Why do Solutions of the Maxwell–Boltzmann Equation Tend to be Gaussians? A Simple Answer

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Abstract. The Maxwell-Boltzmann functional equation has recently attraction renewed interest since besides its importance in Boltzmann's kinetic theory of gases it also characterizes maximizers of certain bilinear estimates for solutions of the free Schrödinger equation. In this note we give a short and simple proof that, under some mild growth restrictions, any measurable complex-valued solution of the Maxwell-Boltzmann equation is a Gaussian. This covers most, if not all, of the applications.

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1 Introduction

The Maxwell-Boltzmann functional equation for a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ states that

$$f(x)f(y) = H(x^2 + y^2, x + y) \quad \text{for almost all } x, y \in \mathbb{R}^d \tag{1}$$

for some measurable function $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$. This equation has attracted a lot of attention in kinetic theory, since it determines the collision invariant of the Boltzmann equation. In this case, it is natural to also assume that f in non-negative and integrable.

In recent years the Maxwell–Boltzmann equation has regained attention for complex-valued functions f, since it determines in some cases the maximizers of the Strichartz inequality in low dimensions $d \leqslant 2$ [4, 5] and of the Ozawa and Tsutsumi bilinear inequality [7] and other bilinear estimates [3, 8], which are space–time inequalities for solutions of the free Schrödinger equation. In

[2] a class of bilinear estimates was proved, which includes the ones of [3, 7, 8] for solutions of the Schrödinger equation.

It was also noticed in [2] that any maximizer f of their bilinear inequality obeys the Maxwell–Boltzmann equation and, furthermore, that $f \in L^1(e^{-\gamma x^2}\,dx)$ for some suitable $\gamma \geqslant 0$. They conclude, with the help of known results [6, 9, 10] on solutions of the Maxwell–Boltzmann equation, that any maximizer of their bilinear inequality must be a Gaussian.

However, the known approaches to show that solutions f of the Maxwell-Boltzmann equation are Gaussians need that f is non-negative and $f \neq 0$. This is the case in [10], where Villani uses an argument due to Desvillettes to reduce the proof for $f \in L^1$ and $f \geqslant 0$ to $f \in C^2$ and $f > 0$. In [6] Lions uses the estimates he established in his work to prove that any solution of Maxwell–Boltzmann is smooth, this is elegant but technical and still needs that $f \geqslant 0$. In [9] the proof needs $f \geqslant 0$ and $f \in L^1_2(\mathbb{R}^d)$, that is, $x \mapsto (1 + |x|^2)f(x)$ is integrable, and, upon closer inspection, it seems to us that it also needs that the Fourier transform of f does not vanish. Using a completely different method, another approach in [1] also shows that solutions f of the Maxwell–Boltzmann functional equation are Gaussians if $f \geqslant 0$ is measurable and finite and $f > 0$ on a set of positive measure.

Our note is intended to give a short proof that complex valued-functions, which obey the Maxwell-Boltzmann equation together with a mild growth condition, are necessarily Gaussians with a rotationally symmetric covariance. More importantly, we believe that the proof we give is very simple. It does not require any technical tools besides some simple linear algebra and the chain rule. The proof we give below is inspired by the proof in [5], where Gaussians were shown to be the only maximizers in the sharp Stichartz inequality in low dimensions.

2 The Maxwell–Boltzmann equation and Gaussians

Theorem 1. *Let $f \in L^1(\mathbb{R}^d, e^{-\gamma x^2}dx)$ for some $\gamma \geqslant 0$ obey the Maxwell–Boltzmann equation*

$$f(x)f(y) = H(x^2 + y^2, x + y) \quad \text{for almost all } x, y \in \mathbb{R}^d \tag{2}$$

for some measurable function $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$. Then there exist $a, A \in \mathbb{C}$ with $\mathrm{Re}(a) < \gamma$ and $b \in \mathbb{C}^d$ such that

$$f(x) = Ae^{ax^2+b\cdot x} \quad \text{for almost every } x \in \mathbb{R}^d. \tag{3}$$

Proof. First, we assume in addition that $f \in L^1(\mathbb{R}^d)$. We will relax this in Step 4 below.

