Ring-Theoretic Properties of Iwasawa Algebras: A Survey[1]

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Abstract. This is a survey of the known properties of Iwasawa algebras, i.e., completed group rings of compact p-adic analytic groups with coefficients the ring $\mathbb{Z}_p$ of p-adic integers or the field $\mathbb{F}_p$ of p elements. A number of open questions are also stated.

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1. Introduction

Noncommutative Iwasawa algebras form a large and interesting class of complete semilocal noetherian algebras, constructed as completed group algebras of compact p-adic analytic groups. They were defined and their fundamental properties were derived in M. Lazard's monumental 1965 paper [23], but in the twenty years from 1970 they were little studied. Interest in them has been revived by developments in number theory over the past fifteen years, see for example [17],[19] and [37]. Prompted by this renewed interest, and helped of course by the better understanding of noncommutative noetherian algebra gained since 1965, a number of recent papers have built on Lazard's initial work. The emerging picture is of a class of rings which in some ways look similar to the classical commutative Iwasawa algebras, (which are rings of formal power series in finitely many commuting variables over the p-adic integers), but which in other respects are very different from their commutative counterparts. And while some progress has been made in understanding these rings, many aspects of their structure and representation theory remain mysterious.

It is the purpose of this article to provide a report of what is known about Iwasawa algebras at the present time, and to make some tentative suggestions for

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future research directions. We approach the latter objective through the listing of a series of open questions, scattered throughout the text. In an attempt to make the paper accessible to readers from as wide a range of backgrounds as possible, we have tried to give fairly complete definitions of all terminology; on the other hand, most proofs are omitted, although we have tried to give some short indication of their key points where possible. An exception to the omission of proofs occurs in the discussion of maximal orders in (4.4)-(4.7) as well as in the discussion of the canonical dimension in (5.4), where we include some original material. These paragraphs can be omitted by a reader who simply wants a quick overview of the subject; moreover, after Sections 2 and 3 the remaining sections are reasonably independent of each other.
Fundamental definitions and examples are given in Section 2; in particular we recall the definition of a *uniform* pro-p group in (2.4), and make the important observation (2.3)(1) that every Iwasawa algebra can be viewed as a crossed product of the Iwasawa algebra of a uniform group by a finite group. This has the effect of focusing attention on the Iwasawa algebra of a uniform group - this is filtered by the powers of its Jacobson radical, and the associated graded algebra is a (commutative) polynomial algebra. This fact and its consequences for the structure of the Iwasawa algebras of uniform groups are explored in Section 3; then in Section 4 we examine how properties of general Iwasawa algebras can be deduced from the uniform case using (2.3)(1). Section 5 concerns dimensions: first, the global (projective) dimension and the injective dimension, whose importance is enhanced because Iwasawa algebras satisfy the *Auslander-Gorenstein condition*, whose definition and properties we recall. In particular, Auslander-Gorenstein rings possess a so-called *canonical dimension function*; we explain this and describe some of the properties of the canonical dimension of an Iwasawa algebra in (5.3)-(5.5). The Krull-Gabriel-Rentschler dimension is discussed in (5.7). Finally, our very sparse knowledge of the two-sided ideals of Iwasawa algebras is summarised in Section 6.

2. Key definitions

Iwasawa algebras are completed group algebras. We begin by recalling which groups are involved, then give the definition of the algebras.

2.1. Compact p-adic analytic groups. Let p be a prime integer and let $\mathbb{Z}_p$ denote the ring of p-adic integers. A group G is *compact p-adic analytic* if it is a topological group which has the structure of a p-adic analytic manifold - that is, it has an atlas of open subsets of $\mathbb{Z}_p^n$, for some $n \geq 0$. Such groups can be characterised in a more intrinsic way, thanks to theorems due to Lazard, dating from his seminal 1965 paper [23]. Namely, a topological group G is compact p-adic analytic if and only if G is profinite, with an open subgroup which is pro-p of finite rank, if and only if G is a closed subgroup of $GL_d(\mathbb{Z}_p)$ for some $d \geq 1$. Nowadays, these equivalences are usually viewed as being consequences of deep properties of finite p-groups; a detailed account from this perspective can be found in [20, Part II].

Examples: (1) Every finite group is p-adic analytic, for every prime p.
(2) The abelian p-adic analytic groups are the direct products of finitely many copies of the additive group of $\mathbb{Z}_p$ with a finite abelian group [20, page 36].
(3) For any positive integer d the groups $GL_d(\mathbb{Z}_p)$ and $SL_d(\mathbb{Z}_p)$ are compact p-adic analytic. More generally, given any root system X_ℓ one can form the *universal Chevalley group* $\mathcal{G}_{\mathbb{Z}_p}(X_\ell)$, [20, page 353]. This is a compact p-adic analytic group. For more information about Chevalley groups, see [13].
(4) Let d and t be positive integers. The *t-th congruence subgroup in* $SL_d(\mathbb{Z}_p)$ is the kernel $\Gamma_t(SL_d(\mathbb{Z}_p))$ of the canonical epimorphism from $SL_d(\mathbb{Z}_p)$ to $SL_d(\mathbb{Z}_p/p^t\mathbb{Z}_p)$. One sees at once from the equivalences above that $\Gamma_t(SL_d(\mathbb{Z}_p))$ is compact p-adic analytic, as indeed are $\Gamma_t(GL_d(\mathbb{Z}_p))$ and $\Gamma_t(\mathcal{G}_{\mathbb{Z}_p}(X_\ell))$ for any root system X_ℓ.

Notation: When discussing a topological group G we shall use $\overline{H}$ to denote the closure of a subset H of G in G; and when we refer to, say, G as being *generated by* elements $\{g_1, \dots, g_d\}$ we mean that $G = \overline{\langle g_1, \dots, g_d \rangle}$. In particular, G is *finitely generated* if $G = \overline{\langle X \rangle}$ for a finite subset X of G. For a subset X of G, X^p denotes the subgroup of G generated by the subset $\{x^p : x \in X\}$ of G.

2.2. Iwasawa algebras. Let G be a compact p-adic analytic group. The *Iwasawa algebra of* G is

$$\Lambda_G \quad := \quad \varprojlim \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G. Closely related to Λ_G is its epimorphic image Ω_G, defined as

$$\Omega_G \quad := \quad \varprojlim \mathbb{F}_p[G/N],$$

where $\mathbb{F}_p$ is the field of p elements. Often, a property of Λ_G can easily be deduced from the corresponding property of Ω_G, and vice versa; where this is routine we will frequently save space by stating only one of the two variants.

2.3. Crossed products. Recall [29, 1.5.8] that a *crossed product* of a ring R by a group A is an associative ring $R * A$ which contains R as a subring and contains a set of units $\overline{A} = \{\overline{a} : a \in A\}$, isomorphic as a set to A, such that

- $R * A$ is a free right R-module with basis $\overline{A}$,
- for all $x, y \in A$, $\overline{x}R = R\overline{x}$ and $\overline{x} \cdot \overline{y}R = \overline{xy}R$.

Suppose that H is an open normal subgroup of the compact p-adic analytic group G. Let $\mathcal{C}_H$ denote the set of open normal subgroups of G which are contained in H; then clearly $\Lambda_G = \varprojlim \mathbb{Z}_p[G/U]$ where U runs over $\mathcal{C}_H$. It follows at once that Λ_G is a crossed product of Λ_H by the finite group G/H and similarly that Ω_G is a crossed product of Ω_H by G/H:

$$\begin{array}{lcl} \Lambda_G & \cong & \Lambda_H * (G/H), \\ \Omega_G & \cong & \Omega_H * (G/H). \end{array} \tag{1}$$

We shall see that, combined with a judicious choice of the subgroup H, the isomorphism (1) reduces many questions about Λ_G and Ω_G to the analysis of certain crossed products of finite groups. Usually, the right subgroup H to choose is a *uniform* one, defined as follows.

2.4. Uniform groups. Let G be a pro-p group. Define $P_1(G) = G$ and $P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]}$ for $i \geq 1$. The decreasing chain of characteristic subgroups

$$G = P_1(G) \supseteq P_2(G) \supseteq \cdots \supseteq P_i(G) \supseteq \cdots \supseteq \cap_{i=1}^{\infty} P_i(G) = 1$$

is called the *lower p-series* of G. The group G is *powerful* if $G/\overline{G^p}$ is abelian (for p odd), or $G/\overline{G^4}$ is abelian (when $p = 2$). Finally, G is *uniform* if it is powerful, finitely generated, and

$$|G : P_2(G)| = |P_i(G) : P_{i+1}(G)|$$

for all $i \geq 1$.
Now we can add one further characterisation, also essentially due to Lazard, to those given in (2.1): a topological group G is compact p-adic analytic if and only if it has an open normal uniform pro-p subgroup of finite index, [20, Corollary 8.34].

Examples: (1) Of course, $(\mathbb{Z}_p)^{\oplus d}$ is uniform for all $d \geq 1$.
(2)The groups $\Gamma_1(GL_d(\mathbb{Z}_p))$ (for p odd) and $\Gamma_2(GL_d(\mathbb{Z}_2))$ are uniform [20, Theorem 5.2].

