The Magnetic Laplacian Acting on Discrete Cusps

Sylvain Golénia and Françoise Truc

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Abstract. We introduce the notion of discrete cusp for a weighted graph. In this context, we prove that the form-domain of the magnetic Laplacian and that of the non-magnetic Laplacian can be different. We establish the emptiness of the essential spectrum and compute the asymptotic of eigenvalues for the magnetic Laplacian.

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1 Introduction

The spectral theory of discrete Laplacians on graphs has drawn a lot of attention for decades. The spectral analysis of the Laplacian associated to a graph is strongly related to the geometry of the graph. Moreover, graphs are discretized versions of manifolds. In [MoT, GM], it is shown that for a manifold with cusps, adding a magnetic field can drastically destroy the essential spectrum of the Laplacian. The aim of this article is to go along this line in a discrete setting.
We recall some standard definitions of graph theory. A *graph* is a triple $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where $\mathcal{V}$ is a countable set (the *vertices*), $\mathcal{E} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+$ is symmetric, and $m : \mathcal{V} \to (0, \infty)$ is a weight. We say that $\mathcal{G}$ is *simple* if $m = 1$ and $\mathcal{E} : \mathcal{V} \times \mathcal{V} \to \{0, 1\}$.
Given $x, y \in \mathcal{V}$, we say that (x, y) is an edge (or x and y are *neighbors*) if $\mathcal{E}(x, y) > 0$. We denote this relationship by $x \sim y$ and the set of neighbors of x by $N_{\mathcal{G}}(x)$. We say that there is a *loop* at $x \in \mathcal{V}$ if $\mathcal{E}(x, x) > 0$. A graph is *connected* if for all $x, y \in \mathcal{V}$, there exists a path γ joining x and y. Here, γ is a sequence $x_0,\ x_1, ..., x_n \in \mathcal{V}$ such that $x = x_0,\ y = x_n$, and $x_j \sim x_{j+1}$ for all $0 \le j \le n-1$. In this case, we set $|\gamma| := n$. A graph $\mathcal{G}$ is *locally finite* if $|N_{\mathcal{G}}(x)|$ is finite for all $x \in \mathcal{V}$. In the sequel, we assume that:

All graphs are locally finite, connected with no loops.

We endow a graph $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ with the metric $\rho_{\mathcal{G}}$ defined by

$$\rho_{\mathcal{G}}(x, y) := \inf\{|\gamma|,\ \gamma \text{ is a path joining } x \text{ and } y\}.$$

The space of complex-valued functions acting on the set of vertices $\mathcal{V}$ is denoted by $C(\mathcal{V}) := \{f : \mathcal{V} \to \mathbb{C}\}$. Moreover, $C_c(\mathcal{V})$ is the subspace of $C(\mathcal{V})$ of functions with finite support.
We consider the Hilbert space

$$\ell^2(\mathcal{V}, m) := \left\{ f \in C(\mathcal{V}), \sum_{x\in\mathcal{V}} m(x)|f(x)|^2 < \infty \right\}$$

with the scalar product $\langle f, g\rangle := \sum_{x\in\mathcal{V}} m(x)\overline{f(x)}g(x)$.
We equip $\mathcal{G}$ with a *magnetic potential* $\theta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}/2\pi\mathbb{Z}$ such that we have $\theta_{x,y} := \theta(x, y) = -\theta_{y,x}$ and $\theta(x, y) := 0$ if $\mathcal{E}(x, y) = 0$. We define the Hermitian form

$$Q_{\mathcal{G},\theta}(f) := \frac{1}{2} \sum_{x,y\in\mathcal{V}} \mathcal{E}(x, y) \left|f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)\right|^2,$$

for all $f \in \mathcal{C}_c(\mathcal{V})$. The associated *magnetic Laplacian* is the unique non-negative self-adjoint operator $\Delta_{\mathcal{G},\theta}$ satisfying $\langle f, \Delta_{\mathcal{G},\theta} f\rangle_{\ell^2(\mathcal{V},m)} = Q_{\mathcal{G},\theta}(f)$, for all $f \in \mathcal{C}_c(\mathcal{V})$. It is the *Friedrichs extension* of $\Delta_{\mathcal{G},\theta}|_{\mathcal{C}_c(\mathcal{V})}$, e.g., [CTT3, RS], where

$$(\Delta_{\mathcal{G},\theta} f)(x) = \frac{1}{m(x)} \sum_{y\in\mathcal{V}} \mathcal{E}(x, y) \left(f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)\right),$$

for all $f \in \mathcal{C}_c(\mathcal{V})$. We set

$$\deg_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y\in\mathcal{V}} \mathcal{E}(x, y),$$

the *degree* of $x \in \mathcal{V}$. We see easily that $\Delta_{\mathcal{G},\theta} \le 2\deg_{\mathcal{G}}(\cdot)$ in the *form sense*, i.e.,

$$0 \le \langle f, \Delta_{\mathcal{G},\theta} f\rangle \le \langle f, 2\deg_{\mathcal{G}}(\cdot) f\rangle, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}). \tag{1}$$

Moreover, setting $\tilde{\delta}_x(y) := m^{-1/2}(x)\delta_{x,y}$ for any $x, y \in \mathcal{V}$, $\langle \tilde{\delta}_x, \Delta_{\mathcal{G},\theta}\tilde{\delta}_x\rangle = \deg_{\mathcal{G}}(x)$, so $\Delta_{\mathcal{G},\theta}$ is bounded if and only if $\sup_{x\in\mathcal{V}} \deg_{\mathcal{G}}(x)$ is finite, e.g. [KL, Go]. Another consequence of (1) is

$$\mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right) \subset \mathcal{D}\left(\Delta_{\mathcal{G},\theta}^{1/2}\right), \tag{2}$$

where $\mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right) := \left\{f \in \ell^2(\mathcal{V}, m), \deg_{\mathcal{G}}(\cdot) f \in \ell^2(\mathcal{V}, m)\right\}$. However, the equality of the form-domains

$$\mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right) = \mathcal{D}\left(\Delta_{\mathcal{G},\theta}^{1/2}\right) \tag{3}$$

is wrong in general for a simple graph, see [Go, BGK]. In fact if $\theta = 0$, (2) is equivalent to a sparseness condition and holds true for planar simple graphs, see [BGK]. We refer to [BGKLM] for a magnetic sparseness condition. On a general weighted graph, if (3) holds true,

$$\sigma_{\rm ess}(\Delta_{\mathcal{G},\theta}) = \emptyset \Leftrightarrow (\Delta_{\mathcal{G},\theta}+1)^{-1} \text{ is compact} \Leftrightarrow \lim_{|x|\to\infty} \deg_{\mathcal{G}}(x) = \infty,$$

where $|x| := \rho_{\mathcal{G}}(x_0, x)$ for a given $x_0 \in \mathcal{V}$. Note that the limit is independent of the choice of x_0. Besides if the latter is true and if the graph is sparse (simple and planar for instance), [BGK] ensures the following asymptotic of eigenvalues,

$$\lim_{n\to\infty} \frac{\lambda_n\left(\Delta_{\mathcal{G},\theta}\right)}{\lambda_n\left(\deg_{\mathcal{G}}(\cdot)\right)} = 1, \tag{4}$$

where $\lambda_n(H)$ denotes the n-th eigenvalue, counted with multiplicity, of a self-adjoint operator H, which is bounded from below.
The technique used in [BGK] does not apply when the graph is a discrete cusp (thin at infinity), see Definition 2.5. The aim of this article is to establish new behaviors for the asymptotic of eigenvalues for the magnetic Laplacian in that case, and also to prove that the form-domain of the non-magnetic Laplacian can be different from that of the magnetic Laplacian, see Theorem 2.14. We found the inspiration by mimicking the continuous case, which was studied in [MoT, GM].
Let us present a flavour of our results (in particular of Theorem 2.14) by introducing the following specific example of *discrete cusp* :