Step 1: Assume that $f \in C^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, i.e., it is integrable and twice continuously differentiable. Assume, furthermore, that f never vanishes. Then there exist $a, A \in \mathbb{C}$ with $\mathrm{Re}(a) < 0$ and $b \in \mathbb{C}^d$ such that

$$f(x) = Ae^{ax^2+b\cdot x} \quad \text{for } x \in \mathbb{R}^d.$$

To prove this, we will need suitable rotations in two-dimensional subspaces of $\mathbb{R}^{2d}$. From our assumption $f(x)f(y) = H(x^2+y^2, x+y)$ one easily deduces that the product $f(x)f(y)$ is invariant under a large class of rotations of $\mathbb{R}^{2d}$, namely the rotations of $\mathbb{R}^{2d}$ which leave $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto x+y$ invariant. To exploit this, we will construct a convenient basis in $\mathbb{R}^{2d}$, in which these rotation have a simple expression.

Let $e_j,\ j = 1\dots,d$ be the standard basis for $\mathbb{R}^d$, that is, e_j has only zero entries, except for the j^{th} slot, in which it has a one, and define the vectors $\alpha_j,\ j=1,\dots,d$ by

$$\alpha_j := \frac{1}{\sqrt{2}} \begin{pmatrix} e_j \\ e_j \end{pmatrix}.$$

so that α_j are unit vectors. Then the equation $x+y=c\in\mathbb{R}^d$ is equivalent to the d equations $c_j = x_j + y_j = \sqrt{2}\,\langle \alpha_j, z\rangle_{\mathbb{R}^{2d}}$, $j=1,\dots,d$ with $z = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2d}}$ the standard scalar product in $\mathbb{R}^{2d}$.

To construct suitable rotations in the orthogonal complement of $\operatorname{span}\{\alpha_1,\dots,\alpha_d\} \subset \mathbb{R}^{2d}$, one has to find a basis for this orthogonal complement. This can be done either in a systematic manner or by simply guessing that the vectors

$$\beta_j := \frac{1}{\sqrt{2}} \begin{pmatrix} e_j \\ -e_j \end{pmatrix}$$

will do the job: It is easy to check that $\{\alpha_1,\dots,\alpha_d,\beta_1,\dots,\beta_d\}$ form an orthonormal basis of $\mathbb{R}^{2d}$.

Fix $j,k\in\{1,\dots,d\}$ with $j\neq k$. We define the rotation by an angle φ in the plane spanned by β_j and β_k by the matrix

$$\begin{aligned} R_{j,k}(\varphi) \coloneqq {}& \sum_{l=1}^{d} \alpha_l\alpha_l^t + \sum_{\substack{m=1,\dots,d\\ m\notin\{j,k\}}} \beta_m\beta_m^t \\ & + \cos(\varphi)\beta_j\beta_j^t - \sin(\varphi)\beta_j\beta_k^t + \sin(\varphi)\beta_k\beta_j^t + \cos(\varphi)\beta_k\beta_k^t\,. \end{aligned} \tag{4}$$

In the following, we will suppress in our notation that the matrix depends on j and k. This rotation keeps all the directions α_l, $l=1,\dots,d$ and β_m, $m\notin\{j,k\}$ invariant. Since the function

$$F(z) := f(x)f(y) = H(x^2+y^2, x+y), \quad z=(x,y)\in\mathbb{R}^{2d}$$

is invariant under such a rotation, we have

$$F(R(\varphi)z) = \text{const}$$

for all fixed $z\in\mathbb{R}^{2d}$ and all $\varphi\in\mathbb{R}$. Thus, since F is twice continuously differentiable, by assumption,

$$0 = \frac{d}{d\varphi}F(R(\varphi)z)\big|_{\varphi=0} = \left\langle \nabla_{2d}F(z), \frac{d}{d\varphi}R(\varphi)\big|_{\varphi=0}\, z \right\rangle_{\mathbb{R}^{2d}}. \tag{5}$$

Here ∇_{2d} is the gradient in $\mathbb{R}^{2d}$. One easily calculates

$$\frac{d}{d\varphi}R(\varphi)\big|_{\varphi=0} = -\beta_j\beta_k^t + \beta_k\beta_j^t.$$

In the splitting $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, the matrix $\beta_j\beta_k^t$ has a simple block structure,

$$\beta_j\beta_k^t = \frac{1}{2}\begin{pmatrix} e_j \\ -e_j \end{pmatrix}(e_k^t \quad -e_k^t) = \frac{1}{2}\begin{pmatrix} e_je_k^t & -e_je_k^t \\ -e_je_k^t & e_je_k^t \end{pmatrix}.$$

