The K-Theory of Versal Flags and Cohomological Invariants of Degree 3

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Abstract. Let G be a split semisimple linear algebraic group over a field and let X be a generic twisted flag variety of G. Extending the Hilbert basis techniques to Laurent polynomials over integers we give an explicit presentation of the Grothendieck ring $K_0(X)$ in terms of generators and relations in the case $G = G^{sc}/\mu_2$ is of Dynkin type A or C (here G^{sc} is the simply-connected cover of G); we compute various groups of (indecomposable, semi-decomposable) cohomological invariants of degree 3, hence, generalizing and extending previous results in this direction.

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1 Introduction

Let G be a split semisimple linear algebraic group over a field F. Let U/G be a *classifying space* of G in the sense of Totaro [18, Rem.1.4], i.e. U is an open G-invariant subset in some representation of G with $U(F) \neq \emptyset$ and $U \to U/G$ is a G-torsor. Consider the generic fiber U' of U over U/G. It is a G-torsor over the quotient field F' of U/G called the *versal* G-torsor [4, Ch.I, §5]. We denote by X the respective flag variety U'/B over F', where B is a Borel subgroup of G, and call it the *versal* flag. The variety X appears in many different contexts, e.g. related to cohomology of homogeneous G-varieties (see [6] for an arbitrary oriented theory; Karpenko [7], [8], [9] for Chow groups; Panin [17] for K-theory) and cohomological invariants of G (see Merkurjev [14] and [5], [15]). It can be viewed as a generic example of the so called *twisted flag variety*.

In the first part of the paper (Sections 2-4) we give an explicit presentation of the ring $K_0(X)$ in terms of generators modulo a *finite* number of relations in cases when $G = G^{sc}/\mu_2$, where G^{sc} is the product of simply-connected simple groups of Dynkin types A or C and μ_2 is a central subgroup of order 2.

Observe that for simply-connected G the ring $K_0(X)$ can be identified with $K_0(G/B)$ (e.g., see Panin [17]), and by Chevalley theorems there is a surjective characteristic map $c\colon R(T_{sc}) \to K_0(G/B)$ from the representation ring of the split maximal torus T_{sc} such that the kernel $\ker(c) = I_{sc}^W$ is generated by augmented classes of fundamental representations. So, all relations in $K_0(X)$ correspond to W-orbits of fundamental weights.

If G is not simply-connected (as in the G^{sc}/μ_2-case), then the situation changes dramatically as by [6, Ex.5.4] we have

$$K_0(X) \simeq R(T)/(I_{sc}^W \cap R(T))$$

and a finite set of generators of $I_{sc}^W \cap R(T)$ is not known in general. Note that by definition we have inclusions of abelian groups $I^W \subseteq I_{sc}^W \cap R(T) \subseteq I_{sc}^W$ which all coincide if taken with $\mathbb{Q}$-coefficients. However, there are examples of semisimple groups (see [15, Ex.3.1] and [1]) where both quotients $(I_{sc}^W \cap R(T))/I^W$ and $I_{sc}^W/(I_{sc}^W \cap R(T))$ are non-trivial.

Our Theorem 3.4 provides a complete list of generators (Definition 3.2) of the ideal $I_{sc}^W \cap R(T)$ assuming the root system of G^{sc} satisfies the generalized flatness condition (see Definition 2.9). In Section 4 we show that this condition holds for types A and C.

In the second part of the paper we study cohomological invariants of degree 3 of G. According to Garibaldi-Merkurjev-Serre [4, p.106], a degree d *cohomological invariant* is a natural transformation of functors

$$a\colon H^1(\,\cdot\,, G) \to H^d(\,\cdot\,, \mathbb{Q}/\mathbb{Z}(d-1))$$

on the category of field extensions over F, where the functor $H^1(\,\cdot\,, G)$ classifies G-torsors, $H^d(\,\cdot\,, \mathbb{Q}/\mathbb{Z}(d-1))$ is the Galois cohomology. Following Merkurjev [14], an invariant is called *decomposable* if it is given by a cup-product of invariants of smaller degrees; the factor group of (normalized) invariants modulo decomposable is called the group of *indecomposable* invariants. For $d = 3$ the latter (denoted by $\mathrm{Inv}^3_{ind}(G)$) has been computed for all simple split groups in [14] and [2]; for some semi-simple groups of type A in [13] and [1]; for adjoint semisimple groups in [12].

Another key subgroup of *semi-decomposable* invariants introduced in [15] consists of invariants given by a cup-product of invariants up to some field extensions. For $d = 3$ it coincides with the group of decomposable invariants for all simple groups [15]. It was also shown that these groups are different for $G = \mathbf{SO}_4$ [15, Ex.3.1] and for some semisimple groups of type A (see [1]).

The relationships between the subgroups $I^W \subseteq I_{sc}^W \cap R(T) \subseteq I_{sc}^W$ and the groups of cohomological invariants are explained in Section 5.

In Sections 6-11, we compute the groups of decomposable, indecomposable and semi-decomposable invariants of degree 3 for new examples of semisimple groups (e.g. G^{sc}/μ_2, products of adjoint groups), hence, extending the results of [14], [2], [1], [15], [13]; to compute semi-decomposable invariants we use the generators of Definition 3.2. We show that

• The factor group $\mathrm{Inv}^3_{sd}(G)$ of semi-decomposable invariants of G modulo decomposable is nontrivial if and only if G is of classical type A, B, C, D. Moreover, we compute both groups $\mathrm{Inv}^3_{sd}(G)$ and $\mathrm{Inv}^3_{ind}(G)$ for an arbitrary product of simply-connected simple groups of the same Dynkin type modulo central subgroups of order 2 (see Corollaries 6.5, 7.2, 8.2, 9.2, and Proposition 11.2).

• If G is of type A, then both these groups can have an arbitrary order and contain any direct product of cyclic p-group (see Corollary 6.6). If G is of type B or C, then it is always a product of cyclic groups of order 2 (see Corollaries 7.2, 8.2).

• The group $\mathrm{Inv}^3_{sd}(G)$ is trivial for the simple group $G = \mathbf{PGO}_8$, i.e., any semi-decomposable invariant is decomposable (Corollary 10.4).

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2 Syzygies and divisibility for Laurent polynomials

Let Λ be a free abelian group of rank n with a fixed basis $\{x_1,\ldots,x_n\}$. Let R be one of the rings $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, $m\geq 2$. Consider the group ring $R[\Lambda]$. It consists of finite linear combinations $\sum_j a_j e^{\lambda_j}$, $a_j\in R$, $\lambda_j\in\Lambda$. We identify $R[\Lambda]$ with the Laurent polynomial ring $R[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ via $e^{x_i}\mapsto x_i$ and $e^{-x_i}\mapsto x_i^{-1}$. By a polynomial we mean always a Laurent polynomial, i.e., an element of $R[\Lambda]$. We denote by Λ_i a free subgroup with the basis $\{x_1,\ldots,x_i\}$, $1\leq i<n$. Hence, $R[\Lambda_i]=R[x_1^{\pm 1},\ldots,x_i^{\pm 1}]$.

Definition 2.1. *Given $f\in R[\Lambda]$, we can express it uniquely as*

$$f=f_kx_n^k+f_{k-1}x_n^{k-1}+\ldots+f_mx_n^m,\quad \textit{where } f_i\in R[\Lambda_{n-1}],\ k,m\in\mathbb{Z},\ k\geq m.$$

The integer k is called the highest degree of f with respect to x_n *and denoted* $\mathrm{hdeg}_n(f)$. *The integer m is called the* lowest degree of f with respect to x_n *and denoted* $\mathrm{ldeg}_n(f)$. *The difference $k-m$ is called the* degree of f with respect to x_n *and denoted* $\mathrm{wdeg}_n(f)$.

By definition, if $\mathrm{wdeg}_n(f)=0$, then f is a product of x_n^k and a polynomial in $x_1,\ldots,x_{n-1}$.

Definition 2.2. *Let f, $p \in R[\Lambda]$ and let* $\mathrm{ldeg}_n(f) \geq d$ *for some* $d \in \mathbb{Z}$. *We say that it is possible to perform a* divison of f by p bounded by d *if there exist monomials* q, $r \in R[\Lambda]$ *such that*

1. $f = pq + r$.
2. *Either* $r = 0$ *or* ($\mathrm{ldeg}_n(r) \geq d$ *and* $\mathrm{hdeg}_n(r) < d + \mathrm{wdeg}_n(p)$).

In this case q is called the quotient, *and r is called the* remainder.

Definition 2.3. *We call $p \in R[\Lambda]$ a* divisor *with respect to x_n if it satisfies the following condition:*
In the presentation of Definition 2.1

$$p = p_k x_n^k + \ldots + p_m x_n^m, \quad p_i \in R[\Lambda_{n-1}],\ k, m \in \mathbb{Z},\ k \geq m,$$

the leading coefficient p_k is a monic *monomial in $x_1, \ldots, x_{n-1}$.*

Lemma 2.4. *Let f, $p \in R[\Lambda]$ and let $\mathrm{ldeg}_n(f) \geq d$ for some $d \in \mathbb{Z}$.*
If p is a divisor with respect to x_n, then it is possible to perform a division of f by p bounded by d.

Proof. We proceed by induction on $\mathrm{hdeg}_n(f)$. If $\mathrm{hdeg}_n(f) < d + \mathrm{wdeg}_n(p)$, then we set $q = 0$ and $r = f$.
Suppose that $\mathrm{hdeg}_n(f) \geq d + \mathrm{wdeg}_n(p)$. Since p is a divisor, we can write it as

$$p = Y x_n^k + p', \text{ where } Y \in R[\Lambda_{n-1}] \text{ is a monic monomial and}$$

p' is either 0 or a polynomial with $\mathrm{hdeg}_n(p') < \mathrm{hdeg}_n(p) = k$ and $\mathrm{ldeg}_n(p') = \mathrm{ldeg}_n(p)$. Observe that Y is invertible in $R[\Lambda_{n-1}]$.
We write f as $f = g x_n^m + f'$, where $m = \mathrm{hdeg}_n(f)$, $g \in R[\Lambda_{n-1}]$, and f' is either 0, or a polynomial with $\mathrm{hdeg}_n(f') < m$ and $\mathrm{ldeg}_n(f') = \mathrm{ldeg}_n(f)$.
Set $q_0 = g Y^{-1} x_n^{m-k}$. Then $Y x_n^k q_0 = g x_n^m$. If both f' and p' are 0, then $Y x_n^k = p$ and $g x_n^m = f$, so $p q_0 = f$, and we are done.
Consider the polynomial $f'' = f' - q_0 p'$. We have $\mathrm{hdeg}_n(q_0) = \mathrm{ldeg}_n(q_0) = m - k$. Recall that either $p' = 0$ or ($\mathrm{hdeg}_n(p') < k$ and $\mathrm{ldeg}_n(p') = \mathrm{ldeg}_n(p)$). So, either $q_0 p' = 0$, or ($\mathrm{hdeg}_n(q_0 p') < m$ and $\mathrm{ldeg}_n(q_0 p') = m - k + \mathrm{ldeg}_n(p) = m - \mathrm{wdeg}_n(p)$).
Recall also that either $f' = 0$, or ($\mathrm{hdeg}_n(f') < m$ and $\mathrm{ldeg}_n(f') = \mathrm{ldeg}_n(f)$). So, if p' and f' are not both 0, then $\mathrm{hdeg}_n(f'') < m$.
Also, if p' and f' are not both 0, then $\mathrm{ldeg}_n(f'') \geq \min(m - \mathrm{wdeg}_n(p), \mathrm{ldeg}_n(f))$. We know that $m = \mathrm{hdeg}_n(f) \geq d + \mathrm{wdeg}_n(p)$, so $m - \mathrm{wdeg}_n(p) \geq d$. Also, $\mathrm{ldeg}_n(f) \geq d$. So, $\mathrm{ldeg}_n(f'') \geq d$, and we can apply the induction hypothesis. By induction, there exist polynomials q_1 and r such that $f'' = p q_1 + r$, and (either $r = 0$ or ($\mathrm{ldeg}_n(r) \geq d$ and $\mathrm{hdeg}_n(r) < d + \mathrm{wdeg}_n(p)$)).
Set $q = q_0 + q_1$. Then

$$\begin{aligned} pq + r &= p q_0 + p q_1 + r = (Y x_n^k + p') q_0 + f'' \\ &= Y x_n^k q_0 + p' q_0 + f' - q_0 p' = g x_n^m + f' = f. \end{aligned}$$

□

Definition 2.5. *(cf. [3, §15.5]) Given a n-tuple of polynomials $\vec{q} = (q_1, \dots, q_n)$, a n-tuple of polynomials $\vec{f} = (f_1, \dots, f_n)$ is called a* syzygy *of $\vec{q}$ if $\sum_i f_i q_i = 0$.*
Observe that syzygies form a submodule of a free module of rank n over $R[\Lambda]$. An element of a submodule generated by

$$S_{ij} = (0, \dots, q_j, \dots, -q_i, \dots, 0),$$

where q_j is at the position i and $-q_i$ is at the position j, $i, j = 1, \dots, n$ and $i \neq j$, is called a trivial syzygy *of $\vec{q}$.*

Lemma 2.6. *Let $\vec{q} = (q_i)$, $q_i \in \mathbb{Z}[\Lambda]$ and let $\bar{q} = (\bar{q}_i)$, where $\bar{q}_i \in \mathbb{Z}/d\mathbb{Z}[\Lambda]$ is the reduction modulo d, $d \geq 2$.*
If $\vec{f'} = (f'_i)$ is a trivial syzygy of $\bar{q}$, then there exists a trivial syzygy $\vec{f} = (f_i)$ of $\vec{q}$ such that its reduction modulo d coincides with $\vec{f'}$, i.e., a trivial syzygy can be always lifted to $\mathbb{Z}$.

