{2v}N_{1,1}N_{2,1}N_{3,1}G(\chi_{1,0})G(\chi_{2,0})G(\chi_{3,0})} \times$$

$$\times (Np^{2v})^{k-2r} \frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}):\Gamma(N^2p^{2v})]^3} \mathfrak{L}_{Np}(-r) \times$$

$$\times \frac{\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi_1)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,N^2p}}$$

Let us show that under the assumptions as above there exist an admissible $\mathbb{C}_p$-valued measure $\tilde{\mu}^\lambda_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p,$$

given for all $x \in X_{N,p}$ by the integral

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde{\mu}^\lambda_{f_1\otimes f_2\otimes f_3}(y),$$

and having the following properties: for all pairs $(r, \chi)$ such that for $\chi \in X_p^{\mathrm{tors}}$ the corresponding Dirichlet characters $\chi_j$ are $Np$-complete, and $r \in \mathbb{Z}$ with $0 \le r \le k-2$, the following equality holds:

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = \tag{5.15}$$

$$i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(2k-3-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda(p)^{2v}}$$

$$\frac{\mathcal{D}^\star(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, 2k-2-r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T}\Big)$$

where $v = \mathrm{ord}_p(C_\chi)$, $\chi_1 \bmod Np^v = \chi$, $\chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi$, $\chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi$, $G(\chi)$ denotes the Gauß sum of a primitive Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$).
Indeed, we may write

$$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = C \cdot x(2) \int_Y x(y) \mathrm{d}\tilde\mu^\lambda(y; f_1 \otimes f_2 \otimes f_3)$$

with an appropriate constant, given by the RHS of (5.14), where $v = \mathrm{ord}_p(C_\chi)$. Moreover, it follows from the properties of the constructed measure

$$\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}(y) := C \cdot \tilde\mu_\lambda(2^{-1}y; f_1 \otimes f_2 \otimes f_3)$$

that

(ii) if $\mathrm{ord}_p\lambda(p) = 0$ then the holomorphic functions in (i), (ii) are bounded $\mathbb{C}_p$-analytic functions: it suffices to use the equality (2.5) with $r = 0$ in order to show that in this case the measure $\tilde\Phi^\lambda$ is bounded because of $|\lambda(p)|_p = 1$);

(iii) in the general case (but assuming that $\lambda(p) \ne 0$) the holomorphic functions in (i) belong to the type $o(\log(x_p^h))$ with $h = [2\mathrm{ord}_p\lambda(p)]+1$ and they can be represented as the Mellin transform of the $h$-admissible measure $\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}$ (in the sense of Amice-Vélu);

(iv) if $h = [2\mathrm{ord}_p\lambda] + 1 \le k-2$ then the function $\mathcal{D}_{(p)}$ is uniquely determined by the above conditions (i). ∎

# A Nearly-holomorphic Siegel-Eisenstein series

## A.1 Fourier expansions of Siegel-Eisenstein series

In this section $\chi$ denotes a Dirichlet character modulo an *arbitrary integer* $N$ (not to be confused with $N$ in the Introduction). We recall some standard facts about the Fourier expansions of the Siegel-Eisenstein series defined by:

$$\begin{aligned} E(\mathcal{Z}, s; k, \chi, N) &= E(\mathcal{Z}, s) \qquad\qquad (A.1)\\ &= \det(y)^s \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\gamma)) j(\gamma, \mathcal{Z})^{-k} |j(\gamma, \mathcal{Z})|^{-2s}, \end{aligned}$$

for $k + 2\mathrm{Re}(s) > m + 1$, $s \in \mathbb{C}$, $k \in \mathbb{Z}$, and by analytic continuation over $s$ for other values of $s \in \mathbb{C}$ (see [Sh83]). It is assumed in the identity (A.1) that $N > 1$, $\chi$ is a Dirichlet character mod $N$ (not necessarily primitive, e.g. trivial modulo $N > 1$), and

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma = \Gamma_0^m(N) \subset \Gamma^m = \mathrm{Sp}(m, \mathbb{Z}).$$

Recall an explicit computation of the Fourier expansion of the series

$$E^\star(\mathcal{Z}, s) = E^\star(\mathcal{Z}, s; k, \chi, N) := E(-\mathcal{Z}^{-1}, s) \det(\mathcal{Z})^{-k}, \qquad (A.2)$$

obtained from (A.1) by applying the involution

$$J_m = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

Note that for $k > m + 1$ and $N = 1$ both series coincide and were studied by Siegel:

$$E(\mathcal{Z}) = E_k^m(\mathcal{Z}) = E(\mathcal{Z}, 0) = E^\star(\mathcal{Z}, 0).$$

The detailed study of the series $E^\star(\mathcal{Z}, s; k, \chi, N)$ was made by G. Shimura [Sh83] and P. Feit ([Fei86], §10).