So

$$\frac{d}{d\varphi}R(\varphi)\big|_{\varphi=0} = \frac{1}{2}\begin{pmatrix} -e_je_k^t + e_ke_j^t & e_je_k^t - e_ke_j^t \\ e_je_k^t - e_ke_j^t & -e_je_k^t + e_ke_j^t \end{pmatrix}$$

and thus, since ${e_k}^t x = \langle e_k, x\rangle_{\mathbb{R}^d} = x_k$, one has

$$\frac{d}{d\varphi}R(\varphi)\big|_{\varphi=0}\, z = \begin{pmatrix} -(x_k - y_k)e_j + (x_j - y_j)e_k \\ (x_k - y_k)e_j - (x_j - y_j)e_k \end{pmatrix} \quad \text{for } z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, writing

$$\nabla_{2d}F(z) = f(x)f(y)\begin{pmatrix} \nabla q(x) \\ \nabla q(y) \end{pmatrix}$$

with $q = \ln f$, which is well-defined, since, by assumption, f never vanishes, we get from (5) the differential equation

$$\begin{aligned} 0 &= -\left\langle \begin{pmatrix} \nabla q(x) \\ \nabla q(y) \end{pmatrix}, \begin{pmatrix} -(x_k - y_k)e_j + (x_j - y_j)e_k \\ (x_k - y_k)e_j - (x_j - y_j)e_k \end{pmatrix} \right\rangle_{\mathbb{R}^{2d}} \\ &= (x_k - y_k)\partial_j q(x) - (x_j - y_j)\partial_k q(x) - (x_k - y_k)\partial_j q(y) + (x_j - y_j)\partial_k q(y) \end{aligned}$$

for all $j \neq k$ and all $x, y \in \mathbb{R}^d$.
Differentiating this with respect to y_j yields

$$0 = \partial_k q(x) - (x_k - y_k)\partial_j^2 q(y) - \partial_k q(y) + (x_j - y_j)\partial_j\partial_k q(y) \tag{6}$$

and differentiating this again with respect to x_j, we arrive at

$$\partial_j\partial_k q(x) + \partial_j\partial_k q(y) = 0$$

for all $x, y \in \mathbb{R}^d$, which setting $x = y$ shows

$$\partial_j\partial_k q(x) = 0 \quad \text{for all } j \neq k, \tag{7}$$

whereas differentiating (6) with respect to x_k gives

$$\partial_k^2 q(x) = \partial_j^2 q(y) \quad \text{for all } j \neq k \tag{8}$$

and for all x, y. The two equations (7) and (8) show that there exists a constant $a \in \mathbb{C}$ such that

$$\nabla\partial_j q(x) = 2ae_j$$

for all $j = 1, \dots, d$. Integrating this gives

$$\partial_j q(x) = 2ax_j + b_j$$

for some constants $b_j \in \mathbb{C}$, i.e.,

$$\nabla q(x) = 2ax + b$$

and integrating this yields $\ln(f(x)) = q(x) = ax^2 + b \cdot x + c$. That is,

$$f(x) = Ae^{ax^2 + b\cdot x} \quad \text{for } x \in \mathbb{R}^d$$

for some constants $a, A \in \mathbb{C}$ and $b \in \mathbb{C}^d$ and in order that this is in $L^1(\mathbb{R}^d)$, we need to have $\mathrm{Re}(a) < 0$.

For the second step let

$$g(x) := \frac{1}{\pi^{d/2}} e^{-x^2}$$

a centered L^1-normalized Gaussian and for $\varepsilon > 0$

$$g_\varepsilon(x) := \varepsilon^{-d} g(x/\varepsilon)$$

its scaled version, which serves as an approximation of the delta-distributions as $\varepsilon \to 0$.

Step 2: Assume that $f \in L^1(\mathbb{R}^d)$ and the convolution $g_\varepsilon * f$ never vanishes for all small enough $\varepsilon > 0$. Then there exist $a, A \in \mathbb{C}$ with $\mathrm{Re}(a) < 0$ and $b \in \mathbb{C}^d$ such that

$$f(x) = Ae^{ax^2 + b\cdot x}.$$

Indeed, $G_\varepsilon(x, y) = g_\varepsilon(x) g_\varepsilon(y)$ is a centered Gaussian in $\mathbb{R}^{2d}$, in particular, invariant under all rotations of $\mathbb{R}^{2d}$. Let $\widetilde{H}(x, y) = H(x^2 + y^2, x + y)$ and set $\widetilde{H}_\varepsilon = G_\varepsilon * \widetilde{H}$, the convolution now on $\mathbb{R}^{2d}$. From the Maxwell-Boltzmann equation for f one gets

$$\Big(g_\varepsilon * f\Big)(x)\Big(g_\varepsilon * f\Big)(y) = (G_\varepsilon * \widetilde{H})(x, y) \quad \text{for all } x, y \in \mathbb{R}^d$$