Example 1.1 *Let $n \geq 3$ be an integer and consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, where*

$$\mathcal{V}_1 := \mathbb{N}, \quad m_1(n) := \exp(-n), \ \text{ and } \mathcal{E}_1(n, n+1) := \exp(-(2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, 1)$ a simple connected finite graph such that $|\mathcal{V}_2| = n$. Set $\theta_1 := 0$ and θ_2 such that $\mathrm{Hol}_{\theta_2} \neq 0$. Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be the twisted Cartesian product *$\mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2$, given by:*

$$\begin{cases} m(x,y) := & m_1(x), \\ \mathcal{E}\left((x,y),(x',y')\right) := & \mathcal{E}_1(x,x') \times \delta_{y,y'} + \delta_{x,x'} \times \mathcal{E}_2(y,y'), \\ \theta\left((x,y),(x',y')\right) := & \delta_{x,x'} \times \theta_2(y,y'), \end{cases}$$

for all $x, x' \in \mathcal{V}_1$ and $y, y' \in \mathcal{V}_2$. Then there exists a constant $\nu > 0$ such that for all $\kappa \in \mathbb{R}/\nu\mathbb{Z}$

$$\sigma_{\rm ess}(\Delta_{\mathcal{G},\kappa\theta}) = \emptyset \Leftrightarrow \mathcal{D}\left(\Delta^{1/2}_{\mathcal{G},\kappa\theta}\right) = \mathcal{D}\left(\deg^{1/2}_{\mathcal{G}}(\cdot)\right) \Leftrightarrow \kappa \neq 0 \text{ in } \mathbb{R}/\nu\mathbb{Z}$$

Moreover:

1) When $\kappa \neq 0$ in $\mathbb{R}/\nu\mathbb{Z}$, we have:

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\mathcal{G},\kappa\theta}\right)}{\mathcal{N}_\lambda\left(\deg_{\mathcal{G}}(\cdot)\right)} = 1,$$

where $\mathcal{N}_\lambda(H) := \dim \operatorname{ran} 1_{]-\infty,\lambda]}(H)$ for a self-adjoint operator H.

2) When $\kappa = 0$ in $\mathbb{R}/\nu\mathbb{Z}$, the absolutely continuous part of the $\Delta_{\mathcal{G},\kappa\theta}$ is

$$\sigma_{\mathrm{ac}}\left(\Delta_{\mathcal{G},\kappa\theta}\right) = \left[e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2\right],$$

with multiplicity 1 and

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\mathcal{G},\kappa\theta} P^{\perp}_{\mathrm{ac},\kappa}\right)}{\mathcal{N}_\lambda\left(\deg_{\mathcal{G}}(\cdot)\right)} = \frac{n-1}{n},$$

where $P_{\mathrm{ac},\kappa}$ denotes the projection onto the a.c. part of $\Delta_{\mathcal{G},\kappa\theta}$.

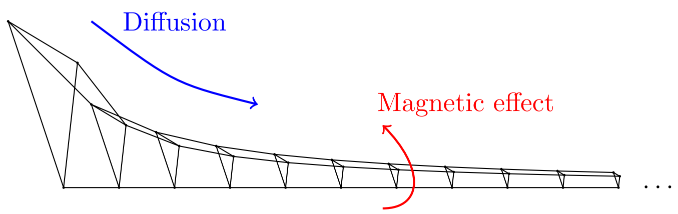
We now describe heuristically the phenomenon. Compared with the first case, the constant $(n-1)/n$ that appears in the second case encodes the fact that a part of the wave packet diffuses. Moreover, switching on the magnetic field is not a gentle perturbation because the form domain of the operator is changed. By Riemann-Lebesgue Theorem, the particle, which is localized in the a.c. part of the operator, escapes from every compact set. More precisely, for a finite subset $X \subset \mathcal{V}$ and all $f \in \mathcal{D}(\Delta_{\mathcal{G},0})$

$$\|1_{\mathrm{X}}(\cdot) e^{\mathrm{i}t\Delta_{\mathcal{G},0}} P_{\mathrm{ac},0} f\| \to 0, \text{ as } t \to \infty.$$

In the first case, when the magnetic potential is active, the spectrum of $\Delta_{\mathcal{G},\kappa\theta}$ is purely discrete. The particle cannot diffuse anymore. More precisely, for a finite subset $X \subset \mathcal{V}$ and an eigenvalue f of $\Delta_{\mathcal{G},\kappa\theta}$ such that $f|_X \neq 0$, there is $c > 0$ such that:

$$\frac{1}{T}\int_0^T \|1_{\mathrm{X}}(\cdot) e^{\mathrm{i}t\Delta_{\mathcal{G},\kappa\theta}} f\|^2\, dt \to c, \text{ as } T \to \infty.$$

The particle is *trapped* by the magnetic field.



Representation of a discrete cusp:
The magnetic field traps the particle by spinning it,
whereas its absence lets the particle diffuse.

We now describe the structure of the paper. In Section 2.1, we recall some properties of the holonomy of a magnetic potential. In Section 2.2 we present our main hypotheses and several notions of (weighted) product for graphs. We introduce the notion of discrete cusp and analyze it under the light of the radius of injectivity. Then in Section 2.3 we give a criteria concerning the absence of essential spectrum. Next, in Section 2.4, we refine the analysis and give our central theorem, a general statement for discrete cusps, computing the form domain and the asymptotic of eigenvalues. We finish the section by proving Theorem 1.1.

Notation: $\mathbb{N}$ denotes the set of non negative integers and $\mathbb{N}^*$ that of the positive integers. We denote by $\mathcal{D}(H)$ the domain of an operator H. Its (essential) spectrum is denoted by $\sigma(H)$ (by $\sigma_{\rm ess}(H)$). We set $\delta_{x,y}$ equals 1 if and only if $x = y$ and 0 otherwise and given a set X, $1_X(x)$ equals 1 if $x \in X$ and 0 otherwise.

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2 Main results

2.1 Holonomy of a magnetic potential

We recall some facts about the gauge theory of magnetic fields, see [CTT3, HS] for more details and also [LLPP] for a different point of view. We recall that a *gauge transform* U is the unitary map on $\ell^2(\mathcal{V}, m)$ defined by

$$(Uf)(x) = u_x f(x),$$

where $(u_x)_{x\in\mathcal{V}}$ is a sequence of complex numbers with $|u_x| \equiv 1$ (we write $u_x = e^{\mathrm{i}\sigma_x}$). The map U acts on the quadratic forms $Q_{\mathcal{G},\theta}$ by $U^\star(Q_{\mathcal{G},\theta})(f) = Q_{\mathcal{G},\theta}(Uf)$, for all $f \in \mathcal{C}_c(\mathcal{V})$. The magnetic potential $U^\star(\theta)$ is defined by:

$$U^\star(Q_{\mathcal{G},\theta}) = Q_{\mathcal{G},U^\star(\theta)}.$$

More explicitly, we get:

$$U^\star(\theta)_{xy} = \theta_{x,y} + \sigma_y - \sigma_x.$$

We turn to the definition of the flux of a magnetic potential, the *Holonomy*.