Proof. Let $\bar{S}_{ij}$ be the reduction modulo d of S_{ij}. We have $\vec{f'} = \sum_{i,j} g'_{ij} \bar{S}_{ij}$ for some $g'_{ij} \in \mathbb{Z}/d\mathbb{Z}[\Lambda]$. Let g_{ij} be liftings of g'_{ij} to $\mathbb{Z}[\Lambda]$. Set $\vec{f} = \sum_{i,j} g_{ij} S_{ij}$. □

Definition 2.7. *We say that a n-tuple of polynomials $(q_1, \dots, q_n)$ satisfies the flatness condition if $q_i \in R[\Lambda_i]$ for each $i = 1, \dots, n$, and q_i is a divisor with respect to x_i.*

Lemma 2.8. *If an n-tuple of polynomials $\vec{r}$ satisfies the flatness condition, then all syzygies of $\vec{r}$ are trivial.*

Proof. First, consider the case where R is a domain (i.e., $R = \mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ with p prime). We use induction on n. If $n = 1$, then the trivial syzygy 0 is the only syzygy. Let $\vec{f} = (f_1, \dots, f_n)$ be a syzygy of $\vec{r} = (r_1, \dots, r_n)$ with $n \geq 2$. By Lemma 2.4, we can divide $f_i = r_n g_i + h_i$ with bound $d = \min\{\operatorname{ldeg}_n(f_i)\}$, where $\operatorname{ldeg}_n(h_i) \geq d$ and $\operatorname{hdeg}_n(h_i) < d + \operatorname{wdeg}_n(r_n)$ for $1 \leq i \leq n$.
Since $(g_1 r_1, \dots, g_{n-1} r_{n-1}, -\sum_{i=1}^{n-1} g_i r_i)$ is a trivial syzygy of $\vec{r}$, it suffices to show that $(h_1, \dots, h_{n-1}, f_n + \sum_{i=1}^{n-1} g_i r_i)$ is a trivial syzygy of $\vec{r}$. If $\sum_{i=1}^{n-1} h_i r_i = (f_n + \sum_{i=1}^{n-1} g_i r_i) r_n$ is nonzero, then by taking wdeg_n of both sides, we obtain

$$\operatorname{wdeg}_n(r_n) > \operatorname{wdeg}_n(\textstyle\sum_{i=1}^{n-1} h_i r_i) = \operatorname{wdeg}((f_n + \sum_{i=1}^{n-1} g_i r_i) r_n) \geq \operatorname{wdeg}(r_n),$$

a contradiction. Thus, we have $f_n + \sum_{i=1}^{n-1} g_i r_i = 0$ and it remains to show that $\vec{h} = (h_1, \dots, h_{n-1}, 0)$ is a trivial syzygy of $\vec{r}$. Let $e = d + \operatorname{wdeg}_n(r_n) - 1$. Write $h_i = h_{id} x_n^d + \dots + h_{ie} x_n^e$ for all $1 \leq i \leq n-1$ and $\vec{h}_j = (h_{1j}, \dots, h_{(n-1)j}, 0)$ for all $d \leq j \leq e$. Then, we have $\vec{h} = \vec{h}_d x_n^d + \dots + \vec{h}_e x_n^e$. By induction, all syzygies $\vec{h}_j$ are trivial, so is $\vec{h}$.
Now we consider the case $R = \mathbb{Z}/m\mathbb{Z}$. We proceed by induction on the number of prime factors in m. If m is a prime, it follows from the previous case.

Write $m = pl$, $l > 1$, where p is a prime. Let $\vec{f} = (f_i)$ be a syzygy of $\vec{r} = (r_i)$ and let $\bar{f} = (\bar{f}_i)$ be the corresponding syzygy of $\bar{r} = (\bar{r}_i)$ over $\bar{R} = \mathbb{Z}/l\mathbb{Z}$ for $1 \le i \le n$. By induction, we have

$$\bar{f} = \sum_{i,j} \bar{g}_{ij} \bar{S}_{ij} \quad \text{for some } \bar{g}_{ij} \in \bar{R}[\Lambda]. \tag{*}$$

Set

$$\vec{f'} = \vec{f} - \sum_{i,j} g_{ij} S_{ij}, \ \text{ where } g_{ij} \text{ is a preimage of } \bar{g}_{ij} \text{ in } R[\Lambda].$$

By (*) we have $f'_i = l f''_i$ for some $f''_i \in R[\Lambda]$. Since $\vec{f'}$ is a syzygy of $\vec{r}$, we have $0 = \sum_i f'_i r_i = l(\sum_i f''_i r_i)$ in $R[\Lambda]$. Thus, $\vec{f''} = (f''_i)$ is a syzygy of $\vec{r}$ modulo p. By the previous case, $\vec{f''}$ is a trivial syzygy modulo p. So $\vec{f''} = \sum_{i,j} g'_{ij} S_{ij} + p\vec{h}$ for some n-tuple of polynomials $\vec{h}$ and preimages g'_{ij}. Then $\vec{f'} = l\vec{f''} = \sum_{ij} l g'_{ij} S_{ij}$ is a trivial syzygy of $\vec{r}$. □

Definition 2.9. *We say that a n-tuple of polynomials $(q_1, \ldots, q_n)$ satisfies a* generalized flatness condition *if there exists a matrix $A \in GL_n(R[\Lambda])$ such that the n-tuple $(r_1, \ldots, r_n) = (q_1, \ldots, q_n)A$ satisfies the flatness condition.*

Lemma 2.10. *Assume that a n-tuple of polynomials $\vec{q} = (q_1, \ldots, q_n)$ satisfies the generalized flatness condition. Then all syzygies of $\vec{q}$ are trivial.*

Proof. Let A be a matrix such that $\vec{r} = \vec{q}A$ satisfies the flatness condition. Let $\vec{f} = (f_1, \ldots, f_n)$ be a syzygy of $\vec{q}$. Then (as a product of matrices)

$$0 = \vec{q} \cdot \vec{f}^t = \vec{q}A \cdot A^{-1}\vec{f}^t = \vec{r} \cdot \vec{g}, \text{ where } \vec{g} = A^{-1}\vec{f}^t.$$

Hence, $\vec{g}$ is a syzygy of $\vec{r}$ and $\vec{f} = A\vec{g}$. By Lemma 2.8 it suffices to prove that if $\vec{g} = S_{ij}$ is a trivial syzygy of $\vec{r}$, then $A\vec{g}$ is a trivial syzygy of $\vec{q}$.
Let M_{ij}, $i \ne j$ denote a matrix where all entries are zeros except 1 at the position (i, j) and -1 at the position (j, i). The matrix M_{ij} is skew-symmetric. By definition, we have $S_{ij} = M_{ij}(\vec{r})^t$. So all trivial syzygies of $\vec{r}$ are linear combinations with coefficients in $R[\Lambda]$ of $M_{ij}\vec{r}^t$. Similarly, all trivial syzygies of $\vec{q}$ are linear combinations of $M_{ij}\vec{q}^t$.
Then we obtain $A\vec{g} = AM_{ij}\vec{r}^t = AM_{ij}A^t\vec{q}^t$. Finally, since the matrix $AM_{ij}A^t$ is skew-symmetric, it is a linear combination with coefficients in $R[\Lambda]$ of matrices $M_{i'j'}$ for various i', j'. □

3 The generators

Consider the weight lattice Λ of a semisimple root system corresponding to a group G. Let T^* be a group of characters of a split maximal torus T of G. We assume that T^* is of index 2 in Λ, i.e., $\Lambda/T^* = \mathbb{Z}/2\mathbb{Z}$.
Consider the $\mathbb{Z}/2\mathbb{Z}$-grading on Λ given by: a weight $\lambda \in \Lambda$ has degree $|\lambda|$ which is its class in the quotient Λ/T^*. We denote by $\Lambda^{(0)} = T^*$ the subgroup of Λ of degree 0 and by $\Lambda^{(1)} = \Lambda \setminus T^*$ the subset of degree 1.

There is an induced grading on the group ring $R[\Lambda]$ so that $R[\Lambda] = R[\Lambda^{(0)}] \oplus R[\Lambda^{(1)}]$. Hence, we can uniquely express any $f \in R[\Lambda]$ as a sum of its homogeneous components, i.e., $f = f^{(0)} + f^{(1)}$. We say that $f \in R[\Lambda]$ is homogeneous of degree 0 or, equivalently, $\deg(f) = 0$ (resp. f is of degree 1 or $\deg(f) = 1$) if $f \in R[\Lambda^{(0)}]$ (resp. $f \in R[\Lambda^{(1)}]$).

Let $\{\omega_1, \ldots, \omega_n\}$ denote the set of fundamental weights (a $\mathbb{Z}$-basis of Λ). Consider the orbit $W(\omega_i)$ of the fundamental weight ω_i by means of the Weyl group W. We denote by $|i|$ the degree of ω_i with respect to the grading and by $|W(\omega_i)|$ the number of elements in the orbit. Let

$$d = \gcd_{\omega_i \in \Lambda^{(1)}} (|W(\omega_i)|) = \gcd_{|i|=1} (|W(\omega_i)|).$$

We set $R = \mathbb{Z}$ and we denote by bar the reduction modulo d, i.e., $\bar{R} = \mathbb{Z}/d\mathbb{Z}$. We define

$$\rho(\omega_i) = \sum_{\lambda \in W(\omega_i)} e^{\lambda} \quad \text{ and } \rho_i = \rho(\omega_i) - |W(\omega_i)| \in R[\Lambda].$$

Since the Weyl group acts trivially on Λ/T^*, we have $\deg(\rho(\omega_i)) = |i|$. Reducing modulo d we obtain $\deg(\bar{\rho}_i) = \deg(\overline{\rho(\omega_i)}) = |i|$.

We will need the following

Lemma 3.1. *Assume that $(\bar{\rho}_1, \ldots, \bar{\rho}_n)$ satisfies the generalized flatness condition with respect to some basis $\{x_i\}$ of Λ. Assume that $f_i \in R[\Lambda]$, $i = 1, \ldots, n$ are such that $\deg(\sum_i f_i \rho_i) = 0$.*
Then there exist polynomials $g_1, \ldots, g_n \in R[\Lambda]$ such that $\sum_i f_i \rho_i = \sum_i g_i \rho_i$ and $\bar{g}_i^{(1-|i|)} = 0$ for each i.

