On the Image of l-Adic Galois Representations for Abelian Varieties of Type I and II

Dedicated to John Coates on the occasion of his 60-th birthday

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Abstract. In this paper we investigate the image of the l-adic representation attached to the Tate module of an abelian variety over a number field with endomorphism algebra of type I or II in the Albert classification. We compute the image explicitly and verify the classical conjectures of Mumford-Tate, Hodge, Lang and Tate for a large family of abelian varieties of type I and II. In addition, for this family, we prove an analogue of the open image theorem of Serre.

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1. Introduction.

Let A be an abelian variety defined over a number field F. Let l be an odd prime number. In this paper we study the images of the l-adic representation $\rho_l : G_F \longrightarrow GL(T_l(A))$ and the *mod l* representation $\overline{\rho}_l : G_F \longrightarrow GL(A[l])$ of the absolute Galois group $G_F = G(\bar{F}/F)$ of the field F, associated with the Tate module, for A of type I or II in the Albert classification list cf. [M]. In our previous paper on the subject cf. [BGK], we computed the images of the Galois representations for some abelian varieties with real (type I) and complex multiplications (type IV) by the field $E{=}End_F(A) \otimes \mathbb{Q}$ and for l which splits completely in the field E *loc. cit.*, Theorem 2.1 and Theorem 5.3.

In the present paper we extend results proven in [BGK] to a larger class (cf. Definition of class $\mathcal{A}$ below) of abelian varieties which includes some varieties with non-commutative algebras of endomorphisms, and to almost all prime numbers l. In order to get these results, we had to implement the Weil restriction functor $R_{L/K}$ for a finite extension of fields L/K. In section 2 of the paper we give an explicit description of the Weil restriction functor for affine group schemes which we use in the following sections. In a very short section 3 we prove two general lemmas about bilinear forms which we apply to Weil pairing in the following section. Further in section 4, we collect some auxiliary facts about abelian varieties. In section 5 we obtain the integral versions of the results of Chi cf. [C2], for abelian varieties of type II and compute Lie algebras and endomorphism algebras corresponding to the λ-adic representations related to the Tate module of A. In section 6 we prove the main results of the paper which concern images of Galois representations ρ_l, $\rho_l \otimes \mathbb{Q}_l : G_F \to GL(V_l(A))$, the mod l-representation $\overline{\rho}_l$ and the associated group schemes $\mathcal{G}_l^{alg}$, G_l^{alg} and $G(l)^{alg}$, respectively.

The main results proven in this paper concern the following class of abelian varieties:

Definition of class $\mathcal{A}$.
We say that an abelian variety A/F, defined over a number field F is of class $\mathcal{A}$, if the following conditions hold:

(i) *A is a simple, principally polarized abelian variety of dimension g*
(ii) *$\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$ and the endomorphism algebra $D = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q}$, is of type I or II in the Albert list of division algebras with involution (cf. [M], p. 201).*
(iii) *the field F is such that for every l the Zariski closure G_l^{alg} of $\rho_l(G_F)$ in $GL_{2g}/\mathbb{Q}_l$ is a connected algebraic group*
(iv) *$g = hed$, where h is an odd integer, $e = [E : \mathbb{Q}]$ is the degree of the center E of D and $d^2 = [D : E]$.*

Let us recall the definition of abelian varieties of type I and II in the Albert's classification list of division algebras with involution [M], p. 201. Let $E \subset D = End_{\bar{F}}(A) \otimes_Z \mathbb{Q}$ be the center of D and E be a totally real extension of $\mathbb{Q}$ of degree e. Abelian varieties of type I are such that $D = E$. Abelian varieties of type II are those for which D is an indefinite quaternion algebra with the center E, such that $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^{e} M_{2,2}(\mathbb{R})$.

We have chosen to work with principal polarizations, however the main results of this paper have their analogs for any simple abelian variety A with a fixed polarization, provided A satisfies the above conditions (ii), (iii) and (iv). The most restrictive of the conditions in the definition of class $\mathcal{A}$ is condition (iv) on the dimension of the variety A. We need this condition to perform computations with Lie algebras in the proof of Lemma 5.33, which are based on an application of the minuscule conjecture cf. [P]. Note that due to results of Serre, the assumption (iii) is not very restrictive. It follows by [Se1] and [Se4] that for an abelian variety A defined over a number field K, there exists a finite extension

K^{conn}/K for which the Zariski closure of the group $\rho_l(G_{K^{conn}})$ in GL is a connected variety for any prime l. Hence, to make A meet the condition (iii), it is enough to enlarge the base field, if necessary. Note that the field K^{conn} can be determined in purely algebraic terms, as the intersection of a family of fields of division points on the abelian variety A cf. [LP2], Theorem 0.1.

Main results

Theorem A. *[Theorem 6.9]*
If A is an abelian variety of class $\mathcal{A}$, then for $l \gg 0$, we have equalities of group schemes:

$$(G_l^{alg})' = \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{2h})$$

$$(G(l)^{alg})' = \prod_{\lambda | l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h}),$$

where G' stands for the commutator subgroup of an algebraic group G, and $R_{L/K}(-)$ denotes the Weil restriction functor.

Theorem B. *[Theorem 6.16]*
If A is an abelian variety of class $\mathcal{A}$, then for $l \gg 0$, we have:

$$\overline{\rho_l}(G'_F) = \prod_{\lambda | l} Sp_{2h}(k_\lambda) = Sp_{2h}(\mathcal{O}_E/l\mathcal{O}_E)$$

$$\rho_l(\overline{G'_F}) = \prod_{\lambda | l} Sp_{2h}(\mathcal{O}_\lambda) = Sp_{2h}(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l),$$

where $\overline{G'_F}$ is the closure of G'_F in the profinite topology in G_F.

As an application of Theorem A we obtain:

Theorem C. *[Theorem 7.12]*
If A is an abelian variety of class $\mathcal{A}$, then

$$G_l^{alg} = MT(A) \otimes \mathbb{Q}_l,$$

for every prime number l, where $MT(A)$ denotes the Mumford-Tate group of A, i.e., the Mumford -Tate conjecture holds true for A.

Using the approach initiated by Tankeev [Ta5] and Ribet [R2], futher developed by V.K. Murty [Mu] combined with some extra work on the Hodge groups in section 7, we obtain:

Theorem D. *[Theorems 7.34, 7.35]*
If A is an abelian variety of class $\mathcal{A}$, then the Hodge conjecture and the Tate conjecture on the algebraic cycle maps hold true for the abelian variety A.

In the past there has been an extensive work on the Mumford-Tate, Tate and Hodge conjectures for abelian varieties. Special cases of the conjectures were verified for some classes of abelian varieties, see for example papers: [Ab], [C2], [Mu], [P], [Po], [R2], [Se1], [Se5], [Ta1], [Ta2], [Ta3]. For an abelian variety A of type I or II the above mentioned papers consider the cases where A is such that $End(A) \otimes \mathbb{Q}$ is either $\mathbb{Q}$ or has center $\mathbb{Q}$. The papers [Ta4], [C1] and [BGK] considered some cases with the center larger than $\mathbb{Q}$. For more complete list of results concerning the Hodge conjecture see [G]. In the current work we prove the conjectures in the case when the center of $End(A) \otimes \mathbb{Q}$ is an arbitrary totally real extension of $\mathbb{Q}$. To prove the conjectures for such abelian varieties we needed to do careful computations using the Weil restriction functor.

Moreover, using a result of Wintenberger (cf. [Wi], Cor. 1, p.5), we were able to verify that for A of class $\mathcal{A}$, the group $\rho_l(G_F)$ contains the group of all the homotheties in $GL_{T_l(A)}(\mathbb{Z}_l)$ for $l \gg 0$, i.e., the Lang conjecture holds true for A cf. Theorem 7.38.

As a final application of the method developed in this paper, we prove an analogue of the open image theorem of Serre cf. [Se1] for the class of abelian varieties we work with.

Theorem E. *[Theorem 7.42]*
If A is an abelian variety of class $\mathcal{A}$, then for every prime number l, the image $\rho_l(G_F)$ is open in the group $C_{\mathcal{R}}(GSp_{(\Lambda,\,\psi)})(\mathbb{Z}_l)$ of $\mathbb{Z}_l$-points of the commutant of $\mathcal{R}$=$End\, A$ in the group $GSp_{(\Lambda,\,\psi)}$ of symplectic similitudes of the bilinear form $\psi : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ associated with the polarization of A. In addition, for $l \gg 0$ we have:

$$\rho_l(\overline{G'_F}) \;=\; C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l).$$

As an immediate corollary of Theorem E we obtain that for any A of class $\mathcal{A}$ and for every l, the group $\rho_l(G_F)$ is open in $\mathcal{G}_l^{alg}(\mathbb{Z}_l)$ (in the l-adic topology), where $\mathcal{G}_l^{alg}$ is the Zariski closure of $\rho_l(G_F)$ in $GL_{2g}/\mathbb{Z}_l$. cf. Theorem 7.48. Recently, the images of Galois representations coming from abelian varieties have also been considered by A.Vasiu (cf. [Va1],[Va2]).

2. Weil restriction functor $R_{E/K}$ for affine schemes and Lie algebras.

In this section we describe the Weil restriction functor and its basic properties which will be used in the paper c.f. [BLR], [V1], [V2, pp. 37-40], [W1] and [W2, pp. 4-9]. For the completeness of the exposition and convenience of the reader we decided to include the results although some of them might be known to specialists. Let E/K be a separable field extension of degree n. Let $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ denote the set of all imbeddings $E \to E^{\sigma_i} \subset \overline{K}$ fixing K. Define M to be the composite of the fields E^{σ_i}

$$M = E^{\sigma_1} \dots E^{\sigma_n}.$$

Let $X = [x_1, x_2, \dots x_r]$ denote a multivariable. For polynomials $f_k = f_k(X) \in E[X]$, $1 \leq k \leq s$, we denote by $I = (f_1, f_2, \dots, f_s)$ the ideal generated by the f_k's and put $I^{\sigma_i} = (f_1^{\sigma_i}(X), f_2^{\sigma_i}(X), \dots, f_s^{\sigma_i}(X))$ for any $1 \leq i \leq n$. Let $A = E[X]/I$. Define E-algebras A^{σ_i} and $\overline{A}$ as follows:

$$A^{\sigma_i} = A \otimes_{E, \sigma_i} M \;\cong\; M[X]/\, I^{\sigma_i} M[X],$$

$$\overline{A} = A^{\sigma_1} \otimes_M \cdots \otimes_M A^{\sigma_n}.$$

Let $X^{\sigma_1}, \dots, X^{\sigma_n}$ denote the multivariables

$$X^{\sigma_i} = [x_{i,1}, x_{i,2}, \dots, x_{i,r}]$$

on which the Galois group $G = G(M/K)$ acts naturally on the right. Indeed for any imbedding σ_i and any $\sigma \in G$ the composition $\sigma_i \circ \sigma$, applied to E on the right, gives uniquely determined imbedding σ_j of E into $\overline{K}$, for some $1 \leq j \leq n$. Hence we define the action of $G(M/K)$ on the elements X^{σ_i} in the following way:

$$(X^{\sigma_i})^{\sigma} = X^{\sigma_j}.$$

We see that

$$\overline{A} \;\cong\; M[X^{\sigma_1}, \dots, X^{\sigma_n}]/\, (I_1 + \cdots + I_n),$$

where $I_k = M[X^{\sigma_1}, \dots, X^{\sigma_n}] I_{(k)}$ and $I_{(k)} = (f_1^{\sigma_k}(X^{\sigma_k}), \dots, f_s^{\sigma_k}(X^{\sigma_k}))$, for any $1 \leq k \leq n$.

Lemma 2.1.

$$\overline{A}^G \otimes_K M \cong \overline{A}.$$

Proof. Let $\alpha_1, \dots, \alpha_n$ be a basis of E over K. It is clear that

$$\sum_{i=1}^{n} \alpha_j^{\sigma_i} X^{\sigma_i} \;\in\; \overline{A}^G.$$

Since $[\alpha_j^{\sigma_i}]_{i,j}$ is an invertible matrix with coefficients in M, we observe that $X^{\sigma_1}, \dots, X^{\sigma_n}$ are in the subalgebra of $\overline{A}$ generated by M and $\overline{A}^G$. But $X^{\sigma_1}, \dots, X^{\sigma_n}$ and M generate $\overline{A}$ as an algebra. $\square$

Remark 2.2. Notice that the elements $\sum_{i=1}^n \alpha_j^{\sigma_i} X^{\sigma_i}$ for $j = 1, \dots, n$ generate $\overline{A}^G$ as a K-algebra. Indeed if C denotes the K-subalgebra of $\overline{A}^G$ generated by these elements and if C were smaller than $\overline{A}^G$, then $C \otimes_K M$ would be smaller than $\overline{A}^G \otimes_K M$, contrary to Lemma 2.1.