Proposition 2.1 *Let us denote by $Z_1(\mathcal{G})$ the space of cycles of $\mathcal{G}$. It is is a free $\mathbb{Z}-$module with a basis of geometric cycles $\gamma = (x_0, x_1) + (x_1, x_2) + \ldots + (x_{N-1}, x_N)$ with, for $i = 0, \cdots, N-1$, $\mathcal{E}(x_i, x_{i+1}) \neq 0$, and $x_N = x_0$. We define the* holonomy map $\mathrm{Hol}_\theta : Z_1(\mathcal{G}) \to \mathbb{R}/2\pi\mathbb{Z}$, *by*

$$\mathrm{Hol}_\theta\left((x_0, x_1) + (x_1, x_2) + \cdots + (x_N, x_0)\right) := \theta_{x_0,x_1} + \cdots + \theta_{x_N,x_0}.$$

Then

1) The map $\theta \mapsto \mathrm{Hol}_\theta$ *is surjective onto* $\mathrm{Hom}_{\mathbb{Z}}(Z_1(\mathcal{G}), \mathbb{R}/2\pi\mathbb{Z})$.

2) $\mathrm{Hol}_{\theta_1} = \mathrm{Hol}_{\theta_2}$ *if and only if there exists a gauge transform* U *so that* $U^\star(\theta_2) = \theta_1$.

In consequence $\mathrm{Hol}_{\theta_1} = \mathrm{Hol}_{\theta_2}$ *if and only if the magnetic Laplacians* $\Delta_{\mathcal{G},\theta_1}$ *and* $\Delta_{\mathcal{G},\theta_2}$ *are unitarily equivalent.*

Lemma 2.2 *Let* $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ *be a connected graph such that* $1 \in \ker \Delta_{\mathcal{G},0}$. *Let* θ *be magnetic potential. Then* $\ker \Delta_{\mathcal{G},\theta} \neq \{0\}$ *if and only if* $\mathrm{Hol}_\theta = 0$.

Remark 2.3 *By construction of the Friedrichs extension, the domain of* $\Delta_{\mathcal{G},0}$ *is given by*

$$\mathcal{D}(\Delta_{\mathcal{G},0}) = \left\{ f \in \ell^2(\mathcal{V}, m), x \mapsto \frac{1}{m(x)} \sum_{y\in\mathcal{V}} \mathcal{E}(x,y)(f(x) - f(y)) \in \ell^2(\mathcal{V}, m) \right\}$$
$$\bigcap \overline{\mathcal{C}_c(\mathcal{V})}^{(\|\cdot\|^2 + Q_{\mathcal{G},0}(\cdot))^{1/2}}.$$

The hypothesis $1 \in \ker \Delta_{\mathcal{G},0}$ *is trivially satisfied if* $\mathcal{G}$ *is a finite graph. In general, it is satisfied if and only if:*

($*$) 1 *belongs to the closure of* $\mathcal{C}_c(\mathcal{V})$ *with respect to the norm* $(\| \cdot \|^2 + Q_{\mathcal{G},0}(\cdot))^{1/2}$.

A sufficient condition to guarantee ($*$) *is that the following two conditions hold true:*

1) $\mathcal{G}$ *is of finite volume, i.e., such that* $\sum_{x\in\mathcal{V}} m(x) < \infty$,

2) $\Delta_{\mathcal{G},0}$ *is essentially self-adjoint on* $\mathcal{C}_c(\mathcal{V})$.

Proof: If $\mathrm{Hol}_\theta = 0$ then $\Delta_{\mathcal{G},\theta}$ is unitarily equivalent to $\Delta_{\mathcal{G},0}$ by Proposition 2.1 and $1 \in \ker(\Delta_{\mathcal{G},0}) \neq \{0\}$ by hypothesis.
Conversely, let $f \neq 0$ with $\Delta_{\mathcal{G},\theta} f = 0$ and hence $Q_{\mathcal{G},\theta}(f) = 0$. This implies that all terms in the expression of $Q_{\mathcal{G},\theta}(f)$ vanish. In particular, if $\mathcal{E}(x, y) \neq 0$ we have

$$f(x) = e^{\mathrm{i}\theta_{x,y}} f(y). \tag{5}$$

Assume that there is a cycle $\gamma = (x_0, x_1, \dots, x_N = x_0)$, such that $\mathrm{Hol}_\theta(\gamma) \neq 0$. Using (5), we obtain that

$$f(x_i) = e^{-\mathrm{i}\mathrm{Hol}_\theta(\gamma)} f(x_i) .$$

for all $i = 0, \dots, N - 1$. Therefore $f|_\gamma = 0$. Then, since $f \neq 0$, there is $x \in \mathcal{V}$ such that $f(x) \neq 0$. Using again (5) and by connectedness between x and γ, it yields that $f(x) = 0$. Contradiction. Therefore if there exists $f \in \ker(\Delta_{\mathcal{G},\theta}) \setminus \{0\}$ then $\mathrm{Hol}_\theta = 0$. ■

We exhibit the following coupling constant effect.

Corollary 2.4 *Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be a connected graph of finite volume, i.e., such that $\sum_{x\in\mathcal{V}} m(x) < \infty$ and let θ be a magnetic potential such that $\mathrm{Hol}_\theta \neq 0$. Assume that the function 1 is in $\ker \Delta_{\mathcal{G},\theta}$. Then there is $\nu \in \mathbb{R}$ such that*

$$\ker \Delta_{\mathcal{G},\lambda\theta} \neq \{0\} \Leftrightarrow \lambda = 0 \text{ in } \mathbb{R}/\nu\mathbb{Z}.$$

Proof: Let $\Phi : (\mathbb{R}, +) \to (\mathrm{Hom}_{\mathbb{Z}}(Z_1(\mathcal{G}), \mathbb{R}/2\pi\mathbb{Z}), +)$ be defined by $\Phi(\lambda) := \mathrm{Hol}_{\lambda\theta}$. It is a homomorphism of group. Hence its kernel is a subgroup of $(\mathbb{R}, +)$. In particular it is either dense with respect to the Euclidean norm or equal to $\nu\mathbb{Z}$ for some $\nu \in \mathbb{R}$, e.g., [Bou, Section V.1.1]. Suppose by contradiction that the kernel is dense. Since for any cycle γ of $\mathcal{G}$, the map $\lambda \mapsto \mathrm{Hol}_{\lambda\theta}(\gamma)$ is continuous from $\mathbb{R}$ to $\mathbb{R}/2\pi\mathbb{Z}$, we infer that $\mathrm{Hol}_{\lambda\theta}(\gamma) = 0$ for all $\lambda \in \mathbb{R}$. Hence, $\Phi(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. This is a contradiction with $\mathrm{Hol}_\theta \neq 0$. We conclude that there is $\nu \in \mathbb{R}$ such that $\ker(\Phi) = \nu\mathbb{Z}$, i.e., using Proposition 2.1, that

$$\{\lambda \in \mathbb{R}, \ker \Delta_{\mathcal{G},\lambda\theta} \neq \{0\}\} = \{\lambda \in \mathbb{R}, \mathrm{Hol}_{\lambda\theta} = 0\} = \nu\mathbb{Z}.$$

