Cohomological Support and the Geometric Join

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Abstract. Let M, N be finitely generated modules over a local complete intersection R. Assume that all the modules $\operatorname{Tor}_i^R(M,N)$ are zero for $i>0$. We prove that the cohomological support of $M \otimes_R N$ (in the sense of Avramov-Buchweitz) is equal to the geometric join of the cohomological supports of M, N. This result gives a new connection between two active areas or research, and immediately produces several interesting corollaries. Naturally, it also raises many intriguing new questions about the homological properties of modules over a complete intersection, some of which are investigated in this work.

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1 Introduction

Let $(R, \mathfrak{m}, k)$ be a local complete intersection and M a finitely generated R-module. Inspired by the ideas of Quillen in modular representations context, Avramov and Buchweitz in [6] defined a geometric object attached to the total Ext module $\bigoplus \operatorname{Ext}_R^i(M,k)$. It was originally called the support variety, or cohomological support of M, and denoted by $V^*(M)$ (see Section 2 for details). In this paper, we shall refer to this object as the cohomological support. It is a closed subscheme of $\mathbb{P}_k^{c-1}$, where c is the codimension of R.
The following is an immediate consequence of the theory of cohomological supports developed in [6]: if M, N are non-zero and Tor-independent, i.e. $\operatorname{Tor}_i^R(M,N)=0$ for $i>0$, then $V^*(M), V^*(N)$ are disjoint and

$$\dim V^*(M \otimes_R N) = \dim V^*(M) + \dim V^*(N) + 1.$$

One of our main results in this paper gives a geometric clarification of this formula, by proving that under the above hypothesis, the cohomological support of $M \otimes_R N$ is actually the join of $V^*(M), V^*(N)$. Here, the join of two disjoint closed subschemes is the closure of the union of all the lines joining two points, one from each subscheme.

THEOREM 1.1 (Theorem 3.1,Theorem 4.7). *Let R be a local complete intersection, and $M, N \in \operatorname{mod}(R)$ with $M, N \neq 0$.*

1. *If $\operatorname{Tor}_{>0}(M,N) = 0$, then*
$$V^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N)).$$
2. *If $\operatorname{Ext}^{>0}(M,N) = 0$, then*
$$V^*(\operatorname{Hom}(M,N)) = \operatorname{Join}(V^*(M), V^*(N)).$$

Thus, we provide a link between the theory of cohomological support to a very classical concept of algebraic geometry. Unsurprisingly, this immediately leads to many corollaries and interesting questions, some of them we also address in this work.
The proof of the first part of Theorem 1.1 occupies Section 3. It combines homological and geometric techniques (the preparatory materials are collected in Section 2). In Section 4, we give some quick corollaries including the Theorem 1.1 (2). In Section 5, we collect some examples regarding the case when $\operatorname{Tor}_i(M,N)$ do not vanish. While computing these examples, we noticed certain patterns involving the asymptotic behaviour of $V^*(\operatorname{Tor}_i(M,N))$, motivating us to pose a couple of questions. Furthermore, in Section 5, we list a few situations which give us positive answers to our questions. In these investigations, we show the following.

THEOREM 1.2 (Corollary 5.10). *Let $(R, \mathfrak{m}, k)$ be a local complete intersection of codimension c and M and N finitely generated R-modules. Fix $m \in \mathbb{N}$. If $\operatorname{Tor}_i(M,N)$ eventually has finite length, then there exists a $\nu \in \mathbb{N}$ such that for any $n \geq \nu$*

$$V^*(\operatorname{Tor}_n(N,M)) \cup V^*(\operatorname{Tor}_{n+2m}(N,M)) \cup \cdots \cup V^*(\operatorname{Tor}_{n+2mc}(N,M)) = \bigcup_{i=\nu}^{\infty} V^*(\operatorname{Tor}_i(M,N)).$$

In particular, there exists an $l \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(M,N)) = \bigcup_{i=0}^{l} V^*(\operatorname{Tor}_i(M,N)).$$

2 Background

2.1 The ring of cohomological operators

The study of cohomological supports over complete intersection rings was initiated by Avramov and Buchweitz in [6]. For the entirety of this section, $(R, \mathfrak{m}, k)$ will be a local complete intersection of codimension c such that $\hat{R} = Q/(f_1, \ldots, f_c)$ where Q is a regular local ring and $\underline{f} = f_1, \ldots, f_c$ a regular sequence not contained in the square of the maximal ideal of Q. Let $\tilde{k}$ be the algebraic closure of k. The cohomological support of a finitely generated R-module M is essentially the support of $\mathrm{Ext}_{\hat{R}}(\hat{M}, k)$ as a module over the polynomial ring $S = k[\chi_1, \ldots, \chi_c]$. We will now elaborate on this defintion.
Let X be a finitely generated $\hat{R}$-module. We recall a construction from [13] which gives an action of S on $\mathrm{Ext}_{\hat{R}}(X, k)$. Let $(F_\bullet, \partial)$ be a free resolution of X over $\hat{R}$. Each $F_n = \hat{R}^{i_n}$ and we may view ∂ as a sequence of matrices with entries in $\hat{R}$. Let $\tilde{F}_n = Q^{i_n}$ and $\tilde{\partial}$ be the lift of ∂ to $\tilde{F}_\bullet$. Since $\partial^2 = 0$, we know that $\tilde{\partial}^2$ is a sequence of matrices whose entries are in the ideal $(f_1, \ldots, f_c)$. Thus we may write

$$\tilde{\partial}^2 = \sum_{i=1}^{c} f_i \tilde{\Phi}_i$$

where $\tilde{\Phi}_i$ is a sequence of matrices with entries in Q. Set $\Phi_i = \tilde{\Phi}_i \otimes \hat{R}$. Eisenbud shows in [13] that multiplication by Φ_i is a degree -2 chain map from $F_\bullet \to F_\bullet$. It follows that Φ induces a degree 2 operator χ_i on

$$\mathrm{Ext}_{\hat{R}}(X, k) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\hat{R}}(X, k).$$

It is also shown in [13] that the oeprators χ_i commute with eachother, i.e. $\chi_i \chi_j = \chi_j \chi_i$. Thus we can extend the action of the χ_i linearly to an action of the polynomial ring $S = k[\chi_1, \ldots, \chi_c]$, turning $\mathrm{Ext}(X, k)$ into a graded S-module, where each χ_i is degree 2. Furthermore, Eisenbud shows in [13] that this action is independent of our choices of $F_\bullet$ and $\tilde{\Phi}_i$ and that $\mathrm{Ext}_{\hat{R}}(X, k)$ is actually a finitely generated S-module. The ring S is known as the ring of cohomological operators and has been the focus of much study including [3, 5, 7, 13, 18]. In fact, there are several equivalent methods for constructing the action of S on $\mathrm{Ext}(X, k)$, the first of which was given in [14]: see [9] for a detailed discussion.
The following result shows that the action is actually invariant up to a change of coordinates of the ideal generated by the regular sequence.

Theorem 2.1 ([13, Proposition 1.7], cf. [9, (3.1)]). *Let $f_1, \ldots, f_c$ and $f'_1, \ldots, f'_c$ be two regular sequences of a regular local ring Q which generate the same ideal. Write*

$$f_i = \sum_{j=1}^{c} q_{i,j} f'_j$$

with each $q_{i,j} \in Q$. Letting $\chi_1, \dots, \chi_c$ and $\chi'_1, \dots, \chi'_c$ be the cohomological operators associated to $f_1, \dots, f_c$ and $f'_1, \dots, f'_c$ respectively, we have

$$\chi'_j = \sum_{i=1}^{c} q_{i,j} \chi_i$$

Thus the matrix $(q_{i,j})$ acts as a change of basis matrix, changing the coordinates of $\mathbb{P}^{c-1}_{\tilde{k}}$. When $k = \tilde{k}$, any change of coordinates of $\mathbb{P}^{c-1}_{\tilde{k}}$ corresponds to choosing a different regular sequence which generates the ideal $(f_1, \dots, f_c)$. This important fact is critical to several proofs in this document, thus we state it precisely.

