Torsion 1-cycles and the coniveau spectral sequence

Shouhei Ma

Received: December 9, 2016
Revised: July 18, 2017

Communicated by Thomas Geisser

Abstract. We relate the torsion part of the Abel-Jacobi kernel in the Griffiths group of 1-cycles to a birational invariant analogous to the degree 4 unramified cohomology and an invariant associated to the generalized Hodge conjecture in degree $2\dim(X) - 3$. We also describe in terms of $\mathcal{H}$-cohomology the Griffiths group of 1-cycles and the group of torsion cycles algebraically equivalent to zero of arbitrary dimension.

2010 Mathematics Subject Classification: 14C25, 14C15, 14C35
Keywords and Phrases: 1-cycles, Griffiths group, Abel-Jacobi map, coniveau spectral sequence, $\mathcal{H}$-cohomology, birational invariant

1 Introduction

In this paper we are interested in a connection between algebraic cycles and birational invariant, or more specifically, a torsion subgroup of the Griffiths group of 1-cycles and a certain cohomology group which can be thought of as a "homology counterpart" of the degree 4 unramified cohomology.

Let X be a smooth complex projective variety of dimension d. Let $\mathrm{Griff}_1(X) = \mathrm{CH}_1(X)_{hom}/A_1(X)$ be the Griffiths group of 1-cycles, the group of 1-cycles homologous to zero modulo algebraic equivalence. Let $\mathcal{T}_1(X) \subset \mathrm{Griff}_1(X)$ be the image of the group of torsion 1-cycles which is homologous to zero and whose Abel-Jacobi invariant is also zero. This subtle invariant, measuring deviation of torsion 1-cycles with null Deligne cycle class from $A_1(X)$, was first introduced by Voisin [21].

On the other hand, for an abelian group A, let $\mathcal{H}^q(A)$ be the Zariski sheaf on X associated to the presheaf $U \mapsto H^q(U, A)$. The cohomology group $H^{d-k}(X, \mathcal{H}^d(A))$ is a birational invariant of smooth projective varieties ([11]) which is analogous to the k-th unramified cohomology. It is known that the natural homomorphism

$$H^{d-k}(X, \mathcal{H}^d(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \to H^{d-k}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))$$

is injective ([11]). We will establish a link between $\mathcal{T}_1(X)$ and this quotient group for $k = 4$.
Recall also that the coniveau filtration of $H^k(X, \mathbb{Z})$ is defined by ([12], [13], [5])

$$N^r H^k(X, \mathbb{Z}) = \operatorname{Ker}(\ H^k(X, \mathbb{Z}) \to \varinjlim_{W \subset X} H^k(X - W, \mathbb{Z})\ ),$$

where W ranges over closed algebraic subsets of X of codimension $\geq r$. We will be concerned with the case $k = 2d-3$, where the coniveau filtration has at most two steps by the Bloch-Ogus theory [5]:

$$0 = N^{d-1} H^{2d-3}(X, \mathbb{Z}) \ \subset \ N^{d-2} H^{2d-3}(X, \mathbb{Z}) \ \subset \ N^{d-3} H^{2d-3}(X, \mathbb{Z}) = H^{2d-3}(X, \mathbb{Z}).$$

We write

$$\Lambda(X) = H^{2d-3}(X, \mathbb{Z}) / N^{d-2} H^{2d-3}(X, \mathbb{Z})$$

and let ${}_{tor}\Lambda(X)$ be its torsion part. The group $\Lambda(X)$ gives a proper analogue of Grothendieck's supérieur cohomological invariant ([12] §9) in degree $2d-3$. His generalized Hodge conjecture ([13]) predicts that $N^{d-2} H^{2d-3}(X, \mathbb{Q})$ would be the largest sub $\mathbb{Q}$-Hodge structure contained in $H^{d-1,d-2}(X) \oplus H^{d-2,d-1}(X)$. If this holds, ${}_{tor}\Lambda(X)$ can be thought of as measuring defect of its naive integral version.

Theorem 1.1 (Theorem 4.1). *Let X be a smooth complex projective variety of dimension d. We have a short exact sequence*

$$0 \to {}_{tor}\Lambda(X) \to H^{d-4}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) / H^{d-4}(X, \mathcal{H}^d(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \to \mathcal{T}_1(X) \to 0.$$

When $\mathrm{CH}_0(X)$ is supported in dimension ≤ 2, e.g., when X is rationally connected, then $\Lambda(X)$ is finite, $\mathcal{T}_1(X)$ coincides with $\mathrm{Griff}_1(X)$, and we have an exact sequence

$$0 \to \Lambda(X) \to H^{d-4}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) \to \mathrm{Griff}_1(X) \to 0.$$

In §5, independently of Theorem 1.1, we study the n-torsion part ${}_nA^p(X)$ of the group $A^p(X) \subset \mathrm{CH}^p(X)$ of codimension p cycles algebraically equivalent to zero with p arbitrary. We deduce an exact sequence (Theorem 5.1)

$$0 \to H^{p-1}(X, \mathcal{K}_p)/n \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}))/n \to {}_nA^p(X) \to 0,$$

where $\mathcal{K}_p$ is the Zariski sheaf on X associated to the Quillen K-theory (thanks to the work of Kerz [15], we may also replace it by the Milnor K-theory sheaf). For $p = d-1$ this is complementary to Theorem 1.1, controlling by $\mathcal{H}$-cohomology the group of torsion 1-cycles in $\mathrm{CH}_1(X)_{hom}$ whose Abel-Jacobi invariant is contained in the algebraic part of the intermediate Jacobian.
The interaction between algebraic cycles and birational invariant has been originated from the work of Bloch-Ogus [5], and developed around the nineties by Parimala, Colliot-Thélène, Barbieri-Viale, Kahn and others (see, e.g., [18], [8], [1], [2], [14] and references in [8]). Especially, the relationship between degree 3 unramified cohomology $H^3_{nr}(X) = H^0(X, \mathcal{H}^3)$ and codimension 2 cycles was extensively studied. After

the Bloch-Kato conjecture was proved by Rost-Voevodsky [20], Colliot-Thélène and Voisin [11] revisited this subject and established a link between $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ and the defect of the integral Hodge conjecture for codimension 2 cycles. Voisin [21] extended this result to the degree 4 unramified cohomology $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z})$, relating it to $\mathcal{T}^3(X)$. Colliot-Thélène and Voisin [11] also found a relation between the defect of the integral Hodge conjecture for 1-cycles and $H^{d-3}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))$. Theorem 1.1 is the "homology" counterpart of Voisin's interpretation of H^4_{nr}, and is also the extension of the second theorem of Colliot-Thélène–Voisin to degree 4. Note that in dimension $d = 4$, Theorem 1.1 recovers Voisin's result. We can also give a refinement of Voisin's result (Remark 4.2).

We would like to thank the referee for many valuable suggestions. In particular, Proposition 4.4 was taught by the referee.

