Basic Operations on Supertropical Quadratic Forms

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Abstract. In the case that a module V over a (commutative) supertropical semiring R is free, the R-module $\mathrm{Quad}(V)$ of all quadratic forms on V is almost never a free module. Nevertheless, $\mathrm{Quad}(V)$ has two free submodules, the module $\mathrm{QL}(V)$ of quasilinear forms with base $\mathfrak{D}_0$ and the module $\mathrm{Rig}(V)$ of rigid forms with base $\mathfrak{H}_0$, such that $\mathrm{Quad}(V) = \mathrm{QL}(V) + \mathrm{Rig}(V)$ and $\mathrm{QL}(V) \cap \mathrm{Rig}(V) = \{0\}$.

In this paper we study endomorphisms of $\mathrm{Quad}(V)$ for which each submodule Rq with $q \in \mathfrak{D}_0 \cup \mathfrak{H}_0$ is invariant; these basic endomorphisms are determined by coefficients in R and do not depend on the base of V. We aim for a description of all basic endomorphisms of $\mathrm{Quad}(V)$, or more generally of its submodules spanned by subsets of $\mathfrak{D}_0 \cup \mathfrak{H}_0$. But, due to complexity issues, this naive goal is highly nontrivial for an arbitrary supertropical semiring R. Our main stress is therefore on results valid under only mild conditions on R, while a complete solution is provided for the case that R is a tangible supersemifield.

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1. Introduction

We continue a study of quadratic forms and modules over semirings, begun in [8] and [10], where now we face a general problem over the so called *supertropical semrings*, as explained in §1.3 and §1.7 below. Exhibiting the contribution of the present paper, our approach is indicated in §1.8. For the reader's convenience we first recall basic terminology and results, mainly from [8] and [10], but also from other sources.

1.1. Modules over a semiring. A (commutative) semiring R is a set R equipped with addition and multiplication such that $(R, +, 0)$ and $(R, \cdot, 1)$ are abelian monoids with natural elements $0 := 0_R$ and $1 := 1_R$ respectively, and multiplication distributes over addition in the standard way. In other words, R satisfies all the properties of a commutative ring except the existence of negation under addition. R is called a *semifield* if every nonzero element of R is invertible, i.e., $R \setminus \{0\}$ is an abelian group.
A *module* V over a semiring R (called also a semimodule) is an abelian monoid $(V, +, 0_V)$ equipped with a scalar multiplication $R \times V \to V$, $(a, v) \mapsto av$, such that all the customary axioms of modules over a ring are satisfied: $a_1(bv) = (a_1b)v$, $(a_1 + a_2)v = a_1v + a_2v$, $a_1(u + v) = a_1u + a_1v$ for all $a_1, a_2, b \in R$, $u, v \in V$. We usually write 0 for both 0_R and 0_V, and 1 for 1_R, and often speak about elements of V as "vectors" and elements of R as "scalars".

1.2. Quadratic forms on a free module. For any module V over a semiring R, a *quadratic form* on V is a function $q : V \to R$ with

$$q(ax) = a^2q(x)$$

for any $a \in R$, $x \in V$, together with a symmetric bilinear form $b : V \times V \to R$ such that

$$q(x + y) = q(x) + q(y) + b(x, y) \tag{1.1}$$

for any $x, y \in V$. Here "symmetric bilinear" has the obvious meaning, but – in contrast to the case where R is a ring – b is often not uniquely determined by q. We call such b a *companion* of q, or say that b *accompanies* q.
In this paper we *assume throughout that V is a free R-module* with base $(\varepsilon_i \mid i \in I)$, i.e., every vector $x \in V$ is a linear combination

$$x = \sum_{i\in I} x_i\varepsilon_i \tag{1.2}$$

with unique family of scalars $(x_i \mid i \in I) \subset R$, only finitely many $x_i \neq 0$, called the *coordinates* of x.
Then, after choosing a companion b of q, a quadratic form $q : V \to R$ can be written as (for notational convenience we choose a total order on I):

$$q(x) = \sum_{i\in I} \alpha_i x_i^2 + \sum_{i<j} \alpha_{ij} x_i x_j, \tag{1.3}$$

where $\alpha_i = q(\varepsilon_i)$ and $\alpha_{ij} = b(\varepsilon_i, \varepsilon_j)$, cf. [8, §1].

Although the case that I is infinite is relevant for applications, we assume in this and the next introductory subsection that $I = \{1, \dots, n\}$ is finite, for simplicity. Then, as customary, the presentation (1.3) of a quadratic from q is written as a triangular scheme

$$q = \begin{bmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ & \alpha_2 & \cdots & \alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix} \tag{1.4}$$

using square brackets. A quadratic form may have presentations by different triangular schemes (cf. [8, §1]). To cope with this difficulty, we use the sign $\cong$ ("equivalent") to indicate such a case.

Note that the entries α_i in (1.4) are uniquely determined by q, since $\alpha_i = q(\varepsilon_i)$. If R is embeddable as subsemiring in a ring R', then also the α_{ij} are uniquely determined by q, since by identifying $R \subset R'$ we have

$$\alpha_{ij} = q(\varepsilon_i + \varepsilon_j) - q(\varepsilon_i) - q(\varepsilon_j).$$

However, this situation is far apart from the semirings in this paper, the so called "*supertropical semirings*", to be described bellow.

1.3. The problem. Assume that $(\varepsilon_i \mid i \in I)$ is a fixed base of a module V. We search for families of scalars

$$(\mu_i \mid 1 \leq i \leq n) \cup (\mu_{ij} \mid 1 \leq i < j \leq n) \tag{1.5}$$

with the following property: For any two equivalent triangular schemes

$$\begin{bmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ & \alpha_2 & \cdots & \alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix} \cong \begin{bmatrix} \alpha_1 & \beta_{12} & \cdots & \beta_{1n} \\ & \alpha_2 & \cdots & \beta_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix}$$

also

$$\begin{bmatrix} \mu_1\alpha_1 & \mu_{12}\alpha_{12} & \cdots & \mu_{1n}\alpha_{1n} \\ & \mu_2\alpha_2 & \cdots & \mu_{2n}\alpha_{2n} \\ & & \ddots & \vdots \\ & & & \mu_n\alpha_n \end{bmatrix} \cong \begin{bmatrix} \mu_1\alpha_1 & \mu_{12}\beta_{12} & \cdots & \mu_{1n}\beta_{1n} \\ & \mu_2\alpha_2 & \cdots & \mu_{2n}\beta_{2n} \\ & & \ddots & \vdots \\ & & & \mu_n\alpha_n \end{bmatrix}.$$

Then the multiplication of the entries of a triangular scheme by the scalars μ_i and μ_{ij} yields a well defined map of the set $\mathrm{Quad}(V)$ of all quadratic forms on V into itself. These maps are the "*basic operations*" on quadratic forms appearing in the title of the paper.

Two quadratic forms q_1, q_2 on V can be added by the rule

$$(q_1 + q_2)(x) = q_1(x) + q_2(x),$$

and a quadratic form q can be multiplied by a scalar $a \in R$ by the rule

$$(aq)(x) = a \cdot q(x).$$

In this way, the set Quad(V) becomes an R-module.
The above presentation (1.3) of a quadratic form q translates to

$$q = \sum_{i\in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{ij}, \tag{1.6}$$

with d_i, h_{ij} defined by

$$d_i(x) = x_i^2, \qquad h_{ij}(x) = x_i x_j, \tag{1.7}$$

where as before the x_i are the coordinates of x, cf (1.2).
We read off from (1.6) that the d_i and h_{ij} generate the R-module Quad(V), which gives us a linear algebraic interpretation of the basic operations as certain endomorphisms of the R-module Quad(V), as follows. An *endomorphism* φ of Quad(V) is called *basic* (w.r. to a given base $(\varepsilon_i \mid i \in I)$ of V) if it maps the submodules Rd_i $(1 \le i \le n)$ and Rh_{ij} $(1 \le i < j \le n)$ to itself, and so

$$\varphi(d_i) = \mu_i d_i, \qquad \varphi(h_{ij}) = \mu_{ij} h_{ij},$$

with scalars μ_i, μ_{ij}, called the *coefficients* of the basic endomorphism φ. It is now immediate that these systems of coefficients are the same families of scalars as occurring for the basic operations (cf. (1.5)), and so basic operations are the same objects as basic endomorphisms in a different disguise.
In general the set of basic endomorphisms of Quad(V) depends on the choice of the base $(\varepsilon_i \mid i \in I)$ of V. But, when R is a supertropical semiring (to be discussed below), the framework of the present paper, it happily turns our that any free module V has only *one base up to scalar multiplication by units* [8, Theorem 0.9], a phenomenon for which we use the catch-phrase "V *has unique base*". Actually, this property holds over a much broader class of semirings than the supertropical ones [11, §1].
If V has unique base, then the set of endomorphisms of Quad(V) is independent of the choice of the base $(\varepsilon_i \mid i \in I)$ of V (also for infinite I). In fact, if $(\varepsilon'_i \mid i \in I)$ is another base, $\varepsilon'_i = u_i \varepsilon_i$, with units u_i of R, then the generators of Quad(V) associated to this base are

$$d'_i = \mu_i^{-2} d_i, \qquad h_{ij}; = \mu_i^{-1}\mu_j^{-1} h_{ij}, \tag{1.8}$$

as easily verified. Thus in the presence of the unique base property, the problem of finding basic endomorphisms of Quad(V) gains extra momentum.

1.4. Supertropical semirings. A semiring R is called *supertropical* ([8, Definition 0.3] and [3, §3]) if $e := 1 + 1$ is an idempotent (i.e., $e = 1 + 1 = 1 + 1 + 1 + 1 = e + e$), and the following axioms hold for all $a, b \in R$:

$$\text{If } ea \neq eb, \text{ then } a + b \in \{a, b\}, \tag{1.9}$$

$$\text{If } ea = eb, \text{ then } a + b = eb. \tag{1.10}$$

Then the ideal eR of R is a semiring with unit element e, which is *bipotent*, i.e., for any $u, v \in eR$ the sum $u + v$ is either u or v. It follows that eR carries a total ordering, compatible with addition and multiplication, which is given by

$$u \le v \quad \Leftrightarrow \quad u + v = v.$$

The addition in a supertropical semiring is determined by the map $a \mapsto ea$ and the total ordering on eR as follows: If $a, b \in R$, then

$$a + b = \begin{cases} b & \text{if } ea < eb, \\ a & \text{if } ea > eb, \\ eb & \text{if } ea = eb. \end{cases} \tag{1.11}$$

In particular (taking $b = 0$ in (1.11) or in (1.10)), for any $a \in R$

$$ea = 0 \quad \Rightarrow \quad a = 0, \tag{1.12}$$

in other terms

$$a + a = 0 \quad \Rightarrow \quad a = 0. \tag{1.12'}$$

Note also that

$$e + 1 = e, \tag{1.13}$$

as follows from (1.11) for $a = e$ and $b = 1$.

For later use we quote another fact, true in any supertropical semiring R:

$$(a + b)^2 = a^2 + b^2 \tag{1.14}$$

for all $a, b \in R$, cf. [8, p.65].

When R is a supertropical semiring, the elements of $\mathcal{T}(R) := R \setminus (eR)$ are called *tangible elements*, and those of $\mathcal{G}(R) := (eR) \setminus \{0\}$ are called *ghost elements*. The zero of R is regarded both as tangible and ghost. The semiring R itself is called *tangible* if R is generated by $\mathcal{T}(R)$ as a semiring. Clearly, this happens iff $e\mathcal{T}(R) = \mathcal{G}(R)$. If $\mathcal{T}(R) \neq \emptyset$, then the set

$$R' := \mathcal{T}(R) \cup e\mathcal{T}(R) \cup \{0\}$$

is the largest subsemiring of R which is tangible supertropical. (We have discarded the "superfluous" ghost elements.) The map

$$\nu_R : R \to eR$$

is a homomorphisms of semirings, which we call the *ghost map* of R. When there is no ambiguity, we write $\mathcal{T}$, $\mathcal{G}$, ν for $\mathcal{T}(R)$, $\mathcal{G}(R)$, ν_R. Sometimes we adhere to the very convenient "*ν-notation*" for $a, b \in R$: $a \leq_\nu b$ means that $ea \leq eb$, $a \cong_\nu b$ ("ν-equivalent") means that $ea = eb$, while $a <_\nu b$ means that $ea < eb$.

We call a supertropical semiring a *supersemifield* if all nonzero tangible elements are invertible in R and all nonzero ghost elements are invertible in the bipotent subsemiring eR, whence both $\mathcal{T}$ and $\mathcal{G}$ are abelian groups under multiplication.

Supertropical semirings have been previously introduced as a tool to refine certain aspects of tropical geometry (e.g. [13]), linear algebra [6, 7], starting with [2], and tropical valuation theory [3]. Up to now supertropical semifields have been prevalent in applications, but more general supertropical semirings are definitely needed for any coherent theory (cf. e.g. [3, 4, 5]). The relevance of quadratic forms over supertropical semirings to classical quadratic forms over rings is explained in [8, §9].

1.5. Partial orderings on R, V, and $\mathrm{Quad}(V)$. Assume that V is *any module over a supertropical semiring* R. Then it is known from more general facts (e.g. [10]), that the binary relation defined by

$$x \leq y \quad \Leftrightarrow \quad \exists z \in V: \ x + z = y \tag{1.15}$$

for any $x, y \in V$ is a partial ordering on V. For the reader's convenience, we provide a direct argument giving this important fact. Reflexivity ($x \leq x$) and transitivity ($x \leq y, y \leq z \Rightarrow x \leq z$) are evident, but antisymmetry is subtler. Given x, y, z, w such that $x + z = y$, $y + w = x$, we need to verify that $x = y$. First we get $x + (z + w) = x$, then $x + e(z + w) = x$. From (1.13) we infer that $ez + z = ez$, and so $y = x + z = x + ez + ew + z = x + ez + ew = x$, as desired. The ordering (1.15) is called the *minimal ordering* of V, since it is the coarsest (partial) ordering on V compatible with addition, such that $0 \leq x$ for all $x \in V$. In particular we have a minimal ordering on R itself. It is immediate that scalar multiplication is compatible with both minimal orderings, i.e., for $a, b \in R$, $x, y \in V$,

$$a \leq b, x \leq y \quad \Rightarrow \quad ax \leq by.$$

Note also that, if V is free with base $(\varepsilon_i \mid i \in I)$, then

$$\sum_{i \in I} x_i \varepsilon_i \leq \sum_{i \in I} y_i \varepsilon_i \quad \Leftrightarrow \quad \forall i \in I: \ x_i \leq y_i.$$

In this paper, the sign "$\leq$" is used for both orderings on R and V. These orderings lead to a "*functional ordering*" on $\mathrm{Quad}(V)$, again denoted by "$\leq$", defined for $q_1, q_2 \in \mathrm{Quad}(V)$ as

$$q_1 \leq q_2 \quad \Leftrightarrow \quad \forall x \in V: \ q_1(x) \leq q_2(x).$$

On the other hand, since $\mathrm{Quad}(V)$ is an R-module, it carries a minimal ordering, denoted here by "$\preceq$". Definition (1.15) now reads as: If q_1, q_2 are quadratic forms on V, then

$$q_1 \preceq q_2 \quad \Leftrightarrow \quad \exists \chi \in \mathrm{Quad}(V): \ q_1 + \chi = q_2.$$

The functional ordering refines the minimal ordering, $q_1 \preceq q_2 \Rightarrow q_1 \leq q_2$. The interplay between these orderings is the major theme in the second half of [10], whose results will be very useful below.

1.6. The submodules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ of $\mathrm{Quad}(V)$. A quadratic form q on a module V over a semring R is called *quasilinear* if the zero bilinear form $b = 0$ is a companion of q, i.e., (cf. (1.1))

$$q(x + y) = q(x) + q(y)$$

for all $x, y \in V$, and q is called *rigid* if q has only one companion. It is obvious that the set $\mathrm{QL}(V)$ of all quasilinear forms on V is an R-submodule of $\mathrm{Quad}(V)$. Assuming that the R-module V is free with base $(\varepsilon_i \mid i \in I)$ and R is supertropical, we have the forms d_i and h_{ij} with $i, j \in I$, $i < j$, cf. (1.7). In consequence

of property (1.14) of R, every d_i is quasilinear. On the other hand a quadratic form q is rigid iff $q(\varepsilon_i) = 0$ for all $i \in I$ [8, Theorem 3.5] [1]. This implies that the set $\mathrm{Rig}(V)$ of all rigid forms on V is a submodule of $\mathrm{Quad}(V)$ and that all forms h_{ij} $(i < j)$ are rigid.
Having this starting point, it is an easy matter to verify that both $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ are free modules with bases

$$\mathfrak{D}_0 := \{d_i \mid i \in I\}, \qquad \mathfrak{H}_0 := \big\{h_{ij} \mid i > j\big\}$$

respectively [10, Proposition 7.2]. For any $\kappa \in \mathrm{QL}(V)$ and $\rho \in \mathrm{Rig}(V)$ we have (as a special cases of (1.3)) the presentations

$$\kappa = \sum_{i \in I} \kappa(\varepsilon_i) d_i. \tag{1.16}$$

$$\rho = \sum_{i<j} b(\varepsilon_i, \varepsilon_j) h_{ij} = \sum_{i<j} \rho(\varepsilon_i + \varepsilon_j) h_{ij},$$

where b is the unique companion of ρ [8, §4]. From these presentations it follows that the functional ordering of $\mathrm{Quad}(V)$ restricts on both $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ to the minimal ordering on these free modules [10, Proposition 7.3].
Every $q \in \mathrm{Quad}(V)$ has a decomposition

$$q = q_{\mathrm{QL}} + \rho \tag{1.17}$$

with $q_{\mathrm{QL}} \in \mathrm{QL}(V)$ and $\rho \in \mathrm{Rig}(V)$, as it is now evident from (1.3). Moreover, for any decomposition (1.17) clearly $q(\varepsilon_i) = q_{\mathrm{QL}}(\varepsilon_i)$, and so we infer from (1.16) that

$$q_{\mathrm{QL}} = \sum_{i \in I} q(\varepsilon_i) d_i,$$

which proves that q_{QL} is uniquely determined by q. We call q_{QL} the *quasilinear part* of q and ρ in (1.17) a *rigid complement* of q_{QL} in q. Most often ρ is not unique [8, §6 and §7].
If I is finite and a triangular scheme for q is given (cf. (1.4)), then q_{QL} is represented by the diagonal part of the scheme, while the upper triangular part gives a rigid complement of q_{QL} in q. We have

$$\mathrm{Quad}(V) = \mathrm{QL}(V) + \mathrm{Rig}(V),$$

$$\mathrm{QL}(V) \cap \mathrm{Rig}(V) = \{0\},$$

but nevertheless $\mathrm{Quad}(V)$ is not a direct sum of the submodules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$, as soon as $\mathrm{Rig}(V) \neq \{0\}$, i.e., $|I| > 1$. Indeed, then different indices i, j give us a relation

$$d_i + d_j = d_i + d_j + h_{ij},$$

since, in consequence of (1.14), $a^2 + b^2 = a^2 + b^2 + ab$ for any $a, b \in R$.

[1] More generally, this remains true if the semiring R has the properties (1.12') and (1.14) [loc. cit].

1.7. A refinement of the problem in §1.3 for R supertropical. As before we assume that the semiring R is supertropical and V is free with base $(\varepsilon_i \mid i \in I)$. We then have the set of generators $\mathfrak{B}_0 = \mathfrak{D}_0 \cup \mathfrak{H}_0$ of $\mathrm{Quad}(V)$ with

$$\mathfrak{D}_0 := \{d_i \mid i \in I\} \quad \text{and} \quad \mathfrak{H}_0 := \big\{h_{ij} \mid i < j\big\},$$

which up to multiplication by scalars does not depend on the choice of the base $(\varepsilon_i \mid i \in I)$, cf. (1.8). We call a submodule Z of $\mathrm{Quad}(V)$ *basic*, if Z is generated by a subset $\mathfrak{B}_0'$ of $\mathfrak{B}_0$, which then is a union $\mathfrak{D}_0' \cup \mathfrak{H}_0'$ with $\mathfrak{D}_0' \subset \mathfrak{D}_0$, $\mathfrak{H}_0' \subset \mathfrak{H}_0$. In this case necessarily $\mathfrak{D}_0' = \mathfrak{D}_0 \cap Z$, $\mathfrak{H}_0' = \mathfrak{H}_0 \cap Z$ and so $\mathfrak{B}_0' = \mathfrak{B}_0 \cap Z$. For these intersections we write $\mathfrak{D}_0(Z), \mathfrak{H}_0(Z)$, $\mathfrak{B}_0(Z)$ respectively.