On the other hand, it is convenient to use the following notation. Let $\phi$ be a Dirichlet character mod $Q > 1$ and consider the Eisenstein series of degree $m \geq 1$

$$\begin{aligned} F_{\alpha,\beta}(\mathcal{Z}, Q, \phi) &:= \det(y)^\beta \sum_{c,d} \phi(\det c) \det(c\mathcal{Z} + d)^{-\alpha,-\beta} \qquad (A.3)\\ &= \det(y)^\beta \sum_{c,d} \phi(\det c) \det(c\mathcal{Z} + d)^{-\alpha} \det(c\overline{\mathcal{Z}} + d)^{-\beta} \\ &= \det(y)^\beta \sum_{c,d} \phi(\det c) \det(c\mathcal{Z} + d)^{\beta-\alpha} |\det(c\mathcal{Z} + d)|^{-2\beta} \qquad (A.4) \end{aligned}$$

where $(c, d)$ runs over all "non-associated coprime symmetric pairs" with $\det(c)$ coprime to $Q$. A more conceptual description would be to sum over

$\mathcal{T}^m(Q)_\infty\backslash\mathcal{T}^m(Q)$, where

$$\mathcal{T}^m(Q)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{Sp}(m,\mathbb{Z})\Big| A\equiv 0 \bmod Q\right\}=\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}\Gamma_0^m(Q)$$

$$\mathcal{T}^m(Q)_\infty=\left(\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}\Gamma_0^m(Q)\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}^{-1}\right)_\infty$$

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{Sp}(m,\mathbb{Z})\Big| c=0, b\equiv 0 \bmod Q\right\}\subset\Gamma^{m,0}(Q)\subset \mathrm{Sp}(m,\mathbb{Z}),$$

where $\Gamma^{m,0}(Q)=\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}\Gamma_0^m(Q)\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}^{-1}\subset \mathrm{Sp}(m,\mathbb{Z})$ is the stabilizer of $\mathcal{M}=\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}\Gamma_0^m(Q)$, and more generally, for any set $\mathcal{M}\subset \mathrm{Sp}(m,\mathbb{Z})$ of symplectic matrices we denote by $\mathcal{M}_\infty$ the set of those matrices $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{Sp}(m,\mathbb{Z})$ satisfying the conditions $c=0$ and $\gamma\mathcal{M}\subset\mathcal{M}$.

Action of $\sigma\in \mathrm{Sp}(m,\mathbb{Z})$ on the Eisenstein series

Note that for any $\sigma\in \mathrm{Sp}(m,\mathbb{Z})$ one has

$$\begin{aligned}E(\mathcal{Z},s;k,\chi,N)|_k\sigma &= \sum_{\gamma\in\Gamma_0^m(N)_\infty\backslash\Gamma_0^m(N)}\phi(\det d_\gamma)(1|_k\gamma\sigma)(\mathcal{Z})(\mathrm{Im}\,(\gamma\sigma(\mathcal{Z}))^s\\ &=\det(y)^s\sum_{\gamma\in\Gamma_0^m(N)_\infty\backslash\Gamma_0^m(N)}\phi(\det d_\gamma)j(\gamma\sigma,\mathcal{Z})^{-k}|j(\gamma\sigma,\mathcal{Z})|^{-2s}\\ &=\det(y)^s\sum_{\tilde\gamma\in(\Gamma_0^m(N))_\infty\backslash\Gamma_0^m(N)\sigma}\phi(\det d_{\sigma^{-1}\tilde\gamma})j(\tilde\gamma,\mathcal{Z})^{-k}|j(\tilde\gamma,\mathcal{Z})|^{-2s},\end{aligned}$$

by writing $\tilde\gamma=\sigma\gamma$, $\sigma^{-1}\tilde\gamma=\gamma$: $P\gamma_1=P\gamma_2\Longleftrightarrow P\tilde\gamma_1=P\tilde\gamma_2$.
In particular, for $\sigma=J_m=\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}$ one has $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)J_m=\left(\begin{smallmatrix} b & -a\\ d & -c\end{smallmatrix}\right)\in\Gamma_0^m(N)J_m$, hence

$$\begin{aligned}&E(\mathcal{Z},s;k,\chi,N)|\begin{pmatrix} 0_m & -1_m\\ 1_m & 0_m\end{pmatrix}=E^\star(\mathcal{Z},s;k,\chi,N)\\ &=\det(y)^s\sum_{\left(\begin{smallmatrix} b & -a\\ d & -c\end{smallmatrix}\right)\in(\Gamma_0^m(N))_\infty\backslash\Gamma_0^m(N)\sigma}\chi(\det d)\det(d\mathcal{Z}-c)^{-k}|\det(d\mathcal{Z}-c)|^{-2s}.\end{aligned}$$