Definition 2.3. *Put $V = spec\, A$, and $W = spec\, \overline{A}^G$. Weil's restriction functor $R_{E/K}$ is defined by the following formula:*

$$R_{E/K}(V) = W.$$

Note that we have the following isomorphisms:

$$W \otimes_K M \;=\; spec\,(\overline{A}^G \otimes_K M) \;\cong\; spec\,\overline{A} \cong$$

$$spec\,(A^{\sigma_1} \otimes_M \cdots \otimes_M A^{\sigma_n}) \;\cong\; (V \otimes_{E,\sigma_1} M) \otimes_M \cdots \otimes_M (V \otimes_{E,\sigma_n} M),$$

hence

$$R_{E/K}(V) \otimes_K M \cong (V \otimes_{E,\sigma_1} M) \otimes_M \cdots \otimes_M (V \otimes_{E,\sigma_n} M).$$

Lemma 2.4. *Let $V' \subset V$ be a closed imbedding of affine schemes over E. Then $R_{E/K}(V') \subset R_{E/K}(V)$ is a closed imbedding of affine schemes over K.*

Proof. We can assume that $V = spec\,(E[X]/I)$ and $V' = spec\,(E[X]/J)$ for two ideals $I \subset J$ of $E[X]$. Put $A = E[X]/I$ and $B = E[X]/J$ and let $\phi\,:\, A \to B$ be the natural surjective ring homomorphism. The homomorphism ϕ induces the surjective E-algebra homomorphism

$$\overline{\phi}\,:\,\overline{A} \to \overline{B}$$

which upon taking fix points induces the K-algebra homomorphism

$$\overline{\phi}^G\,:\,\overline{A}^G \to \overline{B}^G. \tag{2.5}$$

By Remark 2.2 we see that $\overline{B}^G$ is generated as a K-algebra by elements $\sum_{i=1}^n \alpha_j^{\sigma_i} X^{\sigma_i}$ (more precisely their images in $\overline{B}^G$). Similarly $\overline{A}^G$ is generated as a K-algebra by elements $\sum_{i=1}^n \alpha_j^{\sigma_i} X^{\sigma_i}$ (more precisely their images in $\overline{A}^G$). It is clear that $\overline{\phi}^G$ sends the element $\sum_{i=1}^n \alpha_j^{\sigma_i} X^{\sigma_i} \in \overline{A}^G$ into $\sum_{i=1}^n \alpha_j^{\sigma_i} X^{\sigma_i} \in \overline{B}^G$. Hence $\overline{\phi}^G$ is onto. $\square$

Let $\alpha_1, \ldots, \alpha_n$ be a basis of E over K and let $\beta_1, \ldots, \beta_n$ be the corresponding dual basis with respect to $Tr_{E/K}$. Define block matrices:

$$\mathbb{A} = \begin{pmatrix} \alpha_1^{\sigma_1} I_r & \alpha_1^{\sigma_2} I_r & \ldots & \alpha_1^{\sigma_n} I_r \\ \alpha_2^{\sigma_1} I_r & \alpha_2^{\sigma_2} I_r & \ldots & \alpha_2^{\sigma_n} I_r \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_n^{\sigma_1} I_r & \alpha_n^{\sigma_2} I_r & \ldots & \alpha_n^{\sigma_n} I_r \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} \beta_1^{\sigma_1} I_r & \beta_2^{\sigma_1} I_r & \ldots & \beta_n^{\sigma_1} I_r \\ \beta_1^{\sigma_2} I_r & \beta_2^{\sigma_2} I_r & \ldots & \beta_n^{\sigma_2} I_r \\ \vdots & \vdots & \ldots & \vdots \\ \beta_1^{\sigma_n} I_r & \beta_2^{\sigma_n} I_r & \ldots & \beta_n^{\sigma_n} I_r \end{pmatrix}$$

Notice that by definition of the dual basis $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A} = I_{rn}$. Define block diagonal matrices:

$$\mathbb{X} = \begin{pmatrix} X^{\sigma_1} & 0I_r & \dots & 0I_r \\ 0I_r & X^{\sigma_2} & \dots & 0I_r \\ \vdots & \vdots & \dots & \vdots \\ 0I_r & 0I_r & \dots & X^{\sigma_n} \end{pmatrix}, \quad \mathbb{Y} = \begin{pmatrix} Y^{\sigma_1} & 0I_r & \dots & 0I_r \\ 0I_r & Y^{\sigma_2} & \dots & 0I_r \\ \vdots & \vdots & \dots & \vdots \\ 0I_r & 0I_r & \dots & Y^{\sigma_n} \end{pmatrix},$$

where $Y^{\sigma_1}, \dots, Y^{\sigma_n}$ and $X^{\sigma_1}, \dots, X^{\sigma_n}$, are multivariables written now in a form of $r \times r$ matrices indexed by $\sigma_1, \dots, \sigma_n$. Let T_{ij} and S_{ij}, for all $1 \leq i \leq n$, $1 \leq j \leq n$, be $r \times r$ multivariable matrices. Define block matrices of multivariables:

$$\mathbb{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & \dots & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}, \quad \mathbb{S} = \begin{pmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{21} & S_{22} & \dots & S_{2n} \\ \vdots & \vdots & \dots & \vdots \\ S_{n1} & S_{n2} & \dots & S_{nn} \end{pmatrix}$$

Notice that:

$$\mathbb{A}\mathbb{X}\mathbb{B} = \begin{pmatrix} \sum_{j=1}^n (\alpha_1\beta_1)^{\sigma_j} X^{\sigma_j} & \sum_{j=1}^n (\alpha_1\beta_2)^{\sigma_j} X^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_1\beta_n)^{\sigma_j} X^{\sigma_j} \\ \sum_{j=1}^n (\alpha_2\beta_1)^{\sigma_j} X^{\sigma_j} & \sum_{j=1}^n (\alpha_2\beta_2)^{\sigma_j} X^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_2\beta_n)^{\sigma_j} X^{\sigma_j} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{j=1}^n (\alpha_n\beta_1)^{\sigma_j} X^{\sigma_j} & \sum_{j=1}^n (\alpha_n\beta_2)^{\sigma_j} X^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_n\beta_n)^{\sigma_j} X^{\sigma_j} \end{pmatrix}$$

$$\mathbb{A}\mathbb{Y}\mathbb{B} = \begin{pmatrix} \sum_{j=1}^n (\alpha_1\beta_1)^{\sigma_j} Y^{\sigma_j} & \sum_{j=1}^n (\alpha_1\beta_2)^{\sigma_j} Y^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_1\beta_n)^{\sigma_j} Y^{\sigma_j} \\ \sum_{j=1}^n (\alpha_2\beta_1)^{\sigma_j} Y^{\sigma_j} & \sum_{j=1}^n (\alpha_2\beta_2)^{\sigma_j} Y^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_2\beta_n)^{\sigma_j} Y^{\sigma_j} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{j=1}^n (\alpha_n\beta_1)^{\sigma_j} Y^{\sigma_j} & \sum_{j=1}^n (\alpha_n\beta_2)^{\sigma_j} Y^{\sigma_j} & \dots & \sum_{j=1}^n (\alpha_n\beta_n)^{\sigma_j} Y^{\sigma_j} \end{pmatrix}.$$

Observe that the entries of $\mathbb{A}\mathbb{X}\mathbb{B}$ and $\mathbb{A}\mathbb{Y}\mathbb{B}$ are G-equivariant. Hence, there is a well defined homomorphism of K-algebras

$$(2.6)\quad \Phi \;:\; K[\mathbb{T}, \mathbb{S}]/(\mathbb{T}\mathbb{S} - I_{rn},\, \mathbb{S}\mathbb{T} - I_{rn}) \;\to\; \big(M[\mathbb{X}, \mathbb{Y}]/(\mathbb{X}\mathbb{Y} - I_{rn},\, \mathbb{Y}\mathbb{X} - I_{rn})\big)^G$$

$$\mathbb{T} \;\to\; \mathbb{A}\mathbb{X}\mathbb{B}$$

$$\mathbb{S} \;\to\; \mathbb{A}\mathbb{Y}\mathbb{B}$$

The definition of Φ and the form of the entries of matrices $\mathbb{A}\mathbb{X}\mathbb{B}$ and $\mathbb{A}\mathbb{Y}\mathbb{B}$ show (by the same argument as in Lemma 2.4) that the map Φ is surjective. Observe that

$$GL_{rn}/K = spec\ K[\mathbb{T}, \mathbb{S}]/(\mathbb{T}\mathbb{S} - I_{rn},\, \mathbb{S}\mathbb{T} - I_{rn}),$$

$$GL_r/E \;=\; spec\ E[X, Y]/(XY - I_r,\, YX - I_r),$$

where X and Y are $r \times r$ multivariable matrices.

Lemma 2.7. *Consider the group scheme GL_r/E. The map Φ induces a natural isomorphism $R_{E/K}(GL_r) \cong C_E(GL_{rn}/K)$ of closed group subschemes of GL_{rn}/K, where $C_E(GL_{rn}/K)$ is the commutant of E in GL_{rn}/K.*

Proof. Observe that there is a natural M-algebra isomorphism

$$M[\mathbb{X}, \mathbb{Y}]/(\mathbb{X}\mathbb{Y} - I_{rn},\, \mathbb{Y}\mathbb{X} - I_{rn}) \cong A^{\sigma_1} \otimes_M \cdots \otimes_M A^{\sigma_n},$$

where in this case

$$A^{\sigma_j} = M[X,Y]/(XY - I_r,\, YX - I_r) \cong M[X^{\sigma_j}, Y^{\sigma_j}]/(X^{\sigma_j}Y^{\sigma_j} - I_r,\, Y^{\sigma_j}X^{\sigma_j} - I_r).$$

Hence, by Definition 2.3 we get a natural isomorphism of schemes over K :

$$R_{E/K}(GL_r) \cong spec\left(M[\mathbb{X}, \mathbb{Y}]/(\mathbb{X}\mathbb{Y} - I_{rn},\, \mathbb{Y}\mathbb{X} - I_{rn})\right)^G$$

and it follows that Φ induces a closed imbedding of schemes $R_{E/K}(GL_r) \rightarrow GL_{rn}$ over K. Moreover we easily check that $Ker\, \Phi$ is generated by elements $\alpha \circ \mathbb{T} - \mathbb{T} \circ \alpha$ and $\alpha \circ \mathbb{S} - \mathbb{S} \circ \alpha$ for all $\alpha \in E$, where $\circ$ denotes the multiplication in GL_{rn}/K. Note that $C_E(GL_{rn}/K)$ is equal to

$$spec\ K[\mathbb{T}, \mathbb{S}]/(\mathbb{T}\mathbb{S} - I_{rn},\ \mathbb{S}\mathbb{T} - I_{rn},\ \alpha \circ \mathbb{T} - \mathbb{T} \circ \alpha,\ \alpha \circ \mathbb{S} - \mathbb{S} \circ \alpha,\ \forall_{\alpha \in E}). \qquad \square$$

Remark 2.8. Let E/K be an unramified extension of two local fields. Hence the extension of rings of integers $\mathcal{O}_E/\mathcal{O}_K$ has an integral basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_E$ over $\mathcal{O}_K$ such that the corresponding dual basis $\beta_1, \ldots, \beta_n$ with respect to $Tr_{E/K}$ is also a basis of $\mathcal{O}_E$ over $\mathcal{O}_K$ see [A], Chapter 7. Let $R_{\mathcal{O}_E/\mathcal{O}_K}$ be the Weil restriction functor defined analogously to the Weil restriction functor for the extension E/K. Under these assumptions the following Lemmas 2.9 and 2.10 are proven in precisely the same way as Lemmas 2.4 and 2.6.

Lemma 2.9. *Let $V' \subset V$ be a closed imbedding of affine schemes over $\mathcal{O}_E$. Under the assumptions of Remark 2.8 $R_{\mathcal{O}_E/\mathcal{O}_K}(V') \subset R_{\mathcal{O}_E/\mathcal{O}_K}(V)$ is a closed imbedding of affine schemes over $\mathcal{O}_K$.*

Lemma 2.10. *Consider the group scheme $GL_r/\mathcal{O}_E$. Under the assumptions of Remark 2.8 there is a natural isomorphism $R_{\mathcal{O}_E/\mathcal{O}_K}(GL_r) \cong C_{\mathcal{O}_E}(GL_{rn}/\mathcal{O}_K)$ of closed group subschemes of $GL_{rn}/\mathcal{O}_K$, where $C_{\mathcal{O}_E}(GL_{rn}/\mathcal{O}_K)$ is the commutant of $\mathcal{O}_E$ in $GL_{rn}/\mathcal{O}_K$.*

We return to the case of the arbitrary separable field extension E/K of degree n. Every point of $X_0 \in GL_r(E)$ is uniquely determined by the ring homomorphism

$$h_{X_0} : E[X,Y]/(XY - I_r, YX - I_r) \rightarrow E$$

$$X \mapsto X_0, \quad Y \mapsto Y_0,$$

where Y_0 is the inverse of X_0. This gives immediately the homomorphism

$$h_{\mathbb{T}_0} : K[\mathbb{T}, \mathbb{S}]/(\mathbb{T}\mathbb{S} - I_{rn}, \mathbb{S}\mathbb{T} - I_{rn}) \to K$$

$$\mathbb{T} \mapsto \mathbb{T}_0 = \mathbb{A}\mathbb{X}_0\mathbb{B},$$

$$\mathbb{S} \mapsto \mathbb{S}_0 = \mathbb{A}\mathbb{Y}_0\mathbb{B}$$

where

$$\mathbb{X}_0 = \begin{pmatrix} X_0^{\sigma_1} & 0I_r & \dots & 0I_r \\ 0I_r & X_0^{\sigma_2} & \dots & 0I_r \\ \vdots & \vdots & \dots & \vdots \\ 0I_r & 0I_r & \dots & X_0^{\sigma_n} \end{pmatrix}, \quad \mathbb{Y}_0 = \begin{pmatrix} Y_0^{\sigma_1} & 0I_r & \dots & 0I_r \\ 0I_r & Y_0^{\sigma_2} & \dots & 0I_r \\ \vdots & \vdots & \dots & \vdots \\ 0I_r & 0I_r & \dots & Y_0^{\sigma_n} \end{pmatrix},$$

and the action of σ_i on X_0 and Y_0 is the genuine action on the entries of X_0 and Y_0. Obviously $h_{\mathbb{T}_0}$ determines uniquely the point $\mathbb{T}_0 \in GL_{rn}(K)$ with the inverse $\mathbb{S}_0$.

DEFINITION 2.11. *Assume that $Z = \{X_t;\ t \in T\} \subset GL_r(E)$ is a set of points. We define the corresponding set of points:*

$$Z_\Phi \quad = \quad \{\mathbb{T}_t = \mathbb{A}\mathbb{X}_t\mathbb{B};\ t \in T\} \quad \subset \quad GL_{rn}(K),$$

where

$$\mathbb{X}_t = \begin{pmatrix} X_t^{\sigma_1} & 0I_r & \dots & 0I_r \\ 0I_r & X_t^{\sigma_2} & \dots & 0I_r \\ \vdots & \vdots & \dots & \vdots \\ 0I_r & 0I_r & \dots & X_t^{\sigma_n} \end{pmatrix}.$$

We denote by Z^{alg} the Zariski closure of Z in GL_r/E and by Z_Φ^{alg} the Zariski closure of Z_Φ in GL_{rn}/K.