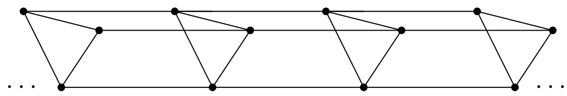
This ends the proof. ■

2.2 The setting

Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, the *Cartesian product of $\mathcal{G}_1$ by $\mathcal{G}_2$* is defined by $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$.

$$\begin{cases} m(x,y) := & m_1(x) \times m_2(y), \\ \mathcal{E}\left((x,y),(x',y')\right) := & \mathcal{E}_1(x,x') \times \delta_{y,y'} m_2(y) + m_1(x)\delta_{x,x'} \times \mathcal{E}_2(y,y'), \\ \theta\left((x,y),(x',y')\right) := & \theta_1(x,x') \times \delta_{y,y'} + \delta_{x,x'} \times \theta_2(y,y'), \end{cases}$$

We denote by $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$. This definition generalizes the unweighted Cartesian product, e.g., [Ha]. It is used in several places in the literature, e.g., [Ch][Section 2.6] and in [BGKLM] for a generalization.



The graph of $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

The terminology is motivated by the following decomposition:

$$\Delta_{\mathcal{G},\theta} = \Delta_{\mathcal{G}_1,\theta_1} \otimes 1 + 1 \otimes \Delta_{\mathcal{G}_2,\theta_2},$$

where $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$. The spectral theory of $\Delta_{\mathcal{G},\theta}$ is well-understood since

$$e^{\mathrm{i}t\Delta_{\mathcal{G},\theta}} = e^{\mathrm{i}t\Delta_{\mathcal{G}_1,\theta_1}} \otimes e^{\mathrm{i}t\Delta_{\mathcal{G}_2,\theta_2}}, \text{ for } t \in \mathbb{R}.$$

We refer to [RS][Section VIII.10] for an introduction to the tensor product of self-adjoint operators.

In this paper, we are motivated by a geometrical situation. A hyperbolic manifold of finite volume is the union of a compact part and of a cusp, e.g., [Th, Theorem 4.5.7]. The cusp part can be seen as the product of $(1,\infty)\times M$, where (M, g_M) is a possibly disconnected Riemannian manifold, endowed with the metric,

$$y^{-1}(dy^2 + g_M).$$

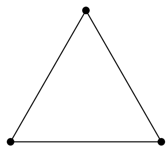
On the cusp part, the infimum of the radius of injectivity is 0.

To analyze the Laplacian on this product one separates the variables and obtain a decomposition which is not of the type of a Cartesian product, e.g., [GM, Eq. (5.22)] for some details. We aim at mimicking this situation and introduce a *modified Cartesian product*. Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I}\subset\mathcal{V}_2$, we define the *product of* $\mathcal{G}_1$ *by* $\mathcal{G}_2$ *through* $\mathcal{I}$ by $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where $\mathcal{V} := \mathcal{V}_1\times\mathcal{V}_2$ and

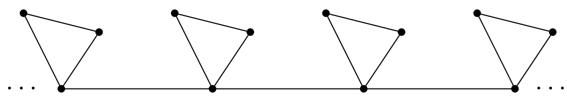
$$\left\{\begin{array}{rl} m(x,y) := & m_1(x)\times m_2(y),\\ \mathcal{E}\left((x,y),(x',y')\right) := & \mathcal{E}_1(x,x')\times\delta_{y,y'}\left(\sum_{z\in\mathcal{I}}\delta_{y,z}\right)+\delta_{x,x'}\times\mathcal{E}_2(y,y'),\\ \theta\left((x,y),(x',y')\right) := & \theta_1(x,x')\times\delta_{y,y'}+\delta_{x,x'}\times\theta_2(y,y'),\end{array}\right.$$

for all $x, x'\in\mathcal{V}_1$ and $y, y'\in\mathcal{V}_2$. We denote $\mathcal{G}$ by $\mathcal{G}_1\times_{\mathcal{I}}\mathcal{G}_2$. If $\mathcal{I}$ is empty, the graph is disconnected and of no interest for our purpose. If $|\mathcal{I}|=1$, $\mathcal{G}_1\times_{\mathcal{I}}\mathcal{G}_2$ is the graph $\mathcal{G}_1$ decorated by $\mathcal{G}_2$, see [SA] for its spectral analysis in the unweighted case. If $\mathcal{I}=\mathcal{V}_2$ and $m=1$, we notice that $\mathcal{G}_1\times_{\mathcal{I}}\mathcal{G}_2=\mathcal{G}_1\times\mathcal{G}_2$.

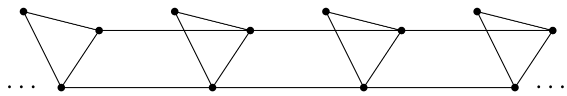
The graph of $\mathbb{Z}$



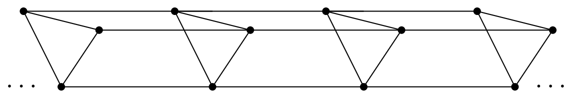
The graph of $\mathbb{Z}/3\mathbb{Z}$



The graph of $\mathbb{Z}\times_{\mathcal{I}}\mathbb{Z}/3\mathbb{Z}$, with $|\mathcal{I}|=1$



The graph of $\mathbb{Z} \times_{\mathcal{I}} \mathbb{Z}/3\mathbb{Z}$, with $|\mathcal{I}| = 2$



The graph of $\mathbb{Z} \times_{\mathcal{I}} \mathbb{Z}/3\mathbb{Z}$*, with* $|\mathcal{I}| = 3$

Under the representation $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$,

$$\deg_{\mathcal{G}}(\cdot) = \deg_{\mathcal{G}_1}(\cdot) \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \tag{6}$$

and

$$\Delta_{\mathcal{G},\theta} = \Delta_{\mathcal{G}_1,\theta_1} \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}. \tag{7}$$

If m is non-trivial, we stress that the Laplacian obtained with our product is usually not unitarily equivalent to the Laplacian obtained with the Cartesian product. However, there is a potential $V : \mathcal{V} \to \mathbb{R}$ such that $\Delta_{\mathcal{G}_1 \times \mathcal{G}_2}$ is unitarily equivalent to $\Delta_{\mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2} + V(\cdot)$, in $\ell^2(\mathcal{V}, m)$.

Definition 2.5 *Set* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$*,* $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$*, and* $\mathcal{I} \subset \mathcal{V}_2$*. We say that* $\mathcal{G} = \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ *is a* discrete cusp *if the following hypotheses are satisfied:*

(H1) $m_1(x)$ *tend to* 0 *as* $|x| \to \infty$*,*

(H2) $\mathcal{G}_2$ *is finite,*

(H3) $\Delta_{\mathcal{G}_1,\theta_1}$ *is bounded (or equivalently* $\sup_{x \in \mathcal{V}_1} \deg_{\mathcal{G}_1}(x) < \infty$*).*

We now motivate the choice of the above hypotheses by discussing the radius of injectivity. We start by defining a different metric on $\mathcal{V}$, this choice is motivated by the works of [CTT2] and [MiT] but it needs a small adaptation for our purpose.