Notation. If A is an abelian group and $n > 0$ a natural number, we denote $A/n = A/nA$ and ${}_nA = \{x \in A | nx = 0\}$. We write ${}_{tor}A$ for the subgroup of torsion elements. If $f : A \to B$ is a homomorphism, ${}_nf : {}_nA \to {}_nB$ and ${}_{tor}f : {}_{tor}A \to {}_{tor}B$ denote its restriction to the respective torsion parts. Whether a sheaf is considered in the Zariski topology or in the classical topology will be clear from the context.

2 Preliminaries

Let X be a smooth complex projective variety of dimension d. Let A be an abelian group, which will be one of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Z}/n$ in the sequel. Let $\mathcal{H}^q(A)$ be the Zariski sheaf on Z associated to the presheaf $U \mapsto H^q(U, A)$ defined by the singular cohomology of Zariski open sets. Bloch-Ogus [5] computed the E_2 page of Grothendieck's coniveau spectral sequence ([12] §10) and obtained a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(A)) \quad \Rightarrow \quad H^{p+q}(X, A)$$

which converges to the coniveau filtration of $H^k(X, A)$ and which coincides with the Leray spectral sequence for the natural continuous map $X_{cl} \to X_{Zar}$ from the classical topology to the Zariski topology. They showed that

$$H^p(X, \mathcal{H}^q(A)) = 0 \quad \text{for} \quad p > q,$$

$$H^p(X, \mathcal{H}^p(\mathbb{Z})) \simeq \mathrm{NS}^p(X),$$

where $\mathrm{NS}^p(X) = \mathrm{CH}^p(X)/A^p(X)$ is the group of codimension p cycles modulo algebraic equivalence. The edge morphism $\mathrm{NS}^p(X) \to H^{2p}(X, \mathbb{Z})$ equals to the cycle map. Note that $\mathcal{H}^q(A) = 0$ for $q > d$ because smooth affine varieties of dimension d have homotopy type of CW complex of real dimension d. Hence we also have $E_2^{p,q} = 0$ for $q > d$. In this way the Bloch-Ogus spectral sequence is confined to the triangle

$$p \geq 0, \quad q \leq d, \quad p \leq q.$$

The group $E_2^{0,k} = H^0(X, \mathcal{H}^k(A))$ is usually called the k-th unramified cohomology group with coefficients in A, and denoted by $H^k_{nr}(X, A)$. As shown in [8], [2], [11], for smooth projective X, it is birationally invariant, and also stably invariant, namely $H^k_{nr}(X,A) \simeq H^k_{nr}(X\times\mathbb{P}^r,A)$. When $A=\mathbb{Z}/n$, $H^k_{nr}(X,\mathbb{Z}/n)$ can be computed in terms of the Galois cohomology of the function field of X (see [8]), which makes $H^k_{nr}(X,\mathbb{Z}/n)$ an effective obstruction in the Noether problem.

In this paper we are mainly interested in the cohomology group which sits in the position of "homology counterpart" of $H^k_{nr}(X,A)$ in the Bloch-Ogus spectral sequence, that is,

$$H^{d-k}(X,\mathcal{H}^d(A)).$$

It is proven by Colliot-Thélène and Voisin [11] that this group is also birationally invariant for smooth projective X. (This can also be seen using the blow-up formula in [2]). Furthermore, we have

Lemma 2.1. *The group $H^{d-k}(X,\mathcal{H}^d(A))$, $d=\dim X$, is stably invariant.*

Proof. In [2] Barbieri-Viale proved the general formula

$$H^p(X\times\mathbb{P}^r,\mathcal{H}^q(A)) \simeq \bigoplus_{i=0}^{r} H^{p-i}(X,\mathcal{H}^{q-i}(A)).$$

Putting $q=d+r$ and $p=d+r-k$ where $d=\dim X$, we obtain

$$\begin{aligned} H^{d+r-k}(X\times\mathbb{P}^r,\mathcal{H}^{d+r}(A)) &\simeq \bigoplus_{i=0}^{r} H^{d+r-k-i}(X,\mathcal{H}^{d+r-i}(A)) \\ &= H^{d-k}(X,\mathcal{H}^d(A)) \end{aligned}$$

because over X the sheaf $\mathcal{H}^{q'}$ is zero for $q'>d$. □

Although $H^{d-k}(X,\mathcal{H}^d(A))$ shares some properties with $H^k_{nr}(X,A)$, these two groups may be rather different, and $H^{d-k}(X,\mathcal{H}^d(A))$ has some defects compared to $H^k_{nr}(X,A)$: for example, $H^{d-k}(X,\mathcal{H}^d(\mathbb{Z}))$ may have torsion; cup product is not defined for $H^{d-k}(X,\mathcal{H}^d(A))$; and currently we do not know how to calculate $H^{d-k}(X,\mathcal{H}^d(\mathbb{Z}/n))$ in terms of the function field of X. It is well-known that $H^{d-1}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))$ and $H^{d-2}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))$ vanish for rationally connected X (cf. Proposition 3.2). According to a conjecture of Voisin, the group $H^{d-3}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))$ which by [11] measures the defect of the Hodge conjecture for degree 2 integral homology classes, should also vanish for such X (see [22] and references therein). Actually she proves in loc. cit. that this vanishing is implied by the Tate conjecture for degree 2 Tate classes on surfaces over finite fields. By contrast, there are known many examples of rationally connected X for which $H^2_{nr}(X,\mathbb{Q}/\mathbb{Z}) = {}_{tor}H^3(X,\mathbb{Z})$ or $H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}) = Z^4(X)$ is nontrivial (cf. [8], [11]).

Bloch-Srinivas [6] proved that the Zariski sheaf $\mathcal{H}^3(\mathbb{Z})$ is torsion-free using the Merkurjev-Suslin Theorem [16]. This was generalized to $\mathcal{H}^q(\mathbb{Z})$ for arbitrary q by Barbieri-Viale [3] and Colliot-Thélène–Voisin [11], as a consequence of the Bloch-Kato conjecture proved by Rost-Voevodsky [20]. As a result, we have a short exact sequence of Zariski sheaves ([11], [1])

$$0\to\mathcal{H}^q(\mathbb{Z})\xrightarrow{n}\mathcal{H}^q(\mathbb{Z})\to\mathcal{H}^q(\mathbb{Z}/n)\to 0 \tag{1}$$

for every $n > 0$. Taking cohomology long exact sequence, we obtain

Proposition 2.2 ([11], [1]). *For every $p, q \geq 0$ and $n > 0$ we have an exact sequence*

$$0 \to H^p(X, \mathcal{H}^q(\mathbb{Z}))/n \to H^p(X, \mathcal{H}^q(\mathbb{Z}/n)) \to {}_nH^{p+1}(X, \mathcal{H}^q(\mathbb{Z})) \to 0. \tag{2}$$

For example, when $(p, q) = (0, k)$, this gives an exact sequence

$$0 \to H^k_{nr}(X, \mathbb{Z})/n \to H^k_{nr}(X, \mathbb{Z}/n) \to {}_nH^1(X, \mathcal{H}^k(\mathbb{Z})) \to 0.$$