Let G be uniform, with $|G : P_2(G)| = p^d$. The non-negative integer d is called the *dimension* of G; it is equal to the cardinality of a minimal set of (topological) generators of G, [20, Definition 4.7 and Theorem 3.6]. More generally, we can define the dimension of an arbitrary compact p-adic analytic group to be the dimension of any open uniform subgroup; this is unambiguous [20, Lemma 4.6], and coincides with the dimension of G as a p-adic analytic manifold, [20, Definition 8.6 and Theorem 8.36].

2.5. Completed group algebras. In fact Λ_G and Ω_G are I-adic completions of the ordinary group algebras $\mathbb{Z}_p[G]$ and $\mathbb{F}_p[G]$, for suitable choices of ideals I. It is most convenient for us to state the result for uniform groups, although it can obviously be extended to the general case using (2.3)(1).

Theorem. *Let G be a uniform pro-p group, and let I denote the augmentation ideal of $\mathbb{F}_p[G]$. Then Ω_G is isomorphic to the I-adic completion of $\mathbb{F}_p[G]$. There is a similar result for $\mathbb{Z}_p[G]$.*

Indeed the theorem follows quite easily from the observations that the lower p-series $P_i(G)$ is coterminal with the family of all open normal subgroups of G, and that the powers of I are coterminal with the ideals of $\mathbb{F}_p[G]$ generated by the augmentation ideals of the subgroups $P_i(G)$, [20, §7.1].

3. The case when G is uniform

Throughout this section, we assume that G is a uniform pro-p group of dimension d. We fix a topological generating set $\{a_1, \ldots, a_d\}$ for G.

3.1. THE "PBW" THEOREM. It follows at once from Theorem 2.5 that the usual group algebra $\mathbb{F}_p[G]$ embeds into Ω_G. For $i = 1, \ldots, d$, let $b_i = a_i - 1 \in \mathbb{F}_p[G] \subseteq \Omega_G$. Then we can form various monomials in the b_i: if $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a d-tuple of nonnegative integers, we define

$$\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d} \in \Omega_G.$$

Note that this depends on our choice of ordering of the b_i's, because Ω_G is noncommutative unless G is abelian. The following basic result shows that Ω_G is a "noncommutative formal power series ring"; it follows from the strong constraints which the hypothesis of uniformity imposes on the quotients $P_i(G)/P_{i+1}(G)$ of G, [20, Theorem 7.23].

THEOREM. *Every element c of Ω_G is equal to the sum of a uniquely determined convergent series*

$$c = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \mathbf{b}^\alpha$$

where $c_\alpha \in \mathbb{F}_p$ for all $\alpha \in \mathbb{N}^d$.

We record an immediate consequence of both this result and of Theorem 2.5:

COROLLARY. *The Jacobson radical J of Ω_G is equal to*

$$J = b_1\Omega_G + \cdots + b_d\Omega_G = \Omega_G b_1 + \cdots + \Omega_G b_d.$$

Hence $\Omega_G/J \cong \mathbb{F}_p$, so in the language of (4.1), Ω_G is a scalar local ring.

Proof. If $c \in \Omega_G$ is such that $c_0 \neq 0$, then $1 - c$ is invertible with inverse $1 + c + c^2 + \cdots \in \Omega_G$. □

Theorem 3.1 says that the monomials $\{\mathbf{b}^\alpha : \alpha \in \mathbb{N}^d\}$ form a topological basis for Ω_G, and is thus analogous to the classical Poincaré-Birkhoff-Witt theorem for Lie algebras $\mathfrak{g}$ over a field k which gives a vector space basis for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ in terms of monomials in a fixed basis for $\mathfrak{g}$ [21]. Nevertheless we should bear in mind that explicit computations in Ω_G are often much more difficult than those in $\mathcal{U}(\mathfrak{g})$, since the Lie bracket of two generators b_i, b_j for Ω_G is in general an infinite power series with obscure coefficients.

3.2. EXAMPLE. Let p be odd for simplicity and let $G = \Gamma_1(SL_2(\mathbb{Z}_p))$ be the first congruence kernel of $SL_2(\mathbb{Z}_p)$. Then

$$a_1 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

is a topological generating set for G. Setting $b_i = a_i - 1$, elementary (but tedious) computations yield

$$\begin{array}{rcll} [b_1, b_2] & \equiv & 2b_2^p & \mod J^{p+1} \\ [b_1, b_3] & \equiv & -2b_3^p & \mod J^{p+1} \\ [b_2, b_3] & \equiv & b_1^p & \mod J^{p+1}. \end{array}$$

Here $J = b_1\Omega_G + b_2\Omega_G + b_3\Omega_G$ denotes the Jacobson radical of Ω_G. Using Proposition 3.3 it is possible to produce more terms in the power series expansion of $[b_1, b_2]$ and $[b_1, b_3]$. However, we consider $[b_2, b_3]$ to be inaccessible to computation.

3.3. Skew power series rings. It is well known that if $\mathfrak{g}$ is a finite dimensional soluble Lie algebra over a field k, then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be thought of as an "iterated skew polynomial ring":

$$\mathcal{U}(\mathfrak{g}) \cong k[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$$

for some appropriate automorphisms σ_i and derivations δ_i (in fact, the σ_is can be chosen to be trivial). This is because any such Lie algebra $\mathfrak{g}$ has a chain of subalgebras

$$0 = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \cdots \subset \mathfrak{h}_n = \mathfrak{g}$$

with $\mathfrak{h}_{i-1}$ an ideal in $\mathfrak{h}_i$, so choosing some $x_i \in \mathfrak{h}_i \backslash \mathfrak{h}_{i-1}$ ensures that

$$\mathcal{U}(\mathfrak{h}_i) \cong \mathcal{U}(\mathfrak{h}_{i-1})[x_i; \delta_i]$$

where δ_i is the derivation on $\mathcal{U}(\mathfrak{h}_{i-1})$ defined by $\delta_i(y) = x_i y - y x_i$.
An analogous result holds for Iwasawa algebras. More precisely, we have the

Proposition. *Suppose that G has closed normal subgroup H such that $G/H \cong \mathbb{Z}_p$. Then Ω_G is a skew power series ring with coefficients in Ω_H:*

$$\Omega_G \cong \Omega_H[[t; \sigma, \delta]].$$

Proof. See [41, §4]. □

Schneider and Venjakob [41] establish a general theory of skew power series rings $S = R[[t; \sigma, \delta]]$ over a pseudocompact ring R. Here σ can be any topological automorphism of R and δ is a σ-derivation in the sense of [29, 1.2.1], satisfying some extra conditions which are required to make the relation

$$ta = \sigma(a)t + \delta(a)$$

extend to a well-defined multiplication on S.
Consequently, the Iwasawa algebra Ω_G of any soluble uniform pro-p group G can be thought of as an iterated skew power series ring over $\mathbb{F}_p$.
For example, in Example 3.2, the topological subring of Ω_G generated by b_1 and b_2 is actually the Iwasawa algebra Ω_B where $B = \overline{\langle a_1, a_2 \rangle}$ is a Borel subgroup of G. Since B is soluble with closed normal subgroup $\overline{\langle a_2 \rangle}$, Ω_B is isomorphic to the skew power series ring $\mathbb{F}_p[[b_2]][[b_1; \sigma, \delta]]$ for some appropriate σ and δ. This justifies the claim that the commutator of b_1 and b_2 is at least partially accessible to computation.
There is surely considerable scope to develop further the "abstract" theory of skew power series algebras initiated in [41] - for instance, one could easily pose skew power series versions of a number of the questions we list later, in Section 6. As a prompt for more work, here are two "general" questions:

Question A. *(1) Are there conditions on R, σ and δ such that $S = R[[t;\sigma,\delta]]$ can be described without involving a derivation - that is, as $S = R'[[t';\sigma']]$, possibly after some Ore localisation?*[1]
(2) Are there conditions on R, σ and δ such that every two-sided ideal of the skew power series ring $S = R[[t;\sigma,\delta]]$ is generated by central elements and "polynomial" elements[2]*?*

3.4. The J-adic filtration. We remind the reader that a *filtration* on a ring R is an ascending sequence

$$\cdots \subseteq F_i R \subseteq F_{i+1} R \subseteq \cdots$$

of additive subgroups such that $1 \in F_0 R$, $F_i R.F_j R \subseteq F_{i+j} R$ for all $i, j \in \mathbb{Z}$, and $\cup_{i\in\mathbb{Z}} F_i R = R$.
Let J denote the Jacobson radical of Ω_G. The J-adic filtration on Ω_G is defined as follows: $F_i \Omega_G = J^{-i}$ for $i \leq 0$ and $F_i \Omega_G = \Omega_G$ f or $i \geq 0$; this is an example of a *negative* filtration. The basic tool which allows one to deduce many ring-theoretic properties of Iwasawa algebras is the following result, which can be deduced from Theorem 3.1, see [20, Theorem 7.24 and remarks on page 160]. We denote the associated graded ring $\bigoplus_{i\in\mathbb{Z}} F_{i+1}\Omega_G / F_i \Omega_G$ by $\operatorname{gr}_J \Omega_G$.