Instead of the basic endomorphisms of $\mathrm{Quad}(V)$ addressed in §1.3, in the present paper we search, more generally, for endomorphisms of a fixed basic submodule Z of $\mathrm{Quad}(V)$ that map each submodule Rq, $q \in \mathfrak{B}_0(Z)$ into itself. Such map φ is called a *basic endomorphism of Z*.

Given a fixed base $(\varepsilon_i \mid i \in I)$ of V, we denote by $I^{[2]}$ the set of all 2-element subsets of I, and write

$$\mathfrak{D}_0(Z) = \{d_i \mid i \in K\}, \qquad \mathfrak{H}_0(Z) = \big\{h_{ij} \mid \{i,j\} \in M\big\}, \tag{1.18}$$

where $K \subset I$, $M \subset I^{[2]}$. Then a basic endomorphisms φ of Z is determined by a family of scalars

$$(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M), \tag{1.19}$$

which we again call the *coefficients* of φ, via the formulas $(i \in K, \{i,j\} \in M)$

$$\varphi(d_i) = \mu_i d_i, \qquad \varphi(h_{ij}) = \mu_{ij} h_{ij}.$$

In the case that $I = \{1, \dots, n\}$ is finite, we may use triangular schemes to present quadratic forms. Now the refined problem means that we focus on quadratic forms which are represented by a scheme as in (1.4), with zero entries at fixed places (i,j), $i \leq j$, namely at (i,i), with $i \notin K$ and $\{i,j\} \notin M$.

The task is to find all systems of scalars $(\mu_i) \cup (\mu_{ij})$ such that any two equivalent schemes of this type remain equivalent after multiplication of entries by the scalars μ_i and μ_{ij} respectively. So this is indeed a very natural expansion of the problem described at the beginning of §1.3.

In what follows we call a basic submodule of $\mathrm{Quad}(V)$ simply a *basic module*. A basic endomorphisms φ of a given basic module is called a *basic projector* on Z, if its coefficients are all 1 or 0, and thus $\varphi(z) = z$ or 0 for any $z \in \mathfrak{B}_0(Z)$. Then $X = \varphi(Z)$ is a basic module with $X \subset Z$ and φ is uniquely determined by X. We call these submodules X of Z the *basic projections* of Z.

For example, $\mathrm{QL}(V)$ is a basic projection of $\mathrm{Quad}(V)$ whose associated basic projector is the endomorphism π_{QL} of $\mathrm{Quad}(V)$ which maps any $q \in \mathrm{Quad}(V)$ to its quasilinear part q_{QL}. But $\mathrm{Rig}(V)$ is not a basic projection of $\mathrm{Quad}(V)$ whenever $\mathrm{Rig}(V) \neq \{0\}$, i.e., $|I| > 1$. Indeed, the existence of an endomorphism of $\mathrm{Quad}(V)$ with $\varphi(d_i) = 0$ for all $i \in I$ and $\varphi(h_{ij}) = h_{ij}$ for $i \neq j$ is prevented by the relations $d_i + d_j = d_i + d_j + h_{ij}$.

1.8. Paper outline and main results. §3 and §4 are devoted to a study of basic projectors to obtain a classification of all basic projections of a basic module Z in combinatorial terms under the mild assumption that eR is "multiplicatively unbounded", i.e., for any $x, y \in \mathcal{G}$ there exists some $z \in \mathcal{G}$ such that $y < xz$ (cf. Corollary 3.6 and Theorem 3.12). In particular it turns out (without the assumption of multiplicatively unboundedness) that *any basic module $X \subset Z$ with $\mathfrak{D}_0(X) = \mathfrak{D}_0(Z)$ is a basic projection of Z*. The associated projectors are constructed in §3 for $Z = \mathrm{Quad}(V)$ under the name of *partial quasilinearizations*. For any subset Λ of the set $I^{[2]}$ of 2-element subsets of I we have a basic projector

$$\pi_{\Lambda,\mathrm{QL}} : q \longmapsto q_{\Lambda,\mathrm{QL}}$$

on $\mathrm{Quad}(V)$ with $\pi_{\Lambda,\mathrm{QL}}(d_i) = d_i$ for $i \in I$, $\pi_{\Lambda,\mathrm{QL}}(h_{ij}) = h_{ij}$ for $\{i,j\} \in \Lambda$, and $\pi_{\Lambda,\mathrm{QL}}(h_{ij}) = 0$ otherwise. This projector then restricts to a basic projector on Z for any basic module $Z \subset \mathrm{Quad}(V)$. Its image is a basic module $X \subset Z$ with

$$\mathfrak{D}_0(X) = \mathfrak{D}_0(Z), \qquad \mathfrak{H}_0(X) = \{h_{ij} \in \mathfrak{H}_0(Z) \mid \{i,j\} \in \Lambda \cap M\}$$

in Notation (1.18).

A basic module Z is a direct sum $X \oplus Y$ of basic modules X and Y iff $\mathfrak{B}_0(Z)$ is a disjoint union of $\mathfrak{B}_0(X)$ and $\mathfrak{B}_0(Y)$ and both X and Y are basic projections of Z (Proposition 4.2). Thus it is not surprising that the classification of the basic projections of Z in §3 in combinatorial terms leads to a description of all (possibly infinite) direct decompositions of Z, again in a combinatorial way. In particular we learn in §4 that Z has (up to permutation of summands) only one decomposition $Z = \bigoplus_{\alpha\in A} Z_\alpha$, such that all Z_α are indecomposable basic modules, and these *components* Z_α of Z can be described combinatorially.

In the important special case that for every $h_{ij} \in \mathfrak{H}_0(Z)$ both d_i and h_{ij} are in Z, and so are elements of $\mathfrak{D}_0(Z)$, this description can be given in terms of graphs. We associate to Z a graph $\Gamma(Z)$ whose sets of vertices and edges are $\mathfrak{D}_0(Z)$ and $\mathfrak{H}_0(Z)$ respectively, an edge h_{ij} connecting the vertices d_i and d_j, and we call the module Z *graphic*. It turns out that the components Z_α of Z are again graphic and the graphs $\Gamma(Z_\alpha)$ are precisely all path components of $\Gamma(Z)$ (Theorem 4.18). Starting from this, we also obtain, under a mild restriction of the supertropical semring R, a description of all components of Z when Z is not graphic (Corollary 4.19).

In the last three sections §5–§8 we work on more general basic endomorphisms than basic projectors. The main result in §5 is that, under still mild conditions on R (in particular if the semiring eR is cancellative), every basic endomorphism φ of a basic module Z yields a basic projector p_φ on Z by the rule $p_\varphi(z) = z$ if $\varphi(z) \neq 0$ and $p_\varphi(z) = 0$ otherwise, for $z \in \mathfrak{B}_0(Z)$. Conversely, given a basic projector π on Z we can describe all basic endomorphisms φ of Z with $p_\varphi = \pi$, called the *satellites* of π (Theorem 5.17 and Corollary 5.18).

In §6 we develop other ways to obtain new basic endomorphisms from old ones. Given scalars $\mu, \upsilon \in R$, we say that υ is *obedient* to μ, if $\upsilon \leq_\nu \mu$ and υ is faithful to μ in the following sense: for all $x, y \in R$, if $\mu x = \mu y$, then $\upsilon x = \upsilon y$. Assume, for simplicity, that Z is graphic and φ is a basic endomorphism of Z with system of coefficients $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$, cf. (1.19). Then it turns out that every tuple of scalars $(\mu_i \mid i \in K) \cup (\upsilon_{ij} \mid \{i,j\} \in M)$, with υ_{ij} obedient to μ_{ij} for all i, j, is again the coeffient system of some basic endomorphism ψ of Z (Theorem 6.9). We call such a basic endomorphism ψ an *$\mathcal{H}$-modification* of φ. To give the flavor we point out what this theorem means in the case that $Z = \mathrm{Quad}(V)$, $\varphi = \mathrm{id}_Z$, $I = \{1, \dots, n\}$. It says that for any family of scalars $(\lambda_{ij} \mid 1 \le i < j \le n)$ with $\lambda_{ij} \le_\nu 1$ for all i, j the assignment

$$\begin{bmatrix} \alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\ & \alpha_2 & \cdots & \alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix} \longmapsto \begin{bmatrix} \alpha_1 & \lambda_{12}\alpha_{12} & \cdots & \lambda_{1n}\alpha_{1n} \\ & \alpha_2 & \cdots & \lambda_{2n}\alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \end{bmatrix}$$

is a well defined basic operation on $\mathrm{Quad}(V)$. The reason is that every λ_{ij} is clearly obedient to 1.

Starting again with the system of coefficients $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$ of φ we call a basic endomorphism ψ of Z a *$\mathcal{D}$-modification* of φ if for the coefficients $(\upsilon_i \mid i \in K) \cup (\upsilon_{ij} \mid \{i,j\} \in M)$ of ψ we have $\upsilon_{ij} = \mu_{ij}$ for all $\{i,j\} \in M$ and $\mu_i \le_\nu \upsilon_i$ for all $i \in K$. While for $\mathcal{H}$-modifications we obtained a best possible result, here our knowledge is less complete. We only know that a tuple $(\upsilon_i) \cup (\mu_{ij})$ is the coefficient system of a basic endomorphism ψ if $\mu_i \le \upsilon_i$ for all $i \in K$ (minimal ordering $\le$ instead of ν-dominance $\le_\nu$), cf. Theorem 6.9.

In the last section §8 we determine, for a tangible supersemifield, R, all basic endomorphisms of any basic module Z. If Z is "linked", i.e., Z is graphic and $\Gamma(Z)$ has no isolated vertices (the main case to be studied), it turns out that the possible coefficient systems $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$, cf. (1.19) are given by the condition

$$\mu_{ij}^2 \le_\nu \mu_i \mu_j,$$

provided that the ghost map $\nu_R : \mathcal{T}(R) \twoheadrightarrow \mathcal{G}(R)$ is *not* bijective. Otherwise there may exist more basic endomorphisms, cf. Theorem 8.5.

2. Partial quasilinearisation

Henceforth, R is a supertropical semiring and V is a free R-module with base $(\varepsilon_i \mid i \in I)$. Let $I^{[2]}$ denote the set of 2-element subsets of I. We choose a total ordering of I and often identify $I^{[2]}$ with the set of pairs $(i,j) \in I \times I$ such that $i < j$. If Λ is a subset of $I^{[2]}$, let Λ^{c} denote the complement $I^{[2]} \setminus \Lambda$.

We define a quasilinear quadratic form d_i on V for every $i \in I$ by

$$d_i(x) = x_i^2 \qquad (2.1)$$

and a rigid quadratic form h_{ij} for every $i, j \in I$ with $i \neq j$ by

$$h_{ij}(x) = x_i x_j. \qquad (2.2)$$

Here, as always, the x_i are the coordinates of the vector $x \in V$, $x = \sum_{i \in I} x_i \varepsilon_i$. We work with the bases $(d_i \mid i \in I)$ and $(h_{ij} \mid (i,j) \in I^{[2]})$ of the free R-modules $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$, respectively.

For any set $\Lambda \subset I^{[2]}$ we introduce the free submodule

$$\mathrm{Rig}(\Lambda, V) := \sum_{\{i,j\} \in \Lambda} R h_{ij}.$$

$\mathrm{Rig}(\Lambda, V)$ is a lower set in the R-module $\mathrm{Quad}(V)$ both in the minimal and the functional ordering of $\mathrm{Quad}(V)$. Clearly

$$\mathrm{Rig}(V) = \mathrm{Rig}(\Lambda, V) \oplus \mathrm{Rig}(\Lambda^{\mathrm{c}}, V). \tag{2.3}$$

In other words, for any rigid form ρ on V we have a unique decomposition $\rho = \rho_1 + \rho_2$ with $\rho_1 \in \mathrm{Rig}(\Lambda, V)$ and $\rho_2 \in \mathrm{Rig}(\Lambda^{\mathrm{c}}, V)$. We call ρ_1 and ρ_2 the Λ-*component* and Λ^{c}-*component* of ρ, respectively.

Definition 2.1. *We call a form $q \in \mathrm{Quad}(V)$* Λ-quasilinear, *if q is quasilinear on the submodule $R\varepsilon_i + R\varepsilon_j$ for every $\{i,j\} \in \Lambda$.*

Λ-quasilinearity of q means that for every $\{i,j\} \in \Lambda$ the set $C_{ij}(q)$ in the companion table of q (cf. [8, §6]) contains zero. Notice that a priori every set $C_{ii}(q)$ contains zero, cf. [8, Example 2.4].

Proposition 2.2. *q is Λ-quasilinear iff the set $\mathrm{Rig}(q)$ of rigid complements of q_{QL} in q contains some $\rho \in \mathrm{Rig}(\Lambda^{\mathrm{c}}, V)$.*

Proof. This is a consequence of the 1-1-correspondence between the off-diagonal companions of q and the rigid complements of q_{QL} in q described in [8, Proposition 4.6]. □

Given $\Lambda \subset I^{[2]}$ we intend to associate to any $q \in \mathrm{Quad}(V)$ a Λ-quasilinear form $q_{\Lambda,\mathrm{QL}} \in \mathrm{Quad}(V)$ in a somewhat canonical way, generalizing the map $q \mapsto q_{\mathrm{QL}}$ from $\mathrm{Quad}(V)$ to $\mathrm{QL}(V)$. The key to do this is provided by the following lemma.

Lemma 2.3. *Let $\Lambda \subset I^{[2]}$ and $q_0 \in \mathrm{QL}(V)$. Further let $\rho, \rho' \in \mathrm{Rig}(V)$ be given, and let ρ_1, ρ'_1 denote the Λ-components of ρ, ρ', respectively.*

(a) *If $q_0 + \rho \leq q_0 + \rho'$, then $q_0 + \rho_1 \leq q_0 + \rho'_1$.*

(b) *If $q_0 + \rho \preceq q_0 + \rho'$, then $q_0 + \rho_1 \preceq q_0 + \rho'_1$.*

To prove the lemma we use part of the following notation, that shall also be helpful later.

Notation 2.4. *Let J be a subset of the index set I.*

(a) *$V_J := \sum\limits_{i \in J} R\varepsilon_i$ is a free submodule of V. It comes with a natural R-linear projection $\pi_J : V \to V_J$, given by $\pi_J(\varepsilon_i) = \varepsilon_i$ for $i \in J$, $\pi_J(\varepsilon_i) = 0$ for $i \in I \setminus J$. We also have the inclusion mapping $i_J : V_J \hookrightarrow V$, with $i_J(\varepsilon_i) = \varepsilon_i$ for every $i \in J$.*

(b) *Any form* $\vartheta \in \mathrm{Quad}(V_J)$ *gives us a form*

$$\vartheta^I := \vartheta \circ \pi_J \in \mathrm{Quad}(V).$$

(c) *Given* $q \in \mathrm{Quad}(V)$, *we define*

$$q_J := (q|V_J)^I = q \circ i_J \circ \pi_J \in \mathrm{Quad}(V).$$

Proof of Lemma 2.3. We write $q_0 = \sum\limits_{i\in I} \alpha_i d_i,\ \rho = \sum\limits_{i<j} \alpha_{ij} h_{ij},\ \rho' = \sum\limits_{i<j} \beta_{ij} h_{ij}$ with $\alpha_i, \alpha_{ij}, \beta_{ij} \in\ R$.

(a): We have

$$\sum_{i\in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{ij} \leq \sum_{i\in I} \alpha_i d_i + \sum_{i<j} \beta_{ij} h_{ij}$$

From this we get for any $i<j$ in I, $(q_0+\rho)_{\{i,j\}} \leq (q_0+\rho')_{\{i,j\}}$, which reads

$$(*) \qquad \alpha_i d_i + \alpha_j d_j + \alpha_{ij} h_{ij} \leq \alpha_i d_i + \alpha_j d_j + \beta_{ij} h_{ij}.$$

Using $(*)$ for every $\{i,j\} \in \Lambda$, we obtain

$$\sum_{i\in I} \alpha_i d_i + \sum_{\{i,j\}\in\Lambda} \alpha_{ij} h_{ij} \leq \sum_{i\in I} \alpha_i d_i + \sum_{\{i,j\}\in\Lambda} \beta_{ij} h_{ij}.$$

This means that $q_0 + \rho_1 \leq q_0 + \rho_1'$.

(b): Same argument, employing $\preceq$ instead of $\leq$. $\square$

Corollary 2.5. *Assume that* $q \in \mathrm{Quad}(V)$ *and* $\rho, \rho' \in \mathrm{Rig}(q)$. *Given* $\Lambda \subset I^{[2]}$, *let now* ρ_2, ρ_2' *denote the* Λ^c*-components of* ρ, ρ'. *Then* $q_{\mathrm{QL}} + \rho_2 = q_{\mathrm{QL}} + \rho_2'$.

Proof. This follows by applying part a) of the lemma to the inequalities $q_{\mathrm{QL}} + \ \rho \leq q_{\mathrm{QL}} + \rho'$ and $q_{\mathrm{QL}} + \rho' \leq q_{\mathrm{QL}} + \rho$, using Λ^c instead of Λ. (We could, equally well, use Lemma 2.3.(b).) $\square$

Definition 2.6. *Given* $q \in \mathrm{Quad}(V)$ *and* $\Lambda \subset I^{[2]}$, *we choose a rigid complement* ρ *of* q_{QL} *in* q *and define*

$$(2.4) \qquad q_{\Lambda,\mathrm{QL}} := q_{\mathrm{QL}} + \rho_2,$$

with ρ_2 *the* Λ^c*-component of* ρ. *Evidently, this form is* Λ*-quasilinear. We call* $q_{\Lambda,\mathrm{QL}}$ *the* Λ-quasilinearisation *of* q.

Corollary 2.5 tells us that $q_{\Lambda,\mathrm{QL}}$ does not depend on the choice of the rigid complement ρ in q. If $\Lambda = I^{[2]}$, then $q_{\Lambda,\mathrm{QL}} = q_{\mathrm{QL}}$, while if $\Lambda = \emptyset$, then $q_{\Lambda,\mathrm{QL}} = q$.

Scholium 2.7. *Let* $I = \{1,2,\dots,n\}$. *We describe a given quadratic form* $q:\ V \to R$ *by a triangular scheme*

$$q = \begin{bmatrix} \alpha_1 \alpha_{12} & \dots & \alpha_{1n} \\ & \ddots & \vdots \\ & & \alpha_n \end{bmatrix},$$

cf. §*1.2. The quadratic form* $q_{\Lambda,\mathrm{QL}}$ *is then given by the triangular scheme, where every entry* α_{ij} *with* $\{i,j\} \in \Lambda$ *is replaced by zero.*

Remark 2.8.

(i) *If* $q_1, q_2 \in \mathrm{Quad}(V)$, *then*

$$(q_1 + q_2)_{\Lambda,\mathrm{QL}} = (q_1)_{\Lambda,\mathrm{QL}} + (q_2)_{\Lambda,\mathrm{QL}}.$$

(ii) *If* $q \in \mathrm{Quad}(V)$ *and* $\lambda \in R$, *then*

$$(\lambda q)_{\Lambda,\mathrm{QL}} = \lambda \cdot q_{\Lambda,\mathrm{QL}}.$$

Let $\mathrm{QL}(\Lambda, V)$ denote the R-submodule of $\mathrm{Quad}(V)$ consisting of all Λ-quasilinear forms on V; in other terms

$$\mathrm{QL}(\Lambda, V) = \mathrm{QL}(V) + \mathrm{Rig}(\Lambda^{\mathrm{c}}, V). \tag{2.5}$$

We have a natural map

$$\pi_{\Lambda,\mathrm{QL}} : \mathrm{Quad}(V) \to \mathrm{QL}(\Lambda, V), \tag{2.6}$$

sending $q \in \mathrm{Quad}(V)$ to its Λ-quasilinearization $q_{\Lambda,\mathrm{QL}}$. It is R-linear by Remark 2.8. Since $\pi_{\Lambda,\mathrm{QL}}$ is additive, it is also plain that $\pi_{\Lambda,\mathrm{QL}}$ respects the minimal ordering on $\mathrm{Quad}(V)$, i.e.,

$$q \preceq q' \quad \Rightarrow \quad q_{\Lambda,\mathrm{QL}} \preceq q'_{\Lambda,\mathrm{QL}}. \tag{2.7}$$

Viewing every map $\pi_{\Lambda,\mathrm{QL}}$ as an endomorphism of the R-module $\mathrm{Quad}(V)$, we may state that

$$\pi_{M,\mathrm{QL}} \circ \pi_{\Lambda,\mathrm{QL}} = \pi_{M,\mathrm{QL}},$$

if $M \subset \Lambda \subset I^{(2)}$. In particular ($\Lambda = M$), $\pi_{\Lambda,\mathrm{QL}} \in \mathrm{End}_R(\mathrm{Quad}(V))$ is a projector. It can be characterized in terms of the minimal ordering of $\mathrm{Quad}(V)$ as follows.