On the mirror edge, namely for $(p, q) = (d - k, d)$, we have an exact sequence

$$0 \to H^{d-k}(X, \mathcal{H}^d(\mathbb{Z}))/n \to H^{d-k}(X, \mathcal{H}^d(\mathbb{Z}/n)) \to {}_nH^{d-(k-1)}(X, \mathcal{H}^d(\mathbb{Z})) \to 0. \tag{3}$$

We remark that the map $H^p(X, \mathcal{H}^q(\mathbb{Z}/n)) \to {}_nH^{p+1}(X, \mathcal{H}^q(\mathbb{Z}))$ in (2) is induced from the connecting map in the snake lemma applied to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \bigoplus_{X^{(p)}} H^{q-p}(\mathbb{C}(x), \mathbb{Z}) \arrow[r] \arrow[d, "n"] & \bigoplus_{X^{(p+1)}} H^{q-p-1}(\mathbb{C}(x), \mathbb{Z}) \arrow[r] \arrow[d, "n"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{X^{(p)}} H^{q-p}(\mathbb{C}(x), \mathbb{Z}) \arrow[r] \arrow[d] & \bigoplus_{X^{(p+1)}} H^{q-p-1}(\mathbb{C}(x), \mathbb{Z}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_{X^{(p)}} H^{q-p}(\mathbb{C}(x), \mathbb{Z}/n) \arrow[r] \arrow[d] & \bigoplus_{X^{(p+1)}} H^{q-p-1}(\mathbb{C}(x), \mathbb{Z}/n) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0. &
\end{tikzcd} \tag{4}$$

Here the horizontals are part of the Bloch-Ogus complexes computing the cohomology of $\mathcal{H}^q(\mathbb{Z})$ and $\mathcal{H}^q(\mathbb{Z}/n)$ (= the q-th row in the E_1 page of the coniveau spectral sequence). The columns are exact by the exactness of (1) for smooth Zariski open sets of $\overline{\{x\}}$.

Colliot-Thélène and Voisin [11], [21] studied the sequence (2) in the following cases and found connection with algebraic cycles.

- $(p, q) = (0, 3)$, with the defect $Z^4(X)$ of the integral Hodge conjecture for codimension 2 cycles [11];
- $(p, q) = (0, 4)$, with the torsion Abel-Jacobi kernel $\mathcal{T}^3(X)$ in $\mathrm{Griff}^3(X)$ [21];
- $(p, q) = (d - 3, d)$, with the defect $Z_2(X)$ of the integral Hodge conjecture for 1-cycles [11].

In §4 we study the case $(p, q) = (d - 4, d)$ and relate it also to 1-cycles.

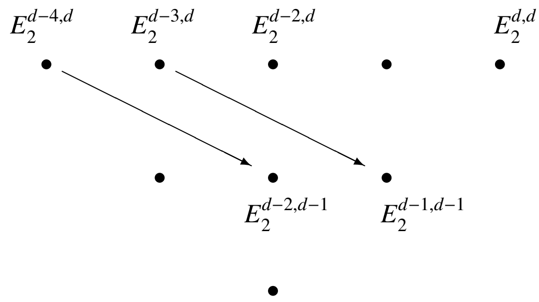

Figure 1: Bloch-Ogus spectral sequence in degree $\geq 2d-4$

3 Griffiths group of 1-cycles

Let X be a smooth projective complex variety of dimension d. We consider the Bloch-Ogus spectral sequence in degree $\geq 2d-4$ and relate it to the Griffiths group of 1-cycles. The situation of the spectral sequence is somewhat similar to that in degree ≤ 4.

Proposition 3.1. *We have an exact sequence*

$$\begin{aligned} &H^{2d-4}(X,\mathbb{Z}) \xrightarrow{e_4} H^{d-4}(X,\mathcal{H}^d(\mathbb{Z})) \\ &\xrightarrow{d_2} H^{d-2}(X,\mathcal{H}^{d-1}(\mathbb{Z})) \to H^{2d-3}(X,\mathbb{Z}) \xrightarrow{e_3} H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \\ &\xrightarrow{d_2} \mathrm{NS}_1(X) \xrightarrow{cl} H^{2d-2}(X,\mathbb{Z}) \xrightarrow{e_2} H^{d-2}(X,\mathcal{H}^d(\mathbb{Z})) \to 0. \end{aligned} \tag{5}$$

Here $\mathrm{NS}_1(X) = \mathrm{CH}_1(X)/A_1(X)$ *is the group of* 1*-cycles modulo algebraic equivalence and cl is the cycle map on* $\mathrm{NS}_1(X)$*. The kernel of* e_i *is the coniveau subgroup* $N^{d-i+1}H^{2d-i}(X,\mathbb{Z})$ *of* $H^{2d-i}(X,\mathbb{Z})$*.*

The same exact sequence holds for $\mathbb{Z}/n$-coefficients with $\mathrm{NS}_1(X)$ replaced by $\mathrm{NS}_1(X)/n$.

Proof. As illustrated in Figure 1,

1. the only nonzero differential from degree $2d-3$ to $2d-2$ is $d_2 : E_2^{d-3,d} \to E_2^{d-1,d-1}$, and there is no nonzero differential from degree $2d-2$ to $2d-1$, so the spectral sequence in degree $2d-2$ degenerates at the E_3 page;

2. similarly, the only nonzero differential from degree $2d-4$ to $2d-3$ is $d_2 : E_2^{d-4,d} \to E_2^{d-2,d-1}$, so the spectral sequence in degree $2d-3$ degenerates at the E_3 page as well.

The observation (1) gives an exact sequence

$$E_2^{d-3,d} \xrightarrow{d_2} E_2^{d-1,d-1} \xrightarrow{cl} E_\infty^{2d-2} \xrightarrow{e_2} E_2^{d-2,d} \to 0,$$

and (2) gives an exact sequence

$$E_\infty^{2d-4} \xrightarrow{e_4} E_2^{d-4,d} \xrightarrow{d_2} E_2^{d-2,d-1} \to E_\infty^{2d-3} \xrightarrow{e_3} E_2^{d-3,d} \xrightarrow{d_2} E_2^{d-1,d-1}.$$

Combining these two exact sequences gives the desired sequence. Since the Bloch-Ogus spectral sequence converges to the coniveau filtration, the first nontrivial step of the coniveau filtration is $\mathrm{Ker}(e_i)$ and it is equal to $N^{d-i+1}H^{2d-i}(X,\mathbb{Z})$. □

Colliot-Thélène and Voisin [11] studied the third line of (5) and found

$$_nZ_2(X) \simeq {}_nH^{d-2}(X,\mathcal{H}^d(\mathbb{Z})). \tag{6}$$

Combining this with (3) with $k = 3$, they obtained the exact sequence

$$0 \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))/n \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}/n)) \to {}_nZ_2(X) \to 0. \tag{7}$$

In §4 we will give a cycle-theoretic interpretation of $H^{d-4}(X,\mathcal{H}^d(\mathbb{Z}/n))$. For completeness we also describe $H^{d-k}(X,\mathcal{H}^d(\mathbb{Z}/n))$ for $k \leq 2$.