Proof. Since $\deg(\sum_i f_i \rho_i) = 0$, we have $\sum_i \bar{f}_i^{(1-|i|)} \bar{\rho}_i = 0$. Hence, by Lemma 2.10 the n-tuple $(\bar{f}_i^{(1-|i|)})$ is a trivial syzygy of $(\bar{\rho}_i)$. By Lemma 2.6 there exists a trivial syzygy (h_i), $h_i \in R[\Lambda]$, of (ρ_i) such that $\bar{h}_i = \bar{f}_i^{(1-|i|)}$. Set $g_i = f_i - h_i$. □

After a possible reindexing, we may assume that the first n' fundamental weights $\{\omega_1, \ldots, \omega_{n'}\}$ have degree 1 and the remaining fundamental weights have degree 0. For $1 \le i \le n'$ we set

$$d_i = \gcd_{i \le j \le n'} (s_j), \qquad \text{where } s_j = |W(\omega_j)|.$$

So we have $d = d_1 \mid d_2 \mid \ldots \mid d_{n'} = s_{n'}$. By a presentation of the gcd, there exist integers (denoted $a_{i,j}$, $1 \le i \le j \le n'$) such that

$$d_i = a_{i,i} s_i + a_{i,i+1} s_{i+1} + \ldots + a_{i,j} s_j + \ldots + a_{i,n'} s_{n'}.$$

For $1 \le i < n'$ we set

$$\tilde{\rho}_i = a_{i,i} \rho_i + a_{i,i+1} \rho_{i+1} + \ldots + a_{i,n'} \rho_{n'}.$$

By definition, the coefficient of $\tilde{\rho}_i$ at $1 = e^0$ is

$$-a_{i,i}s_i - a_{i,i+1}s_{i+1} - \ldots - a_{i,n'}s_{n'} = -d_i.$$

Set $\tilde{\rho}(\omega_i) = \tilde{\rho}_i + d_i \in R[\Lambda^{(1)}]$.

Definition 3.2. *Fix $\lambda_0 \in \Lambda^{(1)}$ and consider the following subsets of elements in $R[T^*]$:*

(1) $\{h_{1,i} = e^{\lambda_0}(\frac{r_i}{s_i}\rho(\omega_i) - \frac{r_i}{d_{i+1}}\tilde{\rho}(\omega_{i+1})) \mid 1 \le i < n',\ r_i = \mathrm{lcm}(s_i, d_{i+1})\}$.

(2) $\{h_{2,i} = \rho(\omega_i)\tilde{\rho}(\omega_1) - ds_i \mid 1 \le i \le n'\}$,

(3) $\{h_{3,i} = \rho_i \mid n' < i \le n\}$.

Remark 3.3. *The elements $h_{1,i}$ will be extensively used (see* (11) *and* (13)*) in the computations of the group of semi-decomposable invariants.*

Let I be the augmentation ideal of $R[\Lambda]$, that is I is the kernel of the map $R[\Lambda] \to R$ given by $e^\lambda \mapsto \lambda$. Let I_{sc}^W denote the ideal in $R[\Lambda]$ generated by elements from $R[\Lambda]^W \cap I$. By the Chevalley theorem I_{sc}^W is generated by the elements ρ_i, $1 \le i \le n$, i.e., any $f \in I_{sc}^W$ can be written as $f = f_1\rho_1 + \ldots + f_n\rho_n$ for some $f_i \in R[\Lambda]$.
Our main result is the following

Theorem 3.4. *Assume that the n-tuple $(\rho_1, \ldots, \rho_n)$ satisfies the generalized flatness condition with respect to some basis $\{x_i\}$ of the weight lattice Λ.*
Then the elements $h_{k,i}$ of Definition 3.2 generate the ideal $I_{sc}^W \cap R[T^*]$ in $R[T^*]$.

Proof. Suppose that $f_1\rho_1 + \ldots + f_n\rho_n \in R[T^*]$ for some $f_i \in R[\Lambda]$. By Lemma 3.1 we may assume that $d \mid f_i^{(1-|i|)}$ for each i.
To prove the theorem we modify $(f_1, \ldots, f_n)$ in several steps. At each step, we subtract a linear combination of the elements $h_{i,j}$ (with coefficients in $R[T^*]$) from $f_1\rho_1 + \ldots + f_n\rho_n$ so that the new polynomials $f_1', \ldots, f_n'$ have fewer non-zero monomials. In the end they will all become 0, so that the original $f_1\rho_1 + \ldots + f_n\rho_n$ will be replaced by a linear combination of $h_{i,j}$.
Step 1. By definition we have for $1 \le i \le n'$

$$\begin{aligned} h_{2,i} &= \rho(\omega_i)\tilde{\rho}(\omega_1) - ds_i = \rho(\omega_i)(\tilde{\rho}_1 + d_1) - ds_i \\ &= \rho(\omega_i)(a_{1,1}\rho_1 + a_{1,2}\rho_2 + \ldots + a_{1,n'}\rho_{n'} + d) - ds_i \\ &= a_{1,1}\rho(\omega_i)\rho_1 + \ldots + (a_{1,i}\rho(\omega_i) + d)\rho_i + \ldots + a_{1,n'}\rho(\omega_i)\rho_{n'}. \end{aligned}$$

Since $|i| = 1$, $d \mid f_i^{(0)}$. Consider the difference

$$f_1'\rho_1 + \ldots + f_n'\rho_n = f_1\rho_1 + \ldots + f_n\rho_n - \tfrac{1}{d}f_i^{(0)}h_{2,i}. \qquad (*)$$

Collecting the coefficients we obtain

- $f'_j = f_j$ for all $j > n'$,
- $f'^{(0)}_j = f^{(0)}_j$ for all $j \leq n'$ and $j \neq i$,
- $f'^{(0)}_i = 0$.

Hence, applying (*) for each i, $1 \leq i \leq n'$ we obtain new coefficients $(f'_1, \dots, f'_n)$ such that $f'^{(0)}_j = 0$ for all $j \leq n'$ and $f'_j = f_j$ for all $j > n'$. (Observe that $f'^{(1)}_j$ for $j \leq n'$ does not necessarily coincides with $f^{(1)}_j$.)
Step 2. We have for $n' < i \leq n$

$$\begin{aligned}\tilde{\rho}(\omega_1)h_{3,i} = \tilde{\rho}(\omega_1)\rho_i &= (a_{1,1}\rho_1 + a_{1,2}\rho_2 + \ldots + a_{1,n'}\rho_{n'} + d)\rho_i \\ &= (a_{1,1}\rho_i)\rho_1 + \ldots + (a_{1,n'}\rho_i)\rho_{n'} + d\rho_i.\end{aligned}$$

Since $|i| = 0$, $d \mid f^{(1)}_i$. Consider the difference

$$f'_1\rho_1 + \ldots + f'_n\rho_n = f_1\rho_1 + \ldots + f_n\rho_n - \tfrac{1}{d} f^{(1)}_i \tilde{\rho}(\omega_1)h_{3,i}. \qquad (**)$$

Collecting the coefficients we obtain

- $f'_j = f_j$ for all $j > n'$ and $j \neq i$,
- $f'^{(0)}_j = f^{(0)}_j$ for all $j \leq n'$.
- $f'^{(1)}_i = 0$.

Hence, applying (**) for each i, $n' < i \leq n$ we obtain new coefficients $(f'_1, \dots, f'_n)$ such that $f'^{(1)}_j = 0$ for all $j > n'$ and $f'^{(0)}_j = 0$ for all $j \leq n'$.
Step 3. As a result of step 2, we have $f_i \in R[T^*]$ for all $i > n'$. Subtracting

$$f_1\rho_1 + \ldots + f_n\rho_n - \sum_{i>n'} f_i h_{3,i}$$

we may assume that $f_i = 0$ for all $i > n'$.
Step 4. Fix i, $1 \leq i \leq n'$. If $i > 1$ we assume in addition that $f_1 = \ldots = f_{i-1} = 0$. So, we have $f_i\rho_i + \ldots + f_{n'}\rho_{n'} \in R[T^*]$, where $f^{(0)}_j = 0$ for all $i \leq j \leq n'$ by previous steps. Hence, we can express it as

$$f_i\rho_i + \ldots + f_{n'}\rho_{n'} = f^{(1)}_i \rho(\omega_i) + \ldots + f^{(1)}_{n'} \rho(\omega_{n'}) - (s_i f^{(1)}_i + \ldots + s_{n'} f^{(1)}_{n'}) \in R[T^*].$$

Since $\deg(\rho(\omega_j)) = 1$ for $i \leq j \leq n'$, we obtain

$$s_i f^{(1)}_i = -s_{i+1} f^{(1)}_{i+1} - \ldots - s_{n'} f^{(1)}_{n'}.$$

The right hand side of this equation is divisible by r_i, hence, $\frac{r_i}{s_i} \mid f^{(1)}_i$.

By definition, we have

$$\begin{aligned} h_{1,i} &= e^{\lambda_0}(\tfrac{r_i}{s_i}\rho(\omega_i) - \tfrac{r_i}{d_{i+1}}\tilde\rho(\omega_{i+1})) = e^{\lambda_0}(\tfrac{r_i}{s_i}\rho_i - \tfrac{r_i}{d_{i+1}}\tilde\rho_{i+1}) \\ &= e^{\lambda_0}(\tfrac{r_i}{s_i}\rho_i - \tfrac{r_i}{d_{i+1}}(a_{i+1,i+1}\rho_{i+1} + a_{i+1,i+2}\rho_{i+2} + \ldots + a_{i+1,n'}\rho_{n'})) \end{aligned}$$

Consider the difference

$$f_i'\rho_i + \ldots + f_{n'}'\rho_{n'} = f_i\rho_i + \ldots + f_{n'}\rho_{n'} - \tfrac{s_i}{r_i} f_i^{(1)} e^{-\lambda_0} h_{1,i}. \qquad (***)$$

Collecting the coefficients we obtain $f_i' = 0$ while keeping $f_j'^{(0)} = f_j^{(0)} = 0$ for all $i < j \le n'$. Hence, applying (***) inductively starting with $i = 1$, we obtain that $f_i = 0$ for all $1 \le i \le n'$. ☐

4 The generalized flatness condition

In the present section we prove that the n-tuple of W-orbits $(\rho_1, \ldots, \rho_n)$ in $R[\Lambda]$ satisfies the generalized flatness condition when Λ is a weight lattice for a semi-simple root system of type A and C. Observe that it is enough to prove the generalized flatness condition for each simple component.

4.1 Type A

Let $\widetilde\Lambda = \mathbb{Z}^{n+1}$ with a standard basis $e_1, \ldots, e_{n+1}$. The weight lattice of type A is then given by $\Lambda = \widetilde\Lambda/(e_1 + \ldots + e_{n+1})$. We denote the class of e_i in Λ by $\bar e_i$. The basis of Λ is given by the fundamental weights $\omega_i = \bar e_1 + \ldots + \bar e_i$, $i = 1, \ldots, n$. The Weyl group (the symmetric group S_{n+1}) acts by permutations of $\{e_1, \ldots, e_{n+1}\}$. Let $x_i = e^{\omega_i}$ in $\mathbb{Z}[\Lambda]$ and let $y_i = e^{e_i}$ in $\mathbb{Z}[\tilde\Lambda]$.
Consider the induced map $\phi\colon \mathbb{Z}[\tilde\Lambda] \to \mathbb{Z}[\Lambda]$ given by $\phi(y_1) = x_1$, $\phi(y_i) = e^{\bar e_i} = x_i x_{i-1}^{-1}$, $1 < i \le n$ and $\phi(y_{n+1}) = x_n^{-1}$. The image of the elementary symmetric function $\sigma_i = \sigma_i(y_1, \ldots, y_{n+1})$ gives the W-orbit $\rho(\omega_i)$.
Let g_i be (the complete sum symmetric function) the sum of all monomials of total degree i in variables $y_1, \ldots, y_{n+2-i}$. We have the following analogue of the Newton relation (see [16, Relation (2)]) for $i > 0$

$$\sum_{j=0}^{i} (-1)^j g_j(y_1, \ldots, y_{n+2-j})\sigma_{i-j}(y_1, \ldots, y_{n+1-j}) = 0 \quad (\text{here } g_0 = \sigma_0 = 1) \quad (1)$$

which implies that the ideal $I_\sigma = (\sigma_1, \ldots, \sigma_{n+1})$ in $\mathbb{Z}[y_1, \ldots, y_{n+1}]$ coincides with the ideal $I_g = (g_1, \ldots, g_{n+1})$. Consider the involution τ of $\mathbb{Z}[y_1, \ldots, y_{n+1}]$ given by $y_i \mapsto 1 - y_i$. We get

$$(\sigma_1 - s_1, \ldots, \sigma_{n+1} - s_{n+1}) = \tau(I_\sigma) = \tau(I_g) = (\tilde g_1, \ldots, \tilde g_{n+1}),$$

where $s_i = \sigma_i(1, \ldots, 1) = |W(\omega_i)|$ and $\tilde g_i$ is the (non-homogeneous) polynomial in variables $y_1, \ldots, y_{n+2-i}$ of degree i such that its coefficient at y_{n+2-i}^i is ± 1. Taking its images in $\mathbb{Z}[\Lambda]$ we obtain

$$(\rho_1, \ldots, \rho_n) = \phi(\tau(I_\sigma)) = \phi(\tau(I_g)) = (r_1, \ldots, r_{n+1}),$$

where $\rho_i = \rho(\omega_i) - s_i$ and $r_i = \phi(\tilde{g}_{n+2-i})$. We claim that r_{n+1} can be written as a linear combination of $r_1, \dots, r_n$. Indeed, taking the sum of relations (1) for all i we obtain

$$1 = \sum_{i=0}^{n+1} (g_i \prod_{j=1}^{n+1-i} (1 - y_j)).$$

After applying τ we obtain $1 = \sum_{i=0}^{n+1} \tilde{g}_i y_1 \dots y_{n+1-i}$. Since $y_1 \dots y_{n+1} = 1$, after taking its image in $\mathbb{Z}[\Lambda]$ we obtain the desired linear combination.