Proposition 2.9. *For any* $q \in \mathrm{Quad}(V)$ *the form* $q_{\Lambda,\mathrm{QL}}$ *is the unique maximal form* $\kappa \preceq q$, *which is* Λ*-quasilinear.*

Proof. If κ is Λ-quasilinear and $\kappa \preceq q$, we conclude by (2.7) that $\kappa = \kappa_{\Lambda,\mathrm{QL}} \preceq q_{\Lambda,\mathrm{QL}}$. □

Problem 2.10. *For which supertropical semirings* R, *sets* $\Lambda \subset I^{[2]}$, *and quadratic forms* q' *on* $R^{(I)}$, *is it true that* $q \leq q'$ *implies* $q_{\Lambda,\mathrm{QL}} \leq q'_{\Lambda,\mathrm{QL}}$*?*

In addition to the cases where we know that $q \leq q'$ means the same as $q \preceq q'$ (cf. [10, Corollaries 9.11 and 9.12]), there is one case where we can give an answer now, for any supertropical semiring R.

Proposition 2.11. *Let* $\Lambda = \{\{k, \ell\} \mid k \in J, \ell \in I \setminus J\}$ *for some subset* J *of* I.

(a) *If* $\vartheta \in \mathrm{Quad}(V)$, *then* ϑ *is* Λ*-quasilinear iff* ϑ *is quasilinear on* $V_J \times V_{I \setminus J}$.

(b) *If* $\vartheta, q \in \mathrm{Quad}(V)$ *and* $\vartheta \leq q$, *then* $\vartheta_{\Lambda,\mathrm{QL}} \leq q_{\Lambda,\mathrm{QL}}$. *Moreover,* $q_{\Lambda,\mathrm{QL}}$ *is the unique maximal* Λ*-quasilinear form* κ *on* V *(in the functional ordering of* $\mathrm{Quad}(V)$*) with* $\kappa \leq q$.

Proof. (a): If ϑ is Λ-quasilinear, ϑ has a companion b with $b(\varepsilon_k, \varepsilon_\ell) = 0$ for $k \in J$, $\ell \in I \setminus J$. It follows that $b(V_J \times V_{I\setminus J}) = 0$, and so ϑ is quasilinear on $V_J \times V_{I\setminus J}$. The converse is trivial.

(b): Let

$$q = \sum_{i\in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{ij}.$$

Then, using Notation 2.4, we may write

$$\begin{aligned} q_{\Lambda,\mathrm{QL}} &= \Big(\sum_{i\in J} \alpha_i d_i + \sum_{\substack{i<j \\ i,j\in J}} \alpha_{ij} h_{ij} \Big) + \Big(\sum_{i\notin J} \alpha_i d_i + \sum_{\substack{i<j \\ i,j\notin J}} \alpha_{ij} h_{ij} \Big) \\ &= (q \mid V_J)^I + (q \mid V_{I\setminus J})^I \\ &= q_J + q_{I\setminus J}. \end{aligned}$$

We have $V = V_J \oplus V_{I\setminus J}$ and obtain for any $x \in V_J$, $y \in V_{I\setminus J}$ the formula

$$q_{\Lambda,\mathrm{QL}}(x+y) = q(x) + q(y).$$

(N.B. This proves again that q is quasilinear on $V_J \times V_{I\setminus J}$.)
If κ is Λ-quasilinear and $\kappa \le q$, then κ is quasilinear on $V_J \times V_{I\setminus J}$ (as just proven again), and $\kappa(x) \le q(x)$, $\kappa(y) \le q(y)$ for $x \in V_J$, $y \in V_{I\setminus J}$, and hence

$$\kappa(x+y) = \kappa(x) + \kappa(y) \le q(x) + q(y) = q_{\Lambda,\mathrm{QL}}(x+y).$$

□

3. Basic modules and basic projections

We repeat that in the whole paper V is a free module over a supertropical semiring R and $\{\varepsilon_i \mid i \in I\}$ is a base (mostly fixed) of V. Let $I^{[2]}$ denote the set of all 2-element subsets of I. The R-module $\mathrm{Quad}(V)$ has the set of generators $\mathfrak{B}_0 := \mathfrak{D}_0 \dot\cup \mathfrak{H}_0$ with

$$\mathfrak{D}_0 := \{d_i \mid i \in I\} \quad \text{and} \quad \mathfrak{H}_0 := \big\{h_{ij} \mid \{i,j\} \in I^{[2]}\big\},$$

where $d_(x) = x_i^2$ and $h_{ij} = x_i x_j$ for $x = \sum_{i\in I} x_i \varepsilon_i$, as said above. Assume now *that eR is multiplicatively unbounded*, i.e., that for any $x, y \in \mathcal{G}$ there exists $z \in \mathcal{G}$ such that $y < xz$ (cf. [10, Definition 6.4]).[2] Then, as proved in [10, §7], the set of generators $\mathfrak{B}_0$ of $\mathrm{Quad}(V)$ is uniquely determined by the R-module $\mathrm{Quad}(V)$ up to multiplication by units of R. More precisely, the set $\mathfrak{B} := R^*\mathfrak{B}_0$, consisting of all products λq with $\lambda \in R^*$, $q \in \mathfrak{B}_0$, coincides with the set of all "basic elements" of $\mathrm{Quad}(V)$ (cf. [10, Definition 6.1]) and also with the set of all "primitive" (loc. cit.) indecomposable elements of $\mathrm{Quad}(V)$ [10, Theorem 7.8, Corollary 7.9]. Moreover, as has been shown in [10, §8], each of the sets

$$\mathfrak{D} := R^*\mathfrak{D}_0, \qquad \mathfrak{H} := R^*\mathfrak{H}_0,$$

[2] This property implies that the set $R\setminus\{0\}$ is closed under multiplication [10, Remark 6.5], i.e., R has no zero divisors.

is uniquely determined by the R-module Quad(V), up to multiplication by units. It now makes sense to extend the notation $\mathfrak{B}_0$, $\mathfrak{D}_0$, $\mathfrak{H}_0$ as follows, and to define "basic submodules" of Quad(V) without referring to a base of R.

Definition 3.1. *Choosing sets of representatives $\mathfrak{D}_0$ and $\mathfrak{H}_0$ of the orbit sets $\mathfrak{D}/R^*$ and $\mathfrak{H}/R^*$, we obtain a set $\mathfrak{B}_0 := \mathfrak{D}_0 \dot{\cup} \mathfrak{H}_0$ which obviously generates the R-module* Quad(V). *We call $\mathfrak{B}_0$ a* basic set of generators *of* Quad(V). [3] *Usually we choose the sets*

$$\mathfrak{D}_0 := \{d_i \mid i \in I\} \quad \textit{and} \quad \mathfrak{H}_0 := \left\{h_{ij} \mid \{i,j\} \in I^{[2]}\right\} \tag{3.1}$$

derived from a base $\{\varepsilon_i \mid i \in I\}$ of the free R-module V, cf. (2.1), (2.2). *Then we call $\mathfrak{B}_0 := \mathfrak{D}_0 \cup \mathfrak{H}_0$ a* geometric basic set of generators *of* Quad(V). [4]

Definition 3.2.

(a) *We call an R-submodue Z of* Quad(V) basic *if Z is spanned by a subset S of $\mathfrak{B}_0$, i.e., every $q \in Z$ has a presentation $q = \sum_{s \in S} \alpha_s s$ with $s \in S$, $\alpha_s \in R$, almost all $\alpha_s = 0$.*

(b) *If this holds, then S is uniquely determined by Z, namely $S = Z \cap \mathfrak{B}_0$, as follows immediately from the fact that every $s \in S$ is primitive and indecomposable. We call S a* basic set of generators of Z, *and also write $Z = \sum RS$ (while RS just means the set of all products RS with $\lambda \in R$, $s \in S$).*

If Z_1 and Z_2 are basic submodules of Quad(V), then the modules $Z_1 + Z_2$ and $Z_1 \cap Z_2$ are again basic in Quad(V). We have

$$(Z_1 + Z_2) \cap \mathfrak{B}_0 = (Z_1 \cap \mathfrak{B}_0) \cup (Z_2 \cap \mathfrak{B}_0)$$

and, of course,

$$(Z_1 \cap Z_2) \cap \mathfrak{B}_0 = (Z_1 \cap \mathfrak{B}_0) \cap (Z_2 \cap \mathfrak{B}_0),$$

whence

$$Z_1 \subset Z_2 \quad \Leftrightarrow \quad Z_1 \cap \mathfrak{B}_0 \subset Z_2 \cap \mathfrak{B}_0.$$

Caution: If $S_1 \cap S_2 = \emptyset$, $Z_1 = \sum RS_1$, $Z_2 = \sum RS_2$, then $Z_1 \cap Z_2 = \{0\}$, but $Z_1 + Z_2$ is not necessarily the direct sum of the modules Z_1 and Z_2.

We already met preeminent basic submodules of Quad(V). Both QL(V) and Rig(V) are basic in Quad(V). If Λ is any subset of the set $I^{[2]}$ of two element subsets of I, then the submodule QL(Λ, V) consisting of the Λ-quasilinear forms on V (cf. Definition 2.1) is basic. Also the submodule Rig(Λ, V) of Rig(V) (cf. (2.3)) is basic in Quad(V).
We are ready for the key definitions of this section.

Definition 3.3. *Assume that Z is a basic submodule of* Quad(V).

[3] Recall that the R-module Quad(V) is not free [10, Proposition 7.10].

[4] Usually not every basic set of generators of Quad(V) is geometric. In the present paper we do not exploit this phenomenon thoroughly.

(a) *We call an endomorphism π of the R-module Z a* basic projector on Z *if π maps every $q \in \mathfrak{B}_0 \cap Z$ either to itself or to zero. Clearly, then $\pi = \pi^2$ and both $X := \pi(Z)$, $Y = \pi^{-1}(0)$ are basic submodules of* $\mathrm{Quad}(V)$ *with $Z = X + Y$, $X \cap Y = \{0\}$. Indeed X is generated by the set $\{q \in \mathfrak{B}_0 \cap Z \mid \pi(q) = q\}$, while Y is generated by $\{q \in \mathfrak{B}_0 \cap Z \mid \pi(q) = 0\}$.*

(b) *We call X a* basic projection *of the R-module Z and Y a* basic projection kernel in Z, *and we also call (Z, X) a* basic projection pair.

(c) *Whenever it is convenient, we identify the basic projector $\pi : Z \to Z$ with the associated projection map $Z \twoheadrightarrow X$.*

(d) *Notice that the projector $\pi : Z \to Z$ is uniquely determined both by the pair (Z, X) and the pair (Z, Y). We write $\pi = \pi_{Z,X}$. We call $Y = \pi^{-1}(0)$ the* kernel *of the projector π, and usually write $Y = \ker(\pi)$. If $q \in X$ then obviously*
$$\pi^{-1}(q) = q + \ker(\pi).$$

(e) *For the sake of brevity we often call a basic submodule Z of* $\mathrm{Quad}(V)$ *simply a* "basic module"*, suppressing the reference to the free R-module V, as long as V is kept fixed.*

The primordial example of a basic projection is provided by $Z = \mathrm{Quad}(V)$, $X = \mathrm{QL}(V)$, $Y = \mathrm{Rig}(V)$. It gives us the quasilinear-rigid decompositions of any quadratic form q on V treated in [8]. More generally the Λ-quasilinearizations of q, defined in §2, are provided by $Z = \mathrm{Quad}(V)$, $X = \mathrm{QL}(\Lambda, V)$, $Y = \mathrm{Rig}(\Lambda, V)$, cf. (2.4), (2.5). [5]

Remark 3.4. *Assume that Z is a basic R-module.*

(a) *If X is a basic projection of Z, then for any basic module W the intersection $X \cap W$ is a basic projection of $Z \cap W$, and $\pi_{Z\cap W, X\cap W}$ is the restriction of $\pi_{Z,X}$ to W. In particular, if $W \subset Z$, then*
$$\pi_{W,X\cap W} = \pi_{X,Z}|W.$$

(b) *If X_1, X_2 are basic projections of Z, then $X_1 \cap X_2$ is also a basic projection of Z and*
$$\pi_{Z,X_1\cap X_2} = \pi_{Z,X_1} \cdot \pi_{Z,X_2} = \pi_{Z,X_2} \cdot \pi_{Z,X_1}.$$
(The products are taken in $\mathrm{End}_R(Z)$.) If Y_i is the kernel of π_{Z,X_i} $(i = 1, 2)$, then $Y_1 + Y_2$ is the kernel of $\pi_{Z,X_1\cap X_2}$.

(c) *More generally, if $(X_i \mid i \in K)$ is a family of basic projections of Z, then $X = \bigcap_{i\in K} X_i$ is a basic projection of Z. Given $q \in \mathfrak{B}_0(Z)$, we have $\pi_{Z,X}(q) = q$ if $q \in X_i$ for every $i \in K$ and $\pi_{Z,X}(q) = 0$ otherwise. We write $\pi_{Z,X} = \prod_{i\in K} \pi_{Z,X_i}$.*

[5] A major reason for our interest in basic projectors is the desire to obtain similar decompositions when q is confined to a fixed proper basic submodule Z of $\mathrm{Quad}(V)$.

(d) *If X_1, X_2 are basic projections of Z with $X_1 \cap X_2 = \{0\}$, then $X_1 + X_2$ is also a basic projection of Z and*

$$\pi_{Z,X_1+X_2} = \pi_{Z,X_1} + \pi_{Z,X_2}.$$

(e) *If X is a basic projection of Z and U is a basic projection of X, then U is a basic projection of Z and*

$$\pi_{Z,U} = \pi_{X,U} \circ \pi_{Z,X}$$

(where now $\pi_{Z,X}$ is identified with the associated projection map $Z \twoheadrightarrow X$).

We strive for a combinatorial description of all basic projection pairs within the fixed R-module $\mathrm{Quad}(V)$.

Notation 3.5. *Given a basic module Z, we set*

$$\mathfrak{D}_0(Z) = \mathfrak{D}_0 \cap Z, \quad \mathfrak{H}_0(Z) = \mathfrak{H}_0 \cap Z, \quad \mathfrak{B}_0(Z) = \mathfrak{B}_0 \cap Z = \mathfrak{D}_0(Z) \cup \mathfrak{H}_0(Z),$$

furthermore

$$\Delta(Z) := \{i \in I \mid d_i \notin \mathfrak{D}_0(Z)\},$$

and

$$\Lambda(Z) := \{\{i,j\} \in I^{[2]} \mid h_{ij} \notin \mathfrak{H}_0(Z)\}.$$

Thus

$$\mathfrak{B}_0(Z) = \{d_i \mid i \notin \Delta(Z)\} \cup \{h_{ij} \mid \{i,j\} \notin \Lambda(Z)\}.$$

Notice that if X and Z are basic R-modules then

$$X \subset Z \quad \Leftrightarrow \quad \Delta(X) \supset \Delta(Z),\ \Lambda(X) \supset \Lambda(Z).$$

Corollary 3.6. *Assume that Z is a basic module and N is a subset of $I^{[2]}$ containing $\Lambda(Z)$. Then the basic submodule X of Z with $\Delta(X) = \Delta(Z)$ and $\Lambda(X) = N$ is a basic projection of Z. The associated basic projector $\pi_{Z,X}$ is the restriction to Z of the N-partial quasi-linearization $\pi_{N,\mathrm{QL}}$ (cf.* (2.6)*).*

Proof. This follows from Remark 3.4.a, since $X = Z \cap \mathrm{QL}(N,V)$ and $\mathrm{QL}(N,V)$ is a basic projection of $\mathrm{Quad}(V)$ with associated projector $\pi_{N,\mathrm{QL}}$. □

By this corollary we know all basic projections X of Z with $\Delta(X) = \Delta(Z)$, i.e., $\mathfrak{D}_0(X) = \mathfrak{D}_0(Z)$.

Lemma 3.7. *Assume that X is a basic projection of a basic module Z, and that $\Lambda(X) = \Lambda(Z) =: \Lambda$. Let $i \neq j$ be indices such that $\{i,j\} \notin \Lambda$, $i \notin \Delta(Z)$, $j \notin \Delta(Z)$. Then $i \notin \Delta(X)$, $j \notin \Delta(X)$.*

Proof. Let $\pi = \pi_{Z,X}$. In Z we have the relation (cf. [10, Eq. (7.10)])

$$d_i + d_j + h_{ij} = d_i + d_j.$$

Now $\{i,j\} \notin \Lambda$, whence $\pi(h_{ij}) = h_{ij}$. Applying π to the relation we obtain

$$\pi(d_i) + \pi(d_j) + h_{ij} = \pi(d_i) + \pi(d_j).$$

Suppose that $d_i \in \Delta(X)$. This means that $\pi(d_i) = 0$ and so

$$\pi(d_j) + h_{ij} = \pi(d_j).$$

But this is impossible in both cases $\pi(d_j) = 0$, $\pi(d_j) = d_j$. Thus $\pi(d_i) \neq 0$, i.e., $i \notin \Delta(X)$. For the same reason $j \notin \Delta(X)$. $\square$

Convention 3.8. *Up to the end of this section, we assume that the supertropical semiring, R, besides multiplicative unboundedness, satisfies the following condition.*

(†) *If $a, b \in R$ and there exists some $c_0 \in R$ such that $ac \leq bc$ for all $c \geq c_0$, then $a \leq b$.*

This condition is a rather mild hypothesis, as the following proposition reveals. We introduce the set

$$\mathrm{canc}(\mathcal{G}) := \{c \in \mathcal{G} \mid \forall a, b \in \mathcal{G} : ac = bc \Rightarrow a = b\},$$

consisting of the "cancellative" elements of $\mathcal{G}$.

Proposition 3.9. *Assume that the set* $\mathrm{canc}(\mathcal{G})$ *is unbounded in $\mathcal{G}$, and furthermore that $\mathcal{T}$ is closed under multiplication and $e\mathcal{T}$ is unbounded in $\mathcal{G}$. Then R has Property* (†).

Proof. Let $a, b \in R$ and assume that $ac \leq bc$ for all $c \geq c_0$ in R, for some $c_0 \in R$. We want to verify that $a \leq b$. If $a <_\nu b$ then $a < b$, and we are done. Henceforth assume that $a \geq_\nu b$. Then $ac \geq_\nu bc$ for all $c \in R$, and we conclude that $ac \cong_\nu bc$ for all $c \geq c_0$. Since $\mathrm{canc}(\mathcal{G})$ is unbounded, this implies $a \cong_\nu b$. If $a, b \in eR$ this means that $a = b$; and if $a \in \mathcal{T}$ and $b \in \mathcal{G}$, then $a \cong_\nu b$ implies $a < b$.
Assume finally that $b \in \mathcal{T}$ and pick $u \in \mathcal{T}$ with $u \geq c_0$, which is possible since $e\mathcal{T}$ is unbounded in $\mathcal{G}$. Now $au \leq bu \in \mathcal{T}$ and $au \cong_\nu bu$. This forces $au = bu$. Thus $au \in \mathcal{T}$. which implies $a \in \mathcal{T}$. From $a \cong_\nu b$ and $a, b \in \mathcal{T}$ we conclude again that $a = b$. Thus $a \leq b$ in all cases. $\square$

Theorem 3.10. *Let Z be a basic module, $\Lambda := \Lambda(Z)$, $\Delta := \Delta(Z)$. Assume that $i \in I$ is an index with $d_i \in Z$.*

(a) *There exists a basic projector π on Z with* $\ker(\pi) = Rd_i$ *iff the following condition holds:*

$$C_{\Lambda,\Delta}(i): \quad \text{If } k \in I \setminus \{i\} \text{ and } \{i, k\} \notin \Lambda, \text{ then } k \in \Delta.$$

(b) *This projector π, if it exists, is compatible with the functional ordering $\leq$ (restricted to Z), i.e., if $q, q' \in Z$, $q' \leq q$, then $\pi(q) \leq \pi(q')$.*

Proof. By Lemma 3.7 condition $C_{\Lambda,\Delta}(i)$ is necessary for the existence of π. Assuming now that $C_{\Lambda,\Delta}(i)$ holds, we want to show that π exists and respects the functional ordering on Z. Without loss of generality we may assume that I is finite, $I = \{1, \dots, n\}$, and $i = 1$, furthermore that $h_{1j} \in Z$ for $2 \leq j \leq r$, but $h_{1j} \notin Z$ for $r < j \leq n$, with some $r \in \{1, \dots, n\}$.