Proposition 2.2. *Assume that k is algebraically closed and set $I = (f_1, \dots, f_c)$ where $f_1, \dots, f_c$ is a regular sequence of a regular local ring Q. Let*

$$\varphi : \mathbb{P}^{c-1}_k \to \mathbb{P}^{c-1}_k$$

be an automorphism, i.e. a change of coordinates. Then there exists a regular sequence $f'_1, \dots, f'_c$ generating I such that $\varphi_*(\chi_i) = \chi'_i$ where $\chi_1, \dots, \chi_c$ and $\chi'_1, \dots, \chi'_c$ are the cohomological operators associated to $f_1, \dots, f_c$ and $f'_1, \dots, f'_c$ respectively.

Proof. Set $\psi = \varphi^{-1}$, and let $\tilde{\varphi}$ and $\tilde{\psi}$ be the lifts of φ and ψ in Q such that $\tilde{\phi} = \tilde{\varphi}^{-1}$. We can regard $\tilde{\varphi}$ and $\tilde{\psi}$ as a matrices whose entries are $q_{i,j} \in Q$ and $p_{i,j} \in Q$ respectively. Set

$$f'_i = \sum_{i=1}^{c} p_{i,j} f_j.$$

By Nakayama's lemma, $f'_1, \dots, f'_c$ generates I, and since there are c elements, $f'_1, \dots, f'_c$ is necessarily a regular sequence. However since $\tilde{\varphi}\tilde{\psi}$ is the identity, we also have

$$f_i = \sum_{j=1}^{c} q_{i,j} f'_j.$$

It follows from Theorem 2.1 that

$$\chi'_j = \sum_{i=1}^{c} q_{i,j} \chi_i = \varphi_*(\chi_j).$$

□

2.2 Cohomological supports

With the machinery of the cohomological operators in place, we may now discuss cohomological supports. Using the notation of the last subsection, we define

$$V(Q, \underline{f}; X) = \{\bar{a} \in \mathbb{A}^c_{\tilde{k}} \mid g(\bar{a}) = 0 \quad \forall g \in \operatorname{ann}_S \operatorname{Ext}(X, k)\}$$

where $\tilde{k}$ is the algebraic closure of k.

Definition 2.3. Let R be a complete intersection ring. Following [6], for a finitely generated R-module M, the *cohomological support*, denoted $V^*(M)$, is the projectivization in $\mathbb{P}_{\tilde{k}}^{c-1}$ of the cone $V(Q, \underline{f}; \hat{M})$. Occasionally, $V_R^*(M)$ will be used to indicate which ring is used to compute the cohomological support.

Proposition 2.4 ([6, Theorem 5.3]). *For any finitely generated R-module M, $V^*(M)$ is independent of the choices of Q, and $\underline{f}$ up to a change of coordinates.*

Remark 2.5. What we call the cohomological support is referred to as the support variety in [6] and other works. In [8], the terminology cohomological support and cohomological variety are both used. Since geometers generally require varieties to be irreducible closed subsets and since $V^*(M)$ is generally not irreducible, we have decided to use the term cohomological support.

Remark 2.6. In [6] and in other works, the authors consider $V^*(M)$ as a cone in $\mathbb{A}_{\tilde{k}}^c$. To facilitate the statement of certain results, we have found it easiest to work in projective space.

The following is a combination of the results [6, Theorem 5.6,Theorem 6.1].

Theorem 2.7. *For finitely generated R-modules M and N, the following are equivalent.*

1. $V^*(M) \cap V^*(N) = \emptyset$
2. $\operatorname{Tor}_{\gg 0}(M, N) = 0$
3. $\operatorname{Ext}^{\gg 0}(M, N) = 0$
4. $\operatorname{Ext}^{\gg 0}(N, M) = 0$

Hence cohomological supports encode homological information about a module. The following result gives another description of cohomological supports.

Theorem 2.8 ([6, Theorem 2.5],[3, Corollary 3.11]). *Suppose that the residue field k is algebraically closed. For any module $M \in \operatorname{mod}(R)$, we have*

$$V^*(M) = \{(a_1, \dots, a_c) \in \mathbb{P}_k^{c-1} \mid \operatorname{pd}_{Q/(\tilde{a}_1 f_1 + \cdots + \tilde{a}_c f_c)} \hat{M} = \infty\}$$

where $\tilde{a}_i$ is a lift in Q of a_i.

From this result and Lemma 2.17, we can easily deduce these corollaries.

Corollary 2.9. *For a finitely generated R-module M, $V^*(M) = \emptyset$ if and only if $\operatorname{pd} M < \infty$. Also $V^*(k) = \mathbb{P}_{\tilde{k}}^{c-1}$.*

Corollary 2.10. *Let $f_1, \dots, f_c$ be a regular sequence of a regular local ring Q, and let $k[\chi_1, \dots, \chi_c]$ be the ring of cohomological operators for $Q/(f_1, \dots, f_c)$. Take n such that $1 \le n \le c$. Let H be the linear space defined by*

$$\chi_{n+1} = \cdots = \chi_c = 0.$$

For any module M over $Q/(f_1, \dots, f_c)$,

$$V^*_{Q/(f_1,\dots,f_n)}(M) = V^*_{Q/(f_1,\dots,f_c)}(M) \cap H.$$

Corollary 2.11. *Suppose $M \in \operatorname{mod}(R)$ and $x \in R$ is regular on M. Then $V^*(M) = V^*(M/xM)$.*

Proof. Let $\tilde{x} \in Q$ be a lift of x. Then $\tilde{x}$ is still regular on M and so M has finite projective dimension over Q/a if and only if $M/\tilde{x}M = M/xM$ does too. The result now follows from Theorem 2.8. □

A generalization of Corollary 2.9 exists involving complexity.

Definition 2.12. For a sequence $(a_n)_{n\geq 0}$ of nonnegative integers, we can define the *complexity*

$$\operatorname{cx}(a_n)_{n\geq 0} = \min\{\deg f \mid f \in \mathbb{Q}[t] \quad a_n \leq f(n) \quad \forall n \gg 0\} + 1.$$

For a module M, we set $\operatorname{cx} M = \operatorname{cx} \beta_n(M)$.

A module has finite projective dimension if and only if $\operatorname{cx} M = 0$. Since R is a complete intersection of codimension c, $\operatorname{cx} k$ equals c.

Proposition 2.13 ([6, Theorem 5.6]). *For any R-module, we have $\dim V^*(M) = \operatorname{cx} M - 1$.*

Remark 2.14. Note that in the previous result, we consider $V^*(M)$ as a closed set of projective space instead of a cone in affine space.

The following are useful results on cohomological supports.

Theorem 2.15 ([10, Corollary 2.3],[8, Theorem 7.8]). *If k is algebraically closed, for each closed set $V \subseteq \mathbb{P}_k^{c-1}$ there is a maximal Cohen-Macaulay module M such that $V^*(M) = V$.*

When working with cohomological supports, it is important to be able to reduce to the case where R is complete and k is algebraically closed. These two results which allow us to do this.

Lemma 2.16. *For any R-module M, we have $V_R^*(M) = V_{\hat{R}}^*(\hat{M})$.*

Lemma 2.17 ([6, Lemma 2.2],[11, App., Théorèm 1, Corollaire])). *There exists a local complete intersection ring $(\tilde{R}, \tilde{\mathfrak{m}}, \tilde{k})$ of codimension c such that $\tilde{R}$ is a flat extension of R, $\mathfrak{m}\tilde{R} = \tilde{\mathfrak{m}}$, and the induced map $k \to \tilde{k}$ is the inclusion of k into its algebraic closure. Furthermore, we have $V_R^*(M) = V_{\tilde{R}}^*(M \otimes \tilde{R})$.*

2.3 Thick subcategories

There is a deep connection between cohomological supports and the thick subcategories of $\operatorname{mod}(R)$. This connection begins with the following result.