Theorem. *The graded ring of Ω_G with respect to the J-adic filtration is isomorphic to a polynomial ring in $d = \dim G$ variables:*

$$\operatorname{gr}_J \Omega_G \cong \mathbb{F}_p[X_1, \ldots, X_d].$$

Moreover, Ω_G is complete with respect to this filtration.

The J-adic filtration is quite different from the filtrations encountered when studying algebras like universal enveloping algebras and Weyl algebras, which are nearly always *positive* (that is, $F_{-1}R = 0$) and often satisfy the finiteness condition $\dim_k F_i R < \infty$ for all $i \in \mathbb{Z}$. In particular, there is no well-behaved notion of the Gel'fand-Kirillov dimension for Iwasawa algebras, a theme we will return to in §5.
However, we are still able to lift many properties of the graded ring back to Ω_G, because the J-adic filtration is *complete*, meaning that Cauchy sequences of elements in Ω_G converge to unique limits. More precisely, recall [26, page 83] that a filtration on a ring R is said to be *Zariskian*, whenever

- The Jacobson radical of $F_0 R$ contains $F_{-1} R$, and
- The Rees ring $\widetilde{R} := \bigoplus_{i\in\mathbb{Z}} F_i R \cdot t^i \subseteq R[t, t^{-1}]$ is noetherian.

Many filtrations are Zariskian. For example, by [26, Chapter II, Proposition 2.2.1], any complete filtration whose associated graded ring is noetherian is necessarily Zariskian. Since any positive filtration is complete, it follows that if a filtration is positive and has noetherian associated graded ring, then it is Zariskian. More importantly for us, for any uniform pro-p group G, the J-adic filtration on Ω_G is clearly complete, thanks to Theorem 2.5; and $\operatorname{gr}_J \Omega_G$ is

[1] Compare with [14].
[2] By the latter, we mean elements of $R[t;\sigma,\delta]$.

noetherian by Theorem 3.4 and Hilbert's basis theorem, so the J-adic filtration is Zariskian.

3.5. The $\mathfrak{m}$-adic filtration on Λ_G. There is an analogue of Theorem 3.4 for the $\mathbb{Z}_p-$version of Iwasawa algebras Λ_G. Recall from (2.3) the lower p-series $P_1(G) \supseteq P_2(G) \supseteq \cdots \supseteq \cap_{i=1}^{\infty} P_i(G) = 1$ of G and define an abelian group

$$\operatorname{gr} G := \bigoplus_{i=1}^{\infty} \frac{P_i(G)}{P_{i+1}(G)}.$$

There is a natural way of turning $\operatorname{gr} G$ into a Lie algebra over $\mathbb{F}_p[t]$, the polynomial ring in one variable over $\mathbb{F}_p$: the Lie bracket on $\operatorname{gr} G$ is induced from the Lie bracket on G described in [20, §4.5], and the action of t is induced from the p-power map. Then $\operatorname{gr} G$ is a free $\mathbb{F}_p[t]$-module of rank equal to $\dim G$. Let $\mathfrak{m} = \ker(\Lambda_G \to \mathbb{F}_p)$ be the $\mathbb{F}_p$-augmentation ideal of Λ_G, or equivalently, the Jacobson radical of Λ_G.

Theorem. *The graded ring of Λ_G with respect to the $\mathfrak{m}$-adic filtration is isomorphic to the universal enveloping algebra of the $\mathbb{F}_p[t]$-Lie algebra* $\operatorname{gr} G$*:*

$$\operatorname{gr}_{\mathfrak{m}} \Lambda_G \cong \mathcal{U}(\operatorname{gr} G).$$

Moreover, Λ_G is complete with respect to this filtration.

Proof. See [39, §3.3] and [23, Chapter III, Theorem 2.3.3]. □

3.6. Lifting information from the graded ring. We recall here some standard properties of a ring R. First, we say that R is *prime* if the product of any two non-zero ideals of R is again non-zero. By Goldie's theorem [29, Theorem 2.3.6], if R is prime and (right) noetherian then it has a simple artinian classical (right) quotient ring $Q(R)$. If S is another ring with classical right quotient ring $Q(R)$, so that $Q(R) = Q(S)$, we say that R and S are *equivalent* if there are units a, b, c and d in $Q(R)$ such that $aRb \subseteq S$ and $cSd \subseteq R$. Now R is a *maximal (right) order* if it is maximal (with respect to inclusion) within its equivalence class, [29, 5.1.1]. (The adjective right is omitted if R is both a maximal right order and a maximal left order.) The commutative noetherian maximal orders are just the noetherian integrally closed domains [29, Lemma 5.3.3].

Let R_R denote the right R-module R. The *Krull dimension* $\mathcal{K}(M)$ of a finitely generated (right) module M over a noetherian ring R is a well-defined ordinal, bounded above by $\mathcal{K}(R_R)$; the precise definition can be found at [29, 6.2.2]. This concept generalises the classical commutative definition; like it, it measures the "size" of a module and is 0 if and only if the module is non-zero and artinian.

The *(right) global dimension* of R is defined to be the supremum of the projective dimensions (denoted $\operatorname{pd}(-)$) of the right R-modules, [29, 7.1.8]. When R is noetherian, its right and left global dimensions are always equal, [29, 7.1.11]. We say that R has *finite (right) injective dimension d* if there is an injective resolution of R_R of length d, but none shorter. If R is noetherian and has finite right and left injective dimensions, then these numbers are equal by [45, Lemma A]. It is also well known [39, Remark 6.4] that if the (right) global dimension of the noetherian ring R is finite, then it equals the (right) injective dimension of R.

It has become apparent over the past 40 years that, when R is noncommutative and noetherian, finiteness of the injective dimension of R is a much less stringent condition than is the case for commutative noetherian rings - the structure of (commutative) Gorenstein rings is rich and beautiful. An additional hypothesis which, when coupled with finite injective dimension, has proved very useful in the noncommutative world is the *Auslander-Gorenstein condition.* To recall the definition, note first that, for every left R-module M and every non-negative integer i, $\mathrm{Ext}^i(M, R)$ is a right R-module through the right action on R. The Auslander-Gorenstein condition on a noetherian ring R requires that, when M is a finitely generated left R-module, i is a non-negative integer and N is a finitely generated submodule of $\mathrm{Ext}^i(M, R)$, then $\mathrm{Ext}^j(N, R)$ is zero for all j strictly less than i; and similarly with "right" and "left" interchanged. We say that R is *Auslander-Gorenstein* if it is noetherian, has finite right and left injective dimensions, and satisfies the Auslander condition. Commutative noetherian rings of finite injective dimension are Auslander-Gorenstein. When R is noetherian of finite global dimension and satisfies the Auslander-Gorenstein condition it is called *Auslander-regular.*

Theorem. *Let R be a ring endowed with a Zariskian filtration FR; then R is necessarily noetherian. Also, R inherits the following properties from* $\mathrm{gr}\, R$*:*

(1) being a domain,
(2) being prime,
(3) being a maximal order,
(4) being Auslander-Gorenstein,
(5) having finite global dimension,
(6) having finite Krull dimension.

Proof. See [26]. □

We immediately obtain from Theorem 3.4, Theorem 3.6 and Corollary 3.1, the

Corollary. *If G is a uniform pro-p group, then Ω_G is a noetherian, Auslander-regular, scalar local domain which is a maximal order in its quotient division ring of fractions.*

4. Extensions over finite index

For an arbitrary p-adic analytic group G, many fundamental properties of Ω_G (and of Λ_G) can be analysed using Corollary 3.6 and (2.3)(1).

4.1. Complete noetherian (semi)local rings. Recall that a ring R is *semilocal* if the factor of R by its Jacobson radical $J(R)$ is semisimple artinian. It is *local* if $R/J(R)$ is simple artinian, and *scalar local* if $R/J(R)$ is a division ring. For a crossed product $R = S * H$ of a finite group H, like that in (2.3)(1), it's not hard to show that $J(S) \subseteq J(R)$, [31, Theorem 1.4.2]. From this, Theorem 2.5 and Corollary 3.6, and their analogues for Λ_G, we deduce (1) of the following. Both it and (2) were known to Lazard.

Theorem. *Let G be a compact p-adic analytic group.*

(1) Ω_G and Λ_G are complete noetherian semilocal rings.
(2) Ω_G and Λ_G are (scalar) local rings if and only if G is a pro-p group.

4.2. Primeness and semiprimeness. Recall that a ring R is *prime* if the product of two nonzero ideals is again nonzero and that R is *semiprime* if it has no nonzero nilpotent ideals. A prime ring is always semiprime, but not necessarily conversely.

The characterisations of these properties for Iwasawa algebras given in the theorem below exactly parallel the results for ordinary group algebras proved in the early 1960s by I.G. Connell and D.S. Passman [32, Theorems 4.2.10 and 4.2.14]. However, the proofs here are quite different from the classical setting; that the stated conditions are necessary is easy to see, but sufficiency in (1) and (2) depends on Corollary 3.6 to handle the uniform case, together with non-trivial results on crossed products of finite groups. Part (3) is much easier - one can simply appeal to the fact (a consequence of Maschke's theorem) that the group ring of a finite group over a commutative coefficient domain of characteristic zero is semiprime, together with the fact that, by definition, Λ_G is an inverse limit of such group rings.