We claim that, given presentations

$$(*) \qquad q = \alpha_1 d_1 + \sum_{1<j\leq r} \alpha_{1j}h_{1j} + \sum_{\substack{1\,<\,k\,\leq\,r\\ r\,<\,\ell\,\leq\,n}} \alpha_{k\ell}h_{k\ell} + \sum_{i>r}\alpha_i d_i + \sum_{i,j>r}\alpha_{ij}h_{ij},$$

$$(**) \qquad q' = \beta_1 d_1 + \sum_{1<j\leq r} \beta_{1j}h_{1j} + \sum_{\substack{1\,<\,k\,\leq\,r\\ r\,<\,\ell\,\leq\,n}} \beta_{k\ell}h_{k\ell} + \sum_{i>r}\beta_i d_i + \sum_{i,j>r}\beta_{ij}h_{ij},$$

of two forms $q, q' \in Z$ with $q \leq q'$, then

(1) $\alpha_{1j} \leq \beta_{1j}$ for $1 < j \leq r$;

(2) $\alpha_{k\ell} \leq \beta_{k\ell}$ for $1 < k \leq r$, $r < \ell \leq n$;

(3) $\sum\limits_{i>r}\alpha_i d_i + \sum\limits_{i,j>r}\alpha_{ij}h_{ij} \leq \sum\limits_{i>r}\beta_i d_i + \sum\limits_{i,j>r}\beta_{ij}h_{ij}$.

If this is proven, then first assuming that $q = q'$, we learn that, if we omit in the presentation $(*)$ the summand $\alpha_1 d_1$, we obtain a quadratic form $\tilde{q} \in Z$ which is independent of the presentation $(*)$, and thus we have a well defined basic projection $\pi : q \mapsto \tilde{q}$ with kernel Rd_1. (Use the claim for $q \leq q'$ and $q' \leq q$.) Then using the claim in general, we learn that if $q \leq q'$, then $\pi(q) \leq \pi(q')$, establishing (a) and (b).

Proof of (3): The projector

$$\pi_J : q \mapsto (q|V_J)^I = q_J$$

(cf. Notation 2.4) with $J = \{i \mid r < i \leq n\}$ obviously respects the functional ordering on $\mathrm{Quad}(V)$. Applying π_J to $q \leq q'$ we obtain (3).

Proof of (1): Given i with $1 < i \leq r$ we insert in $(*)$ and $(**)$ the vector $\varepsilon_1 + c\varepsilon_i$ with c running through R, and obtain from $q \leq q'$ that

$$(***) \qquad \alpha_1 + c\alpha_{1i} \leq \beta_1 + c\beta_{1i}$$

for all $c \in R$. Choosing here $c = 0$ gives $\alpha_1 \leq \beta_1$. If $\beta_{1i} = 0$ then $c\alpha_{1i} \leq \beta_1$ for all $c \in R$. By multiplicative unboundedness of eR, we have some $c_0 \in eR$ such that $c\beta_{1i} >_\nu \beta_1$ for all $c \geq c_0$ and then obtain from $(***)$ that $c\alpha_{1i} \leq c\beta_{1i}$ for all $c \geq c_0$, which by property (†) implies $\alpha_{1i} \leq \beta_{1i}$. Thus $\alpha_{1i} \leq b_{1i}$ in both cases.

Proof of (2): Let $1 < k \leq r$, $r < \ell \leq n$. we insert in $(*)$ and $(**)$ the vector $c\varepsilon_k + \varepsilon_\ell$ with c running through R, and obtain from $q \leq q'$ that

$$\alpha_\ell + c\alpha_{k\ell} \leq \beta_\ell + c\beta_{k\ell}$$

for all $c \in R$. This implies $\alpha_\ell \leq \beta_\ell$ and $\alpha_{k\ell} \leq \beta_{k\ell}$ by the same arguments as before.

□

We are ready to determine all basic projections X of a basic module Z. In view of Corollary 3.6 it suffices to look for those submodules X of Z where $\Lambda(X) = \Lambda(Z)$, equivalently $\mathfrak{H}_0(X) = \mathfrak{H}_0(Z)$.

Definition 3.11. *We say that two elements d_i and d_j, $i \neq j$, of $\mathfrak{D}_0(Z)$ are* linked in Z *if $h_{ij} \in Z$.*

Note that the condition $C_{\Lambda,\Delta}(i)$ from Theorem 3.10 means that d_i is not linked in Z to any $d_j \in Z$.

Theorem 3.12. *Let X and Z be basic modules with $X \subset Z$ and $\mathfrak{H}_0(X) = \mathfrak{H}_0(Z)$. Then X is a basic projection of Z iff any two elements $d_i, d_j \in \mathfrak{D}_0(Z)$ which are linked in Z are both elements of $\mathfrak{D}_0(X)$. In more imaginative terms: we obtain all basic projections X of Z with $\mathfrak{H}_0(X) = \mathfrak{H}_0(Z)$ by discarding from $\mathfrak{D}_0(Z)$ some elements which are not linked in Z to other elements of $\mathfrak{D}_0(Z)$.*

Proof. ($\Leftarrow$): Lemma 3.7 means that, if X is a basic projection of Z, then this condition holds.

($\Rightarrow$): Let $\Lambda = \Lambda(Z)$, $\Delta = \Delta(Z)$, and

$$\begin{aligned} E := E(Z) \quad &= \{ i \in I \mid C_{\Lambda,\Delta}(i) \text{ holds } \} \\ &= \{ i \in I \mid d_i \in Z, d_i \text{ is not linked to any } d_i \in Z \}. \end{aligned}$$

Theorem 3.10 assures that for every $i \in E$ there exists a basic projector ϑ_i on Z with kernel Rd_i. It now follows by Remark 3.4.c that for any subset K of E the basic projection

$$\vartheta_K = \prod_{i \in K} \vartheta_i : Z \to Z \tag{3.2}$$

has kernel $\sum_{i \in K} Rd_i$. Thus

$$\mathfrak{D}_0(\vartheta_K(Z)) = \{ d_i \in \mathfrak{D}_0(Z) \mid i \notin K \}$$

and, of course, $\mathfrak{H}_0(\vartheta_K(Z)) = \mathfrak{H}_0(Z)$. □

Corollary 3.13. *Let Z be any basic module. Among the basic projections X of Z with $\mathfrak{H}_0(X) = \mathfrak{H}_0(Z)$ there is a unique minimal one, $X_{\min}$. All basic submodules X of Z with $X_{\min} \subset X$ are basic projections of Z, of course with $\mathfrak{H}_0(X) = \mathfrak{H}_0(Z)$.*

Theorem 3.14. *With the hypotheses of Convention 3.8, let Z be any basic submodule of* Quad(V). *Then any basic projector π of Z with $\pi(q) = q$ for all $\mathfrak{H}_0(Z)$ respects the functional ordering on Z:*

$$q \leq q' \Rightarrow \pi(q) \leq \pi(q').$$

Proof. Clear from Theorem 3.10.b, since such a projector has the shape ϑ_K given in (3.2). □

4. Direct decompositions; linked versus free modules

In this section, as well as in §5 and §6, it will be good to keep the following simple fact in mind.

Lemma 4.1. *Assume that $\varphi : X \to Y$ is a linear map between R-modules X and Y (with R supertropical as always). Assume that S is a subset of Y which is convex w.r. to the minimal ordering $\preceq$ (i.e., if $s, t \in S$, $y \in Y$ and $s \preceq y \preceq t$, then $y \in S$). Then the preimage $\varphi^{-1}(S)$ is convex in X. In particular (take $S = \{0\}$) the kernel $\varphi^{-1}(0)$ of φ is a convex submodule of X.*

Proof. This is evident, since φ is additive and so $x \preceq x'$ implies $\varphi(x) \preceq \varphi(x')$. $\square$

Note that a submodule S of an R-modules X is convex w.r. to the minimal ordering on X iff S is a lower set in X, since we always have $0 \in S$.

In the sequel *we assume that eR is mutiplicatively unbounded* and Z is a "basic module", i.e., Z is a basic submodule of $\mathrm{Quad}(V)$ for V a fixed free R-module with base $\{\varepsilon_i \mid i \in I\}$, cf. Definitions 3.2 and 3.3. We want to get insight into the presentations of Z as a direct sum of submodules (which then are again basic modules).

We start with two easy facts.

Proposition 4.2. *Assume that X and Y are two basic submodules of a basic module Z with $X \cap Y = \{0\}$ and $X + Y = Z$.*

- (a) *The following are equivalent*
  - (1) $Z = X \oplus Y$;
  - (2) *Both X and Y are basic projections of Z;*
  - (3) *Both X and Y are basic projection kernels of Z.*
- (b) *If (1)-(3) hold the both X and Y are convex submodules of Z.*

Proof. a): Since the elements of $\mathfrak{B}_0(Z)$ are indecomposable, it follows from $X \cap Y = \{0\}$ and $X + Y = Z$ that $\mathfrak{B}_0(Z)$ is the disjoint union of $\mathfrak{B}_0(X)$ and $\mathfrak{B}_0(Y)$. Thus the implication (1) $\Rightarrow$ (2) is obvious. Furthermore, if X is a basic projection of Z, then Y is the associated projection kernel, i.e., $Y = \pi_{Z,X}^{-1}(0)$. The same holds for X, Y interchanged. This makes the implications (2) $\Leftrightarrow$ (3) evident.

(2) $\Rightarrow$ (1): Let $z \in Z$ and $z = x + y$ with $x \in X$, $y \in Y$. Then $\pi_{Z,X}(x) = x$, $\pi_{Z,X}(y) = 0$, $\pi_{Z,Y}(x) = 0$, $\pi_{Z,Y}(y) = y$, whence $\pi_{Z,X}(z) = x$, $\pi_{Z,Y}(z) = y$. Thus x and y are uniquely determined by z, which proves that $Z = X \oplus Y$.

b): Obvious from Lemma 4.1, since $X = \pi_{Z,Y}^{-1}(0)$ and $Y = \pi_{Z,X}^{-1}(0)$. $\square$

Proposition 4.3. *Assume that $\{X_\alpha \mid \alpha \in A\}$ is a family of convex submodules of Z with $X = \sum_{\alpha \in A} X_\alpha$ and $X_\alpha \cap X_\beta = 0$ for $\alpha \neq \beta$. Then each X_α is a basic submodule of Z, and the basic set of generators $\mathfrak{B}_0(Z)$ is the disjoint union of the sets $\mathfrak{B}_0(X_\alpha)$.*

Proof. Since the elements of $\mathfrak{B}_0(Z)$ are indecomposable, it follows from $X = \sum_{\alpha \in A} X_\alpha$ and $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \neq \beta$ that every element of $\mathfrak{B}_0(Z)$ is contained in some X_α, $\alpha \in A$. Let X'_α denote the R-submodule spanned by

$X_\alpha \cap \mathfrak{B}_0(Z)$. This is the maximal basic module of Z containing X_α. Since $\mathfrak{B}_0(Z)$ is the union of the sets $X_\alpha \cap \mathfrak{B}_0(Z)$ it is clear that $Z = \sum_{\alpha \in A} X'_\alpha$.
Picking any $\alpha \in A$, we will be done by verifying that $X_\alpha = X'_\alpha$. Let $x \in X_\alpha$ and write $x = \sum_{\beta \in A} x'_\beta$ with $x'_\beta \in X'_\beta$ (almost all $x'_\beta = 0$). Clearly $x'_\beta \preceq x$ for all β and thus $x'_\beta \in X_\alpha$. If $\beta \neq \alpha$, then $x'_\beta \in X_\alpha \cap X_\beta = \{0\}$, and so $x = x'_\alpha$. $\square$

But if X and Y are convex submodules of the basic module $X + Y = Z$, where $X \cap Y = \{0\}$, then it is not necessarily true that Z is the direct sum of X and Y, as the following key example shows.
As usual, we call an R-module U *decomposable* if there exist submodules $U_1 \neq U$, $U_2 \neq U$ such that $U = U_1 \oplus U_2$; otherwise, we call U *indecomposable.*

Example 4.4. *Let $I = \{1, 2\}$ and let $Z = \mathrm{Quad}(V) = Rd_1 + Rd_2 + Rh_{12}$. Then*

$$Rd_1 \cap (Rd_2 + Rh_{12}) = \{0\},$$

and $Rd_2 + Rh_{12}$ is convex in Z since it is the kernel of a basic projector $\pi_1 : Z \to Rd_1$, namely the composite of $\pi_{\mathrm{QL}} : Z \to Rd_1 + Rd_2$ and the basic projector of the free module $Rd_1 + Rd_2$ onto Rd_2. For the same reason $Rd_1 + Rh_{12}$ is also convex in Z.
We verify directly that Rd_1 is convex in $Rd_1 + Rh_{12}$ and so is convex in Z. Indeed, if $\alpha d_1 + \beta h_{12} \preceq \gamma d_1$ with $\alpha, \beta, \gamma \in R$, then $\beta h_{12} \preceq \gamma d_1$. A fortiori $\beta h_{12} \leq \gamma d_1$, which means that $\beta x_1 x_2 \leq \gamma x_1^2$ for all $x_1, x_2 \in R$. In particular for $x_1 = 1$ we obtain $\beta x_2 \leq \gamma$ for all $x_2 \in R$, which forces $\beta = 0$ by multiplicative unboundedness.
But Z is not the direct sum of the convex submodules Rd_1 and $Rd_2 + Rh_{12}$. In fact $d_1 + d_2 = d_1 + d_2 + h_{12}$, while $d_2 \neq d_1 + h_{12}$. (Insert $x_1\varepsilon_1 + \varepsilon_2$ with some $x_1 >_\nu 1$.) Neither is Z the direct sum of Rd_2 and $Rd_1 + Rh_{12}$. We conclude that Z is indecomposable.

Remark 4.5. *Hypothesis (†) in Convention 3.8 is not needed in this example. If (†) holds in addition to multipicativity unboundedness, then it is immediate from Theorem 3.10 that $Rd_2 + Rh_{12}$ is free. Also $Rd_1 + Rh_{12}$ is free, and so all proper basic submodules of Z are free.*

Definition 4.6.

(a) *We call a basic module Z* linked *if*
  (i) *for every $h_{ij} \in Z$ both d_i and d_j are in Z;*
  (ii) *for each $d_i \in Z$ there exists some $j \neq i$ such that $h_{ij} \in Z$, and so d_i is linked in Z to some $d_j \in Z$ (cf. Definition 3.11).*

(b) *We call*

$$E_{ij} := Rd_i + Rd_j + Rh_{ij} \qquad (i, j \in I,\ i \neq j)$$

an elementary linked module. *Thus Z is linked iff Z is the sum of all elementary linked modules contained in Z.*

(c) *Given any basic module Z we denote the sum of all $E_{ij} \subset Z$ by Z_{link}, which is the maximal linked submodule of Z. We call it the* linked core *of Z.*

Example 4.4 shows that every elementary linked module is indecomposable. The formation of linked cores behaves well with respect to direct sums.

PROPOSITION 4.7. *If $(Z_\alpha \mid \alpha \in A)$ is any family of basic R-modules, then*

$$\left(\bigoplus_{\alpha \in A} Z_\alpha\right)_{\text{link}} = \bigoplus_{\alpha \in A} (Z_\alpha)_{\text{link}}.$$

This is an immediate consequence of the fact that every elementary linked module is indecomposable, together with the following easy lemma.

LEMMA 4.8. *Assume that Z is a basic module and $(X_\alpha \mid \alpha \in A)$ is a family of submodule such that*

$$Z = \bigoplus_{\alpha \in A} X_\alpha.$$

Then for any convex submodule W of Z we have

$$W = \bigoplus_{\alpha \in A'} W \cap X_\alpha$$

where $A' := \{\alpha \in A \mid W \cap X_\alpha \neq 0\}$. In particular, when W is indecomposable, $W \subset X_\alpha$ for some $\alpha \in A'$.

Proof. We have a family $(\pi_\alpha \mid \alpha \in A)$ of basic projectors $\pi_\alpha : Z \to Z$ at hands with $\pi_\alpha(Z) = X_\alpha$. It follows that $\pi_\alpha \pi_\beta = \pi_\beta \pi_\alpha = 0$ for $\alpha \neq \beta$ and $\sum_{\alpha \in A} \pi_\alpha = \mathrm{id}_Z$ (which means that, given $q \in Z$, almost all values $\pi_\alpha(q)$ are zero and $\sum_{\alpha \in A} \pi_\alpha(q) = q$). Now $\pi_\alpha(z) \preceq z$ for every $z \in Z$, $\alpha \in A$, and thus $\pi_\alpha(W) \subset W$ for every $\alpha \in A$. By restriction we obtain a family of projectors

$$\pi'_\alpha = \pi_\alpha | W : W \to W, \qquad \alpha \in A'.$$

Since $\pi'_\alpha \pi'_\beta = \pi'_\beta \pi'_\alpha = 0$ for $\alpha \neq \beta$ and $\sum_{\alpha \in A'} \pi'_\alpha = \mathrm{id}_W$, it is immediate that $W = \bigoplus_{\alpha \in A'} \pi'_\alpha(W) = \bigoplus_{\alpha \in A'} W \cap X_\alpha$. [6] □

PROPOSITION 4.9. *Assume that Convention 3.8 is in force. Then Z is free iff $Z_{\text{link}} = 0$.*

Proof. When Z is free, all its basic submodules are free, and so Z cannot contain any elementary linked module, whence $Z_{\text{link}} = 0$. (N.B. In this argument hypothesis (†) is not needed.) Conversely, if $Z_{\text{link}} = 0$, we know by Theorem 3.12 that $Z \cap \mathrm{Rig}(V)$ is a basic projection of Z with kernel $Z \cap \mathrm{QL}(V)$. (All $d_i \in \mathfrak{D}_0(Z)$ may be discarded from the list $\mathfrak{B}_0(Z)$.) But

[6] In this proof our standard assumption that eR is multiplicatively unbounded is not needed.

also $Z \cap \mathrm{QL}(V)$ is a basic projection of Z, namely the image of the restriction $\pi_{\mathrm{QL}} : \mathrm{Quad}(V) \twoheadrightarrow \mathrm{QL}(V)$ to Z. Thus

$$Z = (Z \cap \mathrm{QL}(V)) \oplus (Z \cap \mathrm{Rig}(V)).$$

Since both $\mathrm{QL}(V)$ and $\mathrm{Rig}(V)$ are free, Z is also free. $\square$

Given a basic submodule X of Z, we call the unique basic module Y with $X + Y = Z$, $X \cap Y = \{0\}$, the *basic complement Y of X in Z*.

Theorem 4.10. *Assume that Convention 3.8 is in force. Then, the basic complement of Z_{link}, Y, in Z is free and $Z = Z_{\mathrm{link}} \oplus Y$.*

Proof. Y does not contain any elementary linked module, and thus is free by the preceding proposition. We will be done by verifying that both Z_{link} and Y are basic projections of Z.
By mapping all the $h_{ij} \in Y$ to zero we obtain a basic projector $Z \twoheadrightarrow Z_1$ with

$$\mathfrak{H}_0(Z_1) = \mathfrak{H}_0(Z) \setminus \mathfrak{H}_0(Y) = \mathfrak{H}_0(Z_{\mathrm{link}}), \qquad \mathfrak{D}_0(Z_1) = \mathfrak{D}_0(Z),$$

cf. Corollary 3.6. We have $Z_1 \supset Z_{\mathrm{link}}$ and

$$\mathfrak{D}_0(Z_1) = \mathfrak{D}_0(Y) \ \dot\cup\ \mathfrak{D}_0(Z_{\mathrm{link}}),$$

but no $d_i \in \mathfrak{D}_0(Y)$ is linked to any $d_j \in \mathfrak{D}_0(Z_1)$, and thus Theorem 3.10 gives us a basic projector $Z_1 \twoheadrightarrow Z_{\mathrm{link}}$. Composing the two projectors we obtain a basic projector $\pi_1 : Z \twoheadrightarrow Z_{\mathrm{link}}$.
On the other hand by mapping all $h_{ij} \in \mathfrak{H}_0(Z_{\mathrm{link}})$ to zero we obtain a basic projector $Z \twoheadrightarrow Z_2$ with

$$\mathfrak{H}_0(Z_2) = \mathfrak{H}_0(Z) \setminus \mathfrak{H}_0(Z_{\mathrm{link}}) = \mathfrak{H}_0(Y), \qquad \mathfrak{D}_0(Z_2) = \mathfrak{D}_0(Y) \ \dot\cup\ \mathfrak{D}_0(Z_{\mathrm{link}}).$$

As no $d_i \in \mathfrak{D}_0(Z_{\mathrm{link}})$ is linked to any $d_j \in \mathfrak{D}_0(Y)$, we obtain a basic projector $Z_2 \twoheadrightarrow Y$, again by Theorem 3.10, which together with $Z \twoheadrightarrow Z_2$ yields a basic projector $\pi_2 : Z \twoheadrightarrow Y$. The projectors π_1 and π_2 together entail $Z = Z_{\mathrm{link}} \oplus Y$. $\square$

We denote the basic complement Y of Z_{link} in Z by Z_{free}, and obtain

$$Z = Z_{\mathrm{link}} \oplus Z_{\mathrm{free}}.$$

Corollary 4.11. *Suppose Z is a basic module and Convention 3.8 is in force, then Z_{free} is the unique maximal basic free submodule of Z which is a direct summand of Z.*

Proof. Let $Z = X \oplus T$ with X free. Then (cf. Proposition 4.7)

$$Z_{\mathrm{link}} = X_{\mathrm{link}} \oplus T_{\mathrm{link}} = T_{\mathrm{link}},$$

and $T = T_{\mathrm{link}} \oplus T_{\mathrm{free}}$. We conclude that

$$Z = X \oplus T_{\mathrm{link}} \oplus T_{\mathrm{free}}$$

and also $Z = Z_{\mathrm{link}} \oplus Z_{\mathrm{free}} = T_{\mathrm{link}} \oplus Z_{\mathrm{free}}$, whence $Z_{\mathrm{free}} = X \oplus T_{\mathrm{free}}$. $\square$

Definition 4.12. *We call any indecomposable direct summand $X \neq 0$ of a basic module Z a* component *of Z.*

We start out to determine the components of a basic R-module Z under Convention 3.8. First an easy preliminary lemma, valid over any supertropical semiring R.