Notice that $J_m(N)\Gamma_0^m(N)=\Gamma_0^m(N)J_m(N)$, where $J_m(N)=\begin{pmatrix} 0_m & -1_m\\ N\cdot 1_m & 0_m\end{pmatrix}$, and

$$J_m(N)\begin{pmatrix} a\ b\\ c\ d\end{pmatrix}=\begin{pmatrix} -c\ -d\\ Na\ Nb\end{pmatrix}=\begin{pmatrix} a_1\ b_1\\ c_1\ d_1\end{pmatrix}J_m(N)=\begin{pmatrix} Nb_1\ -a_1\\ Nd_1\ -c_1\end{pmatrix}.$$

Therefore $(Nd_1, -c_1) = (Na, Nb)$, and $(a, b)$ runs over all "non-associated coprime symmetric pairs" with $\det(a)$ coprime to $N$. We may therefore write $(Nd_1, -c_1) = (Na, Nb)$, and

$$E^\star(N\mathcal{Z}, s; k, \chi, N) \tag{A.5}$$
$$= \det(Ny)^s \sum_{\left(\begin{smallmatrix} b_1 & -a_1 \\ d_1 & -c_1 \end{smallmatrix}\right) \in (\Gamma_0^m(N))_\infty \backslash \Gamma_0^m(N)\sigma} \chi(\det d_1) \det(d_1 N\mathcal{Z} - c_1)^{-k} |\det(d_1 N\mathcal{Z} - c_1)|^{-2s}$$
$$= N^{-m(k+s)} \det(y)^s \sum_{a,b} \chi(\det a) \det(a\mathcal{Z} + b)^{-k-s,-s} \tag{A.6}$$
$$= N^{-m(k+s)} F_{k+s,s}(\mathcal{Z}, N, \chi) \tag{A.7}$$

## A.2 Arithmetical variables of nearly-holomorphic Siegel modular forms and differential operators

Consider a commutative ring $A$, the formal variables $q = (q_{i,j})_{i,j=1,\dots,m}$, $R = (R_{i,j})_{i,j=1,\dots,m}$, and the ring of *formal arithmetical Fourier series*

$$A[\![q^{B_m}]\!][R_{i,j}] = \left\{ f = \sum_{\mathcal{T} \in B_m} a(\mathcal{T}, R) q^{\mathcal{T}} \;\middle|\; a(\mathcal{T}, R) \in A[R_{i,j}] \right\} \tag{A.8}$$

using the semi-group

$$B_m = \left\{ \mathcal{T} = (\mathcal{T}_{ij}) \in \mathrm{M}_m(\mathbb{R}) \;\mid\; \mathcal{T} = {}^t\mathcal{T}, \mathcal{T} \geq 0, \mathcal{T}_{ij}, 2\mathcal{T}_{ii} \in \mathbb{Z} \right\}$$

and the symbols

$$q^{\mathcal{T}} = \prod_{i=1}^{m} q_{ii}^{\mathcal{T}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{T}_{ij}} \subset A[\![q_{11}, \dots, q_{mm}]\!][q_{ij}, q_{ij}^{-1}]_{i,j=1,\cdots,m}$$

(over the complex numbers this notation corresponds to $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}\mathcal{Z}))$, $R = (4\pi \mathrm{Im}(\mathcal{Z}))^{-1}$).

The formal Fourier expansion of a nearly-holomorphic Siegel modular form $f$ with coefficients in $A$ is an element of $A[\![q^{B_m}]\!][R_{i,j}]$. Let

$$\mathcal{M}_k^m(N, \psi) \subset \tilde{\mathcal{M}}_k^m(N, \psi) \subset \mathcal{M}_k^m(N, \psi)^\infty$$

denote the complex vector spaces of holomorphic, nearly-holomorphic, and $\mathcal{C}^\infty$-Siegel modular forms of weight $k$ and character $\psi$ for $\Gamma_0^m(N)$, see [ShiAr], [CourPa] so that $\mathcal{M}_k^m(N, \psi) \subset \mathbb{C}[\![q^{B_m}]\!]$, $\tilde{\mathcal{M}}_k^m(N, \psi) \subset \mathbb{C}[\![q^{B_m}]\!][R_{i,j}]$, and $\mathcal{M}_k^m(N, \psi)^\infty \subset \mathcal{C}^\infty(\mathbb{H}_m)$.

### A.3 Formal Fourier expansions of nearly-holomorphic Siegel-Eisenstein series

In the Siegel modular case $\Gamma^m = \mathrm{Sp}_{2m}(\mathbb{Z}) \supset \Gamma_0^m(N)$ the series

$$\begin{aligned} &E(\mathcal{Z}, s; k, \chi, N) = E(\mathcal{Z}, s) &\text{(A.9)}\\ &= \det(y)^s \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\gamma)) j(\gamma, \mathcal{Z})^{-k} |j(\gamma, \mathcal{Z})|^{-2s} \in \mathcal{M}_k^\infty(\Gamma_0(N), \bar\chi) \end{aligned}$$

is absolutely convergent for $k + 2\mathrm{Re}(s) > m+1$, but can be continued to all $s \in \mathbb{C}$. However, for $N > 1$, the Fourier expansion is known only for *the involuted series* $E(\cdot, s)|W(N)$, where $W(N) = \begin{pmatrix} 0_m & -1_m \\ N\cdot 1_m & 0_m \end{pmatrix}$, and for some critical values $s \in \mathbb{Z}$ (for $N = 1$ both series coincide). Here $\mathcal{Z} \in \mathbb{H}_m$ is in the Siegel upper half-space:

$$\mathbb{H}_m = \left\{ \mathcal{Z} = {}^t\mathcal{Z} \in \mathrm{M}_m(\mathbb{C}) | \operatorname{Im} \mathcal{Z} > 0 \right\}, \quad \text{and} \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in Sp_{2m}(\mathbb{R}) \right\}$$

is the Siegel parabolic subgroup.

