Representations of *-Algebras by Unbounded Operators: C*-Hulls, Local–Global Principle, and Induction

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Abstract. We define a C*-hull for a *-algebra, given a notion of integrability for its representations on Hilbert modules. We establish a local–global principle which, in many cases, characterises integrable representations on Hilbert modules through the integrable representations on Hilbert spaces. The induction theorem constructs a C*-hull for a certain class of integrable representations of a graded *-algebra, given a C*-hull for its unit fibre.

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1 Introduction

Savchuk and Schmüdgen [27] have introduced a method to define and classify the integrable representations of certain *-algebras by an inductive construction. The original goal of this article was to clarify this method and thus make it apply to more situations. This has led me to reconsider some foundational aspects of the theory of representations of *-algebras by unbounded operators. This is best explained by formulating an induction theorem that is inspired by [27].

Let G be a discrete group with unit element $e \in G$. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded unital *-algebra. That is, $A_g \cdot A_h \subseteq A_{gh}$, $A_g^* = A_{g^{-1}}$, and $1 \in A_e$. In particular, the *unit fibre* A_e is a unital *-algebra. Many interesting examples of this situation are studied in [7, 27]. A *Fell bundle* over G is a family of subspaces $(B_g)_{g\in G}$ of a C^*-algebra B (which is not part of the data) such that $B_g \cdot B_h \subseteq B_{gh}$ and $B_g^* = B_{g^{-1}}$. The universal choice for B is the *section* C^**-algebra* of the Fell bundle.

Briefly, our main result says the following. Let B_e be a C^*-algebra such that "integrable" "representations" of A_e are "equivalent" to "representations" of B_e. Under some technical conditions, we construct a Fell bundle $(B_g^+)_{g\in G}$ over G such that "integrable" "representations" of A are "equivalent" to "representations" of its section C^*-algebra. Here the words in quotation marks must be interpreted carefully to make this true.

A *representation* of a *-algebra A on a Hilbert D-module $\mathcal{E}$ is an algebra homomorphism π from A to the algebra of D-module endomorphisms of a dense D-submodule $\mathfrak{E} \subseteq \mathcal{E}$ with $\langle \xi, \pi(a)\eta\rangle = \langle \pi(a^*)\xi, \eta\rangle$ for all $\xi, \eta \in \mathfrak{E}$, $a \in A$. The representation induces a graph topology on $\mathfrak{E}$. We restrict to *closed* representations most of the time, that is, we require $\mathfrak{E}$ to be complete in the graph topology. The difference from usual practice is that we consider representations on *Hilbert modules* over C^*-algebras. A *representation of a* C^**-algebra* B on a Hilbert module $\mathcal{E}$ is a nondegenerate *-homomorphism $B \to \mathbb{B}(\mathcal{E})$, where $\mathbb{B}(\mathcal{E})$ denotes the C^*-algebra of adjointable operators on $\mathcal{E}$.

The notion of "integrability" for representations is a choice. The class of all Hilbert space representations of a *-algebra may be quite wild. Hence it is customary to limit the study to some class of "nice" or "integrable" representations. For instance, for the universal enveloping algebra of the Lie algebra of a Lie group G, we may call those representations "integrable" that come from a unitary representation of G. This example suggests the name "integrable" representations.

We fix a notion of integrability for representations of $A_e \subseteq A$ on all Hilbert modules over all C^*-algebras. A representation of A is called integrable if its restriction to A_e is integrable. The induction theorem describes the integrable representations of A in terms of integrable representations of A_e. For instance, if A_e is finitely generated and commutative, then we may call a representation π on a Hilbert module integrable if the closure $\overline{\pi(a)}$ is a regular, self-adjoint operator for each $a \in A_e$ with $a = a^*$. All examples in [7, 27] are of this type.

An "equivalence" between the integrable representations of a unital *-algebra A and the representations of a C^*-algebra B is a family of bijections – one for each Hilbert module $\mathcal{E}$ over each C^*-algebra D – between the sets of integrable representations of A and of representations of B on $\mathcal{E}$; these bijections must be compatible with isometric intertwiners and interior tensor products. These properties require some more definitions.

First, an *isometric intertwiner* between two representations is a Hilbert module isometry – not necessarily adjointable – between the underlying Hilbert modules that restricts to a left module map between the domains of the representations. For an equivalence between integrable representations of A and representations of B we require an isometry to intertwine two representations of B if and only if it intertwines the corresponding integrable representations of A.

Secondly, a C^*-*correspondence* from D_1 to D_2 is a Hilbert D_2-module $\mathcal{F}$ with a representation of D_1. Given such a correspondence and a Hilbert D_1-module $\mathcal{E}$, the interior tensor product $\mathcal{E} \otimes_{D_1} \mathcal{F}$ is a Hilbert D_2-module. A representation of A or B on $\mathcal{E}$ induces a representation on $\mathcal{E} \otimes_{D_1} \mathcal{F}$. We require our bijections between integrable representations of A and representations of B to be compatible with this interior tensor product construction on representations.

We call B a C^*-*hull for the integrable representations of* A if the integrable representations of A are equivalent to the representations of B as explained above, that is, through a family of bijections compatible with isometric intertwiners and interior tensor products. The Induction Theorem builds a C^*-hull for the integrable representations of A using a C^*-hull for the integrable representations of A_e and assuming a further mild technical condition, which we explain below.

Many results of the general theory remain true if we only require the equivalence of representations to be compatible with interior tensor products and *unitary* *-intertwiners, that is, isomorphisms of representations; we speak of a *weak* C^*-*hull* in this case. The Induction Theorem, however, fails for weak C^*-hulls. We show this by a counterexample. Some results only need the class of integrable representations to have some properties that are clearly necessary for the existence of a C^*-hull or weak C^*-hull, but they do not need the (weak) C^*-hull itself. This is formalised in our notions of *admissible* and *weakly admissible* classes of representations.

For example, let A be commutative. Let $\hat{A}$ be the space of characters of A with the topology of pointwise convergence. If $\hat{A}$ is locally compact and A is countably generated, then $\mathrm{C}_0(\hat{A})$ is a C^*-hull for the integrable representations of A as defined above, that is, those representations where each $\overline{\pi(a)}$ for $a \in A$ with $a = a^*$ is regular and self-adjoint. If, say, $A = \mathbb{C}[x]$ with $x = x^*$, then the C^*-hull is $\mathrm{C}_0(\mathbb{R})$. Here the equivalence of representations maps an integrable representation π of $\mathbb{C}[x]$ to the functional calculus homomorphism for the regular, self-adjoint operator $\overline{\pi(x)}$.

If $\hat{A}$ is not locally compact, then the integrable representations of A defined above still form an admissible class, but they have no C^*-hull. If, say, A is the algebra of polynomials in countably many variables, then $\hat{A} = \mathbb{R}^\infty$, which is not locally compact. The problem of associating C^*-algebras to this *-algebra has recently been studied by Grundling and Neeb [12]. From our point of view, this amounts to choosing a smaller class of "integrable" representations that does admit a C^*-hull.

We have now explained the terms in quotation marks in our Induction Theorem and how we approach the representation theory of *-algebras. Most previous work focused either on representations on Hilbert spaces or on single unbounded operators on Hilbert modules. Hilbert module representations occur both in the assumptions and in the conclusions of the Induction Theorem, and hence we cannot prove it without them. In addition, taking into account Hilbert module representations makes our C*-hulls unique.

Besides the Induction Theorem, the other main strand of this article are Local–Global Principles, which aim at reducing the study of integrability for representations on general Hilbert modules to representations on Hilbert space. We may use a state ω on the coefficient C*-algebra D of a Hilbert module $\mathcal{E}$ to complete $\mathcal{E}$ to a Hilbert space. Thus a representation of A on $\mathcal{E}$ induces Hilbert space representations for all states on D. The *Local–Global Principle* says that a representation of A on $\mathcal{E}$ is integrable if and only if these induced Hilbert space representations are integrable for all states; the *Strong* Local–Global Principle says the same with all states replaced by all *pure* states. We took these names from [15]. Earlier results of Pierrot [21] show that the Strong Local–Global Principle holds for any class of integrable representations that is defined by certain types of conditions, such as the regularity and self-adjointness of $\overline{\pi(a)}$ for certain $a \in A$ with $a = a^*$. For instance, this covers the integrable representations of commutative *-algebras and universal enveloping algebras.

In all examples that we treat, the regularity of $\overline{\pi(a)}$ for certain $a \in A$ is part of the definition of an integrable representation. Other elements of A may, however, act by irregular operators in some integrable representations. Thus affiliation and regularity are important to study the integrable representations in concrete examples, but cannot play a foundational role for the *general* representation theory of *-algebras.

If B is generated in the sense of Woronowicz [32] by some self-adjoint, affiliated multipliers that belong to A, then it is a C*-hull and the Strong Local–Global Principle holds (see Theorem 5.19). A counterexample shows that this theorem breaks down if the generating affiliated multipliers are not self-adjoint: both the Local–Global Principle and compatibility with isometric intertwiners fail in the counterexample. So regularity without self-adjointness seems to be too weak for many purposes. The combination of regularity and self-adjointness is an easier notion than regularity alone. A closed operator T is regular and self-adjoint if and only if $T - \lambda$ is surjective for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, if and only if the Cayley transform of T is unitary, if and only if T has a functional calculus homomorphism on $\mathrm{C}_0(\mathbb{R})$.

Now we describe the Fell bundle in the Induction Theorem and, along the way, the further condition besides compatibility with isometric intertwiners that it needs. Our input data is a graded *-algebra $A = \bigoplus_{g \in G} A_g$ and a C*-hull B_e for A_e. A representation of A is integrable if its restriction to A_e is integrable. We seek a C*-hull for the integrable representations of A.

As in [27], we induce representations from A_e to A, and this requires a positivity condition. We call representations of A_e that may be induced to A *inducible*. We describe a quotient C*-algebra B_e^+ of B_e that is a C*-hull for the inducible, integrable representations of A. It is the unit fibre of our Fell bundle.

If a representation π of A is integrable, then its restriction to A_e is integrable and inducible. Thus it corresponds to a representation $\bar{\pi}_e^+$ of B_e^+. The identity correspondence on B_e^+ corresponds to a particular ("universal") inducible, integrable representation of A_e on B_e^+. Its domain is a dense right ideal $\mathfrak{B}_e^+ \subseteq B_e^+$. The operators $\pi(a)\bar{\pi}_e^+(b)$ on $\mathcal{E}$ for $a \in A_g$, $b \in \mathfrak{B}_e^+$ are adjointable. Their closed linear span is the fibre B_g^+ of our Fell bundle at g provided π_e^+ is faithful. The most difficult point is to prove $B_e^+ \cdot B_g^+ = B_g^+$ for all $g \in G$. This easily implies $B_g^+ \cdot B_h^+ \subseteq B_{gh}^+$ and $(B_g^+)^* = B_{g^{-1}}^+$, so that the subspaces $B_g^+ \subseteq \mathbb{B}(\mathcal{E})$ form a Fell bundle.

To prove $B_e^+ \cdot B_g^+ = B_g^+$, we need compatibility with isometric intertwiners and that induction maps inducible, integrable representations of A_e to *integrable* representations of A. Two counterexamples show that both assumptions are necessary for the Induction Theorem.

Fell bundles are noncommutative partial dynamical systems. More precisely, a Fell bundle $(B_g^+)_{g\in G}$ over G is equivalent to an action of G on B_e^+ by partial Morita–Rieffel equivalences; this is made precise in [4]. In the examples in [7, 27], the group G is almost always $\mathbb{Z}$; the C*-algebras B_e and hence B_e^+ are commutative; and the resulting Fell bundle comes from a partial action of G on the spectrum of B_e^+. In these examples, the section C*-algebra is a partial crossed product. This may also be viewed as the groupoid C*-algebra of the transformation groupoid for the partial action of G on the spectrum of B_e^+. We show that the C*-hull B for the integrable representations of A is a *twisted* groupoid C*-algebra of this transformation groupoid whenever B_e is commutative. We give some criteria when the twist is absent, and examples where the twist occurs. One way to insert such twists is by Rieffel deformation, using a 2-cocycle on the group G. We show that Rieffel deformation is compatible with the construction of C*-hulls.

We describe commutative and noncommutative C*-hulls for the polynomial algebra $\mathbb{C}[x]$ in §4 and §6; the noncommutative C*-hulls for $\mathbb{C}[x]$ make very good counterexamples. We classify and study commutative C*-hulls in §8. Many results about them generalise easily to *locally bounded* representations. Roughly speaking, these are representations where the vectors on which the representation acts by bounded operators form a core. The only *-algebras for which we treat locally bounded representations in some detail are the commutative ones.

Through the Induction Theorem, the representation theory of commutative *-algebras is important even for noncommutative algebras because they may admit a grading by some group with commutative unit fibre. Many examples of this are treated in detail in [7, 27]. We discuss untwisted and twisted Weyl algebras in finitely and infinitely many generators in §13. The twists involved are Rieffel deformations. Since these examples have commutative unit fibres, the resulting C*-hulls are twisted groupoid C*-algebras. As it turns out, all twists of the relevant groupoids are trivial, so that the twists do not change the representation theory of the Weyl algebras up to equivalence.

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2 Representations by unbounded operators on Hilbert modules

Let A be a unital *-algebra, D a C*-algebra, and $\mathcal{E}$ a Hilbert D-module. Our convention is that inner products on Hilbert spaces and Hilbert modules are linear in the second and conjugate-linear in the first variable.

Definition 2.1. A *representation* of A on $\mathcal{E}$ is a pair $(\mathfrak{E}, \pi)$, where $\mathfrak{E} \subseteq \mathcal{E}$ is a dense D-submodule and $\pi\colon A \to \operatorname{End}_D(\mathfrak{E})$ is a unital algebra homomorphism to the algebra of D-module endomorphisms of $\mathfrak{E}$, such that

$$\langle \pi(a)\xi, \eta\rangle_D = \langle \xi, \pi(a^*)\eta\rangle_D \qquad \text{for all } a \in A,\ \xi, \eta \in \mathfrak{E}.$$

We call $\mathfrak{E}$ the *domain* of the representation. We may drop π from our notation by saying that $\mathfrak{E}$ is an A, D-bimodule with the right module structure inherited from $\mathcal{E}$, or we may drop $\mathfrak{E}$ because it is the common domain of the partial linear maps $\pi(a)$ on $\mathcal{E}$ for all $a \in A$.

We equip $\mathfrak{E}$ with the *graph topology*, which is generated by the *graph norms*

$$\|\xi\|_a := \|(\xi, \pi(a)\xi)\| := \|\langle \xi, \xi\rangle + \langle \pi(a)\xi, \pi(a)\xi\rangle\|^{1/2} = \|\langle \xi, \pi(1 + a^*a)\xi\rangle\|^{1/2}$$

for $a \in A$. The representation is *closed* if $\mathfrak{E}$ is complete in this topology. A *core* for $(\mathfrak{E}, \pi)$ is an A, D-subbimodule of $\mathfrak{E}$ that is dense in $\mathfrak{E}$ in the graph topology.

Definition 2.1 for $D = \mathbb{C}$ is the usual definition of a representation of a *-algebra on a Hilbert space by unbounded operators. This situation has been studied extensively (see, for instance, [28]). For $\mathcal{E} = D$ with the canonical Hilbert D-module structure, we get representations of A by *densely defined unbounded multipliers*. The domain of such a representation is a dense right ideal $\mathfrak{D} \subseteq D$. This situation is a special case of the "compatible pairs" defined by Schmüdgen [29].

Given two norms p, q, we write $p \preceq q$ if there is a scalar $c > 0$ with $p \le cq$.

Lemma 2.2. *The set of graph norms partially ordered by $\preceq$ is directed: for all $a_1, \dots, a_n \in A$ there are $b \in A$ and $c \in \mathbb{R}_{>0}$ so that $\|\xi\|_{a_i} \le c\|\xi\|_b$ for any representation $(\mathfrak{E}, \pi)$, any $\xi \in \mathfrak{E}$, and $i = 1, \dots, n$.*

Proof. Let $b = \sum_{j=1}^n a_j^* a_j$. The following computation implies $\|\xi\|_{a_i} \le {}^5\!/\!_4\|\xi\|_b$:

$$\begin{aligned} 0 &\le \langle \xi, \pi(1 + a_i^* a_i)\xi\rangle \\ &\le \langle \xi, \pi(1 + a_i^* a_i)\xi\rangle + \sum_{i\neq j} \langle \pi(a_j)\xi, \pi(a_j)\xi\rangle + \langle \pi(b - 1/2)\xi, \pi(b - 1/2)\xi\rangle \\ &= \langle \xi, \pi(1 + b + (b - 1/2)^2)\xi\rangle = \langle \xi, \pi(5/4 + b^2)\xi\rangle \le 5/4\langle \xi, \pi(1 + b^*b)\xi\rangle. \qquad \square \end{aligned}$$

Definition 2.3 ([20], [17, Chapter 9]). A densely defined operator t on a Hilbert module $\mathcal{E}$ is *semiregular* if its adjoint is also densely defined. It is *regular* if it is closed, semiregular and $1 + t^*t$ has dense range. An *affiliated multiplier* of a C*-algebra D is a regular operator on D viewed as a Hilbert D-module.

The closability assumption in [20, Definition 2.1.(ii)] is redundant by [15, Lemma 2.1]. Regularity was introduced by Baaj and Julg [1], affiliation by Woronowicz [31].

Remark 2.4. Let $(\mathfrak{E}, \pi)$ be a representation of A on $\mathcal{E}$ and let $a \in A$. The operator $\pi(a)$ is automatically semiregular because $\pi(a)^*$ is defined on $\mathfrak{E}$. The closure $\overline{\pi(a)}$ of $\pi(a)$ need not be regular. The regularity of $\overline{\pi(a)}$ for *some* $a \in A$ is often assumed in the definition of *integrable* representations. For non-commutative A, we should expect that $\overline{\pi(a)}$ is irregular for some $a \in A$ even if π is integrable. For instance, a remark after Corollaire 1.27 in [21] says that this happens for certain symmetric elements in the universal enveloping algebra $U(\mathfrak{g})$ for a simply connected Lie group G: they act by irregular operators in certain representations that integrate to unitary representations of G.

The usual norm on $\mathcal{E}$ is the graph norm for $0 \in A$. Hence the inclusion map $\mathfrak{E} \hookrightarrow \mathcal{E}$ is continuous for the graph topology on $\mathfrak{E}$ and extends continuously to the completion $\overline{\mathfrak{E}}$ of $\mathfrak{E}$ in the graph topology.

PROPOSITION 2.5. *The canonical map $\overline{\mathfrak{E}} \to \mathcal{E}$ is injective, and its image is*

$$\overline{\mathfrak{E}} = \bigcap_{a\in A} \operatorname{dom} \overline{\pi(a)}. \tag{2.6}$$

Thus $(\mathfrak{E}, \pi)$ is closed if and only if $\mathfrak{E} = \bigcap_{a\in A} \operatorname{dom} \overline{\pi(a)}$. Each $\pi(a)$ extends uniquely to a continuous operator $\overline{\pi(a)}$ on $\overline{\mathfrak{E}}$. This defines a closed representation $(\overline{\mathfrak{E}}, \overline{\pi})$ of A, called the closure *of $(\mathfrak{E}, \pi)$.*

Proof. The operator $\pi(a)$ for $a \in A$ is semiregular and hence closable by [15, Lemma 2.1]. Equivalently, the canonical map from the completion of $\mathfrak{E}$ in the graph norm for a to $\mathcal{E}$ is injective. Its image is $\operatorname{dom} \overline{\pi(a)}$, the domain of the closure of $\pi(a)$. The graph norms for $a \in A$ form a directed set that defines the graph topology on $\mathfrak{E}$. So the completion of $\mathfrak{E}$ in the graph topology is the projective limit of the graph norm completions for $a \in A$. Since each of these graph norm completions embeds into $\mathcal{E}$, the projective limit in question is just an intersection in $\mathcal{E}$, giving (2.6). For Hilbert space representations, this is [28, Proposition 2.2.12].
The operators $\pi(a) \in \operatorname{End}_D(\mathfrak{E})$ for $a \in A$ are continuous in the graph topology. Thus they extend uniquely to continuous linear operators $\overline{\pi}(a) \in \operatorname{End}_D(\overline{\mathfrak{E}})$. These are again D-linear and the map $\overline{\pi}$ is linear and multiplicative because extending operators to a completion is additive and functorial. The set of $(\xi, \eta) \in \overline{\mathfrak{E}} \times \overline{\mathfrak{E}}$ with $\langle \xi, \overline{\pi}(a)\eta\rangle = \langle \overline{\pi}(a^*)\xi, \eta\rangle$ for all $a \in A$ is closed in the graph topology and contains $\mathfrak{E} \times \mathfrak{E}$, which is dense in $\overline{\mathfrak{E}} \times \overline{\mathfrak{E}}$. Hence this equation holds for all $\xi, \eta \in \overline{\mathfrak{E}}$. So $(\overline{\mathfrak{E}}, \overline{\pi})$ is a representation of A on $\mathcal{E}$. The graph topology on $\overline{\mathfrak{E}}$ for $\overline{\pi}$ extends the graph topology on $\mathfrak{E}$ for π and hence is complete. So $(\overline{\mathfrak{E}}, \overline{\pi})$ is a closed representation. ☐

We shall need a generalisation of (2.6) that replaces A by a sufficiently large subset.

Definition 2.7. A subset $S \subseteq A$ is called a *strong generating* set if it generates A as an algebra and the graph norms for $a \in S$ generate the graph topology in any representation. That is, for any representation on a Hilbert module, any vector ξ in its domain and any $a \in A$, there are $c \geq 1$ in $\mathbb{R}$ and $b_1, \dots, b_n \in S$ with $\|\xi\|_a \leq c \sum_{i=1}^n \|\xi\|_{b_i}$.

An estimate $\|\xi\|_a \leq c \sum_{i=1}^n \|\xi\|_{b_i}$ is usually shown by finding $d_1, \dots, d_m \in A$ with $a^*a + \sum_{j=1}^m d_j^* d_j = c \cdot \sum_{i=1}^n b_i^* b_i$, compare the proof of Lemma 2.2.

Example 2.8. Let $A_h := \{a \in A \mid a = a^*\}$ be the set of *symmetric* elements. Call an element of A *positive* if it is a sum of elements of the form a^*a. The positive elements and, *a fortiori*, the symmetric elements form strong generating sets for A. Any element is of the form $a_1 + \mathrm{i}a_2$ with $a_1, a_2 \in A_h$, and

$$a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2$$

for $a \in A_h$. Thus the positive elements generate A as an algebra. The graph norms for positive elements generate the graph topology by the proof of Lemma 2.2.

Proposition 2.9. *Let $S \subseteq A$ be a strong generating set. Two closed representations $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ of A on the same Hilbert module $\mathcal{E}$ are equal if and only if $\overline{\pi_1(a)} = \overline{\pi_2(a)}$ for all $a \in S$.*

Proof. One direction is trivial. To prove the non-trivial direction, assume $\overline{\pi_1(a)} = \overline{\pi_2(a)}$ for all $a \in S$. Let $(\mathfrak{E}, \pi) = (\mathfrak{E}_i, \pi_i)$ for $i = 1, 2$. The completion of $\mathfrak{E}$ for the graph norm of a is $\operatorname{dom} \overline{\pi(a)}$, compare the proof of Proposition 2.5. Hence the completion of $\mathfrak{E}$ in the sum of graph norms $\sum_{k=1}^n \|\xi\|_{b_k}$ for $b_1, \dots, b_n \in S$ is $\bigcap_{k=1}^n \operatorname{dom} \overline{\pi(b_k)}$. These sums of graph norms for $b_1, \dots, b_n \in S$ form a directed set that generates the graph topology on $\mathfrak{E}$. Hence

$$\overline{\mathfrak{E}} = \bigcap_{a \in S} \operatorname{dom} \overline{\pi(a)}, \tag{2.10}$$

compare the proof of (2.6). So $\mathfrak{E}_1 = \mathfrak{E}_2$. Moreover, $\pi_1(a) = \overline{\pi_1(a)}|_{\mathfrak{E}_1} = \overline{\pi_2(a)}|_{\mathfrak{E}_2} = \pi_2(a)$ for all $a \in S$. Since S generates A as an algebra and $\pi_i(A)\mathfrak{E}_i \subseteq \mathfrak{E}_i$, this implies $\pi_1 = \pi_2$. ☐

Proposition 2.9 may fail for generating sets that are not strong, see Example 4.2.

Corollary 2.11. *Let S be a strong generating set of A and let $(\mathfrak{E}, \pi)$ be a closed representation of A with $\operatorname{dom} \overline{\pi(a)} = \mathcal{E}$ for each $a \in S$. Then $\mathfrak{E} = \mathcal{E}$ and π is a *-homomorphism to the C^*-algebra $\mathbb{B}(\mathcal{E})$ of adjointable operators on $\mathcal{E}$.*

Proof. Equation (2.10) gives $\mathfrak{E} = \mathcal{E}$. Since $\pi(a^*) \subseteq \pi(a)^*$ and $\pi(a^*)$ is defined everywhere, it is adjoint to $\pi(a)$. So $\pi(a) \in \mathbb{B}(\mathcal{E})$ and π is a *-homomorphism to $\mathbb{B}(\mathcal{E})$. ☐

Lemma 2.12. *Let A be a unital* C**-algebra. Any closed representation of A on $\mathcal{E}$ has domain $\mathfrak{E}=\mathcal{E}$ and is a unital *-homomorphism to $\mathbb{B}(\mathcal{E})$.*

Proof. Let $a \in A$. There are a positive scalar $C > 0$ and $b \in A$ with $a^*a+b^*b = C$; say, take $C = \|a\|^2$ and $b = \sqrt{C - a^*a}$. Then

$$\langle\pi(a)\xi, \pi(a)\xi\rangle \leq \langle\pi(a)\xi, \pi(a)\xi\rangle + \langle\pi(b)\xi, \pi(b)\xi\rangle = \langle\xi, \pi(a^*a + b^*b)\xi\rangle = C\langle\xi, \xi\rangle$$

for all $\xi \in \mathfrak{E}$. Thus the graph topology on $\mathfrak{E}$ is equivalent to the norm topology on $\mathcal{E}$. Hence $\mathcal{E} = \mathfrak{E}$ for any closed representation. □

An *isometry* $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ between two Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ is a right D-module map with $\langle I\xi_1, I\xi_2\rangle = \langle\xi_1, \xi_2\rangle$ for all $\xi_1, \xi_2 \in \mathcal{E}_1$.

Definition 2.13. Let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ be representations on Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. An *isometric intertwiner* between them is an isometry $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ with $I(\mathfrak{E}_1) \subseteq \mathfrak{E}_2$ and $I \circ \pi_1(a)(\xi) = \pi_2(a) \circ I(\xi)$ for all $a \in A$, $\xi \in \mathfrak{E}_1$. Equivalently, $I \circ \pi_1(a) \subseteq \pi_2(a) \circ I$ for all $a \in A$, that is, the graph of $\pi_2(a) \circ I$ contains the graph of $I \circ \pi_1(a)$. We neither ask I to be adjointable nor $I(\mathfrak{E}_1) = \mathfrak{E}_2$. Let $\mathrm{Rep}(A, D)$ be the category with closed representations of A on Hilbert D-modules as objects, isometric intertwiners as arrows, and the usual composition. The unit arrow on $(\mathfrak{E}, \pi)$ is the identity operator on $\mathcal{E}$.

Lemma 2.14. *Let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ be representations on Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, and let $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ be an isometric intertwiner. Then I is also an intertwiner between the closures of $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$.*

Proof. Since I intertwines the representations, it is continuous for the graph topologies on $\mathfrak{E}_1$ and $\mathfrak{E}_2$. Hence I maps the domain of the closure $\overline{\pi_1}$ into the domain of $\overline{\pi_2}$. This extension is still an intertwiner because it is an intertwiner on a dense subspace. □

Proposition 2.15. *Let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ be closed representations of A on Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Let $S \subseteq A$ be a strong generating set. An isometry $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ is an intertwiner from $(\mathfrak{E}_1, \pi_1)$ to $(\mathfrak{E}_2, \pi_2)$ if and only if $I \circ \overline{\pi_1(a)} \subseteq \overline{\pi_2(a)} \circ I$ for all $a \in S$.*

Proof. First let I satisfy $I \circ \overline{\pi_1(a)} \subseteq \overline{\pi_2(a)} \circ I$ for all $a \in S$. Then I maps the domain of $\overline{\pi_1(a)}$ into the domain of $\overline{\pi_2(a)}$ for each $a \in S$. Now (2.10) implies $I(\mathfrak{E}_1) \subseteq \mathfrak{E}_2$. Since $\pi_i(a) = \overline{\pi_i(a)}|_{\mathfrak{E}_i}$, we get $I(\pi_1(a)(\xi)) = \pi_2(a)(I(\xi))$ for all $a \in S$, $\xi \in \mathfrak{E}_1$. Since S generates A as an algebra and $\pi_i(A)\mathfrak{E}_i \subseteq \mathfrak{E}_i$, this implies $I \circ \pi_1(a) = \pi_2(a) \circ I$ for all $a \in A$, that is, I is an intertwiner.
Conversely, assume that I is an intertwiner from $(\mathfrak{E}_1, \pi_1)$ to $(\mathfrak{E}_2, \pi_2)$. Equivalently, $I \circ \pi_1(a) \subseteq \pi_2(a) \circ I$ for all $a \in A$. We have $I \circ \overline{\pi_1(a)} = \overline{I \circ \pi_1(a)}$ because I is an isometry, and $\overline{\pi_2(a) \circ I} \subseteq \overline{\pi_2(a)} \circ I$. Thus $I \circ \overline{\pi_1(a)} \subseteq \overline{\pi_2(a)} \circ I$ for all $a \in A$. □

Now we relate the categories $\mathrm{Rep}(A, D)$ for different C^*-algebras D.

Definition 2.16. Let D_1 and D_2 be two C^*-algebras. A C^*-*correspondence* from D_1 to D_2 is a Hilbert D_2-module with a representation of D_1 by adjointable operators (representations of C^*-algebras are tacitly assumed nondegenerate). An *isometric intertwiner* between two correspondences from D_1 to D_2 is an isometric map on the underlying Hilbert D_2-modules that intertwines the left D_1-actions. Let $\mathrm{Rep}(D_1, D_2)$ denote the category of correspondences from D_1 to D_2 with isometric intertwiners as arrows and the usual composition.

By Lemma 2.12, our two definitions of $\mathrm{Rep}(A, D)$ for unital *-algebras and C^*-algebras coincide if A is a unital C^*-algebra. So our notation is not ambiguous. There is no need to define representations of a non-unital *-algebra A because we may adjoin a unit formally. A representation of A extends uniquely to a representation of the unitisation $\tilde{A}$. Thus the nondegenerate representations of A are contained in $\mathrm{Rep}(\tilde{A})$. To get rid of degenerate representations, we may require nondegeneracy on A when defining the integrable representations of $\tilde{A}$, compare Example 5.13.

Let $\mathcal{E}$ be a Hilbert D_1-module and $\mathcal{F}$ a correspondence from D_1 to D_2. The interior tensor product $\mathcal{E} \otimes_{D_1} \mathcal{F}$ is the (Hausdorff) completion of the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ to a Hilbert D_2-module, using the inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \langle \xi_1, \xi_2 \rangle_{D_1} \cdot \eta_2 \rangle_{D_2}, \tag{2.17}$$

see the discussion around [17, Proposition 4.5] for more details. We may use the balanced tensor product $\mathcal{E} \odot_{D_1} \mathcal{F}$ instead of $\mathcal{E} \odot \mathcal{F}$ because the inner product (2.17) descends to this quotient. If we want to emphasise the left action $\varphi\colon D_1 \to \mathbb{B}(\mathcal{F})$ in the C^*-correspondence $\mathcal{F}$, we write $\mathcal{E} \otimes_\varphi \mathcal{F}$ for $\mathcal{E} \otimes_{D_1} \mathcal{F}$.

In addition, let $(\mathfrak{E}, \pi)$ be a closed representation of A on $\mathcal{E}$. We are going to build a closed representation $(\mathfrak{E} \otimes_{D_1} \mathcal{F}, \pi \otimes_{D_1} 1)$ of A on $\mathcal{E} \otimes_{D_1} \mathcal{F}$. First let $X \subseteq \mathcal{E} \otimes_{D_1} \mathcal{F}$ be the image of $\mathfrak{E} \odot \mathcal{F}$ or $\mathfrak{E} \odot_{D_1} \mathcal{F}$ under the canonical map to $\mathcal{E} \otimes_{D_1} \mathcal{F}$.

Lemma 2.18. *For $a \in A$, there is a unique linear operator $\pi(a) \otimes 1\colon X \to X$ with $(\pi(a) \otimes 1)(\xi \otimes \eta) = \pi(a)(\xi) \otimes \eta$ for all $\xi \in \mathfrak{E}$, $\eta \in \mathcal{F}$. The map $a \mapsto \pi(a) \otimes 1$ is a representation of A with domain X.*

Proof. Write $\omega, \zeta \in X$ as images of elements of $\mathfrak{E} \odot \mathcal{F}$:

$$\omega = \sum_{i=1}^{n} \xi_i \otimes \eta_i, \qquad \zeta = \sum_{j=1}^{m} \alpha_j \otimes \beta_j$$

with $\xi_i, \alpha_j \in \mathfrak{E}$, $\eta_i, \beta_j \in \mathcal{F}$ for $1 \le i \le n$, $1 \le j \le m$. Then

$$\left\langle \zeta, \sum_{i=1}^{n} \pi(a)\xi_i \otimes \eta_i \right\rangle = \left\langle \sum_{j=1}^{m} \pi(a^*)\alpha_j \otimes \beta_j, \omega \right\rangle. \tag{2.19}$$

An element $\omega' \in \mathcal{E} \otimes_{D_1} \mathcal{F}$ is determined uniquely by its inner products $\langle \zeta, \omega' \rangle = 0$ for all $\zeta \in X$ because X is dense in $\mathcal{E} \otimes_{D_1} \mathcal{F}$. The right hand side in (2.19) does not depend on how we decomposed ω. Hence $(\pi(a) \otimes 1)\omega := \sum_{i=1}^n \pi(a)\xi_i \otimes \eta_i$ well-defines an operator $\pi(a) \otimes 1 \colon X \to X$. This is a right D_2-module map, and $a \mapsto \pi(a) \otimes 1$ is linear and multiplicative because π is. Equation (2.19) says that $\langle \zeta, (\pi(a) \otimes 1)\omega \rangle = \langle (\pi(a^*) \otimes 1)\zeta, \omega \rangle$ for all $\omega, \zeta \in X$. Thus $\pi \otimes 1$ is a representation. $\square$

Definition 2.20. Let $(\mathfrak{E} \otimes_{D_1} \mathcal{F}, \pi \otimes_{D_1} 1)$ be the closure of the representation on $\mathcal{E} \otimes_{D_1} \mathcal{F}$ defined in Lemma 2.18.

Lemma 2.21. *Let $I \colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ be an isometric intertwiner between two representations $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$, and let $J \colon \mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ be an isometric intertwiner of* C*-correspondences. Then $I \otimes_{D_1} J \colon \mathcal{E}_1 \otimes_{D_1} \mathcal{F}_1 \hookrightarrow \mathcal{E}_2 \otimes_{D_1} \mathcal{F}_2$ is an isometric intertwiner between $(\mathfrak{E}_1 \otimes_{D_1} \mathcal{F}_1, \pi_1 \otimes 1)$ and $(\mathfrak{E}_2 \otimes_{D_1} \mathcal{F}_2, \pi_2 \otimes 1)$.*

Proof. The isometry $I \otimes_{D_1} J$ maps the image X_1 of $\mathfrak{E}_1 \odot \mathcal{F}_1$ to the image X_2 of $\mathfrak{E}_2 \odot \mathcal{F}_2$ and intertwines the operators $\pi_1(a) \otimes 1$ on X_1 and $\pi_2(a) \otimes 1$ on X_2 for all $a \in A$. That is, it intertwines the representations defined in Lemma 2.18. It also intertwines their closures by Lemma 2.14. $\square$

The lemma gives a bifunctor

$$\otimes_{D_1} \colon \operatorname{Rep}(A, D_1) \times \operatorname{Rep}(D_1, D_2) \to \operatorname{Rep}(A, D_2). \tag{2.22}$$

The corresponding bifunctor

$$\otimes_{D_1} \colon \operatorname{Rep}(B, D_1) \times \operatorname{Rep}(D_1, D_2) \to \operatorname{Rep}(B, D_2)$$

for a C*-algebra B is the usual composition of C*-correspondences. This composition is associative up to canonical unitaries

$$\mathcal{E} \otimes_{D_1} (\mathcal{F} \otimes_{D_2} \mathcal{G}) \xrightarrow{\sim} (\mathcal{E} \otimes_{D_1} \mathcal{F}) \otimes_{D_2} \mathcal{G}, \qquad \xi \otimes (\eta \otimes \zeta) \mapsto (\xi \otimes \eta) \otimes \zeta, \tag{2.23}$$

for all triples of composable C*-correspondences.

Lemma 2.24. *If $\mathcal{E}$ carries a representation $(\mathfrak{E}, \pi)$ of a *-algebra A, then the unitary in* (2.23) *is an intertwiner $(\mathfrak{E}, \pi) \otimes_{D_1} (\mathcal{F} \otimes_{D_2} \mathcal{G}) \xrightarrow{\sim} \big((\mathfrak{E}, \pi) \otimes_{D_1} \mathcal{F}\big) \otimes_{D_2} \mathcal{G}$.*

Proof. The bilinear map from $\mathfrak{E} \times \mathcal{F}$ to $\mathfrak{E} \otimes_{D_1} \mathcal{F}$ is separately continuous with respect to the graph topologies on $\mathfrak{E}$ and $\mathfrak{E} \otimes_{D_1} \mathcal{F}$ and the norm topology on $\mathcal{F}$. Since the image of $\mathcal{F} \odot \mathcal{G}$ in the Hilbert module $\mathcal{F} \otimes_{D_2} \mathcal{G}$ is dense in the norm topology, the image of $\mathfrak{E} \odot \mathcal{F} \odot \mathcal{G}$ in $\mathcal{E} \otimes_{D_1} (\mathcal{F} \otimes_{D_2} \mathcal{G})$ is a core for the representation $(\mathfrak{E}, \pi) \otimes_{D_1} (\mathcal{F} \otimes_{D_2} \mathcal{G})$. Since the image of $\mathfrak{E} \odot \mathcal{F}$ in $\mathfrak{E} \otimes_{D_1} \mathcal{F}$ is dense in the graph topology, the image of $\mathfrak{E} \odot \mathcal{F} \odot \mathcal{G}$ in $(\mathcal{E} \otimes_{D_1} \mathcal{F}) \otimes_{D_2} \mathcal{G}$ is a core for the representation $\big((\mathfrak{E}, \pi) \otimes_{D_1} \mathcal{F}\big) \otimes_{D_2} \mathcal{G}$. The unitary in (2.23) intertwines between these cores. Hence it also intertwines between the resulting closed representations by Lemma 2.14. $\square$

Definition 2.25. Let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ be two representations of A on Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$. An adjointable operator $x\colon \mathcal{E}_1 \to \mathcal{E}_2$ is an *intertwiner* if $x(\mathfrak{E}_1) \subseteq \mathfrak{E}_2$ and $x\pi_1(a)\xi = \pi_2(a)x\xi$ for all $a \in A$, $\xi \in \mathfrak{E}_1$. It is a **-intertwiner* if both x and x^* are intertwiners.

Any adjointable intertwiner between two representations of a C^*-algebra B is a *-intertwiner. In contrast, for a general *-algebra, even the adjoint of a unitary intertwiner u fails to be an intertwiner if $u(\mathfrak{E}_1) \subsetneq \mathfrak{E}_2$.

Example 2.26. Let t be a positive symmetric operator on a Hilbert space $\mathcal{H}$. Assume that $\bigcap_{n\in\mathbb{N}} \operatorname{dom} t^n$ is dense in $\mathcal{H}$, so that t generates a representation π of the polynomial algebra $\mathbb{C}[x]$ on $\mathcal{H}$. The Friedrichs extension of t is a positive self-adjoint operator t' on $\mathcal{H}$. It generates another representation π' of $\mathbb{C}[x]$ on $\mathcal{H}$. The identity map on $\mathcal{H}$ is a unitary intertwiner $\pi \hookrightarrow \pi'$. It is not a *-intertwiner unless $t = t'$.

The following proposition characterises when an adjointable isometry $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}$ between two representations on Hilbert D-modules is a *-intertwiner. Since $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_1^\perp$ if I is adjointable, we may as well assume that I is the inclusion of a direct summand.

Proposition 2.27. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert modules over a C^*-algebra D and let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}, \pi)$ be representations of A on $\mathcal{E}_1$ and $\mathcal{E}_1 \oplus \mathcal{E}_2$, respectively. The following are equivalent:*

1. *the canonical inclusion $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_1 \oplus \mathcal{E}_2$ is a *-intertwiner from π_1 to π;*
2. *the canonical inclusion $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_1 \oplus \mathcal{E}_2$ is an intertwiner from π_1 to π and $\mathfrak{E} = \mathfrak{E}_1 + (\mathfrak{E} \cap \mathcal{E}_2)$;*
3. *there is a representation $(\mathfrak{E}_2, \pi_2)$ on $\mathcal{E}_2$ such that $\pi = \pi_1 \oplus \pi_2$.*

Proof. We view $\mathcal{E}_1$ and $\mathcal{E}_2$ as subspaces of $\mathcal{E}_1 \oplus \mathcal{E}_2$, so we may drop the isometry I from our notation. The implication (3)⇒(1) is trivial. We are going to prove (1)⇒(2)⇒(3). First assume that I is a *-intertwiner. Then I is an intertwiner. In particular, $\mathfrak{E}_1 \subseteq \mathfrak{E}$. Write $\xi \in \mathfrak{E}$ as $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathcal{E}_1$, $\xi_2 \in \mathcal{E}_2$. Since I^* is an intertwiner, $\xi_1 = I^*(\xi) \in \mathfrak{E}_1$. Hence $\xi_2 = \xi - \xi_1 \in \mathfrak{E} \cap \mathcal{E}_2$. Thus (1) implies (2).
If (2) holds, then $\mathfrak{E}_1 \subseteq \mathfrak{E}$ is π-invariant and $\pi|_{\mathfrak{E}_1} = \pi_1$ because I is an intertwiner. We claim that $\mathfrak{E}_2 := \mathfrak{E} \cap \mathcal{E}_2$ is π-invariant as well. Let $\xi \in \mathfrak{E}_2$ and $\eta \in \mathfrak{E}_1$. Then $\langle \eta, \pi(a)\xi \rangle = \langle \pi(a^*)\eta, \xi \rangle = \langle \pi_1(a^*)\eta, \xi \rangle = 0$ because $\pi_1(a^*)\eta \in \mathcal{E}_1$ is orthogonal to $\mathcal{E}_2$. Since $\mathfrak{E}_1$ is dense in $\mathcal{E}_1$, this implies $\pi(a)\xi \in \mathcal{E}_1^\perp = \mathcal{E}_2$, and this implies our claim.
The condition (2) implies $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$ as a vector space with $\mathfrak{E}_2 = \mathcal{E}_2 \cap \mathfrak{E}$ because $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$. Then $\mathfrak{E}_2$ is dense in $\mathcal{E}_2$ because $\mathfrak{E}$ is dense in $\mathcal{E}_1 \oplus \mathcal{E}_2$. Thus $(\mathfrak{E}_2, \pi|_{\mathfrak{E}_2})$ is a representation of A on $\mathcal{E}_2$. And $(\mathfrak{E}, \pi)$ is the direct sum of $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi|_{\mathfrak{E}_2})$ because $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$ and $\pi_1 = \pi|_{\mathfrak{E}_1}$. Thus (2) implies (3). □

3 Integrable representations and C*-hulls

From now on, *we tacitly assume representations to be closed*. Proposition 2.5 shows that this is no serious loss of generality.
Let A be a unital *-algebra. We assume that a class of "integrable" (closed) representations of A on Hilbert modules is chosen. Let $\mathrm{Rep}_{\mathrm{int}}(A, D) \subseteq \mathrm{Rep}(A, D)$ be the full subcategory with integrable representations on Hilbert D-modules as objects. Being full means that the set of arrows between two integrable representations of A is still the set of all isometric intertwiners. We sometimes write $\mathrm{Rep}_{\mathrm{int}}(A)$ and $\mathrm{Rep}(A)$ for the collection of all the categories $\mathrm{Rep}_{\mathrm{int}}(A, D)$ and $\mathrm{Rep}(A, D)$ for all C*-algebras D. A C*-hull is a C*-algebra B with natural isomorphisms $\mathrm{Rep}(B, D) \cong \mathrm{Rep}_{\mathrm{int}}(A, D)$ for all C*-algebras D. More precisely:

Definition 3.1. A C*-*hull* for the integrable representations of A is a C*-algebra B with a family of bijections $\Phi = \Phi^{\mathcal{E}}$ from the set of representations of B on $\mathcal{E}$ to the set of integrable representations of A on $\mathcal{E}$ for all Hilbert modules $\mathcal{E}$ over all C*-algebras D with the following properties:

- *compatibility with isometric intertwiners*: an isometry $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ (not necessarily adjointable) is an intertwiner between two representations ϱ_1 and ϱ_2 of B *if and only if* it is an intertwiner between $\Phi(\varrho_1)$ and $\Phi(\varrho_2)$;
- *compatibility with interior tensor products*: if $\mathcal{F}$ is a correspondence from D_1 to D_2, $\mathcal{E}$ is a Hilbert D_1-module, and ϱ is a representation of B on $\mathcal{E}$, then $\Phi(\varrho \otimes_{D_1} 1_{\mathcal{F}}) = \Phi(\varrho) \otimes_{D_1} 1_{\mathcal{F}}$ as representations of A on $\mathcal{E} \otimes_{D_1} \mathcal{F}$.

The compatibility with isometric intertwiners means that the bijections Φ for all $\mathcal{E}$ with fixed D form an isomorphism of categories $\mathrm{Rep}(B, D) \cong \mathrm{Rep}_{\mathrm{int}}(A, D)$ which, in addition, does not change the underlying Hilbert D-modules. The compatibility with interior tensor products expresses that these isomorphisms of categories for different D are natural with respect to C*-correspondences.

Definition 3.2. A *weak* C*-*hull* for the integrable representations of A is a C*-algebra B with a family of bijections Φ between representations of B and integrable representations of A on Hilbert modules that is compatible with unitary *-intertwiners and interior tensor products.

Much of the general theory also works for weak C*-hulls. But the Induction Theorem 9.26 fails for weak C*-hulls, as shown by a counterexample in §9.6.