Lemma 4.13. *Assume that* $\bigoplus_{\alpha \in A} Z_\alpha$ *is a direct decomposition of* Z, *where each* Z_α *is indecomposable and* $\neq 0$. *Then these* Z_α *are precisely all components of* Z.

Proof. Let X be a basic nonzero submodule of Z. By Lemma 4.8

$$X = \bigoplus_{\alpha \in A} X \cap Z_\alpha.$$

If X is indecomposable then $X = X \cap Z_\alpha$ for a unique $\alpha \in A$, whence $X \subset Z_\alpha$. If Y is the basic complement of X in Z so that $Z = X \oplus Y$, then $Z_\alpha = X \oplus (Y \cap Z_\alpha)$. Since Z_α is indecomposable, this implies that $X = Z_\alpha$. □

Definition 4.14.

(a) *We call a basic module* Z graphic *if for any* $h_{ij} \in \mathfrak{H}_0(Z)$ *both* d_i *and* d_j *are in* Z.

(b) *When* Z *is graphic, we define the graph* $\Gamma(Z)$ *(simple, undirected, without loops) as follows.* $\Gamma(Z)$ *has the sets of vertices and edges*

$$\mathrm{Ver}(\Gamma(Z)) = \mathfrak{D}_0(Z) \quad \textit{and} \quad \mathrm{Edg}(\Gamma(Z)) = \mathfrak{H}_0(Z).$$

An edge h_{ij} *connects the vertices* d_i, d_j. [7]

(c) *A basic module* Z *has a unique maximal submodule which is graphic, denoted* Z_{graph}. $\mathfrak{B}_0(Z_{\mathrm{graph}})$ *is obtained from* $\mathfrak{B}_0(Z)$ *by omitting every* $h_{ij} \in Z$ *with* $d_i \notin Z$ *or* $d_j \notin Z$.

Note that Z is linked iff Z is graphic and $\Gamma(Z)$ has no isolates vertices. When Convention 3.8 holds, clearly Z is graphic iff $\mathfrak{H}_0(Z_{\mathrm{free}})$ is empty.

Remark 4.15. $\Gamma(\mathrm{Quad}(V))$ *is the complete graph over the vertex set* $\{d_i \mid i \in I\}$, *and thus may be seen as the graph* $(I, I^{[2]})$. *For any subgraph* Γ' *of* $\Gamma(\mathrm{Quad}(V))$ *there exists a unique graphic module* Z *with* $\Gamma(Z) = \Gamma'$.

Next we describe the components of graphic modules in graph theoretic terms. As before we tacitly assume that $\mathcal{G}$ is multiplicatively unbounded.

Proposition 4.16. *Assume that* Z *is a graphic module and* $(X_\alpha \mid \alpha \in A)$ *is a family of submodules of* Z. *For each* $\alpha \in A$ *let* $J_\alpha := \{i \in I \mid d_i \in X_\alpha\}$. *The following are equivalent:*

(i) $Z = \bigoplus_{\alpha \in A} X_\alpha$;

(ii) *Each* X_α *is graphic and* $\Gamma(Z)$ *is the disjoint union of the graphs* $\Gamma(X_\alpha)$, *for which we write*

[7] It would be more precise to consider $\mathfrak{D}(Z)$ and $\mathfrak{H}(Z)$ respectively as sets of vertices and edges, but the present setting has proved to be more convenient.

$$\Gamma(Z) = \bigsqcup_{\alpha \in A} \Gamma(X_\alpha).$$

When (i) and (ii) hold, the projection $Z \twoheadrightarrow X_\alpha$ is given, for $q \in Z$, by (cf. Notation 2.4)

$$\pi_\alpha(q) = (q|V_{J_\alpha})^I. \tag{4.1}$$

Proof. a) Assume that Z is the direct sum of the X_α. If $h_{ij} \in X_\alpha$ for some $i, j \in I$, then the d_i and d_j are elements of Z, since Z is graphic. The elementary linked submodule E_{ij} of Z is indecomposable (cf. Example 4.4) and $E_{ij} \cap X_\alpha \neq \{0\}$. By the (essentially trivial) Lemma 4.8 we conclude that $E_{ij} \subset X_\alpha$. Thus X_α is graphic. It now follows directly from Definition 4.14 that

$$\Gamma(Z) = \bigsqcup_{\alpha \in A} \Gamma(X_\alpha).$$

b) We now *assume* that $\Gamma(Z)$ is the disjoint union of the graphs $\Gamma(X_\alpha)$, $\alpha \in A$. Firstly this implies that $Z = \sum_{\alpha \in A} X_\alpha$. For each $\alpha \in A$ we *define* a map $\pi_\alpha : Z \to Z$ by formula (4.1). Let $J := \{i \in I \mid d_i \in Z\}$, so that J is the disjoint union of the sets J_α. Given $i \in J$ we have $(d_i|V_{J_\alpha})^I = d_i$ for $i \in J_\alpha$ and $(d_i|V_{J_\alpha})^I = 0$ otherwise. Given different $i, j \in J$ we have a similar story: $(h_{ij}|V_{J_\alpha})^I = h_{ij}$ when $\{i, j\} \subset J_\alpha$ and zero otherwise. This proves that π_α is a basic projector on Z with image X_α, and furthermore that $\pi_\alpha \pi_\beta = 0$ if $\alpha \neq \beta$ and $\sum_{\alpha \in A} \pi_\alpha = \mathrm{id}_Z$. It is now obvious that the sum $\sum_{\alpha \in A} X_\alpha$ is direct with associated projectors π_α. □

Corollary 4.17. *A graphic module Z is indecomposable iff the graph $\Gamma(Z)$ is connected.*

Proof. We argue by contradiction. If $Z = X_1 \oplus X_2$ with $X_1 \neq 0$, $X_2 \neq 0$, then by Proposition 4.16 both X_1, X_2 are graphic and $\Gamma(Z) = \Gamma(X_1) \sqcup \Gamma(X_2)$; so $\Gamma(Z)$ is not connected. Conversely, assume $\Gamma(Z) = \Gamma_1 \sqcup \Gamma_2$ and let X_i denote the graphic submodule of Z with $\Gamma(X_i) = \Gamma_i$ $(i = 1, 2)$. Then $Z = X_1 \oplus X_2$ by Proposition 4.16, and thus Z is decomposable. □

Theorem 4.18. *Assume that Z is a graphic module. Let $(\Gamma_\gamma \mid \gamma \in C)$ denote the set of components of the graph $\Gamma(Z)$, arbitrarily indexed, and for every $\gamma \in C$ let Z_γ denote the graphic module with $\Gamma(Z_\gamma) = \Gamma_\gamma$. Then*

$$Z = \bigoplus_{\gamma \in C} Z_\gamma$$

and the Z_γ are precisely all components of the basic module Z.

Proof. We know by Proposition 4.16, that Z is the direct sum of the Z_γ and by Corollary 4.17 that the Z_γ are indecomposable. It follows from Lemma 4.8 that the Z_γ are all components of Z. □

We emphasize that in our study of the components of a basic module Z up to now hypothesis (†) has not been needed. But if (†) holds, then we know that

$Z = Z_{\text{link}} \oplus Z_{\text{free}}$. Applying Theorem 4.18 to Z_{link}, we obtain the following corollary.

Corollary 4.19. *Assume that Convention 3.8 is in force. Let Z be a basic module, and let $(Z_\gamma \mid \gamma \in C)$ denote the set of components of Z indexed in some way. Then*

$$Z = \bigoplus_{\gamma \in C} Z_\gamma.$$

Furthermore, Z_{link} is the direct sum of those components Z_γ, which are not free, while Z_{free} is the direct sums of all others. They are free of rank one.

5. Basic endomorphisms and their associated projectors

As before R is a supertropical semiring. Let Z be a basic module over R, i.e., a basic submodule of $\mathrm{Quad}(V)$, for V a fixed free R-module V, cf. Definitions 3.2 and 3.3. While in §3 and §4 we studied basic projectors on Z, we now proceed to the more general "basic endomorphisms" of Z. We denote the set of endomorphisms of Z by $\mathrm{End}_R(Z)$ or simply $\mathrm{End}(Z)$. This is an R-algebra in the obvious sense. As before we work with the set of generators $\mathfrak{B}_0 = \mathfrak{D}_0 \cup \mathfrak{H}_0$,

$$\mathfrak{D}_0 := \{d_i \mid i \in I\}, \qquad \mathfrak{H}_0 := \left\{h_{ij} \mid \{i,j\} \in I^{[2]}\right\}$$

of $\mathrm{Quad}(V)$ derived from a fixed base of V, cf. (3.1). By intersection with Z it gives a "basic set of generators" $\mathfrak{B}_0(Z) = \mathfrak{D}_0(Z) \cup \mathfrak{H}_0(Z)$, cf. Notation 3.5.

Definition 5.1. *An endomorphism φ of Z is called* basic, *if $\varphi(Rq) \subset Rq$ for every $q \in \mathfrak{B}_0$, whence*

$$\varphi(d_i) = \mu_i d_i, \qquad \varphi(h_{ij}) = \mu_{ij} d_{ij}$$

for all $d_i \in Z$, $h_{ij} \in Z$, with elements μ_i, μ_{ij} in R, which we name the coefficients *of φ. The set of all basic endomorphisms of Z, denoted by $\mathrm{End}_b(Z)$, is a commutative subalgebra of $\mathrm{End}(Z)$.*

We remark that for any basic submodule Z' of Z, every $\varphi \in \mathrm{End}_b(Z)$ restricts to a basic endomorphism $\varphi|Z'$ of Z'. Note also that the basic projectors on Z are precisely the basic endomorphisms of Z with all coefficients in $\{0, 1\}$. They are idempotents of the R-algebra $\mathrm{End}_b(Z)$.[8]

Example 5.2. *If ρ is an endomorphism of the free R-module V, then for every quadratic form $q : V \to R$, the composite $q \circ \rho : V \to R$ is again a quadratic form on V, and so we obtain an endomorphism*

$$\rho^* : q \mapsto q \circ \rho,$$

of the R-module $\mathrm{Quad}(V)$. *We call these ρ^* the* geometric endomorphisms *of* $\mathrm{Quad}(V)$. *If ρ itself is "basic", i.e., $\rho(\varepsilon_i) = \mu_i \varepsilon_i$ for every $i \in I$ with some $\mu_i \in R$, then an easy computation shows that*

$$\rho^*(d_i) = \mu_i^2 d_i, \qquad \rho^*(h_{ij}) = \mu_i \mu_j h_{ij},$$

[8] There exist other idempotents, e.g. $\varphi = e\,\mathrm{id}_Z$.

whence ρ^* is a basic endomorphism of $\mathrm{Quad}(V)$. *We denote this endomorphism by γ_μ, where $\mu := (\mu_i \mid i \in I)$, and call the γ_μ the* geometric basic endomorphisms *of* $\mathrm{Quad}(V)$.

These endomorphisms γ_μ are the "easy" basic endomorphisms of $\mathrm{Quad}(V)$. Every tuple $\mu = (\mu_i \mid i \in I) \in R^I$ gives such an endomorphism with system of coefficients

$$(\mu_i^2 \mid i \in I) \cup (\mu_i\mu_j \mid \{i,j\} \in I^{[2]}).$$

By restriction we obtain for γ_μ a basic endomorphism $\gamma_{Z,\mu} := \gamma_\mu|Z$, and then have the following upshot of Example 5.2.

Proposition 5.3. *Let Z be a basic module and set $I(Z) := (i \in I \mid d_i \in Z)$. Every tuple $\mu = (\mu_i \mid i \in I(Z))$ yields a unique basic endomorphisms $\gamma_{Z,\mu}$ of Z with*

$$\gamma_{Z,\mu}(d_i) = \mu_i^2 d_i$$

for all $d_i \in Z$ and

$$\gamma_{Z,\mu}(h_{ij}) = \mu_i\mu_j h_{ij}$$

for all $h_{ij} \in Z$.

We call the $\gamma_{Z,\mu}$ the *geometric basic endomorphisms* of Z. They form a subset of $\mathrm{End}_b(Z)$, closed under multiplication. Note that this set does not depend on the choice of the base $\{\varepsilon_i \mid i \in I\}$ of V.
In particular we have the *geometric basic projectors* of Z at hands. These are the endomorphisms $\gamma_{Z,\mu}$ with $\mu_i \in \{0,1\}$ for all $i \in I(Z)$, and thus they correspond uniquely to the subsets $J = \{i \in I(Z) \mid \mu_i = 1\}$ of $I(Z)$.

Notations 5.4. *We denote the basic projection coming from such a set J by $\pi_{Z,J}$. In the most important case, namely $Z = \mathrm{Quad}(V)$, we write π_J instead of $\pi_{\mathrm{Quad}(V),J}$ and so*

$$\pi_{Z,J} = \pi_J|Z$$

for $J \subset I(Z)$.

Proposition 5.5.

(a) *For any $J \subset I$ the geometric basic projector π_J on $\mathrm{Quad}(V)$ can be also described by the formula ($q \in \mathrm{Quad}(V)$)*

$$\pi_J(q) = (q|V_J)^I =: q_J, \tag{5.1}$$

cf. Notation 2.4.
More generally, for any $J \subset I$ and $\varphi \in \mathrm{End}_b(\mathrm{Quad}(V))$, $q \in \mathrm{Quad}(V)$, we have the formula

$$(\varphi\pi_J)(q) = (\varphi(q)|V_J)^I = \varphi(q)_J. \tag{5.2}$$

(b) *Given subsets J, K of I, we have $\pi_J\pi_K = \pi_{J\cap K}$. If $J \cap K$ is empty, then $\pi_{J\cup K} = \pi_J + \pi_K$.*

Proof. (a): (5.1) is the formula (5.2) in the special case that φ is the identity map. In order to verify (5.2) it is suffices to check this formula for every $q \in \mathfrak{B}_0 = \mathfrak{D}_0 \cup \mathfrak{H}_0$. Let $(\mu_i) \cup (\mu_{ij})$ be the family of coefficients of φ. If $i, j \in J$, $i \neq j$, then

$$(\varphi\pi_J)(d_i) = \varphi(d_i) = \mu_i d_i,$$
$$(\varphi\pi_J)(h_{ij}) = \varphi(h_{ij}) = \mu_{ij} h_{ij},$$

while

$$(\varphi\pi_J)(d_k) = 0 \quad \text{if } k \notin J,$$
$$(\varphi\pi_J)(h_{k\ell}) = 0 \quad \text{if } \{k, \ell\} \not\subset J.$$

$(\varphi(q)|V_J)^I$ has exactly the same values for all $q \in \mathfrak{B}_0$.

(b): Obvious by considering the coefficients of the occurring projectors, or (better) the endomorphisms of V including these geometric endomorphisms of $\mathrm{Quad}(V)$. □

In the case of $|J| \leq 2$ we simplify the notation by writing

$$\pi_i := \pi_{\{i\}}, \qquad \pi_{ij} := \pi_{\{i,j\}}. \tag{5.3}$$

These projectors will play a very helpful role later.
All basic endomorphisms of $\mathrm{Quad}(V)$ can be built from basic endomorphisms of the elementary linked submodules of $\mathrm{Quad}(V)$ (cf. Definition 4.6) as follows.

Lemma 5.6 (Pasting Lemma). *Let $Z = \mathrm{Quad}(V)$. Assume that $(\varphi_{ij} \mid \{i, j\} \subset I)$ is a family of basic endomorphisms $\varphi_{ij} \in \mathrm{End}_b(E_{ij})$, $E_{ij} = Rd_i + Rd_j + Rh_{ij}$. Assume further that $\varphi_{ij}|Rd_i = \varphi_{ik}|Rd_i$ for any three distinct $i, j, k \in I$. Then there is a unique $\varphi \in \mathrm{End}_b(Z)$ with $\varphi|E_{ij} = \varphi_{ij}$, for all $\{i, j\} \in I^{[2]}$.*

Proof. We have elements μ_i, μ_{ij} in R with

$$\varphi(d_i) = \mu_i d_i, \qquad \varphi(h_{ij}) = \mu_{ij} h_{ij}$$

for all $\{i, j\} \in I^{[2]}$. The basic endomorphism φ of Z with system of coefficients $(\mu_i) \cup (\mu_{ij})$ has the required properties, and clearly is the unique one, provided that φ exists.
For notational convenience we choose a total ordering on I. If $q \in \mathrm{Quad}(V)$, and two presentations of q are given,

$$q = \sum_{i \in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{ij} = \sum_{i \in I} \alpha_i d_i + \sum_{i<j} \beta_{ij} h_{ij} \tag{A}$$

(of course with only finitely many scalars $\alpha_i, \alpha_{ij}, \beta_{ij} \neq 0$), we need to verify that

$$\sum_{i \in I} \alpha_i \mu_i d_i + \sum_{i<j} \alpha_{ij} \mu_{ij} h_{ij} = \sum_{i \in I} \alpha_i \mu_i d_i + \sum_{i<j} \beta_{ij} \mu_{ij} h_{ij}. \tag{B}$$

For any pair $i < j$ in I we apply the projector π_{ij} to (A), viewed as a map onto E_{ij}, and then the map φ_{ij}. We obtain

$$\alpha_i \mu_i d_i + \alpha_j \mu_j d_j + \alpha_{ij} \mu_{ij} h_{ij} = \alpha_i \mu_i d_i + \alpha_j \mu_j d_j + \beta_{ij} \mu_{ij} h_{ij}.$$

Since this holds for all $i < j$, (B) is now evident.
From B we conclude that there is a well defined map $\varphi : \mathrm{Quad}(V) \to \mathrm{Quad}(V)$ sending any $q = \sum_{i \in I} \alpha_i d_i + \sum_{i<j} \alpha_{ij} h_{ij}$ to $\sum_{i \in I} \alpha_i \mu_i d_i + \sum_{i<j} \alpha_{ij} \mu_{ij} h_{ij}$. It is obvious that this map φ is a basic endomorphism and $\varphi(d_i) = \mu_i d_i$, $\varphi(h_{ij}) = \mu_{ij} h_{ij}$ for all $i < j$. □

Remark 5.7. *If I is finite, then*

$$\varphi = \sum_{i<j} \varphi_{ij} \circ \pi_{ij},$$

with π_{ij} the basic projector from (5.3)*, viewed as a map from* $\mathrm{Quad}(V)$ *onto* E_{ij}*, since* $\pi_{ij}|E_{ij} = \mathrm{id}_{E_{ij}}$*,* $\pi_{k\ell}|E_{ij} = 0$ *for* $\{k, \ell\} \neq \{i, j\}$*.*

Theorem 5.8 (Extension Theorem). *Let φ be a basic endomorphism of a graphic module $Z \subset \mathrm{Quad}(V)$, and for every $d_i \in \mathfrak{D}_0 \setminus \mathfrak{D}_0(Z)$ choose a scalar $v_i \in R$. Then there exists a unique $\psi \in \mathrm{End}_b(\mathrm{Quad}(V))$ such that $\psi|Z = \varphi$, $\psi(d_i) = v_i d_i$ for $d_i \notin Z$, and $\psi(h_{ij}) = 0$ for all $h_{ij} \notin Z$.*

Proof. We choose a family $(\psi_{ij} \mid i < j)$ of endomorphisms $\psi_{ij} \in \mathrm{End}_b(E_{ij})$ as follows. When $E_{ij} \subset Z$ we set $\psi_{ij} = \varphi|E_{ij}$. When $E_{ij} \not\subset Z$, whence $h_{ij} \notin Z$, we define $\psi_{ij} : E_{ij} \to E_{ij}$ as $\psi_{ij}(d_i) = \mu_i d_i$ with $\mu_i = v_i$ if $d_i \notin Z$, and $\psi_{ij}(d_i) = \varphi(d_i)$ if $d_i \in Z$. We define $\psi_{ij}(d_j)$ by the same rule, setting $\psi_{ij}(h_{ij}) = 0$. This map ψ_{ij} is the composite of the quasilinear projector $\pi_{\mathrm{QL}}|E_{ij}$ and an endomorphism of the free module $Rd_i + Rd_j$ with prescribed values for d_i, d_j, and thus is well defined. By construction it is clear that for any three different indices i, j, k we have $\psi_{ij}(d_i) = \psi_{ik}(d_i)$. Thus the pasting Lemma 5.6 applies and yields a basic endomorphism ψ of $\mathrm{Quad}(V)$, which clearly extends φ. □

In particular, we can choose in Theorem 5.8 all $v_i = 0$ to obtain an extension $\psi = \widetilde{\varphi}$ of φ to $\mathrm{Quad}(V)$ with $\widetilde{\varphi}(q) = 0$ for all $q \in \mathfrak{B}_0 \setminus \mathfrak{B}_0(Z)$. We call $\widetilde{\varphi}$ the *extension of φ by zero.*

Convention 5.9. *Up to end of §5 we assume, usually without explicitly stating it, that for R and Z one of the two following conditions holds.*

Hypothesis A: *$\mathcal{G}$ is multiplicatively unbounded and Z is a graphic module.*

Hypothesis B: *$\mathcal{G}$ is multiplicatively unbounded and has property (†), cf. Convention 3.8. Here Z can be any basic module.*

Recall that, since $\mathcal{G}$ is assumed to be multiplicatively unbounded, the set $R \setminus \{0\}$ is closed under multiplication ([10, Remark 6.5]).
We are ready for a central result of this section.