PROPOSITION 2.12. *We have a natural isomorphism of schemes over K :*

$$R_{E/K}(Z^{alg}) \ \cong \ Z_\Phi^{alg}.$$

Proof. Let

$$J_t = (XY - I_r, YX - I_r, X - X_t, Y - Y_t)$$

be the prime ideal of $E[X, Y]$ corresponding to the point $X_t \in GL_r(E)$. Let

$$J = \bigcap_{t \in T} J_t.$$

By definition $Z^{alg} = spec\,(E[X,Y]/J)$. Let

$$\mathbb{J}_t = (\mathbb{T}\mathbb{S} - I_{rn},\, \mathbb{S}\mathbb{T} - I_{rn},\, \mathbb{T} - \mathbb{A}\mathbb{X}_t\mathbb{B},\, \mathbb{S} - \mathbb{A}\mathbb{Y}_t\mathbb{B})$$

be the prime ideal in $K[\mathbb{T},\mathbb{S}]/(\mathbb{T}\mathbb{S} - I_{rn},\, \mathbb{S}\mathbb{T} - I_{rn})$ corresponding to the point $\mathbb{A}\mathbb{X}_t\mathbb{B} \in GL_{rn}(K)$. Define

$$\mathbb{J} = \bigcap_{t \in T} \mathbb{J}_t.$$

By definition $Z_{\Phi}^{alg} = spec\,(K[\mathbb{T},\mathbb{S}]/\mathbb{J})$. Put $A = E[X,Y]/(XY - I_r,\, YX - I_r)$. Observe that the ring $\overline{A}^G$ is generated as a K-algebra by $\mathbb{A}\mathbb{X}\mathbb{B}$ and $\mathbb{A}\mathbb{Y}\mathbb{B}$, since $\overline{A}$ is generated by $\mathbb{X}$ and $\mathbb{Y}$ as an M-algebra. Define

$$\mathbb{J}'_t = (\mathbb{A}\mathbb{X}\mathbb{B} - \mathbb{A}\mathbb{X}_t\mathbb{B},\, \mathbb{A}\mathbb{Y}\mathbb{B} - \mathbb{A}\mathbb{Y}_t\mathbb{B})$$

which is an ideal of $\overline{A}^G$. Put

$$\mathbb{J}' = \bigcap_{t \in T} \mathbb{J}'_t.$$

We have the following isomorphism induced by Φ.

$$K[\mathbb{T},\mathbb{S}]/\mathbb{J}_t \;\cong\; \overline{A}^G/\,\mathbb{J}'_t \;\cong\; K. \tag{2.13}$$

Hence, $\Phi^{-1}(\mathbb{J}'_t) \;=\; \mathbb{J}_t$ and $\Phi^{-1}(\mathbb{J}') \;=\; \mathbb{J}$. This gives the isomorphism

$$K[\mathbb{T},\mathbb{S}]/\mathbb{J} \;\cong\; \overline{A}^G/\,\mathbb{J}'. \tag{2.14}$$

Let $B = E[X,Y]/J$. There is a natural surjective homomorphism of K-algebras coming from the construction in the proof of Lemma 2.4 (see (2.5)):

$$\overline{A}^G/\,\mathbb{J}' \to \overline{B}^G \tag{2.15}$$

induced by the quotient map $A \to B$. We want to prove that (2.15) is an isomorphism. Observe that there is natural isomorphism of K-algebras:

$$\overline{A}^G/\,\mathbb{J}'_t \;\cong\; \overline{A/J_t}^G \;\cong\; K. \tag{2.16}$$

Consider the following commutative diagram of homomorphisms of K-algebras:

$$\begin{tikzcd} \overline{A}^G/\,\mathbb{J}' \arrow[r] \arrow[d] & \overline{B}^G \arrow[d] \\ \prod_{t\in T} \overline{A}^G/\,\mathbb{J}'_t \arrow[r, "\cong"] & \prod_{t\in T} \overline{A/J_t}^G \end{tikzcd} \tag{2.17}$$

The left vertical arrow is an imbedding by definition of $\mathbb{J}'$ and the bottom horizontal arrow is an isomorphism by (2.16). Hence the top horizontal arrow is an imbedding, i.e., the map (2.15) is an isomorphism. The composition of maps (2.14) and (2.15) gives a natural isomorphism of K-algebras

$$K[\mathbb{T},\mathbb{S}]/\mathbb{J} \cong \overline{B}^G. \tag{2.18}$$

But $Z_\Phi^{alg} = spec\,(K[\mathbb{T},\mathbb{S}]/\mathbb{J})$. In addition, $Z^{alg} = spec\,B$, hence $R_{E/K}(Z^{alg}) = spec\,\overline{B}^G$ and Proposition 2.12 follows by (2.18). $\square$

REMARK 2.19. If Z is a subgroup of $GL_r(E)$, then Z_Φ is a subgroup of $GL_{rn}(K)$. In this case Z^{alg} is a closed algebraic subgroup of GL_r/E and Z_Φ^{alg} is a closed algebraic subgroup of GL_{rn}/K.

DEFINITION 2.20. *Let $H = spec\,A$ be an affine algebraic group scheme defined over E and $\mathfrak{h}$ its Lie algebra. We define $\mathfrak{g} = R_{E/K}\mathfrak{h}$ to be the Lie algebra obtained from $\mathfrak{h}$ by considering it over K with the same bracket.*

LEMMA 2.21. *There is the following equality of Lie algebras*

$$\mathcal{L}ie(R_{E/K}H) = R_{E/K}\mathfrak{h}.$$

Proof. Let $n = [E : K]$ and $G = Gal(E/K)$. Since H is an algebraic group $\mathfrak{h} = \mathcal{D}er(A)$ is the Lie algebra of derivations of the algebra A of functions on H [H1]. Let $\phi : Der(A) \to Der(\bar{A})$ be the homomorphism of Lie algebras (considered over E) given by the following formula:

$$\phi(\delta) = \Sigma_{i=1}^n id \otimes \cdots \otimes id \otimes \delta_i \otimes id \otimes \cdots \otimes id,$$

where $\delta_i = \delta \otimes 1$ as an element of $Der(A^{\sigma_i})$. Recall that $A^{\sigma_i} = A \otimes_{E,\sigma_i} M$. If $\sigma \in G$ and $\sigma(a_1 \otimes \cdots \otimes a_n) = \sigma(a_{k_1}) \otimes \cdots \otimes \sigma(a_{k_n})$ one readily sees that $\delta_j(\sigma(a_{k_j})) = \sigma(\delta_{k_j}(a_{k_j}))$ and therefore $\phi(\delta)$ is G-equivariant i.e., $\phi(\delta) \in Der(\bar{A}^G)$. It is easy to see that $\phi(\delta)$ as an element of $Der(\bar{A})$ is nontrivial if δ is nontrivial. Since $\phi(\delta)$ is M-linear and $\bar{A}^G \otimes_K M = \bar{A}$, we see that $\phi(\delta)$ is a nontrivial element of $Der(\bar{A}^G) = \mathcal{L}ie(R_{E/K}H)$. On the other hand, observe that

$$\mathcal{L}ie(R_{E/K}H) \otimes_K \bar{K} = \mathcal{L}ie(R_{E/K}H \otimes_K \bar{K}) =$$

$$= \mathcal{L}ie(\bar{H} \times_K \cdots \times_K \bar{H}) = (\oplus \mathfrak{h}) \otimes_E \bar{K} = \mathfrak{g} \otimes_K \bar{K}.$$

This shows that $\mathcal{L}ie(R_{E/K}H)$ and $R_{E/K}\mathfrak{h}$ have the same dimensions and therefore are equal. $\square$

LEMMA 2.22. *Let $\mathfrak{g}$ be a Lie algebra over E and let $\mathfrak{g}'$ be its derived algebra. Then*

$$R_{E/K}(\mathfrak{g}') = (R_{E/K}(\mathfrak{g}))'$$

Proof. This follows immediately from the fact that $R_{E/K}(\mathfrak{g})$ and $\mathfrak{g}$ have the same Lie bracket (cf. Definition 2.20) $\square$

Lemma 2.23. *If G is a connected, algebraic group over E of characteristic 0, then*

$$R_{E/K}(G') = (R_{E/K}G)'$$

Proof. We have the following identities:

$$Lie((R_{E/K}(G))') = (Lie(R_{E/K}(G)))' = (R_{E/K}(Lie(G)))' =$$

$$= R_{E/K}((Lie(G))') = R_{E/K}(Lie(G')) = Lie(R_{E/K}(G'))$$

The first and the fourth equality follow from Corollary on p.75 of [H1]. The second and fifth equality follow from Lemma 2.21. The third equality follows from Lemma 2.22. The Lemma follows by Theorem on p. 87 of [H1] and Proposition on p. 110 of [H1]. □

3. Some remarks on bilinear forms.

Let E be a finite extension of $\mathbb{Q}$ of degree e. Let $E_l = E \otimes \mathbb{Q}_l$ and $\mathcal{O}_{E_l} = \mathcal{O}_E \otimes \mathbb{Z}_l$. Hence $E_l = \prod_{\lambda | l} E_\lambda$ and $\mathcal{O}_{E_l} = \prod_{\lambda | l} O_\lambda$. Let $\mathcal{O}'_\lambda$ be the dual to $\mathcal{O}_\lambda$ with respect to the trace $Tr_{E_\lambda/\mathbb{Q}_l}$. For $l \gg 0$ we have $\mathcal{O}'_\lambda = \mathcal{O}_\lambda$ see [A], Chapter 7. From now on we take l big enough to ensure that $\mathcal{O}'_\lambda = \mathcal{O}_\lambda$ for all primes λ in $\mathcal{O}_E$ over l and that an abelian variety A we consider, has good reduction at all primes in $\mathcal{O}_F$ over l. The following lemma is the integral version of the sublemma 4.7 of [D].

Lemma 3.1. *Let T_1 and T_2 be finitely generated, free $\mathcal{O}_{E_l}$-modules. For any $\mathbb{Z}_l$-bilinear (resp. nondegenerate $\mathbb{Z}_l$-bilinear) map*

$$\psi_l \,:\, T_1 \times T_2 \to \mathbb{Z}_l$$

such that $\psi_l(ev_1, v_2) = \psi_l(v_1, ev_2)$ for all $e \in \mathcal{O}_{E_l}, v_1 \in T_1, v_2 \in T_2$, there is a unique $\mathcal{O}_{E_l}$-bilinear (resp. nondegenerate $\mathcal{O}_{E_l}$-bilinear) map

$$\phi_l \,:\, T_1 \times T_2 \to \mathcal{O}_{E_l}$$

such that $Tr_{E_l/\mathbb{Q}_l}(\phi_l(v_1, v_2)) = \psi_l(v_1, v_2)$ for all $v_1 \in T_1$ and $v_2 \in T_2$.

Proof. Similary to Sublemma 4.7, [D] we observe that the map

$$Tr_{E_l/\mathbb{Q}_l} \,:\, Hom_{\mathcal{O}_{E_l}}(T_1 \otimes_{\mathcal{O}_{E_l}} T_2\,; \mathcal{O}_{E_l}) \to Hom_{\mathbb{Z}_l}(T_1 \otimes_{\mathcal{O}_{E_l}} T_2\,; \mathbb{Z}_l)$$

is an isomorphism since it is a surjective map of torsion free $\mathbb{Z}_l$-modules of the same $\mathbb{Z}_l$-rank. The surjectivity of $Tr_{E_l/\mathbb{Q}_l}$ can be seen as follows. The $\mathbb{Z}_l$-basis of the module $T_1 \otimes_{\mathcal{O}_{E_l}} T_2$ is given by

$$\mathcal{B} = \big((0, \ldots, 0, \alpha_k^\lambda, 0, \ldots, 0)e_i \otimes e'_j\big)$$

where $(0,\dots,0,\alpha_k^\lambda,0,\dots,0)\in\prod_{\lambda|l}\mathcal{O}_\lambda$ and α_k^λ is an element of a basis of $\mathcal{O}_\lambda$ over $\mathbb{Z}_l$ and e_i (resp. e'_j) is an element of the standard basis of T_1 (resp. T_2) over $\mathcal{O}_{E_l}$. Let $\psi_{k,i,j}^\lambda\in Hom_{\mathbb{Z}_l}(T_1\otimes_{\mathcal{O}_{E_l}}T_2\,;\mathbb{Z}_l)$ be the homomorphism which takes value 1 on the element $(0,\dots,0,\alpha_k^\lambda,0,\dots,0)e_i\otimes e'_j$ of the basis $\mathcal{B}$ and takes value 0 on the remaining elements of the basis $\mathcal{B}$. Let us take $\phi_{i,j}\in Hom_{\mathcal{O}_{E_l}}(T_1\otimes_{\mathcal{O}_{E_l}}T_2\,;\mathcal{O}_{E_l})$ such that

$$\phi_{i,j}(e_r\otimes e'_s)=\begin{cases}1 & \text{if } i=r \text{ and } j=s\\ 0 & \text{if } i\neq r \text{ or } j\neq s\end{cases}$$

Then for each k there exist elements (the dual basis) $\beta_k^\lambda\in\mathcal{O}_\lambda$ such that $Tr_{E_\lambda/\mathbb{Q}_l}(\beta_k^\lambda\alpha_n^\lambda)=\delta_{k,n}$. If we put $\phi_{i,j,k}^\lambda=(0,\dots,0,\beta_k^\lambda,0,\dots,0)\phi_{i,j}$ then clearly $Tr_{E_l/\mathbb{Q}_l}(\phi_{i,j,k}^\lambda(t_1,t_2))=\psi_{i,j,k}^\lambda(t_1,t_2)$. Hence the proof is finished since the elements $\psi_{i,j,k}^\lambda(t_1,t_2)$ form a basis of $Hom_{\mathbb{Z}_l}(T_1\otimes_{\mathcal{O}_{E_l}}T_2\,;\mathbb{Z}_l)$ over $\mathbb{Z}_l$. $\square$

Consider the case $T_1=T_2$ and put $T_l=T_1=T_2$. Assume in addition that ψ_l is nondegenerate. Let

$$\overline{\psi}_l\,:\,T_l/l\,T_l\times T_l/l\,T_l\to\mathbb{Z}/l$$

be the $\mathbb{Z}/l$-bilinear pairing obtained by reducing the form ψ_l modulo l. Define

$$T_\lambda=e_\lambda T_l\cong T_l\otimes_{\mathcal{O}_{E_l}}\mathcal{O}_\lambda$$

$$V_\lambda=T_\lambda\otimes_{\mathcal{O}_\lambda}E_\lambda$$

where e_λ is the standard idempotent corresponding to the decomposition $\mathcal{O}_{E_l}=\prod_\lambda\mathcal{O}_\lambda$. Let $\pi_\lambda\,:\,\mathcal{O}_{E_l}\to\mathcal{O}_\lambda$ be the natural projection. We can define an $\mathcal{O}_\lambda$-nondegenerate bilinear form as follows:

$$\psi_\lambda\,:\,T_\lambda\times T_\lambda\to\mathcal{O}_\lambda$$

$$\psi_\lambda(e_\lambda v_1,e_\lambda v_2)=\pi_\lambda(\phi_l(v_1,v_2))$$

for any $v_1,v_2\in T_l$. Put $k_\lambda=\mathcal{O}_\lambda/\lambda\,\mathcal{O}_\lambda$. This gives the k_λ-bilinear form $\overline{\psi}_\lambda=\psi_\lambda\otimes_{\mathcal{O}_\lambda}k_\lambda$

$$\overline{\psi}_\lambda\,:\,T_\lambda/\lambda T_\lambda\,\times\,T_\lambda/\lambda T_\lambda\to k_\lambda.$$

We also have the E_λ-bilinear form $\psi_\lambda^0:=\psi_\lambda\otimes_{\mathcal{O}_\lambda}E_\lambda$

$$\psi_\lambda^0\,:\,V_\lambda\times V_\lambda\to E_\lambda.$$

Lemma 3.2. *Assume that the form $\overline{\psi}_l$ is nondegenerate. Then the forms $\overline{\psi}_\lambda$, ψ_λ and ψ_λ^0 are nondegenerate for each $\lambda | l$.*

Proof. First we prove that $\overline{\psi}_\lambda$ is nondegenerate for all $\lambda|l$. Assume that $\overline{\psi}_\lambda$ is degenerate for some λ. Without loss of generality we can assume that the left radical of $\overline{\psi}_\lambda$ is nonzero. So there is a nonzero vector $e_\lambda v_0 \in T_\lambda$ (for some $v_0 \in T_l$) which maps to a nonzero vector in $T_\lambda / \lambda T_\lambda$ such that $\psi_\lambda(e_\lambda v_0,\, e_\lambda w) \in \lambda\, \mathcal{O}_\lambda$ for all $w \in T_l$. Now use the decomposition $T_l = \oplus_\lambda T_\lambda$, Lemma 3.1 and the $\mathcal{O}_{E_l}$-linearity of ϕ_l to observe that for each $w \in T_l$

$$\psi_l(e_\lambda v_0,\, w) = Tr_{E_l/\mathbb{Q}_l}(\phi_l(e_\lambda v_0,\, \sum_{\lambda'} e_{\lambda'} w)) = Tr_{E_\lambda/\mathbb{Q}_l} \psi_\lambda(e_\lambda v_0,\, e_\lambda w) \in l\mathbb{Z}_l.$$