Theorem. *Let G be a compact p-adic analytic group.*

(1) [5] *Ω_G and Λ_G are prime if and only if G has no non-trivial finite normal subgroups.*
(2) [5] *Ω_G is semiprime if and only if G has no non-trivial finite normal subgroups of order divisible by p.*
(3) (Neumann, [30]) *Λ_G is always semiprime.*

4.3. Zero divisors. There is a method, familiar from the treatment of ordinary group rings, which allows one to use homological properties to deduce results about the non-existence of zero divisors in certain noetherian rings. In its simplest form, which is all that is needed here, the statement is due to Walker [42]: *if R is a scalar local noetherian semiprime ring of finite global dimension, then R is a domain.*[3] This yields the following result; it was proved by Neumann [30] for Λ_G, but for Ω_G it was necessary to wait first for semiprimeness to be settled, as in Theorem 4.2(2).

Theorem. *Let G be a compact p-adic analytic group. Then Ω_G and Λ_G are domains if and only if G is torsion free.*

Proof. If $1 \neq x \in G$ with $x^n = 1$, then $(1-x)(1+x+\cdots x^{n-1}) = 0$, so the absence of torsion is clearly necessary. Suppose that G is torsion free. Since G

[3] It is a famous and long-standing open question in ring theory whether "semiprime" is necessary in Walker's theorem.

has a pro-p subgroup of finite index by (2.4), its Sylow q-subgroups are finite for primes q not equal to p. Since G is torsion free these subgroups are trivial, so G is a pro-p group. Therefore Ω_G and Λ_G are scalar local and noetherian by Theorem 4.1. The other conditions needed for Walker's theorem are given by Theorems 4.2(2) and (3) and Theorem 5.1. □

4.4. Maximal orders. It might seem natural to suppose, in the light of Theorem 3.6(3), that whenever Λ_G or Ω_G are prime then they are maximal orders. This guess is wrong, though, as the following example shows. First, recall from [29, 5.1.7] that if R is a ring and M is an R-module, then M is said to be *reflexive* if the natural map $M \to M^{**} = \operatorname{Hom}(\operatorname{Hom}(M,R),R)$ is an isomorphism. Also, recall [29, Chapter 4] that the ideal I of R is said to be *localisable* if the set $\mathcal{C}_R(I)$ of elements of R which are regular modulo I is an Ore set in R.

Example: Let $D := A \rtimes \langle\gamma\rangle$, where A is a copy of $\mathbb{Z}_2$ and γ is the automorphism of order 2 sending each 2-adic integer to its negative. Since D is a pro-2 group with no non-trivial finite normal subgroups, Theorems 4.1 and 4.2 show that Ω_D and Λ_D are prime noetherian scalar local rings. But it's not hard to see that neither of these algebras is a maximal order: for Ω_D, observe that it is local with reflexive Jacobson radical J which is not principal, impossible for a prime noetherian maximal order by [28, Théorème IV.2.15]; for Λ_D, the kernel of the canonical map to $\mathbb{Z}_p$ is a reflexive prime ideal which is not localisable by [4, Theorem A and Lemma 4.1], impossible in a maximal order by [28, Corollaire IV.2.14]. We therefore ask:

Question B. *When are Ω_G and Λ_G maximal orders?*

Since the powerful structural results [15], which can be obtained for certain quotient categories of the category of finitely generated modules over a noetherian maximal order, are potentially important tools in arithmetic applications [18], this question is of more than passing interest.

In the next three paragraphs we offer a conjecture for the answer to Question B, and give some evidence in its support.

4.5. Conjectured answer to Question B. We will need some group-theoretic notions. Let H be a closed subgroup of a compact p-adic analytic group G. We say that H is *orbital* if H has finitely many G-conjugates, or equivalently if its normaliser $N = N_G(H)$ has finite index in G. We say that an orbital subgroup H is *isolated* if N/H has no non-trivial finite normal subgroups.

We will say that G is *dihedral-free* if, whenever H is an orbital closed subgroup of G with $\dim H = 1$, H is isomorphic to $\mathbb{Z}_p$. This seems to be the correct generalisation of the definition in [9].

Conjecture. *Let G be a compact p-adic analytic group, and suppose Ω_G is prime. Then Ω_G is a maximal order if and only if G is dihedral-free.*

4.6. Necessary conditions on G. We fix a prime p and assume throughout this paragraph that G is a compact p-adic analytic group.

Proposition. *Suppose Ω_G is a prime maximal order and let H be a closed* normal *subgroup of G with $\dim H = 1$. Then H is pro-p.*

Proof. We may assume that H is isolated, so G/H has no non-trivial finite normal subgroups. Hence, by Theorem 4.2(1), $w_H = \ker(\Omega_G \to \Omega_{G/H})$ is a prime ideal of Ω_G, and it is not hard to see that it is also a reflexive ideal.[4] Now because Ω_G is a maximal order and w_H is a prime reflexive ideal, it must be localisable [28, Corollaire IV.2.14].
But the conditions needed for augmentation ideals to be localisable are known [5, Theorem E]: H/F must be pro-p, where F is the largest finite normal p'-subgroup of H. Since H is normal in G and G has no non-trivial finite normal subgroups by Theorem 4.2(1), $F = 1$ and H is pro-p as required. $\square$

We need the following group-theoretic lemma. We first set ϵ to be 1 for p odd, and $\epsilon = 2$ if $p = 2$, and define, for a closed normal uniform subgroup N of G, $E_G(N)$ to be the centraliser in G of N/N^{p^ϵ}, [5, (2.2)].

Lemma. *Suppose that G is a pro-p group of finite rank with no non-trivial finite normal subgroups. Let N be a maximal open normal uniform subgroup of G. Then*

$$E_G(N) = N.$$

Proof. Recall that $E = E_G(N)$ is an open normal subgroup of G containing N. If E strictly contains N then E/N must meet the centre $Z(G/N)$ non-trivially since G/N is a finite p-group by [20, Proposition 1.11(ii)]. Pick $x \in E \backslash N$ such that $xN \in Z(G/N)$; then $H = \langle N, x \rangle$ is normal in G by the choice of x, and also H is uniform by [5, Lemma 2.3]. This contradicts the maximality of N. $\square$

Recall from Example 4.4 that D denotes the pro-2 completion of the infinite dihedral group.

Corollary. *Let H be a pro-p group of finite rank with no non-trivial finite normal subgroups. Suppose that* $\dim H = 1$. *Then $H \cong \mathbb{Z}_p$, unless $p = 2$ and H is isomorphic to D.*

Proof. Choose a maximal open normal uniform subgroup N of H. By the lemma, $H/N \hookrightarrow \mathrm{Aut}(N/N^{p^\epsilon})$. If p is odd, $|N : N^{p^\epsilon}| = p$, so the latter automorphism group is just $\mathbb{F}_p^\times$. Since H/N is a p-group by [20, Proposition 1.11(ii)] again, $H = N \cong \mathbb{Z}_p$. If $p = 2$ and $H > N$, $H \cong D$. $\square$

This gives us the following weak version of one half of the conjecture. To improve the result from "normal" to "orbital" will presumably require some technical work on induced ideals.

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[4] One quick way to see this uses the canonical dimension from (5.4): since $\mathrm{Cdim}(\Omega_G/w_H) = \dim(G/H) = \dim G - 1$ and since Ω_G is Auslander-Gorenstein, w_H is reflexive by Gabber's Maximality Principle [36, Theorem 2.2].

Corollary. *Suppose Ω_G is a prime maximal order. Then any closed normal subgroup H of G of dimension 1 is isomorphic to $\mathbb{Z}_p$.*

Proof. When p is odd the statement is immediate from the proposition and corollary above. So suppose that $p = 2$. We have to rule out the possibility that $H \cong D$, so suppose for a contradiction that this is the case. Then, as in the proof of the proposition, w_H is a prime reflexive, and hence localisable, ideal of Ω_G. Let R denote the local ring $(\Omega_G)_{w_H}$, which has global dimension one by [28, Théorème IV.2.15]. Let $C = \langle c \rangle$ be a copy of the cyclic group of order 2 in H. Then $\mathbb{F}_2 C \subseteq \Omega_G$ and Ω_G is a projective $\mathbb{F}_2 C$-module by [11, Lemma 4.5]. Thus R is a flat $\mathbb{F}_2 C$-module. Since $c+1 \in J(R)$, the $\mathbb{F}_2 C$-module $R/J(R)$ is a sum of copies of the trivial module, so

$$\infty = \mathrm{pd}_{\mathbb{F}_2 C}(\mathbb{F}_2) = \mathrm{pd}_{\mathbb{F}_2 C}(R/J(R)) \leq \mathrm{pd}_R(R/J(R)) = 1.$$

This contradiction shows that the only possibility for H is $\mathbb{Z}_2$. □

4.7. Sufficient conditions on G. We use the following result, essentially due to R. Martin:

Proposition. [27] *Let R be a prime noetherian maximal order and let F be a finite group. Let $S = R * F$ be a prime crossed product. Then S is a maximal order if and only if*

(a) every reflexive height 1 prime P of S is localisable, and
(b) $\mathrm{gld}(S_P) < \infty$ for all such P.