4.2 Type C

Consider the weight lattice Λ of type C. It is generated by the standard vectors $\{e_1, \dots, e_n\}$ with fundamental weights $\omega_i = e_1 + \dots + e_i$. The Weyl group W acts on the standard vectors by permutations and changing signs. Consider the embedding $\phi\colon \mathbb{Z}[y_1, \dots, y_n] \hookrightarrow \mathbb{Z}[\Lambda]$ given by $\phi(y_i) = e^{e_i} + e^{-e_i}$. The image of the elementary symmetric function $\sigma_i = \sigma_i(y_1, \dots, y_n)$ gives the W-orbit $\rho(\omega_i)$. As in type A let g_i be the sum of all monomials of total degree i in variables $y_1, \dots, y_{n+1-i}$. Then $(\sigma_1, \dots, \sigma_n) = (g_1, \dots, g_n)$ as ideals in $\mathbb{Z}[y_1, \dots, y_n]$. Applying the involution τ we obtain $(\rho_1, \dots, \rho_n) = (r_1, \dots, r_n)$, where $\rho_i = \rho(\omega_i) - s_i$, $r_i = \phi(\tau(g_{n+1-i})$ and $s_i = |W(\omega_i)|$. The n-tuple $(r_1, \dots, r_n)$ satisfies the flatness condition and there is an invertible transformation matrix between ρ_i and r_i.

5 Characters and invariants

In the present section we introduce some notation and recall basic definitions for the group of characters, characteristic classes and invariants which will be used in the subsequent sections. We follow [14], [15] and [1].

5.1 Characters.

Let H and H' be simply connected simple split groups of the same Dynkin type $\mathcal{D}$ over a field F. Assume that there is a central diagonal subgroup μ_k in $H \times H'$. The quotient $G = (H \times H')/\mu_k$ will be called a *group of index* k of type $\mathcal{D}$.

We denote by T_{sc} the split maximal torus of $H \times H'$, by T the split maximal torus of G and by T_{ad} the split maximal torus of the product of the adjoint forms $H_{ad} \times H'_{ad}$. Then there is an exact sequence for the groups of characters

$$0 \to T^*/T^*_{ad} \to T^*_{sc}/T^*_{ad} \to \mathbb{Z}/k\mathbb{Z} \to 0$$

which can be used to describe T^*. Indeed, the quotient T^*_{sc}/T^*_{ad} is the group of characters of the center $Z(G)$ and the map $T^*_{sc}/T^*_{ad} \to \mathbb{Z}/k\mathbb{Z}$ is induced by the diagonal embedding $\mu_k \to Z(G)$. Moreover, T^*_{sc}/T^*_{ad} is the product of groups of characters of the centers of H and H', hence,

$$T^*_{sc}/T^*_{ad} = \Lambda_w/\Lambda_r \oplus \Lambda'_w/\Lambda'_r \to \mathbb{Z}/k\mathbb{Z},$$

is given by taking the sum, where Λ_w (resp. Λ'_w) is the weight lattice and Λ_r (resp. Λ'_r) is the root lattice of H (resp. of H').

5.2 Invariant forms.

Let $W = W_H \times W_{H'}$ be the Weyl group of $H \times H'$. It naturally acts on $T^*_{sc} = \Lambda_w \oplus \Lambda'_w$. Consider the group of W-invariant quadratic forms. It is a direct sum of cyclic groups

$$S^2(T^*_{sc})^W = S^2(\Lambda_w)^{W_H} \oplus S^2(\Lambda'_w)^{W_{H'}} = \mathbb{Z}q \oplus \mathbb{Z}q',$$

where q and q' are generators given by normalized Killing forms. So any form $\phi \in S^2(T^*_{sc})^W$ can be written uniquely as $\phi = dq + d'q'$, $d, d' \in \mathbb{Z}$. The list of Killing forms for all types can be found in [14, §4].
Let $\{\omega_i\}$ and $\{\omega'_i\}$ denote the fundamental weights of H and H', i.e., the $\mathbb{Z}$-bases of Λ_w and Λ'_w. Choose a $\mathbb{Z}$-basis $\{x_i\}$ of T^*. Expressing each ω_i and ω'_i in terms of x_j's and substituting into ϕ allows us to explicitly describe the subgroup

$$Q(G) = S^2(T^*)^W = S^2(T^*) \cap S^2(T^*_{sc})^W.$$

5.3 Characteristic map

Consider the group ring $\mathbb{Z}[T^*_{sc}]$ that is the representation ring $R(T_{sc})$ of T_{sc}. Each element of $\mathbb{Z}[T^*_{sc}]$ can be written as a finite linear combination $\sum_i a_i e^{\lambda_i}$, $a_i \in \mathbb{Z}$, $\lambda_i \in T^*_{sc}$. Fix a basis of T^*_{sc} consisting of fundamental weights $\{\omega_i\}$ and $\{\omega'_j\}$. Following [14, §3c] and [5] we define a $\mathbb{Z}[W]$-module homomorphism

$$c_2 \colon \mathbb{Z}[T^*_{sc}] \to S^{\leq 2}(T^*_{sc}) \to S^2(T^*_{sc})$$

by sending $1 \mapsto 1$, $e^{-\omega_i} \mapsto 1 - \omega_i$ and $e^{\omega_i} \mapsto 1 + \omega_i + \omega_i^2$ (resp. for ω'_j) and then taking the degree 2 homogeneous component.
Let I_{sc} denote the augmentation ideal in $\mathbb{Z}[T^*_{sc}]$, i.e., the kernel of the trace map $\mathbb{Z}[T^*_{sc}] \to \mathbb{Z}$, $e^\lambda \mapsto 1$. Then the image $c_2(I^3_{sc}) = 0$, so c_2 can be restricted to I^2_{sc}[5].
Observe that the filtration by powers of the ideal I_{sc} can be viewed as a γ-filtration on $K_0(BT)$; its image in $K_0(G/B)$ via the characteristic map $c \colon \mathbb{Z}[T^*_{sc}] \to K_0(G/B)$ gives the Grothendieck γ-filtration on $K_0(G/B)$ (e.g., see [19]).

5.4 Invariants

Given $\lambda \in T^*$ denote by $\rho(\lambda) = \sum_{\chi \in W(\lambda)} e^\chi$, where $W(\lambda)$ is the W-orbit of λ. If restricted to invariants, the map c_2 defines a group homomorphism

$$c_2 \colon \mathbb{Z}[T^*]^W \to S^2(T^*)^W$$

with image generated by forms $c_2(\rho(\lambda)) = -\frac{1}{2}\sum_{\chi \in W(\lambda)} \chi^2$ for all $\lambda \in T^*$ [14, §3c]. It was shown in [14] that the image $c_2(\mathbb{Z}[T^*]^W)$ can be identified with the group of degree 3 decomposable invariants $\mathrm{Dec}(G)$ and the quotient $Q(G)/\mathrm{Dec}(G)$ with the group of indecomposable invariants denoted by $\mathrm{Inv}^3_{ind}(G)$. By definition for any two semisimple groups G_1, G_2 we have

$$Q(G_1 \times G_2) = Q(G_1) \times Q(G_2) \text{ and } \mathrm{Dec}(G_1 \times G_2) = \mathrm{Dec}(G_1) \times \mathrm{Dec}(G_2). \quad (2)$$

Let I^W_{sc} denote an ideal in $\mathbb{Z}[T^*_{sc}]$ generated by W-invariants from the augmentation ideal I_{sc}, namely, $I^W_{sc} = (I_{sc} \cap \mathbb{Z}[T^*_{sc}]^W)$. The main result of [15] says that the image $c_2(\mathbb{Z}[T^*] \cap I^W_{sc})$ in $S^2(T^*)^W$ coincides with the subgroup of semi-decomposable invariants $\mathrm{Sdec}(G)$ and that $\mathrm{Dec}(H) = \mathrm{Sdec}(H)$ if H is a simple group. Observe that we have

$$\mathrm{Dec}(G) \subseteq \mathrm{Sdec}(G) \subseteq Q(G). \quad (3)$$

We denote by $\mathrm{Inv}^3_{sd}(G)$ the quotient $\mathrm{Sdec}(G)/\mathrm{Dec}(G)$.

6 Type A

In the present section we consider semisimple groups of type A. The following lemma gives a simple geometric proof for the coincidence between the normalized invariants and semi-decomposable invariants (c.f. [1]):

Lemma 6.1. *Let $G = (\prod_{i=1}^m \mathbf{SL}_{n_i})/\boldsymbol{\mu}_k$, $m, n_i, k \geq 2$, where $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_k$ is a diagonal (central) subgroup. Then, $Q(G) = \mathrm{Sdec}(G)$, i.e., each degree 3 indecomposable invariant of G is semi-decomposable.*

Proof. We follow arguments used in [1, §5]. Assume

$$\boldsymbol{\mu} = \{(\lambda_1, \ldots, \lambda_m) \in \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_m} \mid \lambda_1^k = \cdots = \lambda_m^k = 1,\, \lambda_1 = \cdots = \lambda_m\}$$

is the diagonal subgroup. The corresponding versal flag variety X over the function field F' of the classifying space of G can be replaced by the product of Severi-Brauer varieties $\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_m)$, where A_i are central simple F'-algebras of degree n_i for $1 \leq i \leq m$ such that $k[A_i] = 0$ and $[A_1] = \cdots = [A_m]$ in the Brauer group $\mathrm{Br}(F')$. Let B be the common underlying division algebra of $A_1, \ldots, A_m$ such that the index and the exponent of B are all equal to k. Then, we obtain $\mathrm{CH}^2(X)_{\mathrm{tors}} = \mathrm{CH}^2(\mathrm{SB}(B))_{\mathrm{tors}}$. By [10, Cor.4] we have $\mathrm{CH}^2(\mathrm{SB}(B))_{\mathrm{tors}} = 0$, thus by the main theorem of [15] $Q(G)$ coincides with $\mathrm{Sdec}(G)$. □

Remark 6.2. *Using arguments in [1] one can compute the quotient*

$$\mathrm{Inv}^3_{ind}(G) = Q(G)/\mathrm{Sdec}(G) = \mathrm{CH}^2(X)_{\mathrm{tors}}$$

for an arbitrary semisimple group G of type A. *For instance, when $m = 2$, the same arguments in the proof work if we replace the diagonal subgroup $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_2$ by a central subgroup $\boldsymbol{\mu}_2 \times 1 \subseteq \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2}$ or $1 \times \boldsymbol{\mu}_2 \subseteq \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2}$.*

The following proposition deals with groups of p-primary index for any prime p, which in turn computes the p-primary component of $\mathrm{Inv}^3_{ind}((\mathbf{SL}_m \times \mathbf{SL}_n)/\boldsymbol{\mu}_k)$ for any diagonal (central) subgroup $\boldsymbol{\mu}_k$.

Proposition 6.3. *Let $G = (\mathbf{SL}_m \times \mathbf{SL}_n)/\boldsymbol{\mu}$, $m,n \geq 2$, where $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_k$ is a p-primary diagonal (central) subgroup. Then,*

$$Q(G) = \{dq + d'q' \mid (\tfrac{k-1}{k})(md+nd') \equiv 0 \bmod 2k\}$$

and

$$\mathrm{Dec}(G) \simeq \begin{cases} k\mathbb{Z}q \oplus k\mathbb{Z}q' & \text{if } p \neq 2 \text{ or } p=2, \min\{v_2(m), v_2(n)\} > v_2(k),\\ k\mathbb{Z}(q-q') \oplus k\mathbb{Z}(q+q') & \text{if } p=2, v_2(m)=v_2(n)=v_2(k),\\ k\mathbb{Z}q \oplus 2k\mathbb{Z}q' & \text{if } p=2, v_2(m) > v_2(k) = v_2(n),\end{cases}$$

where q (resp. q') is the normalized Killing form of $\mathbf{SL}_m$ (resp. $\mathbf{SL}_n$).