Definition 2.6 *Given* $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$*, the* weighted length *of an edge* $(x, y) \in \mathcal{E}$ *defined by:*

$$L_{\mathcal{G}}\big((x,y)\big) := \sqrt{\frac{\min\big(m(x), m(y)\big)}{\mathcal{E}(x,y)}}.$$

Given $x, y \in \mathcal{V}$, we define the weighted distance *from x to y with respect to this length by:*

$$\rho_{L_{\mathcal{G}}}(x,y) := \inf_{\gamma} \sum_{i=0}^{|\gamma|-1} L_{\mathcal{G}}\big(\gamma(i), \gamma(i+1)\big),$$

where γ is a path joining x to y and with the convention that $\rho_{L_{\mathcal{G}}}(x,x) := 0$ for all $x \in \mathcal{V}$.

Remark 2.7 *Since $\mathcal{G}$ is assumed connected, $\rho_{L_{\mathcal{G}}}$ is a metric on $\mathcal{V}$. Observe that, by [Ke, Section 3.2.5], $\rho_{L_{\mathcal{G}}}$ belongs to the class of intrinsic metrics if and only if the combinatorial vertex degree is bounded. We refer to [Ke] for a general definition, historical references, properties, and applications. However, since Propositions 2.9 and 2.10 do not hold in general with an arbitrary intrinsic metric, we stick to our specific choice of metric.*

We turn to the definitions of the girth and of the weighted radius of injectivity. This is essentially a weighted version of the standard ones, e.g, [EGL].

Definition 2.8 *Given $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, the* girth *at $x \in \mathcal{V}$ of $\mathcal{G}$ w.r.t. the weighted length $L_{\mathcal{G}}$ is*

$$\mathrm{girth}(x) := \inf\{L_{\mathcal{G}}(\gamma),$$
$$\gamma \textit{ simple cycle of unweighted length } \geq 3 \textit{ and containing } x\},$$

where simple cycle means a closed walk with no repetitions of vertices and edges allowed, other than the repetition of the starting and ending vertex. We use the convention that the girth is $+\infty$ if there is no such cycle.

$$\mathrm{girth}(\mathcal{G}) := \inf_{x \in \mathcal{V}} \mathrm{girth}(x).$$

The radius of injectivity *(at x) of $\mathcal{G}$ with respect to $L_{\mathcal{G}}$ is half the girth (at x). We denote the radius of injectivity by $\mathrm{rad}(\mathcal{G})$ (at x by $\mathrm{rad}(x)$ respectively)*

Note that with this definition, the radius of injectivity of a tree is $+\infty$.

Proposition 2.9 *Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ and $\mathcal{I} \subset \mathcal{V}_2$ Assume that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp. We have:*

1) $\mathrm{rad}(\mathcal{G}_1) > 0$.

2) If $\mathrm{rad}(\mathcal{G}_2) < \infty$, *then* $\mathrm{rad}(\mathcal{G}) = 0$.

Proof: (1) Assume that $\mathrm{rad}(\mathcal{G}_1) = 0$. Then for all $\varepsilon > 0$, there is $x \sim y$ in $\mathcal{V}_1$ such that $L_{\mathcal{G}_1}\big((x,y)\big) < \varepsilon$. In particular, we have $\deg_{\mathcal{G}_1}(x) > \varepsilon^{-2}$ or $\deg_{\mathcal{G}_1}(y) > \varepsilon^{-2}$. This is in contradiction with (H3).
(2) Since $\mathrm{rad}(\mathcal{G}_2) < \infty$, for all $x \in \mathcal{V}_1$, there is a pure cycle contained in $\{x\} \times \mathcal{V}_2$. Moreover, for all $x \in \mathcal{V}_1$ and $a \sim b$ in $\mathcal{V}_2$, since $\mathcal{E}(x,x) = 0$, we have:

$$L_{\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2}\big(((x,a),(x,b))\big) = \sqrt{m_1(x)} L_{\mathcal{G}_2}\big((a,b)\big)$$

By (H1) we obtain that $\mathrm{rad}(\mathcal{G}) = 0$. ■

In contrast with this result we see that under the same hypotheses, the Cartesian product is not small at infinity. More precisely, we have:

Proposition 2.10 *Set* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ *and* $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$. *Assume that (H1), (H2), and (H3) are satisfied. Then* $\mathrm{rad}(\mathcal{G}_1 \times \mathcal{G}_2) > 0$.

Proof: Assume that $\mathrm{rad}(\mathcal{G}_1 \times \mathcal{G}_2) = 0$. For all $\varepsilon > 0$, there are $x_1 \sim y_1$ in $\mathcal{V}_1$ and $x_2 \sim y_2$ in $\mathcal{V}_2$ such that

$$\begin{aligned} \varepsilon &> L_{\mathcal{G}_1 \times \mathcal{G}_2}\big(((x_1, x_2), (x_1, y_2))\big) = L_{\mathcal{G}_2}\big((x_2, y_2)\big) \\ \text{or } \varepsilon &> L_{\mathcal{G}_1 \times \mathcal{G}_2}\big(((x_1, x_2), (y_1, x_2))\big) = L_{\mathcal{G}_1}\big((x_1, y_1)\big). \end{aligned}$$

The first line is in contradiction with (H2) and the second line with (H3). ■

2.3 Absence of essential spectrum

We have a first result of absence of essential spectrum. We refer to [CTT3] for related results based on the non-triviality of Hol_θ in the context of non-complete graphs. See also [BGKLM] for similar ideas.

Proposition 2.11 *Set* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, *and* $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$, *with* $|\mathcal{I}| > 0$. *Assume that (H1), (H2), and* $\mathrm{Hol}_{\theta_2} \neq 0$ *hold true. Then* $\Delta_{\mathcal{G},\theta}$ *has a compact resolvent, and*

$$\mathcal{N}_\lambda \left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right) \geq \mathcal{N}_\lambda(\Delta_{\mathcal{G},\theta}), \text{ for all } \lambda \geq 0.$$

Proof: Note that

$$\Delta_{\mathcal{G},\theta} \geq \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}$$

in the form sense on $\mathcal{C}_c(\mathcal{V})$. Since (H1) and (H2) hold, Lemma 2.2 ensures that 0 is not in the spectrum of $(\Delta_{\mathcal{G}_2,\theta_2})$. Hence the spectrum of the r.h.s. is purely discrete. By the min-max Principle, e.g., [Go, RS], $\Delta_{\mathcal{G},\theta}$ has a compact resolvent. ■

2.4 The asymptotic of the eigenvalues

From now on, we focus on the case when the graph is a discrete cusp and aim at a more precise result. To start off, we give the key-stone of our approach:

Proposition 2.12 *Set* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, *and* $\mathcal{I} \subset \mathcal{V}_2$ *non-empty. Assume that* $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ *is a discrete cusp. We set*

$$M := \sup_{x \in \mathcal{V}_1} \deg_{\mathcal{G}_1}(x) \times \max_{y \in \mathcal{V}_2}(1/m_2(y)) < \infty. \tag{8}$$

We have:

$$\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \leq \deg_{\mathcal{G}}(\cdot) \leq \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) + M, \tag{9}$$

$$\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2} \leq \Delta_{\mathcal{G},\theta} \leq 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}, \tag{10}$$

in the form sense on $\mathcal{C}_c(\mathcal{V})$.

Proof: Use (1), (6), and (7). ■

We work in the spirit of [Go, BGK, BGKLM] and compare the Laplacian directly with the degree.