Example A.1 (Involuted Siegel-Eisenstein series) *Let $\chi$ be a Dirichlet character modulo $N$. Recall that by (A.5)*

$$\begin{aligned} &E^\star(N\mathcal{Z}, s; k, \chi, N) = N^{-m(k+s)} F_{k+s,s}(\mathcal{Z}, N, \chi) &\text{(A.10)}\\ &= N^{-m(k+s)} \det(y)^s \sum_{a,b} \chi(\det a) \det(a\mathcal{Z} + b)^{-k-s,-s}, \ \textit{where} \end{aligned}$$

$$E^\star(N\mathcal{Z}, s) = E(-(N\mathcal{Z})^{-1}, s) \det(N\mathcal{Z})^{-k} = N^{-km/2} E|W(N), \qquad \text{(A.11)}$$

$$\begin{aligned} &G^\star(\mathcal{Z}, s) = G^*(\mathcal{Z}, s; k, \chi, N) = N^{m(k+s)} E^*(N\mathcal{Z}, s)\cdot &\text{(A.12)}\\ &\cdot \tilde\Gamma(k, s) L_N(k+2s, \chi) \left( \prod_{i=1}^{[m/2]} L_N(2k+4s-2i, \chi^2) \right) \end{aligned}$$

*$\kappa = (m+1)/2$, and for $m$ odd the $\Gamma$-factor has the form:*

$$\tilde\Gamma(k, s) = i^{mk} 2^{-m(k+1)} \pi^{-m(s+k)} \Gamma_m(k+s),$$
$$\textit{where } \Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2))).$$

In order to describe the formal Fourier expansions explicitly let us consider the Maass differential operator $\Delta_m$, acting on $\mathcal{C}^\infty$-functions over $V \otimes \mathbb{C}$ of degree $m$, which is defined by the equality:

$$\Delta_m = \det(\partial_{ij}), \qquad \partial_{ij} = 2^{-1}(1 + \delta_{ij}) \partial/\partial_{ij}. \qquad \text{(A.13)}$$

For an integer $n \geq 0$ and a complex number $\beta$ consider the polynomial

$$R_m(\mathcal{Z}; n, \beta) = (-1)^{mn} e^{\mathrm{tr}(\mathcal{Z})} \det(\mathcal{Z})^{n+\beta} \Delta_m^n \left[ e^{-\mathrm{tr}(\mathcal{Z})} \det(\mathcal{Z})^{-\beta} \right], \qquad \text{(A.14)}$$

with $\mathcal{Z} \in V \otimes \mathbb{C}$, where the exponentiation is well defined by

$$\det(y)^\beta = \exp(\beta \log[\det(y)]),$$

for $\det(y) > 0$, $y \in Y \otimes \mathbb{C}$. According to definition (A.14) the degree of the polynomial $R_m(\mathcal{Z}; n, \beta)$ is equal to $mn$ and the term of the highest degree coincides with $\det(\mathcal{Z})^n$. We have also that for $\beta \in \mathbb{Q}$ the polynomial $R_m(\mathcal{Z}; n, \beta)$ has rational coefficients.

Theorem A.2 *Let $m$ be an odd integer such that $2k > m$, and $N > 1$ be an integer, then:*
*For an integer $s$ such that $s = -r \leq 0$, $0 \leq r \leq k - \kappa$, there is the following Fourier expansion*

$$G^\star(\mathcal{Z}, -r) = G^\star(\mathcal{Z}, -r; k, \chi, N) = \sum_{A_m \ni \mathcal{T} \geq 0} b^\star(\mathcal{T}, y, -r) q^{\mathcal{T}} = \sum_{A_m \ni \mathcal{T} \geq 0} a(\mathcal{T}, R) q^{\mathcal{T}}, \qquad \text{(A.15)}$$

*where for $s > (m + 2 - 2k)/4$ in (A.15) the only non-zero terms occur for positive definite $\mathcal{T} > 0$, and for all $s = -r$ with $0 \leq r \leq k - \kappa$, and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, where*

$$\begin{aligned} b^\star(\mathcal{T}, y, -r) &= a(\mathcal{T}, R) = W^\star(y, \mathcal{T}, -r) M(\mathcal{T}, \chi, k - 2r), \qquad \text{(A.16)} \\ W^\star(y, \mathcal{T}, -r) &= 2^{-m\kappa} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k - 2r, r). \end{aligned}$$