Proposition 3.2. *(1) We have exact sequences*

$$0 \to H^{d-2}(X,\mathcal{H}^d(\mathbb{Z}))/n \to H^{d-2}(X,\mathcal{H}^d(\mathbb{Z}/n)) \to {}_nH_1(X,\mathbb{Z}) \to 0, \tag{8}$$

$$0 \to H^{d-2}(X,\mathcal{H}^d(\mathbb{Z}))\otimes\mathbb{Q}/\mathbb{Z} \to H^{d-2}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) \to {}_{tor}H_1(X,\mathbb{Z}) \to 0.$$

When X is rationally connected, we have $H^{d-2}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) = 0$.
(2) We have $H^{d-1}(X,\mathcal{H}^d(\mathbb{Z}/n)) \simeq H_1(X,\mathbb{Z})/n$. This vanishes for rationally connected X.

Proof. These are consequences of (3) with $k = 1, 2$ and degeneration of the Bloch-Ogus spectral sequence in degree $\geq 2d-1$, which gives

$$H^{d-1}(X,\mathcal{H}^d(\mathbb{Z})) \simeq H^{2d-1}(X,\mathbb{Z}) \simeq H_1(X,\mathbb{Z}),$$

$$H^d(X,\mathcal{H}^d(\mathbb{Z})) \simeq H^{2d}(X,\mathbb{Z}) \simeq \mathbb{Z}.$$

When X is rationally connected, it is simply connected and so $H_1(X,\mathbb{Z}) = 0$. Moreover, $H^{d-2}(X,\mathcal{H}^d(\mathbb{Z}))\otimes\mathbb{Q}/\mathbb{Z}$ is zero by [11] Proposition 3.3. □

If Voisin's conjecture (cf. [22]) that $Z_2(X) = 0$ for rationally connected X holds true, we have $H^{d-2}(X,\mathcal{H}^d(\mathbb{Z})) = 0$ for such X by (6). Then $H^{d-2}(X,\mathcal{H}^d(\mathbb{Z}/n))$ would also vanish by (8).
We go back to observing the exact sequence (5). The Griffiths group $\mathrm{Griff}_1(X)$ is the kernel of the cycle map cl defined on $\mathrm{NS}_1(X)$ and hence equals to the image of $d_2 : H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \to \mathrm{NS}_1(X)$. As in §1, we denote

$$\Lambda(X) := H^{2d-3}(X,\mathbb{Z})/N^{d-2}H^{2d-3}(X,\mathbb{Z}). \tag{9}$$

This is a finitely generated abelian group and also isomorphic to

$$\begin{aligned}&\mathrm{Im}(\ e_3 : H^{2d-3}(X,\mathbb{Z}) \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))\ )\\&= \mathrm{Ker}(\ d_2 : H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \to \mathrm{NS}_1(X)\ )\end{aligned}$$

by (5). Looking at the second to third line of (5), we obtain the following description of $\mathrm{Griff}_1(X)$.

Proposition 3.3. *We have an exact sequence*

$$0 \to \Lambda(X) \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \to \mathrm{Griff}_1(X) \to 0. \tag{10}$$

In particular, when $H^{2d-3}(X,\mathbb{Z}) = N^{d-2}H^{2d-3}(X,\mathbb{Z})$ holds, we have

$$\mathrm{Griff}_1(X) \simeq H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})).$$

This is analogous to Bloch-Ogus' description ([5]) of $\mathrm{Griff}^2(X)$

$$0 \to H^3(X,\mathbb{Z})/N^1H^3(X,\mathbb{Z}) \to H^3_{nr}(X,\mathbb{Z}) \to \mathrm{Griff}^2(X) \to 0.$$

It is known that the three terms in (10) are all birationally invariant for smooth projective X. For $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))$ this is proved in [11]; for the other two terms, this results from the corresponding blow-up formulae:

$$\mathrm{CH}_1(\tilde{X}) \simeq \mathrm{CH}_1(X) \oplus \mathrm{CH}_0(Y),$$

$$H^{2d-3}(\tilde{X},\mathbb{Z}) \simeq H^{2d-3}(X,\mathbb{Z}) \oplus H^{2e-1}(Y,\mathbb{Z}),$$

where $\tilde{X} \to X$ is the blow-up along a smooth subvariety $Y \subset X$ of dimension e.

Corollary 3.4. *The group* $\mathrm{Griff}_1(X)$ *is finitely generated if and only if* $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))$ *is.*

Bloch-Srinivas [6] proved that when $\mathrm{CH}_0(X)$ is supported in dimension ≤ 2, $\mathrm{Griff}_1(X)$ is a torsion group. Hence in that case, $\mathrm{Griff}_1(X)$ is finite if and only if $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))$ is. We remark that the result of Bloch-Srinivas can also be derived from (10) and the fact that $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))$ is torsion in that case ([11]), but this may be roundabout.
The assumption $H^{2d-3}(X,\mathbb{Z}) = N^{d-2}H^{2d-3}(X,\mathbb{Z})$ in the second statement of Proposition 3.3 might be hard to check, unless $H^{2d-3}(X,\mathbb{Z}) = 0$. For example, this holds for complete intersections of dimension $d \geq 5$. When this assumption holds, we have an exact sequence

$$0 \to \mathrm{Griff}_1(X)/n \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}/n)) \to {}_nZ_2(X) \to 0$$

by Colliot-Thélène–Voisin's exact sequence (7).
Taking the $\mathrm{Ext}(\mathbb{Z}/n,\cdot)$ long exact sequence associated to (10) (or equivalently, applying the snake lemma to the multiplication by n on (10)), we obtain a description of the torsion part of $\mathrm{Griff}_1(X)$.

Proposition 3.5. *We have exact sequences*

$$\begin{aligned}0 &\to {}_n\Lambda(X) \to {}_nH^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \to {}_n\mathrm{Griff}_1(X)\\ &\to \Lambda(X)/n \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))/n \to \mathrm{Griff}_1(X)/n \to 0,\end{aligned}$$

$$\begin{aligned}0 &\to {}_{tor}\Lambda(X) \to {}_{tor}H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \to {}_{tor}\mathrm{Griff}_1(X)\\ &\to \Lambda(X)\otimes\mathbb{Q}/\mathbb{Z} \to H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))\otimes\mathbb{Q}/\mathbb{Z} \to \mathrm{Griff}_1(X)\otimes\mathbb{Q}/\mathbb{Z} \to 0.\end{aligned}$$

Corollary 3.6. *The group ${}_n\mathrm{Griff}_1(X)$ (resp. $\mathrm{Griff}_1(X)/n$) is finite if and only if ${}_nH^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))$ (resp. $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))/n$) is.*

In the corollary we may also replace $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}))/n$ by $H^{d-3}(X,\mathcal{H}^d(\mathbb{Z}/n))$ thanks to the exact sequence (7).