Theorem 5.10. *Given a basic endomorphism φ of Z there exists a unique basic projector on Z, denoted by p_φ, such that any $q \in \mathfrak{B}_0(Z)$ has the image $p_\varphi(q) = q$ if $\varphi(q) \neq 0$ and $p_\varphi(q) = 0$ if $\varphi(q) = 0$.*

Proof. a) We assume Hypothesis A. We extend φ to a basic endomorphism ψ of $\mathrm{Quad}(V)$ in some way, which is possible by Theorem 5.8. It suffices to prove the theorem for ψ instead of φ. Restricting p_ψ to Z we then obtain the desired p_φ. Thus we furthermore assume that $Z = \mathrm{Quad}(V)$.
We employ the Pasting Lemma 5.6. For any two indices $i \neq j$ in I let $\varphi_{ij} := \varphi|E_{ij}$. We define a basic projector p_{ij} on E_{ij} as follows. If $\varphi(d_i) \neq 0$, $\varphi(d_j) \neq 0$, $\varphi(h_{ij}) \neq 0$ we take $p_{ij} = \mathrm{id}_{E_{ij}}$. If $\varphi(d_i) \neq 0$, $\varphi(d_j) \neq 0$, but $\varphi(h_{ij}) = 0$ we choose for p_{ij} the quasilinear projection $\pi_{\mathrm{QL}}|E_{ij}$, i.e., $p_{ij}(d_i) = d_i$, $p_{ij}(d_j) = d_j$, $p_{ij}(h_{ij}) = 0$. If $\varphi(d_i) = \varphi(d_j) = 0$, whence also $\varphi(h_{ij}) = 0$, we choose $p_{ij} = 0$.
There remains the case that exactly one of the vectors $\varphi(d_i)$, $\varphi(d_j)$ is not zero, say $\varphi(d_i) = \mu_i d_i$, $\mu_i \neq 0$, $\varphi(d_j) = 0$. Let $\varphi(h_{ij}) = \mu_{ij} h_{ij}$. Applying φ to the relation

$$d_i + d_j + h_{ij} = d_i + d_j,$$

we obtain $\mu_i d_i + \mu_{ij} h_{ij} = \mu_i d_i$. Inserting the vector $x = v_i + c v_j$ for any $c \in R$ we obtain that $\mu_i + c\mu_{ij} = \mu_i$ for all $c \in R$. Since $\mathcal{G}$ is multiplicatively unbounded, this forces $\mu_{ij} = 0$, i.e., $\varphi(h_{ij}) = 0$. Now define p_{ij} as the composite of $\pi_{\mathrm{QL}}|E_{ij}$ with the obvious projection from $Rd_i + Rd_j$ to Rd_i. Then $p_{ij}(d_i) = d_i$, $p_{ij}(d_j) = p_{ij}(h_{ij}) = 0$.
By construction $p_{ij}(d_i) = p_{ik}(d_i)$ for any three different indices $i, j, k \in I$. Thus there exists a basic endomorphism p of $\mathrm{Quad}(V)$ with $p|E_{ij} = p_{ij}$ for any $i \neq j$. It is the projector p_φ we were looking for.

b) We assume now Hypothesis B. By Theorem 4.10 there is a decomposition $Z = Y_1 \oplus Y_2$ with $Y_1 = Z_{\mathrm{link}}$ and Y_2 a free module. Given $\varphi \in \mathrm{End}_b(Z)$, we have $\varphi(Y_i) \subset Y_i$ for $i = 1, 2$, and thus $\varphi = \varphi_1 \oplus \varphi_2$ with $\varphi_i \in \mathrm{End}_b(Y_i)$. As just proved there exists a unique basic projector $\pi_1 = p_{\varphi_1}$ on Y_1 with $\pi_1(q) = q \Leftrightarrow \varphi_1(q) \neq 0$ for all $q \in \mathfrak{B}_0(Y_1)$. Since Y_2 is free we trivially also have a unique basic projector π_2 on Y_2 such that for all $q \in \mathfrak{B}_0(Y_2)$ $\pi_2(q) = q \Leftrightarrow \varphi_2(q) \neq 0$. The projector $p_\varphi := \pi_1 \oplus \pi_2$ on Z has the required property addressed in the theorem. $\square$

We call p_φ the *basic projector associated to* φ. We now strive for a characterization of the image and the kernel of p_φ in terms of the image and kernel of φ.

Definition 5.11. *Given an R-submodule M of* $\mathrm{Quad}(V)$*, let M_b denote the unique maximal basic submodule of* $\mathrm{Quad}(V)$ *contained in M, and let M^b denote the unique minimal basic submodule of* $\mathrm{Quad}(V)$ *containing M. M_b and M^b are respectively called the* basic core *and* basic hull *of M.*

It is obvious that

$$\mathfrak{B}_0(M_b) = \mathfrak{B}_0 \cap M,$$

but determining $\mathfrak{B}_0(M^b)$ can be difficult. For example, if $M = Rq$ with $q \in$ Quad(V), then $\mathfrak{D}_0(M^b)$ consists of all d_i showing up in q_{QL}, while $\mathfrak{H}_0(M^b)$ is the minimal subset of $\mathfrak{H}_0$ appearing in a rigid complement of q_{QL}, in other terms, the minimal subset $\Lambda \subset I^{[2]}$ such that q is Λ^c-quasilinear. But there is an extended class of submodules M of Quad(V) at hands, for which the determination of $\mathfrak{B}_0(M^b)$ is trivial.

Definition 5.12. *We call a submodule M of* Quad(V) diagonal *if*

$$M = \sum_{i \in I} \mathfrak{a}_i d_i + \sum_{i<j} \mathfrak{a}_{ij} h_{ij} \tag{5.4}$$

with ideals (= R-submodules) $\mathfrak{a}_i$, $\mathfrak{a}_{ij}$ of R.

We read off from (5.4) that

$$\mathfrak{D}_0(M^b) := \{d_i \mid \mathfrak{a}_i \neq 0\}, \qquad \mathfrak{H}_0(M^b) := \{h_{ij} \mid \mathfrak{a}_{ij} \neq 0\}.$$

Examples 5.13.

(a) *Every basic module Z is diagonal.*

(b) *Every submodule of* Quad(V) *that is convex in the minimal ordering $\preceq$ of* Quad(V) *is easily seen to be diagonal.*

(c) *If M is a diagonal submodule of a basic module Z, then for any $\varphi \in$ End$_b(Z)$ also $\varphi(M)$ is diagonal.*

(d) *Intersections and sums of families of diagonal submodules of* Quad(V) *are diagonal.*

(e) *$R(d_i + d_j)$ is not diagonal if $i \neq j$. (This module has the basic hull $Rd_i + Rd_j$.)*

Proposition 5.14. *If φ is a basic endomorphism of a basic module Z then $\varphi(Z)$ is a diagonal module and $p_\varphi(Z) = \varphi(Z)^b$. The kernel $\varphi^{-1}(0)$ is itself basic and so*

$$p_\varphi^{-1}(0) = \varphi^{-1}(0) = \varphi^{-1}(0)_b.$$

Proof. It is evident that the modules $\varphi(Z)$ and $\varphi^{-1}(0)$ are diagonal by Examples 5.13.(b) and (c). By definition of p_φ we have

$$\begin{aligned} \mathfrak{B}_0(p_\varphi(Z)) &= \{q \in \mathfrak{B}_0(Z) \mid p_\varphi(q) = q\} \\ &= \{q \in \mathfrak{B}_0(Z) \mid \varphi(q) \neq 0\} \quad = \mathfrak{B}_0(\varphi(Z)^b), \end{aligned}$$

which proves that $p_\varphi(Z) = \varphi(Z)^b$. In the same way

$$\begin{aligned} \mathfrak{B}_0(p_\varphi^{-1}(0)) &= \{q \in \mathfrak{B}_0(Z) \mid p_\varphi(q) = 0\} \\ &= \{q \in \mathfrak{B}_0(Z) \mid \varphi(q) = 0\}, \end{aligned}$$

whence $p_\varphi^{-1}(0) = \varphi^{-1}(0)_b$. But $\varphi^{-1}(0)$ is already basic. Indeed, if $\sum_1^n \lambda_i q_i \in \varphi^{-1}(0)$ with $q_i \in \mathfrak{B}_0$, $\lambda_i \neq 0$, then $\sum_1^n \lambda_i \varphi(q_i) = 0$, whence $\lambda_i \varphi(q_i) = 0$, and so all $\varphi(q_i) = 0$, since R has no zero-divisors. $\square$

Therefore, $p_\varphi = \pi$ can also be characterized by the property that, for all $q \in$Z, $\pi(q) \neq 0$ iff $\varphi(q) \neq 0$.

Example 5.15. *Let $\mu = (\mu_i \mid i \in I) \in R^I$. Then the geometric basic endomorphism γ_μ of* $\mathrm{Quad}(V)$ *(cf. Example 5.2) has the associated basic projector π_J (cf. Notation 5.4) with $J = \{i \in I \mid \mu_i \neq 0\}$.*

We turn to the inverse problem of analyzing the set of basic endomorphisms φ with $p_\varphi = \pi$ for a given basic projector π.

Definition 5.16. *Let Z be a basic module.*

(a) *We denote the set of all basic projectors on Z by $\mathcal{P}_b(Z)$.*

(b) *If $\pi \in \mathcal{P}_b(Z)$, then we call an endomorphism $\varphi \in \mathrm{End}_b(Z)$ with $p_\varphi = \pi$ a* satellite *of π; the set of these φ is denoted by* $\mathrm{Stl}(\pi)$.

It will be helpful to work in $\mathrm{End}_b(Z)$ with a partial ordering finer than the minimal ordering, analogous to our setting for Z itself. Given $\varphi, \psi \in \mathrm{End}_b(Z)$, we define

$$\varphi \leq \psi \quad \Leftrightarrow \quad \forall z \in Z: \ \varphi(z) \leq \psi(z),$$

where on the right hand side " $\leq$" stands for the function ordering on $Z \subset \mathrm{Quad}(V)$. We call this finer relation $\leq$ the *function ordering* on $\mathrm{End}_b(Z)$, in contrast to the *minimal ordering* which is denoted by $\preceq$ and defined as

$$\varphi \preceq \psi \quad \Leftrightarrow \quad \exists \chi \in \mathrm{End}_b(Z): \ \varphi + \chi = \psi.$$

Given $\varphi, \psi \in \mathrm{End}_b(Z)$ with coefficients systems $(\mu_i) \cup (\mu_{ij})$ and $(\upsilon_i) \cup (\upsilon_{ij})$, respectively, then $\varphi \leq \psi$ iff $\mu_i \leq \upsilon_i$, $\mu_{ij} \leq \upsilon_{ij}$ for all i with $d_i \in Z$ and all $\{i, j\}$ with $h_{ij} \in Z$.

Theorem 5.17. *Assume that either Hypothesis A or B holds, and that there exists some $\vartheta \in \mathcal{G}$ with $\vartheta < e$. Let π be a nonzero basic projector on Z with $\mathfrak{B}_0(\pi(Z))$ finite (e.g. $\mathfrak{B}_0(Z)$ is finite, i.e., Z is finitely generated). Then a basic endomorphism φ of Z is a satellite of π iff there exist $\alpha, \beta \in R \setminus \{0\}$ such that*

$$\alpha\pi \leq \varphi \leq \beta\pi;$$

in other terms, $\mathrm{Stl}(\pi) = \mathrm{conv}_{\leq}((R\pi) \setminus \{0\})$. *As usual,* $\mathrm{conv}_{\leq}$ *stands for the convex hull in* $\mathrm{End}_b(Z)$ *(equivalently, in* $\mathrm{End}_R(Z)$*) with respect to the function ordering.*

Proof. Let $\{q_j \mid j \in J\}$ denote the set $\mathfrak{B}_0(Z)$. Let

$$\varphi(q_j) = \mu_j q_j \qquad (j \in J)$$

with $\mu_j \in R$, and write

$$K := \{j \in J \mid \pi(q_j) = q_j\},$$

which by assumption is a finite set. By definition, φ is a satellite of π iff

$$K = \{j \in J \mid \mu_j \neq 0\}.$$

When this holds, set

$$\begin{aligned} e\mu_k &= \min\{e\mu_j \mid j \in K\}, \\ e\mu_\ell &= \max\{e\mu_j \mid j \in K\}. \end{aligned}$$

Set $\beta := e\mu_\ell$. If $e\mu_k < e$ set $\alpha := e\mu_k^2$, otherwise take $\alpha = \vartheta$. For these parameters we have $\alpha \leq \mu_j \leq \beta$ for all $j \in K$, and so $\alpha\pi \leq \varphi \leq \beta\pi$. Conversely, if this holds for $\alpha, \beta \neq 0$, then for any $q \in \mathfrak{B}_0(Z)$ we have $\varphi(q) \neq 0$ iff $\pi(q) \neq 0$, i.e., $\pi(q) = q$. Thus φ is a satellite of π. □

The finiteness assumption in Theorem 5.17 is not essential; it can be easily extended to the case in which V has an infinite basis $\{\varepsilon_i \mid i \in I\}$, as follows:

Corollary 5.18. *Assume that either Hypothesis A or B holds, and that there is some $\vartheta < e$ in $\mathcal{G}$. Let $\pi \in \mathcal{P}_b(Z)$ and $\varphi \in \mathrm{End}_b(Z)$. Then φ is a satellite of π iff for every finite subset K of I there exist $\alpha_K, \beta_K \in R \setminus \{0\}$ such that*

$$\alpha_K(\pi|Z_K) \leq \varphi|Z_K \leq \beta_K(\pi|Z_K),$$

where $Z_K := (Z \cap V_K)^I = \pi_K(Z)$ for $V_K = \sum_{i \in K} R\varepsilon_i$, and π_K is the (geometric) basic projector (5.1) *on* $\mathrm{Quad}(V)$ *with image* $\mathrm{Quad}(V_K)^I$ *(cf. Proposition 5.5).*

Theorem 5.19. *Given $\pi_1, \pi_2 \in \mathcal{P}_b(Z)$, both the minimum $\pi_1 \wedge \pi_2$ and the maximum $\pi_1 \vee \pi_2$ exist in $(\mathrm{End}_b(Z), \leq)$, namely*

$$\pi_1 \wedge \pi_2 = \pi_1\pi_2, \qquad \pi_1 \vee \pi_2 = p_{\pi_1+\pi_2}.$$

Proof. This follows from the fact that for any $q \in \mathfrak{B}_0(Z)$ we have

$$(\pi_1\pi_2)(q) = q \quad \text{iff} \quad \pi_1(q) = q \text{ and } \pi_2(q) = q,$$

while

$$p_{\pi_1+\pi_2}(q) = q \quad \text{iff} \quad \pi_1(q) + \pi_2(q) \neq 0 \quad \text{iff} \quad \pi_1(q) = q \text{ or } \pi_2(q) = q.$$

□

Now the following is immediate.

Scholium 5.20. *Suppose $\pi_1, \pi_2 \in \mathcal{P}_b(Z)$. Then*

(a)
$$\pi_1 \leq \pi_2 \quad \Leftrightarrow \quad \pi_1\pi_2 = \pi_1 \quad \Leftrightarrow \quad \pi_1(Z) \subset \pi_2(Z);$$

(b)
$$(\pi_1 \wedge \pi_2)(Z) = \pi_1(Z) \cap \pi_2(Z),$$
$$(\pi_1 \vee \pi_2)(Z) = \pi_1(Z) + \pi_2(Z),$$

and thus $\pi \mapsto \pi(Z)$ embeds $\mathcal{P}_b(Z)$ as a sublattice into the boolean lattice of all basic submodules of Z. □

Thus we have a restriction map

$$\tau_Z : \mathrm{End}_b(Z) \twoheadrightarrow \mathcal{P}_b(Z), \qquad \varphi \mapsto p_\varphi$$

from the commutative R-algebra $\mathrm{End}_b(Z)$ to the distributive lattice $\mathcal{P}_b(Z)$. Concerning the fibers of τ_Z, i.e., the satellite sets, note that if $\pi_1 \leq \pi_2$ then $\pi_1 \, \mathrm{Stl}(\pi_2) \subset \mathrm{Stl}(\pi_1)$. More generally, for any $\pi_1, \pi_2 \in \mathcal{P}_b(Z)$ we have

$$\pi_1 \, \mathrm{Stl}(\pi_2) \subset \mathrm{Stl}(\pi_1\pi_2).$$

It turns out that each fiber of τ_Z is closed under addition and multiplication. More generally the following holds.

Proposition 5.21. *Let $\pi_1, \pi_2 \in \mathcal{P}_b(Z)$. Then*

$$\begin{aligned} \mathrm{Stl}(\pi_1) + \mathrm{Stl}(\pi_2) &\subset \mathrm{Stl}(\pi_1 \vee \pi_2) \quad \textit{and} \\ \mathrm{Stl}(\pi_1) \cdot \mathrm{Stl}(\pi_2) &\subset \mathrm{Stl}(\pi_1 \pi_2). \end{aligned}$$

Proof. Let $\varphi_1 \in \mathrm{Stl}(\pi_1)$, $\varphi_2 \in \mathrm{Stl}(\pi_2)$. Fixing $q \in \mathfrak{B}_0(Z)$, we have

$$\varphi_1(q) = \mu_1 q, \qquad \varphi_2(q) = \mu_2 q$$

with $\mu_1, \mu_2 \neq 0$, and so $\pi_i(q) = q$ if $\mu_i \neq 0$ $(i = 1, 2)$. Now $(\pi_1 \vee \pi_2)(q) = q$ iff $(\pi_1 + \pi_2)(q) \neq 0$ iff $\mu_1 + \mu_2 \neq 0$ iff $(\varphi_1 + \varphi_2)(q) \neq 0$. This proves that $\varphi_1 + \varphi_2 \in \mathrm{Stl}(\pi_1 \vee \pi_2)$.
Furthermore, $(\pi_1 \pi_2)(q) = q$ iff $\pi_1(q) \neq 0$ and $\pi_2(q) \neq 0$ iff $\mu_1 \neq 0$ and $\mu_2 \neq 0$ iff $\mu_1 \mu_2 \neq 0$ iff $(\varphi_1 \varphi_2)(q) \neq q$, since R has no zero divisors. This proves that $\varphi_1 \varphi_2 \in \mathrm{Stl}(\pi_1 \pi_2)$. □

Moreover, it is evident, say by Corollary 5.18, that for every $\pi \in \mathcal{P}_b(Z)$ the set $\mathrm{Stl}(\pi)$ is convex in the function ordering on $\mathrm{End}_b(Z)$, and that $(R \setminus \{0\})\,\mathrm{Stl}(\pi) \subset \mathrm{Stl}(\pi)$.