*Here $a(\mathcal{T}, R) = a(\mathcal{T}, R; r, N, \chi)$ is a homogeneous polynomial with rational coefficients in the variables $R_{ij}$ and $\mathcal{T}_{ij}$, and*

$$M(\mathcal{T}, k - 2r, \chi) = \prod_{\ell | \det(2\mathcal{T})} M_\ell(\mathcal{T}, \chi(\ell)\ell^{-k+2r}) \qquad \text{(A.17)}$$

*is a finite Euler product, in which $M_\ell(\mathcal{T}, x) \in \mathbb{Z}[x]$; we use the notation $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}\mathcal{Z}))$, $R = (4\pi \mathrm{Im}(\mathcal{Z}))^{-1}$ as above, and polynomials $Q(R, \mathcal{T}; k - 2r, r)$ are given by (3.1).*

*Proof*: see [Sh83], [Fei86], Theorem 2.14 and formulas (2.137) in [CourPa]. The use of definitions gives

$$W^\star(y, \mathcal{T}, -r) = 2^{-m\kappa} \det(\mathcal{T})^{k-2r-\kappa} \det(4\pi y)^{-r} R_m(4\pi \mathcal{T} y; r, \kappa - k + r)$$

where $R_m(y; n, \beta)$ is defined by (A.14). Moreover, let us use the polynomials (3.1):

$$Q(R, \mathcal{T}; k - 2r, r) \det(\mathcal{T})^{-r} = \det(4\pi \mathcal{T} y)^{-r} R_m(4\pi \mathcal{T} y; r, \kappa - k + r),$$

it follows

$$\begin{aligned} W^\star(y, \mathcal{T}, -r) &= 2^{-m\kappa} \det(\mathcal{T})^{k-2r-\kappa} \det(4\pi y)^{-r} R_m(4\pi \mathcal{T} y; r, \kappa - k + r) \\ &= 2^{-m\kappa} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k - 2r, r). \quad \blacksquare \end{aligned}$$

# B An integral representation for the triple product

## B.1 Summary of analytic results

In this section we use the following data :

- Three equal weights $k = k_1 = k_2 = k_3$
- Three Dirichlet characters $\bmod N_j$ with $\psi_j(-1) = (-1)^k$
- Three cusp forms $\tilde{f}_j(z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j} e(nz) \in \mathcal{S}_k(\tilde{N}_j, \psi_j)$, $(j = 1, 2, 3)$ with $N_j | \tilde{N}_j$, assumed to be eigenforms for all Hecke operators $T_q$, with $q$ prime to $N$. In our construction we use as $\tilde{f}_j$ some "easy transforms" of primitive cusp forms $f_j \in \mathcal{S}_k(N_j, \psi_j)$ in the Introduction, so that they have the same eigenvalues for all Hecke operators $T_q$, for $q$ prime to $N$. For example, $\tilde{f}_j$ could be chosen as eigenfunctions $\tilde{f}_j = f_j^0$ of the conjugate Atkin's operator $U_p^*$ given by (0.10), in this case we denote by $f_{j,0}$ the corresponding eigenfunctions of $U_p$.
- Assume that $\tilde{N} | Np^v$, where $\tilde{N} := \mathrm{LCM}\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3\}$
- Consider a non necessary primitive Dirichlet character $\chi \bmod Np^v$, and the Dirichlet characters as in (0.12).

Using the notation $z_j = x_j + iy_j \in \mathbb{H}$, one associates to this data the following function

$$\mathcal{E}(z_1, z_2, z_3) = \mathcal{E}(z_1, z_2, z_3; s, k, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3) := \tag{B.1}$$

$$i^{3k} 2^{-3(k+1)-2s-2k+2} \pi^{3(s+k)+2} \Gamma(2s + 2k - 1)\Gamma(s + k - 1)\times$$

$$\times L^{(Np)}(k + 2s, \boldsymbol{\psi}) L^{(Np)}(4s + 2k - 2, \boldsymbol{\psi}^2) \sum_{\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23} \bmod Np^v} \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23})$$

$$F_{k+s,s}(\star, N^2p^{2v}, \boldsymbol{\psi})\Bigg| \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\varepsilon_{12}}{Np^v} & \frac{\varepsilon_{13}}{Np^v} \\ & 1 & 0 & \frac{\varepsilon_{12}}{Np^v} & 0 & \frac{\varepsilon_{23}}{Np^v} \\ & & 1 & \frac{\varepsilon_{13}}{Np^v} & \frac{\varepsilon_{23}}{Np^v} & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} (z_1, z_2, z_3) y_1^s y_2^s y_3^s.$$

Note that the product of the normalizing Gamma-factor and of the two Dirichlet $L$-functions come from the definitions (A.11) and (A.10) of the Siegel-Eisenstein series.