This contradicts the assumption that $\overline{\psi}_l$ is nondegenerate.
Similarly, but in an easier way, we prove that ψ_λ is nondegenerate. From this it immediately follows that ψ_λ^0 is nondegenerate. $\square$

4. Auxiliary facts about abelian varieties.

Let A/F be a principally polarized, simple abelian variety of dimension g with the polarization defined over F. Put $\mathcal{R} = End_{\bar{F}}(A)$ We assume that $End_{\bar{F}}(A) = End_F(A)$, hence the actions of $\mathcal{R}$ and G_F on $A(\overline{F})$ commute. Put $D = End_{\bar{F}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The ring $\mathcal{R}$ is an order in D. Let E_1 be the center of D and let

$$E := \{a \in E_1;\ a' = a\},$$

where $'$ is the Rosati involution. Let $\mathcal{R}_D$ be a maximal order in D containing $\mathcal{R}$. Put $\mathcal{O}_E^0 := \mathcal{R} \cap E$. The ring $\mathcal{O}_E^0$ is an order in E. Take l that does not divide the index $[\mathcal{R}_D : \mathcal{R}]$. Then $\mathcal{R}_D \otimes_{\mathbb{Z}} \mathbb{Z}_l = \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_l$ and $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l = \mathcal{O}_E^0 \otimes_{\mathbb{Z}} \mathbb{Z}_l$

The polarization of A gives a $\mathbb{Z}_l$-bilinear, nondegenerate, alternating pairing

$$\psi_l : T_l(A) \times T_l(A) \to \mathbb{Z}_l. \tag{4.1}$$

Because A has the principal polarization, for any endomorphism $\alpha \in \mathcal{R}$ we get $\alpha' \in \mathcal{R}$, (see [Mi] chapter 13 and 17) where α' is the image of α by the Rosati involution. Hence for any $v, w \in T_l(A)$ we have $\psi_l(\alpha v, w) = \psi_l(v, \alpha' w)$ (see loc. cit.).

Remark 4.2. Notice that if an abelian variety were not principally polarized, one would have to assume that l does not divide the degree of the polarization of A, to get $\alpha' \otimes 1 \in \mathcal{R} \otimes \mathbb{Z}_l$ for $\alpha \in \mathcal{R}$.

By Lemma 3.1 there is a unique nondegenerate, $\mathcal{O}_{E_l}$-bilinear pairing

$$\phi_l : T_l(A) \times T_l(A) \to \mathcal{O}_{E_l} \tag{4.3}$$

such that $Tr_{E_l/\mathbb{Q}_l}(\phi_l(v_1, v_2)) = \psi_l(v_1, v_2)$. As in the general case define

$$T_\lambda(A) = e_\lambda T_l(A) \cong T_l(A) \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_\lambda$$

$$V_\lambda(A) = T_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda.$$

Note that $T_\lambda(A)/\lambda T_\lambda(A) \cong A[\lambda]$ as $k_\lambda[G_F]$-modules.
Again as in the general case define nondegenerate, $\mathcal{O}_\lambda$-bilinear form

$$\psi_\lambda \,:\, T_\lambda(A) \times T_\lambda(A) \to \mathcal{O}_\lambda \tag{4.4}$$

$$\psi_\lambda(e_\lambda v_1, e_\lambda v_2) = \pi_\lambda(\phi_l(v_1, v_2))$$

for any $v_1, v_2 \in T_l(A)$, where $\pi_\lambda : \mathcal{O}_{E_l} \to \mathcal{O}_\lambda$ is the natural projection. The form ψ_λ gives the forms:

$$\overline{\psi}_\lambda \,:\, A[\lambda] \times A[\lambda] \to k_\lambda. \tag{4.5}$$

$$\psi_\lambda^0 \,:\, V_\lambda(A) \times V_\lambda(A) \to E_\lambda. \tag{4.6}$$

Notice that all the bilinear forms $\psi_\lambda, \overline{\psi}_\lambda$ and ψ_λ^0 are alternating forms. For l relatively prime to the degree of polarization the form ψ_l is nondegenerate. Hence by lemma 3.2 the forms $\psi_\lambda, \overline{\psi}_\lambda$ and ψ_λ^0 are nondegenerate.

Lemma 4.7. *Let $\chi_\lambda \,:\, G_F \to \mathbb{Z}_l \subset \mathcal{O}_\lambda$ be the composition of the cyclotomic character with the natural imbedding $\mathbb{Z}_l \subset \mathcal{O}_\lambda$.*

(i) *For any $\sigma \in G_F$ and all $v_1, v_2 \in T_\lambda(A)$*

$$\psi_\lambda(\sigma v_1, \sigma v_2) = \chi_\lambda(\sigma)\psi_\lambda(v_1, v_2).$$

(ii) *For any $\alpha \in \mathcal{R}$ and all $v_1, v_2 \in T_\lambda(A)$*

$$\psi_\lambda(\alpha v_1, v_2) = \psi_\lambda(v_1, \alpha' v_2).$$

Proof. The proof is the same as the proof of Lemma 2.3 in [C2]. □

Remark 4.8. After tensoring appropriate objects with $\mathbb{Q}_l$ in lemmas 3.1 and 4.6 we obtain Lemmas 2.2 and 2.3 of [C2].

Let A/F be an abelian variety defined over a number field F such that $End_{\bar{F}}(A) = End_F(A)$. We introduce some notation. Let G_{l^∞}, G_l, $G_{l^\infty}^0$ denote the images of the corresponding representations:

$$\rho_l \,:\, G_F \to GL(T_l(A)) \cong GL_{2g}(\mathbb{Z}_l),$$

$$\overline{\rho_l} \,:\, G_F \to GL(A[l]) \cong GL_{2g}(\mathbb{F}_l),$$

$$\rho_l \otimes \mathbb{Q}_l \,:\, G_F \to GL(V_l(A)) \cong GL_{2g}(\mathbb{Q}_l).$$

Let $\mathcal{G}_l^{alg}$, (G_l^{alg} resp.) denote the Zariski closure of the image of the representation ρ_l, ($\rho_l \otimes \mathbb{Q}_l$, resp.) in $GL_{2g}/\mathbb{Z}_l$, ($GL_{2g}/\mathbb{Q}_l$, resp). We define $G(l)^{alg}$ to be the special fiber of the $\mathbb{Z}_l-$scheme $\mathcal{G}_l^{alg}$.

Due to our assumptions on the G_F-action and the properties of the forms $\psi_\lambda, \overline{\psi}_\lambda$ and ψ_λ^0 we get:

$$G_{l^\infty} \subset \mathcal{G}_l^{alg}(\mathbb{Z}_l) \subset \prod_{\lambda | l} GSp_{T_\lambda(A)}(\mathcal{O}_\lambda) \subset GL_{T_l(A)}(\mathbb{Z}_l) \tag{4.9}$$

$$G_l \subset G(l)^{alg}(\mathbb{F}_l) \subset \prod_{\lambda | l} GSp_{A[\lambda]}(k_\lambda) \subset GL_{A[l]}(\mathbb{F}_l) \tag{4.10}$$

$$G_{l^\infty}^0 \subset G_l^{alg}(\mathbb{Q}_l) \subset \prod_{\lambda | l} GSp_{V_\lambda(A)}(E_\lambda) \subset GL_{V_l(A)}(\mathbb{Q}_l). \tag{4.11}$$

Before we proceed further let us state and prove some general lemmas concerning l-adic representations. Let $K/\mathbb{Q}_l$ be a local field extension and $\mathcal{O}_K$ the ring of integers in K. Let T be a finitely generated, free $\mathcal{O}_K$-module and let $V = T \otimes_{\mathcal{O}_K} K$. Consider a continuous representation $\rho : G_F \rightarrow GL(T)$ and the induced representation $\rho^0 = \rho \otimes K : G_F \rightarrow GL(V)$. Since G_F is compact and ρ^0 is continuous, the subgroup $\rho^0(G_F)$ of $GL(V)$ is closed. By [Se7], LG. 4.5, $\rho^0(G_F)$ is an analytic subgroup of $GL(V)$.

Lemma 4.12. *Let $\mathfrak{g}$ be the Lie algebra of the group $\rho^0(G_F)$*

(i) *There is an open subgroup $U_0 \subset \rho^0(G_F)$ such that*

$$End_{U_0}(V) = End_{\mathfrak{g}}(V).$$

(ii) *For all open subgroups $U \subset \rho^0(G_F)$ we have*

$$End_U(V) \subset End_{\mathfrak{g}}(V).$$

(iii) *Taking union over all open subgroups $U \subset \rho^0(G_F)$ we get*

$$\bigcup_U End_U(V) = End_{\mathfrak{g}}(V).$$

Proof. (i) Note that for any open subgroup $\tilde{U}$ of $\mathfrak{g}$ we have

$$End_{\tilde{U}}\,(V) \;=\; End_{\mathfrak{g}}\,(V) \tag{4.13}$$

because $K\,\tilde{U} \;=\; \mathfrak{g}$. By [B], Prop. 3, III.7.2, for some open $\tilde{U} \subset \mathfrak{g}$, there is an exponential map

$$exp\,:\,\tilde{U}\;\rightarrow \rho^0(G_F)$$

which is an analytic isomorphism and such that $exp\,(\tilde{U})$ is an open subgroup of $\rho^0(G_F)$. The exponential map can be expressed by the classical power series for $exp\,(t)$. On the other hand by [B], Prop. 10, III.7.6, for some open $U \subset \rho^0(G_F)$, there is a logarithmic map

$$log\,:\,U\;\rightarrow \mathfrak{g}$$

which is an analytic isomorphism and the inverse of exp. The logarithmic map can be expressed by the classical power series for $\ln t$. Hence, we can choose $\tilde{U}_0$ such that $U_0 = exp\,(\tilde{U}_0)$ and $log\,(U_0) = \tilde{U}_0$. This gives

$$End_{U_0}\,(V) \;=\; End_{\tilde{U}_0}\,(V). \tag{4.14}$$

and (i) follows by (4.13) and (4.14).

(ii) Observe that for any open $U \subset \rho^0(G_F)$ we have

$$End_U\,(V) \;\subset\; End_{U_0 \cap U}\,(V).$$

Hence (ii) follows by (i).

(iii) Follows by (i) and (ii). $\square$

Lemma 4.15. *Let A/F be an abelian variety over F such that $End_F\,(A) = End_{\overline{F}}\,(A)$. Then*

$$End_{G_F}\,(V_l(A)) = End_{\mathfrak{g}_l}\,(V_l(A)).$$

Proof. By the result of Faltings [Fa], Satz 4,

$$End_L\,(A) \otimes \mathbb{Q}_l = End_{G_L}\,(V_l(A))$$

for any finite extension L/F. By the assumption $End_F\,(A) = End_L\,(A)$. Hence

$$End_{G_F}\,(V_l(A)) = End_{U'}\,(V_l(A))$$

for any open subgroup U' of G_F. So the claim follows by Lemma 4.12 (iii). $\square$

Let A be a simple abelian variety defined over F and E be the center of the algebra $D = End_F(A) \otimes \mathbb{Q}$. Let $\lambda | l$ be a prime of $\mathcal{O}_E$ over l. Consider the following representations.

$$\rho_\lambda\,:\,G_F \rightarrow GL(T_\lambda(A)),$$
$$\overline{\rho_\lambda}\,:\,G_F \rightarrow GL(A[\lambda]),$$
$$\rho_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda\,:\,G_F \rightarrow GL(V_\lambda(A)),$$

where $\lambda | l$. Let $\mathcal{G}_\lambda^{alg}$, (G_λ^{alg} resp.) denote the Zariski closure of the image of the representation ρ_λ, ($\rho_\lambda \otimes E_\lambda$ resp.) in $GL_{T_\lambda(A)}/\mathcal{O}_\lambda$, ($GL_{V_\lambda(A)}/E_\lambda$ resp.) We define $G(\lambda)^{alg}$ to be the special fiber of the $\mathcal{O}_\lambda$-scheme $\mathcal{G}_\lambda^{alg}$.

Theorem 4.16. *Let A be a simple abelian variety with the property that $\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$. Let $\mathcal{R}_\lambda = \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ and let $D_\lambda = D \otimes_E E_\lambda$. Then*

(i) $End_{\mathcal{O}_\lambda[G_F]}(T_\lambda(A)) \cong \mathcal{R}_\lambda$

(ii) $End_{R_\lambda[G_F]}(V_\lambda(A)) \cong D_\lambda$

(iii) $End_{k_\lambda[G_F]}(A[\lambda]) \cong \mathcal{R}_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda$ *for* $l \gg 0$.