Proof. Let $G = (\mathbf{SL}_m \times \mathbf{SL}_n)/\boldsymbol{\mu}_k$, $m,n,k \geq 2$, $k \mid \gcd(m,n)$, where $\boldsymbol{\mu}_k$ is a diagonal subgroup. Then, by 5.1 the character group of the split maximal torus T of G is given by

$$T^* = \{\sum_{i=1}^{m-1} a_i\omega_i + \sum_{j=1}^{n-1} a'_j\omega'_j \mid \sum_{r=0}^{\frac{m-k}{k}}\sum_{i=1}^{k-1} ia_{i+rk} + \sum_{s=0}^{\frac{n-k}{k}}\sum_{i=1}^{k-1} ia'_{i+sk} \equiv 0 \bmod k\}. \tag{4}$$

Following 5.2 the group of W-invariant quadratic forms $S^2(T^*_{sc})^W$ is generated by the normalized Killing forms

$$q = \sum_{i=1}^{m-1}\omega_i^2 - \sum_{i=1}^{m-2}\omega_i\omega_{i+1} \text{ and } q' = \sum_{j=1}^{n-1}\omega_j'^2 - \sum_{j=1}^{n-2}\omega'_j\omega'_{j+1}.$$

Consider the $\mathbb{Z}$-basis $\{x_1,\ldots,x_{m-1},x'_1,\ldots,x'_{n-1}\}$ of the character group T^* given by

$$x_{i+rk} = \omega_{i+rk} + i\omega'_{n-1},\ x_{k+rk} = \omega_{k+rk} \text{ and } x'_{i+sk} = \omega'_{i+sk} + i\omega'_{n-1},\ x'_{k+sk} = \omega'_{k+sk},$$

where $1 \leq i \leq k-1$, $0 \leq r \leq \frac{m-k}{k}$, $0 \leq s \leq \frac{n-k}{k}$ (for convenience, we set $x_j = x'_j = 0$ for any $j > n-1$ or $j > m-1$). In this basis a form $\phi \in S^2(T^*_{sc})^W$ can be written as

$$\phi = (\tfrac{md(k-1)+nd'(k-1)}{2k^2})x'^2_{n-1} + \psi, \quad d,d' \in \mathbb{Z}$$

where ψ is a quadratic form with integer coefficients. Hence, we obtain

$$Q(G) = \{dq + d'q' \mid (\tfrac{k-1}{k})(md+nd') \equiv 0 \bmod 2k\}. \tag{5}$$

From now on we assume that k is p-primary. We claim that $\mathrm{Dec}(G) \subseteq k\mathbb{Z}q \oplus k\mathbb{Z}q'$. To show this we extend the arguments in [4, p.136]. We use the standard presentation of the root system of type A, namely, that Λ_w (resp. Λ'_w) consists

of vectors in the standard basis $\{e_1, .., e_m\}$ (resp. $\{e'_1, .., e'_n\}$) whose sum of coordinates is zero.
Choose a character $\chi \in T^*$. Assume χ has l (resp. l') distinct coordinates in some order $b_1 > \cdots > b_l$ (resp. $b'_1 > \ldots > b'_{l'}$) which repeat $r_1, .., r_l$ (resp. $s_1, .., s_{l'}$) times with respect to the basis $\{e_i\}$ (resp. $\{e'_j\}$). Then, for the orbit $\rho(\chi)$ of χ under the action of W we obtain

$$c_2(\rho(\chi)) = \tfrac{n!}{s_1!\cdots s_{l'}!} \cdot [r, b] \cdot q + \tfrac{m!}{r_1!\cdots r_l!} \cdot [s, b'] \cdot q', \text{ where} \tag{6}$$

$$[r, b] = \tfrac{(m-2)!}{r_1!\cdots r_l!}\big(m(\sum_{i=1}^{l} r_i b_i^2) - (\sum_{i=1}^{l} r_i b_i)^2\big), [s, b'] = \tfrac{(n-2)!}{s_1!\cdots s_l!}\big(n(\sum_{j=1}^{l'} s_j b_j'^2) - (\sum_{j=1}^{l'} s_j b'_j)^2\big).$$

Observe that by (4) we have $\sum r_i b_i + \sum s_j b'_j \equiv 0 \bmod k$. Let $c = \min\{v_p(r_i)\}$ and $d = \min\{v_p(s_j)\}$. If $v_p(k) \leq d$, then $v_p(k) \leq v_p(\sum r_i b_i)$. So we obtain $v_p(k) \leq v_p(\gcd(m, \sum r_i b_i))$. Hence, by [4, p.137, Lemma 11.4] the coefficient $[r, b]$ is divisible by k. Similarly, if $v_2(k) \leq c$, then $[s, b']$ is divisible by k. Hence, we may assume that $c, d \leq v_p(k)$. By [4, p.137, Lemma 11.3], we have

$$v_p\big(\tfrac{n!}{s_1!\cdots s_{l'}!}\big) \geq v_p(k) - d \geq 0 \text{ and } v_p\big(\tfrac{m!}{r_1!\cdots r_l!}\big) \geq v_p(k) - c \geq 0,$$

which implies that

$$v_p(\gcd(m, \sum r_i b_i)) \geq d \text{ and } v_p(\gcd(n, \sum s_j b'_j)) \geq c.$$

Therefore, again by [4, p.137, Lemma 11.4] we see that each coefficient of q and q' in (6) is divisible by k, which proves the claim.
Finally, we compute the group $\mathrm{Dec}(G)$ case by case. As

$$c_2(\rho(k\omega_1)) = -k^2 q,\ c_2(\rho(2\omega_1 - \omega_2)) = -2mq \tag{7}$$

and similarly, $c_2(\rho(k\omega'_1)) = -k^2 q'$, $c_2(\rho(2\omega'_1 - \omega'_2)) = -2mq'$, we have

$$\gcd(2, p)kq \in \mathrm{Dec}(G) \text{ if } v_p(m) = v_p(k) \tag{8}$$

and similarly, $\gcd(2, p)kq' \in \mathrm{Dec}(G)$ if $v_p(n) = v_p(k)$. Moreover, we get

$$c_2(\rho(\omega_k)) = (kk')q \text{ with } \gcd(k', p) = 1 \text{ if } v_p(m) > v_p(k) \tag{9}$$

and similarly, $c_2(\rho(\omega_k)) = (kk'')q$ with $\gcd(k'', p) = 1$ if $v_p(n) > v_p(k)$ (see also [2, Thm.4.1]). Thus if $p \neq 2$, then by (7) and (9) we obtain

$$\gcd(k^2, kk')q = kq \in \mathrm{Dec}(G), \quad \gcd(k^2, kk'')q = kq' \in \mathrm{Dec}(G).$$

Therefore, by (8) and the claim above, $\mathrm{Dec}(G) = k\mathbb{Z}q \oplus k\mathbb{Z}q'$ if $p \neq 2$.
Now assume that $p = 2$. If $v_2(m) > v_2(k)$ and $v_2(n) > v_2(k)$, then by (7), (9) and the claim above, we have $\mathrm{Dec}(G) = k\mathbb{Z}q \oplus k\mathbb{Z}q'$.
If $v_2(m) = v_2(n) = v_2(k)$, then $c_2(\rho(\omega_{k/2} + \omega'_{k/2})) \equiv -kq - kq' \bmod 2k$. Hence, by (8), $k(q - q')$, $k(q + q') \in \mathrm{Dec}(G)$. Since $\mathrm{Dec}(\mathbf{SL}_m / \boldsymbol{\mu}_k) = 2kq$ if $v_2(m) =$

$v_2(k)$, and $\mathrm{Dec}(\mathbf{SL}_n/\boldsymbol{\mu}_k) = 2kq'$ if $v_2(n) = v_2(k)$ ([2, Thm.4.1]), it follows from the claim above that $\mathrm{Dec}(G) = k\mathbb{Z}(q-q') \oplus k\mathbb{Z}(q+q')$.
Similarly, if $v_2(m) > v_2(k) = v_2(n)$, then by (7) and (9) we have $kq \in \mathrm{Dec}(G)$. Since $\mathrm{Dec}(\mathbf{SL}_n/\boldsymbol{\mu}_k) = 2kq'$, we get $kq', k(q \pm q') \notin \mathrm{Dec}(G)$. Therefore, by the claim we obtain $\mathrm{Dec}(G) = k\mathbb{Z}q \oplus 2k\mathbb{Z}q'$. □

Remark 6.4. *This proposition and its proof generalize [14, Theorem 4.4] and [2, Theorem 4.1] for split simple groups $G' = \mathbf{SL}_m/\boldsymbol{\mu}_k$ of type A.*
Indeed, following the proof of Proposition 6.3 one can easily check that $Q(G') = Q(G)|_{d'=0}$. For instance, in order to obtain $Q(\mathbf{PGL}_m)$ we simply set $d' = 0$ and $k = m$ in (5), then we get $Q(\mathbf{PGL}_m) = 2\gcd(2,m)\mathbb{Z}q$ [14, Theorem 4.4]. Similarly, we can obtain $\mathrm{Dec}(\mathbf{PGL}_m)$ in the same way.
To compute the indecomposable groups for $G' = \mathbf{SL}_{2m}/\boldsymbol{\mu}_2$, we set $d' = 0$, $k = 2$. Then it follows by Proposition 6.3 that $Q(G') = \{dq \mid md \equiv 0 \bmod 4\}$ and

$$\mathrm{Dec}(G') = \begin{cases} 2\mathbb{Z}q & \text{if } v_2(m) > 0, \\ 4\mathbb{Z}q & \text{if } v_2(m) = 0. \end{cases}$$

Hence, we obtain [2, Theorem 4.1], that is

$$\mathrm{Inv}^3_{ind}(G') \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})q & \text{if } v_2(m) \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Finally observe that together with Lemma 6.1, properties (2) *and* (3) *it computes the group $\mathrm{Inv}^3_{sd}(G'')$, where $G'' = (\mathbf{SL}_{2m}/\boldsymbol{\mu}_2) \times \mathbf{SL}_{2n}$, $n, m \geq 1$.*

The following corollary generalizes [15, Example 3.1] to groups of type A (see also [1]). Similarly, by using Lemma 6.1 and Proposition 6.3, one can compute both groups $\mathrm{Inv}^3_{ind}(G)$ and $\mathrm{Inv}^3_{sd}(G)$ for any p-primary diagonal subgroup $\boldsymbol{\mu}_k$.

Corollary 6.5. *Let $G = (\prod_{i=1}^m \mathbf{SL}_{2n_i})/\boldsymbol{\mu}_2$, $n_i \geq 1$, where $\boldsymbol{\mu}_2$ is the diagonal subgroup. Then*

$$\mathrm{Inv}^3_{sd}(G) = \mathrm{Inv}^3_{ind}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus m} & \text{if } n_i \equiv 0 \bmod 4 \ \forall i, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus m-1} & \text{otherwise.} \end{cases}$$

Proof. Let $G = (\mathbf{SL}_{2m} \times \mathbf{SL}_{2n})/\boldsymbol{\mu}_2$. Then, it follows from Proposition 6.3 that

$$\mathrm{Inv}^3_{ind}(G) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})q \oplus (\mathbb{Z}/2\mathbb{Z})q' & m \equiv n \equiv 0 \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})q & m \equiv 0, n \equiv 2 \bmod 4 \text{ or } m \equiv 0 \bmod 4, n \text{ is odd}, \\ (\mathbb{Z}/2\mathbb{Z})(q-q') & m \equiv n \equiv 2 \bmod 4 \text{ or both } m, n \text{ are odd}, \\ (\mathbb{Z}/2\mathbb{Z})(q-2q') & m \equiv 2 \bmod 4, n \text{ is odd.} \end{cases}$$

Hence, the result follows by Lemma 6.1. Applying the same arguments for three and more groups completes the proof. □

In the following we show that the both indecomposable group $\mathrm{Inv}^3_{ind}(G)$ and the semi-decomposable group $\mathrm{Inv}^3_{sd}(G)$ can have an arbitrary order (c.f. [1]). In particular, the order can be arbitrarily large.