Proposition 2.18 ([6, Theorem 5.6]). *If*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

is exact, then

$$V^*(X_i) \subseteq V^*(X_j) \cup V^*(X_l)$$

with $\{i,j,l\} = \{1,2,3\}$*. In particular,* $V^*(M) = V^*(\Omega\, M)$*. Furthermore*

$$V^*(X \oplus Y) = V^*(X) \cup V^*(Y)$$

Definition 2.19. A subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ is thick if

1. $R \in \mathcal{C}$
2. $\mathcal{C}$ is closed under direct summands, that is if $X \oplus Y \in C$ then $X, Y \in \mathcal{C}$
3. $\mathcal{C}$ has the two out of three property, that is if $0 \to X_1 \to X_2 \to X_3 \to 0$ and $X_i, X_j \in \mathcal{C}$, then $X_l \in \mathcal{C}$ with $\{i,j,l\} = \{1,2,3\}$.

Let $\operatorname{Thick} M$ denote the smallest thick category containing M.

The thick subcategories of R are in bijection with the thick subcategories of the triangulated category $\underline{\mathrm{MCM}(R)}$, the stable category of maximal Cohen-Macaulay modules. The category of modules of finite projective dimension is thick. We can generalize this example: by Proposition 2.18, for any $V \subseteq \mathbb{P}^{c-1}_{\tilde{k}}$, the category

$$\{M \in \operatorname{mod}(R) \mid V^*(M) \subseteq V\}$$

is thick. In fact, cohomological supports are used to classify the thick subcategories in the complete intersection in [20], and, in the zero dimensional case, [12, Remark 5.12]. The proceeding result follows from the classification of thick subcategories [20, Remark 10.7].

Theorem 2.20. *For two modules* $M, N \in \operatorname{mod}(R)$*,* $V^*_{R_p}(M_p) \subseteq V^*_{R_p}(N_p)$ *for every* $p \in \operatorname{Sing} R$ *if and only if* $\operatorname{Thick} M \subseteq \operatorname{Thick} N$ *where* $\operatorname{Thick} M$ *and* $\operatorname{Thick} N$ *are the smallest thick subcategories containing* M *and* N*.*

Remark 2.21. The full classification theorem in [20] utilises the full scheme structure of $\mathbb{P}^{c-1}_{k}$ instead of the closed points of $\mathbb{P}^{c-1}_{\tilde{k}}$, a context considered in other works such as [6]. The difference between such approaches are subtle and beyond the scope of this article.

2.4 Geometric join

In this subsection we give attention to another construction central to this paper.

Definition 2.22. Let $U, V \subseteq \mathbb{P}^n_k$ be Zariski closed sets. We define the *join* of U and V to be

$$\operatorname{Join}(U,V) = \overline{\bigcup_{\substack{u\in U \ v\in V\\ u\neq v}} \operatorname{line}(u,v)}$$

where $\operatorname{line}(u,v)$ is the projective line containing u and v.

Remark 2.23. When U and V are disjoint Zariski closed sets, we may simplify this definition to

$$\operatorname{Join}(U,V) = \bigcup_{u \in U \ v \in V} \operatorname{line}(u,v).$$

and we still obtain a closed set, see [15, Example 6.17]. In most contexts in this paper, we will be taking the join of disjoint sets.

Remark 2.24. There is some ambiguity with this definition when V is empty. To that end, we establish the following convention:

$$\operatorname{Join}(U,\emptyset) = U.$$

This is the convention followed in [1]. We justify this convention by considering the join in affine space: the empty set corresponds to the the zero point and the join of the cone and the zero point is simply the original cone.

The following in anopther interesting fact about joins.

Lemma 2.25 ([15, Proposition 6.13,Example 6.14]). *If U and V are irreducible closed sets, then* $\operatorname{Join}(U,V)$ *is also irreducible.*

To visualize the join, consider the following easy examples. The join of two distinct points is a projective line, and the join of two skew lines is a three dimensional projective linear space. In fact, the join of any two linear spaces is the smallest linear space containing both of them.

Theorem 2.26 ([1, 1.1]). *For two closed sets $U, V \subseteq \mathbb{P}^n_k$, we have*

$$\dim \operatorname{Join}(U,V) \leq \dim U + \dim V + 1$$

and if $U \cap V = \emptyset$, then

$$\dim \operatorname{Join}(U,V) = \dim U + \dim V + 1.$$

The converse is not true, and, in fact, it is not known in general when $\dim \operatorname{Join}(U,V) = \dim U + \dim V + 1$ in the case $U \cap V \neq \emptyset$. In particular, an active topic of research is understanding when $\operatorname{Join}(V,V) \neq 2\dim V + 1$.

3 Joins of cohomological supports

In this section, let $(R, \mathfrak{m}, k)$ be a local complete intersection of codimension c. The goal of this section is to prove our first main result of this paper, namely the following theorem.

Theorem 3.1. *Let R be a local complete intersection, and $M, N \in \operatorname{mod}(R)$ with $M, N \neq 0$. If $\operatorname{Tor}_{>0}(M,N) = 0$, then $V^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N))$.*

We prove Theorem 3.1 in a few steps. First we cite a critical fact.

Lemma 3.2 ([19, Proposition 2.1]). *If* $\operatorname{Tor}_{>0}(M,N)=0$*, then* $\operatorname{cx}(M\otimes N)=\operatorname{cx}M+\operatorname{cx}N$.

We now prove a special case of the main result.

Lemma 3.3. *If R has codimension two and* $\operatorname{Tor}_{>0}(M,N)=0$ *with* $M,N\neq 0$*, then*

$$V^*(M\otimes N)=\operatorname{Join}(V^*(M),V^*(N)).$$

Proof. By Lemma 2.16 and Lemma 2.17, we may assume that R is complete and k is algebraically closed. If $\operatorname{pd}M,\operatorname{pd}N<\infty$, then $\operatorname{pd}M\otimes N<\infty$ and the conclusion is clear.

Assume that $\operatorname{pd}M=\operatorname{pd}N=\infty$. Since $V^*(M)$ and $V^*(N)$ are disjoint, nonempty, and lie in $\mathbb{P}^1_k$, we know that $\dim V^*(M)=\dim V^*(N)=0$. Therefore, the complexity of both M and N is one. Since $\operatorname{Tor}_{>0}(M,N)=0$, the complexity of $M\otimes N$ is two, and thus $\dim V^*(M\otimes N)=1$. This means that

$$V^*(M\otimes N)=\mathbb{P}^1_k=\operatorname{Join}(V^*(M),V^*(N)).$$

We now assume that $\operatorname{pd}M<\infty$ and $\operatorname{pd}N=\infty$. Using the conventions in Remark 2.11, we may assume that $M\neq 0$. We wish to show that $V^*(M\otimes N)=V^*(N)$. Letting

$$0\to R^{n_t}\to\cdots\to R^{n_0}\to M\to 0$$

be a free resolution, the sequence

$$0\to N^{n_t}\to\cdots\to N^{n_0}\to M\otimes N\to 0$$

is exact, implying $V^*(M\otimes N)\subseteq V^*(N)$. By Lemma 3.2, we have

$$\operatorname{cx}(M\otimes N)=\operatorname{cx}M+\operatorname{cx}N=\operatorname{cx}N.$$

Therefore $\dim V^*(M\otimes N)$ is the same as $\dim V^*(N)$. So if $V^*(N)$ is irreducible, we are done. In particular, if $\dim V^*(N)=1$, then $V^*(N)=\mathbb{P}^1_k$ and we are done.