Proof. It follows by [Fa], Satz 4 and [Za], Cor. 5.4.5. □

Lemma 4.17. *Let K be a field and let R be a unital K-algebra. Put $D = End_R(M)$ and let L be a subfield of the center of D. Assume that L/K is a finite separable extension. If M is a semisimple R-module then M is also a semisimple $R \otimes_K L$-module with the obvious action of $R \otimes_K L$ on M.*

Proof. Take $\alpha \in L$ such that $L = K(\alpha)$. Let $[L : K] = n$. Let us write $M = \oplus_i M_i$ where all M_i are simple R modules. For any i we put $\tilde{M}_i = \sum_{k=0}^{n-1} \alpha^k M_i$. Then $\tilde{M}_i$ is an $R \otimes_K L$-module. Because M_i is a simple R-module we can write

$$\tilde{M}_i = \bigoplus_{k=0}^{m-1} \alpha^k M_i,$$

for some m. Observe that if $m = 1$, then $\tilde{M}_i$ is obviously a simple $R \otimes_K L$-module. If $m>1$, we pick any simple R-submodule $N_i \subset \tilde{M}_i$, $N_i \neq M_i$. There is an R- isomorphism $\phi : M_i \to N_i$ by semisimplicity of $\tilde{M}_i$. We can write $M = M_i \oplus N_i \oplus M'$, where M' is an R-submodule of M. Define $\Psi \in Aut_R(M) \subset End_R(M)$ by $\Psi|_{M_i} = \phi$, $\Psi|_{N_i} = \phi^{-1}$ and $\Psi|_{M'} = Id_{M'}$. Note that

$$\Psi(\bigoplus_{k=0}^{m-1} \alpha^k M_i) = \bigoplus_{k=0}^{m-1} \alpha^k N_i \tag{4.18}$$

since α is in the center of D. Hence $\tilde{M}_i = \bigoplus_{k=0}^{m-1} \alpha^k N_i$ by the classification of semisimple R-modules. We conclude that $\tilde{M}_i$ is a simple $R \otimes_K L$-module. Indeed, if $N \subset \tilde{M}_i$ were a nonzero $R \otimes_K L$-submodule of $\tilde{M}_i$ then we could pick any simple R-submodule $N_i \subset N$. If $N_i = M_i$ then $N = \tilde{M}_i$. If $N_i \neq M_i$ then by (4.18) $\tilde{M}_i = \bigoplus_{k=0}^{m-1} \alpha^k N_i \subset N$. Since $M = \sum_i \tilde{M}_i$, we see that M is a semisimple $R \otimes_K L$-module. □

Theorem 4.19. *Let A be a simple abelian variety with the property that $\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$. Let $\mathcal{R}_\lambda = \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ and let $D_\lambda = D \otimes_E E_\lambda$. Then G_F acts on $V_\lambda(A)$ and $A[\lambda]$ semisimply and G_λ^{alg} and $G(\lambda)^{alg}$ are reductive algebraic groups. The scheme $\mathcal{G}_\lambda^{alg}$ is a reductive group scheme over $\mathcal{O}_\lambda$ for l big enough.*

Proof. It follows by [Fa], Theorem 3 and our Lemma 4.17. The last statement follows by [LP1], Proposition 1.3, see also [Wi], Theoreme 1. □

5. Abelian varieties of type I and II.

In this section we work with abelian varieties of type I and II in the Albert's classification list of division algebras with involution [M], p. 201, i.e. $E \subset D = End_{\bar{F}}(A) \otimes_Z \mathbb{Q}$ is the center of D and E is a totally real extension of $\mathbb{Q}$ of degree e. To be more precise D is either E (type I) or an indefinite quaternion algebra with the center E, such that $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^{e} M_{2,2}(\mathbb{R})$ (type II). In the first part of this section we prove integral versions of the results of Chi [C2] for abelian varieties of type II. Let l be a sufficiently large prime number that does not divide the index $[\mathcal{R}_D : \mathcal{R}]$ and such that $D \otimes_E E_\lambda$ splits over E_λ for any prime λ in $\mathcal{O}_E$ over l. Hence, $D_\lambda = M_{2,2}(E_\lambda)$. Then by [R, Corollary 11.2 p. 132 and Theorem 11.5 p. 133] the ring R_λ is a maximal order in D_λ. So by [R] Theorem 8.7 p. 110 we get $R_\lambda = M_{2,2}(\mathcal{O}_\lambda)$, hence $R_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda = M_{2,2}(k_\lambda)$. Similarly to [C2] we put

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $e = \frac{1}{2}(1+t)$, $f = \frac{1}{2}(1+u)$, $\mathcal{X} = e\, T_\lambda(A)$, $\mathcal{Y} = (1-e)\, T_\lambda(A)$, $\mathcal{X}' = f\, T_\lambda(A)$, $\mathcal{Y}' = (1-f)\, T_\lambda(A)$. Put $X = \mathcal{X} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $X' = \mathcal{X}' \otimes_{\mathcal{O}_\lambda} E_\lambda$, $Y = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $Y' = \mathcal{Y}' \otimes_{\mathcal{O}_\lambda} E_\lambda$, $\overline{\mathcal{X}} = \mathcal{X} \otimes_{\mathcal{O}_\lambda} k_\lambda$, $\overline{\mathcal{X}}' = \mathcal{X}' \otimes_{\mathcal{O}_\lambda} k_\lambda$, $\overline{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} k_\lambda$, $\overline{\mathcal{Y}}' = \mathcal{Y}' \otimes_{\mathcal{O}_\lambda} k_\lambda$. Because $ueu = 1-e$, the matrix u gives an $\mathcal{O}_\lambda[G_F]$-isomorphism between $\mathcal{X}$ and $\mathcal{Y}$, hence it yields an $E_\lambda[G_F]$-isomorphism between X and Y and a $k_\lambda[G_F]$-isomorphism between $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$. Multiplication by t gives an $\mathcal{O}_\lambda[G_F]$-isomorphism between $\mathcal{X}'$ and $\mathcal{Y}'$, hence it yields an $E_\lambda[G_F]$-isomorphism between X' and Y' and a $k_\lambda[G_F]$-isomorphism between $\overline{\mathcal{X}}'$ and $\overline{\mathcal{Y}}'$. Observe that

$$End_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}) \cong End_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}') \cong \mathcal{O}_\lambda \tag{5.1}$$

$$End_{E_\lambda[G_F]}(X) \cong End_{E_\lambda[G_F]}(X') \cong E_\lambda \tag{5.2}$$

$$End_{k_\lambda[G_F]}(\overline{\mathcal{X}}) \cong End_{k_\lambda[G_F]}(\overline{\mathcal{X}}') \cong k_\lambda. \tag{5.3}$$

So the representations of G_F on the spaces X, Y, X', Y' (resp. $\overline{\mathcal{X}}, \overline{\mathcal{Y}}, \overline{\mathcal{X}}', \overline{\mathcal{Y}}'$) are absolutely irreducible over E_λ (resp. over k_λ). Hence, the bilinear form ψ_λ^0 cf. (4.4) (resp. $\overline{\psi}_\lambda$ cf. (4.5)) when restricted to any of the spaces X, X', Y, Y', (resp. spaces $\overline{\mathcal{X}}, \overline{\mathcal{X}}', \overline{\mathcal{Y}}, \overline{\mathcal{Y}}'$) is either nondegenerate or isotropic.

We obtain the integral version of Theorem A of [C2].

Theorem 5.4. *If A is of type II, then there is a free $\mathcal{O}_\lambda$-module $\mathcal{W}_\lambda(A)$ of rank $2h$ such that*

- (i) *we have an isomorphism of $\mathcal{O}_\lambda[G_F]$- modules $T_\lambda(A) \cong \mathcal{W}_\lambda(A) \oplus \mathcal{W}_\lambda(A)$*
- (ii) *there is an alternating pairing $\psi_\lambda : \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda$*
- (ii') *the induced alternating pairing $\psi_\lambda^0 : W_\lambda(A) \times W_\lambda(A) \to E_\lambda$ is nondegenerate, where $W_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$*
- (ii") *the induced alternating pairing $\overline{\psi}_\lambda : \overline{\mathcal{W}}_\lambda(A) \times \overline{\mathcal{W}}_\lambda(A) \to k_\lambda$ is nondegenerate, where $\overline{\mathcal{W}}_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$.*

The pairings in (ii), (ii') and (ii") are compatible with the G_F-action in the same way as the pairing in Lemma 4.7 (i).

Proof. (ii') is proven in [C2], while (i) and (ii) are straightforward generalizations of the arguments in *loc. cit.* The bilinear pairing ϕ_l is nondegenerate, hence the bilinear pairing $\overline{\phi}_l$ is nondegenerate, since the abelian variety A is principally polarized by assumption. (Actually $\overline{\phi}_l$ is nondegenerate for any abelian variety with polarization degree prime to l). So, by Lemma 3.2 the form $\overline{\psi}_\lambda$ is nondegenerate for all λ hence simultaneously the forms ψ^0_λ and $\overline{\psi}_\lambda$ are nondegenerate. Now we finish the proof of (ii") arguing for $A[\lambda]$ similarly as it is done for V_λ in [C2], Lemma 3.3. $\square$

From now on we work with the abelian varieties of type either I or II. We assume that the field F of definition of A is such that G_l^{alg} is a connected algebraic group.
Let us put

$$T_\lambda = \begin{cases} T_\lambda(A) & \text{if } A \text{ is of type I} \\ \\ \mathcal{W}_\lambda(A)\,, & \text{if } A \text{ is of type II} \end{cases} \tag{5.5}$$

Let $V_\lambda = T_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ and $A_\lambda = V_\lambda/T_\lambda$. With this notation we have:

$$V_l(A) = \begin{cases} \bigoplus_{\lambda|l} V_\lambda & \text{if } A \text{ is of type I} \\ \\ \bigoplus_{\lambda|l}\big(V_\lambda \oplus V_\lambda\big)\,, & \text{if } A \text{ is of type II} \end{cases} \tag{5.6}$$

We put

$$V_l \;=\; \bigoplus_{\lambda|l} V_\lambda \tag{5.7}$$

Let V_{Φ_λ} be the space V_λ considered over $\mathbb{Q}_l$. We define $\rho_{\Phi_\lambda}(g) = \mathbb{T}_\lambda = \mathbb{A}_\lambda\mathbb{X}_\lambda\mathbb{B}_\lambda$, where $X_\lambda \in GL(V_\lambda)$ is such that $\rho_\lambda(g) = X_\lambda$. (cf. the definition of the map Φ in (2.6) for the choice of $\mathbb{A}_\lambda$ and $\mathbb{B}_\lambda$). Proposition 2.12 motivates the definition of ρ_{Φ_λ}. We have the following equality of $\mathbb{Q}_l$-vector spaces:

$$V_l \;=\; \bigoplus_{\lambda|l} V_{\Phi_\lambda} \tag{5.8}$$

The l-adic representation

$$\rho_l \,:\, G_F \longrightarrow GL(V_l(A)) \tag{5.9}$$

induces the following representations (note that we use the notation ρ_l for both representations (5.9) and (5.10) cf. Remark 5.13):

$$\rho_l : G_F \longrightarrow GL(V_l) \tag{5.10}$$

$$\prod \rho_\lambda : G_F \longrightarrow \prod_\lambda GL(V_\lambda) \tag{5.11}$$

$$\prod \rho_{\Phi_\lambda} : G_F \longrightarrow \prod\nolimits_{\Phi_\lambda} GL(V_{\Phi_\lambda}). \tag{5.12}$$

Remark 5.13. In the case of abelian variety of type II we have $V_l(A) = V_l \oplus V_l$ and the action of G_F on the direct sum is the diagonal one as follows from Theorem 5.4. Hence, the images of the Galois group via the representations (5.9), (5.10) and (5.12) are isomorphic. Also the Zariski closures of the images of these three representations are isomorphic as algebraic varieties over $\mathbb{Q}_l$ in the corresponding GL-groups. Similarly, $V_\lambda(A) = V_\lambda \oplus V_\lambda$ with the diagonal action of G_F on the direct sum by Theorem 5.4. Hence, the images of the representations given by G_F-actions on V_λ and $V_\lambda(A)$ are isomorphic and so are their Zariski closures in corresponding GL-groups. For this reason, in the sequel, we will identify the representation of G_F on $V_l(A)$ (respectively on $V_\lambda(A)$) with its representation on V_l (resp. V_λ).

By Remark 5.13 we can consider G_l^{alg} (resp. G_λ^{alg}) to be the Zariski closure in GL_{V_l} (resp. GL_{V_λ}) of the image of the representation ρ_l of (5.10) (resp. ρ_λ of (5.11)). Let $G_{\Phi_\lambda}^{alg}$ denote the Zariski closure in $GL_{V_{\Phi_\lambda}}$ of the image of the representation ρ_{Φ_λ} of (5.12). Let $\mathfrak{g}_l$ be the Lie algebra of G_l^{alg}, $\mathfrak{g}_\lambda$ be the Lie algebra of G_λ^{alg} and let $\mathfrak{g}_{\Phi_\lambda}$ be the Lie algebra of $G_{\Phi_\lambda}^{alg}$. By definition, we have the following inclusions:

$$G_l^{alg} \subset \prod\nolimits_{\lambda | l} G_{\Phi_\lambda}^{alg} \tag{5.14}$$

$$(G_l^{alg})' \subset \prod\nolimits_{\lambda | l} (G_{\Phi_\lambda}^{alg})' \tag{5.15}$$

$$\mathfrak{g}_l \subset \bigoplus\nolimits_{\lambda | l} \mathfrak{g}_{\Phi_\lambda} \tag{5.16}$$

$$\mathfrak{g}_l^{ss} \subset \bigoplus\nolimits_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}^{ss}. \tag{5.17}$$

The map (5.14) gives a map

$$G_l^{alg} \to G_{\Phi_\lambda}^{alg}, \tag{5.18}$$

which induces the natural map of Lie algebras:

$$\mathfrak{g}_l \to \mathfrak{g}_{\Phi_\lambda}. \tag{5.19}$$

Lemma 5.20. *The map (5.19) of Lie algebras is surjective for any prime $\lambda | l$. Hence the following map of Lie algebras:*

$$\mathfrak{g}_l^{ss} \rightarrow \mathfrak{g}_{\Phi_\lambda}^{ss} \tag{5.21}$$

is surjective.

Proof. We know by the result of Tate, [T2] that the $\mathbb{Q}_l[G_F]$-module $V_l(A)$ is of Hodge-Tate type for any prime v of $\mathcal{O}_F$ dividing l. Hence by the theorem of Bogomolov cf. [Bo] we have

$$\mathfrak{g}_l = \mathcal{L}ie\,(\rho_l(G_F)).$$

Since each $\mathbb{Q}_l[G_F]$-module V_{Φ_λ} is a direct summand of the $\mathbb{Q}_l[G_F]$-module V_l, then the $\mathbb{Q}_l[G_F]$-module V_{Φ_λ} is also of Hodge-Tate type for any prime v of $\mathcal{O}_F$ dividing l. It follows by the theorem of Bogomolov, [Bo] that

$$\mathfrak{g}_{\Phi_\lambda} = \mathcal{L}ie\,(\rho_{\Phi_\lambda}(G_F)).$$

But the surjective map of l-adic Lie groups $\rho_l(G_F) \rightarrow \rho_{\Phi_\lambda}(G_F)$ induces the surjective map of l-adic Lie algebras $\mathcal{L}ie\,(\rho_l(G_F)) \rightarrow \mathcal{L}ie\,(\rho_{\Phi_\lambda}(G_F))$. $\square$

Lemma 5.22. *Let A/F be an abelian variety over F of type I or II such that $End_F\,(A) = End_{\overline{F}}\,(A)$. Then*

$$End_{\mathfrak{g}_\lambda}\,(V_\lambda) \;\cong\; End_{E_\lambda[G_F]}\,(V_\lambda) \;\cong\; E_\lambda \tag{5.23}$$

$$End_{\mathfrak{g}_{\Phi_\lambda}}\,(V_{\Phi_\lambda}) \;\cong\; End_{\mathbb{Q}_l[G_F]}\,(V_{\Phi_\lambda}) \;\cong\; E_\lambda. \tag{5.24}$$

Proof. By [F], Theorem 4, the assumption $End_F\,(A) = End_L\,(A)$ for any finite extension L/F, Theorem 4.16 (ii), the equality (5.2) and Theorem 5.4 we get

$$E_\lambda \cong End_{E_\lambda[G_F]}\,(V_\lambda) \cong End_{E_\lambda[G_L]}\,(V_\lambda). \tag{5.25}$$

This implies the equality

$$End_{G_F}\,(V_\lambda) = End_U\,(V_\lambda)$$

for any open subgroup U of G_F. Hence, the equality (5.23) follows by Lemma 4.12 (iii). For any $F \subset L \subset \overline{F}$ we have $M_{2,2}(End_{\mathbb{Q}_l[G_L]}(V_l)) = End_{\mathbb{Q}_l[G_L]}(V_l^2) = End_{\mathbb{Q}_l[G_L]}(V_l(A))$ and

$$End_{\mathbb{Q}_l[G_L]}(V_l(A)) \cong \prod_{\lambda | l} D_\lambda \;\cong\; \prod_{\lambda | l} M_{2,2}(E_\lambda). \tag{5.26}$$

On the other hand

$$\prod_{\lambda | l} E_\lambda \cong \prod_{\lambda | l} End_{E_\lambda [G_L]}(V_\lambda) \subset End_{\mathbb{Q}_l [G_L]}(V_l). \tag{5.27}$$

Hence, comparing the dimensions over $\mathbb{Q}_l$ in (5.26) and (5.27) we get

$$\prod_{\lambda | l} End_{E_\lambda [G_L]}(V_\lambda) \cong End_{\mathbb{Q}_l [G_L]}(V_l). \tag{5.28}$$

By (5.28) we clearly have

$$\prod_{\lambda | l} End_{\mathbb{Q}_l [G_L]}(V_{\Phi_\lambda}) \subset End_{\mathbb{Q}_l [G_L]}(V_l) \cong \prod_{\lambda | l} E_\lambda, \tag{5.29}$$

and

$$End_{E_\lambda [G_L]}(V_\lambda) \subset End_{\mathbb{Q}_l [G_L]}(V_{\Phi_\lambda}). \tag{5.30}$$

It follows by (5.25), (5.29) and by (5.30) that for any finite field extension $F \subset L$ contained in $\overline{F}$ we have

$$End_{\mathbb{Q}_l [G_L]}(V_{\Phi_\lambda}) \cong End_{E_\lambda [G_L]}(V_\lambda) \cong E_\lambda. \tag{5.31}$$

The isomorphisms (5.31) imply that

$$End_{G_F}(V_{\Phi_\lambda}) \cong End_U(V_{\Phi_\lambda}) \tag{5.32}$$

for any open subgroup U of G_F. The isomorphism (5.24) follows by (5.32) and Lemma 4.12 (iii). □

LEMMA 5.33. $\mathfrak{g}_\lambda^{ss} = sp_{2h}(E_\lambda)$.