Proof. Conditions (a) and (b) hold in any prime noetherian maximal order, [28, Théorème IV.2.15]. Conversely, suppose that (a) and (b) hold. We use the Test Theorem [27, Theorem 3.2]. Condition (i) of the Test Theorem is just condition (a). We claim that if P is as in the theorem, then $\mathrm{gld}(S_P) = 1$. It's easy to check that $P \cap R$ is a semiprime reflexive ideal of R, so that the localisation $R_{P\cap R}$ exists and is hereditary by [28, Théorème IV.2.15]. Thus $R_{P\cap R} * F$ has injective dimension 1 by [5, Corollary 5.4]. But S_P is a localisation of $R_{P\cap R} * F$, so - given (b) and the comments in (3.6) - $\mathrm{gld}(S_P) \leq 1$. The reverse inequality is obvious, so our claim follows. Condition (ii) now follows from [27, Proposition 2.7]. Condition (iii) follows from the proof of [27, Lemma 3.5] and condition (iv) follows from [27, Remark 3.6 and Lemma 3.7]. □

Lemma. *Let G be a pro-p group of finite rank with no non-trivial finite normal subgroups. Then every reflexive height 1 prime of Ω_G is localisable.*

Proof. Let P be a reflexive height 1 prime of Ω_G. Choose an open normal uniform subgroup N of G. Then Ω_N is a maximal order by Corollary 3.6. Set $\overline{G} := G/N$. Now let $Q = P \cap \Omega_N$ - it is easy to see [27, Remark 3.6] that this is a height 1 reflexive $\overline{G}$-prime ideal of Ω_N. Indeed, Q is the intersection of a $\overline{G}$-orbit of reflexive prime ideals $\{P_1, \ldots, P_n\}$ of Ω_N.
Since each P_i is localisable by [28, Théorème IV.2.15], Q is localisable. In other words, the subset $\mathcal{C} := \mathcal{C}_{\Omega_N}(Q) = \cap_{i=1}^n \mathcal{C}_{\Omega_N}(P_i)$ is a $\overline{G}$-invariant Ore set in Ω_N. An easy calculation [32, proof of Lemma 13.3.5(ii)] shows that $\mathcal{C}$ is an Ore set

in Ω_G. In other words, the semiprime ideal $A = \sqrt{Q\Omega_G}$ is localisable in Ω_G and

$$(\Omega_N)_Q * \overline{G} \cong (\Omega_G)_A.$$

Since $\overline{G}$ is a p-group, $A = P$ by [31, Proposition 16.4] and the result follows. $\square$

Corollary. *Let G be a torsion free compact p-adic analytic group. Then Ω_G is a prime maximal order.*

Proof. Suppose that G is as stated. Since G has a pro-p open subgroup, the Sylow q-subgroups of G are finite, and hence trivial, for all primes q not equal to p. That is, G is a pro-p group. Thus the corollary follows from the lemma and the proposition, since $\operatorname{gld}\Omega_G$ is finite by Theorem 5.1. $\square$

5. Dimensions

5.1. Global dimension. The situation as regards the global dimension of Ω_G and Λ_G is completely understood, and depends fundamentally on properties of the cohomology of profinite groups - in particular behaviour under finite extensions - due to Serre [34]. The result is due to Brumer [11, Theorem 4.1] who computed the global dimension of the completed group algebra of an arbitrary profinite group G with coefficients in a pseudo-compact ring R. As a consequence of his work, we have

Theorem. *Let G be a compact p-adic analytic group of dimension d. Then Ω_G and Λ_G have finite global dimension if and only if G has no elements of order p, and in this case*

$$\operatorname{gld}(\Omega_G) = d \quad \text{and} \quad \operatorname{gld}(\Lambda_G) = d + 1.$$

5.2. Auslander-Gorenstein rings. Recall that the group algebra of an arbitrary finite group over any field is a Frobenius algebra [44, Proposition 4.2.6], and thus is self-injective. It should therefore come as no surprise that injective dimension is well-behaved for Iwasawa algebras. In fact, much more is true:

Theorem. [5, Theorem J] *Let G be a compact p-adic analytic group of dimension d. Then Ω_G and Λ_G are Auslander-Gorenstein rings of dimensions d and $d+1$ respectively.*

This result was first proved by O. Venjakob [39] and is easy to deduce from Theorem 3.6(4) and Theorem 5.1, as follows. Let H be an open uniform normal subgroup of G. Then Ω_H and Λ_H are Auslander-Gorenstein by Theorem 3.6(4), and the dimensions are given by Theorem 5.1. Now apply (2.3)(1): a simple lemma [5, Lemma 5.4] shows that

$$\operatorname{Ext}^i_{\Omega_G}(M, \Omega_G) \cong \operatorname{Ext}^i_{\Omega_H}(M, \Omega_H) \tag{1}$$

for all $i \geq 0$ and all Ω_G-modules M, with a similar isomorphism for Λ_G, and the result follows.

5.3. Dimension functions for Auslander-Gorenstein rings. We recall from [24] the basics of dimension theory over an Auslander-Gorenstein ring R. Write d for the injective dimension of R. The *grade* $j(M)$ of a finitely generated R-module M is defined as follows:

$$j(M) = \min\{j : \mathrm{Ext}_R^j(M,R) \neq 0\}.$$

Thus $j(M)$ exists and belongs to the set $\{0,\dots,d\} \cup \{+\infty\}$. The *canonical dimension* of M, $\mathrm{Cdim}(M)$ is defined to be

$$\mathrm{Cdim}(M) = d - j(M).$$

It is known [24, Proposition 4.5] that Cdim is an exact, finitely partitive dimension function on finitely generated R-modules in the sense of [29, §6.8.4]. That is,

- $\mathrm{Cdim}(0) = -\infty$;
- if $0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$ is an exact sequence of finitely generated modules, then $\mathrm{Cdim}(M) = \max\{\mathrm{Cdim}(N), \mathrm{Cdim}(T)\}$;
- if $MP = 0$ for a prime ideal P of R, and M is a torsion R/P-module, then $\mathrm{Cdim}(M) \leq \mathrm{Cdim}(R/P) - 1$;
- if $\mathrm{Cdim}(M) = t$ then there is an integer n such that every descending chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \cdots$ of submodules of M has at most n factors M_i/M_{i+1} with $\mathrm{Cdim}(M_i/M_{i+1}) = t$.

The ring R is said to be *grade symmetric* if

$$\mathrm{Cdim}({}_RM) = \mathrm{Cdim}(M_R)$$

for any R–R-bimodule M which is finitely generated on both sides.[5] The triangular matrix ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ over a field k gives an easy example of an Auslander Gorenstein ring which is *not* grade symmetric.

The existence of an exact, finitely partitive, symmetric dimension function for the finitely generated modules over a noncommutative noetherian ring R is a very valuable tool which is often not available: the Gel'fand-Kirillov dimension [29, §8.1] - although symmetric - is often not defined; and although the Krull dimension is always defined [29, §6.2], it is a long-standing open question whether it is symmetric in general. As we shall see in the next paragraph, the canonical dimension function fulfils these requirements for an Iwasawa algebra.

If δ is a dimension function on finitely generated R-modules, we say that R is *Cohen-Macaulay with respect to* δ if $\delta(M) = \mathrm{Cdim}(M)$ for all finitely generated R-modules M.

This definition is consistent with, and therefore generalises, the definition from commutative algebra. To see this, suppose that R is a commutative noetherian ring of dimension d. Suppose that R is Cohen-Macaulay [12, Definition 2.1.1], and let M be a finitely generated R-module with Krull dimension $\mathcal{K}(M)$. Note

[5] Alternatively, we can say in these circumstances that the dimension function Cdim is *symmetric*.

that if R is an *affine* (i.e. finitely generated) k-algebra, this equals the Gel'fand-Kirillov dimension of M. Then

$$j(M) + \mathcal{K}(M) = d, \tag{1}$$

[12, Corollary 2.1.4 and Theorem 1.2.10(e)]. And conversely, if (1) holds for all simple R-modules M, then R is Cohen-Macaulay [12, Theorem 1.2.5].