So suppose $\dim V^*(N)=0$, that is $V^*(N)=\{p_1,\dots,p_n\}$ where p_i are points. The short exact sequence $0\to\Omega N\to R^m\to N\to 0$ yields the short exact sequence

$$0\to M\otimes\Omega N\to M^m\to M\otimes N\to 0.$$

But since $V^*(M)$ is empty, we have $V^*(M\otimes\Omega N)=V^*(M\otimes N)$ by Proposition 2.18. Thus, since $V^*(N)=V^*(\Omega N)$ we may assume that N is maximal Cohen-Macaulay by replacing N with a sufficiently high syzygy. Therefore, by Theorem 3.1 of [10], we may write $N=N_1\oplus\cdots\oplus N_n$ with $V^*(N_i)=\{p_i\}$. Since each $V^*(N_i)$ is irreducible, we have

$$\begin{aligned}V^*(M\otimes N)&=V^*(M\otimes N_1\oplus\cdots\oplus M\otimes N_n)\\&=V^*(M\otimes N_1)\cup\cdots\cup V^*(M\otimes N_n)=\{p_1\}\cup\cdots\cup\{p_n\}=V^*(N)\end{aligned}$$

which completes the proof. □

Proposition 3.4. *If* $\operatorname{Tor}_{>0}(M,N)=0$ *with* $M,N\neq 0$*, then*

$$\operatorname{Join}(V^*(M),V^*(N))\subseteq V^*(M\otimes N).$$

Proof. By Lemma 2.16 and Lemma 2.17, we may assume that R is complete and k is algebraically closed. We proceed by induction on the codimension of R, which we will denote by c. When $c=1$, the cohomological supports lie in $\mathbb{P}^0_k$, a point. Thus the cohomological support of a module is either that point or empty, depending on whether or not the module has finite projective dimension. Since $\operatorname{Tor}_{>0}(M,N)=0$, the result follows from the equality

$$\operatorname{cx} M+\operatorname{cx} N=\operatorname{cx} M\otimes N.$$

If $c=2$, the statement is true by Lemma 3.3.
Now suppose that $c\geq 3$. It suffices to show that for any hyperplane $H\subseteq\mathbb{P}^{c-1}_k$ we have

$$\operatorname{Join}(V^*(M),V^*(N))\cap H\subseteq V^*(M\otimes N)\cap H$$

Since $c\geq 2$, any hyperplane is a linear space with dimension at least one. Therefore, for any $x\in V^*(M)\cap H$ and $y\in V^*(N)$, the projective line between x and y is also in H. Because of this, we have

$$\operatorname{Join}(V^*(M),V^*(N))\cap H=\operatorname{Join}(V^*(M)\cap H,V^*(N)\cap H).$$

Thus we need to show that

$$\operatorname{Join}(V^*(M)\cap H,V^*(N)\cap H)\subseteq V^*(M\otimes N)\cap H.$$

To that end, we fix a hyperplane H. Now may write $R=Q/(f_1,\dots,f_c)$ where Q is a regular local ring and $f_1,\dots,f_c$ is a regular sequence. By Proposition 2.2, we may change our coordinate system and assume that $H=V(\chi_1)$ where $k[\chi_1,\dots,\chi_c]$ is the ring of cohomological operators. Set $T=Q/(f_2,\dots,f_c)$ and $f=f_1$. Note that T is a complete intersection with $\operatorname{codim} T=c-1$, f is regular on T, and $R=T/(f)$. For any module $X\in\operatorname{mod}(R)$, $V^*_R(X)\cap H=V^*_T(X)$ by Corollary 2.10. Therefore we need to show that

$$\operatorname{Join}(V^*_T(M),V^*_T(N))\subseteq V^*_T(M\otimes N).$$

Since $\operatorname{Tor}^R_{>0}(M,N)=0$, by [4, Lemma 9.3.8], we have $\operatorname{Tor}^T_1(M,N)=M\otimes N$ and $\operatorname{Tor}^T_{>1}(M,N)=0$. It follows that $\operatorname{Tor}^T_{>0}(M,\Omega_T N)=0$. After tensoring $0\to\Omega_T N\to T^t\to N\to 0$ with M, we get the exact sequence

$$0\to M\otimes N\to M\otimes\Omega_T N\to M^t\to M\otimes N\to 0.$$

Thus, by induction and the above exact sequence, we have

$$\begin{aligned}\operatorname{Join}(V^*_T(M),V^*_T(N))&=\operatorname{Join}(V^*_T(M),V^*_T(\Omega_T N))\\&\subseteq V^*_T(M\otimes\Omega_T N)\subseteq V^*_T(M\otimes N)\cup V^*_T(M).\end{aligned}$$

Note, that we can only choose $\Omega_T N$ to not be zero. A similar argument using $\Omega_T M$ gives us

$$\operatorname{Join}(V_T^*(M), V_T^*(N)) \subseteq V_T^*(M \otimes N) \cup V_T^*(N).$$

This implies that

$$\operatorname{Join}(V_T^*(M), V_T^*(N)) \subseteq V_T^*(M \otimes N) \cup (V_T^*(M) \cap V_T^*(N)) = V_T^*(M \otimes N)$$

proving the claim. $\square$

Lemma 3.5. *Suppose that $c = \operatorname{codim} R \geq 2$, R is complete, and k is algebraically closed. Fix $M \in \operatorname{mod}(R)$ such that $V_R^*(M) = q$ for some point $q \in \mathbb{P}_k^{c-1}$. For any $p \in \mathbb{P}_k^{c-1}$ distinct from q, there exists an $L \in MCM$ such that $V_R^*(L) = p$ and $V_R^*(M \otimes L) = \operatorname{Join}(p, q)$.*

Proof. As R is complete, we write $R = Q/(f_1, \dots, f_c)$ where Q is a regular local ring and $\underline{f}$ is a regular sequence. After a change of coordinates, we may assume that $p = (1, 0, 0, \dots, 0)$. Set $T = Q/(f_1)$, $X = \Omega_T^{d-1} k$ where $d = \dim Q$, and $L = X/(f_2, \dots, f_c)X$. We prove that L is our desired module.
First, we show that $V^*(L) = p$. Since $\operatorname{pd}_T L = \infty$, it follows from Theorem 2.8 that $p \in V^*(L)$. Take any point $p' \in \mathbb{P}_k^{c-1}$ such that $p' \neq p$. By Theorem 2.15, there exists a $Y \in \operatorname{mod}(R)$ such that $p' = V^*(Y)$. It follows from Theorem 2.8 that $\operatorname{pd}_T Y < \infty$. Furthermore, we have $\operatorname{Tor}_i^R(Y, L) \cong \operatorname{Tor}_i^T(Y, X)$ for all $i > 0$, and thus $\operatorname{Tor}_{\gg 0}^R(Y, L) = 0$. Therefore $V^*(Y) \cap V^*(L) = \emptyset$ and so $p' \notin V^*(L)$. We now have $V^*(X) = \{p\}$ as claimed.
Set $l = \operatorname{Join}(p, q)$. We now show that $V^*(M \otimes L) = l$. By proposition 3.4, we have $l \subseteq V^*(M \otimes L)$. Take any point $r \notin l$. We claim that r is not in $V^*(M \otimes L)$. After changing coordinates, we may assume that $r = (0, 1, 0, \dots, 0)$. Set $S = Q/(f_1, f_2)$ and $X' = X/f_2X$. Let l' be the projective line defined by p and r. Since $r \notin l$, we have $q \notin l'$. Hence Corollary 2.10 implies that $V_S^*(M) = V_R^*(M) \cap l' = \emptyset$. Therefore, $\operatorname{Tor}_{\gg 0}^{R'}(M, X') = 0$. However, since X' is maximal Cohen-Macaulay over S, Lemma 4.4 implies that $\operatorname{Tor}_{>0}^S(M, X') = 0$. Since $M \otimes L \cong M \otimes X'$, and since $\operatorname{codim} S = 2$, Lemma 3.3 implies that

$$V_S^*(M \otimes L) = V_S^*(M \otimes X') = \operatorname{Join}(V_S^*(M), V_S^*(X')) = V_S^*(X').$$

By Corollary 2.10 and Corollary 2.11, we have

$$V_S^*(X') = V_S^*(L) = V_R^*(L) \cap l' = p.$$

Therefore, we have $p = V_S^*(M \otimes L) = V_R^*(M \otimes L) \cap l'$ by Corollary 2.10. Since r is in l', r is not in $V^*(M \otimes L)$, as desired. $\square$

We now proceed with the proof of the main result of this paper.