Proposition 2.13 *Set $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, and $\mathcal{I} \subset \mathcal{V}_2$ non-empty. Assume that $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ is a discrete cusp. Set M as in (8). We have:*

$$\frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y\in\mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} \big(\deg_{\mathcal{G}}(\cdot) - M\big) \leq \Delta_{\mathcal{G},\theta} \leq 2M + 2\deg_{\mathcal{G}}(\cdot), \tag{11}$$

in the form sense on $\mathcal{C}_c(\mathcal{V})$.
Moreover, assuming that $\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2}) > 0$, then $\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right)$. Furthermore, since $\lim_{|x|\to\infty} \deg_{\mathcal{G}}(x) = \infty$, $\Delta_{\mathcal{G},\theta}$ has a compact resolvent and

$$0 < \frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y\in\mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} \leq \liminf_{n\to\infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(\deg_{\mathcal{G}}(\cdot))} \leq \limsup_{n\to\infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(\deg_{\mathcal{G}}(\cdot))} \leq 2.$$

Proof: Use (10) and (1) to get

$$\frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y\in\mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \leq \Delta_{\mathcal{G},\theta} \leq 2M + \frac{2}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot),$$

Then apply (9) to obtain (11). Concerning the statement about the eigenvalue this follows from the standard consequences of the min-max Principle, e.g., [Go]. ■

Here, trying to compare directly $\Delta_{\mathcal{G},\theta}$ to $\deg_{\mathcal{G}}$ to get sharp results about eigenvalues is too optimistic because it is unclear how to obtain constants arbitrarily close to 1 in front of $\deg_{\mathcal{G}}$, as in [Go, BGK]. To obtain some sharp asymptotics for the eigenvalues of $\Delta_{\mathcal{G},\theta}$, as in (15), we will use directly (10) and analyze very carefully the operator $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}$.

Theorem 2.14 *Set* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, *and* $\mathcal{I} \subset \mathcal{V}_2$ *non-empty. Assume that* $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ *is a discrete cusp. We obtain that*

$$\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(m_1^{-1/2}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}^{1/2}\right). \tag{12}$$

Moreover, we have:

1) $\Delta_{\mathcal{G},\theta}$ *has a compact resolvent if and only if* $\mathrm{Hol}_{\theta_2} \neq 0$.

2) If $\mathrm{Hol}_{\theta_2} \neq 0$, *then*

$$\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right)$$

and

$$\lim_{n\to\infty} \frac{\lambda_n\left(\Delta_{\mathcal{G},\theta}\right)}{\lambda_n\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right)} = 1. \tag{13}$$

Furthermore, setting M *as in* (8),

$$\mathcal{N}_{\lambda-2M}\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right) \leq \mathcal{N}_\lambda(\Delta_{\mathcal{G},\theta}) \leq \mathcal{N}_\lambda\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right), \tag{14}$$

for all $\lambda \geq 0$.

Proof: First note that (12) follows directly from (10). Denoting by $\{g_i\}_{i=1,..,|\mathcal{V}_2|}$ the eigenfunctions associated to the eigenvalues $\{\lambda_i\}_{i=1,..,|\mathcal{V}_2|}$ of $\Delta_{\mathcal{G}_2,\theta_2}$, where $\lambda_j \leq \lambda_{j+1}$, we see that the eigenfunctions of $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$ are given by $\{\delta_x \otimes g_i\}$, where $x \in \mathcal{V}_1$ and $i = 1,..,|\mathcal{V}_2|$. Then, using (H1), we observe that

$$\sigma\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}\right) = \overline{m_1^{-1}(\mathcal{V}_1) \times \{\lambda_1, \ldots, \lambda_{|\mathcal{V}_2|}\}} = m_1^{-1}(\mathcal{V}_1) \times \{\lambda_1, \ldots, \lambda_{|\mathcal{V}_2|}\}.$$

Besides, $0 \in \sigma\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}\right)$ if and only if 0 is an eigenvalue of $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$ of infinite multiplicity if and only if $\lambda_1 = 0$ if and only if $\mathrm{Hol}_{\theta_2} = 0$, by Lemma 2.2. Moreover, recalling (H1), we see that all the eigenvalues of $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$ which are not 0 are of finite multiplicity. Therefore, $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$ has a compact resolvent if and only if $\mathrm{Hol}_{\theta_2} \neq 0$. Combining the latter and (10), the min-max Principle yields the first point.
We turn to the second point and assume that $\mathrm{Hol}_{\theta_2} \neq 0$. The equality of the form-domains is given by (11). Taking in account (10), the min-max Principle ensures the asymptotic behavior of λ_n and the inequalities (14). ■

Remark 2.15 *In the case when* $\mathrm{Hol}_{\theta_2} = 0$, *for instance when* $\theta_2 = 0$, *we see that the form-domain is* $m_1^{-1/2} \otimes P^\perp_{\ker(\Delta_{\mathcal{G}_2,\theta_2})}$. *In particular, the form-domain is* NOT *that of* $\deg_{\mathcal{G}}(\cdot)$. *Indeed if the two form-domains are the same, the closed graph theorem yields the existence of* $c_1 > 0$ *and* $c_2 > 0$ *so that*

$$c_1 \deg_{\mathcal{G}}(\cdot) - c_2 \leq m_1^{-1/2} \otimes P^\perp_{\ker(\Delta_{\mathcal{G}_2,\theta_2})},$$

in the form sense on $\mathcal{C}_c(\mathcal{V})$*. However, note that* $0 \in \sigma_{\mathrm{ess}}\left(m_1^{-1/2} \otimes P^{\perp}_{\ker(\Delta_{\mathcal{G}_2,\theta_2})}\right)$*, whereas* $\deg(\cdot)$ *has a compact resolvent. This is a contradiction with the min-max Principle. We obtain:*

$$\begin{aligned}\mathcal{D}\left(\Delta_{\mathcal{G},\theta}^{1/2}\right) = \mathcal{D}\left(\deg^{1/2}(\cdot)\right) &\Leftrightarrow \mathrm{Hol}_{\theta_2} \neq 0\\ &\Leftrightarrow \Delta_{\mathcal{G},\theta} \text{ has a compact resolvent.}\end{aligned}$$

In (13), we exhibit the behaviour of the eigenvalues in terms of an explicit and computable mean. We now aim at comparing the asymptotic with that of the degree, as in [Go, BGK]. The new phenomenon is that we are able to obtain a constant different from 1 in the asymptotic.