### B.2 Fourier expansion of the Eisenstein series (B.1)

Consider again the Dirichlet characters (0.12), and the corresponding function (B.1) of level $Np^v$.
We wish to express the series (B.1), evaluated at $s = -r$, through the series (1.2) in the case of $Np$-complete conductors.

Proposition B.1 *For* $F(\mathcal{Z}) = \sum_{\mathcal{T}} a(\mathcal{T}, R)q^{\mathcal{T}}$ *one has* $F^{\phi}(\mathcal{Z}) = \sum_{\mathcal{T}} g_t(\phi, \mathcal{T})a(\mathcal{T}, R)q^{\mathcal{T}}$, *where* $\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix}$, $\phi(\boldsymbol{\varepsilon}) = \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23})$, $\mathcal{T}$ *denotes the (half integral) block matrix and*

$$g_t(\phi, \mathcal{T}) = \sum_{\boldsymbol{\varepsilon} \in S_{N,p}/Np^v S_{N,p}} \phi(\boldsymbol{\varepsilon}) \exp(2\pi i \mathrm{tr}(\frac{1}{Np^v}\mathcal{T}\boldsymbol{\varepsilon})), \text{ where } \phi(\boldsymbol{\varepsilon}) = \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23}).$$

*Proof.* Indeed,
$F|t_{\boldsymbol{\varepsilon},Np^v} = \sum_{\mathcal{T}} a(\mathcal{T}, R)q^{\mathcal{T}}|t_{\boldsymbol{\varepsilon},Np^v} = \sum_{\mathcal{T}} \exp(2\pi i \mathrm{tr}(\boldsymbol{\varepsilon}\mathcal{T})/Np^v)a(\mathcal{T}, R)q^{\mathcal{T}}$, and it suffices to notice again that

$$\mathrm{tr}(\boldsymbol{\varepsilon}\mathcal{T}) = \mathrm{tr}\left(\begin{pmatrix} 0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}\right) = 2(\varepsilon_{12}t_{12} + \varepsilon_{13}t_{13} + \varepsilon_{23}t_{23}).$$

∎

Using this formula for $F = G^{\star}(\mathcal{Z}, s; k-2r, (Np^v)^2, \boldsymbol{\psi})$ at $s = -r$ (see (A.3)), gives:

$$\mathcal{E}(z_1, z_2, z_3; -r, k, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3) = \tag{B.2}$$

$$\sum_{\boldsymbol{\varepsilon} \in S/Np^v S} \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23}) G^{\star}(\mathcal{Z}, -r; k-2r, (Np^v)^2, \boldsymbol{\psi})|t_{\boldsymbol{\varepsilon},Np^v}(z_1, z_2, z_3)$$

$$= \left(\sum_{\mathcal{T}} \sum_{\boldsymbol{\varepsilon} \in S/Np^v S} \chi_1(\varepsilon_{12})\chi_2(\varepsilon_{13})\chi_3(\varepsilon_{23}) \exp(2\pi i \mathrm{tr}(\boldsymbol{\varepsilon}\mathcal{T})/Np^v)a(\mathcal{T}, R)q^{\mathcal{T}}\right) \circ \mathrm{diag}$$

then the sum over $\boldsymbol{\varepsilon} \in S/Np^v S$ transforms simply to the product

$$G_{Np^v}(\chi_1, 2t_{12})G_{Np^v}(\chi_2, 2t_{13})G_{Np^v}(\chi_3, 2t_{23}),$$

which is easily evaluated by the general formula for a generalized Gauss sum $G_N(\chi, c) = \sum_{b \bmod N} \chi(b)e(bcN^{-1})$. This last sum admits the following known expression in terms of the usual Gauss sums (see for example [PaTV], Section

2, (2.20)): let $\chi_0$ denote the primitive Dirichlet character modulo $N_0$ associated with $\chi$, $N_1 = NN_0^{-1}$, then

$$G_N(\chi, c) = G(\chi_0)N_1 \sum_{d|N_1} \mu(d)\chi_0(d)d^{-1}\delta\Big(\frac{c}{N_1 d^{-1}}\Big)\bar{\chi}_0\Big(\frac{c}{N_1 d^{-1}}\Big).$$

Writing $\chi_{0,j}$ for the primitive Dirichlet character modulo $N_{0,j}$ associated with $\chi_j \bmod Np^v$, and using the notation $Np^v = N_{0,j}N_{1,j}$, gives