Proof of Theorem 3.1. By Lemma 2.16 and Lemma 2.17, we may assume that R is complete and k is algebraically closed. Proposition 3.4 gives us one containment, which leaves us to show the reverse containment:

$$V^*(M \otimes N) \subseteq \operatorname{Join}(V^*(M), V^*(N)).$$

We will proceed by induction using induction on

$$\alpha(M, N) = 2 \operatorname{depth} R - \operatorname{depth} M - \operatorname{depth} N.$$

Assume for the moment that we have shown the base case, i.e. the theorem is true when when $\alpha(M, N) = 0$, which is precisely the case when both M and N are maximal Cohen-Macaulay. Suppose that $\alpha(M, N) > 0$. Then one of the modules, say M, is not maximal Cohen-Macaulay, and

$$\alpha(\Omega\, M, N) = \alpha(M, N) - 1.$$

Tensoring the short exact sequence $0 \to \Omega\, M \to R^s \to M \to 0$ with N yields

$$0 \to \Omega\, M \otimes N \to N^s \to M \otimes N \to 0.$$

By induction, we have the following

$$\begin{aligned} V^*(M \otimes N) &\subseteq V^*(N) \cup V^*(\Omega\, M \otimes N) \\ &\subseteq V^*(N) \cup \operatorname{Join}(V^*(\Omega\, M), V^*(N)) = \operatorname{Join}(V^*(M), V^*(N)) \end{aligned}$$

which yields the desired inclusion.

We now prove the base case. Assume that $\alpha(M, N) = 0$ or, equivalently, that M and N are maximal Cohen-Macaulay modules. First we show the theorem when $V^*(M)$ is simply a single point, say q. Take any $p \notin \operatorname{Join}(V^*(M), V^*(N))$. By Lemma 3.5, there exists maximal Cohen-Macaulay module L such that $V^*(L) = p$ and $V^*(M \otimes L) = \operatorname{Join}(p, q) = \operatorname{Join}(V^*(M), V^*(L))$. However, since $p \notin \operatorname{Join}(V^*(M), V^*(N))$, there are no lines containing p, q and a point in $V^*(N)$. Therefore $V^*(M \otimes L) = \operatorname{Join}(p, q)$ and $V^*(N)$ are disjoint. Since M, N, L are all maximal Cohen-Macaulay, this shows that $\operatorname{Tor}_{>0}(M, L) = 0$ and also $\operatorname{Tor}_{>0}(M \otimes L, N) = 0$. Now let $A_\bullet, B_\bullet, C_\bullet$ be free resolutions of L, M, N respectively. Then, $(M \otimes L) \otimes C_\bullet$ is quasi-isomorphic to $\operatorname{Tot}_\bullet(A_\bullet \otimes B_\bullet \otimes C_\bullet)$ which is quasi-isomorphic to $A_\bullet \otimes (M \otimes N)$. Therefore

$$\operatorname{Tor}_i(L, M \otimes N) \cong \operatorname{Tor}_i(M \otimes L, N) = 0$$

for all $i \gg 0$. This implies that $V^*(M \otimes N)$ does not contain $p = V^*(L)$, yielding the desired containment.

Now we show the general case. Again, take a point $p \notin \operatorname{Join}(V^*(M), V^*(N))$. By Theorem 2.15, there exists an $L \in \operatorname{mod}(R)$ with $V^*(L) = p$. In the previous paragraph, we have shown that $V^*(M \otimes L) = \operatorname{Join}(V^*(M), V^*(L))$. Thus the argument in the previous paragraph still applies, completing the proof. $\square$

4 Corollaries of Theorem 3.1

We now state some interesting corollaries of Theorem 3.1. The following is immediate.

Corollary 4.1. *If N is not zero and* $\operatorname{Tor}_{>0}(M,N)=0$*, then*

$$V^*(M)\subseteq V^*(M\otimes N).$$

From this we derive a plethora of other corollaries.

Corollary 4.2. *If $N\neq 0$ and* $\operatorname{Tor}_{>0}(M,N)=0$*, then the following hold.*

1. $\operatorname{Ext}^{\gg 0}(M\otimes N,L)=0\Rightarrow \operatorname{Ext}^{\gg 0}(M,L)=0$
2. $\operatorname{Ext}^{\gg 0}(L,M\otimes N)=0\Rightarrow \operatorname{Ext}^{\gg 0}(L,M)=0$
3. $\operatorname{Tor}_{\gg 0}(M\otimes N,L)=0\Rightarrow \operatorname{Tor}_{\gg 0}(M,L)=0$

Proof. The previous corollary shows that $V^*(M\otimes N,L)=\emptyset$ implies that $V^*(M,L)=\emptyset$. □

Corollary 4.3. *Suppose* $\operatorname{Tor}_{>0}(M,N)=0$*. If*

$$\operatorname{Sing} R\subseteq \operatorname{supp} N\cup(\operatorname{spec} R\setminus \operatorname{supp} M)$$

then M is in $\operatorname{Thick} M\otimes N$*. In particular, if R is an isolated singularity, then* $\operatorname{Tor}_{>0}(M,N)=0$ *implies that* $M,N\subseteq \operatorname{Thick} M\otimes N$ *when* $M,N\neq 0$*.*

Proof. First note that $\operatorname{Tor}_{>0}(M_p,N_p)=0$ for every $p\in\operatorname{spec} R$. Let $p\in\operatorname{Sing} R$. Then either $p\in\operatorname{supp} N$ or $p\notin\operatorname{supp} M$. Then Corollary 4.1 implies that $V^*_{R_p}(M_p)\subseteq V^*_{R_p}((M\otimes N)_p)$ for all $p\in\operatorname{Sing} R$. The result then follows by Theorem 2.20. □

The next results use the following lemma.

Lemma 4.4. *Suppose R is a complete intersection ring and either M or N is maximal Cohen-Macaualay. If* $\operatorname{Tor}_{\gg 0}(M,N)=0$*, then* $\operatorname{Tor}_{>0}(M,N)=0$*.*

Proof. If not, then for every $n\in\mathbb{N}$, there exists a cosyzygy M' of M such that $\operatorname{Tor}_n(M',N)\neq 0$ and $\operatorname{Tor}_{\gg 0}(M',N)=0$. But this contradicts the fact that complete intersections are AB; see [17, Corollary 3.4]. □

We may also use Theorem 3.1 to construct modules with linear cohomological supports.

Corollary 4.5. *Assume that k is algebraically closed and R is complete. Set $p_i=(0,\dots,1,\dots,0)\in\mathbb{P}^{c-1}_k$ be the point that is one in the ith position and zeros elsewhere. Let L be the affine span of $p_1,\dots,p_n$. Set*

$$X_i=\frac{\Omega^{d-1}_{Q/(f_i)}\,k}{(\Omega^{d-1}_{Q/(f_i)}\,k)(f_1,\dots,\hat{f}_i,\dots,f_c)}$$

where $d = \dim Q$. *Then* $X_1 \otimes \cdots \otimes X_n$ *is maximal Cohen-Macaulay and* $L = V^*(X_1 \otimes \cdots \otimes X_n)$.

Note that by Proposition 2.2, after changing coordinate any linear space of $\mathbb{P}_k^{c-1}$ is of the form of

$$L = V^*(X_1 \otimes \cdots \otimes X_n).$$

Proof. By Theorem 2.8, $V^*(X_i) = \{p_i\}$. We work by induction on n. When $n = 1$, we are done. So assume the statement is true for $n-1$. Let L' be the affine span of $p_1, \ldots, p_{n-1}$. The induction hypothesis implies that $V^*(X_{n-1}) = L'$. Since X_n is maximal Cohen-Macaulay , $\operatorname{Tor}_{>0}(X_1 \otimes \cdots \otimes X_{n-1}, X_n) = 0$ by Lemma 4.4. Then $X_1 \otimes \cdots \otimes X_n$ is maximal Cohen-Macaulay and by Theorem 3.1, we have

$$V^*(X_1 \otimes \cdots \otimes X_n) = \operatorname{Join}(V^*(X_1 \otimes \cdots \otimes X_{n-1}), V^*(X_n)) = \operatorname{Join}(L', p_n) = L$$

proving the claim. $\square$

The main result of this paper also prevents certain tor modules from vanishing.