Proposition 3.3. *Let a class of integrable representations of A have a weak C*-hull B. Let $(\mathfrak{E}_1, \pi_1)$ and $(\mathfrak{E}_2, \pi_2)$ be integrable representations of A on Hilbert D-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, and let ϱ_i be the corresponding representations of B on $\mathcal{E}_i$ for $i = 1, 2$. An adjointable operator $x\colon \mathcal{E}_1 \to \mathcal{E}_2$ is a *-intertwiner from $(\mathfrak{E}_1, \pi_1)$ to $(\mathfrak{E}_2, \pi_2)$ if and only if it is an intertwiner from ϱ_1 to ϱ_2.*

Proof. Working with the direct sum representations on $\mathcal{E}_1 \oplus \mathcal{E}_2$ and the adjointable operator $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$, we may assume without loss of generality that

$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, $\mathfrak{E}_1 = \mathfrak{E}_2 = \mathfrak{E}$, $\pi_1 = \pi_2 = \pi$, and $\varrho_1 = \varrho_2 = \varrho$. The adjointable intertwiners for the representation ϱ of B form a C*-algebra B': the commutant of B in $\mathbb{B}(\mathcal{E})$. We claim that the *-intertwiners for the representation π of A also form a C*-algebra A'. Intertwiners and hence *-intertwiners form an algebra. Thus A' is a *-algebra. We show that it is closed.
Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of adjointable intertwiners for $(\mathfrak{E},\pi)$ that converges in norm to $x \in \mathbb{B}(\mathcal{E})$. Let $\xi \in \mathfrak{E}$. Then $x_i(\xi) \in \mathfrak{E}$ because each x_i is an intertwiner. Since $\pi(a)(x_i\xi) = x_i\pi(a)\xi$ is norm-convergent for each $a \in A$, the sequence $x_i(\xi)$ is a Cauchy sequence for the graph topology on $\mathfrak{E}$. Since representations are tacitly assumed to be closed, this Cauchy sequence converges in $\mathfrak{E}$, so that $x(\mathfrak{E}) \subseteq \mathfrak{E}$. Moreover, $x(\pi(a)\xi) = \pi(a)x(\xi)$ for all $a \in A$, $\xi \in \mathfrak{E}$, so x is again an intertwiner. Thus the algebra of intertwiners is norm-closed. This implies that A' is a C*-algebra.
Since the family of bijections $\mathrm{Rep}_{\mathrm{int}}(A) \cong \mathrm{Rep}(B)$ is compatible with unitary *-intertwiners, a unitary operator on $\mathcal{E}$ is a *-intertwiner for A if and only if it is an intertwiner for B. That is, the unital C*-subalgebras $A', B' \subseteq \mathbb{B}(\mathcal{E})$ contain the same unitaries. A unital C*-algebra is the linear span of its unitaries because any self-adjoint element t of norm at most 1 may be written as

$$t = {}^1\!/\!{}_2\bigl(t + \mathrm{i}\sqrt{1-t^2}\bigr) + {}^1\!/\!{}_2\bigl(t - \mathrm{i}\sqrt{1-t^2}\bigr)$$

and $t \pm \mathrm{i}\sqrt{1-t^2}$ are unitary. Thus $A' = B'$. This is what we had to prove. □

Corollary 3.4. *Let* $\mathrm{Rep}_{\mathrm{int}}(A)$ *have a weak* C*-*hull* B. *Direct sums and summands of integrable representations remain integrable, and the family of bijections* $\mathrm{Rep}_{\mathrm{int}}(A) \cong \mathrm{Rep}(B)$ *preserves direct sums.*

Proof. Let π_1, π_2 be representations of A on Hilbert D-modules $\mathcal{E}_1, \mathcal{E}_2$. Let $S_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E}_1 \oplus \mathcal{E}_2$ for $i = 1, 2$ be the inclusion maps. First we assume that π_1, π_2 are integrable. Let ϱ_i be the representation of B on $\mathcal{E}_i$ corresponding to π_i, and let π be the integrable representation of A on $\mathcal{E}_1 \oplus \mathcal{E}_2$ corresponding to the representation $\varrho_1 \oplus \varrho_2$ of B. The isometries S_i are intertwiners from ϱ_i to $\varrho_1 \oplus \varrho_2$. By Proposition 3.3, they are *-intertwiners from π_i to π. Hence $\pi = \pi_1 \oplus \pi_2$ by Proposition 2.27. Thus $\pi_1 \oplus \pi_2$ is integrable and the family of bijections $\mathrm{Rep}_{\mathrm{int}}(A) \cong \mathrm{Rep}(B)$ preserves direct sums. The same argument works for infinite direct sums.
Now we assume instead that $\pi_1 \oplus \pi_2$ is integrable. Let ϱ be the representation of B corresponding to $\pi_1 \oplus \pi_2$. The orthogonal projection onto $\mathcal{E}_1$ is a *-intertwiner on the representation $\pi_1 \oplus \pi_2$ by Proposition 2.27, and hence also on ϱ by Proposition 3.3. Thus $\varrho = \varrho_1 \oplus \varrho_2$ for some representations ϱ_i of B on $\mathcal{E}_i$. Let π_i' be the integrable representation of A corresponding to ϱ_i. The isometry S_i is a *-intertwiner from ϱ_i to $\varrho_1 \oplus \varrho_2$ and hence from π_i' to $\pi_1 \oplus \pi_2$ by Proposition 3.3. This implies $\pi_i' = \pi_i$, so that π_i is integrable for $i = 1, 2$. □

Definition 3.5. Let B be a weak C*-hull for A. The *universal* integrable representation of A is the integrable representation $(\mathfrak{B}, \mu)$ of A on B that corresponds to the identity representation of B on itself.

Proposition 3.6. *Let B with a family of bijections Φ between representations of B and integrable representations of A on Hilbert modules be a weak* C^**-hull for the integrable representations of A. Let $(\mathfrak{B},\mu)$ be the universal integrable representation of A. Then $\Phi(\mathcal{E}) \cong (\mathfrak{B},\mu) \otimes_B \mathcal{E}$ for any* C^**-correspondence $\mathcal{E}$ from B to D.* (*The proof makes this isomorphism more precise.*)

Proof. Let $\varrho\colon B \to \mathbb{B}(\mathcal{E})$ be a representation of B on a Hilbert module $\mathcal{E}$. Then $u\colon B \otimes_\varrho \mathcal{E} \xrightarrow{\sim} \mathcal{E}$, $b \otimes \xi \mapsto \varrho(b)\xi$, is a unitary *-intertwiner between the interior tensor product of the identity representation of B with $\mathcal{E}$ and the representation ϱ on $\mathcal{E}$. As Φ is compatible with interior tensor products and unitary *-intertwiners, u is a unitary *-intertwiner between $(\mathfrak{B},\mu) \otimes_B \mathcal{E}$ and $\Phi(\varrho)$. Therefore, the image $u(\mathfrak{B} \odot \mathcal{E}) = \varrho(\mathfrak{B})\mathcal{E}$ is a core for $\Phi(\varrho)$, and $a \in A$ acts on this core by $a \mapsto u(\mu(a) \otimes 1)u^*$ or, explicitly, $a \cdot (\varrho(b)\xi) = \varrho(\mu(a)b)\xi$ for all $a \in A$, $b \in \mathfrak{B}$, $\xi \in \mathcal{E}$. □

Put in a nutshell, the whole isomorphism between integrable representations of A and representations of B is encoded in the single representation $(\mathfrak{B},\mu)$ of A on B. This is similar to Schmüdgen's approach in [29]. In the following, we disregard the canonical unitary u in the proof of Proposition 3.6 and write $\Phi(\varrho) = (\mathfrak{B},\mu) \otimes_B \mathcal{E}$.

A (weak) C^*-hull B does not solve the problem of describing the integrable representations of A. It only reduces it to the study of the representations of the C^*-algebra B. This reduction is useful because it gets rid of unbounded operators. If B is of type I, then any Hilbert space representation of B is a direct integral of irreducible representations, and irreducible representations may, in principle, be classified. Thus integrable Hilbert space representations of A are direct integrals of irreducible integrable representations, and the latter may, in principle, be classified. But if B is not of type I, then the integrable Hilbert space representations of A are exactly as complicated as the Hilbert space representations of B, and giving the C^*-algebra B may well be the best one can say about them.

Proposition 3.7. *A class of integrable representations has at most one weak* C^**-hull.*

Proof. Let B_1 and B_2 be weak C^*-hulls for the same class of integrable representations of A. The identity map on B_1, viewed as a representation of B_1 on itself, corresponds first to an integrable representation of A on B_1 and further to a representation of B_2 on B_1. This is a "morphism" from B_2 to B_1, that is, a nondegenerate *-homomorphism $B_2 \to \mathcal{M}(B_1)$. Similarly, we get a morphism from B_1 to B_2. These morphisms $B_1 \leftrightarrow B_2$ are inverse to each other with respect to the composition of morphisms because the maps they induce on representations of B_1 and B_2 on B_1 and B_2 are inverse to each other. An isomorphism in the category of morphisms is an isomorphism of C^*-algebras in the usual sense by [6, Proposition 2.10]. □

Now take any representation $(\mathfrak{B},\mu)$ of A on B. When is this the universal integrable representation of a (weak) C*-hull? Let D be a C*-algebra and $\mathcal{E}$ a Hilbert D-module. For a representation $\varrho\colon B \to \mathbb{B}(\mathcal{E})$, let $\Phi(\varrho) = (\mathfrak{B},\mu) \otimes_\varrho \mathcal{E}$ be the induced representation of A on $\mathcal{E}$ as in the proof of Proposition 3.6. A representation of A is called *B-integrable* if it is in the image of Φ.

Proposition 3.8. *The* C*-*algebra B is a weak* C*-*hull for the B-integrable representations of A if and only if*

(1) *if two representations $\varrho_1,\varrho_2\colon B \rightrightarrows \mathbb{B}(\mathcal{H})$ on the same Hilbert space $\mathcal{H}$ satisfy $\mu\otimes_B \varrho_1 = \mu\otimes_B\varrho_2$ as closed representations of A, then $\varrho_1=\varrho_2$.*

It is a C*-*hull if and only if* (1) *and the following equivalent conditions hold:*

(2) *Let $(\mathfrak{H},\pi)$ be a representation of A on a Hilbert space $\mathcal{H}$ and let $(\mathfrak{H}_0,\pi_0)$ be a subrepresentation on a closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$; that is, $\mathfrak{H}_0 \subseteq \mathfrak{H}$ and $\pi_0(a) = \pi(a)|_{\mathfrak{H}_0}$ for all $a\in A$. If both π_0 and π are B-integrable, then $\mathfrak{H} = \mathfrak{H}_0 \oplus (\mathfrak{H}\cap\mathcal{H}_0^\perp)$ as vector spaces.*

(3) *Isometric intertwiners between B-integrable Hilbert space representations of A are *-intertwiners.*

(4) *B-integrable subrepresentations of B-integrable Hilbert space representations of A are direct summands.*

The conditions (1)–(4) *together are equivalent to*

(5) *let $\varrho\colon B\to\mathbb{B}(\mathcal{H})$ be a Hilbert space representation and let $(\mathfrak{H},\pi)$ be the associated representation of A on $\mathcal{H}$. If $(\mathfrak{H}_0,\pi|_{\mathfrak{H}_0})$ is a B-integrable subrepresentation of $(\mathfrak{H},\pi)$ on a closed subspace $\mathcal{H}_0\subseteq\mathcal{H}$, then the projection onto $\mathcal{H}_0$ commutes with $\varrho(B)$.*

Proof. The map Φ is compatible with interior tensor products by Lemma 2.24. The condition (1) says that Φ is injective on Hilbert space representations. We claim that this implies injectivity also for representations on a Hilbert module $\mathcal{E}$ over a C*-algebra D. Let ϱ_1,ϱ_2 be representations of B on $\mathcal{E}$ with $\mu\otimes_B\varrho_1 = \mu\otimes_B\varrho_2$. Let $D\to\mathbb{B}(\mathcal{H})$ be a faithful representation. Then the representations $\varrho_1\otimes_D 1$ and $\varrho_2\otimes_D 1$ on the Hilbert space $\mathcal{E}\otimes_D\mathcal{H}$ satisfy $\mu\otimes_B\varrho_1\otimes_D 1 = \mu\otimes_B\varrho_2\otimes_D 1$ by Lemma 2.24. Then condition (1) implies $\varrho_1\otimes_D 1 = \varrho_2\otimes_D 1$. Since the representation $\mathbb{B}(\mathcal{E})\to\mathbb{B}(\mathcal{E}\otimes_D\mathcal{H})$ is faithful, this implies $\varrho_1=\varrho_2$. So Φ is injective also for representations on $\mathcal{E}$.
The image of Φ consists exactly of the B-integrable representations of A by definition. A unitary operator u *-intertwines two representations $(\mathfrak{E}_1,\pi_1)$ and $(\mathfrak{E}_2,\pi_2)$ of A if and only if $\pi_2 = u\pi_1u^*$, where $u\pi_1u^*$ denotes the representation with domain $u(\mathfrak{E}_1)$ and $(u\pi_1u^*)(a) = u\pi_1(a)u^*$. Similarly, u intertwines two representations ϱ_1 and ϱ_2 of B if and only if $\varrho_2 = u\varrho_1u^*$. Hence (1) implies that a unitary that *-intertwines two B-integrable representations of A also intertwines the corresponding representations of B. The converse is clear. So B is a weak C*-hull for the B-integrable representations if and only if (1) holds.

The equivalence between (2), (3) and (4) follows from Proposition 2.27 by writing $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$. Assume that B is a C*-hull. An isometric intertwiner for A is also one for B. Then it is a *-intertwiner for A and its range projection is an intertwiner for B by Proposition 3.3. Thus both (3) and (5) follow if B is a C*-hull.

Conversely, assume (1) and (3). We are going to prove that B is a C*-hull for the B-integrable representations of A. We have already seen that B is a weak C*-hull. We must check compatibility with isometric intertwiners.

Let D be a C*-algebra and let $\mathcal{E}_1, \mathcal{E}_2$ be Hilbert D-modules with representations ϱ_1, ϱ_2 of B. The corresponding representations $(\mathfrak{E}_i, \pi_i)$ of A for $i = 1, 2$ are the closures of the representations on $\varrho_i(\mathfrak{B})\mathcal{E}_i$ given by $\pi_i(a)(\varrho_i(b)\xi) := \varrho_i(\mu(a)b)(\xi)$ for $a \in A$, $b \in \mathfrak{B}$, $\xi \in \mathcal{E}_i$. Hence an isometric intertwiner for B is also one for A. Conversely, let $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ be a Hilbert module isometry with $I(\mathfrak{E}_1) \subseteq \mathfrak{E}_2$ and $\pi_2(a)(I\xi) = I(\pi_1(a)\xi)$ for all $a \in A$, $\xi \in \mathfrak{E}_1$. We must prove $\varrho_2(b)I = I\varrho_1(b)$ for all $b \in B$.

Let $\varphi\colon D \hookrightarrow \mathbb{B}(\mathcal{K})$ be a faithful representation on a Hilbert space $\mathcal{K}$. Equip $\mathcal{H}_i := \mathcal{E}_i \otimes_\varphi \mathcal{K}$ with the induced representations $\tilde{\varrho}_i$ of B and $\tilde{\pi}_i$ of A for $i = 1, 2$. Since the family of bijections $\Phi\colon \operatorname{Rep}(B) \xrightarrow{\sim} \operatorname{Rep}_{\mathrm{int}}(A)$ is compatible with interior tensor products, it maps $\tilde{\varrho}_i$ to $\tilde{\pi}_i$. The operator I induces an isometric intertwiner $\tilde{I}$ from $\tilde{\pi}_1$ to $\tilde{\pi}_2$ by Lemma 2.21.

Since $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are B-integrable, we are in the situation of (3). So $\tilde{I}$ is a *-intertwiner from $\tilde{\pi}_1$ to $\tilde{\pi}_2$. Thus $\tilde{I}$ is an intertwiner from $\tilde{\varrho}_1$ to $\tilde{\varrho}_2$ by Proposition 3.3. That is, $\tilde{I}\tilde{\varrho}_1(b) = \tilde{\varrho}_2(b)\tilde{I}$ for all $b \in B$. Equivalently, $(I\varrho_1(b)\xi) \otimes \eta = (\varrho_2(b)I\xi) \otimes \eta$ in $\mathcal{E}_2 \otimes_\varphi \mathcal{H}$ for all $b \in B$, $\xi \in \mathcal{E}$, $\eta \in \mathcal{H}$. Since the representation φ is faithful, this implies $I\varrho_1(b)\xi = \varrho_2(b)I\xi$ for all b, ξ, so that $I\varrho_1(b) = \varrho_2(b)I$ for all b, that is, I intertwines ϱ_1 and ϱ_2. Thus Φ is compatible with isometric intertwiners.

Since (5) holds for C*-hulls, we have proved along the way that (1) and (3) imply (5). It remains to show, conversely, that (5) implies (3) and (1). In the situation of (3), the projection P onto $\mathcal{H}_0$ commutes with B by (5). Thus the representation of B on $\mathcal{H}$ is a direct sum of representations on $\mathcal{H}_0$ and $\mathcal{H}_0^{\perp}$. This is inherited by the induced representation of A and its domain. So (5) implies (3).

In the situation of (1), form the direct sum representation $\varrho_1 \oplus \varrho_2$ on $\mathcal{H} \oplus \mathcal{H}$ and let $\mathcal{H}_0 := \{(\xi, \xi) \mid \xi \in \mathcal{H}\}$. The representation of A corresponding to $\varrho_1 \oplus \varrho_2$ is $\mu \otimes_{\varrho_1} \mathcal{H} \oplus \mu \otimes_{\varrho_2} \mathcal{H}$. Since $\mu \otimes_{\varrho_1} \mathcal{H} = \mu \otimes_{\varrho_2} \mathcal{H}$ by assumption, the domain of $\mu \otimes_{\varrho_1} \mathcal{H} \oplus \mu \otimes_{\varrho_2} \mathcal{H}$ is $\mathfrak{H} \oplus \mathfrak{H}$ for some dense subspace $\mathfrak{H} \subseteq \mathcal{H}$, and $\mathfrak{H}_0 := \{(\xi, \xi) \mid \xi \in \mathfrak{H}\}$ is a dense subspace in $\mathcal{H}_0$ that is invariant for the representation $\mu \otimes_{\varrho_1} \mathcal{H} \oplus \mu \otimes_{\varrho_2} \mathcal{H}$. The restricted representation on this subspace is B-integrable because it is unitarily equivalent to $\mu \otimes_{\varrho_1} \mathcal{H} = \mu \otimes_{\varrho_2} \mathcal{H}$. Therefore, the projection onto $\mathcal{H}_0$ commutes with the representation of B by (5). Thus $\varrho_1 = \varrho_2$. So (5) implies (1). □

The equivalent conditions (2)–(4) may be easier to check than (5) because they do not involve the C*-hull.

Corollary 3.9. *Let A be a *-algebra and let B_i be* C**-algebras with representations $(\mathfrak{B}_i, \mu_i)$ of A for $i = 1, 2$. Assume that for each Hilbert space $\mathcal{H}$, the maps $\Phi_i\colon \operatorname{Rep}(B_i, \mathcal{H}) \to \operatorname{Rep}(A, \mathcal{H})$, $\varrho_i \mapsto (\mathfrak{B}_i, \mu_i) \otimes_{\varrho_i} \mathcal{H}$, are injective and have the same image. Then there is a unique isomorphism $B_1 \cong B_2$ intertwining the representations $(\mathfrak{B}_i, \mu_i)$ of A for $i = 1, 2$.*

Hence a C*-envelope as defined in [7] is unique if it exists.

Proof. Both B_1 and B_2 are weak C*-hulls for the same class of representations of A by Proposition 3.8. Proposition 3.7 gives the isomorphism $B_1 \cong B_2$. □

Remark 3.10. The Hilbert space representations of a C*-algebra only determine its bidual W*-algebra, not the C*-algebra itself. Hence it is remarkable that the conditions in Proposition 3.8 and Corollary 3.9 only need Hilbert space representations. For Corollary 3.9, this works because the bijection between the representations is of a particular form, induced by representations of A.

The condition (1) in Proposition 3.8 is required in several other theories that associate a C*-algebra to a *-algebra, such as the host algebras of Grundling [10, 11], the C*-envelopes of Dowerk and Savchuk [7], or the notion of a C*-algebra generated by affiliated multipliers by Woronowicz [32], see [32, Theorem 3.3] or the proof of Theorem 5.19 below.

Definition 3.11. Let A be a *-algebra. A class of "integrable" representations of A on Hilbert modules over C*-algebras is *admissible* if it satisfies the conditions (1)–(4) below, and *weakly admissible* if it satisfies (1)–(3).

(1) If there is a unitary *-intertwiner from an integrable representation to another representation, then the latter is integrable.

(2) If D and D' are C*-algebras, $\mathcal{F}$ is a correspondence from D to D', and $(\mathfrak{E}, \pi)$ is an integrable representation of A on a Hilbert D-module $\mathcal{E}$, then the representation $(\mathfrak{E}, \pi) \otimes_D \mathcal{F}$ on $\mathcal{E} \otimes_D \mathcal{F}$ is integrable.

(3) Direct sums and summands of integrable representations are integrable.

(4) Any integrable subrepresentation of an integrable representation of A on a Hilbert space is a direct summand.

Lemma 3.12. *Any class of integrable representations with a* (*weak*) C**-hull is* (*weakly*) *admissible.*

Proof. If there is a C*-hull, Proposition 3.8 implies (4) in Definition 3.11. If there is a weak C*-hull, then (1) and (2) in Definition 3.11 follow from the compatibility with unitary *-intertwiners and interior tensor products in the definition of a C*-hull, and (3) follows from Corollary 3.4. □

Proposition 3.13. *Let A be a unital *-algebra and let $\mathcal{E}$ be a Hilbert module over a* C**-algebra D. There is a natural bijection between the sets of representations of A on $\mathcal{E}$ and $\mathbb{K}(\mathcal{E})$. It preserves integrability if the class of integrable* *representations of A is weakly admissible or, in particular, if it has a weak* C^**-hull.*

Proof. We may view $\mathcal{E}$ as an imprimitivity bimodule between $\mathbb{K}(\mathcal{E})$ and the ideal I in D that is spanned by the inner products $\langle\xi,\eta\rangle_D$ for $\xi,\eta\in\mathcal{E}$. Let $\mathcal{E}^*$ be the inverse imprimitivity bimodule, which is a Hilbert module over $\mathbb{K}(\mathcal{E})$ with $\mathbb{K}(\mathcal{E}^*)\cong I$. Then $\mathbb{K}(\mathcal{E})\cong\mathcal{E}\otimes_D\mathcal{E}^*$ and $\mathcal{E}^*\otimes_{\mathbb{K}(\mathcal{E})}\mathcal{E}=I$.
If $(\pi,\mathfrak{E})$ is a representation of A on $\mathcal{E}$, then $(\pi,\mathfrak{E})\otimes_D\mathcal{E}^*$ is a representation of A on $\mathcal{E}\otimes_D\mathcal{E}^*=\mathbb{K}(\mathcal{E})$. This maps $\operatorname{Rep}(A,\mathcal{E})$ to $\operatorname{Rep}(A,\mathbb{K}(\mathcal{E}))$. If $(\varrho,\mathfrak{K})$ is a representation of A on $\mathbb{K}(\mathcal{E})$, then $(\varrho,\mathfrak{K})\otimes_{\mathbb{K}(\mathcal{E})}\mathcal{E}$ is a representation of A on $\mathbb{K}(\mathcal{E})\otimes_{\mathbb{K}(\mathcal{E})}\mathcal{E}\cong\mathcal{E}$. This maps $\operatorname{Rep}(A,\mathbb{K}(\mathcal{E}))$ to $\operatorname{Rep}(A,\mathcal{E})$. We claim that these two maps are inverse to each other. Both preserve integrability by (2) in Definition 3.11.
The map $\operatorname{Rep}(A,\mathcal{E})\to\operatorname{Rep}(A,\mathbb{K}(\mathcal{E}))\to\operatorname{Rep}(A,\mathcal{E})$ sends a representation $(\pi,\mathfrak{E})$ of A on $\mathcal{E}$ to the representation $(\pi,\mathfrak{E})\otimes_D(\mathcal{E}^*\otimes_{\mathbb{K}(\mathcal{E})}\mathcal{E})=(\pi,\mathfrak{E})\otimes_D I$ of A on $\mathcal{E}\cong\mathcal{E}\otimes_D I$ by Lemma 2.24. This is the restriction of π to $\mathfrak{E}\cdot I\subseteq\mathfrak{E}$. Since $\mathcal{E}$ is also a Hilbert module over I, it is nondegenerate as a right I-module. Therefore, if (u_i) is an approximate unit in I, then $\lim\xi u_i=\xi$ for all $\xi\in\mathcal{E}$. Then also $\lim\pi(a)\xi u_i=\pi(a)\xi$ for all $\xi\in\mathfrak{E}$, $a\in A$, so $\lim\xi u_i=\xi$ in the graph topology for all $\xi\in\mathfrak{E}$. Thus $\mathfrak{E}\cdot I=\mathfrak{E}$, and we get the identity map on $\operatorname{Rep}(A,\mathcal{E})$. A similar, easier argument shows that we also get the identity map on $\operatorname{Rep}(A,\mathbb{K}(\mathcal{E}))$. □

4 POLYNOMIALS IN ONE VARIABLE I

Let $A=\mathbb{C}[x]$ with $x=x^*$. A (not necessarily closed) representation of A on a Hilbert D-module $\mathcal{E}$ is determined by a dense D-submodule $\mathfrak{E}\subseteq\mathcal{E}$ and a single symmetric operator $\pi(x)\colon\mathfrak{E}\to\mathfrak{E}$, that is, $\pi(x)\subseteq\pi(x)^*$. Then $\pi(x^n)=\pi(x)^n$.

LEMMA 4.1. *The graph topology on $\mathfrak{E}$ is generated by the increasing sequence of norms $\|\xi\|_n:=\|\langle\xi,(1+\pi(x^{2n}))\xi\rangle\|$ for $n\in\mathbb{N}$.*

Proof. We must show that for any $a\in\mathbb{C}[x]$ there are $C>0$ and $n\in\mathbb{N}$ with $\|\xi\|_a\le C\|\xi\|_n$. We choose n so that a has degree at most n. Then there is $C>0$ so that $C(1+t^{2n})>1+|a(t)|^2$ for all $t\in\mathbb{R}$. Thus the polynomial $b:=C(1+x^{2n})-(1+a^*a)$ is positive on $\mathbb{R}$. So the zeros of b are complex and come in pairs $\lambda_j\pm\mathrm{i}\mu_j$ for $j=1,\dots,n$ with $\lambda_j,\mu_j\in\mathbb{R}$ by the Fundamental Theorem of Algebra. Then $b=\prod_{j=1}^n\big((x-\lambda_j)^2+\mu_j^2\big)=\sum_{k=1}^{2^n}b_k^2$, where each b_k is a product of either $x-\lambda_j$ or μ_j for $j=1,\dots,k$, so $b_k=b_k^*$. Thus $\|\xi\|_a\le C\|\xi\|_n$. □

Thus the monomials $\{x^n\mid n\in\mathbb{N}\}$ form a strong generating set for $\mathbb{C}[x]$. A representation of $\mathbb{C}[x]$ is determined by the closed operators $\overline{\pi(x^n)}$ for $n\in\mathbb{N}$ by Proposition 2.9. In contrast, it is not yet determined by the single closed operator $\overline{\pi(x)}$ because $\{x\}$ is not a *strong* generating set:

Example 4.2. We construct a closed representation of $\mathbb{C}[x]$ on a Hilbert space with $\overline{\pi(x^2)}\subsetneq\big(\overline{\pi(x)}\big)^2$. Let $\mathcal{H}:=L^2(\mathbb{T})$, viewed as the space of $\mathbb{Z}$-periodic functions on $\mathbb{R}$. Let $\mathfrak{H}_0 := \mathrm{C}^\infty(\mathbb{T})$ and let $\pi_0\colon \mathbb{C}[x] \to \mathrm{End}(\mathfrak{H}_0)$ be the polynomial functional calculus for the operator $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$. The graph topology generated by this representation of $\mathbb{C}[x]$ is the usual Fréchet topology on $\mathrm{C}^\infty(\mathbb{T})$. So the representation of $\mathbb{C}[x]$ on $\mathrm{C}^\infty(\mathbb{T})$ is closed. Now for some $\lambda \in \mathbb{T}$, let

$$\mathfrak{H} := \{f \in \mathrm{C}^\infty(\mathbb{T}) \mid f^{(n)}(\lambda) = 0 \text{ for all } n \geq 1\}.$$

This is a closed, $\mathbb{C}[x]$-invariant subspace in $\mathfrak{H}_0$. Let π be the restriction of π_0 to $\mathfrak{H}_0$. This is also a closed representation of $\mathbb{C}[x]$. Its domain $\mathfrak{H}$ is dense in $\mathfrak{H}_0$ in the graph norm of x, but not in the graph norm of x^2. So $\overline{\pi(x)} = \overline{\pi_0(x)}$ and $\overline{\pi(x^2)} \subsetneq \overline{\pi_0(x^2)} = \left(\overline{\pi(x)}\right)^2$.

All notions of integrability for representations of $\mathbb{C}[x]$ that we shall consider imply $\overline{\pi(x^n)} = \overline{\pi(x)}^n$. Under this assumption, an *integrable* representation of $\mathbb{C}[x]$ is determined by the single closed operator $\overline{\pi(x)}$.

Let $B := \mathrm{C}_0(\mathbb{R})$. Let X be the identity function on $\mathbb{R}$, viewed as an unbounded multiplier of B. We define a closed representation $(\mathfrak{B}, \mu)$ of A on $\mathrm{C}_0(\mathbb{R})$ by

$$\mathfrak{B} := \{f \in B \mid \forall n\colon X^n \cdot f \in B\} \quad \text{and} \quad \mu(x^n)f := X^n \cdot f \quad \text{for } f \in \mathfrak{B},\ n \in \mathbb{N}. \tag{4.3}$$

Theorem 4.4. *Let $(\mathfrak{E}, \pi)$ be a representation of $A = \mathbb{C}[x]$ on a Hilbert module $\mathcal{E}$ over a C^*-algebra D. The following are equivalent:*

(1) *$\pi = \mu \otimes_\varrho 1_{\mathcal{E}}$ for a representation $\varrho\colon B \to \mathbb{B}(\mathcal{E})$;*

(2) *$\overline{\pi(a)}$ is regular and self-adjoint for each $a \in A_h := \{a \in A \mid a = a^*\}$;*

(3) *$\overline{\pi(x^n)}$ is regular and self-adjoint for each $n \in \mathbb{N}$;*

(4) *$\overline{\pi(x)}$ is regular and self-adjoint and $\overline{\pi(x^n)} = \overline{\pi(x)}^n$ for all $n \in \mathbb{N}$;*

(5) *$\overline{\pi(x)}$ is regular and self-adjoint and $\mathfrak{E} = \bigcap_{n=1}^\infty \operatorname{dom} \overline{\pi(x)}^n$.*

Call representations with these equivalent properties integrable. The C^*-algebra $\mathrm{C}_0(\mathbb{R})$ is a C^*-hull for the integrable representations of A with $(\mathfrak{B}, \mu)$ as the universal integrable representation.

Proof. If $a \in A_h$, then $\overline{\mu(a)}$ is a self-adjoint, affiliated multiplier of B. Hence $\mu(a) \otimes_D 1$ is a regular, self-adjoint operator on $B \otimes_\varrho \mathcal{E} \cong \mathcal{E}$ for any representation ϱ of B on $\mathcal{E}$ by [17, Proposition 9.10]. Thus (1) implies (2). The implication (2)⇒(3) is trivial. The operator $\overline{\pi(x^n)}$ is always contained in the n-fold power $\overline{\pi(x)}^n$. The latter is symmetric, and a proper suboperator of a symmetric operator cannot be self-adjoint. Thus (3) implies (4). The set $\{x^n \mid n \in \mathbb{N}\}$ is a strong generating set for $\mathbb{C}[x]$ by Lemma 4.1. Equation (2.10) gives $\mathfrak{E} = \bigcap_{n=1}^\infty \operatorname{dom} \overline{\pi(x^n)}$ for any (closed) representation. Thus (4) implies (5).

Assume (5) and abbreviate $t = \overline{\pi(x)}$. The functional calculus for t is a nondegenerate *-homomorphism $\varrho\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{B}(\mathcal{E})$ (see [17, Theorem 10.9]). Let π'

be the representation $\mu \otimes_\varrho 1$ of A on $\mathcal{E}$ associated to ϱ. We claim that $\pi = \pi'$. The functional calculus extends to affiliated multipliers and maps the identity function on $\mathbb{R}$ to the regular, self-adjoint operator t. This means that $\overline{\pi'(x)} = t$. Then $\overline{\pi'(x^n)} \subseteq t^n$. This implies $\overline{\pi'(x^n)} = t^n$ because $\overline{\pi'(x^n)}$ is self-adjoint and t^n is symmetric. Since the set $\{x^n \mid n \in \mathbb{N}\}$ is a strong generating set for $\mathbb{C}[x]$, the domain of π' is $\bigcap \operatorname{dom} \overline{\pi'(x^n)} = \mathfrak{E}$ by condition (5) and (2.10). On this domain, $\pi(x)$ and $\pi'(x)$ act by the same operator because they have the same closure. Thus $\pi = \pi'$ and (5) implies (1). So all five conditions in the theorem are equivalent.

To show that B is a C*-hull for the class of representations described in (1), we check (5) in Proposition 3.8. An integrable representation of A on a Hilbert space $\mathcal{H}$ corresponds to a self-adjoint operator t on $\mathcal{H}$ by (5). An integrable subrepresentation is a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ with a self-adjoint operator t_0 on $\mathcal{H}_0$ whose graph is contained in that of t. Since t_0 is self-adjoint, the subspaces $(t_0 \pm \mathrm{i})(\operatorname{dom}(t_0)) = (t \pm \mathrm{i})(\operatorname{dom}(t_0))$ are equal to $\mathcal{H}_0$. The Cayley transform u of t maps $(t + \mathrm{i})(\operatorname{dom}(t_0))$ onto $(t - \mathrm{i})(\operatorname{dom}(t_0))$. Thus it maps $\mathcal{H}_0$ onto itself. Since $u - 1$ generates the image of $B = \mathrm{C}_0(\mathbb{R})$ under the functional calculus, the projection onto $\mathcal{H}_0$ is B-invariant. $\square$

Example 4.5. Regularity and self-adjointness are independent properties of a symmetric operator. Examples of regular symmetric operators that are not self-adjoint are easy to find, see §6. We are going to construct a representation π of $\mathbb{C}[x]$ on a Hilbert module for which $\overline{\pi(a)}$ is self-adjoint for each $a \in \mathbb{C}[x]$ with $a = a^*$, but $\overline{\pi(x)}$ is not regular. We follow the example after Théorème 1.3 in [21], which Pierrot attributes to Hilsum.

Let $\mathcal{H}$ be the Hilbert space $L^2([0,1])$ and let T_1 and T_2 be the operators $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathcal{H}$ with the following domains. For T_1, we take 1-periodic smooth functions; for T_2, we take the restrictions to $[0,1]$ of smooth functions on $\mathbb{R}$ satisfying $f(x+1) = -f(x)$. Both T_1 and T_2 are essentially self-adjoint. Let $D := \mathrm{C}([-1,1])$ and $\mathcal{E} := \mathrm{C}([-1,1], \mathcal{H})$. Let $\mathfrak{E} \subseteq \mathcal{E}$ be the dense subspace of all functions $f\colon [-1,1] \times [0,1] \to \mathbb{C}$ such that $\frac{\partial^n}{\partial^n x} f(t,x)$ is continuous for each $n \in \mathbb{N}$,

$$\frac{\partial^n}{\partial^n x} f(t,1) = \operatorname{sign}(t) \cdot \frac{\partial^n}{\partial^n x} f(t,0) \tag{4.6}$$

for all $t \in [-1,1]$, $x \in \mathbb{R}$, $t \neq 0$, and

$$\frac{\partial^n}{\partial^n x} f(0,0) = \frac{\partial^n}{\partial^n x} f(0,1) = 0. \tag{4.7}$$

Equivalently, $f(t,\textvisiblespace)$ belongs to the domain of $\overline{T_1}^n = \overline{T_1^n}$ for all $n \in \mathbb{N}$, $t \leq 0$ and to the domain of $\overline{T_2}^n = \overline{T_2^n}$ for all $n \in \mathbb{N}$, $t \geq 0$; indeed, this forces $\frac{\partial^n}{\partial^n x} f$ to be continuous on $[-1,1] \times [0,1]$ and to satisfy the boundary conditions (4.6). These imply (4.7) by continuity. Let $x^n \in \mathbb{C}[x]$ act on $\mathfrak{E}$ by $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$. This defines a closed *-representation of $\mathbb{C}[x]$ on $\mathcal{E}$ with $\mathfrak{E} = \bigcap_{n \in \mathbb{N}} \operatorname{dom} \overline{\pi(x)^n}$.

The closure $\overline{\pi(x)}$ is the irregular self-adjoint operator described in [21]. Let $a \in \mathbb{C}[x]$ with $a = a^*$. Then $\overline{\varrho(a)}$ is (regular and) self-adjoint for any *integrable* representation ϱ of $\mathbb{C}[x]$ by Theorem 4.4. Therefore, the restriction of $\overline{\pi(a)}$ to a single fibre of $\mathcal{E}$ at some $t \in [-1,1] \setminus \{0\}$ is a self-adjoint operator on $L^2([0,1])$ because $\overline{T_1}$ and $\overline{T_2}$ are self-adjoint and $\mathfrak{E} = \bigcap_{n\in\mathbb{N}} \operatorname{dom} \overline{\pi(x)^n}$. The restriction of $\overline{\pi(a)}^*$ at $t = 0$ is contained in the self-adjoint operators $\overline{a(T_1)}$ and $\overline{a(T_2)}$ by continuity. We claim that $\overline{a(T_1)} \cap \overline{a(T_2)} = \overline{\pi(a)}|_{t=0}$. This claim implies that $\overline{\pi(a)}^*$ is contained in $\overline{\pi(a)}$, that is, $\overline{\pi(a)}$ is self-adjoint.

Let $a \in \mathbb{C}[x]$ have degree n. Then the graph norms for a and x^n are equivalent in any representation by the proof of Lemma 4.1. Hence $\overline{a(T_i)}$ and $\overline{T_i^n}$ have the same domain. The domain of $\overline{T_i^n}$ consists of functions $[0,1] \to \mathbb{C}$ whose nth derivative lies in L^2 and whose derivatives of order strictly less than n satisfy the boundary condition for T_i. Hence the domain of $\overline{T_1^n} \cap \overline{T_2^n}$ consists of those functions $[0,1] \to \mathbb{C}$ whose nth derivative lies in L^2 and whose derivatives of order strictly less than n vanish at the boundary points 0 and 1. This is exactly the domain of the closure of $(T_1 \cap T_2)^n = \pi(x^n)|_{t=0}$. On this domain the operators $\overline{a(T_1)} \cap \overline{a(T_2)}$ and $\overline{\pi(a)}|_{t=0}$ both act by the differential operator $a(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x})$.

The algebra $A = \mathbb{C}[x]$ has many Hilbert space representations coming from closed symmetric operators that are not self-adjoint. There is, however, no larger admissible class of integrable representations:

Proposition 4.8. *Assume that an admissible class of integrable representations of $A = \mathbb{C}[x]$ contains all representations coming from self-adjoint Hilbert space operators. Then any integrable representation of A on a Hilbert module comes from a regular, self-adjoint operator.*

Proof. We first prove that there can be no more integrable Hilbert space representations than those coming from self-adjoint operators. Let $(\mathfrak{H}, \pi)$ be an integrable representation on a Hilbert space $\mathcal{H}$. We may extend the closed symmetric operator $t := \overline{\pi(x)}$ on $\mathcal{H}$ to a self-adjoint operator t_2 on a larger Hilbert space $\mathcal{H}_2$. This gives a representation π_2 of A on $\mathcal{H}_2$ as in Theorem 4.4, which is integrable by assumption. The inclusion map $\mathcal{H} \hookrightarrow \mathcal{H}_2$ is an isometric intertwiner from π to π_2. Hence π is a direct summand of π_2 by (4) in Definition 3.11. Thus $\overline{\pi(x^n)}$ is self-adjoint for each $n \in \mathbb{N}$, and π is the representation induced by t.

Now let $(\mathfrak{E}, \pi)$ be an integrable representation of A on a Hilbert D-module $\mathcal{E}$. For any Hilbert space representation $\varphi\colon D \to \mathbb{B}(\mathcal{H})$, the induced representation of A on the Hilbert space $\mathcal{E} \otimes_\varphi \mathcal{H}$ is also integrable by (2) in Definition 3.11. Thus $\overline{\pi(x^n) \otimes_\varphi 1_{\mathcal{H}}}$ is self-adjoint for any Hilbert space representation $\varphi\colon D \to \mathbb{B}(\mathcal{H})$.

A closed, densely defined, symmetric operator T on a Hilbert D-module $\mathcal{E}$ is self-adjoint and regular if and only if, for any state ω on D, the closure of $T \otimes_D 1$ on the Hilbert spaces $\mathcal{E} \otimes_D \mathcal{H}_\omega$ is self-adjoint; here $\mathcal{H}_\omega$ means the GNS-representation for ω. This is called the Local–Global Principle by Kaad and Lesch ([15, Theorem 1.1]); the result was first proved by Pierrot ([21, Théorème 1.18]). We will take up Local–Global Principles more systematically in §5.

Thus $\overline{\pi(x^n)}$ is regular and self-adjoint for each $n \in \mathbb{N}$. So π is obtained from the regular self-adjoint operator $\overline{\pi(x)}$ as in Theorem 4.4. □

Example 4.9. There are many admissible classes of representations of $\mathbb{C}[x]$ that are smaller than the class in Theorem 4.4. There are even many such classes that contain the same Hilbert space representations. For instance, let $B := \mathrm{C}_0((-\infty,0)) \oplus \mathrm{C}_0([0,\infty))$ with the representation of polynomials by pointwise multiplication. This is a C*-hull for a class of representations of $\mathbb{C}[x]$ by Theorem 8.2 below. Since the standard topologies on $\mathbb{R}$ and $(-\infty,0) \sqcup [0,\infty)$ have the same Borel sets, both C*-hulls $\mathrm{C}_0((-\infty,0)) \oplus \mathrm{C}_0([0,\infty))$ and $\mathrm{C}_0(\mathbb{R})$ give the same integrable Hilbert space representations because of the Borel functional calculus. But there are regular, self-adjoint operators on Hilbert modules that do not give a B-integrable representation. The obvious example is the multiplier X of $\mathrm{C}_0(\mathbb{R})$ that generates the universal integrable representation of $\mathbb{C}[x]$.

Can there be an *admissible* class of representations of $\mathbb{C}[x]$ that contains some representation on a Hilbert space that does not come from a self-adjoint operator? We cannot rule this out completely. But such a class would have to be rather strange. By Proposition 4.8, it cannot contain all self-adjoint operators. By Example 2.26, it cannot contain all representations coming from positive symmetric operators because then there would be isometric intertwiners among integrable representations that are not *-intertwiners. The following example rules out symmetric operators with one deficiency index 0:

Example 4.10. Let t be a closed symmetric operator on a Hilbert space $\mathcal{H}$ of deficiency indices $(0,n)$ for some $n \in [1,\infty]$. Then $\operatorname{dom}^\infty(t) := \bigcap_{n=1}^\infty \operatorname{dom}(t^n)$ is a core for each power t^k by [28, Proposition 1.6.1]. Thus there is a closed representation π of $\mathbb{C}[x]$ with domain $\operatorname{dom}^\infty(t)$ and $\overline{\pi(x^k)} = t^k$ for all $k \in \mathbb{N}$. By assumption, the operator $t + \mathrm{i}$ is surjective, but $t - \mathrm{i}$ is not. That is, the Cayley transform $c := (t-\mathrm{i})(t+\mathrm{i})^{-1}$ is a non-unitary isometry. The operator t may be reconstructed from c as in [17, Equation (10.11)]. Here c^* is surjective, so this simplifies to $\operatorname{dom}(t) = (1-c)c^*\mathcal{H} = (1-c)\mathcal{H}$, and $t(1-c)\xi = \mathrm{i}(1+c)\xi$ for all $\xi \in \mathcal{H}$. Thus $c(\operatorname{dom} t) \subseteq \operatorname{dom} t$ and $ct \subseteq tc$ because

$$ct\bigl((1-c)\xi\bigr) = \mathrm{i}c(1+c)\xi = \mathrm{i}(1+c)(c\xi) = t(1-c)(c\xi) = (tc)\bigl((1-c)\xi\bigr).$$

Then $ct^n \subseteq t^n c$ for all $n \in \mathbb{N}$. Thus c is an isometric intertwiner from π to itself by Proposition 2.15. If c^* were an intertwiner as well, then $c^*(\operatorname{dom} t) \subseteq \operatorname{dom} t$ and $c^*(t \pm \mathrm{i})\xi = (t \pm \mathrm{i})c^*\xi$ for all $\xi \in \operatorname{dom}(t)$. So

$$c^*c(t+\mathrm{i})\xi = c^*(t-\mathrm{i})\xi = (t-\mathrm{i})c^*\xi = c(t+\mathrm{i})c^*\xi = cc^*(t+\mathrm{i})\xi.$$

This is impossible because $c^*c \neq cc^*$ and $t+\mathrm{i}$ is surjective. So the isometry c is an intertwiner, but not a *-intertwiner. This is forbidden for admissible classes of integrable representations.

If t has deficiency indices $(n,0)$ instead, then $-t$ has deficiency indices $(0,n)$ and its Cayley transform is an isometric intertwiner that is not a *-intertwiner by the argument above.

5 Local–Global principles

Definition 5.1. Let A be a *-algebra with a weakly admissible class of integrable representations (Definition 3.11).
The *Local–Global Principle* says that a representation π of A on a Hilbert D-module $\mathcal{E}$ is integrable if (and only if) the representations $\pi\otimes_\varrho 1$ are integrable for all Hilbert space representations $\varrho\colon D \to \mathbb{B}(\mathcal{H})$.
The *Strong Local–Global Principle* says that a representation π of A on a Hilbert D-module $\mathcal{E}$ is integrable if (and only if) the representations $\pi\otimes_\varrho 1$ are integrable for all *irreducible* Hilbert space representations $\varrho\colon D \to \mathbb{B}(\mathcal{H})$.

Roughly speaking, the Local–Global Principle says that the class of integrable representations on Hilbert modules is determined by the class of integrable representations on Hilbert spaces. Examples where the Local–Global Principle fails are constructed in §6 and §8. We do not know an example with the Local–Global Principle for which the Strong Local–Global Principle fails.
An irreducible representation $\varrho\colon D \to \mathbb{B}(\mathcal{H})$ is unitarily equivalent to the GNS-representation for a pure state ψ on D. The tensor product $\mathcal{E}\otimes_\varrho\mathcal{H}$ is canonically isomorphic to the completion $\mathcal{E}_\psi$ of $\mathcal{E}$ to a Hilbert space for the scalar-valued inner product $\langle x, y\rangle_\mathbb{C} := \psi(\langle x, y\rangle_D)$. The induced representation $\pi \otimes_D 1$ of A on $\mathcal{H}_\psi$ is the closure of the representation π with domain $\mathfrak{E} \subseteq \mathcal{E} \subseteq \mathcal{E}_\psi$.
Any representation $\varrho\colon D \to \mathbb{B}(\mathcal{H})$ is a direct sum of cyclic representations, and these are GNS-representations of states. Since any weakly admissible class of integrable representations is closed under direct sums, the Local–Global Principle holds if and only if integrability of $\pi\otimes_\varrho 1$ for all GNS-representations ϱ of states on D implies integrability of π.