A simple calculation, using that G_ε is invariant under all rotations of $\mathbb{R}^{2d}$, shows that $\widetilde{H}_\varepsilon$ inherits all rational invariances of $\widetilde{H}$, that is, it is invariant under all rotations of $\mathbb{R}^{2d}$ which leave $\mathbb{R}^2 \times \mathbb{R}^d \ni (x, y) \mapsto x + y$ invariant. Clearly $g_\varepsilon * f$ is infinitely often differentiable and, by assumption it does not vanish for all small enough $\varepsilon > 0$. So Step 1 applies to $g_\varepsilon * f$ and shows that there exist $a_\varepsilon, A_\varepsilon \in \mathbb{C}$ with $\mathrm{Re}(a_\varepsilon) < 0$ and $b_\varepsilon \in \mathbb{C}^d$ such that

$$\Big(g_\varepsilon * f\Big)(x) = A_\varepsilon e^{a_\varepsilon x^2 + b_\varepsilon \cdot x}$$

for all $x \in \mathbb{R}^d$. Taking the limit $\varepsilon \to 0$ shows that f is the L^1-limit of Gaussians, hence it must be a Gaussian.

To finish the proof for $f \in L^1(\mathbb{R}^d)$ it is enough to show

Step 3: Let $f \in L^1$ obey the Maxwell–Boltzmann equation and f not be the zero function. Then for any small enough $\varepsilon > 0$ the convolution $g_\varepsilon * f$ never vanishes.

Indeed, as above one sees

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Big(g_\varepsilon * f\Big)(x)\Big(g_\varepsilon * f\Big)(y)$$

is invariant under all rotations of $\mathbb{R}^{2d}$ which leave $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto x + y$ invariant. So taking the modulus and applying the same argument again, shows that for any $\delta > 0$

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Big(g_\delta * |g_\varepsilon * f|\Big)(x)\Big(g_\delta * |g_\varepsilon * f|\Big)(y)$$

is also invariant under all rotations of $\mathbb{R}^{2d}$ which leave $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto x+y$ invariant.

Since f is not identically zero and $g_\varepsilon * f$ converges to f in L^1, we see that $g_\varepsilon * f$ is not identically zero for all small enough $\varepsilon > 0$. But in this case $|g_\varepsilon * f|$ is non negative and positive on some set of positive measure, thus $\Big(g_\delta * |g_\varepsilon * f|\Big)(x) > 0$ for all $\delta > 0$ and all $x \in \mathbb{R}^d$, i.e., it is infinitely often differentiable and it never vanishes. By Step 1, we see that $|g_\varepsilon * f|$ is a Gaussian and thus, if it vanishes somewhere it must vanish everywhere. So if $|g_\varepsilon * f|$ vanishes somewhere for all small ε, it is identically zero for all small ε, and taking the limit $\varepsilon \to 0$, we see that f must be equal to zero almost everywhere, in contradiction to our assumption. So $|g_\varepsilon * f|$ never vanishes for all small enough $\varepsilon > 0$.

This finished the proof in case $f \in L^1(\mathbb{R}^d)$ obeys the Maxwell–Boltzmann equation. The last step is to relax the integrability assumption on f, which is easy:

Step 4: If $f \in L^1(\mathbb{R}^d, e^{-\gamma x^2}dx)$ obeys the Maxwell–Boltzmann equation, then there exist $a, A \in \mathbb{C}$ with $\mathrm{Re}(a) < \gamma$ and $b \in \mathbb{C}^d$ such that

$$f(x) = Ae^{ax^2+b\cdot x}$$

for almost all $x \in \mathbb{R}^d$.

Indeed, let $f_\gamma(x) = e^{-\gamma x^2} f(x)$. Then $f_\gamma \in L^1(\mathbb{R}^d)$ and it also obeys the Maxwell–Boltzmann equation. So by the above there exist $a_0, A \in \mathbb{C}$ with $\mathrm{Re}(a_0) < 0$ and $b \in \mathbb{C}^d$ such that

$$f_\gamma(x) = Ae^{a_0x^2+b\cdot x} \quad \text{for almost all } x \in \mathbb{R}^d.$$

Then cleary

$$f(x) = Ae^{ax^2+b\cdot x}$$

with $a = a_0 + \gamma$ and $\mathrm{Re}(a) < \gamma$. □

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