Corollary 4.6. *Suppose* $M_1, \ldots, M_{c+1}$ *are nonfree maximal Cohen-Macaulay modules. Then for some* $i \in \{1, \ldots, c\}$,

$$\operatorname{Tor}_n(M_1 \otimes \cdots \otimes M_i, M_{i+1}) \neq 0$$

for infinitely many n.

Proof. Proceeding by contradiction, suppose that

$$\operatorname{Tor}_{\gg 0}(M_1 \otimes \cdots \otimes M_i, M_{i+1}) = 0$$

for each $1 \leq i \leq c$. Inducting on i, we will show two statements: first that

$$V^*(M_1 \otimes \cdots \otimes M_i) = \operatorname{Join}(V^*(M_1), \ldots, V^*(M_i))$$

for each i in $\{1, \ldots, c\}$, and second that $V^*(M_1 \otimes \ldots \otimes M_i)$ contains a linear space of dimension $i-1$. When $i = 1$, the statement is trivial. So suppose the statement is true for i. Since R is a complete intersection and each M_{i+1} is maximal Cohen-Macaulay, Lemma 4.4 implies that $\operatorname{Tor}_{>0}(M_1 \otimes \cdots \otimes M_i, M_{i+1}) = 0$. By Theorem 3.1, it follows that

$$\begin{aligned} V^*(M_1 \otimes \cdots \otimes M_i \otimes M_{i+1}) &= \operatorname{Join}((V^*(M_1), \ldots, V^*(M_i)), V^*(M_{i+1})) \\ &= \operatorname{Join}(V^*(M_1), \ldots, V^*(M_{i+1})) \end{aligned}$$

Furthermore, let L be the dimension i linear space in $V^*(M_1 \otimes \cdots \otimes M_i)$ guaranteed by the induction hypothesis. Take $x \in V^*(M_{i+1})$ which exists since M_{i+1} is not free. Now x is not in L and so

$$\operatorname{Join}(L, x) \subseteq V^*(M_1 \otimes \cdots \otimes M_{i+1}).$$

But $\operatorname{Join}(L, x)$ is a linear space of dimension $i+1$, proving the claim.
Now the contradiction is clear, for there is a c-dimensional linear space contained in $V^*(M_1 \otimes \cdots \otimes M_{c+1})$ which is a closed subset of $\mathbb{P}^{c-1}$. $\square$

We now prove our second main result, Theorem 1.1 (2), which is the analogue of Theorem 3.1 for Ext.

Theorem 4.7. *Suppose R is a complete intersection ring and $M, N \in \operatorname{mod}(R)$. If $\operatorname{Ext}^{>0}(M,N) = 0$, then $V^*(\operatorname{Hom}(M,N)) = \operatorname{Join}(V^*(M), V^*(N))$.*

Proof. Since $\operatorname{Ext}^{>0}(M,N) = 0$, [2, Lemma 2.5] implies that M is maximal Cohen-Macaulay. By [16, Proposition 3.6], we have the following exact sequence

$$\operatorname{Tor}_2(\operatorname{Tr} M, N) \to \operatorname{Hom}(M,R) \otimes N \to \operatorname{Hom}(M,N) \to \operatorname{Tor}_1(\operatorname{Tr} M, N) \to 0$$

where $\operatorname{Tr} M$ is the Auslander-Bridger transpose of M. However, since R is Gorenstein, $\operatorname{Tr} M$ is an inverse syzygy of M, i.e. there exists an exact sequence

$$0 \to M \to F \to G \to \operatorname{Tr} M \to 0$$

with F, G free. So $\operatorname{Tor}_{\gg 0}(\operatorname{Tr} M, N) = 0$. Since $\operatorname{Tr} M$ is again Cohen-Macaulay, it follows from Lemma 4.4 that $\operatorname{Tor}_{>0}(\operatorname{Tr} M, N) = 0$. Therefore, the above short exact sequence gives us the isomorphism $M^* \otimes N \cong \operatorname{Hom}(M,N)$.
Since $V^*(M) = V^*(M^*)$, we know that $\operatorname{Tor}_{\gg 0}(M^*, N) = 0$. Since M is maximal Cohen-Macaulay, Lemma 4.4 implies $\operatorname{Tor}_{>0}(M^*, N) = 0$. Therefore, by Theorem 3.1 we have

$$\begin{aligned} V^*(\operatorname{Hom}(M,N)) &= V^*(M^* \otimes N) \\ &= \operatorname{Join}(V^*(M^*), V^*(N)) = \operatorname{Join}(V^*(M), V^*(N)) \end{aligned}$$

□

This result provides an elementary proof of the following result.

Corollary 4.8. *If $\operatorname{Ext}^{>0}(M,N) = 0$, then $\operatorname{cx} \operatorname{Hom}(M,N) = \operatorname{cx} M + \operatorname{cx} N$.*

Proof. It follows from the previous theorem, Theorem 2.26, and Proposition 2.13 that

$$\begin{aligned} \operatorname{cx} \operatorname{Hom}(M,N) &= \dim \operatorname{Join}(V^*(M), V^*(N)) + 1 \\ &= \dim V^*(M) + \dim V^*(N) + 2 = \operatorname{cx} M + \operatorname{cx} N \end{aligned}$$

□

5 Questions and examples

What happens to Theorem 3.1 if we remove the assumption that all the Tor modules vanish? The following two examples show that in general neither containment holds.

Example 5.1. Let k be a field and set $R = k[x,y]/(xy)$. Now the modules $M = R/(x+y)$ and $N = R/(x-y)$ have finite projective dimension. However, we have

$$\operatorname{Join}(V^*(M), V^*(N)) = \emptyset \not\supseteq V^*(M \otimes N) = \mathbb{P}^0_{\tilde{k}}$$

showing that $V^*(M \otimes N)$ is not always contained in $\operatorname{Join}(V^*(M), V^*(N)$, even if $\operatorname{Tor}_{\gg 0}(M,N) = 0$.

Example 5.2. Set $R = \mathbb{Q}[a,b,c]/(a^2 - b^2, b^3 - c^3)$ and

$$M = \operatorname{coker} \begin{bmatrix} 8ab^2c^2 + 4abc^3 + 6b^2c^3 + 8ac^4 + 6bc^4 + c^5 & 3ab + 4b^2 + 7ac + 7bc + 4c^2 \\ 4ab^2c^2 + 6abc^3 + 9b^2c^3 + ac^4 + 9bc^4 + 4c^5 & 4ab + 5b^2 + 3ac + 5bc + 5c^2 \end{bmatrix}$$

$$N = \frac{R}{(8ab^2c + 8b^2c^2 + 6ac^3 + 5bc^3 + c^4, 3ab + 2b^2 + 3ac + 2bc + 9c^2)}.$$

An easy computation in Macaulay2 shows that

$$\operatorname{cx} M = 0 \qquad \operatorname{cx} N = 2 \qquad \operatorname{cx} M \otimes N = 1.$$

This shows that $\operatorname{Join}(V^*(M), V^*(N)) = V^*(N) \not\subseteq V^*(M \otimes N)$.

We now give an example where none of the modules involved have finite projective dimension.