4 Torsion Abel-Jacobi kernel in $\mathrm{Griff}_1(X)$

Let $J^{2d-3}(X)$ be the intermediate Jacobian of X in degree $2d-3$. The algebraic part $J^{2d-3}(X)_{alg}$ of $J^{2d-3}(X)$ is defined as the image of the Abel-Jacobi map $\lambda_{alg} : A_1(X) \to J^{2d-3}(X)$ from $A_1(X) \subset \mathrm{CH}_1(X)_{hom}$. This is an abelian variety, which corresponds to the weight 1 sub $\mathbb{Q}$-Hodge structure $N^{d-2}H^{2d-3}(X,\mathbb{Q})$ of $H^{2d-3}(X,\mathbb{Q})$. Consider the quotient complex torus

$$J^{2d-3}(X)_{tr} := J^{2d-3}(X)/J^{2d-3}(X)_{alg}.$$

The Abel-Jacobi map $\lambda : \mathrm{CH}_1(X)_{hom} \to J^{2d-3}(X)$ from $\mathrm{CH}_1(X)_{hom}$ induces a homomorphism $\lambda_{tr} : \mathrm{Griff}_1(X) \to J^{2d-3}(X)_{tr}$. We consider its restriction to the torsion part

$${}_{tor}\lambda_{tr} : {}_{tor}\mathrm{Griff}_1(X) \to {}_{tor}J^{2d-3}(X)_{tr}.$$

The situation is summarized in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & {}_{tor}A_1(X) \arrow[r] \arrow[d, "{}_{tor}\lambda_{alg}"'] & {}_{tor}\mathrm{CH}_1(X)_{hom} \arrow[r] \arrow[d, "{}_{tor}\lambda"] & {}_{tor}\mathrm{Griff}_1(X) \arrow[r] \arrow[d, "{}_{tor}\lambda_{tr}"'] & 0 \\ 0 \arrow[r] & {}_{tor}J^{2d-3}(X)_{alg} \arrow[r] & {}_{tor}J^{2d-3}(X) \arrow[r] & {}_{tor}J^{2d-3}(X)_{tr} \arrow[r] & 0. \end{tikzcd}$$

The two rows are exact by the divisibility of $A_1(X)$ and $J^{2d-3}(X)_{alg}$ respectively. The homomorphism ${}_{tor}\lambda_{alg}$ remains surjective because $A_1(X)$ is generated by abelian varieties via correspondences.

Following Voisin [21], we define

$$\mathcal{T}_1(X) := \mathrm{Ker}({}_{tor}\lambda_{tr}) \subset {}_{tor}\mathrm{Griff}_1(X).$$

As shown by Voisin, $\mathcal{T}_1(X)$ coincides with the image of $\mathrm{Ker}({}_{tor}\lambda)$ in $\mathrm{Griff}_1(X)$, which is the definition stated in §1. This equality can be seen from the snake lemma applied to the above commutative diagram, which gives the short exact sequence

$$0 \to {}_{tor}\mathrm{Ker}(\lambda_{alg}) \to {}_{tor}\mathrm{Ker}(\lambda) \to \mathcal{T}_1(X) \to \mathrm{Coker}({}_{tor}\lambda_{alg}) = 0. \tag{11}$$

Our main result is the following.

Theorem 4.1. *We have an exact sequence*

$$0 \to {}_{tor}\Lambda(X) \to H^{d-4}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z}))/H^{d-4}(X, \mathcal{H}^d(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \to \mathcal{T}_1(X) \to 0. \quad (12)$$

If $\mathrm{CH}_0(X)$ *is supported in dimension* ≤ 3*, we have an exact sequence*

$$0 \to {}_{tor}\Lambda(X) \to H^{d-4}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) \to \mathcal{T}_1(X) \to 0.$$

If $\mathrm{CH}_0(X)$ *is supported in dimension* ≤ 2*, then* $\Lambda(X)$ *is finite,* $\mathcal{T}_1(X)$ *coincides with* $\mathrm{Griff}_1(X)$*, and hence we have an exact sequence*

$$0 \to \Lambda(X) \to H^{d-4}(X, \mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) \to \mathrm{Griff}_1(X) \to 0. \quad (13)$$

Proof. When $\mathrm{CH}_0(X)$ is supported in dimension ≤ 3, $H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}))$ is torsion by [11]. Hence it is annihilated when tensored with $\mathbb{Q}/\mathbb{Z}$. When $\mathrm{CH}_0(X)$ is supported in dimension ≤ 2, $\mathrm{Griff}_1(X)$ is torsion by [6], and also $H^{2d-3}(X, \mathbb{Q}) = N^{d-2}H^{2d-3}(X, \mathbb{Q})$ because $H^{d-3}(X, \mathcal{H}^d(\mathbb{Q})) = 0$ by [11]. Hence $\Lambda(X) \otimes \mathbb{Q} = 0$ and $J^{2d-3}(X)_{tr} = 0$. It remains to prove (12).

We apply (2) and the universal coefficient theorem to the 2nd to 4th terms of (5) and its $\mathbb{Z}/n$-coefficients version. This gives

$$\begin{tikzcd}
0 \arrow[r] & H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}))/n \arrow[r] \arrow[d] & H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}/n)) \arrow[r, "\alpha"] \arrow[d, "d_2"] & {}_nH^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \arrow[r] \arrow[d, "-{}_nd_2"] & 0 \\
0 \arrow[r] & H^{d-2}(X, \mathcal{H}^{d-1}(\mathbb{Z}))/n \arrow[r] \arrow[d, "\psi/n"] & H^{d-2}(X, \mathcal{H}^{d-1}(\mathbb{Z}/n)) \arrow[r, "\alpha"] \arrow[d, "\psi_n"] & {}_n\mathrm{NS}_1(X) \arrow[r] \arrow[d, "-{}_ncl"] & 0 \\
0 \arrow[r] & H^{2d-3}(X, \mathbb{Z})/n \arrow[r] & H^{2d-3}(X, \mathbb{Z}/n) \arrow[r, "\beta"] & {}_nH^{2d-2}(X, \mathbb{Z}) \arrow[r] & 0,
\end{tikzcd} \quad (14)$$

where all rows and the middle column are exact, and the other two columns are complex. The right column is (up to sign) restriction of the 5th to 7th terms of (5) to the n-torsion parts.

Commutativity at the lower right, saying that the connecting map α of (4) with $(p, q) = (d-2, d-1)$ is translated to the Bockstein homomorphism β via the edge morphisms of the spectral sequence (up to sign), is essentially proved in [10] Proposition 1 (replace μ_m by $\mathbb{Z}$, and the Gersten-Quillen complex by the Bloch-Ogus complex with coefficients in $\mathbb{Z}$). Commutativity at the upper right, namely compatibility of the connecting map α of (4) with the d_2 differential, holds generally. It can be checked in the following way (we refer to [9] §1 for the notation): (1) consider the multiplication-by-n and reduction-to-$\mathbb{Z}/n$-coefficients on the whole exact couple $(\underset{\rightarrow}{D}^{p,q}, \underset{\rightarrow}{E}^{p,q})$ with which the coniveau spectral sequence started; (2) interpolate

$$\varinjlim_{\vec{Z}} H^{p+q+1}(X - Z_{p+3}, X - Z_{p+1}) \quad (15)$$

into the relevant diagram

$$\begin{tikzcd}
\underset{\to}{E}^{p,q} \arrow[r, "k"'] & \underset{\to}{D}^{p+1,q} \arrow[r, "j"'] \arrow[dr, dashed] & \underset{\to}{E}^{p+1,q} \arrow[r, "k"'] & \underset{\to}{D}^{p+2,q} & \\
& & (15) \arrow[u, dashed] \arrow[dr, dashed] & & \\
& \underset{\to}{D}^{p+2,q-1} \arrow[uu, "i"'] \arrow[r, "j"'] & \underset{\to}{E}^{p+2,q-1} \arrow[r, "k"'] \arrow[u, dashed] & \underset{\to}{D}^{p+3,q-1} \arrow[uu, "i"'] \arrow[r, "j"'] & \underset{\to}{E}^{p+3,q-1}
\end{tikzcd}$$

(multiplication-by-n and reduction-to-$\mathbb{Z}/n$-coefficients are omitted); and then (3) run diagram chasing. The key point is that both the results of $d_2 \circ \alpha$ and $-\alpha \circ d_2$ come from a common element of (15) with $\mathbb{Z}$-coefficients which is to be multiplied by n.