Example 5.2. Define integrable representations of the polynomial algebra $\mathbb{C}[x]$ as in Theorem 4.4. Thus they correspond to regular, self-adjoint operators on Hilbert modules. The main result in [15] says that the integrable representations of $\mathbb{C}[x]$ satisfy the Local–Global Principle. This is where our notation comes from. We already used this to prove Proposition 4.8. The Strong Local–Global Principle for integrable representations of $\mathbb{C}[x]$ is only conjectured in [15]. This conjecture had already been proved by Pierrot in [21, Théorème 1.18] before [15] was written. It is based on the following Hahn–Banach type theorem for Hilbert submodules:

Theorem 5.3 ([21, Proposition 1.16]). *Let D be a* C^**-algebra and let $\mathcal{E}$ be a Hilbert D-module. Let $\mathcal{F} \subsetneq \mathcal{E}$ be a proper, closed Hilbert submodule. There is an irreducible Hilbert space representation $\varrho\colon D \to \mathbb{B}(\mathcal{H})$ with $\mathcal{F}\otimes_\varrho\mathcal{H} \subsetneq \mathcal{E}\otimes_\varrho\mathcal{H}$.*

Corollary 5.4 ([21, Corollaire 1.17]). *Let $\mathcal{E}$ be a Hilbert module over a* C^**-algebra D. Let $\mathcal{F}_1, \mathcal{F}_2 \subsetneq \mathcal{E}$ be two closed Hilbert submodules. If $\mathcal{F}_1 \neq \mathcal{F}_2$, then there is an irreducible Hilbert space representation $\varrho\colon D \to \mathbb{B}(\mathcal{H})$ with $\mathcal{F}_1 \otimes_\varrho \mathcal{H} \neq \mathcal{F}_2 \otimes_\varrho \mathcal{H}$ as closed subspaces in $\mathcal{E} \otimes_\varrho \mathcal{H}$.*

Corollary 5.5 ([21, Théorème 1.18]). *Let T be a closed, semiregular operator on a Hilbert D-module $\mathcal{E}$. The operator T is regular if and only if, for each*

irreducible representation $\varrho\colon D \to \mathbb{B}(\mathcal{H})$ *on a Hilbert space* $\mathcal{H}$*, the closures of* $T \otimes_\varrho 1$ *and* $T^* \otimes_\varrho 1$ *on* $\mathcal{E} \otimes_\varrho \mathcal{H}$ *are adjoints of each other.*
Hence T *is regular and self-adjoint if and only if* $\overline{T \otimes_\varrho 1}$ *is a self-adjoint operator on* $\mathcal{E} \otimes_\varrho \mathcal{H}$ *for each irreducible Hilbert space representation* $\varrho\colon D \to \mathbb{B}(\mathcal{H})$*.*

We now apply the above results of Pierrot. First we deduce a criterion for representations to be equal. Then we prove that certain definitions of integrability automatically satisfy the Strong Local–Global Principle.

Theorem 5.6. *Let* A *be a* **-algebra and let* π_i *for* $i = 1, 2$ *be (closed) representations of* A *on a Hilbert module* $\mathcal{E}$ *over a* C^**-algebra* D*. The following are equivalent:*

(1) $\pi_1 = \pi_2$*;*

(2) $\pi_1 \otimes_\varrho \mathcal{H} = \pi_2 \otimes_\varrho \mathcal{H}$ *for each irreducible Hilbert space representation* ϱ *of* D*;*

(3) $\overline{\pi_1(a)} = \overline{\pi_2(a)}$ *for each* $a \in A$*.*

Proof. The equivalence (3) $\iff$ (1) is Proposition 2.9, and (1) clearly implies (2). Thus we only have to prove that not (3) implies not (2). Assume that there is $a \in A$ with $\overline{\pi_1(a)} \neq \overline{\pi_2(a)}$. The graphs Γ_1 and Γ_2 of $\overline{\pi_1(a)}$ and $\overline{\pi_2(a)}$ are different Hilbert submodules of $\mathcal{E} \oplus \mathcal{E}$. Corollary 5.4 gives an irreducible representation ϱ of D with $\Gamma_1 \otimes_\varrho \mathcal{H} \neq \Gamma_2 \otimes_\varrho \mathcal{H}$. This says that $\overline{\overline{\pi_1(a)} \otimes_\varrho 1_\mathcal{H}} \neq \overline{\overline{\pi_2(a)} \otimes_\varrho 1_\mathcal{H}}$ because $\Gamma_i \otimes_\varrho \mathcal{H}$ is the graph of $\overline{\overline{\pi_i(a)} \otimes_\varrho 1_\mathcal{H}}$. □

How do we specify which representations π of a *-algebra A are integrable? There are two basically different ways. The "universal way" specifies the universal integrable representation. That is, it starts with a representation $(\mathfrak{B}, \mu)$ on a C*-algebra B that satisfies (1) in Proposition 3.8 and takes the class of B-integrable representations. The "operator way" imposes conditions on the operators $\pi(a)$, such as regularity and self-adjointness of $\overline{\pi(a)}$ or strong commutation relations.
In good cases, the same class of integrable representations may be specified in both ways. For instance, Theorem 4.4 shows that several classes of representations of $\mathbb{C}[x]$ are equal. The first is defined by the universal representation on $\mathrm{C}_0(\mathbb{R})$. The second asks $\overline{\pi(a)}$ to be regular and self-adjoint for all $a \in A_h$.
We are going to make the "operator way" more precise so that *all* classes of representations defined in this way satisfy the Strong Local–Global Principle. This is a powerful method to prove Local–Global Principles.

Definition 5.7. Let A be a *-algebra and $\mathrm{Rep}'(A)$ some weakly admissible class of representations of A on Hilbert modules over C*-algebras. A *natural construction of Hilbert submodules* (of rank $n \in \mathbb{N}_{\geq 1}$) associates to each representation π on a Hilbert module $\mathcal{E}$ that belongs to $\mathrm{Rep}'(A)$ a Hilbert submodule $\mathcal{F}(\pi) \subseteq \mathcal{E}^n$, such that

(1) if $u\colon \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ is a unitary *-intertwiner between two representations π_1 and π_2 in $\mathrm{Rep}'(A)$, then $u^{\oplus n}\colon \mathcal{E}_1^n \to \mathcal{E}_2^n$ maps $\mathcal{F}(\pi_1)$ onto $\mathcal{F}(\pi_2)$;

(2) let D_1 and D_2 be C*-algebras and let $\mathcal{G}$ be a D_1, D_2-correspondence; let π be a representation in $\mathrm{Rep}'(A)$ on a Hilbert D_1-module $\mathcal{E}$; then the canonical isomorphism $\mathcal{E}^n \otimes_{D_1} \mathcal{G} \xrightarrow{\sim} (\mathcal{E} \otimes_{D_1} \mathcal{G})^n$ maps $\mathcal{F}(\pi) \otimes_{D_1} \mathcal{G}$ onto $\mathcal{F}(\pi \otimes_{D_1} \mathcal{G})$;

(3) if π_i for i in a set I are representations in $\mathrm{Rep}'(A)$ on Hilbert D-modules $\mathcal{E}_i$ over the same C*-algebra D, then the canonical isomorphism $\left(\bigoplus \mathcal{E}_i\right)^n \xrightarrow{\sim} \bigoplus \mathcal{E}_i^n$ maps $\mathcal{F}\left(\bigoplus \pi_i\right)$ onto $\bigoplus \mathcal{F}(\pi_i)$.

In brief, $\mathcal{F}(\pi) \subseteq \mathcal{E}^n$ is compatible with unitary *-interwiners, interior tensor products, and direct sums.

A smaller class of representations $\mathrm{Rep}''(A) \subseteq \mathrm{Rep}'(A)$ is *defined by a submodule condition* if there are two natural constructions of Hilbert submodules $\mathcal{F}_i(\pi)$, $i = 1, 2$, of the same rank n, such that a representation π in $\mathrm{Rep}'(A)$ belongs to $\mathrm{Rep}''(A)$ if and only if $\mathcal{F}_1(\pi) = \mathcal{F}_2(\pi)$.

A class of representations $\mathrm{Rep}_{\mathrm{int}}(A) \subseteq \mathrm{Rep}(A)$ is *defined by submodule conditions* if it is defined by transfinite recursion by repeating the step in the previous paragraph. More precisely, there are a well-ordered set I with a greatest element M and least element 0 and subclasses $\mathrm{Rep}^i(A) \subseteq \mathrm{Rep}(A)$ for $i \in I$ such that

(1) $\mathrm{Rep}^0(A) = \mathrm{Rep}(A)$ and $\mathrm{Rep}^M(A) = \mathrm{Rep}_{\mathrm{int}}(A)$;

(2) $\mathrm{Rep}^{i+1}(A) \subseteq \mathrm{Rep}^i(A)$ is defined by a submodule condition for each $i \in I$;

(3) $\mathrm{Rep}^i(A) = \bigcap_{i'<i} \mathrm{Rep}^{i'}(A)$ if $i \neq 0$ and $i \neq i' + 1$ for all $i' \in I$.

The following lemma makes this definition meaningful, the following theorem makes it interesting.

Lemma 5.8. *If* $\mathrm{Rep}'(A) \subseteq \mathrm{Rep}(A)$ *is weakly admissible and* $\mathrm{Rep}''(A) \subseteq \mathrm{Rep}'(A)$ *is defined by a submodule condition, then* $\mathrm{Rep}''(A)$ *is also weakly admissible. If* $(\mathrm{Rep}^i(A))_{i\in I}$ *is a set of weakly admissible subclasses, then* $\bigcap_{i\in I} \mathrm{Rep}^i(A)$ *is weakly admissible. Any class of representations defined by submodule conditions is weakly admissible.*

Theorem 5.9. *If* $\mathrm{Rep}_{\mathrm{int}}(A) \subseteq \mathrm{Rep}(A)$ *is defined by submodule conditions, then it satisfies the Strong Local–Global Principle.*

Before we prove these two results, we give examples of classes of representations defined by one or more submodule conditions, and a few counterexamples. These show that a class of integrable representations defined in the operator way is often but not always defined by submodule conditions.

Example 5.10. The *regularity condition* for $a \in A_h$ requires $\overline{\pi(a)}$ to be regular and self-adjoint. Equivalently, the closures of $(\pi(a) \pm \mathrm{i})(\mathfrak{E})$ for both signs are dense in $\mathcal{E}$; this is equivalent to $\overline{\pi(a)}$ having a unitary Cayley transform. Sending π to the image of $\overline{\pi(a) + \mathrm{i}}$ or $\overline{\pi(a) - \mathrm{i}}$ is a natural construction of a Hilbert submodule. Hence the condition that $\overline{\pi(a)}$ is regular and self-adjoint is equivalent to the combination of two submodule conditions of rank 1.
Alternatively, we may proceed as in the definition of regularity for non-self-adjoint operators. Let $\Gamma(T)$ denote the closure of the graph of an operator T. A closed operator T is regular if and only if the direct sum of $\Gamma(T)$ and $U_0(\Gamma(T^*))$ is $\mathcal{E} \oplus \mathcal{E}$, where $U_0(\xi_1, \xi_2) := (\xi_2, -\xi_1)$. If $a \in A_h$, then regularity and self-adjointness of $\overline{\pi(a)}$ together are equivalent to the equality of

$$\mathcal{F}_1(\pi) := \Gamma(\pi(a)) \oplus U_0(\Gamma(\pi(a^*))) \quad \text{and} \quad \mathcal{F}_2(\pi) := \mathcal{E} \oplus \mathcal{E}.$$

We claim that $\mathcal{F}_1$ and $\mathcal{F}_2$ are natural constructions of Hilbert submodules of rank 2. This is trivial for $\mathcal{F}_2$. That $\mathcal{F}_1$ is compatible with unitary intertwiners and direct sums is an easy exercise. The construction $\mathcal{F}_1$ is compatible with interior tensor products because the graph of $\overline{(\pi \otimes_{D_1} 1_{\mathcal{G}})(a)}$ is $\Gamma(\pi(a)) \otimes_{D_1} \mathcal{G}$. For instance, (2) in Theorem 4.4 defines integrable representations of $\mathbb{C}[x]$ by regularity conditions. We generalise this in Theorem 5.17 below.

Example 5.11. The class of representations where $\overline{\pi(a)}$ is regular for some $a \in A$ is always weakly admissible by [17, Proposition 9.10]. The first example in §6 shows a class of representations defined by such a condition that does not satisfy the Local–Global Principle, in contrast to Theorem 5.9. Hence asking for $\overline{\pi(a)}$ to be regular for some $a \in A$ cannot be a submodule condition. The problem is that the inclusion $\Gamma(\pi(a)^*) \otimes_D 1_{\mathcal{G}} \subseteq \Gamma\big((\pi(a) \otimes_D 1_{\mathcal{G}})^*\big)$ for a correspondence $\mathcal{G}$ may be strict.

Example 5.12. Let $a_1, a_2 \in A_h$ and suppose that $t_1 := \overline{\pi(a_1)}$ and $t_2 := \overline{\pi(a_2)}$ are self-adjoint, regular operators for all representations in $\mathrm{Rep}'(A)$; we may achieve this by submodule conditions as in Example 5.10 in previous steps of a recursive definition. We say that t_1 and t_2 *strongly commute* if their Cayley transforms u_1 and u_2 commute. Equivalently, u_1 commutes with t_2, that is, $u_1 t_2 u_1^* = t_2$. The graphs of t_2 and $u_1 t_2 u_1^*$ are natural constructions of Hilbert submodules of rank 2. Therefore, strong commutation of $\overline{\pi(a_1)}$ and $\overline{\pi(a_2)}$ is a submodule condition.

Example 5.13. Let $I \triangleleft A$ be an ideal. A *nondegeneracy condition* for I asks the closed linear span of $\pi(I)\mathfrak{E}$ to be all of $\mathcal{E}$; here $\mathfrak{E}$ is the domain of π. This means that $\mathcal{F}_1(\pi)$, the closed linear span of $\pi(a)\xi$ for $a \in I$, $\xi \in \mathfrak{E}$, is equal to $\mathcal{F}_2(\pi) = \mathcal{E}$. These are natural constructions of Hilbert submodules of rank 1. So a nondegeneracy condition is a submodule condition.
For instance, let I be a non-unital *-algebra and let $A = \tilde{I}$ be its unitisation. Any representation of I extends uniquely to a unital representation of A. The class of nondegenerate representations of I inside the class of all representations of A is defined by a submodule condition.

More generally, let $V_1, V_2 \subseteq A$ be vector subspaces and ask the closed linear spans of $\pi(a)\xi$ for $a \in V_j$, $\xi \in \mathfrak{E}$ to be equal for $j = 1, 2$. This is a submodule condition as well. For instance, the condition $\overline{\pi(a + \mathrm{i})\mathfrak{E}} = \mathcal{E}$ for $a \in A_h$ is of this form. It holds if and only if the Cayley transform of $\overline{\pi(a)}$ is an isometry (possibly without adjoint).

Often we need a mild generalisation of the above construction, see Example 5.14 below. Suppose that we have constructed a representation $\varphi(\pi)$ of a unital *-algebra A' on $\mathcal{E}$ for any representation π in $\mathrm{Rep}'(A)$, such that $\pi \mapsto \varphi(\pi)$ is compatible with unitary *-intertwiners, direct sums, and interior tensor products; the last property means that $\varphi(\pi \otimes_{D_1} 1_{\mathcal{G}}) = \varphi(\pi) \otimes_{D_1} 1_{\mathcal{G}}$ as representations on $\mathcal{E} \otimes_{D_1} \mathcal{G}$. Then we may ask the nondegeneracy condition for an ideal in A' instead. In particular, A' may be a weak C*-hull for some class of representations containing $\mathrm{Rep}'(A)$.

Example 5.14. Let $a_1, \dots, a_n \in A_h$ be commuting, symmetric elements and suppose that $\overline{\pi(a_j)}$ for $j = 1, \dots, n$ are strongly commuting, self-adjoint, regular operators for all representations in $\mathrm{Rep}'(A)$; we may achieve all this by previous submodule conditions as in Examples 5.10 and 5.12. A *closed spectral condition* asks the joint spectrum of $\overline{\pi(a_1)}, \dots, \overline{\pi(a_n)}$ to be contained in a closed subset $X \subseteq \mathbb{R}^n$.

We claim that this is a submodule condition. Under our assumptions, the functional calculus $\Phi\colon \mathrm{C}_0(\mathbb{R}^n) \to \mathbb{B}(\mathcal{E})$ exists. Our spectral condition means that $\Phi(\mathrm{C}_0(\mathbb{R}^n \setminus X))\mathcal{E} = 0$. The construction of Φ is clearly compatible with unitary *-intertwiners and direct sums. It is also compatible with interior tensor products, that is, the functional calculus for $\overline{\pi \otimes 1(a_1)}, \dots, \overline{\pi \otimes 1(a_n)}$ maps $f \mapsto \Phi(f) \otimes 1$. Hence $\Phi(\mathrm{C}_0(\mathbb{R}^n \setminus X))\mathcal{E}$ is a naturally constructed Hilbert submodule of $\mathcal{E}$. So our spectral condition for closed $X \subseteq \mathbb{R}^n$ is a submodule condition.

More generally, let $X \subseteq \mathbb{R}^n$ be locally closed, that is, X is relatively open in its closure $\overline{X}$. Suppose that the spectral condition for $\overline{X}$ holds for all representations in $\mathrm{Rep}'(A)$, say, by previous recursion steps. Then the functional calculus homomorphism for $\overline{\pi(a_1)}, \dots, \overline{\pi(a_n)}$ exists and descends to $\mathrm{C}_0(\overline{X})$. The *spectral condition* for X asks the restriction of this homomorphism to the ideal $\mathrm{C}_0(X) \triangleleft \mathrm{C}_0(\overline{X})$ to be nondegenerate. This is a submodule condition by Example 5.13.

Example 5.15 (see [31, §3]). Let A_μ for some $\mu \in \mathbb{R} \setminus \{0\}$ be the unital *-algebra generated by two elements v, n with the relations $v^*v = vv^* = 1$, $n^*n = nn^*$, $v^*nv = \mu n$. This is the algebra of polynomial functions on the quantum group $\mathrm{E}_\mu(2)$. The relations allow to write any element as a linear combination of $v^k \cdot g(n, n^*)$ for $k \in \mathbb{Z}$ and a polynomial g. It follows that the graph topology of a representation of A_μ is generated by the graph norms of $(n^*n)^k$ for $k \in \mathbb{N}$. Thus a representation is closed if and only if its domain is $\bigcap_{k=0}^\infty \overline{\pi((n^*n)^k)}$, compare the proof of (2.10).

The C*-algebra of $\mathrm{E}_\mu(2)$ is a C*-hull for a certain class of integrable representations of A_μ that is defined by submodule conditions. First, we require $\overline{\pi(n)}$

to be a regular, normal operator; equivalently, $\overline{\pi(n+n^*)}$ and $-\mathrm{i}\overline{\pi(n-n^*)}$ are regular and self-adjoint, and they strongly commute; these are submodule conditions by Examples 5.10 and 5.12. Secondly, we require the spectrum of $\overline{\pi(n)}$ (or the joint spectrum of its real and imaginary part) to be contained in $X_\mu := \{z \in \mathbb{C} \mid |z| \in \mu^{\mathbb{Z}}\} \cup \{0\}$; this is a submodule condition by Example 5.14. Finally, we require $\overline{\pi(n^*n)^k}$ to be regular and self-adjoint for all $k \geq 1$. These are submodule conditions by Example 5.10.

We claim that an integrable representation on $\mathcal{E}$ is equivalent to a pair (V, N) consisting of a unitary operator V and a regular, normal operator N on $\mathcal{E}$ with spectrum contained in X_μ, subject to the relation $V^*NV = \mu N$. First, any such pair (V, N) gives an integrable representation of A_μ with domain $\bigcap_{k=0}^\infty \mathrm{dom}(N^k)$. Conversely, if π is an integrable representation, then let $N := \overline{\pi(n)}$, $V := \overline{\pi(v)}$. These have the properties required above. Since $\overline{\pi((n^*n)^k)}$ is self-adjoint and contained in the symmetric operator $(N^*N)^k$, we must have $\overline{\pi((n^*n)^k)} = (N^*N)^k$. So the domain of the representation of A_μ is $\bigcap_{k=0}^\infty \mathrm{dom}(N^k)$.

The regular, normal operator N with spectrum in X_μ defines a functional calculus ϱ on $\mathrm{C}_0(X_\mu)$. The commutation relation $v^*nv = \mu n$ is equivalent to $V^*\varrho(f)V = \varrho(\alpha(f))$ for the automorphism $\alpha(f)(x) := f(\mu x)$ on $\mathrm{C}_0(X_\mu)$. As a consequence, the crossed product C^*-algebra $\mathrm{C}_0(X_\mu) \rtimes_\alpha \mathbb{Z}$ is a C^*-hull for our class of integrable representations.

By the way, this also follows from our Induction Theorem. For this, we give A_μ the unique $\mathbb{Z}$-grading where v has degree 1 and n has degree 0. Then $(A_\mu)_0 = \mathbb{C}[n, n^*]$, and we call a representation of $\mathbb{C}[n, n^*]$ integrable if n is regular and normal with spectrum contained in X_μ. The C^*-hull for this class of integrable representations of $\mathbb{C}[n, n^*]$ is $\mathrm{C}_0(X_\mu)$. In this case, all representations of $\mathbb{C}[n, n^*]$ are inducible to A_μ, and the induced C^*-hull for A_μ is $\mathrm{C}_0(X_\mu) \rtimes_\alpha \mathbb{Z}$.

Interesting classes of representations defined by submodule conditions occur in Theorems 5.21 and 8.6. The examples in [7, 27] are also defined by submodule conditions, compare Proposition 9.4.

Example 5.16. If the algebra A carries a topology, then we may restrict attention to representations of A that are continuous in some sense. For instance, if G is a topological group and $A = \mathbb{C}[G]$ is the group ring of the underlying discrete group, then representations of A are unitary representations of G, possibly discontinuous. Among them, we may restrict to the continuous representations (compare the definition of a host algebra for G in [11]). If G is an infinite-dimensional Lie group, we may restrict further to representations of $\mathbb{C}[G]$ that are smooth in the sense that the smooth vectors are dense. I do not expect continuity or smoothness to be a submodule condition, and I do not know when the classes of continuous or smooth representations satisfy the Local–Global Principle or its strong variant.

Semiboundedness conditions ask for certain (regular) self-adjoint operators to be bounded above, see [19]. If we specify the upper bound on the spectrum, this is a spectral condition as in Example 5.14. When we let the upper bound go to ∞, however, then direct sums no longer preserve semiboundedness. Therefore, semiboundedness conditions seem close enough to submodule conditions to be tractable, but the details require further thought.

Proof of Lemma 5.8. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be natural constructions of Hilbert submodules of rank n that define $\mathrm{Rep}''(A)$ inside $\mathrm{Rep}'(A)$, and let $\mathrm{Rep}'(A)$ be weakly admissible. Let π_i for $i = 1,2$ be representations on Hilbert D-modules $\mathcal{E}_i$ for a C*-algebra D that belong to $\mathrm{Rep}'(A)$. Let $u\colon \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ be a unitary *-intertwiner from π_1 to π_2. If $\mathcal{F}_1(\pi_1) = \mathcal{F}_2(\pi_1)$, then $\mathcal{F}_1(\pi_2) = u^{\oplus n}(\mathcal{F}_1(\pi_1)) = u^{\oplus n}(\mathcal{F}_2(\pi_1)) = \mathcal{F}_2(\pi_2)$. Thus π_2 belongs to $\mathrm{Rep}''(A)$ if π_1 does. This verifies (1) in Definition 3.11 using (1) in Definition 5.7. Similarly, (2) and (3) in Definition 5.7 show that $\mathrm{Rep}''(A)$ inherits (2) and (3) in Definition 3.11 from $\mathrm{Rep}'(A)$. Thus $\mathrm{Rep}''(A)$ is again weakly admissible.
It is trivial that weak admissibility is hereditary for intersections. By transfinite induction, it follows that any class of representations defined by submodule conditions is weakly admissible. □

Proof of Theorem 5.9. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be natural constructions of Hilbert submodules of rank n that define $\mathrm{Rep}''(A)$ inside $\mathrm{Rep}'(A)$, and assume that $\mathrm{Rep}'(A)$ satisfies the Strong Local–Global Principle. Let π be a representation on a Hilbert D-module $\mathcal{E}$ that does not belong to $\mathrm{Rep}''(A)$. We must find an irreducible representation ϱ of D on a Hilbert space $\mathcal{H}$ such that $\pi \otimes_\varrho \mathcal{H}$ does not belong to $\mathrm{Rep}''(A)$. If the representation does not even belong to $\mathrm{Rep}'(A)$, this is possible because $\mathrm{Rep}'(A)$ satisfies the Strong Local–Global Principle by assumption. So we may assume that π belongs to $\mathrm{Rep}'(A)$ but not to $\mathrm{Rep}''(A)$. Thus $\mathcal{F}_1(\pi)$ and $\mathcal{F}_2(\pi)$ are well defined and different Hilbert submodules of $\mathcal{E}^n$. Corollary 5.4 gives an irreducible representation ϱ of D on a Hilbert space $\mathcal{H}$ such that $\mathcal{F}_1(\pi) \otimes_\varrho \mathcal{H} \neq \mathcal{F}_2(\pi) \otimes_\varrho \mathcal{H}$ as closed subspaces of $\mathcal{E}^n \otimes_\varrho \mathcal{H}$. Identify these with subspaces of $(\mathcal{E} \otimes_\varrho \mathcal{H})^n$. The condition (2) in Definition 5.7 gives

$$\mathcal{F}_1(\pi \otimes_\varrho \mathcal{H}) = \mathcal{F}_1(\pi) \otimes_\varrho \mathcal{H} \neq \mathcal{F}_2(\pi) \otimes_\varrho \mathcal{H} = \mathcal{F}_2(\pi \otimes_\varrho \mathcal{H}).$$

That is, $\pi \otimes_\varrho \mathcal{H}$ does not belong to $\mathrm{Rep}''(A)$. Thus $\mathrm{Rep}''(A)$ inherits the Strong Local–Global Principle from $\mathrm{Rep}'(A)$.
The Strong Local–Global Principle is easily seen to be hereditary for intersections. Hence any class of representations defined by submodule conditions satisfies the Strong Local–Global Principle by transfinite induction. □

Theorem 5.17. *Let A be a *-algebra and let $S \subseteq A_h$. Let $\mathrm{Rep}^S(A)$ be the class of all representations where the elements of S act by regular, self-adjoint operators. This class is defined by submodule conditions and hence satisfies the Strong Local–Global Principle. It is admissible if S is a strong generating set for A.*

Proof. Asking $\overline{\pi(a)}$ to be regular and self-adjoint for a single $a \in S$ is a submodule condition by Example 5.10. In order to ask this simultaneously for a set S,

let $\prec$ be a well-ordering on S, and add an element M with $a \prec M$ for all $a \in S$. Let $\mathrm{Rep}^a(A) \subseteq \mathrm{Rep}(A)$ for $a \in S \cup \{M\}$ be the class of all representations π where $\overline{\pi(b)}$ is regular and self-adjoint for all $b \in S$ with $b \prec a$. These subclasses form a recursive definition of $\mathrm{Rep}^S(A)$ by submodule conditions as in Definition 5.7. Thus $\mathrm{Rep}^S(A)$ is defined by submodule conditions. Then it is weakly admissible and satisfies the Strong Local–Global Principle by Lemma 5.8 and Theorem 5.9.
From now on, we assume that S is a strong generating set. For $\mathrm{Rep}^S(A)$ to be admissible, we must prove that any isometric intertwiner $I\colon (\mathfrak{E}_0, \pi_0) \hookrightarrow (\mathfrak{E}, \pi)$ between two Hilbert space representations in $\mathrm{Rep}^S(A)$ is a *-intertwiner.
If $a \in S$, then $\overline{\pi(a)}$ and $\overline{\pi_0(a)}$ are regular, self-adjoint operators. Hence they generate integrable representations of $\mathbb{C}[x]$ as in Theorem 4.4. The isometry I intertwines these representations of $\mathbb{C}[x]$. Hence it is a *-intertwiner by Theorem 4.4. In particular, I^* maps $\operatorname{dom}\overline{\pi(a)}$ to $\operatorname{dom}\overline{\pi_0(a)}$ for each $a \in S$. Since S is a strong generating set, (2.10) gives $\mathfrak{E}_0 = \bigcap_{a\in S} \operatorname{dom}\overline{\pi_0(a)}$ and similarly for π. So $I^*(\mathfrak{E}) \subseteq \mathfrak{E}_0$. Then $\mathfrak{E} = \mathfrak{E}_0 + (\mathfrak{E} \cap \mathfrak{E}_0^\perp)$ and I is a *-intertwiner by Proposition 2.27. □

Corollary 5.18. *Let $S \subseteq A_h$ be a strong generating set for a *-algebra A and let B with universal representation μ be a weak* C*-*hull. If the closed multipliers $\overline{\mu(a)}$ for $a \in S$ are self-adjoint and affiliated with B, then B is a* C*-*hull.*

Proof. All B-integrable representations belong to $\mathrm{Rep}^S(A)$ because the latter is weakly admissible and contains the universal B-integrable representation. Since $\mathrm{Rep}^S(A)$ is admissible by Theorem 5.17, any smaller class of integrable representations inherits the equivalent conditions (2)–(4) in Proposition 3.8, which characterise C*-hulls among weak C*-hulls. □

Theorem 5.19. *Let A be a *-algebra, B a* C*-*algebra, $(\mathfrak{B}, \mu)$ a representation of A on B, and $T_1, \dots, T_n \in A$. Assume that $\overline{\mu(T_1)}, \dots, \overline{\mu(T_n)}$ are self-adjoint and affiliated with B and generate B in the sense of Woronowicz, see* [32, Definition 3.1]. *Then B is a* C*-*hull for the B-integrable representations of A defined by $(\mathfrak{B}, \mu)$, and these satisfy the Strong Local–Global Principle.*

Proof. To show that B is a C*-hull, we check the condition (5) in Proposition 3.8. Let $\varrho\colon B \to \mathbb{B}(\mathcal{H})$ be a representation of B on a Hilbert space $\mathcal{H}$ and let $(\mathfrak{E}, \pi)$ be the corresponding B-integrable representation of A. Let $(\mathfrak{E}_0, \pi|_{\mathfrak{E}_0})$ be a B-integrable representation on a closed subspace $\mathcal{E}_0 \subseteq \mathcal{E}$ and let $P \in \mathbb{B}(\mathcal{E})$ be the projection onto $\mathcal{E}_0$. We must show that $\varrho(B)$ is contained in the commutant of P. Equivalently, ϱ is a morphism in the notation of [32] to the algebra $K = \mathbb{K}(\mathcal{E}_0) \oplus \mathbb{K}(\mathcal{E}_0^\perp)$ of all compact operators on $\mathcal{E}$ that commute with P.
Let $1 \le i \le n$. Since T_i is self-adjoint and regular as an adjointable operator on the Hilbert B-module B, it generates an integrable representation of the polynomial algebra $\mathbb{C}[x]$ on B as in Theorem 4.4. These integrable representations form an admissible class. Therefore, a B-integrable representation of A

gives an integrable representation of $\mathbb{C}[x]$ when we compose with the canonical map $j_i\colon \mathbb{C}[x] \to A$, $x \mapsto T_i$, and take the closure. And since π and $\pi|_{\mathfrak{E}_0}$ are both B-integrable, $\overline{\pi|_{\mathfrak{E}_0} \circ j_i}$ is a direct summand in $\overline{\pi \circ j_i}$. Equivalently, the unbounded operator $\overline{\pi(T_i)}$ is affiliated with K.

The extension of ϱ to affiliated multipliers maps $\overline{\mu(T_i)}$ to $\overline{\pi(T_i)}$, which is affiliated with K. Hence ϱ is a morphism to K because these affiliated multipliers generate B. Thus B is a C*-hull for the B-integrable representations by Proposition 3.8.

Now we check the Strong Local–Global Principle. Let $(\mathfrak{E}, \pi)$ be a representation of A on a Hilbert D-module $\mathcal{E}$. Assume that the representation $(\mathfrak{E}, \pi) \otimes_\omega \mathcal{H}_\omega$ is integrable for each irreducible representation ω of D on a Hilbert space $\mathcal{H}_\omega$ in the sense that it comes from a representation of B. We must show that the representation $(\mathfrak{E}, \pi)$ is integrable.

The condition that $\overline{\pi(T_i)}$ be self-adjoint and regular is a submodule condition by Example 5.10. Hence the class of representations with this property satisfies the Strong Local–Global Principle by Theorem 5.9. Therefore, $\overline{\pi(T_i)}$ is a regular, self-adjoint operator on $\mathcal{E}$ for $i = 1, \dots, n$.

Let ω be the direct sum of all irreducible representations of D; this is a faithful representation of D on some Hilbert space $\mathcal{H}$. The induced representation j of $\mathbb{K}(\mathcal{E})$ on $\mathcal{K} := \mathcal{E} \otimes_D \mathcal{H}$ is faithful as well. By assumption, the representation $\pi \otimes_D 1$ of A on $\mathcal{K}$ is integrable, so it comes from a representation σ of B. The extension of σ to affiliated multipliers maps $\overline{\mu(T_i)}\ \eta\ B$ to $\overline{(\pi \otimes_D 1)(T_i)}$. Since $\overline{\pi(T_i)}$ is a regular operator on $\mathcal{E}$, it is an affiliated multiplier of $\mathbb{K}(\mathcal{E})$, see [20] or Proposition 3.13. Thus $\overline{(\pi \otimes_D 1)(T_i)}$ is affiliated with the image of $\mathbb{K}(\mathcal{E})$ on $\mathcal{K}$ by [17, Proposition 9.10]. Thus $\sigma(\overline{\mu(T_i)})\ \eta\ \mathbb{K}(\mathcal{E})$ for $i = 1, \dots, n$. Since the affiliated multipliers $\overline{\mu(T_i)}$ generate B in the sense of Woronowicz, σ factors through a morphism $\tau\colon B \to \mathbb{K}(\mathcal{E})$. This is the same as a representation of B on $\mathcal{E}$. Let π' be the representation of A on $\mathcal{E}$ associated to τ. If ϱ is an irreducible Hilbert space representation of D, then $\pi \otimes_\varrho \mathcal{H} = \pi' \otimes_\varrho \mathcal{H}$ by construction of τ. Hence Theorem 5.6 gives $\pi = \pi'$. Since π' is integrable by construction, so is π. $\square$

The first counterexample in §6 exhibits a *symmetric* affiliated multiplier that generates a C*-algebra, such that the Local–Global Principle fails and B is not a C*-hull. Without self-adjointness, we only get the following much weaker statement:

Lemma 5.20. *Let A be a *-algebra, B a* C*-*algebra, $(\mathfrak{B}, \mu)$ a representation of A on B, and $T_1, \dots, T_n \in A$. Assume that $\overline{\mu(T_1)}, \dots, \overline{\mu(T_n)}$ are affiliated with B and generate B in the sense of Woronowicz. Then B is a weak* C*-*hull for the B-integrable representations of A.*

Proof. To show that B is a weak C*-hull, we check (1) in Proposition 3.8. Let ϱ_1, ϱ_2 be representations of B on a Hilbert space $\mathcal{H}$ with $(\mathfrak{B}, \mu) \otimes_{\varrho_1} \mathcal{H} = (\mathfrak{B}, \mu) \otimes_{\varrho_2} \mathcal{H}$. We claim that $\varrho_1 \oplus \varrho_2\colon B \to \mathbb{B}(\mathcal{H}^2) = \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ maps B into the multiplier algebra of the diagonally embedded copy K of $\mathbb{K}(\mathcal{H})$. This is equivalent to $\varrho_1 = \varrho_2$. Since $(\mathfrak{B},\mu)\otimes_{\varrho_1}\mathcal{H} = (\mathfrak{B},\mu)\otimes_{\varrho_2}\mathcal{H}$, the extension of $\varrho_1\oplus\varrho_2$ to affiliated multipliers maps $\overline{\mu(T_i)}\ \eta\ B$ to an operator of the form (X_i, X_i) for $i = 1,\dots,n$; these are affiliated with K. Since these affiliated multipliers generate B, $\varrho_1 \oplus \varrho_2$ is a morphism from B to K. Thus B is a weak C*-hull for A. □

5.1 Universal enveloping algebras

We illustrate our theory by an example. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ and let $A = U(\mathfrak{g})$ be its universal enveloping algebra with the usual involution, where elements of $\mathfrak{g}$ are skew-symmetric. A representation of A on $\mathcal{E}$, possibly not closed, is equivalent to a dense submodule $\mathfrak{E} \subseteq \mathcal{E}$ with a Lie algebra representation $\pi\colon \mathfrak{g} \to \operatorname{End}_D(\mathfrak{E})$ satisfying $\langle \xi, \pi(X)(\eta)\rangle = -\langle \pi(X)(\xi), \eta\rangle$ for all $X \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{E}$.

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $B := \mathrm{C}^*(G)$. A representation of $\mathrm{C}^*(G)$ on a Hilbert module $\mathcal{E}$ is equivalent to a strongly continuous, unitary representation of G on $\mathcal{E}$. Given such a representation, let $\mathcal{E}^\infty \subseteq \mathcal{E}$ be its subspace of smooth vectors. This is the domain of a closed representation of $U(\mathfrak{g})$. We call a representation of $U(\mathfrak{g})$ integrable if it comes from a unitary representation of G in this way.

In particular, G acts continuously on $\mathrm{C}^*(G)$ by left multiplication with unitary multipliers. Let $\mathfrak{B} = \mathrm{C}^*(G)^\infty$ be the right ideal of smooth elements for this G-action, equipped with the canonical $U(\mathfrak{g})$-module structure μ. By the universal property of $\mathrm{C}^*(G)$, the pair $(\mathfrak{B}, \mu)$ is the universal integrable representation. That is, a representation of $U(\mathfrak{g})$ is integrable if and only if it is of the form $(\mathfrak{B}, \mu) \otimes_\varrho \mathcal{E}$ for a representation ϱ of $\mathrm{C}^*(G)$.

Let $X_1, \dots, X_d$ form a basis of $\mathfrak{g}$. The *Laplacian* is $L := -\sum_{i=1}^d X_i^2 \in U(\mathfrak{g})$.

Theorem 5.21 ([21, Théorème 2.12]). *A representation $(\pi, \mathfrak{E})$ of $U(\mathfrak{g})$ is integrable if and only if $\overline{\pi(L^n)}$ is regular and self-adjoint for all $n \in \mathbb{N}$.*

Proof. Since Pierrot does not require representations to be closed, his statement is slightly different from ours. Pierrot shows that there is a continuous representation ϱ of G with differential $X \mapsto \overline{\pi(X)}$ if and only if $T := \overline{\pi(L)}$ is self-adjoint and regular. His proof shows that all elements of $\bigcap_{n=1}^\infty \operatorname{dom} T^n$ are smooth vectors for ϱ. Conversely, all smooth vectors must belong to this intersection. A representation of $U(\mathfrak{g})$ is determined by its domain and the closed operators $\overline{\pi(X)}$ for $X \in \mathfrak{g}$. So a closed representation $(\mathfrak{E}, \pi)$ of $U(\mathfrak{g})$ is integrable if and only if T is self-adjoint and regular and $\mathfrak{E} = \bigcap_{n=1}^\infty \operatorname{dom} T^n$. Moreover, the proof shows that the graph topology for a representation with regular self-adjoint T is determined by the graph norms of L^n for all $n \in \mathbb{N}$. If $\overline{\pi(L^n)}$ is self-adjoint, then it must be equal to T^n because $\overline{\pi(L^n)} \subseteq T^n$ and T^n is symmetric. Therefore, if $\overline{\pi(L)}$ is regular and self-adjoint, then the domain of π is $\bigcap_{n=1}^\infty \operatorname{dom} T^n$ if and only if $\overline{\pi(L^n)}$ is regular and self-adjoint also for all $n \geq 2$. □

Theorem 5.22. *The class of integrable representations of $U(\mathfrak{g})$ has $\mathrm{C}^*(G)$ as a C^*-hull and is defined by submodule conditions. So it satisfies the Strong Local–Global Principle.*

Proof. By Theorem 5.21, a representation is integrable if and only if all elements of the set $\{L^n \mid n \in \mathbb{N}\}$ act by a regular and self-adjoint operator. Hence the assertion follows from Theorem 5.17.
Alternatively, the closed multipliers of $\mathrm{C}^*(G)$ associated to $\mathrm{i}X_1, \dots, \mathrm{i}X_d$ are regular and affiliated with $\mathrm{C}^*(G)$ and generate $\mathrm{C}^*(G)$ by [32, Example 3 in §3]. Hence $\mathrm{C}^*(G)$ is a C^*-hull and the Strong Local–Global Principle holds by Theorem 5.19. ☐

The results of Vassout [30] get close to proving an analogue of Theorem 5.22 for an s-simply connected Lie groupoid G with compact base. This analogue would replace $\mathfrak{g}$ by the space of smooth sections of the Lie algebroid $A(G)$, and $U(\mathfrak{g})$ by the *-algebra of G-equivariant differential operators on G, a subalgebra of the *-algebra of G-pseudodifferential operators. Any symmetric, elliptic element of $U(\mathfrak{g})$ should be a possible replacement for the Laplacian in Theorem 5.22.

6 Polynomials in one variable II

We discuss two classes of "integrable" representations of the *-algebra $\mathbb{C}[x]$ with $x = x^*$ which are weakly admissible, but not admissible, and which violate the Local–Global Principle. Both examples have a weak C^*-hull, on which all powers of the generator x act by an affiliated multiplier. In the first example, these affiliated multipliers generate the weak C^*-hull, but not in the second. Neither Theorem 5.17 nor Theorem 5.19 apply because the generating affiliated multipliers are not self-adjoint. The first example shows that a C^*-algebra generated by affiliated multipliers in the sense of Woronowicz need not be a C^*-hull, though it is always a weak C^*-hull by Lemma 5.20. The second example shows that a weak C^*-hull need not be generated by affiliated multipliers.
Let $S \in \mathbb{B}(\ell^2\mathbb{N})$ be the unilateral shift. Let Q be the closed symmetric operator on $\ell^2\mathbb{N}$ with Cayley transform S. Thus Q has deficiency index $(0, 1)$. The domain of Q is $(1 - S)\ell^2\mathbb{N}$, and $Q(1 - S)\xi := \mathrm{i}(1 + S)\xi$ for all $\xi \in \ell^2\mathbb{N}$ (see also Example 4.10). We may identify $\ell^2\mathbb{N}$ with the Hardy space H^2. Then Q becomes the Toeplitz operator with the unbounded symbol $\mathrm{i}(1 + z)(1 - z)^{-1}$.
Let $\mathcal{T}$ be the Toeplitz C^*-algebra, that is, the C^*-subalgebra of $\mathbb{B}(\ell^2\mathbb{N})$ generated by S. Every element in $\mathcal{T}$ is of the form $T_\varphi + K$, where T_φ is the Toeplitz operator with symbol $\varphi \in \mathrm{C}(S^1)$ and K is a compact operator. Let $\mathcal{T}_0 \triangleleft \mathcal{T}$ be the kernel of the unique *-homomorphism $\mathcal{T} \to \mathbb{C}$ that maps S to 1.

Proposition 6.1. *There is a symmetric, affiliated multiplier Q of $\mathcal{T}_0$ with domain $(1-S)\cdot\mathcal{T}_0$ and $Q\cdot(1-S)\cdot t := \mathrm{i}(1+S)\cdot t$ for all $t \in \mathcal{T}_0$. It generates $\mathcal{T}_0$ in the sense of Woronowicz.*

Proof. We claim that the right ideal $(1-S)S^*\mathcal{T}_0 \subseteq \mathcal{T}_0$ is dense. This would fail for $\mathcal{T}$ because the continuous *-homomorphism $\mathcal{T} \to \mathbb{C}$, $S \mapsto 1$, annihilates this right ideal. First, $(1-S)S^*\mathbb{K}(\ell^2\mathbb{N})$ is dense in $\mathbb{K}(\ell^2\mathbb{N})$ because $(1-S)S^*$ has dense range on $\ell^2\mathbb{N}$. So the closure of $(1-S)S^*\mathcal{T}_0$ contains $\mathbb{K}(\ell^2\mathbb{N})$. Secondly, $(1-S)S^*\mathcal{T}_0/\mathbb{K}(\ell^2\mathbb{N})$ is dense in $\mathcal{T}_0/\mathbb{K}(\ell^2\mathbb{N}) \cong \mathrm{C}_0(S^1 \setminus \{1\})$ because the function $(1-z)\overline{z}$ on S^1 vanishes only at 1.

An affiliated multiplier of $\mathcal{T}_0$ is the same as a regular operator on $\mathcal{T}_0$, viewed as a Hilbert module over itself. Since $(1-S)S^*\mathcal{T}_0$ is dense in $\mathcal{T}_0$, there is a regular, symmetric operator Q' on $\mathcal{T}_0$ that has S as its Cayley transform, see [17, Chapter 10]. The operator Q' has the domain $(1-S)S^*\mathcal{T}_0$ and acts by $Q' \cdot (1-S)S^*t := \mathrm{i}(1+S)S^*t$. Rewriting any $t \in \mathcal{T}_0$ as $t = S^*St$, we may replace S^*t by t here. Thus $Q' = Q$.

Since $Q + \mathrm{i}$ maps $(1-S)t$ to $\mathrm{i}(1+S)t + \mathrm{i}(1-S)t = 2\mathrm{i}t$, it is surjective, and $(Q+\mathrm{i})^{-1} = \frac{1}{2\mathrm{i}}(1-S)$ belongs to $\mathcal{T}_0$. Hence $(Q+\mathrm{i})^* = Q^* - \mathrm{i}$ is the inverse of $\frac{1}{-2\mathrm{i}}(1-S^*)$. So Q^* has domain $(1-S^*)\mathcal{T}_0$ and maps $(1-S^*)t \mapsto \mathrm{i}(1-S^*)t - 2\mathrm{i}t = -\mathrm{i}(1+S^*)t$. As expected, Q^* contains Q: we may write $(1-S)t = S^*St - St = (1-S^*)(-St)$, and Q^* maps this to $-\mathrm{i}(1+S^*)(-St) = \mathrm{i}(S+1)t$.

Next we show that Q^*Q+1 is the inverse of $\frac{1}{4}(1-S)(1-S^*) \in \mathcal{T}_0$. We compute

$$\begin{aligned} &Q^*Q(1-S)(1-S^*)t = \mathrm{i}Q^*(1+S)(1-S^*)t = \mathrm{i}Q^*(1+S-S^*-SS^*)t \\ &= \mathrm{i}Q^*(1-S^*)(2+S-SS^*)t = (1+S^*)(2+S-SS^*)t = (4-(1-S)(1-S^*))t. \end{aligned}$$

This implies $(Q^*Q+1)(1-S)(1-S^*)t = 4t$. Since this is already surjective and Q^*Q+1 is injective, the domain of Q^*Q+1 is exactly $(1-S)(1-S^*)\mathcal{T}_0$, and Q^*Q+1 is the inverse of $\frac{1}{4}(1-S)(1-S^*) \in \mathcal{T}_0$ as asserted.

Let ϱ_1 and ϱ_2 be two Hilbert space representations of $\mathcal{T}_0$ whose extension to affiliated multipliers maps Q to the same unbounded operator. Then they also map the Cayley transform S of Q to the same partial isometry. So $\varrho_1(S) = \varrho_2(S)$, which gives $\varrho_1 = \varrho_2$. Thus Q separates the representations of $\mathcal{T}_0$. Since $(Q^*Q+1)^{-1} \in \mathcal{T}_0$ as well, [32, Theorem 3.3] shows that the affiliated multiplier Q generates $\mathcal{T}_0$. □

The domain of Q^n is the right ideal $(1-S)^n \cdot \mathcal{T}_0$, which is dense in $\mathcal{T}_0$ for the same reason as $(1-S)\cdot\mathcal{T}_0$. Even more, the right ideal $(1-S)^{n+1}\cdot\mathcal{T}_0$ is dense in $(1-S)^n\cdot\mathcal{T}_0$ in the graph norm of Q^n. Thus the intersection $\mathfrak{T}$ of this decreasing chain of dense right ideals $(1-S)^n\mathcal{T}_0$ is still dense in $\mathcal{T}_0$ by [28, Lemma 1.1.2]. This intersection is the domain of a closed representation μ of $\mathbb{C}[x]$ on $\mathcal{T}_0$ with $\overline{\mu(x^n)} = Q^n$. We call a representation of $\mathbb{C}[x]$ on a Hilbert module $\mathcal{E}$ *Toeplitz integrable* if it is of the form $(\mathfrak{T}, \mu) \otimes_\varrho \mathcal{E}$ for some representation $\varrho\colon \mathcal{T}_0 \to \mathbb{B}(\mathcal{E})$.