Example 5.3. Set $R = \mathbb{Q}[a,b,c,d]/(a^2 - b^2, b^2 - c^2, d^2)$ and define the ideal

$$I = \left(\frac{3}{5}a + \frac{8}{7}b + \frac{5}{2}c, 2a + \frac{1}{2}b + 3c, d \right).$$

A simple computation in Macaulay2 shows that

$$V^*(I) = V(3740x_1 + 477x_2)$$

$$V^*(I \otimes I) = V(0) = \mathbb{P}^2_{\tilde{\mathbb{Q}}}$$

Where $\tilde{\mathbb{Q}}[x_1, x_2, x_3]$ is the ring of cohomological operators over the algebraic closure of $\mathbb{Q}$. Since $V^*(I)$ is linear, we have

$$\operatorname{Join}(V^*(I), V^*(I)) = V^*(I) \not\supseteq V^*(I \otimes I).$$

Example 5.4. Let $R = \mathbb{Q}[a,b,c]/(a^2, b^2, c^2)$ and $I = (b)$ and $J = (ab)$. An easy computation yields $V^*(R/I) = V(x_1, x_3)$ and $V^*(R/J) = V(x_1)$ where $\tilde{\mathbb{Q}}[x_1, x_2, x_3]$ is the ring of cohomological operators over the algebraic closure of $\mathbb{Q}$. Now because $V^*(R/J)$ is a linear space containing $V^*(R/I)$, we have

$$\begin{aligned} &\operatorname{Join}(V^*(R/I), V^*(R/J)) = V^*(R/J) \not\subseteq \\ &\qquad \not\subseteq V^*(R/J \otimes R/I) = V^*(R/(I+J)) = V^*(I). \end{aligned}$$

The authors wondered if there was a relation between the asymptotic behaviour of $V^*(\mathrm{Tor}_i(M,N))$ and $\mathrm{Join}(V^*(M),V^*(N))$. Investigations using Macaulay2 compelled them to ask the following questions.

Question 1. *Does*

$$\bigcup_{i=0}^{n} V^*(\mathrm{Tor}_i(M,N))$$

stabilize as n tends to infinity?

Question 2. *For any modules M and N, do we have the following?*

$$\mathrm{Join}(V^*(M),V^*(N)) \subseteq \bigcup_{i=0}^{\infty} V^*(\mathrm{Tor}_i(M,N))$$

It seems that these statements cannot be made any stronger. For instance, Example 5.3 shows that we cannot hope to replace the containment in Question 2 with equality. Similarly, the following example shows that $V^*(\mathrm{Tor}_i(M,N))$ need not stabelize.

Example 5.5. Let k be a field and set $R = k[x,y]/(xy)$. It is easy to show that $\mathrm{Tor}_{\mathrm{odd}}(R/(x),R/(y)) = 0$ and $\mathrm{Tor}_{\mathrm{even}}(R/(x),R/(y)) \cong k$. The projective dimension of the former is obviously finite, and the projective dimension of the latter is infinite. Thus $V^*(\mathrm{Tor}_i(R/(x),R/(y)))$ cannot stabilize.

For closed sets $U,V \in \mathbb{P}^n_k$, it is known that $\dim \mathrm{Join}(U,V) \le \dim U + \dim V + 1$. It is not known when precisely this equality is strict. This question is particularly interesting when $U = V$ and has been the subject of much research. As the following shows, Question 1 and Question 2 are actually related to this topic.

Proposition 5.6. *Suppose Question 1 and Question 2 have positive answers. Then for any modules M and N over a complete intersection ring,*

$$\dim \mathrm{Join}(V^*(M),V^*(N)) \le \max_{i\in\mathbb{N}} \operatorname{cx} \mathrm{Tor}_i(M,N) - 1.$$

Proof. The result is obvious after recalling that $\dim V^*(\mathrm{Tor}_i(M,N)) = \operatorname{cx} \mathrm{Tor}_i(M,N) - 1$. □

The rest of this paper will discuss special cases where we know Question 1 and 2 to be true. First, we note that Question 1 has a positive answer when R is a hypersurface, because over such rings $\mathrm{Tor}_i(M,N)$ is eventually periodic.

Proposition 5.7. *Questions 1 and 2 have positive answers when $\mathrm{Tor}_{\gg 0}(M,N) = 0$.*

Proof. The first question is trivially true in this case. We prove that the second question is true using induction on the smallest n such that $\operatorname{Tor}_{>n}(M,N)=0$. When $n=0$, the the statement follows from Theorem 3.1. So suppose $n>0$. Then we have $\operatorname{Tor}_{>n-1}(\Omega M,N)=0$ and so by induction, we have

$$\begin{aligned}\operatorname{Join}(V^*(M),V^*(N)) &= \operatorname{Join}(V^*(\Omega M),V^*(N))\\ &\subseteq \bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(\Omega M,N))\\ &= \bigcup_{i=2}^{\infty} V^*(\operatorname{Tor}_i(M,N)) \cup V^*(\Omega M\otimes N).\end{aligned}$$

Note that M is not free and so ΩM is not zero. The short exact sequence

$$0\to \Omega M\to R^m\to M\to 0$$

yields

$$0\to \operatorname{Tor}_1(M,N)\to \Omega M\otimes N\to N^m\to M\otimes N\to 0.$$

It follows that $V^*(\Omega M\otimes N)\subseteq V^*(N)\cup V^*(M\otimes N)\cup V^*(\operatorname{Tor}_1(M,N))$ and hence

$$\operatorname{Join}(V^*(M),V^*(N))\subseteq \bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(M,N))\cup V^*(N).$$

Similarly, we have

$$\operatorname{Join}(V^*(M),V^*(N))\subseteq \bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(M,N))\cup V^*(M)$$

but since $V^*(M)\cap V^*(N)=\emptyset$, this shows the desired result. $\square$

It is interesting that the corresponding statement in Question 1 holds for Ext. We thank Mark Walker for bringing the following argument to our attention.

Proposition 5.8. *Let $(R,\mathfrak{m},k)$ be a local complete intersection of codimension c and M and N finitely generated R modules. Fix $m\in\mathbb{N}$. There exists a $\nu\in\mathbb{N}$ such that for any $n\geq\nu$*

$$\begin{aligned}V^*(\operatorname{Ext}^n(N,M))\cup V^*(\operatorname{Ext}^{n+2m}(N,M))\cup\cdots\cup V^*(\operatorname{Ext}^{n+2mc}(N,M))\\ = \bigcup_{i=\nu}^{\infty} V^*(\operatorname{Ext}^i(M,N)).\end{aligned}$$

In particular, there exists an $l\in\mathbb{N}$ such that

$$\bigcup_{i=0}^{\infty} V^*(\operatorname{Ext}^i(M,N)) = \bigcup_{i=0}^{l} V^*(\operatorname{Ext}^i(M,N)).$$

When $m = 1$, the first statement shows that the union of c terms of the sequences $V^*(\mathrm{Ext}^{2i}(M,N))$ and $V^*(\mathrm{Ext}^{2i+1}(M,N))$ stabilises.

Proof. Let $S = R[\chi_1,\dots,\chi_c]$ be the ring of cohomological operators. Recall that $\mathrm{Ext}(M,N)$ is a graded S-module and that each χ_i has degree 2. Take any $m \in \mathbb{N}$, and let

$$K_\bullet = K_\bullet(\chi_1^m,\dots,\chi_c^m;\mathrm{Ext}(M,N))$$

be the Koszul complex of powers of the cohomological operators on the S-module $\mathrm{Ext}(M,N)$. Note that this is a complex of graded S-modules. Therefore, in degree n, we get a complex of R-modules

$$0 \to \mathrm{Ext}^{n+(c+1)m}(M,N) \to \mathrm{Ext}^{n+(c)m}(M,N)^{a_1} \to \mathrm{Ext}^{n+(c-1)m}(M,N)^{a_2} \\ \to \cdots \to \mathrm{Ext}^{n+m}(M,N)^{a_{c-1}} \to \mathrm{Ext}^{n}(M,N) \to 0$$

where $a_i = \binom{c}{i}$.
Since $\mathrm{Ext}(M,N)$ is finitely generated over $R[\chi_1,\dots,\chi_n]$, we know that the homology of $K_\bullet$ is annihilated by some power of the ideal $(\chi_1,\dots,\chi_c)$, and thus is concentrated in low degrees. In particular, the complex $K_\bullet$ is exact in all degrees larger than some constant $\nu \in \mathbb{N}$. Since the differentials of $K_\bullet$ have degree $2m$, this means that for any $n \geq \nu$, the above complex is exact.
We can conclude two statements:

$$V^*(\mathrm{Ext}^{n+(c+1)m}(M,N)) \subseteq \bigcup_{i=0}^{c} V^*(\mathrm{Ext}^{n+im}(M,N))$$

$$V^*(\mathrm{Ext}^{n}(M,N)) \subseteq \bigcup_{i=1}^{c+1} V^*(\mathrm{Ext}^{n+im}(M,N)).$$

Working inductively, and shifting n, we can use these two statements to prove the first claim. The second claim follows by taking $l = \nu + n + (c+1)m$. $\square$

The idea of taking the Koszul complex over the ring of cohomological operators originates in [7]

Remark 5.9. The same argument works in a more general situation. Let R be Noetherian ring which is not necessarily commutative or local. Suppose a Noetherian graded commutative ring S concentrated in non-negative degrees acts on $D(R)$ (see [8] for the relevant definitions and examples). Suppose further that for complexes M,N, $\mathrm{Ext}(M,N)$ is finitely generated as an S module. Letting $x_1,\dots,x_c$ be a generating set of ideal $S^{>0} \subseteq S$, the Koszul complex

$$K_\bullet(x_1^m,\dots,x_c^m;\mathrm{Ext}(M,N))$$

yields a similar conclusion regarding the objects $\mathrm{supp}_S\,\mathrm{Ext}(\mathrm{Ext}^i(M,N),L)$ where L is an object in $D(R)$.