Proof. In the proof we adapt to the current situation the argument from [BGK], Lemma 3.2. The only thing to check is the minuscule conjecture for the λ-adic representations $\rho_F : G_F \to GL(V_\lambda)$. By the work of Pink cf. [P], Corollary 5.11, we know that $\mathfrak{g}_l^{ss} \otimes \bar{\mathbb{Q}}_l$ may only have simple factors of types A, B, C or D. By the semisimplicity of $\mathfrak{g}_l^{ss}$ and Lemma 5.20 the simple factors of $\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes \bar{\mathbb{Q}}_l$ are of the same types. By Proposition 2.12 and Lemmas 2.21, 2.22, 2.23 we get

$$\mathfrak{g}_{\Phi_\lambda}^{ss} \cong R_{E_\lambda/\mathbb{Q}_l} \mathfrak{g}_\lambda^{ss}. \tag{5.34}$$

Since

$$\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \cong \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} E_\lambda \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}} \cong \bigoplus_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}$$

we see that the simple factors of $\mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}$ are of types A, B, C or D. The rest of the argument is the same as in the proof of Lemma 3.2 of [BGK]. □

Lemma 5.35. *There are natural isomorphisms of $\mathbb{Q}_l$-algebras.*

$$End_{\mathfrak{g}_{\Phi_\lambda}^{ss}}\,(V_{\Phi_\lambda}) \cong End_{\mathfrak{g}_\lambda^{ss}}\,(V_\lambda) \cong E_\lambda \tag{5.36}$$

Proof. Since $\mathfrak{g}_\lambda$ is reductive and it acts irreducibly on the module V_λ (cf. Lemma 5.33) by [H2], Prop. p. 102 we have:

$$\mathfrak{g}_\lambda = Z(\mathfrak{g}_\lambda) \oplus \mathfrak{g}_\lambda^{ss} \tag{5.37}$$

and $Z(\mathfrak{g}_\lambda) = 0$ or $Z(\mathfrak{g}_\lambda) = E_\lambda$. This gives

$$End_{\mathfrak{g}_\lambda^{ss}}\,(V_\lambda) \;=\; End_{\mathfrak{g}_\lambda}\,(V_\lambda). \tag{5.38}$$

The Weil restriction functor commutes with the operation of taking the center of a Lie algebra, hence we get $Z(\mathfrak{g}_{\Phi_\lambda}) = 0$ or E_λ and by (5.34):

$$\mathfrak{g}_{\Phi_\lambda} = Z(\mathfrak{g}_{\Phi_\lambda}) \oplus \mathfrak{g}_{\Phi_\lambda}^{ss}.$$

Since $\mathfrak{g}_{\Phi_\lambda} \cong R_{E_\lambda/\mathbb{Q}_l}\mathfrak{g}_\lambda$, it is clear that

$$End_{\mathfrak{g}_{\Phi_\lambda}^{ss}}\,(V_{\Phi_\lambda}) \;=\; End_{\mathfrak{g}_{\Phi_\lambda}}\,(V_{\Phi_\lambda}).$$

The lemma follows now from Lemma 5.22. □

Proposition 5.39. *There is an equality of Lie algebras:*

$$\mathfrak{g}_l^{ss} = \bigoplus_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}^{ss} \tag{5.40}$$

Proof. Put $\overline{V}_l = V_l \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$, $\overline{V}_\lambda = V_\lambda \otimes_{E_\lambda} \overline{\mathbb{Q}}_l$, $\overline{\mathfrak{g}}_l^{ss} = \mathfrak{g}_l^{ss} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$, $\overline{\mathfrak{g}}_{\Phi_\lambda}^{ss} = \mathfrak{g}_{\Phi_\lambda}^{ss} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$. By (5.34) we get

$$\overline{\mathfrak{g}}_{\Phi_\lambda}^{ss} \;\cong\; \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} E_\lambda \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \;\cong\; \prod_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l \;\cong\; \prod_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} sp\,(\overline{V}_\lambda) \tag{5.41}$$

By Corollary 1.2.2 of [C1] we have $\mathfrak{g}_l = \mathbb{Q}_l \oplus \mathfrak{g}_l^{ss}$, hence

$$End_{\mathfrak{g}_l^{ss}}\,(V_l(A)) \;=\; End_{\mathfrak{g}_l}\,(V_l(A)).$$

By Lemmas 5.20 and 5.35

$$\prod_{\lambda | l} E_\lambda \;\cong\; \prod_{\lambda | l} End_{\mathfrak{g}_{\Phi_\lambda}^{ss}}\,(V_{\Phi_\lambda}) \;\cong\; \prod_{\lambda | l} End_{\mathfrak{g}_l^{ss}}(V_{\Phi_\lambda}) \subset End_{\mathfrak{g}_l^{ss}}(V_l). \tag{5.42}$$

But by assumption on l and (5.42)

$$\prod_{\lambda | l} D_\lambda \;\cong\; \prod_{\lambda | l} M_{2,2}(E_\lambda) \;\cong\; M_{2,2}(\prod_{\lambda | l} E_\lambda) \subset M_{2,2}(End_{\mathfrak{g}_l^{ss}}(V_l)) \;=$$

$$= End_{\mathfrak{g}_l^{ss}}(V_l(A)) \;=\; End_{\mathfrak{g}_l}(V_l(A)) \;\cong\; \prod_{\lambda | l} D_\lambda. \tag{5.43}$$

Comparing dimensions in (5.43) we get

$$End_{\mathfrak{g}_l^{ss}}(V_l) \;\cong\; \prod_{\lambda | l} E_\lambda. \tag{5.44}$$

Hence we get

$$End_{\overline{\mathfrak{g}}_l^{ss}}(\overline{V}_l) \;\cong\; End_{\mathfrak{g}_l^{ss}}(V_l) \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \;\cong\; \prod_{\lambda | l} E_\lambda \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \;\cong\; \prod_{\lambda | l} \prod_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \overline{\mathbb{Q}}_l. \tag{5.45}$$

$$End_{\overline{\mathbb{Q}}_l[G_F]}(\overline{V}_\lambda) \;\cong\; End_{E_\lambda[G_F]}(V_\lambda) \otimes_{E_\lambda} \overline{\mathbb{Q}}_l \;\cong\; E_\lambda \otimes_{E_\lambda} \overline{\mathbb{Q}}_l \;\cong\; \overline{\mathbb{Q}}_l. \tag{5.46}$$

$$\overline{V}_l \;\cong\; \bigoplus_{\lambda | l} V_\lambda \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \;\cong\; \bigoplus_{\lambda | l} \bigoplus_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \overline{V}_\lambda. \tag{5.47}$$

By (5.21) the map of Lie algebras $\overline{\mathfrak{g}}_l^{ss} \to \overline{\mathfrak{g}}_{\Phi_\lambda}^{ss}$ is surjective. Isomorphisms (5.45), (5.46) and (5.47) show that the simple $\overline{\mathfrak{g}}_l^{ss}$ modules $\mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l$, for all $\lambda | l$ and all $E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l$, are pairwise nonisomorphic submodules of $\overline{\mathfrak{g}}_l^{ss}$. Hence by [H2], Theorem on page 23

$$\bigoplus_{\lambda | l} \bigoplus_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l \;\subset\; \overline{\mathfrak{g}}_l^{ss}. \tag{5.48}$$

Tensoring (5.17) with $\overline{\mathbb{Q}}_l$ and comparing with (5.48) we get

$$\bigoplus_{\lambda | l} \bigoplus_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\mathbb{Q}}_l \;\cong\; \overline{\mathfrak{g}}_l^{\,ss}. \tag{5.49}$$

Hence for dimensional reasons (5.17), (5.41) and (5.49) imply (5.40). □

Corollary 5.50. *The representations ρ_{Φ_λ}, for $\lambda | l$ are pairwise nonisomorphic. The representations of the Lie algebra $\mathfrak{g}_l^{ss}$ on V_{Φ_λ} are pairwise nonisomorphic over $\mathbb{Q}_l$.*

Proof. It follows by Lemmas 5.20 and 5.22 and equalities (5.8), (5.36), (5.44). □

Corollary 5.51. *There is an equality of ranks of group schemes over* $\mathbb{Q}_l$*:*

$$\text{rank } (G_l^{alg})' = \text{rank } \prod_{\lambda|l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{2h}/E_\lambda). \tag{5.52}$$

Proof. The Corollary follows by Lemma 5.33, equality (5.40), the isomorphism (5.34) and Lemma 2.21. $\square$

Taking into account (4.10), (4.11) and Remark 5.13 we get:

$$G(l)^{alg} \subset \prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(GSp_{A_\lambda[\lambda]}) \cong \prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(GSp_{2h}) \tag{5.53}$$

$$G_l^{alg} \subset \prod_{\lambda|l} R_{E_\lambda/\mathbb{Q}_l}(GSp_{V_\lambda}) \cong \prod_{\lambda|l} R_{E_\lambda/\mathbb{Q}_l}(GSp_{2h}). \tag{5.54}$$

6. Computation of the images of the Galois representations ρ_l and $\overline{\rho}_l$.

In this section we explicitly compute the images of the l-adic representations induced by the action of the absolute Galois group on the Tate module of a large class of abelian varieties of types I and II described in the definition below.

Definition of class $\mathcal{A}$. *We say that an abelian variety A/F, defined over a number field F, is of class $\mathcal{A}$, if the following conditions hold:*

(i) *A is a simple, principally polarized abelian variety of dimension g*

(ii) *$\mathcal{R} = End_{\bar{F}}(A) = End_F(A)$ and the endomorphism algebra $D = \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Q}$, is of type I or II in the Albert list of the division algebras with involution cf. [Mu], p. 201*

(iii) *the field F is such that for every l the Zariski closure G_l^{alg} of $\rho_l(G_F)$ in $GL_{2g}/\mathbb{Q}_l$ is a connected algebraic group*

(iv) *$g = hed$, where h is an odd integer, $e = [E : Q]$ is the degree of the center E of D and $d^2 = [D : E]$.*

Let L be a local field with the ring of integers $\mathcal{O}_L$ with maximal ideal $\mathfrak{m}_L = \mathfrak{m}$ and the residue field $k = \mathcal{O}_L/\mathfrak{m}$.

Lemma 6.1. *Let*

$$\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \tag{6.2}$$

be a closed immersion of two smooth, reductive group schemes over $\mathcal{O}_L$. Let

$$G_1 \hookrightarrow G_2 \tag{6.3}$$

be the base change to L of the arrow (6.2) and let

$$G_1(\mathfrak{m}) \hookrightarrow G_2(\mathfrak{m}) \tag{6.4}$$

be the base change to k of the arrow (6.2). If $rank\, G_1 = rank\, G_2$ then $rank\, G_1(\mathfrak{m}) = rank\, G_2(\mathfrak{m})$.

Proof. By [SGA3, Th. 2.5 p. 12] applied to the special point of the scheme $spec\, \mathcal{O}_L$ there exists an étale neighborhood $S' \to spec\, \mathcal{O}_L$ of the geometric point over the special point such that the group schemes $\mathcal{G}_{1,S'} = \mathcal{G}_1 \times_{spec\, \mathcal{O}_L} S'$ and $\mathcal{G}_{2,S'} = \mathcal{G}_2 \times_{spec\, \mathcal{O}_L} S'$ have maximal tori $\mathcal{T}_{1,S'}$ and $\mathcal{T}_{1,S'}$ respectively. By [SGA3] XXII, Th. 6.2.8 p. 260 we observe (we do not need it here but in the Theorem 6.6 below) that $(\mathcal{G}_{i,S'})' \cap \mathcal{T}_{i,S'}$ is a maximal torus of $(\mathcal{G}_{i,S'})'$. By the definition of a maximal torus and by [SGA3] XIX, Th. 2.5, p. 12 applied to the special point of $spec\, \mathcal{O}_L$, we obtain that the special and generic fibers of each scheme $\mathcal{G}_{i,S'}$ have the same rank. But clearly the generic (resp. special) fibers of schemes $\mathcal{G}_{i,S'}$ and $\mathcal{G}_i$ have the same rank for $i = 1, 2$. Hence going around the diagram

$$\begin{tikzcd} G_1 \arrow[r, hook] \arrow[d, hook] & G_2 \arrow[d, hook] \\ \mathcal{G}_1 \arrow[r, hook] & \mathcal{G}_2 \\ G_1(\mathfrak{m}) \arrow[r, hook] \arrow[u, hook] & G_2(\mathfrak{m}) \arrow[u, hook] \end{tikzcd} \tag{6.5}$$

and taking into account the assumptions that the ranks of the upper corners are the same we get $rank\, G_1(\mathfrak{m}) = rank\, G_2(\mathfrak{m})$. $\square$

THEOREM 6.6. *Let A/F be an abelian variety of class $\mathcal{A}$. Then for all $l \gg 0$, we have equalitiy of ranks of group schemes over $\mathbb{F}_l$:*

$$rank\ (G(l)^{alg})' = rank \prod_{\lambda | l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h}) \tag{6.7}$$