Corollary 2.16 *Let* $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$*,* $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$*, and* $\mathcal{I} \subset \mathcal{V}_2$ *non-empty such that* $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ *is a discrete cusp. Suppose that* $\deg_{\mathcal{G}_2}$ *is constant on* $\mathcal{V}_2$ *and take* θ_2 *such that* $\mathrm{Hol}_{\theta_2} \neq 0$*. Then, for all* $a \in [1, +\infty[$*, there exists* $\widetilde{\mathcal{G}}_1 := (\widetilde{\mathcal{E}}_1, \mathcal{V}_1, \tilde{m}_1)$ *such that*

1) $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}_1 \times_{\mathcal{I}} \mathcal{G}_2$ *is a discrete cusp.*

2) $\mathcal{E}_1$ *and* $\widetilde{\mathcal{E}}_1$ *have the same zero set.*

3) $\deg_{\widetilde{\mathcal{G}}_1}(x) \leq \deg_{\mathcal{G}_1}(x)$ *for all* $x \in \mathcal{V}_1$*.*

4) $\Delta_{\widetilde{\mathcal{G}},\theta}$ *is with compact resolvent, and*

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_\lambda\left(\Delta_{\widetilde{\mathcal{G}},\theta}\right)}{\mathcal{N}_\lambda\left(\deg_{\widetilde{\mathcal{G}}}(\cdot)\right)} = a. \tag{15}$$

Proof: We choose $\widetilde{m}_1$ and $\widetilde{\mathcal{E}}_1$ later. We denote by $\{\lambda_i\}_{i=1,\dots,|\mathcal{V}_2|}$ the eigenvalues of $\Delta_{\mathcal{G}_2,\theta_2}$. Since $\mathrm{Hol}_{\theta_2} \neq 0$, we have $\lambda_i \neq 0$ for all $i = 1, \dots, |\mathcal{V}_2|$. This yields:

$$\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}\right) = \left|\left\{(x,i), \frac{\lambda_i}{\widetilde{m}_1(x)} \leq \lambda\right\}\right| = \sum_{i=1}^{|\mathcal{V}_2|}\left|\left(\frac{1}{\widetilde{m}_1}\right)^{[-1]}\left(\left[0, \frac{\lambda}{\lambda_i}\right]\right)\right|,$$

where $[-1]$ denotes the reciprocal image. On the other hand,

$$\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right) = |\mathcal{V}_2| \times \left|\left(\frac{1}{\widetilde{m}_1}\right)^{[-1]}\left(\left[0, \frac{\lambda}{\deg_{\mathcal{G}_2}}\right]\right)\right|.$$

Moreover, from (9) we get

$$\mathcal{N}_{\lambda-M}(\widetilde{m}_1^{-1}(\cdot) \otimes \deg_{\mathcal{G}_2}) \leq \mathcal{N}_\lambda(\deg_{\widetilde{\mathcal{G}}}(\cdot)) \leq \mathcal{N}_\lambda(\widetilde{m}_1^{-1}(\cdot) \otimes \deg_{\mathcal{G}_2}), \tag{16}$$

for all $\lambda \geq 0$, where M is given by (8).

Step 1: We first aim at $a = 1$ in (15). Thanks to Lemma 2.18, we choose $\widetilde{m}_1$ and $\widetilde{\mathcal{E}}_1$ such that the three first points are satisfied and

$$\left|\left\{x \in \mathcal{V}_1, \frac{1}{\widetilde{m}_1(x)} \leq \lambda\right\}\right| \sim \ln(\lambda), \quad \text{as } \lambda \to \infty,$$

where $\sim$ stands for asymptotically equivalent. We obtain:

$$\frac{\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}\right)}{\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} \sim \frac{\sum_{i=1}^{|\mathcal{V}_2|}(\ln(\lambda) - \ln(\lambda_i))}{|\mathcal{V}_2|(\ln(\lambda) - \ln(\deg_{\mathcal{G}_2}))} \to 1, \quad \text{as } \lambda \to \infty. \tag{17}$$

and for all $c \in \mathbb{R}$,

$$\begin{aligned} \mathcal{N}_{\lambda-c}\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right) &\sim |\mathcal{V}_2| \ln(\lambda - c) \sim |\mathcal{V}_2| \ln(\lambda) \\ &\sim \mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right), \quad \text{as } \lambda \to \infty. \end{aligned} \tag{18}$$

Combining the latter with (16), we infer that for all $c \in \mathbb{R}$

$$\mathcal{N}_{\lambda-c}\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right) \sim \mathcal{N}_\lambda(\deg_{\widetilde{\mathcal{G}}}(\cdot)), \quad \text{as } \lambda \to \infty. \tag{19}$$

Using now (17), this yields that for all $c \in \mathbb{R}$

$$\mathcal{N}_{\lambda-c}\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}\right) \sim \mathcal{N}_\lambda(\deg_{\widetilde{\mathcal{G}}}(\cdot)), \quad \text{as } \lambda \to \infty. \tag{20}$$

Finally recalling (14), we infer that

$$\mathcal{N}_\lambda\left(\Delta_{\widetilde{\mathcal{G}},\theta}\right) \sim \mathcal{N}_\lambda\left(\deg_{\widetilde{\mathcal{G}}}(\cdot)\right), \quad \text{as } \lambda \to \infty.$$

In other words, there are $\widetilde{m}_1$ and $\widetilde{\mathcal{E}}_1$ such that the three first points are satisfied and such that (15) is satisfied with $a = 1$.

Step 2: We turn to the case $a > 1$ in (15). Given $\alpha > 0$,. Thanks to Lemma 2.18, we choose $\widetilde{m}_1$ and $\widetilde{\mathcal{E}}_1$ such that the three first points are satisfied and

$$\left|\left\{x \in \mathcal{V}_1, \frac{1}{\widetilde{m}_1(x)} \leq \lambda\right\}\right| \sim \lambda^\alpha, \quad \text{as } \lambda \to \infty,$$

We obtain:

$$\frac{\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}\right)}{\mathcal{N}_\lambda\left(\frac{1}{\widetilde{m}_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} \underset{\lambda\to\infty}{\sim} \frac{1}{|\mathcal{V}_2|} \sum_{i=1}^{|\mathcal{V}_2|} \left(\frac{\deg_{\mathcal{G}_2}}{\lambda_i}\right)^\alpha =: F(\alpha).$$

First note that

$$\lim_{\alpha\to 1^+} F(\alpha) = 1.$$

Next, the sum of the eigenvalues (counted with multiplicity) of $\Delta_{\mathcal{G}_2,\theta_2}$ is equal to $|\mathcal{V}_2| \deg_{\mathcal{G}_2}$. Therefore, there exists at least one eigenvalue λ_i, with $1 \leq i \leq |\mathcal{V}_2|$ so that $\deg_{\mathcal{G}_2} > \lambda_i$. In particular

$$\lim_{\alpha\to +\infty} F(\alpha) = +\infty.$$

Finally, by continuity of F, we obtain that for all $a > 1$ there is $\alpha > 1$ such that $F(\alpha) = a$. To conclude, repeating the end of step 1, we obtain that for all $a > 1$, there are $\widetilde{m}_1$ and $\widetilde{\mathcal{E}}_1$ such that the three first points are satisfied and such that (15) is satisfied. ■

Remark 2.17 *In [Go, BGK], the asymptotic in $\mathcal{N}_\lambda$ was not discussed since the estimates that they obtain seem too weak to conclude. Being able to compute $\mathcal{N}_\lambda$ in an explicit way, as in* (15)*, is a new phenomenon.*

We have used the following lemma:

Lemma 2.18 *Let $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ be a graph satisfying (H1) and (H3) in Definition 2.5 and let $f : [1, +\infty) \to [1, +\infty)$ be a continuous and strictly increasing function that tends to $+\infty$ at $+\infty$. There exists $\widetilde{\mathcal{G}}_1 := (\widetilde{\mathcal{E}}_1, \mathcal{V}_1, \widetilde{m}_1)$ such that*

1) $\mathcal{E}$ and $\tilde{\mathcal{E}}$ have the same zero set.

2) (H1) and (H3) are satisfied for $\widetilde{\mathcal{G}}_1$.

3) $\deg_{\widetilde{\mathcal{G}}_1}(x) \leq \deg_{\mathcal{G}_1}(x)$ for all $x \in \mathcal{V}_1$.