$$\begin{aligned}
&G_{Np^v}(\chi_1, 2t_{12})\\
&= G(\chi_{0,1})N_{1,1} \sum_{d_1|N_{1,1}} \mu(d_1)\chi_{0,1}(d_1)d_1^{-1}\delta\Big(\frac{2t_{12}}{N_{1,1}d_1^{-1}}\Big)\bar{\chi}_{0,1}\Big(\frac{2t_{12}}{N_{1,1}d_1^{-1}}\Big)\\
&G_{Np^v}(\chi_2, 2t_{13})\\
&= G(\chi_{0,2})N_{1,2} \sum_{d_2|N_{1,2}} \mu(d_2)\chi_{0,2}(d_2)d_2^{-1}\delta\Big(\frac{2t_{12}}{N_{1,2}d_2^{-1}}\Big)\bar{\chi}_{0,2}\Big(\frac{2t_{13}}{N_{1,2}d_2^{-1}}\Big)\\
&G_{Np^v}(\chi_3, 2t_{23})\\
&= G(\chi_{0,3})N_{1,3} \sum_{d_3|N_{1,3}} \mu(d_3)\chi_{0,3}(d_3)d_3^{-1}\delta\Big(\frac{2t_{23}}{N_{1,3}d_3^{-1}}\Big)\bar{\chi}_{0,3}\Big(\frac{2t_{23}}{N_{1,3}d_3^{-1}}\Big)
\end{aligned}$$

Let us take the product of these expressions using the notation

$$\begin{aligned}
2t'_{12} &= \frac{2t_{12}}{N_{1,1}/d_1}(\mathrm{mod} N_{0,1}d_1),\\
2t'_{13} &= \frac{2t_{13}}{N_{1,2}/d_2}(\mathrm{mod} N_{0,2}d_2),\\
2t'_{23} &= \frac{2t_{23}}{N_{1,3}/d_3}(\mathrm{mod} N_{0,3}d_3)
\end{aligned}$$

It follows

$$\begin{aligned}
&G_{Np^v}(\chi_1, 2t_{12})G_{Np^v}(\chi_2, 2t_{13})G_{Np^v}(\chi_3, 2t_{23})\\
&= N_{1,1}N_{1,2}N_{1,3} \sum_{\substack{d_1|N_{1,1}\\ d_2|N_{1,2}\\ d_3|N_{1,3}}} \mu(d_1)\mu(d_2)\mu(d_3)\chi_{0,1}(d_1)\chi_{0,2}(d_2)\chi_{0,3}(d_3)(d_1d_2d_3)^{-1}\\
&G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})\bar{\chi}_{0,1}(2t'_{12})\bar{\chi}_{0,2}(2t'_{13})\bar{\chi}_{0,3}(2t'_{23}).
\end{aligned}$$

The formula (B.3) transforms to

$$\begin{aligned}
&\mathcal{E}(z_1, z_2, z_3; -r, k, \boldsymbol{\psi}, \chi_1, \chi_2, \chi_3) \qquad\qquad \text{(B.3)}\\
&= \Bigg(\sum_{\mathcal{T}} G_{Np^v}(\chi_1, 2t_{12})G_{Np^v}(\chi_2, 2t_{13})G_{Np^v}(\chi_3, 2t_{23})a(\mathcal{T}, R)q^{\mathcal{T}}\Bigg) \circ \mathrm{diag}
\end{aligned}$$

$$= N_{1,1}N_{1,2}N_{1,3}\sum_{\substack{d_1|N_{1,1}\\ d_2|N_{1,2}\\ d_3|N_{1,3}}} \mu(d_1)\mu(d_2)\mu(d_3)\chi_{0,1}(d_1)\chi_{0,2}(d_2)\chi_{0,3}(d_3)(d_1d_2d_3)^{-1}$$

$$G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})\sum_{\substack{\mathcal{T}:t_{12}=d_1t'_{12},\\ t_{13}=d_2t'_{13},t_{23}=d_3t'_{23},}} \bar{\chi}_{0,1}(2t'_{12})\bar{\chi}_{0,2}(2t'_{13})\bar{\chi}_{0,3}(2t'_{23})a(\mathcal{T},R)q_1^{t_{11}}q_2^{t_{22}}q_3^{t_{33}}.$$

Later on we impose the condition that the conductors of $\chi_{0,1},\chi_{0,2},\chi_{0,3}$ are complete (i.e. have the same prime divisors as those of $Np$), when $\chi_{0,j}(d_j)=0$ unless all $d_j=1$, when $\chi_{0,j}(d_j)=1$. In this *complete* case $\chi_{0,j}(n)=\chi_j(n)$ for all $n\in\mathbb{Z}$, hence the equality (B.3) simplifies to the following:

$$\mathcal{E}(z_1,z_2,z_3;-r,k,\boldsymbol{\psi},\chi_1,\chi_2,\chi_3) \tag{B.4}$$

$$= \left(\sum_{\mathcal{T}} G_{Np^v}(\chi_1,2t_{12})G_{Np^v}(\chi_2,2t_{13})G_{Np^v}(\chi_3,2t_{23})a(\mathcal{T},R)q^{\mathcal{T}}\right)\circ \mathrm{diag}$$

$$= N_{1,1}N_{1,2}N_{1,3}G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})$$

$$\left(\sum_{\mathcal{T}} \bar{\chi}_1(2t_{12})\bar{\chi}_2(2t_{13})\bar{\chi}_3(2t_{23})a(\mathcal{T},R)q^{\mathcal{T}}\right)\circ \mathrm{diag}$$

$$= N_{1,1}N_{1,2}N_{1,3}(\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})$$

$$\left(\sum_{\mathcal{T}} a(\mathcal{T},R)\bar{\chi}_1(t_{12})\bar{\chi}_2(t_{13})\bar{\chi}_3(t_{23})q^{\mathcal{T}}\right)\circ \mathrm{diag}\,.$$