Now we apply the snake lemma to the lower two rows of (14). The resulting connecting map $\mathrm{Ker}({}_n cl) \to \mathrm{Coker}(\psi/n)$ is rewritten as

$$\lambda_n : {}_n\mathrm{Griff}_1(X) \to \Lambda(X)/n.$$

Then the upper side of (14) induces the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}))/n \arrow[r] \arrow[d] & H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}/n)) \arrow[r] \arrow[d, "d_2"'] & {}_nH^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \arrow[r] \arrow[d, "\pi_n"'] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\psi/n) \arrow[r] & \mathrm{Ker}(\psi_n) \arrow[r] & \mathrm{Ker}(\lambda_n) \arrow[r] & 0.
\end{tikzcd}$$

Since the middle vertical is surjective by the exactness of the middle column of (14), the right vertical π_n is also surjective. Since π_n is restriction of the map $d_2 : H^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \to \mathrm{NS}_1(X)$ of (5) to the n-torsion part, its kernel is

$${}_n\mathrm{Ker}(\, d_2 : H^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \to \mathrm{NS}_1(X)\,) \simeq {}_n\Lambda(X).$$

Thus we obtain an exact sequence

$$0 \to {}_n\Lambda(X) \to {}_nH^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \to \mathrm{Ker}(\lambda_n) \to 0.$$

Taking direct limit with respect to n, we have an exact sequence

$$0 \to {}_{tor}\Lambda(X) \to {}_{tor}H^{d-3}(X, \mathcal{H}^d(\mathbb{Z})) \to \mathrm{Ker}(\lambda_\infty) \to 0$$

where

$$\lambda_\infty := \varinjlim_n \lambda_n : {}_{tor}\mathrm{Griff}_1(X) \to \Lambda(X) \otimes \mathbb{Q}/\mathbb{Z}.$$

The construction of λ_∞ as a connecting map coincides with that of Voisin's map $cl_{d-1,tors,tr}$ in [21] p.354 (more precisely, its restriction to ${}_{tor}\mathrm{Griff}_1(X)$). By [21] Proposition 4.4, we see that $\lambda_\infty = {}_{tor}\lambda_{tr}$ and so $\mathrm{Ker}(\lambda_\infty) = \mathcal{T}_1(X)$. Finally, our assertion follows by taking direct limit of (3) with $k = 4$. □

Remark 4.2. *(1) This proof was inspired by the argument of Voisin [21] in the H^4_{nr} case. On the other hand, if we add the present argument to Voisin's proof, we obtain an exact sequence*

$$0 \to {}_{tor}(N^1H^5(X,\mathbb{Z})/N^2H^5(X,\mathbb{Z})) \to H^4_{nr}(X,\mathbb{Q}/\mathbb{Z})/H^4_{nr}(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to \mathcal{T}^3(X) \to 0.$$

Since $H^5(X,\mathbb{Z})/N^1H^5(X,\mathbb{Z})$ has no torsion ([11], [3]), the first term is equal to ${}_{tor}(H^5(X,\mathbb{Z})/N^2H^5(X,\mathbb{Z}))$. This gives a refinement of Voisin's result which was proved under the assumption that $H^5(X,\mathbb{Z})/N^2H^5(X,\mathbb{Z})$ has no torsion.
(2) We can rewrite (12) *in a form compatible with* (10)*:*

$$\begin{tikzcd}
0 \arrow[r] & {}_{tor}\Lambda(X) \arrow[r] \arrow[d, hook] & {}_{tor}H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \arrow[r] \arrow[d, hook] & \mathcal{T}_1(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \Lambda(X) \arrow[r] & H^{d-3}(X,\mathcal{H}^d(\mathbb{Z})) \arrow[r] & \mathrm{Griff}_1(X) \arrow[r] & 0.
\end{tikzcd}$$

It seems plausible that the map λ_n coincides with the connecting map in Proposition 3.5, but we have not checked this.
(3) The exact sequence (13) *also follows from* (10) *and the vanishing $H^{d-i}(X,\mathcal{H}^d(\mathbb{Q})) = 0$ for $i = 3,4$.*

In the proof of Theorem 4.1 we also obtained a description of the kernel of the "mod n" transcendental Abel-Jacobi map $\lambda_n : {}_n\mathrm{Griff}_1(X) \to \Lambda(X)/n$ in terms of $H^{d-4}(X,\mathcal{H}^d(\mathbb{Z}/n))$.

Corollary 4.3. *We have an exact sequence*

$$0 \to {}_n\Lambda(X) \to H^{d-4}(X,\mathcal{H}^d(\mathbb{Z}/n))/(H^{d-4}(X,\mathcal{H}^d(\mathbb{Z}))/n) \to \mathrm{Ker}(\lambda_n) \to 0.$$

The referee pointed out that in some case, algebraic equivalence for torsion 1-cycles with null Deligne class reduces to rational equivalence. Consider the condition

$$\bigoplus_{W\in X^{(d-2)}} H^1(\tilde{W},\mathbb{Z}) \to N^{d-2}H^{2d-3}(X,\mathbb{Z})/\text{torsion} \quad \text{is surjective,} \tag{16}$$

where W runs through irreducible surfaces in X and $\tilde{W}$ is a desingularization of W. As noticed by Grothendieck [13], this seems a subtle problem while its $\mathbb{Q}$-coefficients version always holds.

Proposition 4.4. *Suppose that (i) $\mathrm{CH}_0(X)$ is supported in dimension 0, (ii) $\Lambda(X)$ is torsion-free, and (iii) the condition* (16) *holds. Then the map ${}_{tor}\mathrm{Ker}(\lambda) \to \mathcal{T}_1(X)$ in* (11) *is isomorphic. Thus in this case we have*

$$H^{d-4}(X,\mathcal{H}^d(\mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Griff}_1(X) = \mathcal{T}_1(X) \simeq {}_{tor}\mathrm{Ker}(\lambda),$$

the last being a subgroup of ${}_{tor}\mathrm{CH}_1(X)_{hom}$.