Corollary 6.6. *For an arbitrary integer $k \geq 2$ there exists a semisimple group G of type* A *such that*

$$|\operatorname{Inv}^3_{ind}(G)| = |\operatorname{Inv}^3_{sd}(G)| = k.$$

Moreover, for any homocyclic p-group C there exists a semisimple group H of type A *such that $\operatorname{Inv}^3_{ind}(H) = \operatorname{Inv}^3_{sd}(H) = C$.*

Proof. Let p^r ($r \geq 1$) be a prime factor of k and let $n = \gcd(2,p)p^r$. We denote $G[p^r] = (\mathbf{SL}_n \times \mathbf{SL}_n)/\boldsymbol{\mu}_{p^r}$, where $\boldsymbol{\mu}_{p^r}$ is the diagonal subgroup. By Proposition 6.3 we have

$$\operatorname{Inv}^3_{ind}(G[p^r]) = \operatorname{Inv}^3_{sd}(G[p^r]) = \mathbb{Z}/p^r\mathbb{Z}.$$

Set $G = (\mathbf{SL}_{\gcd(2,k)k} \times \mathbf{SL}_{\gcd(2,k)k})/\boldsymbol{\mu}_k$. Then, the same argument as in [15, §3b] shows that $\operatorname{Inv}^3_{ind}(G[p^r])$ is a p-primary component of $\operatorname{Inv}^3_{ind}(G)$ and the first statement follows from Lemma 6.1.

Let $C = (\mathbb{Z}/p^r\mathbb{Z})^{\oplus m}$ be a homocyclic p-group of rank m for some prime p. It suffices to consider the case $m \geq 2$. Let $H = (\mathbf{SL}_n)^m/\boldsymbol{\mu}_{p^r}$, where $n = \gcd(2,p)p^{2r}$ and $\boldsymbol{\mu}_{p^r}$ is the diagonal subgroup. Then, the arguments used in the proof of Proposition 6.3 yield

$$Q(G) = \{\sum_{i=1}^{m} d_i q_i \mid (n/p^r)\sum_{i=1}^{m} d_i \equiv 0 \bmod \gcd(2,p)p^r\},$$

where q_i is the corresponding normalized Killing form of $\mathbf{SL}_n$. Similarly, we have $\operatorname{Dec}(G) = \bigoplus_{i=1}^m p^r\mathbb{Z}q_i$. Then the second statement follows by Remark 6.2. □

7 Type B

In the present section we show that any semi-decomposable invariant of semisimple groups of type B is decomposable, except in the case of a product of groups of type $B_2 = C_2$ modulo the diagonal subgroup $\boldsymbol{\mu}_2$. We first consider the index 2 case.

Proposition 7.1. *Let $G = (\mathbf{Spin}_{2m+1} \times \mathbf{Spin}_{2n+1})/\boldsymbol{\mu}_2$, $m, n \geq 2$, where $\boldsymbol{\mu}_2$ is the diagonal subgroup. Then, we have $\operatorname{Inv}^3_{ind}(G) = \mathbb{Z}/2\mathbb{Z}$ and*

$$\operatorname{Inv}^3_{sd}(G) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } m = n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., each semi-decomposable invariant is decomposable unless $m = n = 2$.

Proof. Following 5.1 the character group of the split maximal torus T of G is given by

$$T^* = \{\sum_{i=1}^{m} a_i\omega_i + \sum_{i=1}^{n} a'_i\omega'_i \mid a_m \equiv a'_n \bmod 2\}.$$

Following 5.2 the group $S^2(T_{sc}^*)^W$ is generated by the normalized Killing forms

$$q=\sum_{i=1}^{m-1}(\omega_i^2-\omega_i\omega_{i+1})+2\omega_m^2-\omega_{m-1}\omega_m, q'=\sum_{i=1}^{n-1}(\omega_i'^2-\omega_i'\omega_{i+1}')+2\omega_m'^2-\omega_{m-1}'\omega_m'.$$

Choose a $\mathbb{Z}$-basis $\{\omega_1,\ldots,\omega_{m-1},\omega_1',\ldots,\omega_{m-1},v_1,v_1'\}$ of T^* where $v_1=\omega_m+\omega_n'$ and $v_1'=\omega_m-\omega_n'$. For any $\phi\in S^2(T^*)^W$ there exist $d,d'\in\mathbb{Z}$ such that $\phi=dq+d'q'$, thus in this basis we have

$$\phi=\tfrac{1}{2}(d+d')(v_1^2+v_1'^2)+\psi$$

for some quadratic form ψ with integer coefficients. Hence, we obtain

$$Q(G)=\{dq+d'q'\mid d+d'\equiv 0 \bmod 2\}=\mathbb{Z}(q-q')\oplus\mathbb{Z}(q+q'). \tag{10}$$

We claim that $\mathrm{Dec}(G)=2\mathbb{Z}q\oplus 2\mathbb{Z}q'$. The result for the group of indecomposable invariants then follows immediately. To prove the claim, since $q=\frac{1}{2}(\sum_{i=1}^m e_i^2)$ and $q'=\frac{1}{2}(\sum_{j=1}^n e_j'^2)$ in terms of the standard basis of $T_{sc}^*=\mathbb{Z}^m\oplus\mathbb{Z}^n$, we conclude that $c_2(\rho(\omega_1))=2q$ and $c_2(\rho(\omega_1'))=2q'$ are contained in $\mathrm{Dec}(G)$. On the other hand, as $\mathrm{Dec}(G)$ is generated by $c_2(\rho(\lambda))$ for all $\lambda\in T^*$ and the Weyl group of G contains normal subgroups $(\mathbb{Z}/2\mathbb{Z})^m$ and $(\mathbb{Z}/2\mathbb{Z})^n$ generated by sign switching, we see that the coefficient at each e_i in the expansion of $c_2(\rho(\lambda))$ is divisible by 2 (c.f. [4, Lemma 14.2]).

We now compute the group $\mathrm{Sdec}(G)$. Assume that $m=n=2$. Consider the element

$$y=e^{\omega_2}z\in\mathbb{Z}[T^*]\cap I_{sc}^W \text{ with } z=\bar\rho(\omega_2)-\bar\rho(\omega_2'), \tag{11}$$

where $\bar\rho(\omega_i)$ denotes the augmented orbit $\rho(\omega_i)-|W(\omega_i)|$. As $(e^{\omega_2}-1)z\in I_{sc}^3$, we see that $c_2(y)=c_2(z)$. Since $c_2(\bar\rho(\omega_2))=q$ and $c_2(\bar\rho(\omega_2'))=q'$, we conclude that $q-q'\in\mathrm{Sdec}(G)$. Therefore, $\mathrm{Inv}^3_{sd}(G)=\mathbb{Z}/2\mathbb{Z}$.

Assume that $m,n\geq 3$. We will show that $q-q'$ which is a generator of $\mathrm{Inv}^3_{ind}(G)$ does not belong to $\mathrm{Sdec}(G)$. Let $x\in\mathbb{Z}[T^*]\cap I_{sc}^W$. Similar to [15, §3c] write

$$x=\sum_{i=1}^m(d_i+\delta_i)\bar\rho(\omega_i)+\sum_{j=1}^n(d_j'+\delta_j')\bar\rho(\omega_j')$$

for some $d_i,d_j'\in\mathbb{Z}$ and $\delta_i,\delta_j'\in I_{sc}$. As $c_2(I_{sc}^3)=0$, we have

$$c_2(x)=\sum_{i=1}^m d_ic_2(\bar\rho(\omega_i))+\sum_{j=1}^n d_j'c_2(\bar\rho(\omega_j')).$$

On the other hand, we have

$$c_2(\bar\rho(\omega_i))=2m_iq \text{ and } c_2(\bar\rho(\omega_j'))=2m_j'q'$$

for all $1\leq i\leq m$ and $1\leq j\leq n$ and for some $m_i,m_j'\in\mathbb{Z}$. Hence, $c_2(x)\equiv 0 \bmod 2$, thus $q-q'\notin\mathrm{Sdec}(G)$. Similarly, if $m=2$ and $n\geq 3$, then $c_2(x)\equiv(d_1+d_2)q \bmod 2$, thus $q-q'\notin\mathrm{Sdec}(G)$, which completes the proof. □

The previous proposition yields the following. Combining these results we obtain both indecomposable and semi-decomposable subgroups for an arbitrary split semisimple group of type B modulo central subgroups μ_2.

COROLLARY 7.2. *(1) Let $G = \prod_{i=1}^{m} \mathbf{SO}_{2n_i+1}$, $n_i \geq 2$. Then,*

$$\mathrm{Inv}^3_{ind}(G) = \mathrm{Inv}^3_{sd}(G) = 0.$$

(2) Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}_2$, $n_i \geq 2$, $m \geq 2$, where $\boldsymbol{\mu}_2$ is the diagonal subgroup. Then,

$$\mathrm{Inv}^3_{ind}(G) = (\mathbb{Z}/2\mathbb{Z})^{\oplus m-1} \text{ and } \mathrm{Inv}^3_{sd}(G) = (\mathbb{Z}/2\mathbb{Z})^{\oplus k-1},$$

where k is the number of n_i's such that $n_i = 2$.
(3) Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1}) \times (\prod_{i=1}^{m'} \mathbf{SO}_{2n'_i+1})$, $n_i, n'_i \geq 2$. Then,

$$\mathrm{Inv}^3_{ind}(G) = (\mathbb{Z}/2\mathbb{Z})^{\oplus k} \text{ and } \mathrm{Inv}^3_{sd}(G) = 0,$$

where k is the number of n_i's such that $n_i \geq 3$.

Proof. (1) We set $d' = 0$ in (10). Then, we obtain $Q(\mathbf{SO}_{2m+1}) = 2\mathbb{Z}q$ ([14, §4b]). It immediately follows from the proof of Proposition 7.1 that we also have $\mathrm{Dec}(\mathbf{SO}_{2m+1}) = 2\mathbb{Z}q$.
(2) This follows by the same argument as in Proposition 7.1.
(3) As $\mathrm{Dec}(\mathbf{Spin}_{2n+1}) = \mathrm{Sdec}(\mathbf{Spin}_{2n+1})$ for any $n \geq 2$, the same argument as in (1) shows that $\mathrm{Sdec}(G) = \mathrm{Dec}(G)$. By [4, Theorem 13.4], $Q(\mathbf{Spin}_{2n+1}) = 2\,\mathrm{Dec}(\mathbf{Spin}_{2n+1})$ for any $n \geq 3$ and $\mathrm{Inv}^3_{ind}(\mathbf{Spin}_5) = 0$, thus the same argument as in (1) proves the result for the indecomposable group. □

8 TYPE C

In the present section we compute the groups of indecomposable and semi-decomposable invariants for semisimple groups of type C. In particular, we show that for groups $G = (\prod_{i=1}^{m} \mathbf{SP}_{2n_i})/\boldsymbol{\mu}_2$, where $m \geq 2$, $n_i \not\equiv 0 \bmod 4$ for all $i = 1, \ldots, m$, and $\boldsymbol{\mu}_2$ is the diagonal subgroup, any indecomposable invariant is semi-decomposable.
We consider the index 2 case, which generalizes the example [15, Example 3.1] (the case $n = m = 1$) to groups of type C.

PROPOSITION 8.1. *Let $G = (\mathbf{SP}_{2m} \times \mathbf{SP}_{2n})/\boldsymbol{\mu}_2$, $m, n \geq 1$ where $\boldsymbol{\mu}_2$ is the diagonal subgroup. Then, we have*

$$\mathrm{Inv}^3_{ind}(G) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } m \equiv n \equiv 0 \bmod 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Inv}^3_{sd}(G) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } m \equiv n \equiv 0 \text{ or } m \not\equiv 0 \not\equiv n \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if both n and m are not divisible by 4, then each indecomposable invariant is semi-decomposable.