Proof. By [LP1] Prop.1.3 and by [Wi], Th.1 and 2.1, for $l \gg 0$ the group scheme $\mathcal{G}_l^{alg}$ over $spec\, \mathbb{Z}_l$ is smooth and reductive. For such an l the structure morphism $(\mathcal{G}_l^{alg})' \to spec\, \mathbb{Z}_l$ is the base change of the smooth morphism

$\mathcal{G}_l^{alg} \rightarrow D_{\mathbb{Z}_l}(D_{\mathbb{Z}_l}(\mathcal{G}_l^{alg}))$ via the unit section of $D_{\mathbb{Z}_l}(D_{\mathbb{Z}_l}(\mathcal{G}_l^{alg}))$, see [SGA3] XXII, Th. 6.2.1, p. 256 where $D_S(G) = \underline{Hom}_{S-gr}(G, \mathbb{G}_{m,S})$ for a scheme S. Hence, the group scheme $(\mathcal{G}_l^{alg})'$ is also smooth over $\mathbb{Z}_l$. By [SGA3] *loc. cit*, the group scheme $(\mathcal{G}_l^{alg})'$ is semisimple. We finish the proof by taking $L = \mathbb{Q}_l$, $\mathcal{G}_1 = (\mathcal{G}_l^{alg})'$, $\mathcal{G}_2 = \prod_{\lambda|l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(Sp_{2h})$ in Lemma 6.1 and applying Corollary 5.51. □

Remark 6.8. If G is a group scheme over S_0 then the derived subgroup G' is defined as the kernel of the natural map

$$G \rightarrow D_{S_0}(D_{S_0}(G))$$

[V], [SGA3]. Since this map is consistent with the base change, we see that for any scheme S over S_0 we get

$$G' \times_{S_0} S = (G \times_{S_0} S)'.$$

Theorem 6.9. *Let A/F be an abelian variety of class $\mathcal{A}$. Then for all $l \gg 0$, we have equalities of group schemes:*

$$(G_l^{alg})' = \prod_{\lambda|l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{2h}) \tag{6.10}$$

$$(G(l)^{alg})' = \prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h}) \tag{6.11}$$

Proof. The proof is similar to the proof of Lemma 3.4 of [BGK]. We prove the equality (6.11). The proof of the equality (6.10) is analogous. Let

$$\underline{\rho}_l : G(l)^{alg} \rightarrow GL_{2g}$$

denote the representation induced by the inclusion $G(l)^{alg} \subset GL_{2g}$. By the result of Faltings cf. [Fa], the representation $\underline{\rho}_l$ is semisimple and the commutant of $\underline{\rho}_l(G(l)^{alg})$ in the matrix ring $M_{2g,2g}$ is $End_{\bar{F}}(A) \otimes_{\mathbb{Z}} \mathbb{F}_l$. The representation $\underline{\rho}_l$ factors through the imbedding (5.53). Projecting onto the λ component in (5.53) we obtain the representation

$$\underline{\rho}_{\Phi_\lambda} : G(l)^{alg} \rightarrow R_{k_\lambda/\mathbb{F}_l}(GSp_{A[\lambda]}) \cong R_{k_\lambda/\mathbb{F}_l}(GSp_{2h}). \tag{6.12}$$

This map corresponds to the map

$$G(l)^{alg} \otimes_{\mathbb{F}_l} k_\lambda \rightarrow GSp_{2h}. \tag{6.13}$$

By Remark 6.8 restriction of the the map (6.13) to the derived subgroups gives the following map:

$$(G(l)^{alg})' \otimes_{\mathbb{F}_l} k_\lambda \to Sp_{2h} \tag{6.14}$$

which in turn gives the representation

$$\underline{\rho}_{\Phi_\lambda} : (G(l)^{alg})' \to R_{k_\lambda/\mathbb{F}_l}(Sp_{2h}).$$

Now by (5.3) we have the natural isomorphisms:

$$\prod_{k_\lambda \hookrightarrow \overline{\mathbb{F}}_l} \overline{\mathbb{F}}_l \;\cong\; k_\lambda \otimes_{\mathbb{F}_l} \overline{\mathbb{F}}_l \;\cong\; End_{k_\lambda \otimes_{\mathbb{F}_l} \overline{\mathbb{F}}_l[G_F]}(A_\lambda[\lambda] \otimes_{\mathbb{F}_l} \overline{\mathbb{F}}_l) \;\cong$$

$$\cong\; End_{k_\lambda \otimes_{\mathbb{F}_l} \overline{\mathbb{F}}_l[G_F]}(A_\lambda[\lambda] \otimes_{k_\lambda} k_\lambda \otimes_{\mathbb{F}_l} \overline{\mathbb{F}}_l) \;\cong$$

$$\cong \prod_{k_\lambda \hookrightarrow \overline{\mathbb{F}}_l} End_{\overline{\mathbb{F}}_l[G_F]}(A_\lambda[\lambda] \otimes_{k_\lambda} \overline{\mathbb{F}}_l). \tag{6.15}$$

Note that $Z(Sp_{2h}) \cong \mu_2$ and this isomorphism holds over any field of definition. The isomorphisms (6.15) imply by the Schur's Lemma:

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{alg})')) \subset R_{k_\lambda/\mathbb{F}_l}(\mu_2).$$

Hence

$$Z((G(l)^{alg})') \subset \prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(\mu_2) = Z(\prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h})).$$

Observe that both groups $(G(l)^{alg})'$ and $\prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h})$ are reductive. Now the proof is finished in the same way as the proof of Lemma 3.4 in [BGK]. □

THEOREM 6.16. *Let A/F be an abelian variety of class $\mathcal{A}$. Then for $l \gg 0$, we have:*

$$\overline{\rho_l}(G'_F) \;=\; \prod_{\lambda|l} Sp_{2h}(k_\lambda) \;=\; Sp_{2h}(\mathcal{O}_E/l\mathcal{O}_E), \tag{6.17}$$

$$\rho_l(\overline{G'_F}) \;=\; \prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda) \;=\; Sp_{2h}(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l), \tag{6.18}$$

where $\overline{\rho_l}$ is the representation ρ_l mod l and $\overline{G'_F}$ is the closure of the commutator subgroup $G'_F \subset G_F$ computed with respect to the natural profinite topology of G_F.

Proof. To prove the equality (6.17), note that the group scheme $\prod_{\lambda|l} R_{k_\lambda/\mathbb{F}_l}(Sp_{2h})$ is simply connected, since its base change to $\overline{\mathbb{F}}_l$ is

$\prod_{\lambda|l}\prod_{k_\lambda\hookrightarrow\overline{\mathbb{F}}_l} Sp_{2h}/\overline{\mathbb{F}}_l$, which is clearly simply connected. From now on the argument is the same as in the proof of Theorem 3.5 in [BGK]. Namely: it follows by (6.11) that $(G(l)^{alg})'$ is simply connected. So $(G(l)^{alg})'(\mathbb{F}_l) = (G(l)^{alg})'(\mathbb{F}_l)_u$. Hence, by a theorem of Serre (cf. [Wi], Th.4) we get

$$(G(l)^{alg})'(\mathbb{F}_l) \subset (\overline{\rho_l}(G_F))' = \overline{\rho_l}(G_F').$$

On the other hand, by definition of the group $G(l)^{alg}$, it is clear that

$$\overline{\rho_l}(G_F') = (\overline{\rho_l}(G_F))' \subset (G(l)^{alg})'(\mathbb{F}_l).$$

As for the second equality in (6.18) we have

$$\rho_l\big(\overline{G_F'}\big) = \overline{(\rho_l(G_F))'} \subset \prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda), \tag{6.19}$$

where $\overline{(\rho_l(G_F))'}$ denotes the closure of $(\rho_l(G_F))'$ in the natural (λ-adic in each factor) topology of the group $\prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda)$. Using equality (6.17) and Lemma 6.20 stated below, applied to $X = \overline{(\rho_l(G_F))'}$, we finish the proof. □

Lemma 6.20. *Let X be a closed subgroup in $\prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda)$ such that its image via the reduction map*

$$\prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda) \to \prod_{\lambda|l} Sp_{2h}(k_\lambda)$$

is all of $\prod_{\lambda|l} Sp_{2h}(k_\lambda)$. Then $X = \prod_{\lambda|l} Sp_{2h}(\mathcal{O}_\lambda)$.

Proof. The proof is similar to the proof of Lemma 3 in [Se] chapter IV, 3.4. □

7. Applications to classical conjectures.

Choose an imbedding of F into the field of complex numbers $\mathbb{C}$. Let $V = H^1(A(\mathbb{C}), \mathbb{Q})$ be the singular cohomology group with rational coefficients. Consider the Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0} \oplus H^{0,1},$$

where $H^{p,q} = H^p(A; \Omega^q_{A/\mathbb{C}})$ and $\overline{H^{p,q}} = H^{q,p}$. Observe that $H^{p,q}$ are invariant subspaces with respect to $D = End_{\overline{F}}(A) \otimes \mathbb{Q}$ action on $V \otimes_{\mathbb{Q}} \mathbb{C}$. Hence, in particular $H^{p,q}$ are E-vector spaces. Let

$$\psi : V \times V \to \mathbb{Q}$$

be the $\mathbb{Q}$-bilinear, nondegenerate, alternating form coming from the Riemann form of A. Since A has a principal polarization by assumption, the form ψ is given by the standard matrix

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Define the cocharacter

$$\mu_\infty : \mathbb{G}_m(\mathbb{C}) \to GL(V \otimes_{\mathbb{Q}} \mathbb{C}) = GL_{2g}(\mathbb{C})$$

such that, for any $z \in \mathbb{C}^\times$, the automorphism $\mu_\infty(z)$ is the multiplication by z on $H^{1,0}$ and the identity on $H^{0,1}$.

Definition 7.1. *The Mumford-Tate group of the abelian variety A/F is the smallest algebraic subgroup $MT(A) \subset GL_{2g}$, defined over $\mathbb{Q}$, such that $MT(A)(\mathbb{C})$ contains the image of μ_∞. The Hodge group $H(A)$ is by definition the connected component of the identity in $MT(A) \cap SL_V \cong MT(A) \cap SL_{2g}$.*

We refer the reader to [D] for an excellent exposition on the Mumford-Tate group. In particular, $MT(A)$ is a reductive group *loc. cit.* Since, by definition

$$\mu_\infty(\mathbb{C}^\times) \subset GSp_{(V,\,\psi)}(\mathbb{C}) \cong GSp_{2g}(\mathbb{C}),$$

it follows that the group $MT(A)$ is a reductive subgroup of the group of symplectic similitudes $GSp_{(V,\,\psi)} \cong GSp_{2g}$ and that

$$H(A) \subset Sp_{(V,\,\psi)} \cong Sp_{2g}. \tag{7.2}$$

Remark 7.3. Let V be a finite dimensional vector space over a field K such that it is also an R-module for a K-algebra R. Let G be a K-group subscheme of GL_V. Then by the symbol $C_R(G)$ we will denote the commutant of R in G. The symbol $C_R^\circ(G)$ will denote the connected component of identity in $C_R(G)$. Let $\beta : V \times V \to K$ be a bilinear form and let $G_{(V,\beta)} \subset GL_V$ be the subscheme of GL_V of all isometries with respect to the bilinear form β. It is easy to check that $C_R(G_{(V,\beta)}) \otimes_K L \cong C_{R\otimes_K L}(G_{(V\otimes_K L,\, \beta\otimes_K L)})$. Note that $MT(A) \subset C_D(GSp_{(V,\,\psi)})$ by definitions.

Definition 7.4. *The algebraic group $L(A) = C_D^\circ(Sp_{(V,\,\psi)})$ is called the Lefschetz group of a principally polarized abelian variety A. Note that the group $L(A)$ does not depend on the form ψ cf. [R2].*

By [D], Sublemma 4.7, there is a unique E-bilinear, nondegenerate, alternating pairing

$$\phi : V \times V \to E$$

such that $Tr_{E/\mathbb{Q}}(\phi) = \psi$. Taking into account that the actions of $H(A)$ and $L(A)$ on V commute with the E-structure, we get

$$H(A) \subset L(A) \subset R_{E/\mathbb{Q}} Sp_{(V,\,\phi)} \subset Sp_{(V,\,\psi)}. \tag{7.5}$$

But $R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}) = C_E(Sp_{(V,\,\psi)})$ hence $C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) = C_D(Sp_{(V,\,\psi)})$ so

$$H(A) \subset L(A) = C_D^\circ(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \subset C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})). \tag{7.6}$$

Definition 7.7. *If $L/\mathbb{Q}$ is a field extension of $\mathbb{Q}$ we put*

$$MT(A)_L := MT(A) \otimes_{\mathbb{Q}} L, \quad H(A)_L := H(A) \otimes_{\mathbb{Q}} L, \quad L(A)_L := L(A) \otimes_{\mathbb{Q}} L.$$

Conjecture 7.8 (Mumford-Tate cf. [Se5], C.3.1). *If A/F is an abelian variety over a number field F, then for any prime number l*

$$(G_l^{alg})^{\circ} = MT(A)_{\mathbb{Q}_l}, \tag{7.9}$$

where $(G_l^{alg})^{\circ}$ denotes the connected component of the identity.