Proof. The second assertion follows from the first and Theorem 4.1. By (11), the first assertion is equivalent to the vanishing of ${}_{tor}\mathrm{Ker}(\lambda_{alg})$. We first show that the conditions (ii) and (iii) imply that $\mathrm{Ker}(\lambda_{alg})$ is a divisible group. Since $A_1(X)$ is divisible, it suffices to prove that ${}_nA_1(X) \to {}_nJ^{2d-3}(X)_{alg}$ is surjective for every $n > 0$. As in the proof of [4] Lemma 1.4, this follows if we could find a (possibly reducible) surface $W \subset X$ such that ${}_n\mathrm{Pic}^0(\tilde{W}) \to {}_nJ^{2d-3}(X)_{alg}$ is surjective where $\tilde{W}$ is a desingularization of W, for this homomorphism is factorized as

$$ {}_n\mathrm{Pic}^0(\tilde{W}) = {}_nA_1(\tilde{W}) \to {}_nA_1(X) \to {}_nJ^{2d-3}(X)_{alg}. $$

Since $\mathrm{Pic}^0(\tilde{W}) \to J^{2d-3}(X)_{alg}$ is defined by the morphism of $\mathbb{Z}$-Hodge structure of weight 1

$$ j_* : H^1(\tilde{W}, \mathbb{Z}) \to N^{d-2}H^{2d-3}(X, \mathbb{Q}) \cap H^{2d-3}(X, \mathbb{Z}) $$

where $j : \tilde{W} \to X$, our assertion holds if j_* is surjectve. Note that j_* factors through

$$ H^1(\tilde{W}, \mathbb{Z}) \to N^{d-2}H^{2d-3}(X, \mathbb{Z})/\mathrm{torsion} \subset N^{d-2}H^{2d-3}(X, \mathbb{Q}) \cap H^{2d-3}(X, \mathbb{Z}). $$

By the condition (ii), the second inclusion is equality. By the condition (iii), we can find finitely many irreducible surfaces $W_1, \cdots, W_k \subset X$ such that $\oplus_i H^1(\tilde{W}_i, \mathbb{Z})$ maps surjectively onto $N^{d-2}H^{2d-3}(X, \mathbb{Z})/\mathrm{torsion}$. It now suffices to take $W = \sum_i W_i$.
We next show that the condition (i) implies that ${}_{tor}\mathrm{Ker}(\lambda_{alg})$ is annihilated by some natural number. We prove this for ${}_{tor}\mathrm{Ker}(\lambda)$. By the Bloch-Srinivas principle [6], we have a decomposition of the diagonal

$$ N\Delta_X \sim \Gamma_1 + \Gamma_2 \quad \in \mathrm{CH}^d(X \times X) $$

for some $N > 0$, where Γ_1 is supported on $D \times X$ for some divisor $D \subset X$ and Γ_2 is supported on $X \times \{p_1, \cdots, p_k\}$ for some points $p_1, \cdots, p_k \in X$. The correspondence by Γ_2 clearly annihilates every 1-cycle. On the other hand, if $\tilde{D} \to D$ is a desingularization of D, the correspondence by Γ_1 factors through the pullback to $\tilde{D}$. The pullback of a torsion 1-cycle with null Deligne class to $\tilde{D}$ is a torsion 0-cycle with null Albanese invariant, which must be rationally equivalent to zero by a theorem of Roitman [19]. Hence $N \cdot {}_{tor}\mathrm{Ker}(\lambda) = 0$. We thus obtain ${}_{tor}\mathrm{Ker}(\lambda_{alg}) = 0$. □

For example, this proposition applies to Fano complete intersections of dimension $d \geq 5$ for they are rationally connected and have $H^{2d-3}(X, \mathbb{Z}) = 0$. We note that to deduce divisibility of $\mathrm{Ker}(\lambda_{alg})$, one may also replace the conditions (ii), (iii) by (iv) $N^{d-2}H^{2d-3}(X, \mathbb{Q}) = 0$ because $\mathrm{Ker}(\lambda_{alg}) = A_1(X)$ in that case.

5 Torsion cycles in $A^k(X)$

The previous sections dealt with $\mathrm{Griff}_1(X)$ and its torsion subgroup. In this section, as another application of (2), we give a description of the torsion part of $A^p(X)$ in terms of $\mathcal{H}$-cohomology for arbitrary p. This is independent of §3 and §4. Let $\mathcal{K}_p$ and $\mathcal{K}_p^M$ be the Zariski sheaves on X associated to the Quillen K-theory and the Milnor K-theory respectively. As a consequence of the Bloch-Kato conjecture proved by Rost-Voevodsky [20] and the work of Kerz [15], we have an isomorphism $\mathcal{K}_p^M/n \simeq \mathcal{H}^p(\mathbb{Z}/n)$ for every $n > 1$ (see [11] p.745 and [3] p.9).

Theorem 5.1. *Let X be a smooth complex projective variety. We have exact sequences*

$$0 \to H^{p-1}(X, \mathcal{K}_p)/n \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}))/n \to {}_nA^p(X) \to 0,$$

$$0 \to H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}/\mathbb{Z} \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \to {}_{tor}A^p(X) \to 0.$$

The same exact sequences with $\mathcal{K}_p$ replaced by $\mathcal{K}_p^M$ also hold.

Remark 5.2. *When $p = 2$, the second sequence reduces to the famous formula ([16], [17])*

$${}_{tor}A^2(X) \simeq H^1(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \simeq N^1H^3(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \simeq {}_{tor}J^3(X)_{alg}.$$

Proof. We will combine three short exact sequences. Firstly, the $(p-1, p)$ case of (2) gives an exact sequence

$$0 \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}))/n \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}/n)) \to {}_n\mathrm{NS}^p(X) \to 0. \tag{17}$$

As explained at (4), the map $H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}/n)) \to {}_n\mathrm{NS}^p(X)$ is induced from the boundary map in the snake lemma for the middle and left columns of the diagram

$$\begin{array}{ccccc}
\bigoplus_{X^{(p-2)}} H^2(\mathbb{C}(x), \mathbb{Z}) & \longrightarrow & \bigoplus_{X^{(p-1)}} H^1(\mathbb{C}(x), \mathbb{Z}) & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z} \\
\downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} \\
\bigoplus_{X^{(p-2)}} H^2(\mathbb{C}(x), \mathbb{Z}) & \longrightarrow & \bigoplus_{X^{(p-1)}} H^1(\mathbb{C}(x), \mathbb{Z}) & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{X^{(p-2)}} H^2(\mathbb{C}(x), \mathbb{Z}/n) & \longrightarrow & \bigoplus_{X^{(p-1)}} H^1(\mathbb{C}(x), \mathbb{Z}/n) & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z}/n \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array} \tag{18}$$

Secondly, Colliot-Thélène, Sansuc and Soulé derived an exact sequence ([10] and [7] §3.2)

$$0 \to H^{p-1}(X, \mathcal{K}_p)/n \to H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}/n)) \to {}_n\mathrm{CH}^p(X) \to 0. \tag{19}$$