Thus we have expressed the series (B.1) through the series (1.2) in the case of $Np$-complete conductors:

$$\mathcal{E}(z_1,z_2,z_3;-r,k,Np^v,\boldsymbol{\psi},\chi_1,\chi_2,\chi_3) \tag{B.5}$$

$$= N_{1,1}N_{1,2}N_{1,3}(\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})F_{\chi,r}^{\bar{\chi}_1,\bar{\chi}_2,\bar{\chi}_3}\circ \mathrm{diag}$$

$$= N_{1,1}N_{1,2}N_{1,3}(\bar{\chi}_1\bar{\chi}_2\bar{\chi}_3)(2)G(\chi_{0,1})G(\chi_{0,2})G(\chi_{0,3})2^{-r}\Phi_r(\chi).$$

### B.3 The integral representation

Consider three auxilliary modular forms as in (0.16):

$$\tilde{f}_j(z)=\sum_{n=1}^{\infty}\tilde{a}_{n,j}e(nz)\in S_k(\Gamma_0(N_jp^{\nu_j}),\psi_j)\quad (1\le i\le 3)$$

with the same eigenvalues, as those of (0.1), for all Hecke operators $T_q$, with $q$ prime to $Np$.

Theorem B.2 *Under the assumptions and notations as in section B.1, the following integral representation holds:*

$$\int\int\int_{(\Gamma_0(N^2p^{2v})\backslash\mathbb{H})^3} \overline{\tilde f_1(z_1)\tilde f_2(z_2)\tilde f_3(z_3)}\mathcal{E}(z_1,z_2,z_3;s,k,N^2p^{2v},\boldsymbol{\psi},\chi_1,\chi_2,\chi_3))\times$$
$$\prod_j y_j^k(\frac{dx_jdy_j}{y_j^2})$$
$$= i^{-3k+3}(2\pi)^{-4s}\Gamma(s+2k-2)\Gamma(s+k-1)^3$$
$$(Np^v)^{k+2s}\frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}):\Gamma(N^2p^{2v})]^3}\times\mathfrak{L}_{Np}(s)$$
$$L^{(Np)}(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho,s+2k-2,\psi_1\psi_2\chi_1),$$

*where*

$$(2\pi)^{-4s}\Gamma(s+2k-2)\Gamma(s+k-1)^3=2^{-4}\Gamma_{\mathbb{C}}(s+2k-2)\Gamma_{\mathbb{C}}(s+k-1)^3,$$
$$\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$$

*is the motivic Gamma-factor,*

$$\mathfrak{L}_{Np}(s)=\mathfrak{L}_{Np}(s;\tilde f_1\otimes\tilde f_2\otimes\tilde f_3):=\sum_{n|(Np)^\infty}G_{Np^v}(\overline{\psi_1\psi_2\chi_1},2n)\frac{\tilde a_{n,1}\tilde a_{n,2}\tilde a_{n,3}}{n^{2s+2k-2}}. \tag{B.6}$$

Remark. *In the special case when the character $\psi_1\psi_2\chi$ has $Np$-complete conductor, or if it is primitive $\bmod Np^v$, and $\tilde f_1,\tilde f_2,\tilde f_3$ are primitive normalized cusp eigenforms, one can show that $\mathfrak{L}_{Np}(s)=(\psi_1\psi_2\chi_1)(2)G(\overline{\psi_1\psi_2\chi_1})$.*

Theorem B.2 follows from a computation, similar to that in [BoeSP], Theorem 4.2, (triple product, no twisting character) and [Boe-Schm], Section 2 (standard $L$-function, with twisting character). Details will appear elsewhere.

Corollary B.3 *Under the notations and assumptions, for all critical values $s=2k-2-r$, $r=0,\cdots,k-2$ the following integral representation holds*

$$(2\pi)^{4r}\Gamma(-r+2k-2)\Gamma(-r+k-1)^3L^{(N)}(f_1^\rho\otimes f_2^\rho\otimes f_3^\rho,2k-2-r,\psi_1\psi_2\chi_1)$$
$$(Np^v)^{k-2r}\frac{N^2p^{2v}\varphi(N^2p^{2v})\varphi(Np^v)}{[\Gamma_0(N^2p^{2v}):\Gamma(N^2p^{2v})]^3}\times\mathfrak{L}_{Np}(s)$$
$$=\left\langle\tilde f_1\otimes\tilde f_2\otimes\tilde f_3,\mathcal{E}(z_1,z_2,z_3;-r,k,k,N^2p^{2v},\boldsymbol{\psi},\chi_1,\chi_2,\chi_3)\right\rangle_{T,N^2p^{2v}}. \blacksquare$$

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