Theorem 7.10 (Deligne [D], I, Prop. 6.2). *If A/F is an abelian variety over a number field F and l is a prime number, then*

$$(G_l^{alg})^{\circ} \subset MT(A)_{\mathbb{Q}_l}. \tag{7.11}$$

Theorem 7.12. *The Mumford-Tate conjecture holds true for abelian varieties of class $\mathcal{A}$ defined in the beginning of Section 6.*

Proof. By [LP1], Theorem 4.3, it is enough to verify (7.9) for a single prime l only. We use the equality (6.10) for a big enough prime l. The proof goes similarly to the proof of Theorem 3.6 in [BGK]. In the proof we will make some additional computations, which provide an extra information on the Hodge group $H(A)$. The Hodge group $H(A)$ is semisimple (cf. [G], Prop. B.63) and the center of $MT(A)$ is $\mathbb{G}_m$ (cf. [G], Cor. B.59). Since $MT(A) = \mathbb{G}_m H(A)$, we get

$$(MT(A)_{\mathbb{Q}_l})' = (H(A)_{\mathbb{Q}_l})' = H(A)_{\mathbb{Q}_l}. \tag{7.13}$$

By (7.11), (7.13) and (6.10)

$$\prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda, \psi_\lambda^0)}) \cong \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{2h}) \subset H(A)_{\mathbb{Q}_l}. \tag{7.14}$$

On the other hand by (7.6)

$$H(A)_{\mathbb{Q}_l} \subset L(A)_{\mathbb{Q}_l} \subset C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \otimes_{\mathbb{Q}} \mathbb{Q}_l. \tag{7.15}$$

Since $R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}) = C_E(Sp_{(V,\,\psi)})$, by Remark 7.3, formulae (7.14) and (7.15) we get:

$$\prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda, \psi_\lambda^0)}) \subset \prod_{\lambda | l} C_{D_\lambda}(R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda(A),\, \psi_\lambda^0)})). \tag{7.16}$$

For A of type I, $D_\lambda = E_\lambda$ and $V_\lambda(A) = V_\lambda$ hence, trivially, the inclusion (7.16) is an equality. Assume that A is of type II. Since $V_\lambda(A) = V_\lambda \oplus V_\lambda$ and

$D_\lambda = M_{2,2}(E_\lambda)$, evaluating both sides of the inclusion (7.16) on the $\overline{\mathbb{Q}}_l$-points, we get equality with both sides equal to

$$\prod_{\lambda | l} \prod_{E_\lambda \hookrightarrow \overline{\mathbb{Q}}_l} (Sp_{(V_\lambda, \phi_\lambda |_{V_\lambda})})(\overline{\mathbb{Q}}_l)$$

which is an irreducible algebraic variety over $\overline{\mathbb{Q}}_l$. Then we use Prop. II, 2.6 and Prop. II, 4.10 of [H] in order to conclude that the groups $H(A)_{\overline{\mathbb{Q}}_l}$, $L(A)_{\overline{\mathbb{Q}}_l}$ and $C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l$ are connected. Hence all the groups $H(A)$, $L(A)$ and $C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}))$ are connected, and we have

$$\text{(7.17)} \qquad \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda, \phi_\lambda |_{V_\lambda})}) \cong \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{2h}) =$$

$$= H(A)_{\mathbb{Q}_l} = L(A)_{\mathbb{Q}_l} = C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

By (6.10), (7.17) and [Bo], Corollary 1. p. 702 we get

$$\text{(7.18)} \qquad MT(A)_{\mathbb{Q}_l} = \mathbb{G}_m H(A)_{\mathbb{Q}_l} = \mathbb{G}_m (G_l^{alg})' \subset G_l^{alg}.$$

The Theorem follows by (7.11) and (7.18). □

Corollary 7.19. *If A is an abelian variety of class $\mathcal{A}$, then*

$$\text{(7.20)} \qquad H(A)_{\mathbb{Q}} = L(A)_{\mathbb{Q}} = C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})) = C_D(Sp_{(V,\,\psi)}).$$

Proof. Taking Lie algebras of groups in (7.17) we deduce by a simple dimension argument that

$$\text{(7.21)} \qquad \mathcal{L}ie\, H(A) = \mathcal{L}ie\, L(A) = \mathcal{L}ie\, C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})).$$

In the proof of Theorem 7.12 we have showed that the groups $H(A)$, $L(A)$ and $C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)}))$ are connected. Hence, by Theorem p. 87 of [H1] we conclude that

$$\text{(7.22)} \qquad H(A) = L(A) = C_D(R_{E/\mathbb{Q}}(Sp_{(V,\,\phi)})). \qquad \square$$

Corollary 7.23. *If A is an abelian variety of class $\mathcal{A}$, then for all l:*

$$\text{(7.24)} \qquad H(A)_{\mathbb{Q}_l} = \prod_{\lambda | l} C_{D_\lambda}(R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda(A),\, \phi \otimes_{\mathbb{Q}} E_\lambda)})).$$

In particular, for $l \gg 0$ we get

$$\text{(7.25)} \qquad H(A)_{\mathbb{Q}_l} = \prod_{\lambda | l} R_{E_\lambda/\mathbb{Q}_l}(Sp_{(V_\lambda,\, \phi \otimes_{\mathbb{Q}} E_\lambda)}).$$

Proof. Equality (7.24) follows immediately from Corollary 7.19. Equality (7.25) follows then from (7.17). □

We have:

$$H^1(A(\mathbb{C}); \mathbb{R}) \cong V \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{\sigma : E \hookrightarrow \mathbb{R}} V \otimes_{E,\sigma} \mathbb{R}.$$

Put $V_\sigma(A) = V \otimes_{E,\sigma} \mathbb{R}$ and let ϕ_σ be the form

$$\phi \otimes_{E,\sigma} \mathbb{R} : V_\sigma(A) \otimes_{\mathbb{R}} V_\sigma(A) \to \mathbb{R}.$$

Lemma 7.26. *If A is simple, principally polarized abelian variety of type II, then for each $\sigma : E \hookrightarrow \mathbb{R}$ there is an $\mathbb{R}$-vector space $W_\sigma(A)$ of dimension $\frac{g}{e} = \frac{4\, dim A}{[D:\mathbb{Q}]}$ such that:*

(i) $V_\sigma(A) \cong W_\sigma(A) \oplus W_\sigma(A)$,

(ii) *the restriction of $\phi \otimes_{\mathbb{Q}} \mathbb{R}$ to $W_\sigma(A)$ gives a nondegenerate, alternating pairing*

$$\psi_\sigma \,:\, W_\sigma(A) \times W_\sigma(A) \to \mathbb{R}.$$

Proof. Using the assumption that $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_{2,2}(\mathbb{R})$ the proof is similar to the proof of Theorem 5.4. □

We put

$$W_{\infty,\sigma} = \begin{cases} V_\sigma(A) & \text{if } A \text{ is of type I} \\ & \\ W_\sigma(A)\,, & \text{if } A \text{ is of type II} \end{cases}$$

and

$$\psi_\sigma = \begin{cases} \phi_\sigma & \text{if } A \text{ is of type I} \\ & \\ \phi_\sigma|_{W_\sigma(A)}\,, & \text{if } A \text{ is of type II.} \end{cases}$$

Observe that

$$dim_{\mathbb{R}}\, W_{\infty,\sigma} = \begin{cases} \frac{2g}{e} = \frac{2\, dim A}{[D:\mathbb{Q}]} & \text{if } A \text{ is of type I} \\ & \\ \frac{g}{e} = \frac{4\, dim A}{[D:\mathbb{Q}]}\,, & \text{if } A \text{ is of type II.} \end{cases}$$

Corollary 7.27. *If A is an abelian variety of class $\mathcal{A}$, then*

$$H(A)_{\mathbb{R}} = L(A)_{\mathbb{R}} = \prod_{\sigma: E \hookrightarrow \mathbb{R}} Sp_{(W_{\infty,\sigma},\, \psi_\sigma)} \tag{7.28}$$

$$H(A)_{\mathbb{C}} = L(A)_{\mathbb{C}} = \prod_{\sigma E \hookrightarrow \mathbb{R}} Sp_{(W_{\infty,\sigma} \otimes_{\mathbb{C}} \mathbb{C},\, \psi_\sigma \otimes_{\mathbb{R}} \mathbb{C})}. \tag{7.29}$$

Proof. It follows from Lemma 7.26 and Corollary 7.19. □

We recall the conjectures of Tate and Hodge in the case of abelian varieties. See [G], [K] and [T1] for more details.

Conjecture 7.30 (Hodge). *If A/F is a simple abelian variety over a number field F, then for every $0 \leq p \leq g$ the natural cycle map induces an isomorphism*

$$A^p(A) \cong H^{2p}(A(\mathbb{C}); \mathbb{Q}) \cap H^{p,p}, \tag{7.31}$$

where $A^p(A)$ is the $\mathbb{Q}$-vector space of codimension p algebraic cycles on A modulo the homological equivalence.

Conjecture 7.32 (Tate). *If A/F is a simple abelian variety over a number field F and l is a prime number, then for every $0 \leq p \leq g$ the cycle map induces an isomorphism:*

$$A^p(A) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H^{2p}_{et}(\overline{A}; \mathbb{Q}_l(p))^{G_F} \tag{7.33}$$

where $\overline{A} = A \otimes_F \overline{F}$.

Theorem 7.34. *The Hodge conjecture holds true for abelian varieties of class $\mathcal{A}$.*

Proof. By [Mu], Theorem 3.1 the Hodge conjecture follows from the equality (7.20) of Corollary 7.19. □

Theorem 7.35. *The Tate conjecture holds true for abelian varieties of class $\mathcal{A}$.*

Proof. It is known (see Proposition 8.7 of [C1]) that Mumford-Tate conjecture implies the equivalence of Tate and Hodge conjectures. Hence the Tate conjecture follows by Theorems 7.12 and 7.34. □

Conjecture 7.36 (Lang). *Let A be an abelian variety over a number field F. Then for $l \gg 0$ the group $\rho_l(G_F)$ contains the group of all homotheties in $GL_{T_l(A)}(\mathbb{Z}_l)$.*

Theorem 7.37 (Wintenberger [Wi], Cor. 1, p. 5). *Let A be an abelian variety over a number field F. The Lang conjecture holds for such abelian varieties A for which the Mumford-Tate conjecture holds or if $\dim A < 5$.*

Theorem 7.38. *The Lang's conjecture holds true for abelian varieties of class $\mathcal{A}$.*

Proof. It follows by Theorem 7.12 and Theorem 7.37. □

We are going to use Theorem 7.12 and Corollary 7.19 to prove an analogue of the open image Theorem of Serre cf. [Se8]. We start with the following remark which is a plain generalization of remark 7.3.

Remark 7.39. Let $B_1 \subset B_2$ be two commutative rings with identity. Let Λ be a free, finitely generated B_1-module such that it is also an R-module for a B_1-algebra R. Let G be a B_1-group subscheme of GL_Λ. Then $C_R(G)$ will denote the commutant of R in G. The symbol $C_R^\circ(G)$ will denote the connected component of identity in $C_R(G)$. Let $\beta : \Lambda \times \Lambda \to B_1$ be a bilinear form and let $G_{(\Lambda,\beta)} \subset GL_\Lambda$ be the subscheme of GL_Λ of the isometries with respect to the form β. Then we check that $C_R(G_{(\Lambda,\beta)}) \otimes_{B_1} B_2 \cong C_{R\otimes_{B_1} B_2}(G_{(\Lambda\otimes_{B_1} B_2, \beta\otimes_{B_1} B_2)})$.

Consider the bilinear form:

$$\psi \ : \ \Lambda \times \Lambda \ \to \mathbb{Z} \tag{7.40}$$

associated with the variety A. Abusing notation sligthly, we will denote by ψ the Riemann form $\psi \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e., we put:

$$\psi \ : \ V \times V \ \to \mathbb{Q}.$$

Consider the group scheme $C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})$ over $Spec\,\mathbb{Z}$. Since $C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)}) \otimes_{\mathbb{Z}} \mathbb{Q} = C_D(Sp_{(V,\,\psi)})$ (see Remark 7.39), there is an open imbedding in the l-adic topology:

$$C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) \subset C_D(Sp_{(V,\,\psi)})(\mathbb{Q}_l). \tag{7.41}$$

Note that the form ψ_l of (4.1) is obtained by tensoring (7.40) with $\mathbb{Z}_l$.

Theorem 7.42. *If A is an abelian variety of class $\mathcal{A}$, then for every prime number l, $\rho_l(G_F)$ is open in the group*

$$C_{\mathcal{R}}(GSp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) = C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l).$$

In addition, for $l \gg 0$ we have:

$$\rho_l(\overline{G_F'}) \ = \ C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l). \tag{7.43}$$

Proof. For any ring with identity R the group $GSp_{2g}(R)$ is generated by subgroups $Sp_{2g}(R)$ and

$$\{\begin{pmatrix} aI_g & 0 \\ 0 & I_g \end{pmatrix}; \, a \in R^{\times}\}.$$

One checks easily that the group $\mathbb{Z}_l^{\times} Sp_{2g}(\mathbb{Z}_l)$ has index 2 (index 4 resp.) in $GSp_{2g}(\mathbb{Z}_l)$, for $l > 2$ (for $l = 2$ resp.). Here the symbol $\mathbb{Z}_l^{\times}$ stands for the subgroup of homotheties in $GL_{2g}(\mathbb{Z}_l)$. Since by assumption A has a principal polarization, $Sp_{2g}(\mathbb{Z}_l) \cong Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l)$. By [Bo], Cor. 1. on p. 702, there is an open subgroup $U \subset \mathbb{Z}_l^{\times}$ such that $U \subset \rho_l(G_F)$. Hence $U\, C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) = C_{\mathcal{R}}(U\, Sp_{(\Lambda,\,\psi)}(\mathbb{Z}_l))$ is an open subgroup of $C_{\mathcal{R}}(GSp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) = C_{\mathcal{R}}(GSp_{(\Lambda,\,\psi)}(\mathbb{Z}_l))$. By [Bo], Th. 1, p. 701, the group $\rho_l(G_F)$ is open in $G_l^{alg}(\mathbb{Q}_l)$. By Theorem 7.12, Corollary 7.19 and Remark 7.3

$$U\, C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) \ \subset \ \mathbb{Q}_l^{\times}\, C_D(Sp_{(V,\,\psi)})(\mathbb{Q}_l) \ =$$

$$= \ \mathbb{G}_m(\mathbb{Q}_l) H(A)(\mathbb{Q}_l) \subset MT(A)(\mathbb{Q}_l) = G_l^{alg}(\mathbb{Q}_l). \tag{7.44}$$

Hence, $U\,C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) \cap \rho_l(G_F)$ is open in $U\,C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l)$ and we get that $\rho_l(G_F)$ is open in $C_{\mathcal{R}}(GSp_{(\Lambda,\,\psi)})(\mathbb{Z}_l)$. Using Remark 7.39 and the universality of the fiber product, we observe that

$$C_{\mathcal{R}}(Sp_{(\Lambda,\,\psi)})(\mathbb{Z}_l) = C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l). \tag{7.45}$$

For $l \gg 0$ we get

$$C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\psi_l)}) \;\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(C_{\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\psi_l)})) \;\cong$$

$$\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(\prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(Sp_{(T_\lambda(A),\,\psi_\lambda)})). \tag{7.46}$$

Evaluating the group schemes in (7.46) on $Spec\,\mathbb{Z}_l$ we get

$$C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(Sp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l) \;\cong\; C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(\prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbb{Z}_l}(Sp_{(T_\lambda(A),\,\psi_\lambda)}))(\mathbb{Z}_l) \;\cong$$

$$\cong\; \prod_{\lambda | l} C_{\mathcal{R}_\lambda} Sp_{(T_\lambda(A),\,\psi_\lambda)}(\mathcal{O}_\lambda) \;\cong\; \prod_{\lambda | l} Sp_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda) \cong \prod_{\lambda | l} Sp_{2h}(\mathcal{O}_\lambda). \tag{7.47}$$

Hence by (7.45), (7.46), (7.47), (6.18) and Theorem 7.38, we conclude that for $l \gg 0$ the equality (7.43) holds. □

Theorem 7.48. *If A is an abelian variety of class $\mathcal{A}$, then for every prime number l, the group $\rho_l(G_F)$ is open in the group $\mathcal{G}_l^{alg}(\mathbb{Z}_l)$ in the l-adic topology.*

Proof. By Theorem 7.42 the group $\rho_l(G_F)$ is open in $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l)$ in the l-adic topology, so $\rho_l(G_F)$ has a finite index in the group $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l)$. By the definition of $\mathcal{G}_l^{alg}$, we have:

$$\rho_l(G_F) \subset \mathcal{G}_l^{alg}(\mathbb{Z}_l) \subset C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l).$$

Hence, $\rho_l(G_F)$ has a finite index in $\mathcal{G}_l^{alg}(\mathbb{Z}_l)$, and the claim follows since $C_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_l}(GSp_{(T_l(A),\,\psi_l)})(\mathbb{Z}_l)$ is a profinite group. □

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