Proposition 6.2. *The class of Toeplitz integrable representations of $\mathbb{C}[x]$ is weakly admissible with the weak C^*-hull $\mathcal{T}_0$. It is not admissible, so $\mathcal{T}_0$ is not a C^*-hull. The Toeplitz integrable representations violate the Local–Global Principle.*

A representation $(\mathfrak{E}, \pi)$ of $\mathbb{C}[x]$ on a Hilbert module $\mathcal{E}$ over a C^*-algebra D is Toeplitz integrable if and only if it has the following properties:

(1) $\overline{\pi(x+\mathrm{i})^n\mathfrak{E}} = \mathcal{E}$ *for all* $n \in \mathbb{N}_{\geq 1}$*;*

(2) $\overline{\pi(x)}$ *is regular.*

Toeplitz integrable representations on $\mathcal{E}$ are in bijection with regular, symmetric operators T on $\mathcal{E}$ for which $T + \mathrm{i}$ is surjective.

Proof. We checked condition (1) in Proposition 3.8 in the proof of Proposition 6.1. Thus $\mathcal{T}_0$ is a weak C*-hull for the Toeplitz integrable representations, and this class is weakly admissible. Any self-adjoint operator on a Hilbert space generates a Toeplitz integrable representation of $\mathbb{C}[x]$ because $\mathcal{T}_0/\mathbb{K}(\ell^2\mathbb{N}) \cong \mathrm{C}_0(\mathbb{R})$; so does Q itself. Thus both Example 4.10 and Proposition 4.8 show that the class of Toeplitz integrable representations is not admissible. So $\mathcal{T}_0$ is not a C*-hull.

We claim that the representation $(\mathfrak{T}, \mu)$ of $\mathbb{C}[x]$ on $\mathcal{T}_0$ has the properties (1) and (2) in the proposition. First, $(\mu(x) + \mathrm{i})^n$ acts by $(2\mathrm{i})^n(1 - S)^{-n}$ on its dense domain $\mathfrak{T} := \bigcap_{k=1}^{\infty}(1 - S)^k\mathcal{T}_0$. Since $(1 - S)^{k+1}\mathcal{T}_0$ is norm dense in $(1-S)^k\mathcal{T}_0$, the closure of $(\mu(x)+\mathrm{i})^n$ is equal to $(2\mathrm{i})^n(1-S)^{-n}$ with its natural domain $(1 - S)^n\mathcal{T}_0$, and this operator is surjective. Secondly, $\mu(x) = Q$ is regular.

The property (1) is a sequence of submodule conditions, see Example 5.13. Hence it is inherited by interior tensor products by Lemma 5.8. So is the property (2) by [17, Proposition 9.10]. Hence both (1) and (2) are necessary for a representation $(\mathfrak{E}, \pi)$ to be Toeplitz integrable.

Conversely, let $(\mathfrak{E}, \pi)$ be a representation of $\mathbb{C}[x]$ on $\mathcal{E}$ that satisfies (1) and (2). Then the closed, symmetric operator $T := \overline{\pi(x)}$ on $\mathcal{E}$ is regular by (2). So its Cayley transform s is an adjointable partial isometry such that $(1 - s)s^*$ has dense range (see [17, Chapter 10]). Even more, s is an isometry because $\overline{(T + \mathrm{i})\mathfrak{E}} = \mathcal{E}$. Thus s generates a unital representation ϱ of $\mathcal{T}$. The restriction of ϱ to $\mathcal{T}_0$ is nondegenerate because $(1-s)s^*$ has dense range. Let $\pi' := \mu \otimes_\varrho 1$ be the representation of $\mathbb{C}[x]$ associated to ϱ. Then

$$\overline{\pi'((x + \mathrm{i})^n)} = (2\mathrm{i})^n(1 - s)^{-n} \supseteq \overline{\pi((x + \mathrm{i})^n)}.$$

Assumption (1) implies that $\mathfrak{E}$ is dense in the domain of $(2\mathrm{i})^n(1 - s)^{-n}$ in the graph norm of $(2\mathrm{i})^n(1 - s)^{-n}$. Hence even $\overline{\pi'((x + \mathrm{i})^n)} = (2\mathrm{i})^n(1 - s)^{-n} = \overline{\pi((x + \mathrm{i})^n)}$. Since the domains of $\overline{\pi(x)^n}$ form a decreasing sequence, induction on n now shows that $\overline{\pi'(x^n)} = \overline{\pi(x^n)}$. The set $\{x^n\}$ is a strong generating set for $\mathbb{C}[x]$ by Lemma 4.1. Thus $\pi = \pi'$ by Proposition 2.9. This finishes the proof that Toeplitz integrable representations of $\mathbb{C}[x]$ are characterised by the properties (1) and (2) and that they are in bijection with regular, symmetric operators T for which $T + \mathrm{i}$ is surjective.

For a counterexample to the Local–Global Principle, let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ and $D = \mathrm{C}(\bar{\mathbb{N}})$. Let $\mathcal{E} \subseteq \mathrm{C}(\bar{\mathbb{N}}, \ell^2\mathbb{N})$ consist of all continuous functions $f\colon \bar{\mathbb{N}} \to \ell^2\mathbb{N}$ with $f(\infty)\perp\delta_0$. The unilateral shift S on $\mathrm{C}(\bar{\mathbb{N}}, \ell^2\mathbb{N})$ restricts to a non-adjointable isometry s on this subspace. Let T be the inverse Cayley transform of s. This is a closed, symmetric operator on $\mathcal{E}$ that is irregular because its Cayley transform is not adjointable. If $\varrho\colon D \to$

$\mathbb{B}(\mathcal{H})$ is a Hilbert space representation, then the induced representation of $\mathbb{C}[x]$ is associated to the closed operator $T \otimes_\varrho 1$. The operator $(T \otimes_\varrho 1) + \mathrm{i}$ remains surjective, and $T \otimes_\varrho 1$ is regular because it acts on a Hilbert space. So $T \otimes_\varrho 1$ generates a Toeplitz integrable representation for all representations ϱ of D. Since T itself does not generate a Toeplitz integrable representation, the Local–Global Principle is violated. ☐

Condition (1) in Proposition 6.2 is a submodule condition. If regularity without self-adjointness were a submodule condition as well, then the Toeplitz integrable representations of $\mathbb{C}[x]$ would be defined by submodule conditions; so the failure of the Local–Global Principle for them would contradict Theorem 5.19.

The identical inclusion $\mathcal{T}_0 \hookrightarrow \mathcal{M}(\mathbb{K}(\ell^2\mathbb{N}))$ is a representation of the weak C^*-hull $\mathcal{T}_0$ on $\mathbb{K}(\ell^2\mathbb{N})$ and thus corresponds to a Toeplitz integrable representation of $\mathbb{C}[x]$ on $\mathbb{K}(\ell^2\mathbb{N})$. This is simply the restriction of $(\mathfrak{T}, \mu)$ to the Hilbert $\mathcal{T}_0$-submodule $\mathbb{K}(\ell^2\mathbb{N}) \subset \mathcal{T}_0$, with domain $\mathfrak{T} \cap \mathbb{K}(\ell^2\mathbb{N})$ and the same action μ of $\mathbb{C}[x]$. Call a representation *purely Toeplitz integrable* if it is of the form $(\mathfrak{T} \cap \mathbb{K}(\ell^2\mathbb{N}), \mu) \otimes_\varrho \mathcal{E}$ for some representation $\varrho\colon \mathbb{K}(\ell^2\mathbb{N}) \to \mathbb{B}(\mathcal{E})$.

Proposition 6.3. *The purely Toeplitz integrable representations of* $\mathbb{C}[x]$ *form a weakly admissible class that is not admissible, and* $\mathbb{K}(\ell^2\mathbb{N})$ *is a weak* C^**-hull for it, but not a* C^**-hull. This class violates the Local–Global Principle. The closed multiplier* $Q = \overline{\mu(x)}$ *of* $\mathcal{T}_0$ *is affiliated with* $\mathbb{K}(\ell^2\mathbb{N})$ *but does not generate* $\mathbb{K}(\ell^2\mathbb{N})$.

A representation $(\mathfrak{E}, \pi)$ *of* $\mathbb{C}[x]$ *on a Hilbert module* $\mathcal{E}$ *over a* C^**-algebra* D *is purely Toeplitz integrable if and only if it has the following property in addition to those in Proposition* 6.2*:*

(3) *the closure of* $\bigcup_{n=1}^{\infty} (\pi(x-\mathrm{i})^n \mathfrak{E})^\perp$ *is* $\mathcal{E}$*.*

Proof. Since $\mathbb{K}(\ell^2\mathbb{N})$ has fewer representations than $\mathcal{T}_0$, the condition (1) in Proposition 3.8 for $\mathbb{K}(\ell^2\mathbb{N})$ follows from the corresponding property for $\mathcal{T}_0$, which we have already checked in the proof of Proposition 6.1. Hence $\mathbb{K}(\ell^2\mathbb{N})$ is a weak C^*-hull for the purely Toeplitz representations of $\mathbb{C}[x]$.

Since Q gives a purely Toeplitz representation of $\mathbb{C}[x]$ on $\ell^2(\mathbb{N})$, the class of purely Toeplitz integrable representations is not admissible by Example 4.10. Therefore, its weak C^*-hull is not a C^*-hull. The same counterexample as in the proof of Proposition 6.2 shows that the Local–Global Principle fails for the purely Toeplitz representations.

Any closed operator on $\ell^2\mathbb{N}$ is affiliated with $\mathbb{K}(\ell^2\mathbb{N})$. In particular, so is Q. In the identical representation of $\mathbb{K}(\ell^2\mathbb{N})$ on the Hilbert space $\ell^2\mathbb{N}$, the image of Q is affiliated with $\mathcal{T}_0$ by Proposition 6.1. But the representation of $\mathbb{K}(\ell^2\mathbb{N})$ is not by a morphism to $\mathcal{T}_0$ because the inclusion map $\mathbb{K}(\ell^2\mathbb{N}) \hookrightarrow \mathcal{T}_0$ is degenerate. Hence Q does not generate $\mathbb{K}(\ell^2\mathbb{N})$ in the sense of Woronowicz.

The element $P_n := 1 - S^n(S^*)^n \in \mathbb{K}(\ell^2\mathbb{N}) \subseteq \mathcal{T}_0$ is the orthogonal projection onto the span of $\delta_0, \dots, \delta_{n-1}$. A representation of $\mathcal{T}_0$ maps P_n to an orthogonal projection whose image is the orthogonal complement of the image of S^n. This is also the orthogonal complement of the image of $\pi(x-\mathrm{i})^n$. These orthogonal complements form an increasing chain of complementable submodules, and π is purely Toeplitz if and only if their union is all of $\mathcal{E}$. This proves our characterisation of purely Toeplitz representations. $\square$

7 Bounded and locally bounded representations

Let A be a *-algebra. A *bounded representation* of A on a Hilbert module $\mathcal{E}$ is a *-homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{E})$. Corollary 2.11 says that a closed representation is bounded once $\overline{\pi(a)}$ is globally defined for a in a strong generating set of A. Finite-dimensional representations are always bounded. In particular, characters are bounded. Thus commutative *-algebras have many bounded representations. Many other *-algebras, such as the Weyl algebra, have no bounded representations. In this section, we are going to study C*-hulls related to bounded representations. These are only relevant if A has many bounded representations.

Any bounded representation π of A is bounded in some C*-seminorm q on A, that is, $\|\pi(a)\| \le q(a)$ for all $a \in A$. Then π extends to the (Hausdorff) completion $\mathcal{A}_q$ of A in the seminorm q, which is a unital C*-algebra.

If p, q are two C*-seminorms on A, then $\max\{p, q\}$ is a C*-seminorm as well. Thus the set $\mathcal{N}(A)$ of C*-seminorms on A is directed. For $q, q' \in \mathcal{N}(A)$ with $q \le q'$, let $\varphi_{q,q'}\colon \mathcal{A}_{q'} \to \mathcal{A}_q$ be the *-homomorphism induced by the identity map on A. The C*-algebras $\mathcal{A}_q$ and the *-homomorphisms $\varphi_{q,q'}$ for $q \le q'$ in $\mathcal{N}(A)$ form a projective system of C*-algebras. Each *-homomorphism $\varphi_{q,q'}$ is unital and surjective because its image contains A, which is unital and dense in $\mathcal{A}_{q'}$.

The C*-seminorms in $\mathcal{N}(A)$ define a locally convex topology on A, where a net converges if and only if it converges in any C*-seminorm. Let $\mathcal{A}$ with the canonical map $j\colon A \to \mathcal{A}$ be the completion of A in this topology. This is a C*-algebra if and only if there is a *largest* C*-seminorm on A. In general, $\mathcal{A}$ is the projective limit of the diagram of unital C*-algebras $(\mathcal{A}_q, \varphi_{q,q'})$ described above. Thus $\mathcal{A}$ is a unital pro-C*-algebra, see [22].

As a concrete example, we describe $\mathcal{A}$ for a commutative *-algebra A.

Definition 7.1. Let $\hat{A}$ be the set of *-homomorphisms $A \to \mathbb{C}$, which we briefly call *characters*. Each $a \in A$ gives a function $\hat{a}\colon \hat{A} \to \mathbb{C}$, $\hat{a}(\chi) := \chi(a)$. We equip $\hat{A}$ with the coarsest topology making these functions continuous. That is, a net $(\chi_i)_{i\in I}$ in $\hat{A}$ converges to $\chi \in \hat{A}$ if and only if $\lim \chi_i(a) = \chi(a)$ for all $a \in A$. Let τ_c be the *compactly generated topology* associated to this topology, that is, a subset in $\hat{A}$ is closed in τ_c if and only if its intersection with any compact subset in $\hat{A}$ is closed.

If $a \in A$, then its *Gelfand transform* $\hat{a}$ is a continuous function on $\hat{A}$. This defines a *-homomorphism $A \to \mathrm{C}(\hat{A})$. If the usual topology on $\hat{A}$ is locally compact or metrisable, then it is already compactly generated and hence equal to τ_c. The topology τ_c may have more closed subsets and hence more continuous functions to $\mathbb{C}$. So $\mathrm{C}(\hat{A}) \subseteq \mathrm{C}(\hat{A}, \tau_c)$.

Proposition 7.2. *Let A be a commutative *-algebra. The directed set $\mathcal{N}(A)$ of C^*-seminorms on A is isomorphic to the directed set of compact subsets of $\hat{A}$, where $K \subseteq \hat{A}$ corresponds to the C^*-seminorm*

$$\|a\|_K := \sup\{|\hat{a}(\chi)| \mid \chi \in K\}.$$

The C^*-completion of A in this C^*-seminorm is $\mathrm{C}(K)$. And $\mathcal{A} \cong \mathrm{C}(\hat{A}, \tau_c)$, where the inclusion map $j\colon A \to \mathcal{A}$ is the Gelfand transform $A \to \mathrm{C}(\hat{A}, \tau_c)$, $a \mapsto \hat{a}$.

Proof. Let q be a C^*-seminorm on A. Let $\hat{A}_q \subseteq \hat{A}$ be the subspace of all q-bounded characters, that is, $\chi \in \hat{A}_q$ if and only if $|\chi(a)| \le q(a)$ for all $a \in A$. These are precisely the characters that extend to characters on the C^*-completion $\mathcal{A}_q$. Conversely, since A is dense in $\mathcal{A}_q$, any character on $\mathcal{A}_q$ is the unique continuous extension of a q-bounded character on A. And the subspace topology on $\hat{A}_q \subseteq \hat{A}$ is equal to the canonical topology on the spectrum of $\mathcal{A}_q$: a net of q-bounded characters that converges uniformly on A also converges uniformly on $\mathcal{A}_q$. Thus

$$\mathcal{A}_q \cong \mathrm{C}(\hat{A}_q)$$

by the Gelfand–Naimark Theorem, and so $\hat{A}_q \subseteq \hat{A}$ is compact for each $q \in \mathcal{N}(A)$.

If $q \le q'$, then $\hat{A}_q \subseteq \hat{A}_{q'}$ and $\varphi_{qq'}\colon \mathcal{A}_{q'} \twoheadrightarrow \mathcal{A}_q$ is the restriction map for the subspace $\hat{A}_q \subseteq \hat{A}_{q'}$. The pro-$\mathrm{C}^*$-algebra $\mathcal{A}$ is the limit of this diagram of commutative C^*-algebras. Since all the maps $\hat{A}_q \subseteq \hat{A}_{q'}$ are injective, $\mathcal{A}$ is the algebra of continuous functions on $\bigcup_{q\in\mathcal{N}(A)} \hat{A}_q$ with the inductive limit topology. That is, a subset of $\bigcup_{q\in\mathcal{N}(A)} \hat{A}_q$ is closed if and only if its intersection with each $\hat{A}_q$ is closed, where $\hat{A}_q$ carries the (compact) subspace topology from $\hat{A}$.

Any character χ on A is bounded with respect to some C^*-seminorm; for instance, $\|a\|_\chi := |\chi(a)|$. Thus $\bigcup_{q\in\mathcal{N}(A)} \hat{A}_q = \hat{A}$ as a set. If $K \subseteq \hat{A}$ is compact, then $\hat{a} \in \mathrm{C}(\hat{A})$ for $a \in A$ must be uniformly bounded on K, so that

$$\|a\|_K := \sup\{|\hat{a}(\chi)| \mid \chi \in K\}$$

is a C^*-seminorm on A. Thus $K \subseteq \hat{A}_q$ for some $q \in \mathcal{N}(A)$. Hence the inductive limit topology on $\bigcup_{q\in\mathcal{N}(A)} \hat{A}_q$ is τ_c. ☐

We return to the general noncommutative case. The class of q-bounded representations for a fixed $q \in \mathcal{N}(A)$ is easily seen to be weakly admissible. The class of bounded representations with variable q is not weakly admissible unless A

has a largest C*-seminorm because it is not closed under direct sums. We are going to define the larger class of "locally bounded" representations to rectify this. Roughly speaking, a representation is locally bounded if and only if it comes from a representation of the pro-C*-algebra $\mathcal{A}$. Before we define locally bounded representations, we characterise q-bounded representations by some slightly weaker estimates.

PROPOSITION 7.3. *Let A be a *-algebra and let q be a C^*-seminorm on A. Let $(\mathfrak{E}, \pi)$ be a representation of A on a Hilbert module $\mathcal{E}$ over some C^*-algebra D and let $\xi \in \mathfrak{E}$. The following are equivalent:*

(1) *there is $C > 0$ with $\|\langle \xi, \pi(a)\xi\rangle\| \le Cq(a)$ for all $a \in A$;*

(2) *there is $C > 0$ with $\|\pi(a)\xi\| \le Cq(a)$ for all $a \in A$;*

(3) *$\|\pi(a)\xi\| \le \|\xi\| q(a)$ for all $a \in A$.*

The set of vectors ξ with these equivalent properties is a norm-closed A, D-submodule of $\mathcal{E}$. The representation of A on this submodule extends to the C^*-completion $\mathcal{A}_q$.

Proof. The implications (3)⇒(2)⇒(1) are trivial. Conversely, assume (1) and let $a \in A$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in A that converges in $\mathcal{A}_q$ towards the positive square-root of $q(a)^2 - a^*a$. Then the sequence $(a^*a + b_n^*b_n)$ in A converges in the norm q to $q(a)^2 \in A$. If (1) holds, then

$$\lim_{n\to\infty} \langle \xi, \pi(a^*a + b_n^*b_n)\xi\rangle = q(a)^2 \langle \xi, \xi\rangle.$$

Since $0 \le \langle \pi(a)\xi, \pi(a)\xi\rangle \le \langle \pi(a)\xi, \pi(a)\xi\rangle + \langle \pi(b_n)\xi, \pi(b_n)\xi\rangle = \langle \xi, \pi(a^*a + b_n^*b_n)\xi\rangle$ for all n, this implies $\|\pi(a)\xi\| \le \lim \|\langle \xi, \pi(a^*a + b_n^*b_n)\xi\rangle\| = q(a)^2 \|\xi\|^2$. Thus (1) implies (3).

The set $\mathcal{E}_q$ of vectors $\xi \in \mathfrak{E}$ satisfying (2) is a vector subspace and closed under left multiplication by elements of A and right multiplication by elements of D. On this subspace, the graph and norm topologies coincide because of (3). The subspace $\mathcal{E}_q$ is closed in the norm topology by the Principle of Uniform Boundedness. The *-representation of A on this submodule is globally defined and bounded by the C*-seminorm q. Hence it extends to a representation of $\mathcal{A}_q$. □

DEFINITION 7.4. Let $(\mathfrak{E}, \pi)$ be a representation of A on a Hilbert module $\mathcal{E}$. A vector $\xi \in \mathfrak{E}$ is *bounded* if it satisfies the equivalent conditions in Proposition 7.3 for some $q \in \mathcal{N}(A)$. The representation is *locally bounded* if the bounded vectors are dense in $\mathfrak{E}$ in the graph topology.

By Proposition 7.3, the q-bounded vectors in $\mathfrak{E}$ for a fixed $q \in \mathcal{N}(A)$ form a closed A, D-submodule $\mathcal{E}_q \subseteq \mathcal{E}$, on which the representation of A extends to the C*-completion $\mathcal{A}_q$ and hence to a representation of $\mathcal{A}$. Since $\mathcal{N}(A)$ is directed and $\mathcal{E}_q \subseteq \mathcal{E}_{q'}$ if $q \le q'$, the family of sub-bimodules $\mathcal{E}_q \subseteq \mathfrak{E}$ is directed. The set of bounded vectors is the increasing union

$$\mathfrak{E}_{\mathrm{b}} := \bigcup_{q\in\mathcal{N}(A)} \mathcal{E}_q.$$

Since $\pi|_{\mathcal{E}_q}$ extends to $\mathcal{A}$ for each q, there is a representation $\bar{\pi}$ of the pro-C*-algebra $\mathcal{A}$ on $\mathfrak{E}_{\mathrm{b}} \subseteq \mathcal{E}$. The representation $(\mathfrak{E},\pi)$ is locally bounded if and only if $(\mathfrak{E}_{\mathrm{b}},\bar{\pi}\circ j)$ is a core for it. Thus $(\mathfrak{E},\pi)$ is the closure of the "restriction" $\bar{\pi}\circ j$ of $\bar{\pi}$ to A.

We do not claim that $\bar{\pi}$ is closed, and neither do we claim that $\bar{\pi}\circ j$ is locally bounded for any representation of $\mathcal{A}$: we need the representation of $\mathcal{A}$ to be locally bounded as well:

Definition 7.5. A representation $(\pi,\mathfrak{E})$ of a pro-C*-algebra $\mathcal{A}$ is *locally bounded* if the vectors $\xi \in \mathfrak{E}$ for which $\mathcal{A} \to \mathcal{E}$, $a \mapsto \pi(a)\xi$, is continuous form a core.

Proposition 7.6. *Composition with $j\colon A \to \mathcal{A}$ induces an equivalence between the categories of locally bounded representations of A and $\mathcal{A}$ which is compatible with isometric intertwiners and interior tensor products.*

Proof. The *-homomorphism j induces an isomorphism between the directed sets of C*-seminorms on A and $\mathcal{A}$. Therefore, a representation $\bar{\pi}$ of $\mathcal{A}$ is locally bounded if and only if the vectors ξ with $\|\bar{\pi}(a)\xi\| \le q(a)\|\xi\|$ for all $a \in \mathcal{A}$, for some $q \in \mathcal{N}(A)$, form a core. Since $j(A)$ is dense in $\mathcal{A}$, this is equivalent to $\|\pi(a)\xi\| \le q(a)\|\xi\|$ for all $a \in A$. Thus the closure of $\bar{\pi}\circ j$ is locally bounded if and only if $\bar{\pi}$ is.

An isometric intertwiner $\bar{\pi}_1 \hookrightarrow \bar{\pi}_2$ also intertwines the closures of $\bar{\pi}_1\circ j$ and $\bar{\pi}_2\circ j$ by Lemma 2.14. Conversely, an isometric intertwiner between two locally bounded representations of A must map q-bounded vectors to q-bounded vectors for any $q \in \mathcal{N}(A)$. Thus it remains an isometric intertwiner between the canonical extensions of the representations to $\mathcal{A}$. So the equivalence between the locally bounded representations of A and $\mathcal{A}$ is compatible with isometric intertwiners. It is also compatible with interior tensor products, that is, the closure of $(\bar{\pi}\otimes_D 1_{\mathcal{G}})\circ j$ is $\overline{\bar{\pi}\circ j}\otimes_D 1_{\mathcal{G}}$. □

Proposition 7.7. *All irreducible, locally bounded Hilbert space representations are bounded.*

Proof. If π is irreducible, then the closed A-submodule $\mathcal{E}_q$ for a C*-seminorm q is either $\{0\}$ or $\mathcal{E}$. The latter must happen for some q if π is locally bounded. □

Thus local boundedness is not an interesting notion for irreducible representations.

If A has no C*-seminorms, then $\mathcal{A} = \{0\}$ and A has no locally bounded representations, so that the following discussion will be empty. Even if the map

$j\colon A \to \mathcal{A}$ is not injective, there are examples where all "integrable" representations of A come from $\mathcal{A}$. An important case is the unit fibre for the canonical $\mathbb{Z}$-grading on the Virasoro algebra studied in [27, §9.3]. In this case, A is not commutative, but all irreducible, integrable representations are characters and hence locally bounded.

Proposition 7.8. *If π is a locally bounded representation, then $\overline{\pi(a)}$ is regular and self-adjoint for each $a \in A_h$.*

Proof. The map of left multiplication by $j(a) \pm \mathrm{i}$ on $\mathcal{A}$ is invertible because $j(a) \in \mathcal{A}$ is symmetric and $\mathcal{A}$ is a pro-C*-algebra. Therefore, $\bar{\pi}(j(a)) \pm \mathrm{i} \subseteq \overline{\pi(a) \pm \mathrm{i}}$ has dense range on $\mathcal{E}$. Thus $\overline{\pi(a)}$ is regular and self-adjoint, see [17, Chapter 10]. □

Corollary 7.9. *Let A be a *-algebra. The class $\mathrm{Rep}_{\mathrm{b}}(A)$ of locally bounded representations of A is admissible.*

Proof. Being locally bounded is clearly invariant under unitary *-intertwiners and direct sums. It is also invariant under direct summands because a *-intertwiner maps bounded vectors to bounded vectors. If $\xi \in \mathcal{E}$ is bounded, then $\xi \otimes \eta \in \mathcal{E} \otimes_D \mathcal{F}$ is bounded for any C*-correspondence $\mathcal{F}$. Thus a locally bounded representation on $\mathcal{E}$ induces one on $\mathcal{E} \otimes_D \mathcal{F}$.
Since A_h is a strong generating set for A by Example 2.8, the class of representations for which all $a \in A_h$ act by a regular and self-adjoint operator is admissible by Theorem 5.17. This class contains the locally bounded representations by Proposition 7.8. Hence this subclass is also admissible. □

Any pro-C*-algebra $\mathcal{A}$ contains a dense unital C*-subalgebra $\mathcal{A}_{\mathrm{b}}$ of bounded elements, see [22, Proposition 1.11]. For instance, if A is commutative, so that $\mathcal{A} \cong \mathrm{C}(\hat{A}, \tau_c)$ by Proposition 7.2, then $\mathcal{A}_{\mathrm{b}} = \mathrm{C}_{\mathrm{b}}(\hat{A}, \tau_c)$ consists of the *bounded* continuous functions.
Let $(\mathfrak{E}, \pi)$ be a locally bounded representation of A. This comes from a locally bounded representation $(\mathfrak{E}_{\mathrm{b}}, \bar{\pi})$ of $\mathcal{A}$ by Proposition 7.6. The closure of the restriction of $\bar{\pi}$ to $\mathcal{A}_{\mathrm{b}}$ is a representation of a unital C*-algebra. Hence it is a unital *-homomorphism $\varrho\colon \mathcal{A}_{\mathrm{b}} \to \mathbb{B}(\mathcal{E})$ by Lemma 2.12.

Proposition 7.10. *Two locally bounded representations π_1 and π_2 of A on a Hilbert module $\mathcal{E}$ are equal if and only if they induce the same representation of $\mathcal{A}_{\mathrm{b}}$.*

Proof. Of course, π_1 and π_2 induce the same representation of $\mathcal{A}_{\mathrm{b}}$ if $\pi_1 = \pi_2$. Conversely, assume that π_1 and π_2 induce the same representation ϱ of $\mathcal{A}_{\mathrm{b}}$. If $a \in A_h$, then the Cayley transform c_a of $j(a) \in \mathcal{A}$ is a unitary element of $\mathcal{A}_{\mathrm{b}}$. The Cayley transforms of $\overline{\pi_1(a)}$ and $\overline{\pi_2(a)}$ are both equal to $\varrho(c_a)$. Hence $\overline{\pi_1(a)} = \overline{\pi_2(a)}$. Since this holds for all $a \in A_h$, Proposition 2.9 gives $\pi_1 = \pi_2$. □

The C^*-algebra $\mathcal{A}_\mathrm{b}$ usually has many representations that do not come from locally bounded representations of A. Hence it is not a C^*-hull. It is, however, a useful tool to decide when a representation μ of A on a C^*-algebra B is a weak C^*-hull, that is, when A separates the Hilbert space representations of B:

PROPOSITION 7.11. *Let μ be a locally bounded representation of A on a C^*-algebra B and let $\varrho\colon \mathcal{A}_\mathrm{b} \to \mathcal{M}(B) = \mathbb{B}(B)$ be the associated representation of $\mathcal{A}_\mathrm{b}$. The image of ϱ is dense in $\mathcal{M}(B)$ in the strict topology if and only if B is a weak C^*-hull for the class of B-integrable representations of A defined by μ.*

Proof. Combine Proposition 7.10 and the following proposition for $D = \mathcal{A}_\mathrm{b}$. □

PROPOSITION 7.12. *Let μ be a representation of A on a C^*-algebra B. Let D be a C^*-algebra and $\varphi\colon D \to \mathcal{M}(B)$ a *-homomorphism. Assume that two representations ϱ_1, ϱ_2 of B on a Hilbert space $\mathcal{H}$ satisfy $\mu \otimes_{\varrho_1} 1_\mathcal{H} = \mu \otimes_{\varrho_2} 1_\mathcal{H}$ if and only if $\bar\varrho_1 \circ \varphi = \bar\varrho_2 \circ \varphi$, where $\bar\varrho_1$ and $\bar\varrho_2$ denote the unique strictly continuous extensions of ϱ_1, ϱ_2 to $\mathcal{M}(B)$. Then B is a weak C^*-hull for a class of integrable representations of A if and only if $\varphi(D)$ is dense in $\mathcal{M}(B)$ in the strict topology.*

Proof. We use the criterion for weak C^*-hulls in (1) in Proposition 3.8. Assume first that $\varphi(D)$ is strictly dense in $\mathcal{M}(B)$. Let ϱ_1, ϱ_2 be two Hilbert space representations of B that satisfy $\mu \otimes_{\varrho_1} 1_\mathcal{H} = \mu \otimes_{\varrho_2} 1_\mathcal{H}$. Extend ϱ_1, ϱ_2 to strictly continuous representations $\bar\varrho_1, \bar\varrho_2$ of $\mathcal{M}(B)$. By assumption, $\bar\varrho_1 \circ \varphi = \bar\varrho_2 \circ \varphi$, that is, $\bar\varrho_1$ and $\bar\varrho_2$ are equal on $\varphi(D)$. Since they are strictly continuous and $\varphi(D)$ is strictly dense, we get $\bar\varrho_1 = \bar\varrho_2$ and hence $\varrho_1 = \varrho_2$. Thus the condition (1) in Proposition 3.8 is satisfied, making B a weak C^*-hull of A.
Conversely, assume that $\varphi(D)$ is not strictly dense in $\mathcal{M}(B)$. We claim that the image of D is not weakly dense in the bidual W^*-algebra B^{**}. Any positive linear functional on B extends to a strictly continuous, positive linear functional on $\mathcal{M}(B)$ by extending its GNS-representation to a strictly continuous representation of $\mathcal{M}(B)$. By the Jordan decomposition, the same remains true for self-adjoint linear functionals and hence for all bounded linear functionals on B. Furthermore, such extensions are unique because B is strictly dense in $\mathcal{M}(B)$. Hence restriction to B maps the space of strictly continuous linear functionals on $\mathcal{M}(B)$ isomorphically onto the dual space B^* of B, which is also the space of weakly continuous linear functionals on B^{**}. If $\varphi(D)$ is not strictly dense in $\mathcal{M}(B)$, then the Hahn–Banach Theorem gives a non-zero functional in B^* that vanishes on the image of D. When viewed as a weakly continuous functional on B^{**}, it witnesses that $\varphi(D)$ is not weakly dense in B^{**}.
Let $\varrho\colon B \to \mathbb{B}(\mathcal{H})$ be the direct sum of all cyclic representations of B. Then ϱ extends to an isomorphism of W^*-algebras from B^{**} onto the double commutant $\varrho(B)''$ of B in $\mathbb{B}(\mathcal{H})$. The extension of ϱ to $\mathcal{M}(B)$ restricts to a representation $\bar\varrho \circ \varphi\colon D \to \mathbb{B}(\mathcal{H})$. Since we assume that the image of D is not strictly

dense in $\mathcal{M}(B)$, our claim shows that $\bar{\varrho}\circ\varphi(D)$ is not weakly dense in $\varrho(B)''$. By the bicommutant theorem, this is equivalent to $\bar{\varrho}\circ\varphi(D)' \neq \varrho(B)'$.
Since these commutants are C*-algebras, they are the linear spans of the unitaries that they contain. So there is a unitary operator U in $\bar{\varrho}\circ\varphi(D)'$ that is not contained in $\varrho(B)'$. So $\varrho_2 := U\varrho U^* \neq \varrho$, but $\bar{\varrho}_2\circ\varphi = \bar{\varrho}\circ\varphi$. By assumption, the latter implies $\mu\otimes_\varrho 1_{\mathcal{H}} = \mu\otimes_{\varrho_2} 1_{\mathcal{H}}$. So A fails to separate the representations ϱ, ϱ_2 of B although they are not equal. Hence B is not a weak C*-hull of A. $\square$

Remark 7.13. Proposition 7.12 applies whenever we can somehow produce enough bounded operators from a representation of A so that these bounded operators and the original representation have the same unitary *-intertwiners. For instance, it applies if the elements of a strong generating set for A act by regular operators, so that we may take their bounded transforms.

The quotient map $\mathcal{A}_q \twoheadrightarrow \mathcal{A}_{q'}$ for $q \geq q'$ in $\mathcal{N}(A)$ identifies the primitive ideal space $\mathrm{Prim}(\mathcal{A}_{q'})$ with a closed subspace of $\mathrm{Prim}(\mathcal{A}_q)$. Let $\mathrm{Prim}\,\mathcal{A} := \bigcup_{q\in\mathcal{N}(A)} \mathrm{Prim}(\mathcal{A}_q)$. Let $a \in \mathcal{A}$ and $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A})$. Then the norm $\|a\|_{\mathfrak{p}}$ of the image of a in $\mathcal{A}_q/\mathfrak{p}$ is the same for all $q \in \mathcal{N}(A)$ with $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}_q)$. Hence the function $\mathfrak{p} \mapsto \|a\|_{\mathfrak{p}}$ on $\mathrm{Prim}(\mathcal{A})$ is well defined.

Definition 7.14. An element $a \in \mathcal{A}$ *vanishes at* ∞ if for every $\varepsilon > 0$ there is $q \in \mathcal{N}(A)$ such that $\|a\|_{\mathfrak{p}} < \varepsilon$ for $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}) \setminus \mathrm{Prim}(\mathcal{A}_q)$. An element $a \in \mathcal{A}$ is *compactly supported* if there is $q \in \mathcal{N}(A)$ with $a \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}) \setminus \mathrm{Prim}(\mathcal{A}_q)$. Let $\mathrm{C}_0(\mathcal{A})$ and $\mathrm{C}_\mathrm{c}(\mathcal{A})$ be the subsets of elements that vanish at ∞ and have compact support, respectively.

It may happen that $\mathrm{C}_0(\mathcal{A}) = \{0\}$. In the following, we are interested in the case where $\mathrm{C}_0(\mathcal{A})$ is dense in $\mathcal{A}$. For instance, $\mathrm{C}_0(\mathbb{R})$ is dense in $\mathrm{C}(\mathbb{R})$.

Lemma 7.15. *The subset* $\mathrm{C}_0(\mathcal{A})$ *is a closed ideal in* $\mathcal{A}_\mathrm{b}$. *The subspace* $\mathrm{C}_\mathrm{c}(\mathcal{A})$ *is a two-sided* **-ideal in* $\mathcal{A}$. *It is norm-dense in* $\mathrm{C}_0(\mathcal{A})$. *More generally, if* D *is a* C**-algebra and* $\varphi\colon D \to \mathcal{A}$ *is a* **-homomorphism, then* $\varphi^{-1}(\mathrm{C}_\mathrm{c}(\mathcal{A}))$ *is dense in* $\varphi^{-1}(\mathrm{C}_0(\mathcal{A}))$.

Proof. The quotient maps $\mathcal{A} \twoheadrightarrow \mathcal{A}_q \twoheadrightarrow \mathcal{A}_q/\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}_q)$ are *-homomorphisms. Thus $\mathrm{C}_0(\mathcal{A})$ is a *-subalgebra of $\mathcal{A}$. An element $a \in \mathcal{A}$ is bounded if and only if the norms of its images in $\mathcal{A}_q$ for $q \in \mathcal{N}(A)$ are uniformly bounded. The norm of a in $\mathcal{A}_q$ is the maximum of $\|a\|_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}_q)$. Hence a is bounded if and only if the function $\|a\|_{\mathfrak{p}}$ on $\mathrm{Prim}(\mathcal{A})$ is bounded. Thus $\mathrm{C}_0(\mathcal{A})$ consists of bounded elements, and it is an ideal in $\mathcal{A}_\mathrm{b}$. We claim that the limit a of a norm-convergent sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathrm{C}_0(\mathcal{A})$ again vanishes at ∞. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that $\|a - a_n\|_{\mathfrak{p}} \leq \|a - a_n\| < \varepsilon/2$ for all $n \geq n_0$ and all $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A})$. Since a_n vanishes at ∞, there is $q \in \mathcal{N}(A)$ with $\|a_n\|_{\mathfrak{p}} < \varepsilon/2$ for $\mathfrak{p} \notin \mathrm{Prim}(\mathcal{A}_q)$. Thus $\|a\|_{\mathfrak{p}} < \varepsilon$ for $\mathfrak{p} \notin \mathrm{Prim}(\mathcal{A}_q)$. Thus $\mathrm{C}_0(\mathcal{A})$ is a closed ideal in $\mathcal{A}_\mathrm{b}$.
The condition $a \in \mathfrak{p}$ for fixed $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A})$ defines a closed two-sided *-ideal in $\mathcal{A}$. Hence $\mathrm{C}_\mathrm{c}(\mathcal{A})$ is a two-sided *-ideal in $\mathcal{A}$. Let $a \in \mathrm{C}_0(\mathcal{A})$ and $\varepsilon > 0$.

Let $f_\varepsilon \in \mathrm{C_b}([0,\infty))$ be increasing and satisfy $f_\varepsilon(t) = 0$ for $0 \le t < \varepsilon$ and $f_\varepsilon(t) = 1$ for $2\varepsilon \le t$. Then $\|a - af_\varepsilon(a^*a)\| \le 2\varepsilon$, and $f_\varepsilon(a^*a) \in \mathfrak{p}$ if $\|a^*a\|_{\mathfrak{p}} \le \varepsilon$. Hence $af_\varepsilon(a^*a) \in \mathrm{C_c}(\mathcal{A})$ for all $\varepsilon > 0$. Thus $\mathrm{C_c}(\mathcal{A})$ is dense in $\mathrm{C_0}(\mathcal{A})$. Similarly, if $\varphi\colon D \to \mathcal{A}$ is a *-homomorphism, $x \in D$, and $\varphi(x) \in \mathrm{C_0}(\mathcal{A})$, then $\varphi(xf_\varepsilon(x^*x)) \in \mathrm{C_c}(\mathcal{A})$ and $\lim_{\varepsilon\to 0} xf_\varepsilon(x^*x) = x$ in the norm topology on D. □

Theorem 7.16. *Let A be a *-algebra and let $\mathcal{A}$ be its pro-C^*-algebra completion. If $\mathrm{C_0}(\mathcal{A})$ is dense in $\mathcal{A}$, then $\mathrm{C_0}(\mathcal{A})$ is a C^*-hull for the class of locally bounded representations of A.*

We shall prove a more general theorem that still applies if $\mathrm{C_0}(\mathcal{A})$ is not dense in $\mathcal{A}$. Then probably there is no C^*-hull for the class of all locally bounded representations. We may, however, find C^*-hulls for smaller classes of representations. We describe such classes of representations by a generalisation of the spectral conditions in Example 5.14. The spectral condition for a locally closed subset in $\mathbb{R}^n$ implicitly uses a subquotient of $\mathrm{C_0}(\mathbb{R}^n)$. We are going to describe subquotients of the pro-C^*-algebra $\mathcal{A}$. We then associate a class $\mathrm{Rep_b}(A,\mathcal{K})$ of locally bounded representations of A to a subquotient $\mathcal{K}$. If $\mathrm{C_0}(\mathcal{K})$ is dense in $\mathcal{K}$, then $\mathrm{C_0}(\mathcal{K})$ is a C^*-hull for $\mathrm{Rep_b}(A,\mathcal{K})$. Theorem 7.16 is the special case $\mathcal{K} = \mathcal{A}$.

Let $\mathcal{J} \triangleleft \mathcal{A}$ be a closed, two-sided *-ideal in the pro-C^*-algebra $\mathcal{A}$. Being closed, the ideal $\mathcal{J}$ is complete in the subspace topology, so it is also a pro-C^*-algebra. Thus $\mathcal{J} = \varprojlim \mathcal{J}_q$, where $\mathcal{J}_q \triangleleft \mathcal{A}_q$ is the image of $\mathcal{J}$ in the quotient $\mathcal{A}_q$. The quotient $\mathcal{A}/\mathcal{J}$ is complete if $\mathcal{A}$ is metrisable, that is, its topology is defined by a sequence of C^*-seminorms. It need not be complete in general, however. Therefore, we replace the quotient $\mathcal{A}/\mathcal{J}$ by its completion $\mathcal{B}$, which is a pro-C^*-algebra as well. It is the projective limit of the quotients $\mathcal{A}_q/\mathcal{J}_q$ for all $q \in \mathcal{N}(A)$. A *subquotient* of $\mathcal{A}$ is a closed, two-sided *-ideal $\mathcal{K} \triangleleft \mathcal{B}$ with $\mathcal{B}$ as above.

Let $\mathrm{Rep_b}(A,\mathcal{K})$ consist of all representations π of A with the following properties:

(1) π is locally bounded, so it comes from a locally bounded representation π' of $\mathcal{A}$;

(2) the representation π' annihilates $\mathcal{J}$;

(3) the representation $\bar\pi$ of $\mathcal{B}$ induced by π' is nondegenerate on $\mathcal{K}$, that is, $\bar\pi(\mathcal{K})(\mathfrak{E})$ is a core for $\bar\pi$.

Define the C^*-algebra $\mathrm{C_0}(\mathcal{K})$ and its dense ideal $\mathrm{C_c}(\mathcal{K})$ by replacing $\mathcal{A}$ by $\mathcal{K}$ in Definition 7.14. Equivalently, $\mathrm{C_0}(\mathcal{K}) = \mathrm{C_0}(\mathcal{B}) \cap \mathcal{K}$.

We may choose $\mathcal{J} = 0$ and $\mathcal{K} = \mathcal{A}$. Then $\mathrm{Rep_b}(A,\mathcal{A}) = \mathrm{Rep_b}(A)$ simply consists of all locally bounded representations of A. Hence Theorem 7.16 is the special case $\mathcal{K} = \mathcal{A}$ of the following theorem:

Theorem 7.17. *If $\mathrm{C_0}(\mathcal{K})$ is dense in $\mathcal{K}$, then $\mathrm{Rep_b}(A,\mathcal{K})$ has the C^*-hull $\mathrm{C_0}(\mathcal{K})$.*

Proof. First we claim that $\mathrm{Rep}_{\mathrm{b}}(A,\mathcal{K})$ is equivalent to the class of nondegenerate, locally bounded representations of the pro-C*-algebra $\mathcal{K}$ as in Definition 7.5. If $\mathcal{K}=\mathcal{A}$, this is Proposition 7.6. A locally bounded representation π' of $\mathcal{A}$ descends to a representation π'' of the quotient $\mathcal{A}/\mathcal{J}$ if and only if it annihilates $\mathcal{J}$; the induced representation of $\mathcal{A}/\mathcal{J}$ remains locally bounded with respect to the family of C*-seminorms from the quotient mappings $\mathcal{A}/\mathcal{J} \twoheadrightarrow \mathcal{A}_q/\mathcal{J}_q$. Hence it extends uniquely to a locally bounded representation $\bar{\pi}''$ of the completion $\mathcal{B}$. Thus locally bounded representations of A for which the corresponding representation of $\mathcal{A}$ annihilates $\mathcal{J}$ are equivalent to locally bounded representations of $\mathcal{B}$.

We claim that a nondegenerate, locally bounded representation ϱ of $\mathcal{K}$ extends uniquely to $\mathcal{B}$. Let q be a continuous seminorm on $\mathcal{B}$, also write q for its restriction to $\mathcal{K}$. The q-bounded vectors for ϱ form a nondegenerate $\mathcal{K}_q$-module. The module structure extends uniquely to the multiplier algebra of $\mathcal{K}_q$, and $\mathcal{B}_q$ maps to this multiplier algebra because $\mathcal{K}_q \lhd \mathcal{B}_q$. Letting q vary gives a locally bounded representation of $\mathcal{B}$ that remains nondegenerate on $\mathcal{K}$. Conversely, any such representation of $\mathcal{B}$ is obtained in this way from its restriction to $\mathcal{K}$. Thus representations of A that belong to $\mathrm{Rep}_{\mathrm{b}}(A,\mathcal{K})$ are equivalent to nondegenerate, locally bounded representations of the pro-C*-algebra $\mathcal{K}$. The equivalence above is compatible with isometric intertwiners, direct sums and interior tensor products, compare Proposition 7.6.

Lemma 7.15 shows that $\mathrm{C_c}(\mathcal{K}) := \mathrm{C_c}(\mathcal{B}) \cap \mathcal{K}$ is dense in $\mathcal{K}$. This is an ideal in $\mathcal{B}$ as an intersection of two ideals. Hence left multiplication defines a representation of $\mathcal{B}$ on $\mathcal{K}$ with core $\mathrm{C_c}(\mathcal{K})$, which is locally bounded by construction. Through the canonical homomorphisms $A \to \mathcal{A} \to \mathcal{B}$, this becomes a representation of A. This representation clearly belongs to $\mathrm{Rep}_{\mathrm{b}}(A,\mathcal{K})$. We claim that it is the universal representation for the class $\mathrm{Rep}_{\mathrm{b}}(A,\mathcal{K})$. So let π be any representation in $\mathrm{Rep}_{\mathrm{b}}(A,\mathcal{K})$. Then π comes from a unique nondegenerate, locally bounded representation $\bar{\pi}$ of $\mathcal{K}$. We must show that it comes from a unique nondegenerate representation ϱ of $\mathrm{C}_0(\mathcal{K})$.

Let ϱ be the restriction of $\bar{\pi}$ to $\mathrm{C}_0(\mathcal{K})$. Then $\varrho(\mathrm{C_c}(\mathcal{K}))\mathcal{E} \subseteq \mathcal{E}_{\mathrm{b}} \subseteq \mathfrak{E}$. We are going to prove that this is a core. The bilinear map $\mathcal{K} \odot \mathcal{E}_{\mathrm{b}} \to \mathfrak{E}$ is separately continuous with respect to the pro-C*-algebra topology on $\mathcal{K}$ and the inductive limit topology on $\mathcal{E}_{\mathrm{b}} = \bigcup_{q\in\mathcal{N}(A)} \mathcal{E}_q$. We have assumed that it has dense range. Since $\mathrm{C_c}(\mathcal{K})$ is dense in $\mathcal{K}$, the image of $\mathrm{C_c}(\mathcal{K}) \odot \mathcal{E}_{\mathrm{b}}$ is a core. Thus $\varrho(\mathrm{C_c}(\mathcal{K}))\mathcal{E}$ is dense in $\mathfrak{E}$ in the graph topology. The representation ϱ is nondegenerate, and the associated representation of A is π. So π comes from a representation of $\mathrm{C}_0(\mathcal{K})$.