Proof. Let $\{e_1, \cdots, e_m, e_1', \cdots, e_n'\}$ be a standard basis of $T_{sc}^* = \mathbb{Z}^m \oplus \mathbb{Z}^n$. Then T^* consists of all linear combinations of standard basis elements with even sums of coefficients. Consider the $\mathbb{Z}$-basis $\{x_1, \dots, x_m, x_1', \dots, x_n'\}$ of T^* given by

$$x_1 = e_1 + e_1',\ x_1' = e_1 - e_1',\ x_i = e_1 - e_i,\ x_j' = e_1' - e_j',\ i, j, > 1,$$

The standard basis can be expressed in terms of this basis over $\mathbb{Q}$ as

$$e_i = e_1 - x_i,\ e_j' = e_1' - x_j',\ \text{where } e_1 = \tfrac{1}{2}(x_1 + x_1') \text{ and } e_1' = \tfrac{1}{2}(x_1 - x_1').$$

The group $S^2(T_{sc}^*)^W$ is generated by $q = \sum_{i=1}^m e_i^2$ and $q' = \sum_{j=1}^n e_j'^2$. Therefore, for any $\phi \in S^2(T_{sc}^*)^W$ there exist $d, d' \in \mathbb{Z}$ such that $\phi = dq + d'q' =$

$$d\Big(\tfrac{1}{4}(x_1+x_1')^2 + \sum_{i>1}(\tfrac{1}{2}(x_1+x_1') - x_i)^2\Big) + d'\Big(\tfrac{1}{4}(x_1-x_1')^2 + \sum_{j>1}(\tfrac{1}{2}(x_1-x_1') - x_j')^2\Big).$$

The form ϕ has integer coefficients at x_1x_1', x_ix_j', x_i^2, $x_j'^2$, $i, j > 1$ and it has coefficient $\frac{1}{4}(dm + d'n)$ at x_1^2 and at $x_1'^2$. Hence,

$$Q(G) = \{dq + d'q' \mid dm + d'n \equiv 0 \bmod 4\}. \tag{12}$$

Consider the subgroup $\mathrm{Dec}(G)$ of decomposable invariants of G. As in the proof of [4, Lemma 14.2], since the Weyl group of G contains normal subgroups $(\mathbb{Z}/2\mathbb{Z})^m$ and $(\mathbb{Z}/2\mathbb{Z})^n$ generated by sign switching, we conclude that the coefficient at each e_i in the expansion of q_χ is divisible by 2, hence, $\mathrm{Dec}(G) \subseteq 2\mathbb{Z}q \oplus 2\mathbb{Z}q'$. Since $c_2(\bar\rho(2e_1)) = 4q$ and $c_2(\bar\rho(2e_1')) = 4q'$, we have $4\mathbb{Z}q \oplus 4\mathbb{Z}q' \subseteq \mathrm{Dec}(G)$.
Assume $n \equiv m \equiv 0 \bmod 2$. Since $c_2(\bar\rho(x_2)) = 2(m-1)q$ and $c_2(\bar\rho(x_2')) = 2(n-1)q'$, we obtain $\mathrm{Dec}(G) = 2\mathbb{Z}q \oplus 2\mathbb{Z}q'$ and $Q(G) = \{dq + d'q' \mid d\frac{m}{2} + d'\frac{n}{2} \equiv 0 \bmod 2\}$. Hence,

$$\mathrm{Inv}^3_{ind}(G) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})q \oplus (\mathbb{Z}/2\mathbb{Z})q' & \text{if } n \equiv m \equiv 0 \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})q & \text{if } m \equiv 0 \not\equiv n \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})q' & \text{if } m \not\equiv 0 \equiv n \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})(q+q') & \text{if } m \not\equiv 0 \not\equiv n \bmod 4. \end{cases}$$

Assume both n and m are odd. If $n \equiv -m \bmod 4$, then $Q(G) = \{dq + d'q' \mid d \equiv d' \bmod 4\} \simeq \mathbb{Z}/4\mathbb{Z}(q+q')$. Since $c_2(\bar\rho(x_1)) = 2nq + 2mq'$, $\mathrm{Dec}(G) = 2\mathbb{Z}(q+q') \oplus 2\mathbb{Z}(q-q')$ and, therefore, $\mathrm{Inv}^3_{ind}(G) \simeq (\mathbb{Z}/2\mathbb{Z})(q+q')$. Similarly, if $n \equiv m \bmod 4$, then $\mathrm{Inv}^3_{ind}(G) \simeq (\mathbb{Z}/2\mathbb{Z})(q-q')$.
Finally, assume n is odd and m is even. If $m \equiv 0 \bmod 4$, then $Q(G) = \mathbb{Z}q \oplus 4\mathbb{Z}q'$. Since $c_2(\bar\rho(x_2)) = 2(m-1)q$, $\mathrm{Dec}(G) = 2\mathbb{Z}q \oplus 4\mathbb{Z}q'$, hence, $\mathrm{Inv}^3_{ind}(G) \simeq (\mathbb{Z}/2\mathbb{Z})q$. If $m \not\equiv 0 \bmod 4$, then d' is even and

$$Q(G) = \{dq + d'q' \mid d + \tfrac{d'}{2} \equiv 0 \bmod 2\} = \{(\bar 0, \bar 0), (\bar 2, \bar 0), (\bar 1, \bar 2), (-\bar 1, \bar 2)\},$$

where $(\bar d, \bar d')$ denotes $dq + d'q'$ modulo 4. Since $\mathrm{Dec}(G) = 2\mathbb{Z}q + 4\mathbb{Z}q'$, we have $\mathrm{Inv}^3_{ind}(G) \simeq (\mathbb{Z}/2\mathbb{Z})(q + 2q')$.

As for semi-decomposable invariants, consider the element (cf. with $h_{1,i}$ of Definition 3.2)

$$z = \tfrac{n}{gcd(m,n)}\bar{\rho}(\omega_1) - \tfrac{m}{gcd(m,n)}\bar{\rho}(\omega_1'). \tag{13}$$

By definition, we have $y = e^{e_1}z \in \mathbb{Z}[T^*] \cap I_{sc}^W$ and we obtain

$$c_2(y) = c_2((1+(e^{e_1}-1))z) = c_2(z) = \tfrac{n}{gcd(m,n)}q - \tfrac{m}{gcd(m,n)}q'$$

where the second equality holds since $(e^{e_1}-1)z \in I_{sc}^3$. The element $c_2(y) \in \mathrm{Sdec}(G)$ coincides with the generator of $\mathrm{Inv}^3_{ind}(G) = \mathbb{Z}/2\mathbb{Z}$ except if $n \equiv m \equiv 0 \bmod 4$. So $\mathrm{Inv}^3_{sd}(G) = \mathrm{Inv}^3_{ind}(G) = \mathbb{Z}/2\mathbb{Z}$ except if $n \equiv m \equiv 0 \bmod 4$.
Assume that $m \equiv n \equiv 0 \bmod 4$. Then $\{q, q'\}$ are generators of the group of indecomposable invariants. Consider an arbitrary element $x \in \mathbb{Z}[T^*] \cap I_{sc}^W$ and the ring homomorphism

$$\phi\colon \mathbb{Z}[T^*_{sc}] \to \mathbb{Z}[T^*_{sc}/T^*] = \mathbb{Z}[t]/(t^2-2t)$$

given by $\phi(1-e^{-w_{odd}}) = \phi(1-e^{-w'_{odd}}) = t$ and $\phi(1-e^{-w_{even}}) = \phi(1-e^{-w'_{even}}) = 0$. Write

$$x = \sum_{i=1}^{m}(d_i+\delta_i)\bar{\rho}(\omega_i) + \sum_{j=1}^{n}(d_i'+\delta_i')\bar{\rho}(\omega_i'), \quad d_i, d_j' \in \mathbb{Z},\ \delta_i, \delta_j' \in I_{sc}.$$

Since $\ker\phi \supset \mathbb{Z}[T^*] \cap I_{sc}^W$, we obtain

$$0 = \phi(x) = \sum_{odd\ i} 2^i \binom{m}{i}(d_i + 2s_i)t + \sum_{odd\ j} 2^j \binom{n}{j}(d_j' + 2s_j')t.$$

Observe that if $2^r \mid m$ and i is odd, then $2^r \mid \binom{m}{i}$. Dividing by the 2-primary part 2^r of the greatest common divisor of all the coefficients we obtain

$$(m/2^{r-1})d_1 + (n/2^{r-1})d_1' \equiv 0 \bmod 2,$$

where $2^{r-1} = g.c.d.(v_2(n), v_2(m))$ is the g.c.d. of the 2-primary parts. Therefore,

$$\begin{cases} d_1 + d_1' \equiv 0 \bmod 2 & \text{if } v_2(m) = v_2(n) \\ d_1 \equiv 0 \bmod 2 & \text{if } v_2(m) < v_2(n). \end{cases} \tag{14}$$

We then have

$$c_2(x) = (\sum_{i=1}^{m} 2^{i-1}\binom{m-1}{i-1}d_i)q + (\sum_{j=1}^{n} 2^{j-1}\binom{n-1}{j-1}d_j')q'.$$

So $c_2(x) \equiv d_1 q + d_1' q' \bmod \mathrm{Dec}(G)$, where d_1 and d_1' satisfy (14).
Since

$$c_2(y) = \begin{cases} q + q' \bmod \mathrm{Dec}(G) & \text{if } v_2(n) = v_2(m) \\ q' \bmod \mathrm{Dec}(G) & \text{if } v_2(m) < v_2(n) \end{cases}$$

we conclude that $c_2(y)$ is also a generator of $\mathrm{Inv}^3_{sd}(G) \simeq \mathbb{Z}/2\mathbb{Z}$. □

We now present a generalization of the previous proposition, which in turn determine both indecomposable and semi-decomposable subgroups for an arbitrary split semisimple group of type C modulo central subgroups μ_2.

Corollary 8.2. *(1) Let* $G = \prod_{i=1}^m \mathbf{PGSp}_{2n_i}$, $m, n_i \geq 1$. *Then,*

$$\mathrm{Inv}^3_{ind}(G) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus k},$$

where k *is the number of* n_i*'s which are divisible by* 4, *and* $\mathrm{Sdec}(G) = \mathrm{Dec}(G)$, *i.e., each semi-decomposable invariant is decomposable.*
(2) Let $G = (\prod_{i=1}^m \mathbf{SP}_{2n_i})/\boldsymbol{\mu}_2$, $m, n_i \geq 1$, *where* $\boldsymbol{\mu}_2$ *is the diagonal subgroup. Then,*

$$\mathrm{Inv}^3_{ind}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus m} & \textit{if } \forall n_i \equiv 0 \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus m-1} & \textit{otherwise,} \end{cases}$$

and

$$\mathrm{Inv}^3_{sd}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus m-1} & \textit{if } \forall n_i \equiv 0 \textit{ or } \forall n_i \not\equiv 0 \bmod 4, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus m-2} & \textit{otherwise.} \end{cases}$$

(3) Let $G = (\prod_{i=1}^m \mathbf{PGSp}_{2n_i}) \times (\prod_{i=1}^{m'} \mathbf{SP}_{2n'_i})$, $n_i, n'_i \geq 1$. *Then,*

$$\mathrm{Inv}^3_{ind}(G) = (\mathbb{Z}/2\mathbb{Z})^{\oplus k} \textit{ and } \mathrm{Sdec}(G) = \mathrm{Dec}(G),$$

where k *is the number of* n_i*'s which are divisible by* 4.

Proof. (1) Let $G_1 = \mathbf{PGSp}_{2m}$ and $G_2 = \mathbf{PGSp}_{2n}$, $m, n \geq 1$. It suffices to consider the case $G = G_1 \times G_2$ since the same arguments can be easily adapted to prove the case of three and more groups. We simply set $d' = 0$ (resp. $d = 0$) in 12. Then, we have $Q(G_1) = 4/\gcd(4, m)\mathbb{Z}q$ (resp. $Q(G_2) = 4/\gcd(4, n)\mathbb{Z}q'$). Similarly, by the proof of Proposition 8.1 we get $\mathrm{Dec}(G_1) = 4/\gcd(2, m)\mathbb{Z}q$ (resp. $\mathrm{Dec}(G_2) = 4/\gcd(2, n)\mathbb{Z}q'$) ([14, §4b]). By (2) the answer for $\mathrm{Inv}^3_{ind}(G)$ then follows.
As for semi-decomposable invariants, by (3) and $\mathrm{Dec}(G) = Q(G)$ for $n \not\equiv 0 \bmod 4$, it suffices to consider the case $n \equiv m \equiv 0 \bmod 4$. We follow arguments used in [15, §3c].
Let $x \in \mathbb{Z}[T^*] \cap I_{sc}^W$ be an arbitrary element. Write

$$x = \sum_{i=1}^m (d_i + \delta_i)\bar{\rho}(\omega_i) + \sum_{j=1}^n (d'_j + \delta'_j)\bar{\rho}(\omega'_j)$$

for some $d_i, d'_j \in \mathbb{Z}$ and $\delta_i, \delta'_j \in I_{sc}$. Consider the ring homomorphism induced by the quotient map $T^*_{sc} \to T^*_{sc}/T^*$

$$\phi\colon \mathbb{Z}[T^*_{sc}] \longrightarrow \mathbb{Z}[T^*_{sc}/T^*] = \mathbb{Z}[\Lambda_w/\Lambda_r] \otimes \mathbb{Z}[\Lambda'_w/\Lambda'_r] = \mathbb{Z}[t, t']/(t^2 - 2t, t'^2 - 2t').$$

It is given by

$$\phi(1 - e^{-\omega_{odd}}) = t,\ \phi(1 - e^{-\omega'_{odd}}) = t' \text{ and } \phi(1 - e^{-\omega_{even}}) = \phi(1 - e^{-\omega'_{even}}) = 0.$$