5.4. Canonical dimension for Ω_G. We continue in this paragraph to assume that G is a compact p-adic analytic group of dimension d. Fix an open uniform normal subgroup H of G, and let M be a finitely generated Ω_G-module. By Theorem 5.2 and paragraph (5.3), and with the obvious notation, $\mathrm{Cdim}_G(-)$ and $\mathrm{Cdim}_H(-)$ are well-defined dimension functions, and in fact (5.2)(1) shows that

$$\mathrm{Cdim}_H(M) = \mathrm{Cdim}_G(M). \tag{1}$$

In particular, in studying the canonical dimension we may as well assume that $G = H$ is uniform, which we now do. Hence, by Theorem 3.4, the graded ring of Ω_G is a polynomial $\mathbb{F}_p$-algebra in d variables.
Choose a good filtration for M ($F_nM = MJ^{-n}$ for $n \leq 0$ will do) and form the associated graded module $\mathrm{gr}\, M$. Because the J-adic filtration is Zariskian, it follows from [8, Remark 5.8] that

$$j(\mathrm{gr}\, M) = j(M). \tag{2}$$

Moreover, from this and the concluding remarks of (5.3) we see that

$$\mathcal{K}(\mathrm{gr}\, M) = \mathrm{Cdim}(\mathrm{gr}\, M) = d - j(M). \tag{3}$$

(This shows, incidentally, that $\mathcal{K}(\mathrm{gr}\, M)$ is actually independent of the choice of good filtration on M.)[6] Combining (2) and (3), we find that

$$\mathrm{Cdim}(M) = d - j(M) = \mathrm{Cdim}(\mathrm{gr}\, M) = \mathcal{K}(\mathrm{gr}\, M) = \mathrm{GK}(\mathrm{gr}\, M)$$

for any choice of good filtration on M. This proves the last part of the

Proposition. *Let G be a compact p-adic analytic group.*

(1) Ω_G is grade-symmetric.
(2) Ω_G is ideal-invariant with respect to Cdim.
(3) Suppose that G is uniform. Then for all finitely generated Ω_G-modules M,

$$\mathrm{Cdim}(M) \quad = \quad \mathrm{GK}(\mathrm{gr}\, M).$$

Proof. (1) In view of (5.4)(1) we can and do assume that G is uniform. Write J for the Jacobson radical of Ω_G and let M be a finitely generated Ω_G-module. Then by the definition of the Gel'fand Kirillov dimension [29, §8.1.11], $\mathrm{GK}(\mathrm{gr}\, M)$ is the growth rate $\gamma(f)$ of the function

$$f(n) = \dim \frac{M}{MJ^n};$$

[6] Consider (3) with M the trivial Ω_G-module $\mathbb{F}_p$. Then $\mathcal{K}(\mathrm{gr}\, M) = 0$, so $j(M) = d$ and therefore the injective dimension of Ω_G actually equals d, providing another proof of the numerical part of Theorem 5.1.

note that this function is eventually polynomial because the finitely generated $\operatorname{gr}\Omega_G$-module $\operatorname{gr} M$ has a Hilbert polynomial.
Now let N be an Ω_G-bimodule, finitely generated on both sides. Then NJ is a sub-bimodule, and N/NJ is finite dimensional over $\mathbb{F}_p$ because N is a finitely generated right Ω_G-module. Hence N/NJ is also a finite dimensional *left* Ω_G-module and as such is killed by some power of J, J^a say. Thus $J^a N \subseteq NJ$ and similarly there exists an integer $b \geq 1$ such that $NJ^b \subseteq JN$. An easy induction on n shows that

$$J^{abn} N \subseteq NJ^{bn} \subseteq J^n N \tag{1}$$

for all $n \geq 0$. Letting $f(n) = \dim \frac{N}{NJ^n}$ and $g(n) = \dim \frac{N}{J^n N}$, we obtain

$$g(n) \leq f(bn) \leq g(abn)$$

for all $n \geq 0$. It follows that $\operatorname{Cdim}(N_{|\Omega_G}) = \gamma(f) = \gamma(g) = \operatorname{Cdim}({}_{\Omega_G|}N)$, proving part (1).
For part (2), recall [29, 6.8.13] that a ring R is said to be *ideal-invariant* with respect to a dimension function δ if $\delta(M \otimes_R I) \leq \delta(M)$ for all finitely generated right R-modules M and all two-sided ideals I of R and if the left-hand version of this statement also holds.
In fact, we will show that

$$\operatorname{Cdim}(M \otimes_{\Omega_G} N) \leq \operatorname{Cdim}(M) \tag{4}$$

for any finitely generated Ω_G-module M and any Ω_G-bimodule N, finitely generated on both sides.[7] Let M and N be as above, and let H be an open uniform normal subgroup of G. Since there is an Ω_H-epimorphism $M \otimes_{\Omega_H} N \twoheadrightarrow M \otimes_{\Omega_G} N$, (5.2)(1) shows that we can replace G by H in proving (4); that is, we now assume that G is uniform.
Choose the integer a as above so that $J^{an} N \subseteq NJ^n$ for all $n \geq 0$. Fix n and let

$$f(n) = \dim \frac{M}{MJ^n} \quad \text{and} \quad g(n) = \dim \left(\frac{M \otimes_{\Omega_G} N}{(M \otimes_{\Omega_G} N).J^n} \right).$$

Note that $(M \otimes_{\Omega_G} N).J^n$ equals the image of $M \otimes_{\Omega_G} NJ^n$ in $M \otimes_{\Omega_G} N$ so the right-exactness of tensor product gives

$$M \otimes_{\Omega_G} \left(\frac{N}{J^{an} N} \right) \twoheadrightarrow M \otimes_{\Omega_G} \left(\frac{N}{NJ^n} \right) \cong \frac{M \otimes_{\Omega_G} N}{(M \otimes_{\Omega_G} N).J^n}.$$

Now we have a natural isomorphism of right Ω_G-modules

$$M \otimes_{\Omega_G} \frac{N}{J^{an} N} \cong \frac{M}{MJ^{an}} \otimes_{\Omega_G} N$$

and picking a finite generating set of size t for the left Ω_G-module N shows that

$$\dim \left(\frac{M}{MJ^{an}} \otimes_{\Omega_G} N \right) \leq \left(\dim \frac{M}{MJ^{an}} \right) \cdot t.$$

[7] Compare this with [29, Proposition 8.3.14].

Hence

$$g(n) = \dim\left(\frac{M\otimes_{\Omega_G} N}{(M\otimes_{\Omega_G} N).J^n}\right) \leq \dim\left(M\otimes_{\Omega_G}\left(\frac{N}{J^{an}N}\right)\right) \leq f(an)\cdot t$$

for all $n \geq 0$, so $\mathrm{Cdim}(M\otimes_{\Omega_G} N) = \gamma(g) \leq \gamma(f) = \mathrm{Cdim}(M)$ as required. □

The above proposition is due to the first author; it was inspired by a result of S. J. Wadsley [43, Lemma 3.1].

5.5. Characteristic varieties. Assume in this paragraph that G is uniform. Let M be a finitely generated Ω_G-module. There is another way of seeing that $\mathcal{K}(\mathrm{gr}\, M)$ does not depend on the choice of good filtration for M, as follows. It is well known [26, Chapter III, Lemma 4.1.9] that

$$J(M) := \sqrt{\mathrm{Ann}_{\mathrm{gr}\,\Omega_G}(\mathrm{gr}\, M)}$$

is independent of this choice. Standard commutative algebra now gives

$$\mathcal{K}(\mathrm{gr}\, M) = \mathcal{K}\left(\frac{\mathrm{gr}\,\Omega_G}{J(M)}\right),$$

as claimed.

The graded ideal $J(M)$ is called the *characteristic ideal* of M, and the affine variety $\mathrm{Ch}(M)$ defined by it is called the *characteristic variety* of M. Thus we obtain yet another expression for the canonical dimension of M:

$$\mathrm{Cdim}(M) = \dim \mathrm{Ch}(M). \tag{2}$$

The characteristic variety is defined in an entirely analogous fashion for finitely generated modules over enveloping algebras and Weyl algebras $A_n(\mathbb{C})$. In that setting it enjoys many pleasant properties, in addition to the simple formula (2). In particular, there exists a *Poisson structure* on $\mathrm{Ch}(M)$, which gives more information about M through the geometric properties of the characteristic variety. For example, the fact that the characteristic variety of a finitely generated $A_n(\mathbb{C})$-module is integrable can be used to prove the Bernstein inequality.

Question C. *Is there a way of capturing more information about M in the characteristic variety* $\mathrm{Ch}(M)$*?*

The naive method (mimicking the construction of the Poisson structure in the enveloping algebra case) seems to fail because derivations are not sufficient when studying algebras in positive characteristic: they kill too much. Presumably, if the answer to the above question is affirmative, then differential operators in characteristic p will play a role.

5.6. No GK-dimension. The theory outlined in the previous sections will sound very familiar to the experts. However, Iwasawa algebras are *not* Cohen Macaulay with respect to the GK dimension. This is easily seen by decoding the definition of GK dimension in the case when $G \cong \mathbb{Z}_p$: in this case, Ω_G is isomorphic to the one-dimensional power series ring $\mathbb{F}_p[[t]]$, which (being uncountable) contains polynomial algebras over $\mathbb{F}_p$ of arbitrarily large dimension.