6. Modifications of basic endomorphisms

Up to now the only explicit examples of basic endomorphisms, which we have met, are the basic projectors (§3, §4) and the geometric basic endomorphisms (cf. §5, Example 5.2 and Proposition 5.3). We now look for procedures to obtain new basic endomorphisms from old ones. We start with a definition and a lemma valid in any supertropical semiring R. We intensely use the ν-notation, cf. §1.4.

Definition 6.1. *Given $\mu, \upsilon \in R$, we say that υ is* obedient to *μ if*

(1) $\upsilon \leq_\nu \mu$,
(2) $\forall x, y \in R$*:* $\mu x = \mu y \Rightarrow \upsilon x = \upsilon y$.

Remarks 6.2.

(a) *If υ is obedient to μ and ω is obedient to υ, then ω is obedient to μ.*
(b) *If υ is obedient to μ then, for any $\omega \in R$, $\upsilon\omega$ is obedient to $\mu\omega$.*
(c) *We conclude from a) and b) that if υ_1 is obedient to μ_1 and υ_2 is obedient to μ_2, then $\upsilon_1\upsilon_2$ is obedient to $\mu_1\mu_2$.*
(d) *Every $\lambda \leq_\nu 1$ in R is obedient to* 1.
(e) *When R is a supersemifield, every $\upsilon \in R$ is obedient to any $\mu \geq_\nu \upsilon$, where $\mu \in \mathcal{T}$.*
(f) *If υ is obedient to μ and $\upsilon \in \mathcal{T}$, then also $e\upsilon$ is obedient to μ, but most often υ is disobedient to $e\upsilon$.*

Lemma 6.3. *Assume that $\upsilon \in R$ is obedient to $\mu \in R$ and that $a + \mu b = a + \mu c$ for $a, b, c \in R$. Then $a + \upsilon b = a + \upsilon c$.*

Proof. 1) Assume first that $a <_\nu \mu b$. Then

$$a + \mu b = \mu b = a + \mu c$$

which implies $a <_\nu \mu c$. Indeed, otherwise we would have $a \geq_\nu \mu c$, whence

$$a + \mu b = a + \mu c \cong_\nu a <_\nu \mu b,$$

a contradiction. We conclude that $a + \mu c = \mu c$, which implies $\mu b = \mu c$ and then $\upsilon b = \upsilon c$ by Property (2) in Definition 6.1, whence $a + \upsilon b = a + \upsilon c$.

2) The remaining case is $a \geq_\nu \mu b$, in which

$$a \cong_\nu a + \mu b = a + \mu c,$$

whence $a \geq_\nu \mu c$. Thus by Property (1) in Definition 6.1

$$(*) \qquad a \geq_\nu \upsilon b, \qquad a \geq_\nu \upsilon c.$$

We run through three subcases.

2.a: Suppose $a \in \mathcal{G}$. Then $(*)$ implies directly that $a + \upsilon b = a = a + \upsilon c$.

2.b: Suppose $a \in \mathcal{T}$ and $a >_\nu \mu b$. Then

$$a = a + \mu b = a + \mu c,$$

which forces $a >_\nu \mu c$, since $a \in \mathcal{T}$. A fortiori $a >_\nu \upsilon b$ and $a >_\nu \upsilon c$; thus

$$a + \upsilon b = a = a + \upsilon c.$$

2.c: Suppose $a \in \mathcal{T}$ and $a \cong_\nu \mu b$. Then

$$a + \mu b = ea = a + \mu c,$$

which forces $e\mu b = e\mu c$, since $a \in \mathcal{T}$. By Property (2) we obtain $e\upsilon b = e\upsilon c$. Recall from $(*)$ that $\upsilon b \leq_\nu a$. Thus, either $\upsilon b \cong_\nu \upsilon c \cong_\nu a$, which gives $a + \upsilon b = ea = a + \upsilon c$, or $\upsilon b \cong_\nu \upsilon c <_\nu a$, which gives $a + \upsilon b = a = a + \upsilon c$.

We have proved that in all cases $a + \upsilon b = a + \upsilon c$. $\square$

The lemma ensures the following fact about quadratic forms, valid for any module V over a supertropical semiring.

Lemma 6.4. *Assume that $\upsilon \in R$ is obedient to $\mu \in R$. Let q_0, q_1, q_1' be quadratic forms on the R-module V such that $q_0 + \mu q_1 = q_0 + \mu q_1'$ then*

$$q_0 + \upsilon q_1 = q_0 + \upsilon q_1'.$$

Proof. For every $x \in V$ we have $q_0(x) + \mu q_1(x) = q_0(x) + \mu q_1'(x)$ and we conclude by Lemma 6.3 that $q_0(x) + \upsilon q_1(x) = q_0(x) + \upsilon q_1'(x)$. $\square$

In the following we assume as before that V is a free quadratic R-module with base $\{\varepsilon_i \mid i \in I\}$, and that $|I| > 1$, discarding a trivial case.

Theorem 6.5. *Assume that Z is a graphic submodule of* $\mathrm{Quad}(V)$. *As before, we write*

$$\mathfrak{D}_0(Z) := \{d_i \mid i \in K\}, \qquad \mathfrak{H}_0(Z) := \big\{h_{ij} \mid \{i,j\} \in M\big\}, \tag{6.1}$$

with $K \subset I$, $M \subset I^{[2]}$. Assume that φ is a basic endomorphism of Z, having the system of coefficients

$$(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M),$$

and furthermore that a family $(\upsilon_{ij} \mid \{i,j\} \in M)$ of elements of R is given such that υ_{ij} is obedient to μ_{ij} for every $\{i,j\} \in M$. Then there exists a basic endomorphism ψ of Z having the coefficient system

$$(\mu_i \mid i \in K) \cup (\upsilon_{ij} \mid \{i,j\} \in M).$$

Proof. a) We first prove the theorem for the special case that $I = \{1,2\}$ and

$$Z = \mathrm{Quad}(V) = E_{12} = Rd_1 + Rd_2 + Rh_{12}.$$

In this case $\varphi \in \mathrm{End}_b(Z)$ has three coefficients μ_1, μ_2, μ, where $\mu = \mu_{12}$, and we are given an element $\upsilon = \upsilon_{12}$ of R obedient to μ. Then

$$\varphi(d_1) = \mu_1 d_1, \qquad \varphi(d_2) = \mu_2 d_2, \qquad \varphi(h_{12}) = \mu h_{12},$$

and we claim that there is a basic endomorphism ψ of E_{12} with $\psi(d_i) = \varphi(d_i)$, for $i = 1,2$, but $\psi(h_{12}) = \upsilon h_{12}$.
Given $q \in \mathrm{Quad}(V)$ with two presentations

(1) $q = \alpha_1 d_1 + \alpha_2 d_2 + \alpha h_{12} = \alpha_1 d_1 + \alpha_2 d_2 + \beta h_{12}$,

the existence of ψ means that in this situation (1) always

(2) $\mu_1\alpha_1 d_1 + \mu_2\alpha_2 d_2 + \upsilon\alpha h_{12} = \mu_1\alpha_1 d_1 + \mu_2\alpha_2 d_2 + \upsilon\beta h_{12}$

holds. Applying φ to (1) we obtain

(3) $\mu_1\alpha_1 d_1 + \mu_2\alpha_2 d_2 + \mu\alpha h_{12} = \mu_1\alpha_1 d_1 + \mu_2\alpha_2 d_2 + \mu\beta h_{12}$.

Using Lemma 6.4 with

$$q_0 = \mu_1\alpha_1 d_1 + \mu_2\alpha_2 d_2, \qquad q_1 = \alpha h_{12}, \qquad q_1' = \beta h_{12},$$

we see that indeed (2) is a consequence of (3).

b) We employ the Pasting Lemma 5.6 to prove the theorem for $Z = \mathrm{Quad}(V)$ in general. Given $\{i,j\} \in I^{[2]}$, let

$$\varphi_{ij} := \varphi | E_{ij} \in \mathrm{End}_b(E_{ij}),$$

having the coefficients μ_i, μ_j, μ_{ij}. By step a), for every $\{i,j\} \in I^{[2]}$, there exists $\psi_{ij} \in \mathrm{End}_b(E_{ij})$ with coefficients $\mu_i, \mu_j, \upsilon_{ij}$. Suppose i,j,k are different indices, then

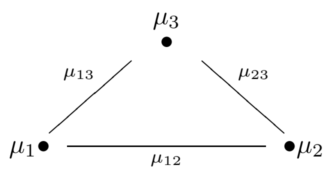
$$\psi_{ij}(d_i) = \mu_i d_i = \psi_{ik}(d_i).$$

Thus Lemma 5.6 applies and gives us a basic endomorphism ψ of $\mathrm{Quad}(V)$ with $\psi|E_{ij} = \psi_{ij}$ for every $\{i,j\} \in I^{[2]}$. This basic endomorphism has the desired system of coefficients $(\mu_i \mid i \in I) \cup (\upsilon_{ij} \mid \{i,j\} \in I^{[2]})$.

c) Finally, we prove the theorem in its full generality. Given an endomorphism $\varphi \in \mathrm{End}_b(Z)$, we extend it to a basic endomorphism $\widetilde{\varphi} \in \mathrm{End}_b(Z)$ with coefficients $(\widetilde{\mu}_i \mid i \in I) \cup (\widetilde{\mu}_{ij} \mid \{i,j\} \in I^{[2]})$, where $\widetilde{\mu}_i = \mu_i$ when $i \in K$, otherwise $\widetilde{\mu}_i = 0$, and $\widetilde{\mu}_{ij} = \mu_{ij}$ for $\{i,j\} \in M$, otherwise $\widetilde{\mu}_{ij} = 0$ (extension by zero, cf. Theorem 5.8). We further extend the family $(\upsilon_{ij} \mid \{i,j\} \in M)$ to a family $(\widetilde{\upsilon}_{ij} \mid \{i,j\} \in M)$ by setting $\widetilde{\upsilon}_{ij} = \upsilon_{ij}$ when $\{i,j\} \in M$ and $\widetilde{\upsilon}_{ij} = 0$ otherwise. It is evident that $\widetilde{\upsilon}_{ij}$ is obedient to $\widetilde{\mu}_{ij}$ for any $\{i,j\} \in I^{[2]}$.
By step b) there exists a basic endomorphism $\widetilde{\psi}$ of $\mathrm{Quad}(V)$ with coefficients $(\widetilde{\mu}_i \mid i \in I) \cup (\widetilde{\upsilon}_{ij} \mid \{i,j\} \in I^{[2]})$. The restriction $\psi := \widetilde{\psi}|Z$ has the desired system of coefficients $(\mu_i \mid i \in K) \cup (\upsilon_{ij} \mid \{i,j\} \in M)$. $\square$

Definition 6.6. *The above endomorphism $\psi \in \mathrm{End}_b(Z)$ is called an $\mathcal{H}$-*modification *of $\varphi \in \mathrm{End}_b(Z)$.*

Comment. *We may interpret the pair (Z, φ) as a weighted graph by assigning weights to the edges and to the vertices of the graph $\Gamma(Z)$ of Z, namely weight μ_i to the vertex $d_i \in \mathrm{Ver}\Gamma(Z)$ and weight μ_{ij} to the edge $h_{ij} \in \mathrm{Edg}\Gamma(Z)$. For example, for $Z = \mathrm{Quad}(V)$ and $I = \{1,2,3\}$ we have the weighted triangle*



If, say, R is cancellative, we obtain all $\mathcal{H}$-modifications of φ by lowering the ν-values of the edges.

Definition 6.7. *Let Z be a graphic submodule of $\mathrm{Quad}(V)$. Let $K \subset I$, and let φ be a basic endomorphism of Z, having the system of coefficients $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$, according to the notation in Theorem 6.5. We say, that a basic endomorphism ψ of Z with coefficients $(\upsilon_i \mid i \in K) \cup (\upsilon_{ij} \mid \{i,j\} \in M)$ is a $\mathcal{D}$-*modification *of φ, if $\mu_{ij} = \upsilon_{ij}$ for $\{i,j\} \in M$, but $\mu_i \leq_\nu \upsilon_i$ for $i \in K$, in other terms $\psi(h_{ij}) = \varphi(h_{ij})$ for all $h_{ij} \in Z$, while $\psi(d_i) \geq_\nu \varphi(d_i)$ for all $d_i \in Z$.*

Open Problem 6.8. *In contrast to the situation of $\mathcal{H}$-modifications we do not know whether for every list of coefficients $(\upsilon_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$, where $\upsilon_i \geq_\nu \mu_i$ for all $i \in K$, a $\mathcal{D}$-modification of φ exists .*

In general we only have the following weaker result.

Theorem 6.9. *Given a basic endomorphism φ on a graphic submodule Z with coefficients $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$, as in the notation of Theorem 6.5, and a family $(\upsilon_i \mid i \in K)$ in R such that $\mu_i \leq \upsilon_i$ (minimal ordering instead of ν-dominance), there exists a basic endomorphism ψ of Z with coefficients $(\upsilon_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$.*

Proof. We prove the theorem for the case that $I = \{1, 2\}$ and $Z = \mathrm{Quad}(V)$. Then, the general case can be obtained as in the proof of Theorem 6.5.
Let
$$Z = \mathrm{Quad}(V) = E_{12} = Rd_1 + Rd_2 + Rh_{12}.$$
and let $\varphi(d_1) = \mu_1 d_1$, $\varphi(d_2) = \mu_2 d_2$, $\varphi(h_{12}) = \mu h_{12}$. For any given $\omega_1, \omega_2 \in R$ we obtain a basic endomorphism χ of Z with coefficients $(\omega_1, \omega_2, 0)$ by composing the quasilinear projection
$$\pi_{\mathrm{QL}} : E_{12} \to Rd_1 + Rd_2$$
with the endomorphism $d_1 \mapsto \omega_1 d_2$, $d_2 \mapsto \omega_2 d_2$ of the free module $Rd_1 + Rd_2$. Then $\psi := \varphi + \chi$ has the coefficients $(\mu_1 + \omega_1, \mu_2 + \omega_2, \mu)$. The elements $\mu_i + \omega_i$ with ω_i running through R, are precisely all $\upsilon_i \geq \mu_i$. $\square$

It is not difficult to provide $\mathcal{D}$-modifications which are not covered by Theorem 6.9.

Examples 6.10. *Set $Z = E_{12} = Rd_1 + Rd_2 + Rh_{12}$.*

(a) *Choose $t_1, t_2 \in \mathcal{T}_e$ with $t_1 \neq 1$, $t_2 \neq 1$. Then $(t_1^2, t_2^2, t_1 t_2)$ is the coefficient system of a geometric basic endomorphism φ_0 of Z, which has an $\mathcal{H}$-modification φ with $(t_1^2, t_2^2, 1)$. Now φ is a $\mathcal{D}$-modification of* $\mathrm{id}_Z \,\hat{=}(1, 1, 1)$. *It is easy to find a supertropical semiring R where such t_i exist with $t_i^2 \neq 1$ $(i = 1, 2)$, cf. e.g.* [3, Construction 3.16].

(b) *Let $\mu \in \mathcal{T}_e$ where $\mu \neq 1$. Then $\mu\,\mathrm{id}_Z$ has the $\mathcal{H}$-modification $(\mu, \mu, 1)$, which is a $\mathcal{D}$-modification of* id_Z.

In the same vein we obtain the following interesting class of basic endomorphisms.

Proposition 6.11. *Assume that $Z = E_{12} = Rd_1 + Rd_2 + Rh_{12}$. Given a triple $(u_1, u_2, v) \in R^3$ with $u_1 \cong_\nu u_2 \cong_\nu 1$, $v <_\nu 1$, there is a well defined basic endomorphism φ of Z which maps any quadratic form*
$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} u_1\alpha_1 & v\alpha \\ & u_2\alpha_2 \end{bmatrix}.$$

Proof. Let $\psi := u_1\pi_1 + u_2\pi_2$, where π_i is the basic projector onto Rd_i, and $\chi := v\,\mathrm{id}_Z$. Then ψ and χ map q to $\left[\begin{smallmatrix} u_1\alpha_1 & 0 \\ & u_2\alpha_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} v\alpha_1 & v\alpha \\ & v\alpha_2 \end{smallmatrix}\right]$, respectively, and so $\varphi := \psi + \chi$ maps q to $\left[\begin{smallmatrix} u_1\alpha_1 & v\alpha \\ & u_2\alpha_2 \end{smallmatrix}\right]$, as desired, since $u_i + v = u_i$, for $i = 1, 2$. $\square$

7. The characteristic projectors

We assume in the whole section that Z is a basic module over a supertropical semiring R such that for every $\varphi \in \mathrm{End}_b(Z)$ there exists a basic projector $p = p_\varphi$ associated to φ, i.e., for any generator $q \in \mathfrak{B}_0(Z)$ of Z we have $p(q) = q$ if $\varphi(q) \neq 0$ and $p(q) = 0$ if $\varphi(q) = 0$. We know from §5 that this holds under one of the hypotheses A, B listed in Convention 5.9, but there may be also other cases where it is true. Our goal is to associate a projector $\widetilde{p}$ on the R-module Z to φ which better reflects the nature of φ than p_φ.

We will use the following simple fact.

Lemma 7.1. *Let $\varphi \in \mathrm{End}_b(Z)$ and $p = p_\varphi$. Then $p\varphi = \varphi$.*

Proof. Since $\mathrm{End}_b(Z)$ is commutative, $p\varphi = \varphi p$. Let $q \in \mathfrak{B}_0(Z)$. If $\varphi(q) \neq 0$ the $p(q) = q$, whence $\varphi(p(q)) = \varphi(q)$. If $\varphi(q) = 0$ then $p(q) = 0$, whence again $\varphi(p(q)) = \varphi(q)$. Thus $\varphi p = \varphi$. $\square$

We define a function $\chi : R \to R$ taking values in $\{0, 1, e\}$, as follows.

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{T}, \\ e & \text{if } x \in \mathcal{G}, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that for every $x \in R$

$$\chi(x)x = x \tag{7.1}$$

and that

$$\chi(x)^2 = \chi(x). \tag{7.2}$$

We emphasize that in this section *it is not necessary to assume that $\mathcal{T} = R \setminus eR$ or $\mathcal{G} = eR \setminus \{0\}$ is closed under multiplication.*
As previously (cf. (6.1)) we use the labeling

$$\mathfrak{D}_0(Z) := \{d_i \mid i \in K\}, \qquad \mathfrak{H}_0(Z) := \big\{h_{ij} \mid \{i,j\} \in M\big\},$$

with $K \subset I$, $M \subset I^{[2]}$.

Theorem 7.2. *Let φ be a basic endomorphism of Z with coefficients*

$$(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M).$$

Then there exists a basic endomorphism $\widetilde{p}$ of Z with coefficients

$$(\chi(\mu_i) \mid i \in K) \cup (\chi(\mu_{ij}) \mid \{i,j\} \in M).$$

Proof. Assume that two presentations of a quadratic form $q \in Z$ are given,

$$\begin{aligned} q &= \sum_{i \in K} \alpha_i d_i + \sum_{\{i,j\} \in M} \alpha_{ij} h_{ij} \\ &= \sum_{i \in K} \alpha_i d_i + \sum_{\{i,j\} \in M} \beta_{ij} h_{ij}. \end{aligned}$$

Then

$$\begin{aligned} \varphi(q) &= \sum_{i \in K} \mu_i \alpha_i d_i + \sum_{\{i,j\} \in M} \mu_{ij} \alpha_{ij} h_{ij} \\ &= \sum_{i \in K} \mu_i \alpha_i d_i + \sum_{\{i,j\} \in M} \mu_{ij} \beta_{ij} h_{ij}. \end{aligned}$$

We introduce the quadratic forms

$$q' = \sum_{i\in K} \chi(\mu_i)\alpha_i d_i + \sum_{\{i,j\}\in M} \chi(\mu_{ij})\alpha_{ij} h_{ij},$$
$$q'' = \sum_{i\in K} \chi(\mu_i)\alpha_i d_i + \sum_{\{i,j\}\in M} \chi(\mu_{ij})\beta_{ij} h_{ij}.$$

Then

$$(*) \qquad \varphi(q') = \varphi(q'')$$

since $\mu_i(\chi(\mu_i)) = \mu_i$, $\mu_{ij}(\chi(\mu_{ij})) = \mu_{ij}$ in consequence of (7.1). Applying p to equation $(*)$, we infer from Lemma 7.1 that $q' = q''$, completing the proof. $\square$

Employing property (7.2) of χ it follows that $(\widetilde{p})^2 = \widetilde{p}$, and so $\widetilde{p}$ is a projector of the R-module Z. We call $\widetilde{p}$ the *characteristic projector of* φ (or, *associated to* φ), and denote it by $\widetilde{p}_\varphi$.
Theorem 7.2 implies the following.