The uniqueness of ϱ means that $\mathrm{C}_0(\mathcal{K})$ is a weak C*-hull for some class of integrable representations of A. We check this using Proposition 7.11. For $q \in \mathcal{N}(A)$, the image of $\mathcal{A}_{\mathrm{b}}$ in $\mathcal{M}(\mathcal{K}_q)$ contains $\mathcal{K}_q$ and hence is strictly dense. This implies that the image of $\mathcal{A}_{\mathrm{b}}$ in $\mathcal{M}(\mathrm{C}_0(\mathcal{K}))$ is strictly dense. So $\mathrm{C}_0(\mathcal{K})$ is a weak C*-hull for a class of representations of A by Proposition 7.11 It is even a C*-hull because the class of locally bounded representations is admissible by Corollary 7.9. ☐

8 Commutative C*-hulls

Let A be a commutative *-algebra. We are going to describe all *commutative* weak C*-hulls for A. Actually, we describe all locally bounded weak C*-hulls, and these turn out to be the same as the commutative ones. We study when a C*-hull satisfies the (Strong) Local–Global Principle and when the class of *all* locally bounded representations has a C*-hull. We compare the class of locally bounded representations with the class of representations defined by requiring all $a \in A_h$ to act by a regular, self-adjoint operator.

Proposition 8.1. *Let A be a *-algebra and let $B = \mathrm{C}_0(X)$ be a commutative C*-algebra. Any representation of A on B has $\mathrm{C}_c(X)$ as a core and is locally bounded. There is a natural bijection between representations of A on B, *-homomorphisms $A \to \mathrm{C}(X)$, and continuous maps $\hat{A} \to X$.*

Proof. Let $(\mathfrak{B}, \mu)$ be a representation. Since $\mathfrak{B}$ is dense in B, for any $x \in X$ there is $f \in \mathfrak{B}$ with $f(x) \neq 0$. Then there is an open neighbourhood of x on which f is non-zero. A compact subset K of X may be covered by finitely many such open neighbourhoods. This gives finitely many functions $f_1, \dots, f_n \in \mathfrak{B}$ so that $\sum f_i \cdot \overline{f_i}(x) > 0$ for all $x \in K$. This sum again belongs to the right ideal $\mathfrak{B}$, and hence $\mathfrak{B}$ contains $\mathrm{C}_c(X)$. There is an approximate unit $(u_i)_{i \in I}$ for $\mathrm{C}_0(X)$ that belongs to $\mathrm{C}_c(X)$. If $b \in \mathfrak{B}$, then $\lim \mu(a) b u_i = \mu(a) b$ for all $a \in A$. That is, $\lim b u_i = b$ in the graph topology. Since $b u_i \in \mathrm{C}_c(X)$, $\mathrm{C}_c(X)$ is a core for $(\mathfrak{B}, \mu)$.
Given $a \in A$, we define a function $f_a \colon X \to \mathbb{C}$ by $f_a(x) := (\mu(a)b)(x) \cdot b(x)^{-1}$ for any $b \in \mathrm{C}_c(X)$ with $b(x) \neq 0$. This does not depend on the choice of b, and f_a is continuous in the open subset where $b \neq 0$. Thus $f_a \in \mathrm{C}(X)$. The map $A \to \mathrm{C}(X)$, $a \mapsto f_a$, is a *-homomorphism. Conversely, any *-homomorphism $A \to \mathrm{C}(X)$ gives a representation of A on $\mathrm{C}_0(X)$ with core $\mathrm{C}_c(X)$ by $\mu(a)b = f_a \cdot b$ for all $a \in A$, $b \in \mathrm{C}_c(X)$. The maps that go back and forth between representations on $\mathrm{C}_0(X)$ and *-homomorphisms $A \to \mathrm{C}(X)$ are inverse to each other.
A *-homomorphism $f \colon A \to \mathrm{C}(X)$ gives a continuous map $X \to \hat{A}$ by mapping $x \in X$ to the character $a \mapsto f(a)(x)$. Conversely, a continuous map $g \colon X \to \hat{A}$ induces a *-homomorphism $g^* \colon A \to \mathrm{C}(X)$, $g^*(a)(x) := g(x)(a)$, and these two constructions are inverse to each other.
Let $f \colon X \to \hat{A}$ be a continuous map. Then f maps compact subsets in X to compact subsets of $\hat{A}$. If $K \subseteq X$ is compact, then any element in $\mathrm{C}_0(K \setminus \partial K) \subseteq \mathrm{C}_0(X)$ is $\|\sqcup\|_{f(K)}$-bounded for the C*-seminorm on A associated to the compact subset $f(K) \subseteq \hat{A}$. Thus all elements in $\mathrm{C}_c(X)$ are bounded. Since $\mathrm{C}_c(X)$ is a core for the representation of A associated to f, this representation is locally bounded. □

Theorem 8.2. *Let A be a commutative *-algebra, let $B = \mathrm{C}_0(X)$ be a commutative C*-algebra, let $f \colon X \to \hat{A}$ be a continuous map, and let $(\mathfrak{B}, \mu)$ be the corresponding representation of A on B. Call a representation of A on a*

*Hilbert module $\mathcal{E}$ X-*integrable *if it is isomorphic to* $(\mathfrak{B},\mu)\otimes_\varrho\mathcal{E}$ *for a representation ϱ of B on $\mathcal{E}$. The following are equivalent:*

(1) *$f\colon X\to\hat{A}$ is injective;*

(2) *B is a weak* C^**-hull for the X-integrable representations;*

(3) *B is a* C^**-hull for the X-integrable representations.*

Furthermore, any locally bounded weak C^**-hull of A is commutative.*

Proof. If f is not injective, then there are $x\neq y$ in X with $f(x)=f(y)$. The evaluation maps at x and y are different 1-dimensional representations of B that induce the same representation of A. Hence the condition (1) in Proposition 3.8 is violated and so B is not a weak C^*-hull. Conversely, assume that f is injective.

The representation of A on B associated to f is locally bounded by Proposition 8.1 and hence induces a representation of the unital C^*algebra $\mathrm{C}_\mathrm{b}(\hat{A},\tau_c)$ of bounded elements in $\mathcal{A}\cong\mathrm{C}(\hat{A},\tau_c)$, see Proposition 7.2. Explicitly, this representation composes functions with f. Since f is injective, $D:=f^*(\mathrm{C}_\mathrm{b}(\hat{A},\tau_c))\subseteq\mathrm{C}_\mathrm{b}(X)$ separates the points of X. We show that D is strictly dense in $\mathrm{C}_\mathrm{b}(X)\cong\mathcal{M}(B)$.

If $K\subseteq X$ is compact, then the image of $f^*(\mathrm{C}_\mathrm{b}(\hat{A},\tau_c))|_K$ in the quotient $\mathrm{C}(K)$ of $\mathrm{C}_0(X)$ separates the points of K. Since this image is again a C^*-algebra, it is equal to $\mathrm{C}(K)$ by the Stone–Weierstraß Theorem. Let $f\in\mathrm{C}_\mathrm{b}(X)$. For any compact subset $K\subseteq X$, there is $d_K\in D$ with $d_K|_K=f$. By functional calculus, we may arrange that $\|d_K\|_\infty\le\|f\|$. The net (d_K) indexed by the directed set of compact subsets $K\subseteq X$ is uniformly bounded and converges towards f in the topology of uniform convergence on compact subsets. Hence it converges towards f in the strict topology (compare [11, Lemma A.1]). This finishes the proof that $f^*(\mathcal{A}_\mathrm{b})$ is strictly dense in $\mathcal{M}(\mathrm{C}_0(X))$. Proposition 7.11 shows that B is a weak C^*-hull for the B-integrable representations of A.

Any X-integrable representation of A is locally bounded. The class $\mathrm{Rep}_\mathrm{b}(A)$ of locally bounded representations of A is admissible by Corollary 7.9. Hence the smaller class of X-integrable representations inherits the equivalent conditions (2)–(4) in Proposition 3.8. Thus $\mathrm{C}_0(X)$ is even a C^*-hull.

Let B with the universal representation $(\mathfrak{B},\mu)$ be a locally bounded weak C^*-hull. Then the image of $\mathrm{C}_\mathrm{b}(\hat{A},\tau_0)$ in the multiplier algebra of B is strictly dense by Proposition 7.11. Thus $\mathcal{M}(B)$ is commutative, and then so is B. Thus a locally bounded weak C^*-hull is commutative. ☐

Theorem 8.3. *Let A be a commutative *-algebra, let $B=\mathrm{C}_0(X)$ be a commutative* C^**-algebra, and let $f\colon X\to\hat{A}$ be an injective continuous map. Let $\mathrm{Rep}_\mathrm{int}(A,X)$ be the class of X-integrable representations. The following statements are equivalent if $\hat{A}$ is metrisable:*

(1) *$f\colon X\to\hat{A}$ is a homeomorphism onto its image;*

(2) $\mathrm{Rep}_{\mathrm{int}}(A, X)$ *is defined by submodule conditions;*

(3) $\mathrm{Rep}_{\mathrm{int}}(A, X)$ *satisfies the Strong Local–Global Principle;*

(4) $\mathrm{Rep}_{\mathrm{int}}(A, X)$ *satisfies the Local–Global Principle;*

(5) *if* $\lim f(x_n) = f(x)$ *for a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *and* $x \in X$*, then already* $\lim x_n = x$.

The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) *hold without assumptions on* $\hat{A}$.

I do not know whether (1)–(4) are equivalent in general. The condition (5) is there to allow to go back from (4) to (1) at least for metrisable $\hat{A}$.

Proof. First we check (5)⇒(1) if $\hat{A}$ is metrisable. If f is not a homeomorphism onto its image, then there is a subset $U \subseteq X$ that is open, such that $f(U)$ is not open in the subspace topology on $f(X) \subseteq \hat{A}$. Since $\hat{A}$ is metrisable, there is $x \in U$ and a sequence in $f(X) \setminus f(U)$ that converges towards $f(x)$. This lifts to a sequence $(x_n)_{n\in\mathbb{N}}$ in $X \setminus U$ such that $\lim f(x_n) = f(x)$. We cannot have $\lim x_n = x$ because x_n never enters the open neighbourhood U of x.
The implication (2)⇒(3) is Theorem 5.9, and (3)⇒(4) is trivial. We are going to verify (1)⇒(2) and (4)⇒(5). This will finish the proof of the theorem.
Assume (1). Let π be a representation in $\mathrm{Rep}_{\mathrm{int}}(A, X)$. Then π is locally bounded, and the operators $\overline{\pi(a)}$ for $a \in A_h$ are regular and self-adjoint by Proposition 7.8. Furthermore, their Cayley transforms belong to the image of $\mathcal{A}_{\mathrm{b}} \cong \mathrm{C}_{\mathrm{b}}(\hat{A}, \tau_c)$, which is commutative. Hence the operators $\overline{\pi(a)}$ for $a \in A_h$ strongly commute with each other. The class $\mathrm{Rep}_{\mathrm{int}}(A)$ of representations of A with the property that all $\overline{\pi(a)}$, $a \in A_h$, are regular and self-adjoint and strongly commute with each other is defined by submodule conditions by Examples 5.10 and Example 5.12.
Let $Y := \prod_{a\in A_h} S^1$. Given a representation in $\mathrm{Rep}_{\mathrm{int}}(A)$, there is a unique representation $\varrho\colon \mathrm{C}(Y) \to \mathbb{B}(\mathcal{E})$ that maps the ath coordinate projection to the Cayley transform of $\overline{\pi(a)}$. We map $\hat{A}$ to Y by sending $\chi \in \hat{A}$ to the point $(c_{\chi(a)})_{a\in A_h} \in Y$. Here $c_{\chi(a)}$ is the Cayley transform of the number $\chi(a) \in \mathbb{R}$ or, equivalently, the value of the Cayley transform of the unbounded function $\hat{a} \in \mathrm{C}(\hat{A})$ at χ. This is a homeomorphism onto its image because for a net of characters (χ_i) and a character χ on A, we have $\lim \chi_i(a) = \chi(a)$ if and only if $\lim c_{\chi_i(a)} = c_{\chi(a)}$. Thus the composite map $X \to \hat{A} \to Y$ is a homeomorphism onto its image as well. This forces the image to be locally closed because Y is compact and X locally compact, and a subspace of a locally compact space is locally compact if and only if its underlying subset is locally closed (see [2, I.9.7, Propositions 12 and 13]).
Let $\overline{X} \subseteq Y$ be the closure of the image of X in Y. Then X is open in $\overline{X}$. All representations in $\mathrm{Rep}_{\mathrm{int}}(A)$ carry a unital *-homomorphism $\mathrm{C}(Y) \to \mathbb{B}(\mathcal{E})$. Asking for this to factor through the quotient $\mathrm{C}(\overline{X})$ of $\mathrm{C}(Y)$ is a submodule condition as in Example 5.14. Asking for the induced *-homomorphism

$\mathrm{C}(\overline{X}) \to \mathbb{B}(\mathcal{E})$ to remain nondegenerate on $\mathrm{C}_0(X)$ is another submodule condition as in Example 5.14.

The class $\mathrm{Rep}'(A)$ defined by these two more submodule conditions is weakly admissible by Lemma 5.8. The universal X-integrable representation belongs to $\mathrm{Rep}'(A)$; by weak admissibility, this is inherited by all X-integrable representations. Conversely, we claim that any representation in $\mathrm{Rep}'(A)$ is X-integrable. If $\pi \in \mathrm{Rep}'(A)$, then the unital *-homomorphism $\mathrm{C}(Y) \to \mathbb{B}(\mathcal{E})$ descends to a nondegenerate *-homomorphism $\varrho\colon \mathrm{C}_0(X) \to \mathbb{B}(\mathcal{E})$. By construction, the extension of ϱ to multipliers maps the Cayley transform of $f^*(a) \in \mathrm{C}(X)$ for $a \in A_h$ to the Cayley transform of $\overline{\pi(a)}$. Let π' be the X-integrable representation of A associated to ϱ. The regular, self-adjoint operators $\overline{\pi'(a)}$ and $\overline{\pi(a)}$ have the same Cayley transform for all $a \in A_h$. Hence $\overline{\pi'(a)} = \overline{\pi(a)}$ for all $a \in A_h$. The subset A_h is a strong generating set for A by Example 2.8. Hence Proposition 2.9 gives $\pi' = \pi$. Thus $\mathrm{Rep}_{\mathrm{int}}(A, X)$ is the class of representations defined by the submodule conditions above. This finishes the proof that (1)⇒(2).

Now we prove (4)⇒(5) by contradiction. Let $(x_n)_{n\in\mathbb{N}}$ and x be as in (5). Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ and view the sequence (x_n) and x as a map $\xi\colon \bar{\mathbb{N}} \to X$. This map is not continuous, but composition with f gives a continuous map $\bar{\mathbb{N}} \to \hat{A}$. Hence Proposition 8.1 gives a representation $(\mathfrak{D}, \mu)$ of A on $\mathrm{C}(\bar{\mathbb{N}})$. This is not X-integrable because the map $\bar{\mathbb{N}} \to X$ is not continuous. We claim, however, that the representation $(\mathfrak{D}, \mu) \otimes_\varrho \mathcal{H}$ is X-integrable for any GNS-representation ϱ on a Hilbert space $\mathcal{H}$. A state on $\mathrm{C}(\bar{\mathbb{N}})$ is the same as a Radon measure on $\bar{\mathbb{N}}$. Since $\bar{\mathbb{N}}$ is countable, any Radon measure is atomic. Thus the resulting GNS-representation is a direct sum of irreducible representations associated to characters. Each character on $\mathrm{C}(\bar{\mathbb{N}})$ gives an X-integrable representation because $\xi(\bar{\mathbb{N}}) \subseteq f(X)$. Hence $(\mathfrak{D}, \mu)$ is a counterexample to the Local–Global Principle. So (4) cannot hold if (5) fails. □

Example 8.4. Let $A = \mathbb{C}[x]$, so that $\hat{A} = \mathbb{R}$. Let X be $\mathbb{R}$ with the discrete topology, and let $f\colon X \to \mathbb{R}$ be the identity map. This is a continuous bijection, but not open. Hence the class of X-integrable representations violates the Local–Global Principle by Theorem 8.3. Nevertheless, $\mathrm{C}_0(X)$ is a C^*-hull for the class of X-integrable representations of A by Theorem 8.2. An X-integrable representation of A is integrable as in Theorem 4.4, and so it comes from a single regular, self-adjoint operator $T := \overline{\pi(x)}$. The representation of $\mathbb{C}[x]$ associated to T is X-integrable if and only if $\mathcal{E} = \bigoplus_{\lambda\in\mathbb{R}} \mathcal{E}_\lambda$, where $\mathcal{E}_\lambda := \{\xi \in \mathcal{E} \mid T\xi = \lambda\xi\}$ for $\lambda \in \mathbb{R}$ is the λ-eigenspace of T.

Another example of a C^*-hull for $\mathbb{C}[x]$ where $X \to \mathbb{R}$ is bijective but not a homeomorphism onto its image is discussed in Example 4.9.

Theorem 8.5. *There is a* C^**-hull for* $\mathrm{Rep}_{\mathrm{b}}(A)$ *if and only if the compactly generated topology* τ_c *on* $\hat{A}$ *is locally compact, and then the* C^**-hull is* $\mathrm{C}_0(\hat{A}, \tau_c)$.

Proof. Assume first that $(\hat{A}, \tau_c)$ is locally compact. The pro-C*-algebra completion $\mathcal{A}$ that acts on locally bounded representations of A is $\mathrm{C}(\hat{A}, \tau_c)$ by Proposition 7.2. The primitive ideal space of $\mathrm{C}(K)$ for a compact subspace $K \subseteq \hat{A}$ is simply K, and $\|a\|_{\mathfrak{p}} = |a(\mathfrak{p})|$ for $a \in \mathrm{C}(\hat{A}, \tau_c)$ and $\mathfrak{p} \in \operatorname{Prim} \mathrm{C}(K) \cong K$. Therefore, a function $f \in \mathrm{C}(\hat{A}, \tau_c)$ vanishes at ∞ in the sense of Definition 7.14 if and only if it vanishes at ∞ in the usual sense. The subalgebra $\mathrm{C}_0(\mathcal{A}) = \mathrm{C}_0(\hat{A}, \tau_c)$ is dense in $\mathcal{A}$ because τ_c is locally compact. Now Theorem 7.16 shows that $\mathrm{C}_0(\mathcal{A}) = \mathrm{C}_0(\hat{A}, \tau_c)$ is a C*-hull for the class of locally bounded representations of A.

Conversely, let B be a (weak) C*-hull for the locally bounded representations of A. Then B is commutative by Theorem 8.2. Let Y be the spectrum of B. The representation of A on $B \cong \mathrm{C}_0(Y)$ corresponds to a continuous map $f\colon Y \to \hat{A}$ by Proposition 8.1. Let $D = \mathrm{C}_0(X)$ be a commutative C*-algebra. Any representation of A on D is locally bounded. So the bijection $\mathrm{Rep}_{\mathrm{b}}(A, D) \cong \mathrm{Rep}(B, D)$ is a bijection between the spaces of continuous maps $X \to \hat{A}$ and $X \to Y$. More precisely, this bijection is composition with f.

For the one-point space X, this bijection says that $f\colon Y \to \hat{A}$ is bijective. The bijection for all compact X means that f becomes a homeomorphism if we replace the topologies on Y and $\hat{A}$ by the associated compactly generated ones. The topology on Y is already compactly generated because Y is locally compact. Hence f is a homeomorphism from Y to $(\hat{A}, \tau_c)$. So τ_c is locally compact. ☐

Let $\mathrm{Rep}_{\mathrm{int}}(A)$ be the class of all representations with the property that $\overline{\pi(a)}$ is regular and self-adjoint for all $a \in A_h$. We are going to compare $\mathrm{Rep}_{\mathrm{int}}(A)$ and $\mathrm{Rep}_{\mathrm{b}}(A)$. Proposition 7.8 gives $\mathrm{Rep}_{\mathrm{b}}(A) \subseteq \mathrm{Rep}_{\mathrm{int}}(A)$.

Theorem 8.6. *The class* $\mathrm{Rep}_{\mathrm{int}}(A)$ *is admissible and defined by submodule conditions. Hence it satisfies the Strong Local–Global Principle. The operators* $\overline{\pi(a)}$ *for* $a \in A_h$ *strongly commute for all* $\pi \in \mathrm{Rep}_{\mathrm{int}}(A)$.

Let $S \subseteq A_h$ *be a strong generating set for* A. *If* $\overline{\pi(a)}$ *is regular and self-adjoint for all* $a \in S$, *then already* $\pi \in \mathrm{Rep}_{\mathrm{int}}(A)$.

Proof. The class $\mathrm{Rep}^S(A)$ of representations defined by requiring $\overline{\pi(a)}$ to be regular and self-adjoint for all $a \in S$ for a strong generating set S is admissible and defined by submodule conditions by Theorem 5.17. The class $\mathrm{Rep}_{\mathrm{int}}(A)$ is defined by submodule conditions as well by Example 5.10. So is the subclass $\mathrm{Rep}'(A)$ of all representations in $\mathrm{Rep}_{\mathrm{int}}(A)$ for which the operators $\overline{\pi(a)}$ for all $a \in A_h$ strongly commute (Example 5.12). Hence our three classes of representations satisfy the Strong Local–Global Principle by Theorem 5.9.

The classes $\mathrm{Rep}_{\mathrm{int}}(A)$ and $\mathrm{Rep}'(A)$ have the same Hilbert space representations by [28, Theorem 9.1.2]. Since S is a strong generating set, the domain of any representation π in $\mathrm{Rep}^S(A)$ is $\bigcap_{a\in S} \operatorname{dom}(\overline{\pi(a)})$ by (2.10). This contains $\bigcap_{a\in S, n\in\mathbb{N}} \operatorname{dom}(\overline{\pi(a)}^n)$. Now [28, Theorem 9.1.3] shows that $\mathrm{Rep}^S(A)$ and $\mathrm{Rep}_{\mathrm{int}}(A)$ contain the same Hilbert space representations. Since our three classes of representations satisfy the (Strong) Local–Global Principle and have the same Hilbert space representations, they are equal. □

Theorem 8.7. *If A is commutative and countably generated, then*

$$\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A).$$

Proof. Proposition 7.8 gives $\mathrm{Rep}_{\mathrm{b}}(A) \subseteq \mathrm{Rep}_{\mathrm{int}}(A)$. Conversely, let $(\mathfrak{E}, \pi)$ be a representation on a Hilbert module $\mathcal{E}$ in $\mathrm{Rep}_{\mathrm{int}}(A)$; that is, $\overline{\pi(a)}$ is regular and self-adjoint for each $a \in A$. Let $(a_i)_{i\in\mathbb{N}}$ be a countable generating set for A. We may assume without loss of generality that $a_i = a_i^*$ for all $i \in \mathbb{N}$ and that (a_i) is a basis for A and hence a strong generating set. Let $\xi \in \mathfrak{E}$. We are going to approximate ξ by bounded vectors for π. This will show that π is locally bounded.

For each $i \in \mathbb{N}$, there is a canonical homomorphism $\alpha_i\colon \mathbb{C}[x] \to A$ mapping $x \mapsto a_i$. The closure of $\pi \circ \alpha_i$ is an integrable representation of $\mathbb{C}[x]$ as in condition (2) in Theorem 4.4. Hence it corresponds to a representation $\varrho_i\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{B}(\mathcal{E})$, the functional calculus of $\overline{\pi(a_i)}$. The operators $\overline{\pi(a)}$ for $a \in A_h$ strongly commute by Theorem 8.6. Thus the Cayley transform of a_i commutes with $\overline{\pi(a)}$ and, in particular, maps the domain of $\overline{\pi(a)}$ to itself. The same remains true for $\varrho_i(f)$ for all $f \in \mathrm{C}_0(\mathbb{R})$ because we get them by the (bounded) functional calculus for the Cayley transform of $\overline{\pi(a_i)}$. So $\varrho_i(f)(\mathfrak{E}) \subseteq \mathfrak{E}$ by (2.6) and $\varrho_i(f)\pi(a) = \pi(a)\varrho_i(f)$ for all $f \in \mathrm{C}_0(\mathbb{R})$, $a \in A$ as operators on $\mathfrak{E}$. Now we show that $\pi \circ \alpha_i$ is locally bounded. If $f \in \mathrm{C_c}(\mathbb{R})$ is supported in a compact subset $K \subseteq \mathbb{R}$, then

$$\|\pi(h(a_i))\varrho_i(f)\xi\| = \|\varrho_i(h \cdot f)\xi\| \le C \sup\{|h(x)| \mid x \in K\}$$

for all $h \in \mathbb{C}[x]$; thus $\varrho_i(f)\xi$ is bounded for the representation $\pi \circ \alpha_i$. There is an approximate unit (f_n) for $\mathrm{C}_0(\mathbb{R})$ that lies in $\mathrm{C_c}(\mathbb{R})$. Then $\lim \varrho_i(f_n)\xi = \xi$ for all $\xi \in \mathfrak{E}$, even in the graph topology for π because $\pi(a)\varrho_i(f_n)\xi = \varrho_i(f_n)\pi(a)\xi$ for all $a \in A$, $f_n \in \mathrm{C}_0(\mathbb{R})$, $\xi \in \mathfrak{E}$. Therefore, the bounded vectors of the form $\varrho(f)\xi$ with $f \in \mathrm{C_c}(\mathbb{R})$, $\xi \in \mathfrak{E}$ form a core for $\pi \circ \alpha_i$. So $\pi \circ \alpha_i$ is locally bounded.

We now refine this construction to approximate ξ by bounded vectors for the whole representation π. We construct ϱ_i as above. Fix $i, k \in \mathbb{N}$ and let $\xi' := \bigl(1 + \pi(a_0^2) + \dots + \pi(a_k^2)\bigr)\xi \in \mathfrak{E}$. The argument above gives $f_{i,k} \in \mathrm{C_c}(\mathbb{R})$ with $0 \le f_{i,k} \le 1$ and $\|\varrho_i(f_{i,k})\xi' - \xi'\| < 2^{-k}$. Thus $\|\varrho_i(f_{i,k})\xi - \xi\|_{a_j} < 2^{-k}$ in the graph norm for a_j for $0 \le j \le k$. For $k, l \in \mathbb{N}$, let

$$\xi_{k,l} := \varrho_0(f_{0,k})\varrho_1(f_{1,k+1}) \cdots \varrho_l(f_{l,k+l})\xi.$$

The operators $\varrho_i(f_{i,j})$ are norm-contracting, map $\mathfrak{E}$ into itself, and commute with each other and with the unbounded operators $\pi(a)$ for all $a \in A$. Hence

$$\begin{aligned}\|\xi_{k,l} - \xi_{k,l+d}\|_{a_j} &\le \sum_{i=1}^{d} \|\xi_{k,l+i-1} - \xi_{k,l+i}\|_{a_j} \\ &\le \sum_{i=1}^{d} \|\varrho_{l+i}(f_{l+i,k+l+i})\xi - \xi\|_{a_j} \le \sum_{i=1}^{d} 2^{-k-l-i} = 2^{-k-l}\end{aligned}$$

for all $k, l, d \in \mathbb{N}$, $0 \le j \le k + l + 1$. Since we assumed (a_j) to be a strong generating set, the graph norms for a_j generate the graph topology. So the estimate above shows that $(\xi_{k,l})_{l\in\mathbb{N}}$ with fixed k is a Cauchy sequence in $\mathfrak{E}$ in the graph topology. Thus it converges to some $\xi_k \in \mathfrak{E}$. Letting $\xi_{k,-1} := \xi$, the above estimate remains true for $l = -1$ and gives $\|\xi_{k,l} - \xi\|_{a_j} \le 2^{-k+1}$ for all $j \le k$, uniformly in $l \in \mathbb{N}$. This implies $\|\xi_k - \xi\|_{a_j} \le 2^{-k+1}$ for $j \le k$, so that $\lim \xi_k = \xi$ in the graph topology. It remains to show that each ξ_k is a bounded vector.

Fix $k, i \in \mathbb{N}$ and let $b \in A$. Choose $R_i > 0$ so that $f_{i,k+i}$ is supported in $[-R_i, R_i]$. If $l \ge i$, then $\pi(b)\xi_{k,l} \in \varrho_i(\mathrm{C}_0(-R_i, R_i))\mathcal{E}$ because $\varrho_i(f_{i,k+i})$ occurs in the definition of $\xi_{k,l}$. As above, this implies $\|\pi(a_i)\pi(b)\xi_k\| \le R_i\|\pi(b)\xi_k\|$ for all $b \in A$. Thus

$$q(a) := \sup_{b\in A} \frac{\|\pi(a)\pi(b)\xi_k\|}{\|\pi(b)\xi_k\|}$$

is finite for $a = a_i$. Since a_i is a basis for A and q is subadditive, we get $q(a) < \infty$ for all $a \in A$. Since $q(a)$ is the operator norm of $\pi(a)|_{\pi(A)\xi_k}$, it is a C^*-seminorm on A. By construction, $\|\pi(a)\xi_k\| \le q(a)$ for all $a \in A$, that is, ξ_k is bounded. □

Proposition 8.8. *If* $\mathrm{Rep}_{\mathrm{int}}(A)$ *has a weak* C^**-hull, then* $\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A)$*.*

Proof. Let B with the universal representation $(\mathfrak{B}, \mu)$ be a weak C^*-hull for $\mathrm{Rep}_{\mathrm{int}}(A)$. First we claim that B is commutative. Let $\omega\colon B \hookrightarrow \mathbb{B}(\mathcal{H})$ be a faithful representation. This corresponds to an integrable representation π of A. Since the equivalence $\mathrm{Rep}_{\mathrm{int}}(A, \mathcal{H}) \cong \mathrm{Rep}(B, \mathcal{H})$ is compatible with unitary *-intertwiners, the commutant of $\omega(B)$ is the C^*-algebra of *-intertwiners of π by Proposition 3.3. The commutant of this is a commutative von Neumann algebra by [28, Theorem 9.1.7]. So the bicommutant of $\omega(B)$ is commutative. This forces B to be commutative.

Any representation of A on a commutative C^*-algebra is locally bounded by Theorem 8.2. If the universal representation for $\mathrm{Rep}_{\mathrm{int}}(A)$ is locally bounded, then all representations in $\mathrm{Rep}_{\mathrm{int}}(A)$ are locally bounded, so that $\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A)$. Thus $\mathrm{Rep}_{\mathrm{int}}(A)$ only has a weak C^*-hull if $\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A)$. □

Example 8.9. Let A be the *-algebra $\mathbb{C}[(x_i)_{i\in\mathbb{N}}]$ of polynomials in countably many symmetric generators. Then $\hat{A} \cong \prod_{\mathbb{N}} \mathbb{R}$ with the product topology. This is metrisable. So τ_c is the usual product topology. Since this is not locally compact, $\mathrm{Rep}_{\mathrm{b}}(A)$ has no C*-hull, not even a weak one (Theorem 8.5). Since A is countably generated, $\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A)$ by Theorem 8.7. A commutative (weak) C*-hull for some class of representations of A is equivalent to an injective, continuous map $X \to \hat{A}$ for a locally compact space X by Theorem 8.2.

Let G be a topological group. A *host algebra* for a G is defined in [12] as a C*-algebra B with a continuous representation λ of G by unitary multipliers, such that for each Hilbert space $\mathcal{H}$, the map that sends a representation $\varrho\colon B \to \mathbb{B}(\mathcal{H})$ to a unitary representation $\varrho \circ \lambda$ of G is injective. We claim that commutative C*-hulls for the polynomial algebra $\mathbb{C}[(x_i)_{i\in\mathbb{N}}]$ are equivalent to host algebras of the topological group $\mathbb{R}^{(\mathbb{N})} := \bigoplus_{\mathbb{N}} \mathbb{R}$.

Let $\mathrm{C}^*(G_d)$ be the C*-algebra of G viewed as a discrete group. Representations of $\mathrm{C}^*(G_d)$ are equivalent to representations of the discrete group underlying G by unitary multipliers. Since any representation of $\mathrm{C}^*(G_d)$ is bounded, any weakly admissible class of representations of $\mathrm{C}^*(G_d)$ is admissible by Corollary 7.9. Call a representation of $\mathrm{C}^*(G_d)$ continuous if the corresponding representation of G is continuous. This class is easily seen to be weakly admissible, hence admissible. The unital *-homomorphism $\mathrm{C}^*(G_d) \to \mathcal{M}(B)$ associated to the unitary representation λ for a host algebra B is continuous by assumption. Thus B-integrable representations of $\mathrm{C}^*(G_d)$ are continuous. The injectivity requirement in the definition of a host algebra is exactly the condition (1) in Proposition 3.8, and this is equivalent to B being a C*-hull. Thus a host algebra for G is the same as a C*-hull or weak C*-hull for a class of continuous representations of $\mathrm{C}^*(G_d)$.

In applications, we would rather study continuous representations of G through the Lie algebra of G instead of through the inseparable C*-algebra $\mathrm{C}^*(G_d)$. The Lie algebra of $G = \mathbb{R}^{(\mathbb{N})}$ is the Abelian Lie algebra $\mathbb{R}^{(\mathbb{N})}$, and its universal enveloping algebra is the polynomial algebra $A = \mathbb{C}[(x_i)_{i\in\mathbb{N}}]$. Call a representation of A integrable if it belongs to $\mathrm{Rep}_{\mathrm{int}}(A) = \mathrm{Rep}_{\mathrm{b}}(A)$.

Let $\mathcal{E}$ be a Hilbert module. We claim that an integrable representation of A on $\mathcal{E}$ is equivalent to a strictly continuous, unitary representation of the group $\mathbb{R}^{(\mathbb{N})}$ on $\mathcal{E}$. Indeed, a unitary representation of $\mathbb{R}$ is equivalent to a representation of $\mathrm{C}^*(\mathbb{R}) \cong \mathrm{C}_0(\mathbb{R})$, and these are equivalent to integrable representations of $\mathbb{C}[x]$ as in Theorem 4.4. In an integrable representation of $\mathbb{C}[(x_i)_{i\in\mathbb{N}}]$, the operators $\overline{\pi(x_i)}$ for $i \in \mathbb{N}$ strongly commute by Theorem 8.6. Hence the resulting representations of $\mathrm{C}_0(\mathbb{R})$ commute. Equivalently, the resulting continuous representations of $\mathbb{R}$ commute, so that we may combine them to a representation of the Abelian group $\mathbb{R}^{(\mathbb{N})}$. Conversely, a continuous unitary representation of $\mathbb{R}^{(\mathbb{N})}$ provides nondegenerate representations of $\mathrm{C}_0(\mathbb{R}^m)$ for all $m \in \mathbb{N}$ by restricting the representation to $\mathbb{R}^m \subseteq \mathbb{R}^{(\mathbb{N})}$. These correspond to a compatible family of representations of the polynomial algebras $\mathbb{C}[x_1, \dots, x_m]$ for $m \in \mathbb{N}$. The intersection of their domains is dense by [28, Lemma 1.1.2]. So these representations combine to a representation of $A = \mathbb{C}[(x_i)_{i\in\mathbb{N}}]$. Hence an integrable representation of A on a Hilbert module as in Theorem 8.6 is equivalent to a continuous representation of $\mathbb{R}^{(\mathbb{N})}$.

9 From graded *-algebras to Fell bundles

Let G be a discrete group with unit element e.

Definition 9.1. A *G-graded *-algebra* is a unital algebra A with a linear direct sum decomposition $A = \bigoplus_{g\in G} A_g$ with $A_g \cdot A_h \subseteq A_{gh}$, $A_g^* = A_{g^{-1}}$, and $1 \in A_e$ for all $g,h \in G$. Thus $A_e \subseteq A$ is a unital *-subalgebra.

The articles [7, 27] study many examples of G-graded *-algebras.
We fix some notation used throughout this section. Let $\mathcal{E}$ be a Hilbert module over a C*-algebra D. Let $(\mathfrak{E},\pi)$ be a representation of A on $\mathcal{E}$. Let $\pi_g\colon A_g \to \operatorname{End}_D(\mathfrak{E})$ for $g \in G$ be the restrictions of π, so $\pi = \bigoplus_{g\in G} \pi_g$. Since π is a *-homomorphism,

$$\pi_g(a_g)\pi_h(a_h) = \pi_{gh}(a_g \cdot a_h), \qquad \pi_{g^{-1}}(a_g^*) \subseteq \pi_g(a_g)^*$$

for all $a_g \in A_g$, $a_h \in A_h$. The last condition means that $\langle \xi, \pi_g(a_g)\eta\rangle = \langle \pi_{g^{-1}}(a_g^*)\xi, \eta\rangle$ for all $\xi,\eta \in \mathfrak{E}$. In particular, $\pi_e\colon A_e \to \operatorname{End}(\mathfrak{E})$ is a representation of A_e.

Lemma 9.2 (compare [27, Lemma 12]). *The families of norms $\|\xi\|_a := \|\pi(a)\xi\|$ for $a \in A$ and for $a \in A_e$ generate equivalent topologies on $\mathfrak{E}$. Hence the representation $\pi_e\colon A_e \to \operatorname{End}_D(\mathfrak{E})$ is closed if and only if π is closed.*

Proof. Any element of A is a sum $a = \sum_{g\in G} a_g$ with $a_g \in A_g$ and only finitely many non-zero terms. We estimate $\|\xi\|_a \le \sum_{g\in G} \|\xi\|_{a_g}$, and $\|\xi\|_{a_g} \le \frac{5}{4}\|\xi\|_{a_g^* a_g}$ by the proof of Lemma 2.2. Since $a_g^* a_g \in A_e$, the graph topologies for π_e and π are equivalent. □

9.1 Integrability by restriction

Definition 9.3. Let a weakly admissible class of integrable representations of A_e on Hilbert modules be given. We call a representation of A on a Hilbert module *integrable* if its restriction to A_e is integrable.

Here "restriction of π" means the representation π_e with the same domain $\mathfrak{E}$ as π. This is closed by Lemma 9.2.

Proposition 9.4. *If integrability for representations of A_e is defined by submodule conditions, then the same holds for A. If the Local–Global Principle holds for the integrable representations of A_e, it also holds for the integrable representations of A. If the class of integrable representations of A_e is admissible or weakly admissible, the same holds for A.*

Proof. The first two statements and the claim about weak admissibility are trivial because integrability for a representation of A only involves its restriction to A_e. Lemma 9.2 shows that restriction from A to A_e does not change the domain. Hence (2) in Proposition 3.8 is inherited by A if it holds for A_e. That is, admissibility of the integrable representations passes from A_e to A. □

It is unclear whether A also inherits the *Strong* Local–Global Principle from A_e. This may often be bypassed using Theorem 5.9.

9.2 Inducible representations and induction

Let $\mathcal{F}$ be a Hilbert D-module and let $\mathfrak{F} \subseteq \mathcal{F}$ and $\varphi_e\colon A_e \to \mathrm{End}_D(\mathfrak{F})$ be a representation of A_e on $\mathcal{F}$. We try to induce φ_e to a representation of A as in [27]. Thus we consider the algebraic tensor product $A \odot \mathfrak{F}$ and equip it with the obvious right D-module structure and the unique sesquilinear map that satisfies

$$\langle a_1 \otimes \xi_1, a_2 \otimes \xi_2 \rangle = \delta_{g,h} \langle \xi_1, \varphi_e(a_1^* a_2)\xi_2 \rangle$$

for all $g, h \in G$, $a_1 \in A_g$, $a_2 \in A_h$, $\xi_1, \xi_2 \in \mathfrak{F}$. This map is sesquilinear and descends to the quotient space $A \odot_{A_e} \mathfrak{F}$. It is symmetric and D-linear in the sense that $\langle x, y \rangle = \langle y, x \rangle^*$ and $\langle x, yd \rangle = \langle x, y \rangle d$. Let π be the action of A on $A \odot_{A_e} \mathfrak{F}$ by left multiplication. This representation is formally a *-homomorphism in the sense that $\langle x, \pi(a)y \rangle = \langle \pi(a^*)x, y \rangle$ for all $a \in A$, $x, y \in A \odot_{A_e} \mathfrak{F}$. The only thing that is missing to get a representation of A on a Hilbert D-module is positivity of the inner product. This requires a subtle extra condition.

Proposition 9.5. *The following are equivalent:*

(1) *the sesquilinear map on $A \odot_{A_e} \mathfrak{F}$ defined above is positive semidefinite;*

(2) *for all $g \in G$, $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in A_g$, $\xi_1, \dots, \xi_n \in \mathfrak{F}$, the element $\sum_{k,l=1}^n \langle \xi_k, \varphi_e(a_k^* a_l)\xi_l \rangle \in D$ is positive;*

(3) *for all $g \in G$, $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in A_g$, $\xi_1, \dots, \xi_n \in \mathfrak{F}$, the matrix $\bigl(\langle \xi_k, \varphi_e(a_k^* a_l)\xi_l \rangle\bigr)_{k,l} \in \mathbb{M}_n(D)$ is positive.*

Proof. The condition (2) for fixed $g \in G$ says that the sesquilinear map on $A_g \odot_{A_e} \mathfrak{F}$ is positive semidefinite. Since the subspaces $A_g \odot_{A_e} \mathfrak{F}$ for different g are orthogonal, this is equivalent to positive semidefiniteness on $A \odot_{A_e} \mathfrak{F}$. Thus (1) $\iff$ (2).
We prove (2) $\iff$ (3). Fix $g \in G$, $n \in \mathbb{N}$, $a_1, \dots, a_n \in A_g$ and $\xi_1, \dots, \xi_n \in \mathfrak{F}$. Let $y = (y_{kl}) \in \mathbb{M}_n(D)$ be the matrix in (3). By [17, Lemma 4.1], $y \geq 0$ in $\mathbb{M}_n(D) \subseteq \mathbb{B}(D^n)$ if and only if $\langle d, yd \rangle \geq 0$ for all $d = (d_1, \dots, d_n) \in D^n$. That is, $\sum_{k,l=1}^n d_k^* y_{kl} d_l \geq 0$ for all $d_1, \dots, d_n \in D$. Since $\mathfrak{F}$ is a right D-module, this condition for all $\xi_i \in \mathfrak{F}$, $d_i \in D$ is equivalent to (2). □

Definition 9.6. A representation φ_e of A_e is *inducible* (to A) if it satisfies the equivalent conditions in Proposition 9.5.

If A_e were a C*-algebra, it would be enough to assume $\langle \xi, \varphi_e(a^* a)\xi \rangle \geq 0$ for all $g \in G$, $a \in A_g$, $\xi \in \mathfrak{F}$, which amounts to the condition $a^* a \geq 0$ in A_e for all $g \in G$, $a \in A_g$. This is part of the definition of a Fell bundle over a group.

For more general *-algebras, the positivity conditions for different $n \in \mathbb{N}$ in Proposition 9.5 may differ, compare [9].

Let $A \otimes_{A_e} \mathcal{F}$ be the Hilbert module completion of $A \odot_{A_e} \mathfrak{F}$ for the inner product above. The *-algebra A acts on $A \odot_{A_e} \mathfrak{F}$ by left multiplication, $a_1 \cdot (a_2 \otimes \xi) := (a_1 a_2) \otimes \xi$ for $a_1, a_2 \in A$, $\xi \in \mathfrak{F}$. As in the proof of Lemma 2.18, this module structure descends to the image of $A \odot_{A_e} \mathfrak{F}$ in $A \otimes_{A_e} \mathcal{F}$ and gives a well defined representation π of A on $A \otimes_{A_e} \mathcal{F}$. Its closure is called the *induced representation* from φ_e, and its domain is denoted by $A \otimes_{A_e} \mathfrak{F}$.

The decomposition $A \odot_{A_e} \mathfrak{F} = \bigoplus_{g \in G} A_g \odot_{A_e} \mathfrak{F}$ is A_e-invariant and orthogonal for the above inner product. Hence

$$A \otimes_{A_e} \mathcal{F} \cong \bigoplus_{g \in G} A_g \otimes_{A_e} \mathcal{F}, \tag{9.7}$$

where $A_g \otimes_{A_e} \mathcal{F}$ is the closure of the image of $A_g \odot_{A_e} \mathfrak{F}$ or, equivalently, the Hilbert D-module completion of $A_g \odot_{A_e} \mathfrak{F}$ with respect to the restriction of the inner product. Each summand $A_g \otimes_{A_e} \mathcal{F}$ carries a closed representation of A_e with domain $A_g \otimes_{A_e} \mathfrak{F}$, and $\pi|_{A_e}$ is the direct sum of these representations.

Lemma 9.8. *Let π be any representation of A. Then $\pi|_{A_e}$ is inducible.*

Proof. For $g \in G$, $a_1, \dots, a_n \in A_g$, $\xi_1, \dots, \xi_n \in \mathfrak{E}$, let $y := \sum_{k=1}^n \pi(a_k)\xi_k$. Then

$$\sum_{k,l=1}^n \langle \xi_k, \pi|_{A_e}(a_k^* a_l)\xi_l \rangle = \sum_{k,l=1}^n \langle \pi(a_k)\xi_k, \pi(a_l)\xi_l \rangle = \langle y, y \rangle \geq 0. \qquad \square$$

Lemma 2.24 about the associativity of $\otimes$ has a variant for induction:

Lemma 9.9. *Let D_1, D_2 be C^*-algebras, let $\mathcal{E}$ be a Hilbert D_1-module and let $\mathcal{F}$ be a C^*-correspondence between D_1, D_2. Let $(\varphi_e, \mathfrak{E})$ be an inducible representation of A on $\mathcal{E}$. Then the representation $\varphi_e \otimes_{D_1} \mathcal{F}$ on $\mathcal{E} \otimes_{D_1} \mathcal{F}$ is inducible and there is a canonical unitary *-intertwiner of representations of A,*

$$(A \otimes_{A_e} \mathcal{E}) \otimes_{D_1} \mathcal{F} \cong A \otimes_{A_e} (\mathcal{E} \otimes_{D_1} \mathcal{F}).$$

Proof. Let $\mathfrak{E} \otimes_{D_1} \mathcal{F} \subseteq \mathcal{E} \otimes_{D_1} \mathcal{F}$ be the domain of $\varphi_e \otimes_{D_1} \mathcal{F}$. Let $g_1, \dots, g_n \in G$, $a_1, \dots, a_n \in A_{g_i}$, and $\omega_1, \dots, \omega_n \in \mathfrak{E} \otimes_{D_1} \mathcal{F}$. Let $\zeta := \sum_{k=1}^n a_k \otimes \omega_k \in A \odot (\mathfrak{E} \otimes_{D_1} \mathcal{F})$. To show that $\varphi_e \otimes_{D_1} \mathcal{F}$ is inducible, we must prove that $\langle \zeta, \zeta \rangle \in D_2$ is positive. Vectors in $\mathfrak{E} \odot \mathcal{F}$ form a core for $\varphi_e \otimes_{D_1} \mathcal{F}$ by construction. Hence there is a sequence of vectors of the form

$$\omega_{j,\tau} := \sum_{i=1}^{\ell_j} \xi_{\tau,j,i} \otimes \eta_{\tau,j,i}, \qquad \xi_{\tau,j,i} \in \mathfrak{E}, \eta_{\tau,j,i} \in \mathcal{F},$$

which, for $\tau \to \infty$, converges to ω_j in the graph norms of the elements $\delta_{g_m,g_k} a_m^* a_k \in A_e$ for all $m, k = 1, \dots, n$. Let $\zeta_\tau := \sum_{j=1}^n a_j \otimes \omega_{j,\tau}$. Then

$$\lim_{\tau \to \infty} \langle \zeta_\tau, \zeta_\tau \rangle = \lim_{\tau \to \infty} \langle \zeta_\tau, \zeta \rangle = \lim_{\tau \to \infty} \langle \zeta, \zeta_\tau \rangle = \langle \zeta, \zeta \rangle$$

in norm and

$$\langle\zeta_\tau,\zeta_\tau\rangle = \left\langle \sum_{i,j} a_j \otimes \xi_{\tau,j,i}\otimes \eta_{\tau,j,i}, \sum_{m,k} a_k\otimes \xi_{\tau,k,m}\otimes \eta_{\tau,k,m}\right\rangle$$
$$= \sum_{i,j,k,m} \delta_{g_j,g_k} \langle \eta_{\tau,j,i}, \langle \xi_{\tau,j,i}, \varphi_e(a_j^* a_k)\xi_{\tau,k,m}\rangle_{D_1}\cdot \eta_{\tau,k,m}\rangle_{D_2}.$$

This is also the inner product of ζ_τ with itself in the tensor product $(A\otimes_{A_e}\mathcal{E})\otimes\mathcal{F}$. This is positive because φ_e is inducible and the usual tensor product of the Hilbert D_1-module $A\otimes_{A_e}\mathcal{E}$ with the D_1,D_2-correspondence $\mathcal{F}$ is a Hilbert D_2-module. Hence $\langle\zeta_\tau,\zeta_\tau\rangle\geq 0$ for all τ. Since the positive elements in D_2 form a closed subset, this implies $\langle\zeta,\zeta\rangle\geq 0$. Thus $\varphi_e\otimes_{D_1}\mathcal{F}$ is inducible. The argument above also shows that the linear span of vectors of the form $a\otimes\xi\otimes\eta$ with $a\in A$, $\xi\in\mathfrak{E}$, $\eta\in\mathcal{F}$ is a core for the representation of A_e on $A\otimes_{A_e}(\mathcal{E}\otimes_{D_1}\mathcal{F})$. Such vectors also form a core for the representation of A on $(A\otimes_{A_e}\mathcal{E})\otimes_{D_1}\mathcal{F}$. The left actions of A and the D_2-valued inner products coincide on such vectors. Hence there is a unique unitary *-intertwiner that maps the image of $a\otimes\xi\otimes\eta$ in $(A\otimes_{A_e}\mathcal{E})\otimes_{D_1}\mathcal{F}$ to its image in $A\otimes_{A_e}(\mathcal{E}\otimes_{D_1}\mathcal{F})$. $\square$

9.3 C*-Hulls for the unit fibre

We assume that the chosen class of integrable representations of A_e has a (weak) C*-hull B_e. We want to construct a Fell bundle whose section C*-algebra is a (weak) C*-hull for the integrable representations of A. At some point, we need B_e to be a full C*-hull (compatible with isometric intertwiners) and one more extra condition. But we may begin the construction without these assumptions. First we build the unit fibre B_e^+ of the Fell bundle. It is a (weak) C*-hull for the inducible, integrable representations of A_e.
Let $(\mathfrak{B}_e,\mu_e)$ be the universal integrable representation of A_e on B_e. Let x_- for a self-adjoint element x in a C*-algebra denote its negative part.