Thus $GK(\Omega_G) = \infty$ for any infinite G, since any such G will contain a closed subgroup isomorphic to $\mathbb{Z}_p$.
If one tries to brush this problem away by replacing the GK dimension by the canonical dimension, then one has to be careful not to fall into the following trap.
Recall [29, Lemma 8.1.13(ii)] that if $R \subseteq S$ are affine k-algebras over a field k, then for any finitely generated S-module M,

$$\mathrm{GK}(N) \leq \mathrm{GK}(M) \tag{3}$$

whenever N is a finitely generated R-submodule of M. This enables one to "pass to subalgebras of smaller dimension" and use inductive arguments on the GK dimension - a ploy used, for example, in the computation of the Krull dimension of $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ by S.P. Smith [29, Theorem 8.5.16]. Another consequence of this property of GK dimension is that it is impossible to find an embedding $R \hookrightarrow S$ of k-algebras such that $\mathrm{GK}(R) > \mathrm{GK}(S)$.
Unfortunately, (3) fails for Iwasawa algebras, if one tries to replace the GK dimension by the canonical dimension. This is due to the following pathological example:

Example. [38, Chapter VII, page 219] *There exists a continuous embedding of $\mathbb{F}_p$-algebras*

$$\Omega_G \hookrightarrow \Omega_H$$

where $\dim G = 3$ *and* $\dim H = 2$.

Proof. Let $G = \mathbb{Z}_p^3$ and $H = \mathbb{Z}_p^2$. By Theorem 3.1 we can identify Ω_G with the three-dimensional power series ring $\mathbb{F}_p[[x, y, z]]$ and Ω_H with the two-dimensional power series ring $\mathbb{F}_p[[a, b]]$.
Because $\mathbb{F}_p[[a]]$ is uncountable, we can find an element $u = u(a) \in a\mathbb{F}_p[[a]]$ such that the $\mathbb{F}_p$-algebra generated by a and u is isomorphic to the two-dimensional polynomial ring $\mathbb{F}_p[a, u]$. Define $\theta : \mathbb{F}_p[[x, y, z]] \to \mathbb{F}_p[[a, b]]$ to be the unique continuous $\mathbb{F}_p$-algebra map such that

$$\theta(x) = b, \quad \theta(y) = ab, \quad \theta(z) = ub.$$

We have

$$\theta\left(\sum_{\lambda,\mu,\nu\in\mathbb{N}} r_{\lambda,\mu,\nu} x^\lambda y^\mu z^\nu\right) = \sum_{n=0}^{\infty} b^n \left(\sum_{\lambda+\mu+\nu=n} r_{\lambda,\mu,\nu} a^\mu u^\nu\right).$$

This shows that θ is an injection, as required. $\square$

One can of course concatenate these embeddings and produce a continuous embedding of Ω_G into $\mathbb{F}_p[[a, b]]$ for abelian uniform pro-p groups G of arbitrarily large dimension. Here is the actual counterexample to the analogue of (3).

Example. *There exist uniform pro-p groups $H \subset G$, a finitely generated Ω_G-module M and a finitely generated Ω_H-submodule N of M such that* $\mathrm{Cdim}(M) = 2$, *but* $\mathrm{Cdim}(N) = 3$.

Proof. Let $R = \mathbb{F}_p[[a,b,c,d]]$ and $S = \mathbb{F}_p[[b,c,d]]$. Let I be the ideal of R generated by $c - ab$ and $d - u(a)b$ where $u(a)$ is chosen as in the previous example and let $M = R/I$. By construction, the graded ideal $\operatorname{gr} I$ is generated by the symbols of c and d, so

$$\mathrm{Cdim}(M) = \mathcal{K}(\operatorname{gr} M) = 2.$$

Now if $r \in I \cap S$, then $\theta(r) = 0$, letting $\theta : \mathbb{F}_p[[b,c,d]] \hookrightarrow \mathbb{F}_p[[a,b]]$ be as above. Hence $r = 0$, so $S \hookrightarrow R/I = M$. Therefore the cyclic S-submodule N of M generated by $1 + I$ is actually free, so $\mathrm{Cdim}(N) = 3$. □

5.7. Krull dimension. The Krull-(Gabriel-Rentschler) dimension of Ω_G was first studied by one of the authors in [1]. An immediate upper bound of $\dim G$ can be obtained using Theorem 3.6, or if one prefers, using [7, Corollary 1.3]. Here is a result covering a large number of cases.

Theorem. [1, Theorem A and Corollary C] *Let G be a compact p-adic analytic group, and let $\mathfrak{g}$ be the $\mathbb{Q}_p$-Lie algebra of an open uniform subgroup of G. Let $\mathfrak{r}$ denote the soluble radical of $\mathfrak{g}$ and suppose that the semisimple part $\mathfrak{g}/\mathfrak{r}$ of $\mathfrak{g}$ is a direct sum of some number of copies of $\mathfrak{sl}_2(\mathbb{Q}_p)$. Then*

$$\mathcal{K}(\Omega_G) = \dim G.$$

In particular, $\mathcal{K}(\Omega_G)$ equals $\dim G$ whenever G is soluble-by-finite. The main idea in the proof is to obtain a lower bound on the Krull dimension of Ω_G for *any* compact p-adic analytic group G. Namely, with $\mathfrak{g}$ as in the theorem, and writing $\lambda(\mathfrak{g})$ for the length of the longest chain of subalgebras of $\mathfrak{g}$, we have

$$\lambda(\mathfrak{g}) \le \mathcal{K}(\Omega_G).$$

Question D. *With the above notation, is $\mathcal{K}(\Omega_G) = \lambda(\mathfrak{g})$ in general?*

It is easy to see that $\lambda(\mathfrak{g}) = \lambda(\mathfrak{n}) + \lambda(\mathfrak{g}/\mathfrak{n})$ whenever $\mathfrak{n}$ is an ideal of $\mathfrak{g}$. Let N be a closed uniform subgroup of G with Lie algebra $\mathfrak{n}$.

Question E. *Is $\mathcal{K}(\Omega_G) = \mathcal{K}(\Omega_N) + \mathcal{K}(\Omega_{G/N})$?*

Aside from its intrinsic interest, an affirmative answer to Question E would obviously reduce Question D to the study of *almost simple* groups G, (where we say that a uniform pro-p group G is *almost simple* provided its Lie algebra has no non-trivial ideals).

The classical split simple Lie algebras are the first examples to study. Given such a Lie algebra $\mathfrak{g}$, choose a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t}$. Then it is easy to produce a chain of subalgebras of $\mathfrak{g}$ of length $\dim \mathfrak{b} + \dim \mathfrak{t}$.

Question F. *For G almost simple and split, is $\mathcal{K}(\Omega_G) = \dim \mathfrak{b} + \dim \mathfrak{t}$?*

Question F has an affirmative answer in the two smallest cases: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q}_p)$ and $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{Q}_p)$. In particular,

Theorem. [1, Theorem B]. *Let G be a uniform pro-p group with $\mathbb{Q}_p$-Lie algebra $\mathfrak{sl}_3(\mathbb{Q}_p)$. Then Ω_G is a scalar local complete noetherian domain of global dimension 8, with*

$$\mathcal{K}(\Omega_G) = 7.$$

The main idea of the proof of this last result is to show that Ω_G has no finitely generated modules whose canonical dimension equals precisely 1; that is, there is a "gap" at Cdim $= 1$.[8] The extra $\dim \mathfrak{t}$ term in our conjectured formula for $\mathcal{K}(\Omega_G)$ comes from the fact that Ω_G is scalar local - this fact is used crucially in the proof of the lower bound for the Krull dimension of Ω_G.

6. Two-sided ideal structure

6.1. One of the first questions asked when studying a noetherian algebra R is "what are its two-sided ideals?" It is usually sensible to focus first on the *prime* ideals of R.

One way of answering the above question is to give a reduction to the commutative case. This is a recurring theme in noncommutative algebra. For example, if $R = k[G]$ is the group algebra of a polycyclic group G over a field k, the paper [33] by J. E. Roseblade achieves this, "to within a finite group".[9] Similar results hold for universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of finite dimensional soluble Lie algebras over a field k: see [21] and [29, Chapter 13]. As for the case when $\mathfrak{g}$ is semisimple, one can view the huge body of research on the primitive ideals of $\mathcal{U}(\mathfrak{g})$ as an analysis of the failure of the naive hope that these primitive ideals should be generated by their intersection with the centre of $\mathcal{U}(\mathfrak{g})$, [21]. And for quantised function algebras of semisimple groups, and many related quantum algebras, there are "stratification theorems" which describe their prime and primitive spectra as finite disjoint unions of affine commutative pieces, [10, Theorem II.2.13].

Unfortunately, no such results are currently known for Iwasawa algebras - see below for a summary of what little *is* known. Alleviation of this state of gross ignorance would seem to be the most pressing problem in the subject.

Because of the crossed product decomposition (2.3)(1) and the going up and down theorems for crossed products of finite groups [31, Theorem 16.6], one should naturally first concentrate on the case when G is uniform.

6.2. Ideals arising from subgroups and from centres. Since centrally generated one-sided ideals are necessarily two-sided, it helps to know the centre of the ring in question. However the centre of Iwasawa algebras is not very big:

Theorem. [2, Corollary A] *Let G be a uniform pro-p group and let Z be its centre. Then the centre of Ω_G equals Ω_Z and the centre of Λ_G equals Λ_Z.*

Thus when the centre of G is trivial (and this happens frequently), Ω_G has no non-trivial centrally generated ideals. This is one place where the analogy with enveloping algebras of semisimple Lie algebras breaks down.