As explained in [10], [7], the map $H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}/n)) \to {}_n\mathrm{CH}^p(X)$ is induced from the boundary map in the snake lemma for the middle and left columns of the diagram

$$\begin{array}{ccccc}
\bigoplus_{X^{(p-2)}} K_2\mathbb{C}(x) & \longrightarrow & \bigoplus_{X^{(p-1)}} \mathbb{C}(x)^{\times} & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z} \\
\downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} \\
\bigoplus_{X^{(p-2)}} K_2\mathbb{C}(x) & \longrightarrow & \bigoplus_{X^{(p-1)}} \mathbb{C}(x)^{\times} & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{X^{(p-2)}} H^2(\mathbb{C}(x), \mathbb{Z}/n) & \longrightarrow & \bigoplus_{X^{(p-1)}} H^1(\mathbb{C}(x), \mathbb{Z}/n) & \longrightarrow & \bigoplus_{X^{(p)}} \mathbb{Z}/n. \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array} \tag{20}$$

The upper two rows come from the Gersten-Quillen resolution of $\mathcal{K}_p$. The columns are exact: the middle comes from the Kummer theory, and the left from the Merkurjev-Suslin theorem [16].
Note that in (19) we may replace $\mathcal{K}_p$ by $\mathcal{K}_p^M$ because the Gersten resolution has been established also for the Milnor K-theory [15] and $K_iF = K_i^MF$ for $i \leq 2$, so the upper rows also compute $H^{p-1}(X, \mathcal{K}_p^M)$. Indeed, we have $H^{p-1}(X, \mathcal{K}_p) \simeq H^{p-1}(X, \mathcal{K}_p^M)$.
Thirdly, since $A^p(X)$ is divisible, the sequence

$$0 \to {}_nA^p(X) \to {}_n\mathrm{CH}^p(X) \to {}_n\mathrm{NS}^p(X) \to 0 \tag{21}$$

remains exact.
Combining the three exact sequences (17), (19), (21), we obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & H^{p-1}(X,\mathcal{K}_p)/n \arrow[r, equal] & H^{p-1}(X,\mathcal{K}_p)/n \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^{p-1}(X,\mathcal{H}^p(\mathbb{Z}))/n \arrow[r] & H^{p-1}(X,\mathcal{H}^p(\mathbb{Z}/n)) \arrow[r] \arrow[d] & {}_n\mathrm{NS}^p(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {}_nA^p(X) \arrow[r] & {}_n\mathrm{CH}^p(X) \arrow[r] \arrow[d] & {}_n\mathrm{NS}^p(X) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

Commutativity at the lower right can be checked by comparing the boundary maps in the snake lemmas for (18) and (20). They are connected through the map $\mathbb{C}(x)^\times \to H^1(\mathbb{C}(x), \mathbb{Z})$ which is induced from the boundary maps of the exponential sequences on smooth Zariski open sets of $\overline{\{x\}}$. Now diagram chasing (or the snake lemma) induces vertical morphisms on the left column of this diagram, and shows that it is exact. □

By Proposition 3.5 and Theorem 5.1 with $p = d - 1$, we conclude that the torsion part of $\mathrm{CH}_1(X)_{hom}$ is controlled by the degree $2d-3$ terms in the E_2 page of the Bloch-Ogus spectral sequence with $\mathbb{Z}$-coefficients.

References

[1] Barbieri-Viale, L. *Cicli di codimensione 2 su varietà unirazionali complesse.* in "K-Theory (Strasbourg 1992)", Astérisque 226 (1994), 13–41.

[2] Barbieri-Viale, L. *$\mathcal{H}$-cohomologies versus algebraic cycles.* Math. Nachr. 184 (1997), 5–57.

[3] Barbieri-Viale, L. *On the Deligne-Beilinson cohomology sheaves.* Ann. K-Theory 1, no.1 (2016), 3–17.

[4] Bloch, S. *Lectures on algebraic cycles.* (Second edition) Cambridge University Press, 2010.

[5] Bloch, S.; Ogus, A. *Gersten's conjecture and the homology of schemes.* Ann. Sci. École Norm. Sup. (4) 7 (1974), 181–201.

[6] Bloch, S.; Srinivas, V. *Remarks on correspondences and algebraic cycles.* Amer. J. Math. 105 (1983), no. 5, 1235–1253.

[7] Colliot-Thélène, J.-L. *Cycles algébriques de torsion et K-théorie algébrique.* in "Arithmetic algebraic geometry (Trento, 1991)", 1–49, Lecture Notes in Math., 1553, Springer, 1993.

[8] Colliot-Thélène, J.-L. *Birational invariants, purity and the Gersten conjecture.* in "K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)", 1–64, Proc. Symp. Pure Math., 58, Part 1, Amer. Math. Soc., 1995.

[9] Colliot-Thélène, J.-L.; Hoobler, R.-T.; Kahn, B. *The Bloch-Ogus-Gabber theorem.* in "Algebraic K-theory (Toronto, ON, 1996)", 31–94, Fields Inst. Commun., 16, Amer. Math. Soc., 1997.

[10] Colliot-Thélène, J.-L.; Sansuc, J.-J.; Soulé, C. *Torsion dans le groupe de Chow de codimension deux.* Duke Math. J. 50 (1983), no. 3, 763–801.

[11] Colliot-Thélène, J.-L.; Voisin, C. *Cohomologie non ramifiée et conjecture de Hodge entière.* Duke Math. J. 161 (2012), no. 5, 735–801.

[12] Grothendieck, A. *Le groupe de Brauer. III.* in "Dix exposés sur la cohomologie des schémas", 88–188, North-Holland, 1968.

[13] Grothendieck, A. *Hodge's general conjecture is false for trivial reasons.* Topology 8 (1969) 299–303.

[14] Kahn, B. *Applications of weight-two motivic cohomology.* Doc. Math. 1 (1996), No. 17, 395–416.

[15] Kerz, M. *The Gersten conjecture for Milnor K-theory.* Invent. Math. 175 (2009), no. 1, 1–33.

[16] Merkurjev, A. S.; Suslin, A. A. *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.* Math. USSR-Izv. 21 (1983), no. 2, 307–340.

[17] Murre, J. P. *Un résultat en théorie des cycles algébriques de codimension deux.* C. R. Acad. Sci. Paris Sér. I 296 (1983), no. 23, 981–984.

[18] Parimala, R. *Witt groups of affine three-folds.* Duke Math. J. 57 (1988), no. 3, 947–954.

[19] Roitman, A. A. *The torsion of the group of* 0*-cycles modulo rational equivalence.* Ann. of Math. (2) 111 (1980), no. 3, 553–569.

[20] Voevodsky, V. *On motivic cohomology with $\mathbb{Z}/l$-coefficients.* Ann. of Math. (2) 174 (2011), no. 1, 401–438.

[21] Voisin, C. *Degree 4 unramified cohomology with finite coefficients and torsion codimension 3 cycles.* in "Geometry and arithmetic", 347–368, EMS Ser. Congr. Rep., Eur. Math. Soc., 2012.

[22] Voisin, C. *Remarks on curve classes on rationally connected varieties.* in "A celebration of algebraic geometry", 591–599, Clay Math. Proc., 18, Amer. Math. Soc., 2013.

Shouhei Ma
Department of Mathematics
Tokyo Institute of Technology
Tokyo 152-8551
Japan
ma@math.titech.ac.jp