Under certain conditions, we prove a dual version of Proposition 5.8.

Corollary 5.10. *Let $(R, \mathfrak{m}, k)$ be a local complete intersection of codimension c and M and N finitely generated R-modules. Fix $m \in \mathbb{N}$. If $\operatorname{Tor}_i(M,N)$ eventually has finite length, then there exists a $\nu \in \mathbb{N}$ such that for any $n \geq \nu$*

$$V^*(\operatorname{Tor}_n(N,M)) \cup V^*(\operatorname{Tor}_{n+2m}(N,M)) \cup \cdots \cup V^*(\operatorname{Tor}_{n+2mc}(N,M)) = \bigcup_{i=\nu}^{\infty} V^*(\operatorname{Tor}_i(M,N)).$$

In particular, there exists an $l \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(M,N)) = \bigcup_{i=0}^{l} V^*(\operatorname{Tor}_i(M,N)).$$

This gives an affirmative answer to Question 1 in the isolated singularity case.

Proof. By replacing M and N with high enough syzygies, we may assume that M and N are maximal Cohen-Macaulay. We claim two things. First, we claim that $\operatorname{Ext}^i(M,N^*)$ also eventually has finite length where $N^* = \operatorname{Hom}(N,R)$. Second, we claim that eventually

$$\operatorname{Ext}^d(\operatorname{Ext}^i(M,N^*),R) \cong \operatorname{Tor}_{i+d}(M,N).$$

By [6, Theorem 5.6], it follows that eventually

$$V^*(\operatorname{Ext}^i(M,N^*)) = V^*(\operatorname{Ext}^d(\operatorname{Ext}^i(M,N^*),R)),$$

and the corollary is now clear.

To prove the first claim, note that $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) \cong \operatorname{Tor}_i(M,N)_\mathfrak{p}$ eventually vanishes for every $\mathfrak{p} \in \operatorname{spec} R\backslash\mathfrak{m}$. Since $R_\mathfrak{p}$ is also complete intersection, [17, Theorem 2.1, Proposition 4.3] and implies that $\operatorname{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p},N^*_\mathfrak{p})$ vanishes for all $i > \dim R_\mathfrak{p}$. Therefore, $\operatorname{Ext}^i(M,N^*)$ is eventually finite length.

Now we prove the second claim. Let $F_\bullet$ and $G_\bullet$ be resolutions of M and N respectively, and let $I^\bullet$ be an injective resolution of R. We have the following quasi-isomorphism

$$F_\bullet \otimes G_\bullet \cong \operatorname{Hom}(\operatorname{Hom}(F_\bullet \otimes G_\bullet, R), R) \simeq \operatorname{Hom}(\operatorname{Hom}(F_\bullet, \operatorname{Hom}(G_\bullet, R)), I_\bullet) \simeq \operatorname{Hom}(\operatorname{Hom}(F_\bullet, N^*), I^\bullet)$$

where the second to last quasi-isomorphism is because R is Gorenstein and N is maximal Cohen-Macaulay. Letting

$$E^0_{i,j} = \operatorname{Hom}(\operatorname{Hom}(F_i, N^*), I^{-j})$$

we get a spectral sequence converging to $\operatorname{Tor}_{i-j}(M,N)$. Note that since $I^\bullet$ is a bounded complex, the sequence does indeed converge. Furthermore, we have

$$E^2_{i,j} = \operatorname{Ext}^{-j}(\operatorname{Ext}^i(M,N^*),R).$$

However, since eventually $\mathrm{Ext}^i(M, N^*)$ has finite length, for $i \gg 0$, $E^2_{i,j} = 0$ for all $j \neq -d$. Furthermore, since $E^0_{i,j} = 0$ for all j not in $[-d, 0]$, it follows that eventually

$$\mathrm{Tor}_{i+d}(M, N) = E^2_{i,-d} = \mathrm{Ext}^d(\mathrm{Ext}^i(M, N^*), R).$$

$\square$

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References

[1] Bjørn Ådlandsvik, *Joins and higher secant varieties*, Math. Scand. 61 (1987), no. 2, 213–222. MR 947474 (89j:14030)

[2] Tokuji Araya, *A homological dimension related to AB rings*, (2012), arxiv:1204.4513v1.

[3] L. L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. 96 (1989), no. 1, 71–101. MR 981738 (90g:13027)

[4] Luchezar L. Avramov, *Infinite free resolutions [mr1648664]*, Six lectures on commutative algebra, Mod. Birkhäuser Class., Birkhäuser Verlag, Basel, 2010, pp. 1–118. MR 2641236

[5] Luchezar L. Avramov and Ragnar-Olaf Buchweitz, *Homological algebra modulo a regular sequence with special attention to codimension two*, J. Algebra 230 (2000), no. 1, 24–67. MR 1774757 (2001g:13032)

[6] ———, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), no. 2, 285–318. MR 1794064 (2001j:13017)

[7] Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 67–114 (1998). MR 1608565 (99c:13033)

[8] Luchezar L. Avramov and Srikanth B. Iyengar, *Constructing modules with prescribed cohomological support*, Illinois J. Math. 51 (2007), no. 1, 1–20. MR 2346182 (2008j:13036)

[9] Luchezar L. Avramov and Li-Chuan Sun, *Cohomology operators defined by a deformation*, J. Algebra 204 (1998), no. 2, 684–710. MR 1624432 (2000e:13021)

[10] Petter Andreas Bergh, *On support varieties for modules over complete intersections*, Proc. Amer. Math. Soc. 135 (2007), no. 12, 3795–3803 (electronic). MR 2341929 (2008g:13022)

[11] Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1983, Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets. [Commutative algebra. Chapter 8. Dimension. Chapter 9. Complete Noetherian local rings]. MR 722608 (86j:13001)

[12] Jon F. Carlson and Srikanth B. Iyengar, *Thick subcategories of the bounded derived category of a finite group*, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2703–2717. MR 3301878

[13] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. 260 (1980), no. 1, 35–64. MR 570778 (82d:13013)

[14] Tor H. Gulliksen, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. 34 (1974), 167–183. MR 0364232 (51 #487)

[15] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR 1416564 (97e:14001)

[16] Robin Hartshorne, *Coherent functors*, Adv.in Math. 140 (1998), 44–94.

[17] Craig Huneke and David A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. 93 (2003), no. 2, 161–184. MR 2009580 (2004k:13039)

[18] Vikram Bhagvandas Mehta, *Endomorphisms of complexes and modules over gorenstein rings*, ProQuest LLC, Ann Arbor, MI, 1976, Thesis (Ph.D.)–University of California, Berkeley. MR 2626509

[19] Claudia Miller, *Complexity of tensor products of modules and a theorem of Huneke-Wiegand*, Proc. Amer. Math. Soc. 126 (1998), no. 1, 53–60. MR 1415354 (98c:13022)

[20] Greg Stevenson, *Subcategories of singularity categories via tensor actions*, Compos. Math. 150 (2014), no. 2, 229–272. MR 3177268

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