[8]A similar idea was used by Smith [35] in giving an upper bound for the Krull dimension of $\mathcal{U}(\mathfrak{g})$ when $\mathfrak{g}$ is a complex semisimple Lie algebra. We note in passing that $\mathcal{K}(\mathcal{U}(\mathfrak{g}))$ when $\mathfrak{g}$ is complex semisimple has been recently proved to be equal to $\dim \mathfrak{b}$ by Levasseur [25], answering a long-standing question in the affirmative.

[9]See [31, Chapter 5] for more details.

One can also produce two-sided ideals by using normal subgroups. Certainly when H is a closed normal subgroup of G, the augmentation ideal

$$w_H := \ker(\Omega_G \to \Omega_{G/H})$$

is a two-sided ideal of Ω_G and we can tell whether it is prime or semiprime using Theorem 4.2. As for Λ_G, H yields two augmentation ideals: the inverse image v_H of w_H under the natural projection $\Lambda_G \twoheadrightarrow \Omega_G$ and "the" augmentation ideal

$$I_H = \ker(\Lambda_G \to \Lambda_{G/H}).$$

The behaviour of these ideals regarding localisation is quite well understood:

Theorem. *Let H be a closed normal subgroup of the compact p-adic analytic group G and let F be the largest finite normal subgroup of H of order coprime to p. Then*

(1) [5] *w_H and v_H are localisable if and only if H/F is pro-p,*
(2) [4] *I_H is localisable if and only if H is finite-by-nilpotent.*

These results were prompted by the formulation of the Iwasawa Main Conjecture by Coates *et al* in [19]. Localisation techniques play an important role in the construction of *characteristic elements* for suitable Λ_G-modules. For number-theoretic reasons, it is assumed in [19] that the subgroup H actually satisfies $G/H \cong \mathbb{Z}_p$: in arithmetic applications, G arises as the Galois group of a certain extension K of $\mathbb{Q}$ containing the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}^{\mathrm{cyc}}$, and H is taken to be $\mathrm{Gal}(K/\mathbb{Q}^{\mathrm{cyc}})$. The characteristic elements all lie inside the K_1-group of the localisation of Λ_G at the Ore set

$$\mathcal{C}_{\Lambda_G}(v_N) \times \{1, p, p^2, \ldots\},$$

where N is the largest closed normal pro-p subgroup of G which is open in H. For more details, see [19, §2], [6] and [5, Theorem G].

Notwithstanding the above, the most embarrassing aspect of the state of our knowledge about ideals of Iwasawa algebras is the lack of examples. In particular, we've noted that central elements and closed normal subgroups give rise to ideals. This suggests the following improperly-posed question, for which we'll suggest more precise special cases in the succeeding paragraphs.

Question G. *Is there a mechanism for constructing ideals of Iwasawa algebras which involves neither central elements nor closed normal subgroups?*

One way to begin the study of prime ideals is to look first at the smallest non-zero ones - that is, the prime ideals of height one. With one eye on the commutative case and another on the results of (4.4) on maximal orders, one can ask when they are all principal. Here are two slightly more precise ways to ask this question:

Question H. *When is Ω_G a unique factorisation ring in the sense of* [16]*?*

Question I. *When G is uniform, is every reflexive prime ideal of Ω_G principal?*

6.3. The case when G is almost simple. Recall that the compact p-adic analytic group G is *almost simple* if every non-trivial closed normal subgroup of G is open (5.7). For such groups the constructions of (6.2) do not produce anything interesting because Ω_G/w_H is artinian and hence finite dimensional over $\mathbb{F}_p$ for any closed normal subgroup $H \neq 1$. So Question G specialises here to

Question J. *Let G be an almost simple uniform pro-p group and let P be a nonzero prime ideal of Ω_G. Must P be the unique maximal ideal of Ω_G?*

We remind the reader that $x \in \Omega_G$ is *normal* if $x\Omega_G = \Omega_G x$. Another closely related question is

Question K. *Let G be as in Question J, with $G \not\cong \mathbb{Z}_p$. Must any nonzero normal element of Ω_G be a unit?*

In [22], M. Harris claimed that, for G as in Question J, any closed subgroup H of G with $2\dim H > \dim G$ gives rise to a non-zero two-sided ideal in Ω_G, namely the annihilator of the "Verma module" constructed by induction from the simple Ω_H-module. Unfortunately his paper contains a gap, so Question J remains open. Some slight evidence towards a positive answer is provided by

Theorem. [3, Theorem A] *Suppose that G is an almost simple uniform pro-p group and that the Lie algebra of G contains a copy of the two-dimensional non-abelian Lie algebra. Then for any two-sided ideal I of Ω_G,*

$$\mathcal{K}(\Omega_G/I) \neq 1.$$

Recall [29, §6.4.4] that if R is a noetherian ring with $\mathcal{K}(R) < \infty$, the *classical Krull dimension* $\dim R$ of R is the largest length of a chain of prime ideals of R. We always have $\dim R \leq \mathcal{K}(R)$; an easy consequence of the above result is

$$\dim(\Omega_G) < \dim G$$

whenever G satisfies conditions of the Theorem.

6.4. The case when G is nilpotent. Towards the opposite end of the "spectrum of commutativity" from the almost simple groups lie the nilpotent groups. Motivated by analogous results for enveloping algebras of nilpotent Lie algebras [21, Chapter 4] and for group algebras $k[G]$ of finitely generated nilpotent groups G [33, Theorem E], we ask

Question L. *Let G be a nilpotent uniform pro-p group with centre Z and let I be a nonzero ideal of Ω_G. Does I contain a non-zero central element? That is, is $I \cap \Omega_Z$ nonzero?*

S. J. Wadsley has shown that Question L has an affirmative answer in the case when G is one of the simplest possible nonabelian nilpotent uniform pro-p groups:

Theorem. [43, Theorem 4.10] *Let G be a uniform Heisenberg pro-p group with centre Z and let I be a nonzero two-sided ideal of Ω_G. Then $I \cap \Omega_Z \neq 0$.*

A uniform pro-p group G is said to be *Heisenberg* provided its centre Z is isomorphic to $\mathbb{Z}_p$ and G/Z is abelian. The main idea of the proof of the above result is to show that for any integer t, any finitely generated Ω_G-module M satisfying $\mathrm{Cdim}(M) \leq \dim G/Z - t$ is actually finitely generated over "most" subalgebras Ω_H satisfying $Z \leq H$ and $\dim G/H = t$ [43, Theorem 3.10].

In a more precise version of Question L, one might also hope that, when G is nilpotent, "small" prime ideals I in Ω_G are *controlled* by Ω_Z; that is

$$I = (I \cap \Omega_Z)\Omega_G.$$

Question O suggests a more general version of this.

Moreover, one might even hope that arbitrary ideals of these Iwasawa algebras of nilpotent groups are constructed by means of a sequence of centrally generated ideals - that is, one can ask:

Question M. *Suppose that G is a nilpotent uniform pro-p group. If I is an ideal of Ω_G strictly contained in $J(\Omega_G)$, is there a non-zero central element in $J(\Omega_G)/I$?* [10]

6.5. The case when G is soluble. Given the parallels pointed out in (3.3) between the Iwasawa algebras of uniform soluble groups and the enveloping algebras of finite dimensional complex soluble Lie algebras, it is natural to wonder whether properties known for the latter case might also be valid in the former. We give two sample questions of this sort. Recall for the first that a prime ideal P of the ring R is *completely prime* if R/P is a domain.

Question N. *Let G be a soluble uniform pro-p group.*

(i) Is every prime ideal of Ω_G completely prime? [11]

(ii) Is the prime spectrum of Ω_G the disjoint union of finitely many commutative strata (along the lines of [10, Theorem II.2.13]*, but with necessarily non-affine strata)?*

The simple possible nonabelian soluble case has been studied by O. Venjakob:

Theorem. [40, Theorem 7.1] *Let $G = X \rtimes Y$ be a nonabelian semidirect product of two copies of $\mathbb{Z}_p$. Then the only prime ideals of Ω_G are $0, w_X$ and $J(\Omega_G)$, and each one is completely prime. Moreover, w_X is generated by a normal element.*

An example of such a nonabelian semidirect product is provided by the group $B = \overline{\langle a_1, a_2 \rangle}$ considered in Example 3.2.

Following J. E. Roseblade and D. S. Passman [33, §1.5], we define the *Zalesskii subgroup* A of the soluble uniform pro-p group G to be the centre of the largest nilpotent closed normal subgroup H of G. We say that an ideal I of Ω_G is *faithful* if G acts faithfully on the quotient Ω_G/I. If Question L has a positive answer, then it's possible that a more general statement is true:

[10] Compare with [21, Proposition 4.7.1(i)].

[11] Compare with [21, Theorem 3.7.2].

Question O. *Let G be a soluble uniform pro-p group. Is every faithful prime ideal of Ω_G controlled by the Zalesskii subgroup A of G?*

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