Definition 9.10. Let B_e^+ be the quotient C*-algebra of B_e by the closed two-sided ideal generated by elements of the form

$$\left(\sum_{k,l=1}^n b_k^*\cdot\mu_e(a_k^*a_l)\cdot b_l\right)_- \qquad \text{for } g\in G,\ a_1,\dots,a_n\in A_g,\ b_1,\dots,b_n\in\mathfrak{B}_e. \tag{9.11}$$

Let $\mathfrak{B}_e^+$ be the image of $\mathfrak{B}_e$ in B_e^+ and let $\mu_e^+\colon A_e\to\operatorname{End}_{B_e^+}(\mathfrak{B}_e^+)$ be the induced representation of A_e on this quotient.

The following proposition shows that the representation $(\mathfrak{B}_e^+,\mu_e^+)$ of A_e on B_e^+ is the universal inducible, integrable representation of A_e.

Proposition 9.12. *Let $(\mathfrak{F},\varphi_e)$ be an integrable representation of A_e on a Hilbert module $\mathcal{F}$. Let $\bar\varphi_e\colon B_e\to\mathbb{B}(\mathcal{F})$ be the corresponding representation* *of B_e. Then φ_e is inducible if and only if $\bar{\varphi}_e$ factors through the quotient map $B_e \twoheadrightarrow B_e^+$. Thus B_e^+ is a* C*-*hull for the inducible, integrable representations of A_e.*

Proof. Assume first that φ_e is inducible. Let $\xi \in \mathcal{F}$ and let $g \in G$, $a_1, \dots, a_n \in A_g$ and $b_1, \dots, b_n \in \mathfrak{B}_e$ be as in (9.11). Let $\xi_k := \varphi_e(b_k)\xi$. Since φ_e is inducible, Proposition 9.5 implies

$$0 \le \sum_{k,l=1}^{n} \langle \xi_k, \varphi_e(a_k^* a_l)\xi_l \rangle = \sum_{k,l=1}^{n} \langle \xi, \bar{\varphi}_e(b_k)^* \varphi_e(a_k^* a_l) \bar{\varphi}_e(b_l)\xi \rangle$$

$$= \left\langle \xi, \bar{\varphi}_e \left(\sum_{k,l=1}^{n} b_k^* \mu_e(a_k^* a_l) b_l \right) \xi \right\rangle.$$

Since $\xi \in \mathcal{F}$ is arbitrary, this means that $\bar{\varphi}_e \left(\sum_{k,l=1}^{n} b_k \mu_e(a_k^* a_l) b_l \right) \ge 0$ in $\mathbb{B}(\mathcal{F})$. Equivalently, $\bar{\varphi}_e$ annihilates the negative part of $\sum_{k,l=1}^{n} b_k \mu_e(a_k^* a_l) b_l$. So $\bar{\varphi}_e$ descends to a homomorphism on the quotient B_e^+. Conversely, the representation $(\mathfrak{B}_e^+, \mu_e^+)$ is inducible by Proposition 9.5. If $\bar{\varphi}_e^+ \colon B_e^+ \to \mathbb{B}(\mathcal{F})$ is a representation, then the representation $\mu_e^+ \otimes_{\bar{\varphi}_e^+} 1_{\mathcal{F}} \cong \varphi_e$ on $B_e^+ \otimes_{B_e^+} \mathcal{F} \cong \mathcal{F}$ is inducible by Lemma 9.9. That is, φ_e is inducible if $\bar{\varphi}_e$ factors through the quotient map $B_e \twoheadrightarrow B_e^+$.
Summing up, the representation $\bar{\varphi}_e$ associated to an integrable representation φ_e of A_e descends to B_e^+ if and only if φ_e is inducible. The quotient map induces a fully faithful embedding $\mathrm{Rep}(B_e^+, D) \hookrightarrow \mathrm{Rep}(B_e, D)$. The argument above shows that its image consists of those representations of B_e that correspond to inducible, integrable representations of A_e under the correspondence $\mathrm{Rep}(B_e, D) \cong \mathrm{Rep}_{\mathrm{int}}(A_e, D)$. Hence B_e^+ is a (weak) C*-hull for the class of inducible, integrable representations of A_e. □

Definition 9.13. Let $B_g^+ := A_g \otimes_{A_e} B_e^+$. This is a well defined Hilbert B_e^+-module because the representation $(\mathfrak{B}_e^+, \mu_e^+)$ of A_e on B_e^+ is inducible. Let $(\mathfrak{B}_g^+, \mu_{e,g}^+)$ be the induced representation of A_e on B_g^+. It has the image of $A_g \odot_{A_e} \mathfrak{B}_e^+$ as a core, with the representation $\mu_{e,g}^+(a_e)(a_g \otimes b) := (a_e a_g) \otimes b$ for $a_e \in A_e$, $a_g \in A_g$, $b \in \mathfrak{B}_e^+$.

By definition, the right B_e^+-module structure and the inner product on B_g^+ are the unique extensions of the following pre-Hilbert module structure on $A_g \odot_{A_e} \mathfrak{B}_e^+$: $(a_g \otimes b_1) \cdot b_2 := a_g \otimes (b_1 \cdot b_2)$ for all $a_g \in A_g$, $b_1 \in \mathfrak{B}_e^+$, $b_2 \in B_e^+$, and

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := b_1^* \mu_e^+(a_1^* a_2) b_2 \tag{9.14}$$

for $a_1, a_2 \in A_g$, $b_1, b_2 \in \mathfrak{B}_e^+$. This is positive definite by Proposition 9.12. By definition, B_g^+ is the norm completion of this pre-Hilbert B_e^+-module, and $\mathfrak{B}_g^+$ is the completion of $A_g \odot_{A_e} \mathfrak{B}_e^+$ in the graph topology for the representation $\mu_{e,g}^+$ of A_e.

The Hilbert B_e^+-modules B_g^+ are the fibres of our Fell bundle.

The Fell bundle structure on $(B_g^+)_{g\in G}$ only exists under extra assumptions. Before we turn to these, we construct representations of the Hilbert B_e^+-modules B_g^+ from an integrable representation π of A on $\mathcal{E}$. Let $\pi_g := \pi|_{A_g}$ and let $\bar\pi_e\colon B_e \to \mathbb{B}(\mathcal{E})$ be the representation of the C*-hull corresponding to π_e. Since π_e is inducible by Lemma 9.8, $\bar\pi_e$ descends to a representation $\bar\pi_e^+\colon B_e^+ \to \mathbb{B}(\mathcal{E})$ by Proposition 9.12.

Let $a \in A_g$ and $b \in \mathfrak{B}_e^+$. The operator $\pi_g(a)\bar\pi_e^+(b)$ is defined on all of $\mathcal{E}$ because $\bar\pi_e^+(b)$ maps $\mathcal{E}$ into the domain $\mathfrak{E}$ of π_e, which is also the domain of $\pi_g(a)$ by Lemma 9.2. Its adjoint contains the densely defined operator $\bar\pi_e^+(b^*)\pi_{g^{-1}}(a^*)$, and the operator

$$\bar\pi_e^+(b^*)\pi_{g^{-1}}(a^*)\pi_g(a)\bar\pi_e^+(b) = \bar\pi_e^+(b^*)\pi_e(a^*a)\bar\pi_e^+(b) = \bar\pi_e^+(b^*\cdot\mu_e^+(a^*a)\cdot b)$$

is bounded. Hence $\bar\pi_e^+(b^*)\pi_{g^{-1}}(a^*)$ extends to a bounded operator on $\mathcal{E}$, which is adjoint to $\pi_g(a)\bar\pi_e^+(b)$. Thus $\pi_g(a)\bar\pi_e^+(b) \in \mathbb{B}(\mathcal{E})$. Define

$$\bar\pi_g^+\colon A_g \odot \mathfrak{B}_e^+ \to \mathbb{B}(\mathcal{E}), \qquad a\otimes b \mapsto \pi_g(a)\bar\pi_e^+(b).$$

As above, we check that

$$\bar\pi_g^+(x_1)^*\bar\pi_g^+(x_2) = \bar\pi_e^+(\langle x_1, x_2\rangle), \qquad \bar\pi_g^+(x\cdot b) = \bar\pi_g^+(x)\bar\pi_e^+(b) \tag{9.15}$$

for all $x_1, x_2, x \in A_g \odot \mathfrak{B}_e^+$, $b \in B_e^+$, where the inner product is the one that defines B_g^+. Thus $\bar\pi_g^+$ extends uniquely to a bounded linear map

$$\bar\pi_g^+\colon B_g^+ \to \mathbb{B}(\mathcal{E}),$$

which still satisfies (9.15). That is, it is a representation of the Hilbert module B_g^+ with respect to $\bar\pi_e^+$.

Lemma 9.16. *If $\bar\pi_e^+\colon B_e^+ \hookrightarrow \mathbb{B}(\mathcal{E})$ is faithful (hence isometric), then so is $\bar\pi_g^+$.*

Proof. Let $\xi \in B_g^+$. Then

$$\|\xi\| = \|\langle\xi,\xi\rangle_{B_e^+}\|^{1/2} = \|\bar\pi_e^+(\langle\xi,\xi\rangle_{B_e^+})\|^{1/2} = \|\bar\pi_g^+(\xi)^*\bar\pi_g^+(\xi)\|^{1/2} = \|\bar\pi_g^+(\xi)\|. \qquad \square$$

Next we want to prove that

$$\bar\pi_g^+(B_g^+)\cdot\bar\pi_h^+(B_h^+) \subseteq \bar\pi_{gh}^+(B_{gh}^+) \quad\text{and}\quad \bar\pi_g^+(B_g^+)^* = \bar\pi_{g^{-1}}^+(B_{g^{-1}}^+) \tag{9.17}$$

for all $g, h \in G$ and for all integrable representations π of A. This would give $(\bar\pi_g^+(B_g^+))_{g\in G}$ a Fell bundle structure, which would lift to $(B_g^+)_{g\in G}$ itself if $\bar\pi_e^+$ is faithful. Lemma 9.23 below gives (9.17) provided the closed linear span of $\bar\pi_e^+(B_e^+)\cdot\bar\pi_g^+(B_g^+)$ is $\bar\pi_g^+(B_g^+)$ for all $g \in G$. But this only holds if we impose two extra assumptions. First, compatibility of integrability and induction gives B_g^+ a canonical left B_e^+-module structure. Secondly, compatibility of the weak C*-hull B_e^+ with isometric intertwiners ensures that the representation $\bar\pi_g^+$ is compatible with this left B_e^+-module structure.

9.4 Integrability and induction

Definition 9.18. We say that *integrability is compatible with induction* if induction of inducible representations preserves integrability; that is, if φ_e is an inducible, integrable representation of A_e on $\mathcal{E}$ and π is the representation of A on $A \otimes_{A_e} \mathcal{E}$ induced by φ_e, then the representation $\pi_e := \pi|_{A_e}$ of A_e is again integrable.

We shall use this assumption in Section 9.5 to prove (9.17). But first, we study some sufficient conditions for integrability to be compatible with induction.
A direct sum of representations is integrable if and only if each summand is integrable by Corollary 3.4. Hence integrability is compatible with induction if and only if an inducible, integrable representation φ_e on $\mathcal{F}$ induces integrable representations of A_e on $A_g \otimes_{A_e} \mathcal{F}$ for all $g \in G$.

Proposition 9.19. *Integrability is compatible with induction if and only if the representations $(\mathfrak{B}_g^+, \mu_{e,g}^+)$ of A_e on B_g^+ are integrable for all $g \in G$.*

Proof. The representations $(\mathfrak{B}_g^+, \mu_{e,g}^+)$ of A_e on B_g^+ are integrable for all $g \in G$ if and only if their direct sum is integrable. Denote this by $(A \otimes_{A_e} \mathfrak{B}_e^+, \mu^+)$. If integrability is compatible with induction, then $(A \otimes_{A_e} \mathfrak{B}_e^+, \mu^+)$ must be integrable because it is the induced representation of the universal integrable (inducible) representation $(\mathfrak{B}_e^+, \mu_e^+)$ of A_e on B_e^+. Conversely, by Lemma 9.9, induction maps the representation $(\mathfrak{B}_e^+, \mu_e^+) \otimes_\varrho \mathcal{F}$ of A_e associated to a representation $\varrho\colon B_e^+ \to \mathbb{B}(\mathcal{F})$ to the representation $(A \otimes_{A_e} \mathfrak{B}_e^+, \mu^+) \otimes_\varrho \mathcal{F}$, which is integrable if $(A \otimes_{A_e} \mathfrak{B}_e^+, \mu^+)$ is, see Definition 3.11.(2). $\square$

The (Strong) Local–Global Principle is useful to check that integrability is compatible with induction:

Proposition 9.20. *Assume that the integrable representations of A_e satisfy the Strong Local–Global Principle and that induction maps irreducible, inducible, integrable Hilbert space representations of A_e to integrable Hilbert space representations of A. Then integrability is compatible with induction.*
The same conclusion holds if the integrable representations of A_e satisfy the Local–Global Principle and induction maps all inducible, integrable Hilbert space representations of A_e to integrable Hilbert space representations of A.

Proof. Let B_e^+ with the representation $(\mathfrak{B}_e^+, \mu_e^+)$ of A_e be the C*-hull for the inducible, integrable representations of A_e. By Proposition 9.19, it suffices to prove that the canonical representation of A_e on $A \otimes_{A_e} \mathfrak{B}_e^+$ is integrable.
By the Strong Local–Global Principle, this follows if the induced representation $\tilde{\pi}$ of A_e on $(A \otimes_{A_e} \mathfrak{B}_e^+) \otimes_\varrho \mathcal{H}$ is integrable for each irreducible representation ϱ of B_e^+ on a Hilbert space $\mathcal{H}$. The representation ϱ is equivalent to an irreducible, inducible, integrable representation π of A_e on $\mathcal{H}$, and $\tilde{\pi}$ is the representation induced by π. By assumption, $\tilde{\pi}$ is integrable. This finishes the proof in the case of the Strong Local–Global Principle. The argument in the other case is the same without the word "irreducible." $\square$

Proposition 9.21. *Assume the following. First, the integrable representations of A_e satisfy the Strong Local–Global Principle. Secondly, all* irreducible, *integrable Hilbert space representations of A_e are finite-dimensional. Third, all finite-dimensional inducible representations of A_e are integrable. And fourth, each A_g is finitely generated as a right A_e-module. Then integrability is compatible with induction.*

Proof. First, since B_e^+ is a quotient of B_e, its irreducible representations form a subset of the irreducible representations of B_e. Thus the irreducible, inducible, integrable Hilbert space representations of A_e are finite-dimensional as well. By Proposition 9.20, it suffices to check that the induced representation of A_e on $A_g \otimes_{A_e} \mathcal{H}$ is integrable when $\mathcal{H}$ is a Hilbert space with an irreducible, inducible, integrable representation. By our assumptions, $\mathcal{H}$ is finite-dimensional and A_g is finitely generated as an A_e-module. Hence $A_g \otimes_{A_e} \mathcal{H}$ is finite-dimensional. This representation is a direct summand in a representation of A on $A \otimes_{A_e} \mathcal{H}$ and hence inducible by Lemma 9.8. By assumption, the induced representation of A_e on $A_g \otimes_{A_e} \mathcal{H}$ is integrable. □

9.5 The Fell bundle structure

If integrability is compatible with induction, the representation $\mu_{e,g}^+$ of A_e on B_g^+ is integrable. It is inducible as well by Lemma 9.8 because it is a direct summand in a representation of A. Hence there is a unique (nondegenerate) representation $\bar{\mu}_{e,g}^+$ of B_e^+ on B_g^+ such that $\bar{\mu}_{e,g}^+(\mathfrak{B}_e^+)B_g^+$ is a core for $\mu_{e,g}^+$, and $\mu_{e,g}^+(a_e)(\bar{\mu}_{e,g}^+(b)x) = \bar{\mu}_{e,g}^+(\mu_e^+(a_e)b)x$ for all $a \in A_e$, $b \in \mathfrak{B}_e^+$, $x \in B_g^+$. Our next goal is to show that the representations $\bar{\pi}_e^+ \colon B_e^+ \to \mathbb{B}(\mathcal{E})$ and $\bar{\pi}_g^+ \colon B_g^+ \to \mathbb{B}(\mathcal{E})$ constructed using (9.15) are compatible in the sense that

$$\bar{\pi}_e^+(b_e) \cdot \bar{\pi}_g^+(b_g) = \bar{\pi}_g^+(\bar{\mu}_{e,g}^+(b_e)b_g) \qquad \text{for all } b_e \in B_e^+,\ b_g \in B_g^+. \tag{9.22}$$

This is not automatic. The following lemma is the most subtle point in the proof of the Induction Theorem.

Lemma 9.23. *Equation* (9.22) *holds if B_e^+ is a* C*-*hull, not just a weak* C*-*hull. Then also $\bar{\pi}_e^+(B_e^+) \cdot \bar{\pi}_g^+(B_g^+) = \bar{\pi}_g^+(B_g^+)$ for all $g \in G$.*

Proof. Let $\mathcal{F} := B_g^+ \otimes_{B_e^+} \mathcal{E}$. The linear map $B_g^+ \odot \mathcal{E} \to \mathcal{E}$, $b \otimes \xi \mapsto \bar{\pi}_g^+(b)\xi$, for $b \in B_g^+$, $\xi \in \mathcal{E}$, preserves the inner products by (9.15). Hence it extends to a well defined isometry $I \colon \mathcal{F} \hookrightarrow \mathcal{E}$. The representation $\bar{\mu}_{e,g}^+$ of B_e^+ on B_g^+ induces a representation $\bar{\mu}_{e,g}^+ \otimes 1_{\mathcal{E}}$ of B_e^+ on $\mathcal{F}$. The meaning of (9.22) is that I intertwines the representations $\bar{\mu}_{e,g}^+ \otimes 1$ and $\bar{\pi}_e^+$ of B_e^+ on $\mathcal{F}$ and $\mathcal{E}$. These representations correspond to the integrable representations $\mu_{e,g}^+ \otimes 1$ and π_e of A_e on $\mathcal{F}$ and $\mathcal{E}$, respectively. Since B_e^+ is a C*-hull, it suffices to prove that I intertwines these representations of A_e.

We identify $\mathcal{E} \cong B_e^+ \otimes_{\bar{\pi}_e^+} \mathcal{E}$ and describe π_e as $\mu_e^+ \otimes_{\bar{\pi}_e^+} 1_{\mathcal{E}}$ as in Proposition 3.6. Then Lemma 9.9 gives a canonical unitary *-intertwiner

$$\mathcal{F} := (A_g \otimes_{A_e} B_e^+) \otimes_{B_e^+} \mathcal{E} \cong A_g \otimes_{A_e} (B_e^+ \otimes_{B_e^+} \mathcal{E}) \cong A_g \otimes_{A_e} \mathcal{E}$$

of representations of A_e. An inspection of the proof shows that I corresponds to the isometry $I'\colon A_g \otimes_{A_e} \mathcal{E} \hookrightarrow \mathcal{E}$ defined by $I'(a \otimes \xi) := \pi_g(a)\xi$ for all $a \in A_g$, $\xi \in \mathfrak{E}$. Since I' is an A_e-intertwiner, so is I. This finishes the proof of (9.22). Then $\bar{\pi}_e^+(B_e^+) \cdot \bar{\pi}_g^+(B_g^+) = \bar{\pi}_g^+(B_g^+)$ follows because $\bar{\mu}_{e,g}^+$ is nondegenerate. $\square$

Lemma 9.24. *Assume $\bar{\pi}_e^+(B_e^+) \cdot \bar{\pi}_g^+(B_g^+) = \bar{\pi}_g^+(B_g^+)$ for all $g \in G$. Then* (9.17) *holds.*

Proof. We write $\doteq$ to denote that two sets of operators have the same closed linear span. By definition, $\bar{\pi}_g^+(B_g^+) \doteq \pi_g(A_g)\bar{\pi}_e^+(\mathfrak{B}_e^+)$, and $\bar{\pi}_e^+(\mathfrak{B}_e^+)^* \doteq \bar{\pi}_e^+(B_e^+)$ because $\mathfrak{B}_e^+$ is dense in B_e^+. Our assumption $\bar{\pi}_g^+(B_g^+) \doteq \bar{\pi}_e^+(B_e^+) \cdot \bar{\pi}_g^+(B_g^+)$ implies $\bar{\pi}_g^+(B_g^+) \doteq \bar{\pi}_e^+(\mathfrak{B}_e^+)^*\pi_g(A_g)\bar{\pi}_e^+(\mathfrak{B}_e^+)$. We have seen above (9.15) that

$$\bar{\pi}_e^+(b^*)\pi_{g^{-1}}(a^*) = \bar{\pi}_e^+(b)^*\pi_{g^{-1}}(a^*)$$

for $b \in \mathfrak{B}_e^+$, $a \in A_g$ extends to a bounded operator on $\mathcal{E}$ that is adjoint to the bounded operator $\pi_g(a)\bar{\pi}_e^+(b)$. Therefore,

$$\bigl(\bar{\pi}_e^+(b_1)^*\pi_g(a)\bar{\pi}_e^+(b_2)\bigr)^* = \bar{\pi}_e^+(b_2)^*\pi_{g^{-1}}(a^*)\bar{\pi}_e^+(b_1)$$

for all $b_1, b_2 \in \mathfrak{B}_e^+$, $a \in A_g$; both sides are globally defined bounded operators because $\bar{\pi}_e^+(\mathfrak{B}_e^+)$ maps $\mathcal{E}$ into $\mathfrak{E}$. The closed linear spans on the two sides of this equality are $\bar{\pi}_g^+(B_g^+)^*$ and $\bar{\pi}_{g^{-1}}^+(B_{g^{-1}}^+)$, respectively. Thus $\bar{\pi}_g^+(B_g^+)^* = \bar{\pi}_{g^{-1}}^+(B_{g^{-1}}^+)$. As above, the operators $\overline{\bar{\pi}_e^+(b)\pi_g(a)}$ for $b \in (\mathfrak{B}_e^+)^*$, $g \in G$, $a \in A_g$ are bounded and generate $(B_{g^{-1}}^+)^* = B_g^+$. Hence

$$\begin{aligned}\bar{\pi}_g^+(B_g^+) \cdot \bar{\pi}_h^+(B_h^+) &\doteq \bar{\pi}_e^+((\mathfrak{B}_e^+)^*)\pi_g(A_g) \cdot \pi_h(A_h)\bar{\pi}_e^+(\mathfrak{B}_e^+)\\ &\subseteq \bar{\pi}_e^+(\mathfrak{B}_e^+)^* \cdot \pi_{gh}(A_{gh})\bar{\pi}_e^+(\mathfrak{B}_e^+) \doteq \bar{\pi}_{gh}^+(B_{gh}^+).\end{aligned}$$

We used here that π is a homomorphism on A and that $A_g \cdot A_h \subseteq A_{gh}$. $\square$

Lemma 9.25. *Assume that B_e is a* C^**-hull and that integrability is compatible with induction. There is a unique Fell bundle structure on $(B_g^+)_{g\in G}$ such that the maps $\bar{\pi}_g^+\colon B_g^+ \to \mathbb{B}(\mathcal{E})$ form a Fell bundle representation for any integrable representation π of A on a Hilbert module $\mathcal{E}$.*

Proof. Lemmas 9.23 and 9.24 show that (9.17) holds under our assumptions. Hence the multiplication and involution in $\mathbb{B}(\mathcal{E})$ restrict to a Fell bundle structure on the subspaces $\bar{\pi}_g^+(B_g^+) \subseteq \mathbb{B}(\mathcal{E})$ for $g \in G$, such that the inclusions $\bar{\pi}_g^+(B_g^+) \hookrightarrow \mathbb{B}(\mathcal{E})$ give a Fell bundle representation.

The induced representation λ of A on the Hilbert B_e^+-module $A \otimes_{A_e} B_e^+$ gives a faithful representation of B_e^+ because $A \otimes_{A_e} B_e^+ \supseteq A_e \otimes_{A_e} B_e^+ = B_e^+$ contains the identity representation. Hence the resulting representations $\bar{\lambda}_g$ of B_g^+ are also faithful, even isometric, by Lemma 9.16. So the Fell bundle structure on $\bar{\lambda}_g(B_g^+)$ lifts to B_g^+, so that the maps $\bar{\lambda}_g\colon B_g^+ \to \mathbb{B}(\mathcal{E})$ form a Fell bundle representation.

Let π be any integrable representation of A. The exterior direct sum $\pi \oplus \lambda$ on the Hilbert $D \oplus B_e^+$-module $\mathcal{E}' := \mathcal{E} \oplus (A \otimes_{A_e} B_e^+)$ is still integrable. The resulting maps from B_g^+ to $\mathbb{B}(\mathcal{E}')$ simply give block matrices $\bar{\pi}_g^+(b) \oplus \bar{\lambda}_g(b)$ for $b \in B_g^+$. The compressions to the direct summands $\mathcal{E}$ and $A \otimes_{A_e} B_e^+$ therefore restrict to Fell bundle representations with respect to the Fell bundle structure on $(\bar{\pi}_g^+ \oplus \bar{\lambda}_g)(B_g^+)$ defined above. Since λ is faithful, the projection $(\bar{\pi}_g^+ \oplus \bar{\lambda}_g)(B_g^+) \to \bar{\lambda}_g(B_g^+) \cong B_g^+$ is a Fell bundle isomorphism. Hence the map $B_g^+ \xrightarrow{\sim} (\bar{\pi}_g^+ \oplus \bar{\lambda}_g)(B_g^+) \to \bar{\pi}_g^+(B_g^+)$ is a Fell bundle representation. □

Let $(\beta_g)_{g\in G}$ be a Fell bundle over a discrete group G (see [8]). Then $\beta := \bigoplus_{g\in G} \beta_g$ is a G-graded *-algebra using the given multiplications and involutions among the subspaces β_g. The *section* C^**-algebra* $\mathrm{C}^*(\beta)$ of the Fell bundle is defined as the completion of β in the maximal C^*-seminorm. By construction, a representation of $\mathrm{C}^*(\beta)$ is equivalent to a representation of the Fell bundle. This holds also for representations on Hilbert modules.

Theorem 9.26. *Let A be a graded *-algebra for which A_e has a C^*-hull. Assume that integrability is compatible with induction as in Definition* 9.18. *The section C^*-algebra B of the Fell bundle $(B_g^+)_{g\in G}$ constructed above is a C^*-hull for the integrable representations of A.*

Proof. Representations of B are in natural bijection with Fell bundle representations: restricting a representation of B to the subspaces B_g^+ gives a Fell bundle representation, and conversely a Fell bundle representation gives a representation of the *-algebra $\bigoplus_{g\in G} B_g^+$, which extends uniquely to the C^*-completion. Lemma 9.25 says that any integrable representation $\pi = \bigoplus_{g\in G} \pi_g$ of A induces a Fell bundle representation $(\bar{\pi}_g^+)_{g\in G}$ of $(B_g^+)_{g\in G}$ and thus a representation of B. By construction, this family of maps $\mathrm{Rep}_{\mathrm{int}}(A) \to \mathrm{Rep}(B)$ is compatible with interior tensor products and unitary *-intertwiners. We are going to show that this is a family of bijections.

First we describe an integrable representation $(\mathfrak{B}, \mu)$ of A on B. By construction, $A \otimes_{A_e} \mathfrak{B}_e^+ = \bigoplus_{g\in G} \mathfrak{B}_g^+$ is dense in B. This subspace carries a representation of A by left multiplication. We extend this to the right ideal in B generated by $A \otimes_{A_e} \mathfrak{B}_e^+$ to get a representation of A on B. Let $(\mathfrak{B}, \mu)$ be its closure.

The representations $\bar{\mu}_{e,g}^+$ of B_e^+ on B_g^+ are defined so that $\bar{\mu}_{e,g}^+(\mathfrak{B}_e^+)B_g^+$ is another core for the representation $(\mathfrak{B}_g^+, \mu_{e,g}^+)$ of A_e on B_g^+. Therefore, $\mathfrak{B}_e^+ \cdot B$ is a core for the restriction of the representation $(\mathfrak{B}, \mu)$ to A_e. This core shows that $(\mathfrak{B}, \mu|_{A_e}) = (\mathfrak{B}_e^+, \mu_e^+) \otimes_{B_e} B$, where the interior tensor product is with respect to the canonical embedding $B_e^+ \hookrightarrow B$. Therefore, the restriction of $(\mathfrak{B}, \mu)$ to A_e is integrable and the corresponding representation $\bar{\mu}_e^+$ of B_e^+ is simply the inclusion map $B_e^+ \hookrightarrow B$. Thus the representation $(\mathfrak{B}, \mu)$ of A on B is also integrable.

The integrable representation $(\mathfrak{B}, \mu)$ of A on B yields a representation $\bar{\mu}_g^+$ of the Fell bundle $(B_g^+)_{g\in G}$ in $\mathcal{M}(B) = \mathbb{B}(B)$. By construction, the image of $a_g \otimes b \in A_g \odot_{A_e} \mathfrak{B}_e^+$ in B_g^+ acts by $\mu(a_g)\bar{\mu}_e^+(b) = \mu(a_g) \cdot b$. That is, B_g^+ is represented by the canonical inclusion map $B_g^+ \hookrightarrow B$. The representation of B associated to this Fell bundle representation is the identity map on B.

Interior tensor product with $(\mathfrak{B}, \mu)$ gives a family of maps $\mathrm{Rep}(B) \to \mathrm{Rep}_{\mathrm{int}}(A)$ that is compatible with unitary *-intertwiners and interior tensor products. Since the composite family of maps $\mathrm{Rep}(B) \to \mathrm{Rep}_{\mathrm{int}}(A) \to \mathrm{Rep}(B)$ is compatible with interior tensor products and maps the identity representation of B to itself, the composite map on $\mathrm{Rep}(B)$ is the identity.

Let $(\mathfrak{E}, \pi)$ be an integrable representation of A on a Hilbert D-module $\mathcal{E}$ for some C*-algebra D. This yields a representation $(\bar{\pi}_g^+)_{g\in G}$ of the Fell bundle $(B_g^+)_{g\in G}$ and an associated representation $\bar{\pi}$ of B. We claim that the integrable representation $(\mathfrak{E}', \pi') := (\mathfrak{B}, \mu) \otimes_{\bar{\pi}} \mathcal{E}$ is equal to $(\mathfrak{E}, \pi)$. Both representations have the same restriction to A_e because

$$(\mathfrak{B}, \mu|_{A_e}) \otimes_{\bar{\pi}} \mathcal{E} \cong (\mathfrak{B}_e^+, \mu_e^+) \otimes_{B_e} B \otimes_{\bar{\pi}} \mathcal{E} \cong (\mathfrak{B}_e^+, \mu_e^+) \otimes_{\bar{\pi}|_{B_e}} \mathcal{E} \cong (\mathfrak{E}, \pi).$$

Hence both representations have the same domain by Lemma 9.2. And $\bar{\pi}_e^+(\mathfrak{B}_e^+)\mathcal{E}$ is a core for both. On $\bar{\pi}_e^+(\mathfrak{B}_e^+)\mathcal{E}$, $a_g \in A_g$ acts by mapping $\bar{\pi}_e^+(b_e)\xi$ to $\pi_g(a_g)\bar{\pi}_e^+(b_e)\xi = \bar{\pi}(a_g \otimes b_e)\xi$ in both representations, where we view $a_g \otimes b_e \in B_g^+ \subseteq B$. Since $(\mathfrak{E}, \pi)$ and $(\mathfrak{E}', \pi')$ have a common core, they are equal.

This finishes the proof that our two families of maps $\mathrm{Rep}_{\mathrm{int}}(A) \leftrightarrow \mathrm{Rep}(B)$ are inverse to each other. Thus B is a weak C*-hull for the integrable representations of A. Since A_e is a C*-hull, the integrable representations of A_e are admissible. So are the integrable representations of A by Proposition 9.4. Thus B is a C*-hull. □

Remark 9.27. The fibres B_e^+ of the Fell bundle in Theorem 9.26 are described in Definitions 9.10 and 9.13, including the right Hilbert B_e^+-module structure on B_g^+. The rest of the Fell bundle structure needs technical extra assumptions. The simplest way to get it is by inducing the universal inducible, integrable representation of A on B_e^+ to an integrable representation of A on the Hilbert B_e^+-module $A \otimes_{A_e} B_e^+$. The Fell bundle $(B_g^+)_{g\in G}$ is represented faithfully in $\mathbb{B}(A \otimes_{A_e} B_e^+)$ by Lemma 9.16. The multiplication, involution, and norm in our Fell bundle are simply the multiplication, involution and norm in the C*-algebra $\mathbb{B}(A \otimes_{A_e} B_e^+)$. The dense image of $A_g \odot_{A_e} \mathfrak{B}_e^+$ in B_g^+ acts on $A \otimes_{A_e} B_e^+$ by $a_g \otimes b \mapsto \pi_g(a_g) \cdot \bar{\pi}_e^+(b)$, where $\bar{\pi}_e^+(b)$ is the representation of the C*-hull B_e^+ associated to the induced representation of A_e on $A \otimes_{A_e} B_e^+$, which is integrable by assumption.

9.6 Two counterexamples

Two assumptions limit the generality of the Induction Theorem 9.26. First, integrability must be compatible with induction. Secondly, B_e should be a C*-hull and not a weak C*-hull. Equivalently, all isometric intertwiners between integrable Hilbert space representations of A_e are *-intertwiners. We show by two simple counterexamples that both assumptions are needed. In particular, there is no version of the Induction Theorem for weak C*-hulls.

Both counterexamples involve the group $G = \mathbb{Z}/2 = \{0,1\}$. A G-graded *-algebra is a *-*superalgebra*, that is, a *-algebra with a decomposition $A = A_0 \oplus A_1$ such that

$$A_0\cdot A_0+A_1\cdot A_1 \subseteq A_0,\quad A_0\cdot A_1+A_1\cdot A_0 \subseteq A_1,\quad A_0^* = A_0,\quad A_1^* = A_1,\quad 1 \in A_0.$$

In both examples, $A_0 = \mathbb{C}[x]$ with $x = x^*$.

In the first example, A is the crossed product for the action of $\mathbb{Z}/2$ on $A_0 = \mathbb{C}[x]$ through the involution $x \mapsto -x$. That is,

$$A = \mathbb{C}\langle x, \varepsilon \mid \varepsilon^2 = 1,\ x\varepsilon = -\varepsilon x,\ x = x^*,\ \varepsilon = \varepsilon^*\rangle, \qquad x \in A_0,\ \varepsilon \in A_1.$$

Since $A_1 = \varepsilon A_0 \cong A_0$ as a right A_0-module, any representation of A_0 is inducible.

Let $B_0 = \mathrm{C}_0((0,\infty))$ with the representation of A_0 from the inclusion map $(0,\infty) \hookrightarrow \mathbb{R} = \widehat{A_0}$ (see Proposition 8.1). This gives a C*-hull for a class of representations of A_0 that is defined by submodule conditions and satisfies the Strong Local–Global Principle by Theorems 8.2 and 8.3. The class of $(0,\infty)$-integrable representations consists of those representations of $\mathbb{C}[x]$ that are generated by a regular, self-adjoint, *strictly positive* operator.

In a representation of A, the element $\varepsilon \in A$ acts by a unitary involution that conjugates $\overline{\pi(x)}$ to $-\overline{\pi(x)}$. Hence $\overline{\pi(x)}$ cannot be strictly positive. Thus the zero-dimensional representation is the only representation of A whose restriction to A_0 is $\mathrm{C}_0((0,\infty))$-integrable. The C*-hull for this class is $\{0\}$. Theorem 9.26 does not apply here because induced representations of inducible, integrable representations of A_0 are *never* integrable when they are non-zero.

The second example is the commutative *-superalgebra

$$A = \mathbb{C}\langle x, \varepsilon \mid \varepsilon^2 = 1 + x^2,\ x\varepsilon = \varepsilon x,\ x = x^*,\ \varepsilon = \varepsilon^*\rangle, \qquad x \in A_0,\ \varepsilon \in A_1.$$

Thus $A_1 = \varepsilon\mathbb{C}[x] \cong A_0$ with the usual A_0-bimodule structure and the inner product $\langle \varepsilon a_1, \varepsilon a_2\rangle = (1 + x^2) \cdot \overline{a_1} \cdot a_2$. Since $(1 + x^2)|a|^2$ is positive in $\mathbb{C}[x]$ for any $a \in \mathbb{C}[x]$, any representation of A_0 is inducible.

Let $(\mathfrak{E}, \pi)$ be a representation of A_0 on a Hilbert module $\mathcal{E}$ over a C*-algebra D. The induced representation $A_1 \otimes_{A_0} (\mathfrak{E}, \pi)$ lives on the Hilbert D-module completion $\mathcal{E}_1$ of $\mathfrak{E}$ for the inner product $\langle \xi_1, \xi_2\rangle_1 := \langle \xi_1, \pi(1 + x^2)\xi_2\rangle$. Its domain is $\mathfrak{E}$, viewed as a dense D-submodule in $\mathcal{E}_1$, and the representation of A_0 is π again. The operator $\pi(x + \mathrm{i})$ on $\mathfrak{E}$ extends to an isometry $I\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}$ because

$$\langle \pi(x + \mathrm{i})\xi_1, \pi(x + \mathrm{i})\xi_2\rangle = \langle \xi_1, \pi(x - \mathrm{i})\pi(x + \mathrm{i})\xi_2\rangle = \langle \xi_1, \pi(1 + x^2)\xi_2\rangle = \langle \xi_1, \xi_2\rangle_1$$

for all $\xi_1, \xi_2 \in \mathfrak{E}$. This isometry commutes with $\pi(a)$ for all $a \in A$, so it is an isometric intertwiner $A_1 \otimes_{A_0} (\mathfrak{E}, \pi) \hookrightarrow (\mathfrak{E}, \pi)$.

Now let B_0 with the universal representation $(\mathfrak{B}_0, \mu_0)$ be one of the two noncommutative weak C*-hulls $\mathcal{T}_0$ or $\mathbb{K}(\ell^2\mathbb{N})$ of $\mathbb{C}[x]$ described in §6. In

a Toeplitz integrable representation, $\pi(x+\mathrm{i})$ has dense range. Even more, $\pi(x+\mathrm{i})\mathfrak{E}$ is dense in $\mathfrak{E}$ in the graph topology. Thus I is a unitary *-intertwiner $A_1 \otimes_{A_0} (\mathfrak{E},\pi) \xrightarrow{\sim} (\mathfrak{E},\pi)$ for any integrable representation $(\mathfrak{E},\pi)$ of A_0.
Since all representations of A_0 are inducible, the unit fibre of the Fell bundle should be B_0. The other fibre B_1 is $A_1 \otimes_{A_0} B_0$, which we have identified with B_0. The unitary $A_1 \otimes_{A_0} B_0 \cong B_0$ is a *-intertwiner between the representations of A_0 as well. Therefore, integrability is compatible with induction. And the left B_0-module structure $\bar{\mu}_{0,1}$ on B_1 in (9.22) is simply left multiplication.
Next we describe the induced representation of A on the Hilbert B_0-module

$$A \otimes_{A_0} B_0 = A_0 \otimes_{A_0} B_0 \oplus A_1 \otimes_{A_0} B_0 \cong B_0 \oplus B_0.$$

The representations of A and A_0 on $A \otimes_{A_0} B_0$ have the same domain by Lemma 9.2, and for A_0 the domain is $\mathfrak{B}_0 \oplus \mathfrak{B}_0$. We claim that A acts on this domain by

$$x \mapsto \begin{pmatrix} \mu_0(x) & 0 \\ 0 & \mu_0(x) \end{pmatrix}, \qquad \varepsilon \mapsto \begin{pmatrix} 0 & \mu_0(x-\mathrm{i}) \\ \mu_0(x+\mathrm{i}) & 0 \end{pmatrix}.$$

We have already seen this for $x \in A_0$. Left multiplication by ε maps $b \in \mathfrak{B}_0 \subseteq B_0$ first to $\varepsilon \otimes b \in A_1 \otimes_{A_0} B_0$, which is mapped by the isometry I to $\mu_0(x+\mathrm{i})b \in \mathfrak{B}_0 \subseteq B_0$. And it maps the element $\mu_0(x+\mathrm{i})b \in B_0$ for $b \in \mathfrak{B}_0$, which corresponds to $\varepsilon \otimes b$ in the odd fibre, to $\varepsilon^2 \otimes b = \mu_0(x^2+1)b = \mu_0(x-\mathrm{i})\mu_0(x+\mathrm{i})b \in B_0$. This proves the formula for the action of ε.
The representation $\bar{\mu}_0$ of B_0 on $A \otimes_{A_0} B_0$ is the representation of the weak C*-hull that corresponds to the representation of $A_0 \subseteq A$ described above. This is

$$\bar{\mu}_0\colon B_0 \to \mathbb{M}_2(B_0), \qquad b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Hence $\varepsilon \otimes b \in A_1 \otimes_{A_0} B_0$ for $b \in \mathfrak{B}_0$ acts by the matrix

$$\begin{pmatrix} 0 & \mu_0(x-\mathrm{i}) \\ \mu_0(x+\mathrm{i}) & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & \mu_0(x-\mathrm{i})b \\ \mu_0(x+\mathrm{i})b & 0 \end{pmatrix}.$$

The map $\mu_0(x+\mathrm{i})b \mapsto \mu_0(x-\mathrm{i})b$ is the Cayley transform of $\mu_0(x)$. For our two weak C*-hulls, this is the unilateral shift $S \in \mathcal{M}(B_0)$ by construction. Thus the odd fibre $B_1 \cong B_0$ of our Fell bundle should act by

$$\bar{\mu}_1\colon B_0 \to \mathbb{M}_2(B_0), \qquad b \mapsto \begin{pmatrix} 0 & Sb \\ b & 0 \end{pmatrix}.$$

The map $\bar{\mu}_0$ is a *-representation, and (9.15) gives

$$\bar{\mu}_1(b_1)^*\bar{\mu}_1(b_2) = \bar{\mu}_0(b_1^*b_2), \qquad \bar{\mu}_1(b_1)\bar{\mu}_0(b_2) = \bar{\mu}_1(b_1b_2)$$

for all $b_1, b_2 \in B_0$. This is also obvious from our explicit formulas. But

$$\bar{\mu}_0(b_1)\bar{\mu}_1(b_2) = \begin{pmatrix} 0 & b_1Sb_2 \\ b_1b_2 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\mu}_0(b_1b_2) = \begin{pmatrix} 0 & Sb_1b_2 \\ b_1b_2 & 0 \end{pmatrix}$$

differ if, say, $b_1 = S^*$, $b_2 = 1$. In fact, $\bar{\mu}_0(B_0)\cdot\bar{\mu}_1(B_0)$ is not contained in $\bar{\mu}_1(B_0)$. Hence there is no Fell bundle structure on $(B_g)_{g\in\mathbb{Z}/2}$ for which $(\bar{\mu}_g)_{g\in\mathbb{Z}/2}$ would be a Fell bundle representation.

10 Locally bounded unit fibre representations

We now specialise the Induction Theorem 9.26 to the case where the universal integrable representation of the unit fibre A_e is locally bounded. In this case, we may first construct a pro-C*-algebraic Fell bundle whose unit fibre is the pro-C*-algebra completion of A_e. This is relevant because pro-C*-algebras are much closer to ordinary C*-algebras than general *-algebras. We will see the importance of this in the commutative case, where the pro-C*-algebraic Fell bundle gives us a twisted partial group action on the space $\hat{A}_e^+$ of positive characters.

As before, let G be a group and let $A = \bigoplus_{g\in G} A_g$ be a G-graded *-algebra. We are interested in the locally bounded representations of A_e, and representations of A that restrict to locally bounded representations on A_e. The class $\mathrm{Rep}_\mathrm{b}(A_e)$ of locally bounded representations of A_e is admissible by Corollary 7.9. So any weak C*-hull for some smaller class of representations will be an ordinary C*-hull.

Let $\mathcal{A}_e$ be the pro-C*-algebra completion of the unit fibre A_e, that is, the completion of A_e in the topology defined by the directed set $\mathcal{N}(A_e)$ of all C*-seminorms on A_e. Locally bounded representations of A_e are equivalent to locally bounded representations of $\mathcal{A}_e$ by Proposition 7.6.

When is a locally bounded representation inducible?

Proposition 10.1. *A locally bounded representation $(\mathfrak{E},\varphi)$ of A_e on a Hilbert module $\mathcal{E}$ is inducible if and only if $\varphi(a^*a) \ge 0$ for all $a \in A_g$, $g \in G$.*

The difference to the general criterion for inducibility in Proposition 9.5 is that we do not consider matrices.