Scholium 7.3. *For any* $q \in \mathfrak{B}_0(Z)$

$$\widetilde{p}_\varphi(q) = \begin{cases} q & \text{if } \varphi(q) \in \mathcal{T}q, \\ eq & \text{if } \varphi(q) \in \mathcal{G}q, \\ 0 & \text{if } \varphi(q) = 0. \end{cases}$$

Corollary 7.4. $\widetilde{p}_\varphi$ *is a satellite of* p_φ.

Proof. Let $\widetilde{p} := \widetilde{p}_\varphi$, $p := p_\varphi$, $q \in \mathfrak{B}_0(Z)$. Trivially $\widetilde{p}(q) \neq 0$ iff $\varphi(q) \in \mathcal{T}q \cup \mathcal{G}q$ and $p(q) \neq 0$ iff $\varphi(q) \neq 0$. Since $\mathcal{T} \cup \mathcal{G} = R \setminus \{0\}$ and the R-module Rq is free, it is evident that $\varphi(q) \in \mathcal{T}q \cup \mathcal{G}q$ iff $\varphi(q) \neq 0$. Thus $\widetilde{p}(q) \neq 0$ iff $p(q) \neq 0$. $\square$

There remains the problem of finding all basic endomorphisms of Z with coefficients in $\{0, 1, e\}$. They then appear as characteristic projectors of basic endomorphisms, in particular of themselves. We denote the set of these projectors by $\mathcal{P}'_b(Z)$. Our goal is to determine all elements of $\mathcal{P}'_b(Z)$ in the case that Z is graphic, but first we have some general observations on $\mathcal{P}'_b(Z)$.
Since $\{0, 1, e\}$ is a subsemiring of R consisting of idempotents, it is evident that $\mathcal{P}'_b(Z)$ is a subsemiring of the commutative semiring $\mathrm{End}_b(Z)$ consisting of idempotents. In the good case that $0, 1, e$ are the only idempotents of R, $\mathcal{P}'_b(b)$ is the set of all idempotents of $\mathrm{End}_b(Z)$. In the degenerate case that $e = 1$, i.e., $eR = R$, we have $\mathcal{P}'_b(Z) = \mathcal{P}_b(Z)$.
We now search for the elements of $\mathcal{P}'_b(Z)$ in the case that $Z = \mathrm{Quad}(V)$ with $I = \{1, 2\}$, so

$$Z = E_{12} = Rd_1 + Rd_2 + Rh_{12}.$$

Identifying each $p \in \mathcal{P}'_b(Z)$ with its triple of coefficients, we have to find out which triples (μ_1, μ_2, μ) with entries in $\{0, 1, e\}$ are elements of $\mathcal{P}'_b(Z)$. We may assume that $e \neq 1$.

The set $\mathcal{P}_b(Z)$ of basic projectors of $Z = E_{12}$ is a subset of $\mathcal{P}'_b(Z)$, closed under multiplication. It consists of the five triples

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,0),\ (1,1,1).$$

The finite sums of these triples constitute the subsemiring $\mathcal{P}''_b(Z)$ of $\mathcal{P}'_b(Z)$, which is generated by $\mathcal{P}_b(Z)$. It contains beside $\mathcal{P}_b(Z)$ the nine triples

$$(e,0,0),\ (0,e,0),\ (e,1,0),\ (1,e,0),$$
$$(e,e,0),\ (e,1,1),\ (1,e,1),\ (e,e,1),\ (e,e,e).$$

But the triples $(1,1,e)$, $(e,1,e)$, $(1,e,e)$ which are not in $\mathcal{P}''_b(Z)$ (as long as $e \neq 1$) are nevertheless in $\mathcal{P}'_b(Z)$, as a consequence of Theorem 6.5 on $\mathcal{H}$-modifications, since $(1,1,1)$, $(e,1,1)$, $(1,e,1)$ are in $\mathcal{P}'_b(Z)$ and e is obedient to 1.
We have verified

Lemma 7.5. *$\mathcal{P}'_b(E_{12})$ contains all triples (μ_1,μ_2,μ) with entries in $\{0,1,e\}$ such that $\mu_1 \neq 0$, $\mu_2 \neq 0$ whenever $\mu \neq 0$, equivalently, such that $\mu^2 \leq_\nu \mu_1\mu_2$.*

Under very mild conditions on the supertropical semiring R we now see that the triples from Lemma 7.5 exhaust the set $\mathcal{P}'_b(Z)$.

Lemma 7.6. *Assume that eR contains an element $z > e$. Then all triples $(\mu_1,0,\mu)$, $(0,\mu_2,\mu)$ with $\mu \neq 0$ are* not *in $\mathcal{P}'_b(Z)$.*

Proof. Suppose that $(e,0,e) \in \mathcal{P}'_b(Z)$, where $Z := E_{12}$. We conclude from the relation $d_1 + d_2 = d_1 + d_2 + h_{12}$ that $ed_1 = ed_1 + eh_{12}$, which means that

$$ex_1^2 = ex_1^2 + ex_1x_2$$

for any $x_1, x_2 \in R$. Inserting here $x_1 = e$ and $x_2 = z \in \mathcal{G}$ with $z > e$, we obtain that $e = e + z = z$, a contradiction. Thus $(e,0,0) \notin \mathcal{P}'_b(Z)$. We conclude by Theorem 6.9 that the triples $(0,0,e)$ and $(1,0,e)$ both are not in $\mathcal{P}'_b(Z)$ since otherwise $(e,0,e)$ would be a $\mathfrak{D}$-modification of one of these triples (since $0 < e$ and $1 < e$), and so an element of $\mathcal{P}'_b(Z)$. This proves that *no* triple $(\mu_1,0,\mu)$ with $\mu \neq 0$ is in $\mathcal{P}'_b(Z)$. By symmetry also no triple $(0,\mu_2,\mu)$ with $\mu \neq 0$ is in $\mathcal{P}'_b(Z)$. □

Theorem 7.7. *Assume that there exists some $z \in eR$ with $z > e$. Furthermore assume that the basic module Z is graphic and $(\mu_i \mid i \in K) \cup (\mu_{ij} \mid \{i,j\} \in M)$ is a tuple of elements in $\{0,1,e\}$. Then there exists a projector $p \in \mathcal{P}'_b(Z)$ with these coefficients iff, whenever $\{i,j\} \in M$ and $\mu_i = 0$ or $\mu_j = 0$, also $\mu_{ij} = 0$, in other terms iff, $\mu_{ij}^2 \leq_\nu \mu_i\mu_j$ for all $\{i,j\} \in M$.*

Proof. Since the Pasting Lemma 5.6 and the Extension Theorem 5.8 hold without any restriction on the supertropical semiring R, we can repeat the first two paragraphs in the proof of Theorem 5.10, showing that it suffices to verify Theorem 7.7 in the very special case that $Z = E_{12}$, which has be done above (Lemmas 7.5 and 7.6). □

8. The basic endomorphisms over a tangible supertropical semifield

In this section we address the "absolute" existence problem for basic endomorphisms, focussing on the case of $Z = \mathrm{Quad}(V)$, $I = \{1, 2\}$, i.e.,

$$Z = \mathrm{Quad}(V) = Rd_1 + Rd_2 + Rh_{12}.$$

At the moment R may be any supertropical semiring. Given a triple $(\mu_1, \mu_2, \mu) \in R^3$, we enquire whether a basic endomorphism φ of Z with coefficients μ_1, μ_2, μ exists, i.e., with

$$\varphi(d_1) = \mu_1 d_1, \quad \varphi(d_2) = \mu_2 d_2, \quad \varphi(h_{12}) = \mu h_{12}.$$

Since $d_1 + d_2 = d_1 + d_2 + h_{12}$, we have the necessary condition for existence of φ that

$$\mu_1 d_1 + \mu_2 d_2 = \mu_1 d_1 + \mu_2 d_2 + \mu h_{12}, \tag{8.1}$$

which means an equivalence of triangular schemes [9]

$$\begin{bmatrix} \mu_1 & \mu \\ & \mu_2 \end{bmatrix} \cong \begin{bmatrix} \mu_1 & 0 \\ & \mu_2 \end{bmatrix}.$$

Problem 8.1. *For which supertropical semirings R is Condition* (8.1) *sufficient for the existence of a basic endomorphism of $Z = \mathrm{Quad}(V) = Rd_1 + Rd_2 + Rh_{12}$ with coefficients μ_1, μ_2, μ?*

While in the last sections, starting from §3, we have been eager to impose only mild restrictions on the supertropical semiring R, we now solve this problem for the class of tangible supersemifields. We start with a general lemma, implementing a previous argument in different context.

Lemma 8.2. *A triple $(\mu_1, \mu_2, \mu) \in R^3$ serves as the coefficients of a basic endomorphism of $Z = Rd_1 + Rd_2 + Rh_{12}$ precisely if, for any $q \in Z$, any two different presentations*

$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta \\ & \alpha_2 \end{bmatrix} \tag{A}$$

ensue an equivalence

$$\begin{bmatrix} \mu_1\alpha_1 & \mu\alpha \\ & \mu_2\alpha_2 \end{bmatrix} \cong \begin{bmatrix} \mu_1\alpha_1 & \mu\beta \\ & \mu_2\alpha_2 \end{bmatrix}. \tag{B}$$

Proof. If (A) implies (B), then the map $\varphi : Z \to Z$ given by

$$\varphi\left(\begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix}\right) = \begin{bmatrix} \mu_1\alpha_1 & \mu\alpha \\ & \mu_2\alpha_2 \end{bmatrix},$$

or in other terms as

$$\varphi(\alpha_1 d_1 + \alpha_2 d_2 + \alpha h_{12}) = \alpha_1\mu_1 d_1 + \alpha_2\mu_2 d_2 + \alpha\mu h_{12},$$

[9] In the terminology of [8, §1]: The "formal" quadratic forms $\left[\begin{smallmatrix} \mu_1 & \mu \\ & \mu_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} \mu_1 & 0 \\ & \mu_2 \end{smallmatrix}\right]$ present the same "functional" quadratic form.

is well defined and obviously is an endomorphism of Z, which is basic with coefficients μ_1, μ_2, μ. Conversely, if such basic endomorphism exists, then (A) implies (B). $\square$

From now on R is a tangible supersemifield, *which we tacitly assume to be nontrivial.*[10] Condition (8.1) means that the quadratic form $q = \left[\begin{smallmatrix}\mu_1 & \mu \\ & \mu_2\end{smallmatrix}\right]$ is quasilinear, $q = q_{\mathrm{QL}}$, equivalently $0 \in C_{12}(q)$. By [8, §7] (cf. there [8, Proposition 7.9], [8, Theorem 7.11]) this holds iff either $\mu^2 \leq_\nu \mu_1\mu_2$ or R is discrete and $\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2$, $\mu_1, \mu_2 \in \mathcal{G}$. We are ready to prove

Theorem 8.3. *Assume that R is a tangible supersemifield and $(\mu_1, \mu_2, \mu) \in R^3$. Then a basic endomorphism of $Z := \mathrm{Quad}(V) = Rd_1 + Rd_2 + Rh_{12}$, $I = \{1, 2\}$, with coefficients (μ_1, μ_2, μ) exists iff*

$$\mu^2 \leq_\nu \mu_1\mu_2,$$

excluding the degenerate case that R is discrete and $\mathcal{T}_e = \{1\}$, in which every $(\mu_1, \mu_2, \mu) \in \mathcal{G}^3$ with $\mu^2 = \pi^{-1}\mu_1\mu_2$ yields a basic endomorphism,

Proof. a) As commented above, we may restrict to the case of triples (μ_1, μ_2, μ) with either $\mu^2 \leq_\nu \mu_1\mu_2$ or R is discrete and $\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2$, where $\mu_1, \mu_2 \in \mathcal{G}$.

b) Assume that $\mu^2 \leq_\nu \mu_1\mu_2$. Given $q \in Z$ with two presentations (A) and $\alpha \neq \beta$, we verify that (B) is valid, assisted again by [8, Theorems 7.11 and 7.12]. We start with the case that $\alpha^2 \leq_\nu \alpha_1\alpha_2$ and $\beta^2 \leq_\nu \alpha_1\alpha_2$. Then

$$(\mu\alpha)^2 \leq_\nu (\mu_1\alpha_1)(\mu_2\alpha_2) \quad \text{and} \quad (\mu\beta)^2 \leq_\nu (\mu_1\alpha_1)(\mu_2\alpha_2),$$

and thus both forms in (B) are quasilinear, whence

$$\begin{bmatrix}\mu_1\alpha_1 & \mu\alpha \\ & \mu_2\alpha_2\end{bmatrix} \cong \begin{bmatrix}\mu_1\alpha_1 & 0 \\ & \mu_2\alpha_2\end{bmatrix} \cong \begin{bmatrix}\mu_1\alpha_1 & \mu\beta \\ & \mu_2\alpha_2\end{bmatrix}.$$

By [8, Theorem 7.11] the remaining case to be considered is that R is discrete and $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$, possibly with α and β interchanged. Then, by the same theorem, $\beta \cong_\nu \alpha$ if $\alpha_2 \in \mathcal{T}$ or $\alpha_1 \in \mathcal{T}$, while $\beta \leq_\nu \alpha$ if both α_1, α_2 are in $\mathcal{G}$. If $\mu^2 <_\nu \mu_1\mu_2$, then

$$(\mu\alpha)^2 \leq_\nu (\mu_1\alpha_1)(\mu_2\alpha_2) \quad \text{and} \quad (\mu\beta)^2 \leq_\nu (\mu_1\alpha_1)(\mu_2\alpha_2),$$

and as before we conclude that (B) is valid.
Let $\mu^2 \cong_\nu \mu_1\mu_2$, then

$$(\mu\alpha)^2 \cong_\nu \pi^{-1}(\mu_1\alpha_1)(\mu_2\alpha_2).$$

If $\beta <_\nu \alpha$ then $\mu\beta <_\nu \mu\alpha$, and $\mu_1\alpha_1, \mu_2\alpha_2 \in \mathcal{G}$, since $\alpha_1, \alpha_2 \in \mathcal{G}$. Thus (B) is valid, cf. [8, Theorem 7.12]. If $\beta \cong_\nu \alpha$ then $\mu\beta \cong_\nu \mu\alpha$, and (B) holds again by [8, Theorem 7.12] (regardless whether $\mu_i\alpha_i$ is ghost or tangible). We conclude that, whenever $\mu^2 \leq_\nu \mu_1\mu_2$, the triple (μ_1, μ_2, μ) yields a basic endomorphism. Finally, we consider the case that R is discrete and $\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2$, where $\mu_1, \mu_2 \in \mathcal{G}$. We choose $\alpha_1, \alpha_2, \alpha, \beta$ in R such that

$$\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2 \cong_\nu \beta^2,$$

[10] This means that $\mathcal{G} \neq \{e\}$. If $\mathcal{G} = \{e\}$, all problems discussed here seem to be trivial.

whence $\alpha \cong_\nu \beta$, but, if possible, $\mu\alpha \neq \mu\beta$. Then

$$(\mu\alpha)^2 \cong_\nu \pi^{-2}(\mu_1\alpha_1)(\mu_2\alpha_2) \cong_\nu (\mu\beta)^2.$$

Thus (B) fails when $\mu\alpha \neq \mu\beta$, and we conclude that (μ_1, μ_2, μ) does *not* provide a basic endomorphism.

For $\mu \in \mathcal{T}$ we may choose $\alpha \in \mathcal{T}$ and $\beta = e\alpha$. When $\mu \in \mathcal{G}$, we need the group $\mathcal{T}_e$ to be nontrivial, since otherwise $\alpha \cong_\nu \beta$ implies $\alpha = \beta$. Therefore we choose $\alpha, \beta \in \mathcal{T}$ with $\alpha \neq \beta$, $\alpha \cong_\nu \beta$, $\alpha^2 \cong_\nu \pi^{-1}\alpha_1\alpha_2$. Thus only in the subcase that $\mu \in \mathcal{G}$, $\mathcal{T}_e = \{1\}$ the triple (μ_1, μ_2, μ) yields a basic endomorphism. □

We now obtain a partial answer, to Problem 8.1 as follows.

Theorem 8.4. *Assume that R is a tangible supersemifield. Let $Z = \mathrm{Quad}(V)$ with $I = \{1, 2\}$, i.e.,*

$$Z := \mathrm{Quad}(V) = Rd_1 + Rd_2 + Rh_{12}.$$

Assume that $(\mu_1, \mu_2, \mu) \in R^3$ is a triple with

$$\mu_1 d_1 + \mu_2 d_2 + \mu h_{12} = \mu_1 d_1 + \mu_2 d_2. \tag{8.2}$$

(a) *If R is dense, then there exists a basic endomorphism of Z with coefficients (μ_1, μ_2, μ).*

(b) *If R is discrete there always exist triples in R^3 satisfying* (8.2) *which do* not *yield basic endomorphisms. These are the $(\mu_1, \mu_2, \mu) \in R^3$ with the following properties:*

I. *If $\mathcal{T}_e \neq \{1\}$:*

$$\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2, \qquad \mu_1, \mu_2 \in \mathcal{G};$$

II. *If $\mathcal{T}_e = \{1\}$:*

$$\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2, \qquad \mu_1, \mu_2 \in \mathcal{G},\ \mu \in \mathcal{T}.$$

Proof. Recall that (8.2) holds iff either $\mu^2 \leq_\nu \mu_1\mu_2$ or R is discrete $\mu^2 \cong_\nu \pi^{-1}\mu_1\mu_2$, and $\mu_1, \mu_2 \in \mathcal{G}$. Comparing this with the conditions in Theorem 8.3 under which (μ_1, μ_2, μ) yields a basic endomorphism of Z, we obtain the claim. □

Theorem 8.3 readily generalizes to an existence theorem for *basic endomorphisms of a linked module Z*, as follows. (We consider first the case that $Z = \mathrm{Quad}(V)$ and adhere to the standard notation (3.1) for $\mathfrak{D}_0$ and $\mathfrak{H}_0$.)

Theorem 8.5. *Assume that R is a tangible supersemifield.*

(a) *If $\mathcal{T}_e \neq \{1\}$, then a family*

$$(\mu_i \mid i \in I) \cup (\mu_{ij} \mid i < j)$$

in R serves as the system of coefficients of a basic endomorphism of $\mathrm{Quad}(V)$ iff

$$\mu_{ij}^2 \leq_\nu \mu_i\mu_j \tag{8.3}$$

for all $i, j \in I$ with $i < j$. The same holds when $\mathcal{T}_e = \{1\}$ and R is dense.

(b) *When R is discrete and $\mathcal{T}_e = \{1\}$, Condition* (8.3) *has to be replaced by the more complicated condition, that for all $i < j$ either $\mu_{ij}^2 \leq_\nu \mu_i\mu_j$ or*

$$\mu_{ij}^2 \cong_\nu \pi^{-1}\mu_i\mu_j, \qquad (\mu_i, \mu_j, \mu_{ij}) \in \mathcal{G}^3.$$

(c) *Mutatis mutandis, all this remains true if we replace* $\mathrm{Quad}(V)$ *by any linked submodule Z of* $\mathrm{Quad}(V)$.

Proof. This follows from Theorem 8.3 by the same line of thought as in parts (b) and (c) of the proof of Theorem 6.5, using the Pasting Lemma 5.6. □

When Z is a basic R-module, and R is a tangible supersemifield, then $Z = Z_{\mathrm{link}} \oplus Y$ with Y a free R-module (cf. Theorem 4.10). Every basic endomorphism φ of Z is a direct sum $\varphi = \varphi_1 \oplus \varphi_2$, with φ_1 and φ_2 basic endomorphisms of Z_{link} and Y respectively. Since Y is free there are no restrictions on the coefficients system of φ_2, and therefore we know all the endomorphisms of Z. We omit the details.

Remark 8.6. *As a consequence of Theorems 8.5 and 6.9, Problem 6.8 about $\mathcal{D}$-modifications has a positive answer when R is a tangible supersemifield. Every change of coefficients $\mu_i \mapsto \upsilon_i$ ($i \in K$), prescribed in Definition 6.7 for a given basic endomorphism φ is realized by a $\mathcal{D}$-modification of φ. Here we do not need to bother about the case that $\mathcal{T}_e = \{1\}$, since then Problem 6.8 vanishes: If $\upsilon_i \leq_\nu \mu_i$, then $\upsilon_i \leq \mu_i$.*

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