4) We have:

$$\left|\left\{x \in \mathcal{V}_1, \frac{1}{\widetilde{m}_1(x)} \leq \lambda\right\}\right| \sim f(\lambda), \qquad \text{as } \lambda \to \infty,$$

where $\sim$ stands for asymptotically equivalent.

Proof: Without any loss of generality, one may suppose that $f(1) = 1$. Let $\phi : \mathbb{N}^* \to \mathcal{V}_1$ be a bijection. Set:

$$\tilde{m}_1(\phi(n)) := \frac{1}{f^{[-1]}(n)},$$

where $[-1]$ denotes the reciprocal image. Note that (H1) is satisfied. Moreover,

$$\left|\left\{x \in \mathcal{V}_1, \frac{1}{\widetilde{m}_1(x)} \leq \lambda\right\}\right| = |\{n \in \mathbb{N}^*, n \leq f(\lambda)\}| = \lfloor f(\lambda) \rfloor + 1 \sim f(\lambda),$$

as $\lambda \to \infty$. Finally, we set:

$$\widetilde{\mathcal{E}}_1(x,y) := \mathcal{E}_1(x,y)\frac{\min(\tilde{m}_1(x), \tilde{m}_1(y))}{\max(m_1(x), m_1(y))}.$$

The first point is clear. For (H3), note that $\deg_{\widetilde{\mathcal{G}}_1}(x) \leq \deg_{\mathcal{G}_1}(x)$ for all $x \in \mathcal{V}_1$. ■

We end this section by proving the results stated in the introduction.

Proof of Theorem 1.1: Let us consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, where

$$\mathcal{V}_1 := \mathbb{N}, \quad m_1(n) := \exp(-n), \text{ and } \mathcal{E}_1(n, n+1) := \exp(-(2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, 1)$ a simple connected finite graph such that $|\mathcal{V}_2| = n$. Set $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2$, $\theta_1 := 0$ and θ_2 such that $\mathrm{Hol}_{\theta_2} \neq 0$.
In the spirit of [GM], we denote by P^{le}_κ the projection on $\ker(\Delta_{\mathcal{G}_2,\kappa\theta_2})$ and by P^{he}_κ is the projection on $\ker(\Delta_{\mathcal{G}_2,\kappa\theta_2})^\perp$. Here *le* stands for *low energy* and *he* for *high energy*.
We have that $\Delta_{\mathcal{G},\kappa\theta} := \Delta^{\mathrm{le}}_{\mathcal{G},\kappa\theta} \oplus \Delta^{\mathrm{he}}_{\mathcal{G},\kappa\theta}$, where

$$\Delta^{\mathrm{le}}_{\mathcal{G},\kappa\theta} := \Delta_{\mathcal{G}_1,0} \otimes P^{\mathrm{le}}_\kappa,$$

on $(1 \otimes P^{\mathrm{le}}_\kappa)\ell^2(\mathcal{V}, m)$, and

$$\Delta^{\mathrm{he}}_{\mathcal{G},\kappa\theta} := \Delta_{\mathcal{G}_1,0} \otimes P^{\mathrm{he}}_\kappa + \frac{1}{m_1(\cdot)} \otimes P^{\mathrm{he}}_\kappa \Delta_{\mathcal{G}_2,\kappa\theta_2},$$

on $(1 \otimes P^{\mathrm{he}}_\kappa)\ell^2(\mathcal{V}, m)$.
By Lemma 2.2, Corollary 2.4, and Remark 2.15, there exists $\nu > 0$ such that

$$\begin{aligned} P^{\mathrm{le}}_\kappa = 0 \quad &\Leftrightarrow \quad \mathrm{Hol}_{\kappa\theta_2} \neq 0 \\ &\Leftrightarrow \quad \kappa \neq 0 \text{ in } \mathbb{R}/\nu\mathbb{Z} \quad \Leftrightarrow \quad \mathcal{D}\left(\Delta^{1/2}_{\mathcal{G},\kappa\theta}\right) = \mathcal{D}\left(\deg^{1/2}_{\mathcal{G}}(\cdot)\right). \end{aligned}$$

The proof of Theorem 2.14 gives the first point. Assume that $\kappa \in \mathbb{R}/\nu\mathbb{Z}$. Let $U : \ell^2(\mathbb{N}, m_1) \to \ell^2(\mathbb{N}, 1)$ be the unitary map given by $Uf(n) := \sqrt{m_1(n)}f(n)$. We see that:

$$U\Delta^{\mathrm{le}}_{\mathcal{G},\kappa\theta}U^{-1} = \Delta_{\mathbb{N},0} + (e^{-1/2} - 1)\delta_0 + e^{1/2} + e^{-1/2} - 2 \text{ in } \ell^2(\mathbb{N}),$$

where $\Delta_{\mathbb{N},0}$ is related to the simple graph of $\mathbb{N}$. By using for instance some Jacobi matrices techniques, it is well-known that the essential spectrum of $\Delta^{\mathrm{le}}_{\mathcal{G},\kappa\theta}$ is purely absolutely continuous and equal to

$$\sigma_{\mathrm{ac}}(\Delta^{\mathrm{le}}_{\mathcal{G},\kappa\theta}) = [e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2],$$

with multiplicity one, e.g., [We]. It has a unique eigenvalue and it is negative.

We turn to the high energy part. Denote by $\{\lambda_i\}_{i=1,\dots,n}$, with $\lambda_i \leq \lambda_{i+1}$, the eigenvalues of $\Delta_{\mathcal{G}_2,\kappa\theta_2}$. Recall that $\lambda_1 = 0$ due to the fact that $\mathrm{Hol}_{\kappa\theta_2} = 0$. By (10),

$$\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\kappa\theta_2} P_\kappa^{\mathrm{he}} \leq \Delta_{\mathcal{G},\kappa\theta}(1 \otimes P_\kappa^{\mathrm{he}}) \leq 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\kappa\theta_2} P_\kappa^{\mathrm{he}}.$$

Hence, $\Delta_{\mathcal{G},\kappa\theta}(1 \otimes P_\kappa^{\mathrm{he}})$ has a compact resolvent and

$$\begin{aligned}\mathcal{N}_{\lambda-2M}\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\kappa\theta_2} P_\kappa^{\mathrm{he}}\right) \leq \mathcal{N}_\lambda\left(\Delta_{\mathcal{G},\kappa\theta}(1 \otimes P_\kappa^{\mathrm{he}})\right) \\ \leq \mathcal{N}_\lambda\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\kappa\theta_2} P_\kappa^{\mathrm{he}}\right),\end{aligned}$$

for all $\lambda \geq 0$. Finally:

$$\frac{\mathcal{N}_\lambda(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\kappa\theta_2} P_\kappa^{\mathrm{he}})}{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} \sim \frac{\sum_{i=2}^n \ln(\lambda) - \ln(\lambda_i)}{n(\ln(\lambda) - \ln(\deg_{\mathcal{G}_2}))} \to \frac{n-1}{n}, \text{ as } \lambda \to \infty.$$

We conclude with (18) for $a = 1$. ■

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Sylvain Golénia
Univ. Bordeaux
Bordeaux INP, CNRS, IMB,
UMR 5251
F-33400 Talence, France

Françoise Truc
Grenoble University
Institut Fourier
Unité mixte de recherche
CNRS-UJF 5582
BP 74, 38402-Saint Martin
d'Hères Cedex France