Proof. The subspace $\mathcal{E}_\mathrm{b} \subseteq \mathfrak{E}$ of bounded vectors is a core for φ. As in the proof of Proposition 9.12, it suffices to prove the positivity of the inner product for a finite linear combination $\sum_{k=1}^n a_k \otimes \xi_k$ with $a_k \in A_g$, $\xi_k \in \mathcal{E}_\mathrm{b}$ for a fixed $g \in G$. Since there are only finitely many ξ_k, there is a C*-seminorm q on A_e so that all ξ_k are q-bounded. Thus we may replace $\mathcal{E}$ by the Hilbert submodule $\mathcal{E}_q$ of q-bounded vectors, where the representation of A_e extends to the C*-completion $D := (\mathcal{A}_e)_q$ for q. Since we assume $\varphi(a^*a) \ge 0$ for all $a \in A_g$, this representation factors through the quotient of D by the closed ideal I generated by the negative parts $(a^*a)_-$ for all $a \in A_g$, $g \in G$. The D/I-valued inner product $\langle a_1, a_2\rangle := a_1^* a_2 \bmod I$ on A_g is positive definite by construction; since D/I is a C*-algebra, we may use the usual notion of positivity here, which does not involve matrices. Then the inner product on the tensor product $A_g \otimes_{D/I} \mathcal{E}_q$ is also positive definite. This is what we had to prove. □

A pro-C*-algebra has a functional calculus for self-adjoint elements. Hence we may construct the negative parts $(a^*a)_- \in \mathcal{A}_e$ for $a \in A_g$, $g \in G$. We let $\mathcal{A}_e^+$ be the completed quotient of $\mathcal{A}_e$ by the closed two-sided ideal generated by these elements. This is another pro-C*-algebra, and it is the largest quotient in which $a^*a \geq 0$ for all $a \in A_g$, $g \in G$. By Proposition 10.1, a locally bounded representation of A_e is inducible if and only if the corresponding locally bounded representation of $\mathcal{A}_e$ factors through $\mathcal{A}_e^+$.

Corollary 10.2. *There is an equivalence between the inducible, locally bounded representations of A_e and the locally bounded representations of the pro-*C**-algebra $\mathcal{A}_e^+$, which is compatible with isometric intertwiners and interior tensor products.*

Proof. Proposition 10.1 says that the equivalence in Proposition 7.6 maps the subclass in $\mathrm{Rep_b}(A_e)$ of inducible, locally bounded representations of A_e onto the subclass $\mathrm{Rep_b}(\mathcal{A}_e^+)$ in $\mathrm{Rep_b}(\mathcal{A}_e)$. □

Let $\mathcal{N}(A_e)^+$ be the directed set of C*-seminorms on $\mathcal{A}_e^+$. This is isomorphic to the subset of $\mathcal{N}(A_e)$ consisting of all C*-seminorms q on A_e for which $a^*a \geq 0$ holds in the C*-completion $(\mathcal{A}_e)_q$ for all $a \in A_g$, $g \in G$. We would like to complete A to a *-algebra $\bigoplus_{g\in G} \mathcal{A}_g^+$ with unit fibre $\mathcal{A}_e^+$, where each $\mathcal{A}_g^+$ is a Hilbert bimodule over $\mathcal{A}_e^+$. But such a construction does not work in the following example.

Example 10.3. It can happen that the class of locally bounded representations of A_e is not compatible with induction. Let $\mathrm{End}^*(\mathbb{C}[\mathbb{N}])$ be the *-algebra of all $\infty \times \infty$-matrix with only finitely many entries in each row and each column, with the usual matrix multiplication and involution. Let A be the $\mathbb{Z}/2$-graded *-algebra of block 2×2-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a \in \mathbb{C},\ b \in \mathbb{C}[\mathbb{N}],\ c \in \mathbb{C}[\mathbb{N}],\ d \in \mathrm{End}^*(\mathbb{C}[\mathbb{N}]),$$

with the grading where a, d are even and b, c are odd. Here b and c are infinite column and row vectors with only finitely many non-zero entries, respectively. Thus $A \cong \mathrm{End}^*(\mathbb{C}[\mathbb{N}])$ with the grading induced by the grading on $\mathbb{C}[\mathbb{N}]$ where $\mathbb{C} \cdot \delta_0$ is the even part and the span of δ_i for $i > 0$ is the odd part.
The character $(a, d) \mapsto a$ is a bounded representation of the unit fibre A_0. Induction gives the standard representation of A on the Hilbert space $\mathbb{C} \oplus \ell^2(\mathbb{N}) \cong \ell^2(\mathbb{N})$ by matrix-vector multiplication. This representation is irreducible because already the ideal of finite matrices $\mathbb{M}_\infty(\mathbb{C})$ in A acts irreducibly. It is not bounded, that is, some elements in $\mathrm{End}^*(\mathbb{C}[\mathbb{N}])$ act by unbounded operators on $\ell^2(\mathbb{N})$. Hence it is not locally bounded by Proposition 7.7.

To rule out this problem, we now *assume that induction from A_e to A and restriction back to A_e maps bounded representations of A_e again to bounded representations of A_e*, briefly, that boundedness is compatible with induction.

This implies that local boundedness is compatible with induction because a locally bounded representation contains bounded subrepresentations whose union is a core for it. Our assumption is equivalent to the boundedness of the induced representations of A_e on the Hilbert $(A_e^+)_q$-modules $A_g \otimes_{A_e} (A_e^+)_q$ for all $g \in G$ and $q \in \mathcal{N}(A_e)^+$. That is, there is another norm $q' \in \mathcal{N}(A_e)^+$ such that

$$q(a^*b^*ba) = \|ba\|_q^2 \le \|b\|_{q'}^2 \|a\|_q^2 = q'(b)^2 \cdot q(a^*a)$$

for all $a \in A_g$, $b \in A_e$. Let $\mathcal{A}_g^+$ be the completion of A_g in the topology generated by the family of norms $q(a^*a)$ for $q \in \mathcal{N}(A_e)^+$.

Lemma 10.4. *The multiplication maps and the involutions in $(A_g)_{g\in G}$ extend to continuous maps $\mathcal{A}_g^+ \times \mathcal{A}_h^+ \to \mathcal{A}_{gh}^+$ and $\mathcal{A}_g^+ \to \mathcal{A}_{g^{-1}}^+$ for $g, h \in G$.*

Proof. Given $q \in \mathcal{N}(A_e)^+$, let $q' \in \mathcal{N}(A_e)^+$ be such that $q(a^*b^*ba) \le q'(b)^2 \cdot q(a^*a)$ for all $a \in A_h$, $b \in A_e$. If $b \in A_g$, $a \in A_h$, then

$$\|ba\|_q^2 := q(a^*b^*ba) = q(a^*(b^*b)^{1/2}(b^*b)^{1/2}a) \le q'((b^*b)^{1/2})^2 \cdot q(a^*a) = \|b\|_{q'}^2 \|a\|_q^2$$

That is, the multiplication is jointly continuous with respect to the topology defining $(\mathcal{A}_g^+)_{g\in G}$ and hence extends to a jointly continuous map $\mathcal{A}_g^+ \times \mathcal{A}_h^+ \to \mathcal{A}_{gh}^+$.

Furthermore, $q(aa^*)^2 = q(aa^*aa^*) \le q'(a^*a)\cdot q(aa^*)$ and hence $q(aa^*) \le q'(a^*a)$ for all $a \in A_h$. That is, $\|a^*\|_q^2 \le \|a\|_{q'}^2$ for all $a \in A_h$. Thus the involution is continuous as well. $\square$

The completion $\mathcal{A}^+ := \bigoplus_{g\in G} \mathcal{A}_g^+$ of A is again a *-algebra by Lemma 10.4. By construction of $\mathcal{A}_e^+$, the inner products $a^*a \in \mathcal{A}_e^+$ are positive for $a \in A_g$, $g \in G$; this remains so for $a \in \mathcal{A}_g^+$ because the subset of positive elements in $\mathcal{A}_e^+$ is closed. Thus $(\mathcal{A}_g^+)_{g\in G}$ has the usual properties of a Fell bundle over G, except that the fibres are only Hilbert bimodules over a pro-C*-algebra. We interpret $(\mathcal{A}_g^+)_{g\in G}$ as a partial action of G on $\mathcal{A}_e^+$ by Hilbert bimodules as in [4].

Usually, the norms $q(a^*a)$ and $q(aa^*)$ on A_g are not equivalent for a fixed $q \in \mathcal{N}(A)^+$. This prevents us from completing $\mathcal{A}^+$ to a pro-C*-algebra. It also means that the integrable representations of A are not locally bounded on A, but only on A_e. This happens in interesting examples such as the Weyl algebra discussed in §13. This phenomenon for Fell bundles is related to the known problem that crossed products for group actions on pro-C*-algebras only work well if the action is strongly bounded, that is, the invariant continuous C*-seminorms are cofinal in the set of all continuous C*-seminorms, see [14].

Proposition 10.5. *Suppose that boundedness for representations of A_e is compatible with induction to A. Representations of A that restrict to locally bounded representations on A_e are equivalent to representations of the *-algebra $\mathcal{A}^+ = \bigoplus_{g\in G} \mathcal{A}_g^+$ that restrict to locally bounded representations on $\mathcal{A}_e^+$; this equivalence is compatible with isometric intertwiners and interior tensor products.*

Proof. Let π be a representation of A for which π_e is a locally bounded representation of A_e. The representation π_e is inducible by Lemma 9.8. Hence π_e is the closure of the restriction of a locally bounded representation $\bar{\pi}_e^+$ of $\mathcal{A}_e^+$ by Corollary 10.2. The representation π_g of A_g for $g \in G$ is continuous with respect to the topology defining $\mathcal{A}_g^+$ and the graph topology on the domain of π_g because $\pi_g(a)^*\pi_g(a) = \pi_e(a^*a)$. Hence it extends uniquely to $\mathcal{A}_g^+$, and this gives a representation $\bar{\pi}^+$ of $\bigoplus \mathcal{A}_g^+$ such that π is the closure of $\bar{\pi}^+ \circ j$. It is easy to see that this equivalence between the locally bounded representations of A and the representations of $\bigoplus_{g\in G} \mathcal{A}_g^+$ that are locally bounded on $\mathcal{A}_e^+$ is compatible with isometric intertwiners and interior tensor products. $\square$

We will explore the consequences of this in the case of commutative A_e in §11. In that case, boundedness is automatically compatible with induction, and the pro-C*-algebraic Fell bundle $\mathcal{A}_e^+$ gives rise to a twisted groupoid with object space $\hat{A}_e^+$. Thus the C*-hull produced by the Induction Theorem 9.26 is a twisted groupoid C*-algebra when A_e is commutative and the integrable representations of A_e are locally bounded.

Here we briefly consider the situation of Theorem 7.16 where $\mathrm{C}_0(\mathcal{A}_e^+)$ is dense in $\mathcal{A}_e^+$ and provides a C*-hull for the class of locally bounded representations. Then we define

$$\mathrm{C}_0(\mathcal{A}_g^+) := \{a \in A_g \mid a^*a \in \mathrm{C}_0(\mathcal{A}_e^+)\}.$$

That is, $a \in \mathrm{C}_0(\mathcal{A}_g^+)$ if and only if for all $\varepsilon > 0$ there is $q \in \mathcal{N}(A_e)^+$ such that $\|a^*a\|_{\mathfrak{p}} < \varepsilon$ for all $\mathfrak{p} \in \mathrm{Prim}(\mathcal{A}_e^+) \setminus \mathrm{Prim}(\mathcal{A}_e^+)_q$. Since the involutions $A_g \to A_{g^{-1}}$ and $A_{g^{-1}} \to A_g$ are both continuous, they are homeomorphisms. Thus $a \in \mathrm{C}_0(\mathcal{A}_g^+)$ if and only if $aa^* \in \mathrm{C}_0(\mathcal{A}_e^+)$. The proof of Lemma 7.15 shows that $\mathcal{A}_g^+ \cdot \mathrm{C}_c(\mathcal{A}_e^+)$ and $\mathrm{C}_c(\mathcal{A}_e^+) \cdot \mathcal{A}_g^+$ are dense in $\mathrm{C}_0(\mathcal{A}_g^+)$.

Theorem 10.6. *Assume that boundedness is compatible with induction from A_e to A and that $\mathrm{C}_0(\mathcal{A}_e^+)$ is dense in $\mathcal{A}_e^+$. Then $\mathrm{C}_0(\mathcal{A}_g^+)_{g\in G}$ is a Fell bundle over G whose section* C*-*algebra is a* C*-*hull for the class of all representations of A that restrict to a locally bounded representation of A_e.*

Proof. The assumption that boundedness is compatible with induction allows us to build the pro-C*-algebraic Fell bundle $(\mathcal{A}_g^+)_{g\in G}$. Call a representation of $A = \bigoplus_{g\in G} A_g$ or $\mathcal{A}^+ := \bigoplus_{g\in G} \mathcal{A}_g^+$ integrable if the restriction to the unit fibre A_e or $\mathcal{A}_e^+$ is locally bounded, respectively. These classes of integrable representations are equivalent by Proposition 10.5.

Since $\mathrm{C}_0(\mathcal{A}_e^+)$ is dense in $\mathcal{A}_e^+$, it is a C*-hull for the locally bounded representations of $\mathcal{A}_e^+$ by Theorem 7.17. Equivalently, it is a C*-hull for the inducible, locally bounded representations of A_e. Let $\mathrm{C}_0(\mathcal{A}^+) := \bigoplus_{g\in G} \mathrm{C}_0(\mathcal{A}_g^+)$. Representations of $\mathrm{C}_0(\mathcal{A}^+)$ are equivalent to representations of the Fell bundle $\mathrm{C}_0(\mathcal{A}_g^+)$. Thus we must prove that the class of all representations of $\mathrm{C}_0(\mathcal{A}^+)$ is equivalent to the class of integrable representations of $\mathcal{A}^+$. More precisely, the equivalence maps a representation ϱ of $\mathrm{C}_0(\mathcal{A}^+)$ on a Hilbert module $\mathcal{E}$ to the representation π of $\mathcal{A}^+$ with the core $\varrho(\mathrm{C}_c(\mathcal{A}_e^+))\mathcal{E}$ and $\pi(a)\varrho(b)\xi := \varrho(a\cdot b)\xi$ for all $a \in \mathcal{A}^+$, $b \in \mathrm{C_c}(\mathcal{A}^+_e)$, $\xi \in \mathcal{E}$; here $a \cdot b$ is the product in $\mathcal{A}^+$, which belongs to $\mathrm{C}_0(\mathcal{A}^+)$ if $b \in \mathrm{C_c}(\mathcal{A}^+_e)$.

In the converse direction, we may simply restrict a locally bounded representation of $\mathcal{A}^+$ to the *-subalgebra $\mathrm{C}_0(\mathcal{A}^+)$. This restriction is nondegenerate because $\mathrm{C}_0(\mathcal{A}^+_e) \subseteq \mathrm{C}_0(\mathcal{A}^+)$ acts nondegenerately in any integrable representation of $\mathcal{A}^+$: this is part of the equivalence between representations of $\mathrm{C}_0(\mathcal{A}^+_e)$ and locally bounded representations of $\mathcal{A}^+_e$ in Theorem 7.17. We claim that the maps from representations of $\mathrm{C}_0(\mathcal{A}^+)$ to integrable representations of $\mathcal{A}^+$ and back are inverse to each other.

Let π be an integrable representation of $\mathcal{A}^+$ on a Hilbert module $\mathcal{E}$. The representations π and $\pi|_{\mathcal{A}^+_e}$ have the same domain by Lemma 9.2. Since $\pi|_{\mathcal{A}^+_e}$ is locally bounded, $\pi(\mathrm{C_c}(\mathcal{A}^+_e))\mathfrak{E}$ is a core for $\pi|_{\mathcal{A}^+_e}$. Since $\mathrm{C_c}(\mathcal{A}^+_e)\cdot\mathcal{A}^+_g = \mathcal{A}^+_g\cdot\mathrm{C_c}(\mathcal{A}^+_e)$ for all $g \in G$, this subspace is $\pi(\mathcal{A}^+)$-invariant and thus a core for π. The representation ϱ of $\mathrm{C}_0(\mathcal{A}^+)$ is the closure of the restriction of π to $\mathrm{C}_0(\mathcal{A}^+) \subseteq \mathcal{A}^+$. By definition, the representation of $\mathcal{A}^+$ has the core $\varrho(\mathrm{C_c}(\mathcal{A}^+_e))\mathcal{E}$ and acts there by $\pi'(a)\varrho(b)\xi = \varrho(a{\cdot}b)\xi$. The subspace $\varrho(\mathrm{C_c}(\mathcal{A}^+_e))\mathfrak{E}$ is a core for this representation because the map $\xi \mapsto \pi'(a)\varrho(b)\xi$ is continuous in the norm topology on $\mathcal{E}$ and $\mathfrak{E}$ is dense in $\mathcal{E}$. If $\xi \in \mathfrak{E}$, then $\varrho(b)\xi = \pi(b)\xi$ and hence $\pi'(a)\pi(b)\xi = \pi(a)\pi(b)\xi$ for all $a \in \mathcal{A}^+$, $b \in \mathrm{C_c}(\mathcal{A}^+_e)$, $\xi \in \mathfrak{E}$. This implies $\pi = \pi'$, as desired.

Now start with a representation ϱ of $\mathrm{C}_0(\mathcal{A}^+)$. Let π be the associated integrable representation of $\mathcal{A}^+$. It has the core $\varrho(\mathrm{C_c}(\mathcal{A}^+_e))\mathcal{E}$ and acts there by $\pi(a)\varrho(b)\xi = \varrho(a \cdot b)\xi$ for all $a \in \mathcal{A}^+$, $b \in \mathrm{C_c}(\mathcal{A}^+_e)$, $\xi \in \mathcal{E}$. In particular, if $a \in \mathrm{C}_0(\mathcal{A}^+)$, then $\pi(a)\varrho(b)\xi = \varrho(a \cdot b)\xi = \varrho(a)\varrho(b)\xi$. Since $\mathrm{C}_0(\mathcal{A}^+_e) \cdot \mathrm{C}_0(\mathcal{A}^+_g)$ is dense in $\mathrm{C}_0(\mathcal{A}^+_g)$ for all $g \in G$, the restriction of ϱ to $\mathrm{C}_0(\mathcal{A}^+_e)$ remains nondegenerate. Therefore, the set of $\varrho(b)\xi$ for $b \in \mathrm{C_c}(\mathcal{A}^+_e)$, $\xi \in \mathcal{E}$ is dense in $\mathcal{E}$. Hence ϱ is the restriction of π to $\mathrm{C}_0(\mathcal{A}^+) \subseteq \mathcal{A}^+$, as desired. □

The proof of Theorem 10.6 does not use the constructions in Section 9 and so provides an alternative proof of the Induction Theorem in case the chosen class of integrable representations of A_e is the class of all locally bounded representations.

11 Fell bundles with commutative unit fibre

In this section, we apply the Induction Theorem in the case where A_e and the chosen C*-hull B_e are commutative. This is the only case considered in [27]. Extra assumptions in [27] ensure that the C*-hull for the integrable representations of A is the crossed product for a partial action of G on the space $\hat{A}^+_e \subseteq \hat{A}_e$ of positive characters. Without these assumptions, we shall get a "twisted" crossed product for a partial action.

So let G be a discrete group and $A = \bigoplus_{g \in G} A_g$ a G-graded *-algebra such that A_e is commutative. We have already classified the possible commutative C*-hulls for A_e in §8. In particular, all commutative weak C*-hulls are already C*-hulls by Theorem 8.2, and they correspond to injective, continuous maps from locally compact spaces to the spectrum $\hat{A}_e$ of A_e.

Explicitly, let X be a locally compact space and let $j\colon X \to \hat{A}_e$ be an injective, continuous map. Let $B_e = \mathrm{C}_0(X)$ and define a representation of A_e on B_e with domain $\mathrm{C_c}(X)$ by $(a \cdot f)(x) = \hat{a}(j(x)) \cdot f(x)$ for all $a \in A_e$, $f \in \mathrm{C_c}(X)$, $x \in X$, where $\hat{a}(\chi) = \chi(a)$ for $\chi \in \hat{A}_e$. Let μ_e be the closure of this representation of A_e on B_e. The C*-algebra B_e with the universal representation μ_e is a C*-hull for a class $\mathrm{Rep}_{\mathrm{int}}(A_e, X)$ of representations of A_e by Theorem 8.2, and any commutative C*-hull is of this form.

Let $\mathrm{Rep}_{\mathrm{int}}(A, X)$ be the class of representations of A that restrict to a representation in $\mathrm{Rep}_{\mathrm{int}}(A_e, X)$ on A_e, as in Definition 9.3. If $\mathrm{Rep}_{\mathrm{int}}(A_e, X)$ is compatible with induction to A as in Definition 9.18, then Theorem 9.26 gives a Fell bundle whose section C*-algebra is a C*-hull for $\mathrm{Rep}_{\mathrm{int}}(A, X)$. We are going to characterise exactly when this happens and describe the C*-hull for $\mathrm{Rep}_{\mathrm{int}}(A, X)$ as a twisted groupoid C*-algebra.

Any representation of A_e on a commutative C*-algebra is locally bounded by Proposition 8.1. Hence the constructions in §10 specialise to our commutative case. Actually, we shall make these results more explicit through independent proofs. First we describe the C*-hull B_e^+ for the inducible representations in $\mathrm{Rep}_{\mathrm{int}}(A_e, X)$ as in Proposition 10.1:

Lemma 11.1. *Call a character $\chi \in \hat{A}_e$* positive *if $\chi(a^*a) \geq 0$ for all $a \in A_g$ and all $g \in G$. These form a closed subset $\hat{A}_e^+$ of $\hat{A}_e$, and $B_e^+ = \mathrm{C}_0\bigl(j^{-1}(\hat{A}_e^+)\bigr)$.*

Proof. The positive characters form a closed subset in $\hat{A}_e$ by definition of the topology on $\hat{A}_e$. We have constructed B_e^+ in Proposition 9.12 as a quotient of B_e, such that a representation is inducible if and only if it factors through B_e^+. Thus B_e^+ corresponds to a certain closed subset of $\hat{A}_e$. Its points are the inducible characters of A_e. Let χ be a character. Any vector in $A_g \otimes_{A_e,\chi} \mathbb{C}$ is of the form $a \otimes 1$ for some $a \in A_g$, that is, there is no need to take linear combinations. Hence the sesquilinear form on $A_g \otimes_{A_e,\chi} \mathbb{C}$ for all $g \in G$ is positive semidefinite if and only if $\chi(a^*a) \geq 0$ for all $a \in A_g$ and all $g \in G$, that is, χ is positive. Thus B_e^+ is the quotient corresponding to those $x \in \hat{A}_e$ for which $j(x) \in \hat{A}_e$ is positive. $\square$

Theorem 11.2. *Let $g \in G$ and $\chi \in \hat{A}_e^+$. Then $\dim A_g \otimes_{A_e,\chi} \mathbb{C} \leq 1$. The set*

$$\mathcal{D}_{g^{-1}} := \{\chi \in \hat{A}_e^+ \mid \dim A_g \otimes_{A_e,\chi} \mathbb{C} = 1\}$$

is relatively open in $\hat{A}_e^+$. The left A_e-module structure on $A_g \otimes_{A_e,\chi} \mathbb{C} \cong \mathbb{C}$ for $\chi \in \mathcal{D}_{g^{-1}}$ is by a character $\vartheta_g(\chi)$ that belongs to $\mathcal{D}_g$. The map ϑ_g is a homeomorphism from $\mathcal{D}_{g^{-1}}$ onto $\mathcal{D}_g$, and these maps form a partial action *of G on $\hat{A}_e^+$, that is, $\vartheta_e = \mathrm{id}_{\hat{A}_e^+}$ and $\vartheta_g \circ \vartheta_h \subseteq \vartheta_{gh}$ for all $g, h \in G$.*

Proof. As in the proof of Lemma 11.1, $A_g \otimes_{A_e,\chi} \mathbb{C}$ is the Hausdorff completion of A_g in the norm coming from the inner product $\langle a_1, a_2\rangle := \chi(a_1^*a_2)$. We write $\lambda \cdot a$ for $a \otimes \lambda$ for $a \in A_g$, $\lambda \in \mathbb{C}$ throughout this proof, and we write $a \equiv b$ if $a, b \in A$ have the same image in $A_g \otimes_{A_e,\chi} \mathbb{C}$. Let $a, b \in A_g$ satisfy $\chi(a^*a) \neq 0$ and $\chi(b^*b) \neq 0$. We must show that a and b are parallel in $A_g \otimes_{A_e,\chi} \mathbb{C}$.

The following computation makes [7, Footnote 3] explicit:

$$(a^*ab^*b)^2 = a^*ab^*(ba^*)(ab^*)b = a^*a(b^*a)(b^*ba^*b) = a^*ab^*ba^*bb^*a$$

because A_e is commutative and the terms in parentheses belong to A_e. Hence

$$\chi(a^*a)^2\chi(b^*b)^2 = \chi(a^*a)\chi(b^*b)\chi(a^*b)\chi(b^*a).$$

Since $\chi(a^*a) \neq 0$ and $\chi(b^*b) \neq 0$, this implies

$$\chi(a^*a)\chi(b^*b) = \chi(a^*b)\chi(b^*a) = |\chi(a^*b)|^2 \neq 0. \tag{11.3}$$

The inner product on $A_g \otimes_{A_e,\chi} \mathbb{C}$ annihilates $a\cdot c \otimes 1 - a \otimes \chi(c)$, which we write as $a\cdot c - \chi(c)a$, for all $a \in A_g$, $c \in A_e$. Hence

$$\begin{aligned} a = \frac{\chi(a^*b)\chi(b^*a)}{\chi(a^*a)\chi(b^*b)}a \equiv \frac{aa^*bb^*a}{\chi(a^*a)\chi(b^*b)} &= \frac{bb^*aa^*a}{\chi(a^*a)\chi(b^*b)} \\ &= \frac{\chi(b^*a)\chi(a^*a)}{\chi(a^*a)\chi(b^*b)}b = \frac{\chi(b^*a)}{\chi(b^*b)}b. \end{aligned} \tag{11.4}$$

Thus all non-zero $a, b \in A_g \otimes_{A_e,\chi} \mathbb{C}$ are parallel, that is, $\dim A_g \otimes_{A_e,\chi} \mathbb{C} \leq 1$. The space $A_g \otimes_{A_e,\chi} \mathbb{C}$ is non-zero if and only if there is $a \in A_g$ with $\chi(a^*a) \neq 0$. Thus

$$\mathcal{D}_{g^{-1}} = \{\chi \in \hat{A}_e^+ \mid \chi(a^*a) \neq 0 \text{ for some } a \in A_g\}. \tag{11.5}$$

The latter set is relatively open in $\hat{A}_e^+$.

Let $\chi \in \mathcal{D}_{g^{-1}}$. Then $\dim A_g \otimes_{A_e,\chi} \mathbb{C} = 1$. Hence the representation of A_e on it is by a character, which we denote by $\vartheta_g(\chi)$. This character is an inducible representation by Lemma 9.8, and hence positive by Lemma 11.1. There is $b \in A_g$ with $\chi(b^*b) > 0$. If $a \in A_e$, then (11.4) implies $ab \equiv \frac{\chi(b^*ab)}{\chi(b^*b)}b$. Thus

$$\vartheta_g(\chi)(a) = \frac{\chi(b^*ab)}{\chi(b^*b)} \tag{11.6}$$

for all $a \in A_e$. Hence $\vartheta_g(\chi)\big((b^*)^*b^*\big) \neq 0$, so that $\vartheta_g(\chi) \in \mathcal{D}_g$ by (11.5). Thus ϑ_g maps $\mathcal{D}_{g^{-1}}$ to $\mathcal{D}_g$. Equation (11.6) also implies that the map ϑ_g is continuous on the open set of characters in A_e^+ with $\chi(b^*b) > 0$. Since these open sets for different $b \in A_g$ cover $\mathcal{D}_{g^{-1}}$, the map ϑ_g is continuous on all of $\mathcal{D}_{g^{-1}}$.

Let $g, h \in G$ and let $\chi \in \mathcal{D}_{h^{-1}}$ and $\vartheta_h(\chi) \in \mathcal{D}_{g^{-1}}$. Then there is $b_h \in A_h$ with $\chi(b_h^*b_h) > 0$, and $b_g \in A_g$ with $\vartheta_h(\chi)(b_g^*b_g) > 0$. Thus $\chi(b_h^*b_g^*b_gb_h) = \chi(b_h^*b_h)\cdot\vartheta_h(\chi)(b_g^*b_g) > 0$, and so (11.6) for $b = b_gb_h \in A_{gh}$ describes ϑ_{gh}. Hence

$$\vartheta_{gh}(\chi)(a) = \frac{\chi(b_h^*b_g^*ab_gb_h)}{\chi(b_h^*b_g^*b_gb_h)} = \frac{\vartheta_h(\chi)(b_g^*ab_g)}{\vartheta_h(\chi)(b_g^*b_g)} = \vartheta_g\big(\vartheta_h(\chi)\big)(a).$$

Thus $\vartheta_{gh} \subseteq \vartheta_g\vartheta_h$ for all $g, h \in G$. In addition, $\vartheta_e = \mathrm{id}_{\hat{A}_e^+}$. So the maps ϑ_g form a partial action of G on $\hat{A}_e^+$, see [8]. In particular, ϑ_g is a homeomorphism from $\mathcal{D}_{g^{-1}}$ onto $\mathcal{D}_g$ with inverse $\vartheta_{g^{-1}}$. □

In the examples considered in [7, 27], the space $\hat{A}_e^+$ is locally compact and the C*-hull for the integrable representations of A is the crossed product for the partial action of G on $\hat{A}_e^+$ described above. In general, however, certain twists are possible. The partial action of G on $\hat{A}_e^+$ may be encoded in a transformation groupoid $G \ltimes \hat{A}_e^+$, which has object space $\hat{A}_e^+$, arrow space $\bigsqcup_{g\in G} \mathcal{D}_{g^{-1}}$ with the disjoint union topology, range and source maps $s(g,\chi) := \chi$, $r(g,\chi) := \vartheta_g(\chi)$ for $g \in G$, $\chi \in \mathcal{D}_{g^{-1}}$, and multiplication $(g, \vartheta_h(\chi)) \cdot (h,\chi) := (g\cdot h, \chi)$ for all $g,h \in G$, $\chi \in \mathcal{D}_{h^{-1}} \cap \vartheta_h^{-1}(\mathcal{D}_{g^{-1}})$. The unit arrow on χ is $(1,\chi)$, and the inverse of (g,χ) is $(g^{-1}, \vartheta_g(\chi))$. This is an étale topological groupoid because r and s restrict to homeomorphisms on the open subsets $\mathcal{D}_{g^{-1}}$ of the arrow space. The object space $\hat{A}_e^+$ need not be locally compact.

We are going to construct another topological groupoid Σ that is a central extension of $G \ltimes \hat{A}_e^+$ by the circle group $\mathbb{T}$. That is, Σ comes with a canonical functor to $G \ltimes \hat{A}_e^+$ whose kernel is the group bundle $\hat{A}_e^+ \times \mathbb{T}$. Such an extension is also called a *twisted groupoid* in [25, Section 4], following a definition by Kumjian [16]. A twisted groupoid with locally compact object space has a twisted groupoid C*-algebra. For a suitable injective continuous map $X \to \hat{A}_e^+$, we are going to identify the C*-hull of the X-integrable representations of A with the twisted groupoid C*-algebra of the restriction of Σ to $j(X^+) \subseteq \hat{A}_e^+$.

A point in Σ is a triple $(g,\chi,[a])$, where $g \in G$, $\chi \in \mathcal{D}_{g^{-1}}$, and $[a]$ is a unit vector in the 1-dimensional Hilbert space $A_g \otimes_{A_e,\chi} \mathbb{C}$. We represent unit vectors in $A_g \otimes_{A_e,\chi} \mathbb{C}$ by elements $a \in A_g$ with $\chi(a^*a) = 1$; two elements $a,b \in A_g$ with $\chi(a^*a) = \chi(b^*b) = 1$ represent the same unit vector $[a] = [b]$ if and only if $\chi(a^*b) = 1$. We get the same set of equivalence classes if we allow $a \in A$ with $\chi(a^*a) > 0$ and set $[a] = [b]$ if $\chi(a^*b) > 0$: then $a_1 := \chi(a^*a)^{-1/2}a$ and $b_1 := \chi(b^*b)^{-1/2}b$ satisfy $[a] = [a_1]$, $[b] = [b_1]$, and $[a] = [b]$ if and only if $\chi(a_1^*b_1) = 1$ by (11.3). The circle group $\mathbb{T}$ acts on Σ by multiplication: $\lambda \cdot (g,\chi,[a]) := (g,\chi,[\lambda a])$. The orbit space projection for this circle action is the coordinate projection $F\colon \Sigma \twoheadrightarrow G \ltimes \hat{A}_e^+$, $(g,\chi,[a]) \mapsto (g,\chi)$. Next we equip Σ with a topology so that this coordinate projection is a locally trivial principal $\mathbb{T}$-bundle.

For $a \in A$, let $U_a := \{\chi \in \hat{A}_e^+ \mid \chi(a^*a) \neq 0\}$. This is an open subset in $\hat{A}_e^+$, and $\chi(a^*a) > 0$ if $\chi \in U_a$ because χ is positive. The map $\sigma_a\colon \{g\} \times U_a \to \Sigma$, $(g,\chi) \mapsto (g,\chi,[a])$, for $a \in A_g$ is a local section for the coordinate projection F. If $a,b \in A_g$, and $\chi \in U_a \cap U_b$, then

$$[a] = \left[\frac{\chi(b^*a)}{\chi(a^*a)^{1/2}\chi(b^*b)^{1/2}} b\right],$$

by (11.4). Since the functions sending χ to $\chi(b^*a)$, $\chi(a^*a)$ and $\chi(b^*b)$ are continuous on A_e, the two trivialisations induce the same topology on the restriction of Σ to $\{g\} \times (U_a \cap U_b)$. For any $\chi \in \mathcal{D}_{g^{-1}}$, there is $a \in A_g$ with $\chi(a^*a) > 0$. Thus the open subsets U_a cover $\mathcal{D}_{g^{-1}}$. Consequently, there is a unique topology on Σ that makes the local sections σ_a for all $a \in A_g$ continuous, and this topology turns Σ into a locally trivial $\mathbb{T}$-bundle over $G \ltimes \hat{A}_e^+$.

We define a groupoid with object space A_e^+, arrow space Σ, and

$$r(g,\chi,[a]) := \vartheta_g(\chi), \qquad s(g,\chi,[a]) := \chi,$$
$$(g,[\vartheta_h(\chi)],[a])\cdot(h,\chi,[b]) := (g\cdot h,\chi,[a\cdot b]);$$

we must show that this multiplication is well defined. We have $ab \in A_{gh}$ and

$$\chi(b^*a^*ab) = \vartheta_h(\chi)(a^*a)\cdot\chi(b^*b) \neq 0$$

by (11.6), so $(g\cdot h,\chi,[a\cdot b]) \in \Sigma$. If $\chi(b^*b_1) > 0$ and $\vartheta_h(\chi)(a^*a_1) > 0$, then $\chi(b^*a^*a_1b_1) > 0$ by computations as in the proof of Theorem 11.2. Hence the multiplication is well defined. It is clearly associative. The unit arrow on χ is $1_\chi := (1,\chi,[1])$, and $(g,\chi,[a])^{-1} = (g^{-1},\vartheta_g(\chi),[a^*])$. The multiplication, unit map and inversion are continuous and the range and source maps are open surjections (even locally trivial). So Σ is a topological groupoid.
The identity map on objects and the coordinate projection $F\colon \Sigma \to G \ltimes \hat{A}_e^+$ on arrows form a functor, which is a locally trivial, open surjection on arrows. The kernel of F consists of those $(g,\chi,[a]) \in \Sigma$ for which $F(g,\chi,[a])$ is a unit arrow in $G \ltimes \hat{A}_e^+$. Then $g = 1$, and $a \in A_e$ is equivalent to $[a] = [\chi(a)\cdot 1]$ because $\chi(a^*\chi(a)1) > 0$. The map $(g,\chi,[a]) \mapsto (\chi,\chi(a))$ is an isomorphism of topological groupoids from the kernel of F onto the trivial group bundle $\hat{A}_e^+ \times \mathbb{T}$. Thus we have an extension of topological groupoids

$$\hat{A}_e^+ \times \mathbb{T} \rightarrowtail \Sigma \twoheadrightarrow G \ltimes \hat{A}_e^+.$$

The three groupoids above are clearly Hausdorff.
To construct C*-algebras, we need groupoids with a locally compact object space. Therefore, we replace $\hat{A}_e$ by a locally compact space X with an injective, continuous map $j\colon X \to \hat{A}_e$. Then $B_e = \mathrm{C}_0(X)$ is a C*-hull for a class $\mathrm{Rep}_{\mathrm{int}}(A_e, X)$ of representations of A_e. By Lemma 11.1, the C*-hull for the class of X-integrable, inducible representations of A_e is $B_e^+ = \mathrm{C}(X^+)$ with $X^+ := j^{-1}(\hat{A}_e^+) \subseteq X$.

Proposition 11.7. *Let $j\colon X \to \hat{A}_e$ be an injective, continuous map. The class $\mathrm{Rep}_{\mathrm{int}}(A, X)$ is compatible with induction if and only if $j(X^+) \subseteq \hat{A}_e^+$ is invariant under the partial maps ϑ_g in Theorem 11.2 and the resulting partial maps on X^+ are continuous in the topology of X^+. We briefly say that the partial action of G on $\hat{A}_e^+$* restricts *to X^+.*

Proof. By Proposition 9.19, it suffices to check that the induced representation of A_e on $A_g \otimes_{A_e} B_e^+$ is X-integrable for $g \in G$ if and only if the partial map $\vartheta_g \circ j$ on X factors through j and the resulting partial map $j^{-1} \circ \vartheta_g \circ j$ on X is again continuous. View the Hilbert module $A_g \otimes_{A_e} B_e^+$ as a continuous field of Hilbert spaces over X^+. The fibres of this field have dimension at most 1 by Theorem 11.2, and the set where the fibre is non-zero is the open subset $j^{-1}(\mathcal{D}_{g^{-1}})$. Hence $\mathbb{K}(A_g \otimes_{A_e} B_e^+) \cong \mathrm{C}_0(j^{-1}(\mathcal{D}_{g^{-1}}))$. The representation of A_e on $A_g \otimes_{A_e} B_e^+$ is equivalent to a representation on $\mathbb{K}(A_g \otimes_{A_e} B_e^+)$ by

Proposition 3.13. This is equivalent to a continuous map $j^{-1}(\mathcal{D}_{g^{-1}}) \to \hat{A}_e$ by Proposition 8.1. This map is $\vartheta_g \circ j$ by a fibrewise computation. Hence the induced representation of A_e on $A_g \otimes_{A_e} B_e^+$ is X-integrable if and only if $\vartheta_g \circ j$ has values in $j(X)$ and the partial maps $j^{-1} \circ \vartheta_g \circ j$ on X are continuous. □

From now on, we assume that the partial action of G on $\hat{A}_e^+$ restricts to X^+. By Proposition 11.7, this assumption is necessary and sufficient for X-integrability to be compatible with induction. The "restriction" of the partial action on $\hat{A}_e^+$ to X^+ is a partial action of G on X^+ by partial homeomorphisms. Its transformation groupoid $G \ltimes X^+$ is constructed like $G \ltimes \hat{A}_e^+$. Its set of arrows is the subset of $G \ltimes \hat{A}_e^+$ of arrows with range and/or source in $j(X^+)$, and the topology on the arrow space is the unique one that makes the inclusion $G \ltimes X^+ \to G \ltimes \hat{A}_e^+$ and the range and source maps $G \ltimes X^+ \to X^+$ continuous. There is also a unique topology on the restriction Σ_X of Σ to $j(X^+)$ so that there is an extension of topological groupoids

$$X^+ \times \mathbb{T} \rightarrowtail \Sigma_X \twoheadrightarrow G \ltimes X^+.$$

Since X^+ is locally compact, the groupoids in this extension are locally compact, Hausdorff groupoids. Since $G \ltimes X^+$ is étale, it carries a canonical Haar system, namely, the family of counting measures. There is also a unique normalised Haar system on $X^+ \times \mathbb{T}$. These produce a unique Haar system on Σ_X by [5, Theorem 5.1], so that the groupoid C*-algebra $\mathrm{C}^*(\Sigma_X)$ is defined. The *twisted groupoid* C*-*algebra* $\mathrm{C}^*(G \ltimes X^+, \Sigma_X)$ of $G \ltimes X^+$ with respect to the *twist* Σ_X is defined in [24]. It is related to the groupoid C*-algebra of Σ_X in [5, Corollary 7.2].

THEOREM 11.8. *Let G be a discrete group and let A be a G-graded *-algebra with commutative A_e. Let $j\colon X \to \hat{A}_e$ be an injective, continuous map, such that the partial action of G on $\hat{A}_e^+$ in Theorem* 11.2 *restricts to X^+ as in Proposition* 11.7. *Then* $\mathrm{C}^*(G \ltimes X^+, \Sigma_X)$ *is a* C*-*hull for* $\mathrm{Rep}_{\mathrm{int}}(A, X)$.

Proof. The C*-algebra $\mathrm{C}^*(G \ltimes X^+, \Sigma_X)$ may be defined as the full section C*-algebra of a certain Fell line bundle over the étale, locally compact groupoid $G \ltimes X^+$. The Fell line bundle involves the space of sections of the Hermitian complex line bundle $\mathcal{L} := \Sigma_X \times_{\mathbb{T}} \mathbb{C}$ associated to the principal $\mathbb{T}$-bundle $\Sigma_X \twoheadrightarrow G \ltimes X^+$ and the multiplication maps $\mathcal{L}_g \times \mathcal{L}_h \to \mathcal{L}_{gh}$ induced by the multiplication of Σ_X (see [5]). By construction, the Hilbert B_e^+-module $B_g^+ = A_g \otimes_{A_e} B_e^+$ is isomorphic to the continuous sections of this line bundle $\mathcal{L}$ over the subset $\{g\} \times \mathcal{D}^X_{g^{-1}}$ of $\{g\} \times X^+$: an element $a \otimes b$ is mapped to the continuous section that sends (g, x) for $x \in X$ with $j(x) \in \mathcal{D}_{g^{-1}}$ to $b(x) \cdot \chi(a^*a)^{1/2}[a]$. The multiplication in Σ_X is defined so that the multiplication maps $B_g^+ \otimes_{B_e^+} B_h^+ \to B_{gh}^+$ are exactly the multiplication maps in the Fell line bundle associated to Σ_X.

Thus the Fell bundle $(B_g^+)_{g\in G}$ constructed in Theorem 9.26 is isomorphic to the Fell bundle $(\beta_g)_{g\in G}$, where β_g is the space of C_0-sections of $\mathcal{L}$ over $\{g\} \times \mathcal{D}_{g^{-1}}$

and the multiplication and involution come from the Fell line bundle structure on $\mathcal{L}$ over the groupoid $G \ltimes X^+$. The full section C*-algebra of this Fell bundle is canonically isomorphic to the section C*-algebra of the corresponding Fell bundle over the groupoid $G \ltimes X^+$ by results of [3]. The small issue to check here is that it makes no difference whether we use C_0-sections or compactly supported continuous sections of $\mathcal{L}$ over $\{g\} \times \mathcal{D}_{g^{-1}}$. Both have the same C*-completion. This is a special case of general results about Fell bundles over étale locally compact groupoids. □

If $(B_g)_{g\in G}$ is any Fell bundle over G, then $\bigoplus_{g\in G} B_g$ is a *-algebra, to which we may apply our machinery although all its representations are bounded. Thus any Fell bundle over G may come up for some choice of the G-graded *-algebra A. Thus the section C*-algebra of a Fell bundle $(B_g)_{g\in G}$ with commutative unit fibre is always a twisted groupoid C*-algebras of a twist of an étale groupoid, namely, the transformation groupoid of a certain partial action on the spectrum of the unit fibre associated to the Fell bundle. This result is already known, even for Fell bundles over inverse semigroups with commutative unit fibre, see [3].

If $\Sigma_X \cong (G \ltimes X^+) \times \mathbb{T}$ as a groupoid, then $C^*(G \ltimes X^+, \Sigma_X) \cong C^*(G \ltimes X^+)$. This is the same as the crossed product for the partial action of G on X^+. This happens in all the examples in [7, 27]. The possible twists have two levels. First, Σ_X may be non-trivial as a principal circle bundle over $G \ltimes X^+$. Secondly, if it is trivial as a principal circle bundle, the multiplication may create a non-trivial twist.

The circle bundle $\Sigma_X \twoheadrightarrow G \ltimes X^+$ is trivial if and only if its restriction to $\{g\} \times \mathcal{D}_{g^{-1}}$ is trivial for each $g \in G$. For a circle bundle, this means that there is a nowhere vanishing section. For instance, if there is $a \in A_g$ that generates A_g as a right A_e-module, then $U_a = \mathcal{D}_{g^{-1}}$ and σ_a is a global trivialisation of $\Sigma_X|_{\{g\}\times\mathcal{D}_{g^{-1}}}$.

The complex line bundles over a space X are classified by the second cohomology group $H^2(X, \mathbb{Z})$. If $\mathcal{L}$ is a line bundle, then the spaces of C_0-sections of $\mathcal{L}^{\otimes n}$ for $n \in \mathbb{Z}$ form a Fell bundle over $\mathbb{Z}$, and the direct sum of these spaces of sections is a $\mathbb{Z}$-graded *-algebra such that the given line bundle $\mathcal{L}$ appears in the resulting twisted groupoid. If $H^2(X, \mathbb{Z}) \neq 0$, the space X is at least 2-dimensional. There are indeed non-trivial complex line bundles over all *compact* oriented 2-dimensional manifolds. The resulting *-algebra, however, has only *-representations by bounded operators if X is compact. Examples where unbounded operators appear must involve a non-trivial line bundle over a non-compact space. These first appear in dimension 3. It is easy to write down a $\mathbb{Z}$-graded *-algebra A where B_e^+ is, say, $S^2 \times \mathbb{R}$ and B_g^+ involves the Bott line bundle over S^2. These examples seem artificial, however.

Now assume that Σ_X is trivial as a principal circle bundle over $(G \ltimes X^+)^1$, that is, $\Sigma_X \cong (G \ltimes X^+)^1 \times \mathbb{T}$ as a $\mathbb{T}$-space. We may choose this homeomorphism to be the obvious one on the open subset $(1 \ltimes X^+) \times \mathbb{T}$ corresponding to $1 \in G$.

The multiplication must be of the form

$$(g_1, \vartheta_{g_2}(x), \lambda_1) \cdot (g_2, x, \lambda_2) = (g_1 \cdot g_2, x, \varphi(g_1, g_2, x) \cdot \lambda_1 \cdot \lambda_2)$$

for some continuous $\mathbb{T}$-valued function φ with $\varphi(1, g, x) = 1 = \varphi(g, 1, x)$ for all g, x; here φ is defined on the space of all triples $(g_1, g_2, x_2) \in G \times G \times X^+$ with $x_2 \in \mathcal{D}_{g_2^{-1}}$ and $\vartheta_{g_2}(x_2) \in \mathcal{D}_{g_1^{-1}}$; this space is homeomorphic to the space $(G \ltimes X^+)^2$ of pairs of composable arrows in $G \ltimes X^+$. The associativity of the multiplication in Σ_X is equivalent to the cocycle condition

$$\varphi(g_1, g_2 \cdot g_3, x) \cdot \varphi(g_2, g_3, x) = \varphi(g_1 \cdot g_2, g_3, x) \cdot \varphi(g_1, g_2, \vartheta_{g_3}(x)) \tag{11.9}$$

for all $g_1, g_2, g_3 \in G$, $x \in X^+$ for which $\vartheta_{g_3}(x)$, $\vartheta_{g_2} \circ \vartheta_{g_3}(x)$, and $\vartheta_{g_1} \circ \vartheta_{g_2} \circ \vartheta_{g_3}(x)$ are defined. A different trivialisation of the circle bundle $\Sigma_X \twoheadrightarrow (G \ltimes X^+)^1$ modifies φ by the coboundary

$$\partial\psi(g_1, g_2, x) := \psi(g_2, x)\psi(g_1 \cdot g_2, x)^{-1}\psi(g_1, \vartheta_{g_2}(x)) \tag{11.10}$$

of a continuous function $\psi\colon (G \ltimes X^+)^1 \to \mathbb{T}$ normalised by $\psi(1, x) = 0$ for all $x \in X^+$. Thus isomorphism classes of twists of $G \ltimes X^+$ are in bijection with the groupoid cohomology $H^2(G \ltimes X, \mathbb{T})$, that is, the quotient of the group of continuous maps $\varphi\colon (G \ltimes X^+)^2 \to \mathbb{T}$ satisfying (11.9) by the group of 2-coboundaries $\partial\psi$ of continuous 1-cochains $\psi\colon (G \ltimes X^+)^1 \to \mathbb{T}$, where $\partial\psi$ is defined in (11.10). In the easiest case, the function φ above does not depend on x. Then $\varphi\colon G \times G \to \mathbb{T}$ is a normalised 2-cocycle on G in the usual sense. These cocycles appear, for instance, in the classification of projective representations of the group G. This is related to the twists above because the Hilbert space representations of the twisted group algebra for a 2-cocycle $\varphi\colon G \times G \to \mathbb{T}$ are exactly the projective representations $\pi\colon G \to \mathrm{U}(\mathcal{H})$ with $\pi(g)\pi(h) = \varphi(g, h)\pi(gh)$ for all $g, h \in G$.

The group $\mathbb{Z}$ has no nontrivial 2-cocycles. They do appear, however, for the group $\mathbb{Z}^2$. A well known example is the noncommutative torus. Its usual gauge action corresponds to a $\mathbb{Z}^2$-grading, where $U^n V^m$ for the canonical generators U, V has degree $(n, m) \in \mathbb{Z}^2$. In this case, $A_e = \mathbb{C} = B_e = B_e^+$, and $\hat{A}_e^+$ has only one point. The transformation groupoid $G \ltimes \hat{A}_e^+$ is simply $G = \mathbb{Z}^2$. This is discrete, so Σ is always trivial as a principal circle bundle. Thus the only non-trivial aspect of Σ is a 2-cocycle $\varphi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$. The cohomology group $H^2(\mathbb{Z}^2, \mathbb{T})$ is isomorphic to $\mathbb{T}$, and the resulting twisted group algebras of $\mathbb{Z}^2$ are exactly the noncommutative tori.

Proposition 11.11. *If there are subsets $S_g \subseteq A_g$ such that S_g generates A_g as a right A_e-module, $S_g \cdot S_h \subseteq S_{gh}$, and $\chi(a^*b) \geq 0$ for all $a, b \in S_g$, $g \in G$, $\chi \in j(X) \subseteq \hat{A}_e$, then the twist Σ_X is trivial and so the* C^**-hull of A is* $\mathrm{C}^*(G \ltimes X^+)$.

Proof. If $\chi \in \mathcal{D}_{g^{-1}}$, then there is $b \in A_g$ with $\chi(b^*b) \neq 0$. Since S_g generates A_g as a right A_e-module, we may write $b = \sum_{i=1}^n a_i \cdot c_i$ with $a_i \in S_g$, $c_i \in A_e$. Then

$$\chi(b^*b) = \sum_{i,j=1}^n \chi(c_i^*)\chi(c_j)\chi(a_i^* a_j).$$

Hence there are i, j with $\chi(a_i^* a_j) \neq 0$. Then $\chi(a_i^* a_i) \neq 0$ by (11.3). This shows that $\bigcup_{a \in S_g} U_a = \mathcal{D}_{g^{-1}}$. We have $(g, \chi, [a]) = (g, \chi, [b])$ for all $a, b \in S_g$, $\chi \in j(X) \cap U_a \cap U_b$ because $\chi(a^* b) \geq 0$ for all $\chi \in j(X)$. Hence the local sections σ_a of $\Sigma_X|_{\{g\} \times \mathcal{D}_{g^{-1}}}$ for $a \in S_g$ coincide on the intersections of their domains and thus combine to a global trivialisation. This trivialisation is multiplicative as well. $\square$

If $\hat{A}_e^+$ itself is locally compact, then we may take $X^+ = X = \hat{A}_e^+$ with the inclusion map j. Since $\hat{A}_e^+$ is closed in $\hat{A}_e$, this happens if $\hat{A}_e$ is locally compact.

Theorem 11.12. *Assume that $\hat{A}_e^+$ is locally compact in the topology τ_c. Call a representation of A integrable if its restriction to A_e is locally bounded. Let Σ be the twisted groupoid constructed above. Then $\mathrm{C}^*(G \ltimes (\hat{A}_e^+, \tau_c), \Sigma)$ is a C^*-hull for the integrable representations of A.*

Proof. If $X^+ = (\hat{A}_e^+, \tau_c)$, then integrability is compatible with induction by Proposition 11.7 because the construction of the topology τ_c is natural and compatible with restriction to open subsets. Theorem 11.8 shows that $\mathrm{C}^*(G \ltimes (\hat{A}_e^+, \tau_c), \Sigma)$ is a C^*-hull for the class of representations of A whose restriction to A_e is $\hat{A}_e^+$-integrable. The locally bounded representations of A_e are equivalent to the locally bounded representations of the pro-C^*-algebra $\mathrm{C}(\hat{A}_e, \tau_c)$ by Propositions 7.2 and 7.6. Restrictions of representations of A to A_e are automatically inducible by Lemma 9.8. By a pro-C^*-algebraic variant of Lemma 11.1, the inducible, locally bounded representations of A_e are equivalent to those representations of $\mathrm{C}(\hat{A}_e, \tau_c)$ that factor through the quotient $\mathrm{C}(\hat{A}_e^+, \tau_c)$. Since $\hat{A}_e^+$ is locally compact, $\mathrm{C}_0(\hat{A}_e^+, \tau_c)$ is dense in the pro-C^*-algebra $\mathrm{C}(\hat{A}_e^+, \tau_c)$. Hence $\mathrm{C}_0(\hat{A}_e^+, \tau_c)$ is a C^*-hull for the inducible, locally bounded representations of A_e by Theorem 7.17. $\square$

Assume that A_e is countably generated. Then the usual topology on $\hat{A}_e$ is metrisable and hence compactly generated, so that τ_c is the standard topology on $\hat{A}_e$. A representation of A_e is locally bounded if and only if all symmetric elements of A_e act by regular, self-adjoint operators by Theorem 8.7. Thus a representation π of A is integrable as in Theorem 11.12 if and only if $\overline{\pi(a)}$ is regular and self-adjoint for all $a \in A_e$ with $a = a^*$. This class of integrable representations has the C^*-hull $\mathrm{C}^*(G \ltimes \hat{A}_e^+, \Sigma)$ if $\hat{A}_e^+$ is locally compact.
In particular, if A_e is finitely generated, then $\hat{A}_e$ is mapped homeomorphically onto a closed subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ by evaluating characters on a finite set of symmetric generators. Thus $\hat{A}_e$ is locally compact. The discussion above gives:

Corollary 11.13. *Assume that A_e is finitely generated. Call a representation of A integrable if its restriction to A_e is locally bounded. Then $\hat{A}_e^+$ is locally compact and $\mathrm{C}^*(G \ltimes \hat{A}_e^+, \Sigma)$ for the twisted groupoid Σ constructed above is a C^*-hull for the integrable representations of A. Moreover, a representation π of A is integrable if and only if $\overline{\pi(a)}$ is regular and self-adjoint for all $a \in A_e$ with $a = a^*$.*

Corollary 11.13 covers all the examples considered in [7, 27], except for the enveloping algebra $\mathcal{W}$ of the Virasoro algebra that is studied in [27, §9.3].
The *-algebra $\mathcal{W}$ is $\mathbb{Z}$-graded. Its unit fibre $\mathcal{W}_0$ is noncommutative. The first step in the study of its representations in [27, §9.3] is to replace $\mathcal{W}$ by a certain $\mathbb{Z}$-graded quotient $A := \mathcal{W}/\mathcal{I}$, whose unit fibre $A_0 = \mathcal{W}_0/(\mathcal{I} \cap \mathcal{W}_0)$ is commutative by construction. The motivation is that all "integrable" representations of $\mathcal{W}$ factor through A. The main result in [27, §9.3] shows that the partial action of $\mathbb{Z}$ on $\hat{A}_e^+$ is free and that the disjoint union $Y := X_1 \sqcup X_2 \sqcup X_3$ of the three families of characters described in (61)–(63) of [27] is a fundamental domain, that is, it meets each orbit of the partial action exactly once. Each subset X_i is closed in $\hat{A}_e$ and locally compact and second countable in the subspace topology. Hence so is Y. Since $\mathbb{Z}$ acts by partial homeomorphisms and Y is a fundamental domain, there is a continuous bijection

$$X := \bigsqcup_{n \in \mathbb{Z}} (\mathcal{D}_{-n} \cap Y) \to \hat{A}_e^+, \qquad (n, y) \mapsto \vartheta_n(y).$$

Each $\mathcal{D}_{-n} \cap Y$ is an open subset of Y, so that X is locally compact. I have not checked whether this continuous bijection is a homeomorphism. If so, then $\hat{A}_e^+$ would be locally compact and the results in [27] for the Virasoro algebra would be contained in Theorem 11.12 after passing to the quotient $\mathcal{W}/\mathcal{I}$. If not, we would use the locally compact space X. The partial action of $\mathbb{Z}$ on $\hat{A}_e^+$ is clearly continuous on X as well, so that Theorem 11.8 applies.

12 Rieffel deformation

Let G be a discrete group. Given a normalised 2-cocycle on G, Rieffel deformation is a deformation functor that modifies the multiplication on a G-graded *-algebra by the 2-cocycle. There is a similar process for Fell bundles over G, which we may transfer to section C*-algebras. This is how Rieffel deformation is usually considered. The setting of graded algebras or Fell bundles is easier. We now define Rieffel deformation more precisely and show that it is compatible with the construction of C*-hulls in Theorem 9.26. This deformation process has also recently been treated in [23].
A *normalised* 2*-cocycle* on a group G is a function $\Lambda\colon G \times G \to \mathrm{U}(1)$ with $\Lambda(e, g) = 1 = \Lambda(g, e)$ for all $g \in G$ and

$$\Lambda(g, h \cdot k)\Lambda(h, k) = \Lambda(g \cdot h, k)\Lambda(g, h) \tag{12.1}$$

for all $g, h, k \in G$. Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra. Let A^Λ be the same G-graded vector space with the deformed multiplication and involution

$$\sum_{g \in G} a_g * \sum_{h \in G} b_h := \sum_{g, h \in G} \Lambda(g, h) a_g b_h, \qquad \big(\sum a_g\big)^\dagger := \sum \overline{\Lambda(g^{-1}, g)} a_g^*,$$

where $a_g, b_g \in A_g$ for all $g \in G$. We call A^Λ the *Rieffel deformation* of A with respect to Λ.

Lemma 12.2. *The deformed multiplication and involution on A^Λ give a G-graded *-algebra with $a * b = ab$ if $a \in A_e$ or $b \in A_e$, and $a^\dagger * b = a^* b$ for all $g \in G$, $a, b \in A_g$.*

Proof. The multiplication remains associative by the 2-cocycle condition (12.1). The normalisation of Λ and (12.1) for g, g^{-1}, g give $\Lambda(g, g^{-1}) = \Lambda(g^{-1}, g)$ for all $g \in G$. Thus

$$(a_g^\dagger)^\dagger = \overline{\Lambda(g, g^{-1})} \cdot \overline{\overline{\Lambda(g^{-1}, g)}}(a_g^*)^* = a_g$$

for $a_g \in A_g$. The normalisation condition and (12.1) for g, h, h^{-1} and gh, h^{-1}, g^{-1} for $g, h \in G$ give

$$\begin{aligned}\Lambda(gh, h^{-1})\Lambda(g, h) &= \Lambda(h, h^{-1}),\\ \Lambda(g, g^{-1})\Lambda(gh, h^{-1}) &= \Lambda(gh, h^{-1}g^{-1})\Lambda(h^{-1}, g^{-1}).\end{aligned}$$

Hence $\Lambda(g, g^{-1})\Lambda(h, h^{-1}) = \Lambda(g, h)\Lambda(gh, h^{-1}g^{-1})\Lambda(h^{-1}, g^{-1})$. This implies the condition $(a_g * b_h)^\dagger = b_h^\dagger * a_g^\dagger$ for $a_g \in A_g$, $b_h \in A_g$:

$$\begin{aligned}(a_g * b_h)^\dagger &= \overline{\Lambda(g, h)} \cdot \overline{\Lambda(gh, (gh)^{-1})} \cdot (a_g b_h)^*\\ &= \overline{\Lambda(g, g^{-1}) \cdot \Lambda(h, h^{-1})} \cdot \Lambda(h^{-1}, g^{-1}) \cdot b_h^* a_g^* = b_h^\dagger * a_g^\dagger.\end{aligned}$$

Thus the deformed multiplication and involution give a *-algebra. The formula $a^\dagger * b = a^* b$ for $g \in G$, $a, b \in A_g$ is trivial, and $a * b = ab$ if $a \in A_e$ or $b \in A_e$ follows from the normalisation of Λ. ☐

The same formulas work if $(B_g)_{g\in G}$ is a Fell bundle over G. Let $(B_g^\Lambda)_{g\in G}$ be the same Banach space bundle as B_g with the multiplication and involution $a_g * b_h := \Lambda(g, h)a_g b_h$ and $a_g^\dagger = \overline{\Lambda(g^{-1}, g)}a_g^*$ for $g, h \in G$, $a_g \in B_g$, $b_h \in B_h$. By Lemma 12.2, the deformation does not change ab for $a \in B_e$ or $b \in B_e$ and $a^* b$ and ab^* for $a, b \in B_g$. Hence $B_g^\Lambda = B_g$ as Hilbert B_e-bimodules, so that the positivity and completeness conditions for a Fell bundle are not affected by the deformation. We call $(B_g^\Lambda)_{g\in G}$ the *Rieffel deformation* of the Fell bundle $(B_g)_{g\in G}$ with respect to Λ.

For a C^*-algebra of the form $B = \mathrm{C}^*(B_g)$ for a Fell bundle $(B_g)_{g\in G}$ over G, we define its Rieffel deformation with respect to Λ as $B^\Lambda := \mathrm{C}^*(B_g^\Lambda)$ for the deformed Fell bundle.

If G is an Abelian group, then $\mathrm{C}^*(B_g)$ for a Fell bundle over G carries a canonical continuous action of $\hat{G}$, called the dual action. Conversely, any C^*-algebra with a continuous $\hat{G}$-action β is of the form $B = \mathrm{C}^*(B_g)$, where $(B_g)_{g\in G}$ is the spectral decomposition of the action,

$$B_g = \{b \in B \mid \beta_\chi(b) = \chi(g) \cdot b \text{ for all } \chi \in \hat{G}\}.$$

Thus Rieffel deformation takes a C^*-algebra with a continuous $\hat{G}$-action to another C^*-algebra with a continuous $\hat{G}$-action. This is how it is usually formulated. Since $\hat{G}$ is compact, there are no analytic difficulties with oscillatory integrals as in [26].

Theorem 12.3. *Let $A = \bigoplus_{g\in G} A_g$ be a G-graded *-algebra and let B_e be a* C^**-hull for a class of integrable representations of A_e. Assume that integrability is compatible with induction for A. Let Λ be a normalised 2-cocycle on G. Then integrability is also compatible with induction for A^Λ, and the* C^**-hull for the integrable representations of A^Λ is the Rieffel deformation with respect to Λ of the* C^**-hull for the integrable representations of A.*

Proof. The compatibility condition in Definition 9.18 is equivalent to the integrability of $A_g \otimes_{A_e} (\mathfrak{E}, \pi)$ for all $g \in G$, which only involves a single A_g with its A_e-bimodule structure and the A_e-valued inner product $\langle a, b\rangle = a^*b$ for $a, b \in A_g$. This is not changed by Rieffel deformation by Lemma 12.2. Hence A^Λ inherits the compatibility condition from A, and Theorem 9.26 applies to both A and A^Λ.
The Hilbert B_e^+-bimodule B_g^+ depends only on A_g with the extra structure above and the universal inducible, integrable representation $(\mathfrak{B}_e^+, \mu_e^+)$ of A_e by Remark 9.27. Since none of this is changed by Rieffel deformation, the Fell bundle obtained from A^Λ has the same fibres $(B_g^+)^\Lambda$ as B_g^+. Rieffel deformation changes the multiplication maps $A_g \times A_h \to A_{gh}$ and the involution $A_g \to A_{g^{-1}}$ for fixed $g, h \in G$ only by a scalar. Inspecting the construction above, we see that the multiplication maps $B_g^+ \times B_h^+ \to B_{gh}^+$ and the involution $B_g^+ \to B_{g^{-1}}^+$ in the Fell bundle are changed by exactly the same scalars. Hence the Fell bundle for A^Λ is $(B_g^+)^\Lambda$. Now the assertion follows from Theorem 9.26. □

13 Twisted Weyl algebras

We illustrate our theory by studying C^*-hulls of twisted n-dimensional Weyl algebras for $1 \le n \le \infty$. We begin with the case $n = 1$, where no twists occur. Then we consider the case of finite n without twists and with twists. Finally, we consider the case $n = \infty$ with and without twists.
The (1-dimensional) *Weyl algebra* A is the universal *-algebra with one generator a and the relation $aa^* = a^*a+1$. There is a unique $\mathbb{Z}$-grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$ with $a \in A_1$. The *-subalgebra A_0 is isomorphic to the polynomial algebra $\mathbb{C}[N]$ with $N = a^*a$, which is commutative. The other subspaces $A_k \subseteq A$ for $k \in \mathbb{N}$ are isomorphic to A_0 as left or right A_0-modules because $A_k = A_0 \cdot a^k = a^k \cdot A_0$ and $A_{-k} = (a^*)^k \cdot A_0 = A_0 \cdot (a^*)^k$ for all $k \ge 0$. The spectrum $\hat{A}_0$ of A_0 is $\mathbb{R}$, where the character $\mathbb{C}[N] \to \mathbb{C}$ for $t \in \mathbb{R}$ evaluates a polynomial at t. A character is positive if and only if it is positive on $(a^*)^k a^k$ and $a^k(a^*)^k$ for all $k \ge 1$. This happens if and only if $t \in \mathbb{N}$ by [27, Example 10].
Since $Na = a^*aa = (aa^* - 1)a = a \cdot (a^*a - 1) = a \cdot (N - 1)$, the partial automorphism ϑ_1 of $\hat{A}_0^+ = \mathbb{N}$ associated to the A_0-bimodule A_1 acts on $\hat{A}_e^+$ by the automorphism $N \mapsto N - 1$, which corresponds to translation by -1. By induction, we get $N \cdot a^k = a \cdot (N - 1) \cdot a^{k-1} = \cdots = a^k \cdot (N - k)$. The domain of ϑ_k is as big as it could possibly be, that is, it contains all $n \in \mathbb{N}$ with $n \ge k$ by (11.5) (see also [27, Example 16]). For any $k, l \in \mathbb{N}$ there is a unique $n \in \mathbb{Z}$ with $k - n = l$. Thus the transformation groupoid $\mathbb{Z} \ltimes_\vartheta \mathbb{N}$ is simply the pair

groupoid on $\mathbb{N}$. There can be no twist in this case. First, the pair groupoid simply has no non-trivial twists. And secondly, the generators $a^k, (a^*)^k$ for $k \geq 0$ satisfy the positivity condition in Proposition 11.11, which also rules out a twist.

Since no proper non-empty subset of $\mathbb{N}$ is invariant under the partial action ϑ of $\mathbb{Z}$, a commutative C*-hull for A_0 for which integrability is compatible with induction gives either $B_0^+ = \mathrm{C}_0(\mathbb{N})$ or $B_0^+ = \{0\}$. In the second case, A has no non-zero integrable representations. In the first case, the C*-hull for the integrable representations of A is the groupoid C*-algebra $\mathbb{K}(\ell^2\mathbb{N})$ of the pair groupoid $\mathbb{N} \times \mathbb{N}$.

The universal representation of A on $\mathbb{K}(\ell^2\mathbb{N})$ is equivalent to a representation π of A on $\ell^2\mathbb{N}$ by Proposition 3.13. The domain of this representation is the space $\mathcal{S}(\mathbb{N})$ of rapidly decreasing sequences, with $\pi(a)(\delta_k) = \sqrt{k}\delta_{k-1}$ for $k \in \mathbb{N}$, so $\pi(a^*)(\delta_k) = \sqrt{k+1}\delta_{k+1}$, $\pi(N)(\delta_k) = k\delta_k$. By Theorem 4.4, a representation π of A_0 on a Hilbert module $\mathcal{E}$ is integrable if and only $\overline{\pi(N^k)}$ is regular and self-adjoint for each $k \in \mathbb{N}$ or, equivalently, $\pi(N)$ is regular and self-adjoint and $\overline{\pi(N^k)} = \overline{\pi(N)}^k$ for all $k \in \mathbb{N}$. By definition, a representation of A is integrable if and only if its restriction to A_0 is integrable.

The $\mathbb{Z}$-grading on the C*-hull $\mathbb{K}(\ell^2\mathbb{N})$ is “inner”: it is induced by the $\mathbb{Z}$-grading on $\ell^2\mathbb{N}$ where δ_k has degree k. Equivalently, the dual action of $\mathbb{T}$ on $\mathbb{K}(\ell^2\mathbb{N})$ associated to the $\mathbb{Z}$-grading is the inner action associated to the unitary representation $U\colon \mathbb{T} \to \mathrm{U}(\ell^2\mathbb{N})$, where $U_z(\delta_k) := z^k\delta_k$ for all $z \in \mathbb{T}$, $k \in \mathbb{N}$.

Now let $m \in \mathbb{N}$ and let $\Theta = (\Theta_{jk})$ be an antisymmetric $m \times m$-matrix. Let $\lambda_{jk} = \exp(2\pi\mathrm{i}\Theta_{jk})$. Let $A^{m,\Theta}$ be the *-algebra with generators $a_1, \dots, a_m$ and the commutation relations $a_j a_j^* = a_j^* a_j + 1$ for $1 \leq j \leq m$ and

$$a_j a_k = \lambda_{jk} a_k a_j, \qquad a_j^* a_k = \lambda_{jk}^{-1} a_k a_j^* \tag{13.1}$$

for $1 \leq j \neq k \leq m$. Since $\lambda_{jk} = \lambda_{kj}^{-1}$, the relations (13.1) for (j,k) and (k,j) are equivalent; so it suffices to require (13.1) for $1 \leq j < k \leq m$. The *-algebra $A^{m,\Theta}$ is $\mathbb{Z}^m$-graded by giving a_j degree $e_j \in \mathbb{Z}^m$, where $e_1, \dots, e_m$ is the standard basis of $\mathbb{Z}^m$.

We first consider the case $\Theta = 0$ and write $A^m := A^{m,0}$. This is the *m-dimensional Weyl algebra*, which is the tensor product of m copies of the 1-dimensional Weyl algebra, with the induced $\mathbb{Z}^m$-grading. Thus the zero fibre A_0^m for $0 \in \mathbb{Z}^m$ is isomorphic to the polynomial algebra $\mathbb{C}[N_1, \dots, N_m]$ in the m generators $N_j = a_j^* a_j$. Its spectrum is $\mathbb{R}^m$. Each A_k^m for $k \in \mathbb{Z}^m$ is isomorphic to A_0^m both as a left and a right A_0^m-module; the generator is the product of $a_j^{k_j}$ for $k_j \geq 0$ or $(a_j^*)^{-k_j}$ for $k_j < 0$ from $j = 1, \dots, m$. Here the order of the factors does not matter because $\Theta = 0$. We may identify A_k^m with the exterior tensor product of the A^1-bimodules $A_{k_1}^1 \otimes A_{k_2}^1 \otimes \dots \otimes A_{k_m}^1$. Hence the space of positive characters on A^m is $\mathbb{N}^m$, and the partial action of $\mathbb{Z}^m$ on $\mathbb{N}^m$ is the exterior product of the partial actions of $\mathbb{Z}$ on $\mathbb{N}$ for the 1-dimensional Weyl algebras. That is, $k \in \mathbb{Z}^m$ acts on $\mathbb{N}^m$ by translation by $-k$

with the maximal possible domain. Thus the transformation groupoid $\mathbb{Z}^m \ltimes \hat{A}_0^+$ is isomorphic to the pair groupoid of the discrete set $\mathbb{N}^m$.

Once again, the only $\mathbb{Z}^m$-invariant subsets of $\hat{A}_0^+$ are the empty set and $\mathbb{N}^m$, so that the only inducible commutative C*-hulls of A_0 for which integrability is compatible with induction are $\{0\}$ and $\mathrm{C}_0(\mathbb{N}^m)$. The first case is boring, and the second case leads to the C*-hull $\mathbb{K}(\ell^2\mathbb{N}^m)$ of the m-dimensional Weyl algebra.

As for $m = 1$, the universal representation of A^m is equivalent to a representation on $\ell^2\mathbb{N}^m$. This has the domain $\mathcal{S}(\mathbb{N}^m)$, and the representation is determined by

$$\pi(a_j)(\delta_{(k_1,\dots,k_m)}) = \sqrt{k_j}\delta_{(k_1,\dots,k_j-1,\dots,k_m)}$$

for $(k_1,\dots,k_m) \in \mathbb{N}^m$ and $j = 1,\dots,m$. Hence $\pi(N_j)(\delta_{(k_1,\dots,k_m)}) = k_j\delta_{(k_1,\dots,k_m)}$. A representation of A is integrable if and only if its restriction to A_0 is integrable in the sense that it integrates to a representation of $\mathrm{C}_0(\mathbb{R}^m)$. This automatically descends to a representation of $\mathrm{C}_0(\mathbb{N}^m)$ by Lemma 9.8. There are several ways to characterise when a representation of $\mathbb{C}[N_1,\dots,N_m]$ integrates to a representation of $\mathrm{C}_0(\mathbb{R}^m)$. One is that $\overline{\pi(N_j)}$ for $j = 1,\dots,m$ are strongly commuting, regular, self-adjoint operators and $\overline{\pi(N_j^k)} = \overline{\pi(N_j)}^k$ for all $j = 1,\dots,m$, $k \in \mathbb{N}$, compare [28, Theorem 9.1.13].

The groups $\mathbb{Z}^m$ for $m \geq 2$ have non-trivial 2-cocycles, and $A^{m,\Theta}$ is, by definition, a Rieffel deformation of $A^{m,0}$ for the normalised 2-cocycle

$$\Lambda\big((x_1,\dots,x_m),(y_1,\dots,y_m)\big) := \prod_{j=1}^{m}\prod_{k=j+1}^{m} \lambda_{jk}^{x_k y_j}. \tag{13.2}$$

We could also use the cohomologous antisymmetric 2-cocycle

$$\prod_{j=1}^{m}\prod_{k=j+1}^{m} \sqrt{\lambda_{jk}}^{\,x_k y_j - x_j y_k} = \prod_{j,k=1}^{m} \exp(-\pi\mathrm{i}\Theta_{jk}x_j y_k).$$

Theorem 12.3 says that the C*-hull $B^{m,\Theta}$ of $A^{m,\Theta}$ is the Rieffel deformation of the C*-hull $B^{m,0}$ of $A^{m,0}$ for the same 2-cocycle.

In the classification of Fell bundles with commutative unit fibre, the important cohomology is that of the transformation groupoid $G \ltimes X^+$, not of G itself. Here $G \ltimes X^+$ is the pair groupoid of $\mathbb{N}^m$.

Lemma 13.3 (compare [18, Lemma 2.9]). *The cohomology of the pair groupoid of a discrete set X with coefficients in an Abelian group H vanishes in all positive degrees.*

Proof. The set of composable n-tuples in the pair groupoid of X is X^{n+1}. The groupoid cohomology with coefficients H is the cohomology of the chain complex with cochains $C_n := H^{X^{n+1}}$, the space of all maps $X^{n+1} \to H$, and with the boundary map $\partial\colon C_n \to C_{n+1}$,

$$\partial\varphi(x_0,\dots,x_n) := \sum_{i=0}^{n} (-1)^i \varphi(x_0,\dots,\widehat{x_i},\dots,x_n);$$

here the hat means that the entry x_i is deleted. Pick some point $x_0 \in X$ and let $h\varphi(x_1,\dots,x_n) := \varphi(x_0,x_1,\dots,x_n)$ for all $n \in \mathbb{N}$, $x_1,\dots,x_n \in X$, $\varphi \in C_{n+1}$. Then $\partial \circ h(\varphi) + h \circ \partial(\varphi) = \varphi$ for all $\varphi \in C_n$, $n \geq 1$. Thus the cohomology vanishes in positive degrees. $\square$

Any twist of the pair groupoid on $\mathbb{N}^m$ is trivial by Lemma 13.3. Therefore, the C*-hull $B^{m,\Theta}$ is isomorphic to $\mathbb{K}(\ell^2\mathbb{N}^m)$, the untwisted groupoid C*-algebra of the pair groupoid. The proof of Lemma 13.3 is explicit and so allows to construct this isomorphism. We explain another way to construct it, using properties of Rieffel deformation. Since the $\mathbb{Z}$-grading on the C*-hull $\mathbb{K}(\ell^2\mathbb{N})$ is inner or, equivalently, the corresponding action of $\mathbb{T}$ is inner, the same holds for the $\mathbb{Z}^m$-grading on the C*-hull $\mathbb{K}(\ell^2\mathbb{N}^m)$ and the corresponding $\mathbb{T}^m$-action on $\mathbb{K}(\ell^2\mathbb{N}^m)$. Explicitly, the $\mathbb{T}^m$-action is induced by the unitary representation of $\mathbb{T}^m$ on $\ell^2\mathbb{N}^m$ defined by

$$U_{(z_1,\dots,z_m)}\delta_{(k_1,\dots,k_m)} := z_1^{-k_1}\cdots z_m^{-k_m}\delta_{(k_1,\dots,k_m)}.$$

Rieffel deformation of C*-algebras for *inner* actions does not change the isomorphism type of the C*-algebra. Hence the C*-hull for the integrable representations of $A^{m,\Theta}$ is also isomorphic to $\mathbb{K}(\ell^2\mathbb{N}^m)$.

We make this more explicit in our Fell bundle language. Let $U\colon \mathbb{T}^m \to \mathcal{UM}(B)$ be a strictly continuous homomorphism to the group of unitary multipliers of a C*-algebra B and let $\alpha_z(b) := U_z b U_z^*$ for $z \in \mathbb{T}^m$, $b \in B$ be the resulting inner action. Let $(B_k)_{k\in\mathbb{Z}^m}$ be the spectral decomposition of this action, that is, $b \in B_k$ if and only if $\alpha_z(b) = z^k \cdot b$ for all $z \in \mathbb{T}^m$. In particular, $U \in \mathcal{UM}(B_0)$ because $\mathbb{T}^m$ is commutative.

Assume for simplicity that the 2-cocycle Λ is a bicharacter as above. For fixed $k \in \mathbb{Z}^m$, we view $\Lambda(k,\textvisiblespace)\colon \mathbb{Z}^m \to \mathbb{T}$ as an element $\tilde{\Lambda}(k)$ of the dual group $\mathbb{T}^m$. The map $\tilde{\Lambda}\colon \mathbb{Z}^m \to \mathbb{T}^m$ is a group homomorphism. The maps $\psi_k\colon B_k \to B_k$, $b \mapsto U^*_{\tilde{\Lambda}(k)} \cdot b$, for $k \in \mathbb{N}$ are Banach space isomorphisms that modify the multiplication as follows:

$$\psi_k(b_1)\psi_l(b_2) = U^*_{\tilde{\Lambda}(k)} b_1 U^*_{\tilde{\Lambda}(l)} b_2 = U^*_{\tilde{\Lambda}(k+l)} \alpha_{\tilde{\Lambda}(l)}(b_1) b_2 = \psi_{k+l}(\Lambda(k,l)\cdot b_1 b_2).$$

They keep the involution unchanged. This is exactly what Rieffel deformation does. Hence the maps ψ_k form an isomorphism between the undeformed and Rieffel deformed Fell bundles. This finishes the proof that the Rieffel deformed algebra for an inner action is canonically isomorphic to the original algebra.

The universal representation of $A^{m,\Theta}$ on $\mathbb{K}(\ell^2\mathbb{N}^m)$ again corresponds to a representation of $A^{m,\Theta}$ on $\ell^2(\mathbb{N}^m)$. We may construct it by carrying over the isomorphism $B^{m,\Theta} \cong B^{m,0}$ between the C*-hulls. This is the inverse of the

isomorphism above, so it multiplies on the left by the unitary $U_{\tilde{\Lambda}(k)}$ of degree 0 on elements of degree k. We do exactly the same on elements of $A^{m,\Theta}$ and so let $x \in A_k^{m,\Theta}$ for $k \in \mathbb{Z}^m$ act on $\ell^2\mathbb{N}^m$ by the operator $U_{\tilde{\Lambda}(k)}\pi^{m,0}(x)$, where $\pi^{m,0}$ is the universal representation of the untwisted Weyl algebra $A^{m,0}$ on $\ell^2\mathbb{N}^m$ described above. The same computation as above shows that this defines a *-representation of $A^{m,\Theta}$. We compute it explicitly.
First, the action of elements of $A_0^{m,\Theta}$ on $\ell^2\mathbb{N}^m$ is not changed. The domain of a representation of $A^{m,\Theta}$ is equal to the domain of its restriction to $A_0^{m,\Theta}$. Hence the domain of our representation is the Schwartz space $\mathcal{S}(\mathbb{N}^m)$, as in the untwisted case. Let $1 \le j \le m$. The generator a_j has degree $e_j \in \mathbb{Z}^m$, and

$$\tilde{\Lambda}(e_j) = (\lambda_{1,j}, \dots, \lambda_{j-1,j}, 1, \dots, 1) \in \mathbb{T}^m$$

for our first definition of Λ in (13.2). Thus

$$\begin{aligned}\pi^{m,\Theta}(a_j)\delta_{(k_1,\dots,k_m)} &= U_{\tilde{\Lambda}(e_j)}\pi^{m,0}(a_j)\delta_{(k_1,\dots,k_m)}\\ &= \left(\prod_{l=1}^{j-1}\lambda_{j,l}^{k_l}\right)\sqrt{k_j}\,\delta_{(k_1,\dots,k_j-1,\dots,k_m)}.\end{aligned}$$

These operators on $\mathcal{S}(\mathbb{N}^m)$ satisfy the defining relations of $A^{m,\Theta}$.

The *infinite-dimensional Weyl algebra* A^∞ is the universal *-algebra with generators a_j for $j \in \mathbb{N}$ and relations $a_j^*a_j = a_ja_j^*+1$, $a_ja_k = a_ka_j$, and $a_j^*a_k = a_ka_j^*$ for $0 \le j < k$. Let $\mathbb{Z}[\mathbb{N}]$ be the free Abelian group on countably many generators. The Weyl algebra A^∞ is $\mathbb{Z}[\mathbb{N}]$-graded, where a_j has degree $e_j \in \mathbb{Z}[\mathbb{N}]$, the jth generator of $\mathbb{Z}[\mathbb{N}]$.
The *-algebra A^∞ is a tensor product of infinitely many 1-dimensional Weyl algebras. The zero fibre A_0^∞ is the polynomial algebra in the generators $N_j = a_j^*a_j$ for $j \in \mathbb{N}$. Hence its spectrum is the infinite product $\hat{A}_0^\infty = \mathbb{R}^\mathbb{N}$, which is not locally compact. The tensor product structure of A^∞ shows that a character is positive if and only if each component is. That is, $(A^\infty)_0^+ \cong \mathbb{N}^\mathbb{N}$ is a product of countably many copies of the discrete space $\mathbb{N}$. Since $\mathbb{N}$ is not compact, this is not locally compact either. Hence to build a commutative C*-hull for A_0^∞, we must choose some locally compact space X with a continuous, injective map $f\colon X \to \mathbb{N}^\mathbb{N}$. Here we have simplified notation by assuming that already $f(X) \subseteq \mathbb{N}^\mathbb{N}$; otherwise, the first step in our construction would replace X by $X^+ := f^{-1}(\mathbb{N}^\mathbb{N})$. For X-integrability to be compatible with induction, we also need $f(X)$ to be invariant under the partial action of $\mathbb{Z}[\mathbb{N}]$, and we need the restricted partial action to lift to a continuous partial action on X.
The partial action of the group $\mathbb{Z}[\mathbb{N}]$ on $\mathbb{N}^\mathbb{N}$ is the obvious one by translations. It is free, and two points (n_k) and (n'_k) in $\mathbb{N}^\mathbb{N}$ belong to the same orbit if and only if there is k_0 such that $n_k = n'_k$ for all $k \ge k_0$. Briefly, such points are called *tail equivalent*. This partial action is minimal, that is, an open, $\mathbb{Z}[\mathbb{N}]$-invariant subset is either empty or the whole space. Hence $\mathbb{N}^\mathbb{N}$ has no

$\mathbb{Z}[\mathbb{N}]$-invariant, locally closed subsets. Thus $\mathbb{N}^{\mathbb{N}}$ has no $\mathbb{Z}[\mathbb{N}]$-invariant, locally compact subspaces.

Let K be any compact subset of $\mathbb{N}^{\mathbb{N}}$. Then the projection $p_j\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ to the jth coordinate must map K to a compact subset of $\mathbb{N}$. So there is an upper bound M_j with $p_j(K) \subseteq [0, M_j]_{\mathbb{N}} := [0, M_j] \cap \mathbb{N}$. Then $K \subseteq \prod_{j\in\mathbb{N}}[0, M_j]_{\mathbb{N}}$, and the right hand side is compact. The closure of $\prod_{j\in\mathbb{N}}[0, M_j]_{\mathbb{N}}$ under tail equivalence is

$$X_{(M_k)} := \bigcup_{j\in\mathbb{N}} \left(\mathbb{N}^j \times \prod_{k>j}[0, M_k]_{\mathbb{N}}\right),$$

the restricted product of copies of $\mathbb{N}$ with respect to the compact-open subsets $[0, M_j]_{\mathbb{N}}$. There is a unique topology on $X_{(M_k)}$ where each subset $\mathbb{N}^j \times \prod_{k>j}[0, M_k]_{\mathbb{N}}$ is open and carries the product topology. This topology is locally compact, and the partial action of $\mathbb{Z}[\mathbb{N}]$ on $X_{(M_k)}$ by translation is continuous.

Lemma 13.4. *The Local–Global Principle fails for the $X_{(M_k)}$-integrable representations of A^∞.*

Proof. Since the map $X \to \hat{A}_0^\infty$ is not a homeomorphism onto its image and $\hat{A}_0^\infty$ is metrisable, the Local–Global Principle fails for the X-integrable representations of A_0^∞ by Theorem 8.3. Applying induction from A_0^∞ to A^∞ to a counterexample for the Local–Global Principle for A_0^∞ produces such a counterexample also for A^∞. Explicitly, choose a sequence (n_k) such that $n_k > M_k$ for infinitely many k. Let $x_k := n_k\delta_k \in \mathbb{N}^{\mathbb{N}}$, that is, $x_k \in \mathbb{N}^{\mathbb{N}}$ has only one non-zero entry, which is n_k in the kth place. This sequence belongs to $X_{(M_k)}$ and converges to 0 in the product topology on $\hat{A}_0^\infty$, but not in the topology of $X_{(M_k)}$. The resulting representation of A_0^∞ on $\mathrm{C}(\bar{\mathbb{N}})$ is not $X_{(M_k)}$-integrable, but it becomes integrable when we tensor with any Hilbert space representation of $\mathrm{C}(\bar{\mathbb{N}})$, see the proof of Theorem 8.3. Now induce this (inducible) representation of A_0^∞ to a representation of A^∞ on $\mathrm{C}(\bar{\mathbb{N}})$. This gives a counterexample for the Local–Global Principle for A^∞. □

I do not know a class of integrable representations of A^∞ with a C*-hull for which the Local–Global Principle holds.

Let S be the set of all words in the letters a_j, a_j^*. If $\chi \in \mathbb{N}^{\mathbb{N}}$ is a positive character and $x, y \in S \cap A_k^\infty$ for some $k \in \mathbb{Z}[\mathbb{N}]$, then $\chi(x^*y) \geq 0$. Hence Proposition 11.11 shows that there is no twist in our case, that is, the C*-hull of the $X_{(M_k)}$-integrable representations of A^∞ is the groupoid C*-algebra of the transformation groupoid $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$. This C*-hull is canonically isomorphic to one of the host algebras for A^∞ constructed in [11], namely, to the one that is denoted $\mathcal{L}[\mathbf{n}]$ in [11] with $\mathbf{n}_k = M_k + 1$ for all $k \in \mathbb{N}$. We remark in passing that the construction of a *full* host algebra from these host algebras in [11] is wrong: the resulting C*-algebra has too many Hilbert space representations, so it is not a host algebra any more, see the erratum [13].

The compact subset $T := \prod_{k\in\mathbb{N}}[0, M_k]_{\mathbb{N}}$ that we started with is a complete transversal in $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$, that is, the range map in $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$ restricted to $s^{-1}(T)$ is an open surjection onto $X_{(M_k)}$. Hence the groupoid $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$ is Morita equivalent to its restriction to the compact subset T. This restriction is the tail equivalence relation on T. Its groupoid C^*-algebra is well known: it is the UHF-algebra for $\prod_{k\in\mathbb{N}}(M_k + 1)$, that is, the infinite tensor product of the matrix algebras $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{M_k+1}$. The C^*-algebra of $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$ itself is the C^*-stabilisation of this UHF-algebra.

Thus the $X_{(M_k)}$-integrable representations of A^∞ are equivalent to the representations of the (stabilisation of the) UHF-algebra of type $\prod_{k\in\mathbb{N}}(M_k+1)$. This depends very subtly on the choice of the sequence (M_k). There is no canonical *-homomorphism between these UHF-algebras if we increase (M_k): for some $(M_k) \le (M'_k)$, there is not even a non-zero map between their K-theory groups. Instead, there are canonical morphisms, that is, there is a canonical nondegenerate *-homomorphism $\mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M'_k)}) \to \mathcal{M}(\mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M_k)}))$ if $(M_k) \le (M'_k)$. They are constructed as follows. The inclusion map $X_{(M_k)} \hookrightarrow X_{(M'_k)}$ is continuous with dense range, but not proper. Thus it induces an injective, nondegenerate *-homomorphism $\mathrm{C}_0(X_{(M'_k)}) \to \mathrm{C}_\mathrm{b}(X_{(M_k)})$. Therefore, if a representation of A^∞_0 is $X_{(M_k)}$-integrable, then it is also $X_{(M'_k)}$-integrable. If a representation of A^∞ has $X_{(M_k)}$-integrable restriction to A^∞_0, then its restriction to A^∞_0 is also $X_{(M'_k)}$-integrable. When we apply this to the universal representation of A^∞ on the C^*-hull $\mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M_k)})$, this gives the desired canonical morphism $\mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M'_k)}) \to \mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M_k)})$ if $(M_k) \le (M'_k)$. It is injective, say, because $\mathrm{C}^*(\mathbb{Z}[N] \ltimes X_{(M'_k)})$ is simple.

Now let $\Theta = (\Theta_{jk})_{j,k\in\mathbb{N}}$ be a skew-symmetric matrix. It corresponds first to a matrix $\lambda_{jk} := \exp(2\pi\mathrm{i}\Theta_{jk})$ and then to a 2-cocycle Λ on $\mathbb{Z}[\mathbb{N}]$ as in (13.2). The Rieffel deformation of A^∞ by Θ is the universal *-algebra $A^{\infty,\Theta}$ with the same generators $(a_j)_{j\in\mathbb{N}}$ and the relations $a_j a_j^* = a_j^* a_j + 1$, $a_j a_k = \lambda_{jk} a_k a_j$, and $a_j^* a_k = \lambda_{jk}^{-1} a_k a_j^*$ for $0 \le j < k$. We define $X_{(M_k)}$ for a sequence (M_k) and the $X_{(M_k)}$-integrable representations of $A^{\infty,\Theta}$ as above. By Theorem 12.3, this has a C^*-hull, namely, the Rieffel deformation of the C^*-hull for the $X_{(M_k)}$-integrable representations of the undeformed Weyl algebra. The Rieffel deformation gives a twist of the groupoid $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$, and the C^*-hull is the twisted groupoid C^*-algebra of $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$ for this twist.

Proposition 13.5. *Let $(M_k) \in \mathbb{N}^\mathbb{N}$. The C^*-hulls for the $X_{(M_k)}$-integrable representations of the twisted Weyl algebras $A^{\infty,\Theta}$ are isomorphic for all Θ.*

Proof. The C^*-hull of $A^{\infty,\Theta}$ is a twisted groupoid algebra of the transformation groupoid $\mathbb{Z}[\mathbb{N}] \ltimes X_{(M_k)}$, which is isomorphic to the tail equivalence relation R on $X_{(M_k)}$. We are going to prove that any twist $X_{(M_k)} \times \mathbb{T} \rightarrowtail \Sigma \twoheadrightarrow R$ is trivial. Hence the C^*-hull of $A^{\infty,\Theta}$ is isomorphic to the untwisted groupoid C^*-algebra of R for all Θ.

The arrow space of R is totally disconnected because $X_{(M_k)}$ is totally disconnected and R is étale. Hence any locally trivial principal bundle over R is trivial. Thus $\Sigma \cong R \times \mathbb{T}$ as a topological space, and the twist is described by a continuous 2-cocycle $\varphi\colon R^{(2)} := R \times_{s,X,r} R \to \mathbb{T}$. We must show that φ is a coboundary.

Let R_d for $d \in \mathbb{N}$ be the equivalence relation on $X_{(M_k)}$ defined by $(n_k)\ R_d\ (n'_k)$ if and only if $n_k = n'_k$ for all $k \geq d$. This is an increasing sequence of open subsets $R_d \subseteq R$ with $R = \bigcup R_d$, and each R_d is also an equivalence relation. The equivalence relation R_d is isomorphic to the product of the pair groupoid on $\mathbb{N}^d$ and the space $X_{(M_{k+d})}$ for the shifted sequence $(M_{k+d})_{k\in\mathbb{N}}$. Thus the cohomology of R_d with coefficients in $\mathbb{T}$ is isomorphic to the cohomology of the pair groupoid on $\mathbb{N}^d$ with values in the Abelian group of continuous map $X_{(M_{k+d})} \to \mathbb{T}$. This cohomology vanishes in positive degrees by Lemma 13.3. Therefore, for each $d \in \mathbb{N}$ there is $\psi_d\colon R_d \to \mathbb{T}$ such that $\varphi|_{R_d}\colon R_d \times_{s,r} R_d \to \mathbb{T}$ is the coboundary $\partial\psi_d$. The restriction of ψ_{d+1} to R_d and ψ_d both have coboundary $\varphi|_{R_d}$. Hence $\psi_{d+1}^{-1}|_{R_d} \cdot \psi_d$ is a 1-cocycle on R_d. Again by Lemma 13.3, there is $\chi\colon X \to \mathbb{T}$ with $\psi_{d+1}^{-1}|_{R_d} \cdot \psi_d = \partial_{R_d}\chi$. We replace ψ_{d+1} by $\psi'_{d+1} := \psi_{d+1} \cdot \partial_{R_{d+1}}\chi$, where $\partial_{R_{d+1}}\chi$ means the coboundary of χ on the groupoid R_{d+1}. This still satisfies $\partial\psi'_{d+1} = \partial\psi_{d+1} = \varphi|_{R_{d+1}}$, and $\psi'_{d+1}|_{R_d} = \psi_d$. Proceeding like this, we get continuous maps $\psi'_d\colon R_d \to \mathbb{T}$ for all $d \in \mathbb{N}$ that satisfy $\psi'_{d+1}|_{R_d} = \psi'_d$ and $\partial\psi'_d = \varphi|_{R_d}$ for all $d \in \mathbb{N}$. These combine to a continuous map $\psi'\colon R \to \mathbb{T}$ with $\partial\psi' = \varphi$. □

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