Since $x \in \mathbb{Z}[T^*]$, $\phi(x) = 0$. Moreover, we have $\phi(I_{sc}) = (t, t')$ and $\phi(\bar{\rho}(\omega_i)) = |W(\omega_i)| \cdot t$, $\phi(\bar{\rho}(\omega'_i)) = |W(\omega'_i)| \cdot t'$. Combining these facts we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{m} (d_i + \phi(\delta_i))\phi(\bar{\rho}(\omega_i)) + \sum_{j=1}^{n} (d'_j + \phi(\delta'_j))\phi(\bar{\rho}(\omega'_j)) \\ &= \sum_{odd\ i} |W(\omega_i)| \cdot (d_i + s_i t + s'_i t')t + \sum_{odd\ j} |W(\omega'_j)| \cdot (d'_j + r_j t + r'_j t')t', \end{aligned}$$

for some $s_i, s'_i, r_j, r'_j \in \mathbb{Z}$. Since $|W(\omega_i)| = 2^i \binom{m}{i}$, collecting the coefficients at t and t', we get

$$d_1 + 2s_1 \equiv d'_1 + 2r'_1 \equiv 0 \bmod 2.$$

Hence, both d_1 and d'_1 are even.
We now compute $c_2(x)$. Since $c_2(I^3_{sc}) = 0$ and $c_2(\bar{\rho}(\omega_i)) = 2^{i-1}\binom{m-1}{i-1}q$, we obtain

$$c_2(x) = \sum_{i=1}^{m} d_i c_2(\bar{\rho}(\omega_i)) + \sum_{j=1}^{n} d'_j c_2(\bar{\rho}(\omega'_j)) = 2sq + 2rq', \quad \text{for some } r, s \in \mathbb{Z}.$$

Therefore, $\mathrm{Sdec}(G_1) \subseteq 2\mathbb{Z}q = \mathrm{Dec}(G_1)$ and $\mathrm{Sdec}(G_2) \subseteq 2\mathbb{Z}q' = \mathrm{Dec}(G_2)$.
(2) This immediately follows from the same argument as in Proposition 8.1.
(3) As $\mathrm{Dec}(\mathbf{SP}_{2n}) = \mathrm{Sdec}(\mathbf{SP}_{2n}) = Q(\mathbf{SP}_{2n})$ for any $n \geq 1$ ([4, Theorem 14.3]), the same argument as in (1) shows that $\mathrm{Sdec}(G) = \mathrm{Dec}(G)$ and the result for the indecomposable subgroup. □

9 TYPE D

In this section we calculate the groups of indecomposable and semi-decomposable invariants for an arbitrary product of simply-connected simple groups of type D modulo the (diagonal) central subgroups. We first consider the groups of index 2 and 4.

PROPOSITION 9.1. *Let $G = (\mathbf{Spin}_{2m} \times \mathbf{Spin}_{2n})/\boldsymbol{\mu}$, where $m, n \geq 4$ and $m + n$ is even, and $\boldsymbol{\mu}$ is a diagonal subgroup of G. Then*

$$\mathrm{Inv}^3_{ind}(G) = \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2, \end{cases} \quad \text{and} \quad \mathrm{Inv}^3_{sd}(G) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ 0 & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2. \end{cases}$$

Proof. Observe that there is a unique diagonal subgroup $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_4$ in the case where m and n are odd and there are two different diagonal subgroups $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_2$: $\boldsymbol{\mu} \subseteq \boldsymbol{\mu}_4^2$ if both m and n are odd and $\boldsymbol{\mu} \subseteq \boldsymbol{\mu}_2^4$ otherwise. First, assume that $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_4$. Then, by 5.1 the character group of the split maximal torus T of G is given by

$$\Big\{ \sum_{i=1}^{m} a_i\omega_i + \sum_{j=1}^{n} a'_i\omega'_i \,\Big|\, a_{m-1} + 3a_m + 2\sum_{i=1}^{\frac{m-1}{2}} a_{2i-1} \equiv 3a'_{n-1} + a'_n + 2\sum_{j=1}^{\frac{n-1}{2}} a'_{2j-1} \bmod 4 \Big\}. \tag{15}$$

Write $\sum_{i=1}^m a_i\omega_i+\sum_{j=1}^n a_i'\omega_i' = \sum_{i=1}^m b_ie_i+\sum_{j=1}^n b_i'e_i'$ in terms of the standard basis vectors $\{e_1,\cdots,e_m,e_1',\cdots,e_n'\}$ of $T_{sc}^*=\mathbb{Z}^m\oplus\mathbb{Z}^n$. Then, the relation in (15) is equivalent to

$$2(b_{m-2}+b_{m-1}+b_m+\sum_{i=1}^{\frac{m-3}{2}} b_{2i-1}-b_{2i}) \equiv 2(b_{n-2}'+b_{n-1}'-b_n'+\sum_{j=1}^{\frac{n-3}{2}} b_{2j-1}-b_{2j}) \bmod 4. \tag{16}$$

Using (15) we choose the following basis $\{x_1,\ldots,x_m,x_1',\ldots,x_n'\}$ of T^*:

$$\begin{aligned} &x_{2i-1}=\omega_{2i-1}+2\omega_n',\ x_{2k}=\omega_{2k},\ x_{m-1}=\omega_{m-1}+\omega_n',\ x_m=\omega_m+3\omega_n',\\ &x_{2j-1}'=\omega_{2j-1}'+2\omega_n',\ x_{2l}'=\omega_{2l}',\ x_{n-1}'=\omega_{n-1}'+\omega_n',\ x_n'=4\omega_n' \end{aligned} \tag{17}$$

for $1\le i\le (m-1)/2$, $1\le k\le (m-3)/2$, $1\le j\le (n-1)/2$, and $1\le l\le (n-3)/2$.

Let ψ be a quadratic form on (17) with integer coefficients. As the group $S^2(T_{sc}^*)^W$ is generated by the normalized Killing forms

$$\begin{aligned} q &:= \sum_{i=1}^m \omega_i^2-2\sum_{i=1}^{m-2}\omega_i\omega_{i+1}-2\omega_{m-2}\omega_m=(\sum_{i=1}^m e_i^2)/2,\\ q' &:= \sum_{j=1}^n \omega_j'^2-2\sum_{j=1}^{n-2}\omega_j'\omega_{j+1}'-2\omega_{n-2}'\omega_n'=(\sum_{j=1}^n e_j'^2)/2, \end{aligned}$$

from the equation $\phi=dq+d'q'$ we get

$$\phi=(\tfrac{md+nd'}{8})x_n'^2+\psi,$$

where ψ is a quadratic form on (17) with integer coefficients. Therefore,

$$Q(G)=\{dq+d'q' \mid md+nd'\equiv 0 \bmod 8\}. \tag{18}$$

We show that $\mathrm{Dec}(G)=4\mathbb{Z}(q-q')+4\mathbb{Z}(q+q')$. First, by (15) we see that all elements $c_2(\rho(2\omega_1))=-8q$, $c_2(\rho(2\omega_1'))=-8q'$, $c_2(\rho(\omega_1+\omega_1'))=-4nq-4mq'$ are contained in $\mathrm{Dec}(G)$, thus $4(q-q'),4(q+q')\in\mathrm{Dec}(G)$. On the other hand, since $\mathrm{Dec}(G)$ is generated by $c_2(\rho(\lambda))$ for all $\lambda\in T^*$ and W contains normal subgroups $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$, we see that the coefficient at each e_i in the expansion of $c_2(\rho(\lambda))$ is divisible by 2, thus, $\mathrm{Dec}(G)\subseteq 4\mathbb{Z}q\oplus 4\mathbb{Z}q'$ (note that by (18) $4q,4q'\notin\mathrm{Dec}(G)$). Hence,

$$\mathrm{Inv}^3_{ind}(G)=\mathbb{Z}/4\mathbb{Z}(\tfrac{n}{\gcd(m,n)}q-\tfrac{m}{\gcd(m,n)}q').$$

Now we show that $\mathrm{Sdec}(G)=2\mathbb{Z}(\frac{n}{\gcd(m,n)}q-\frac{m}{\gcd(m,n)}q')$. First of all, as

$$c_2(\bar\rho(\omega_i))=2m_iq \text{ and } c_2(\bar\rho(\omega_j'))=2m_j'q' \tag{19}$$

for all $1\le i\le m$ and $1\le j\le n$ and for some $m_i,m_j'\in\mathbb{Z}$, by the same argument as in the proof of Proposition 7.1 we obtain $\frac{n}{\gcd(m,n)}q-\frac{m}{\gcd(m,n)}q'\notin\mathrm{Sdec}(G)$. On the other hand, consider an element

$$z=\tfrac{n}{gcd(m,n)}\bar\rho(\omega_1)-\tfrac{m}{gcd(m,n)}\bar\rho(\omega_1').$$

Then, by (16) we obtain $y := e^{e_1}z \in \mathbb{Z}[T^*] \cap I_{sc}^W$. Therefore, as in the proof of Proposition 8.1, we have $c_2(y) = 2(\frac{n}{\gcd(m,n)}q - \frac{m}{\gcd(m,n)}q') \in \mathrm{Sdec}(G)$. Hence, the result for $\mathrm{Inv}^3_{sd}(G)$ follows.
Now we assume that $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_2$, so that $\boldsymbol{\mu} \subseteq \boldsymbol{\mu}_4^2$ if m and n are odd and $\boldsymbol{\mu} \subseteq \boldsymbol{\mu}_2^4$ otherwise. In both cases, the corresponding character group of T is given by

$$T^* = \{ \sum_{i=1}^{m} a_i\omega_i + \sum_{j=1}^{n} a_i'\omega_i' \,|\, a_{m-1} + a_m \equiv a_{n-1}' + a_n' \bmod 2\}.$$

By applying the same argument in the above $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_4$ case, we obtain

$$Q(G) = \{dq + d'q' \,|\, d + d' \equiv 0 \bmod 2\}.$$

Since $c_2(\rho(\omega_1)) = -2q$ and $c_2(\rho(\omega_1')) = -2q'$, we have $2q, 2q' \in \mathrm{Dec}(G)$. Moreover, by (19), we get $q - q' \notin \mathrm{Sdec}(G)$, thus $\mathrm{Sdec}(G) = \mathrm{Dec}(G) = 2\mathbb{Z}q \oplus 2\mathbb{Z}q'$ and $\mathrm{Inv}^3_{ind}(G) = \mathbb{Z}/2\mathbb{Z}(q - q')$. □

We obtain the following generalization of the previous proposition. Together with Remark 9.3, they determine the groups of indecomposable and semi-decomposable invariants for an arbitrary product of simply-connected simple groups of type D modulo the central subgroups $\boldsymbol{\mu}_2$.

COROLLARY 9.2. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$, $n_i \geq 4$, $m \geq 1$, where either all n_i are even or odd, and $\boldsymbol{\mu}$ is a diagonal subgroup. Then,*

$$\mathrm{Inv}^3_{ind}(G) = \begin{cases} (\mathbb{Z}/4\mathbb{Z})^{m-1} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ (\mathbb{Z}/2\mathbb{Z})^{m-1} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2, \end{cases} \quad \text{and} \quad \mathrm{Inv}^3_{sd}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{m-1} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ 0 & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2. \end{cases}$$

Proof. By the same argument as in Proposition 9.1, we obtain

$$Q(G) = \begin{cases} \{\sum_{i=1}^m d_iq_i \mid \sum n_id_i \equiv 0 \bmod 8\} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ \{\sum_{i=1}^m d_iq_i \mid \sum d_i \equiv 0 \bmod 2\} & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2, \end{cases}$$

where $q_1, \ldots, q_m$ are the corresponding normalized Killing forms of $\mathbf{Spin}_{2n_i}$. Similarly,

$$\mathrm{Dec}(G) = \begin{cases} \bigoplus_{i=2}^m 4\mathbb{Z}(q_1 - q_i) \oplus 4\mathbb{Z}(q_1 + q_2) & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_4, \\ \bigoplus_{i=1}^m 2\mathbb{Z}q_i & \text{if } \boldsymbol{\mu} \simeq \boldsymbol{\mu}_2, \end{cases}$$

thus, the factor groups follow. Following Proposition 9.1, we see that

$$\mathrm{Sdec}(G)/\,\mathrm{Dec}(G) = \bigoplus_{i=2}^{m} \mathbb{Z}/2\mathbb{Z}(\tfrac{n_i}{\gcd(n_1,n_i)}q_1 - \tfrac{n_1}{\gcd(n_1,n_i)}q_i)$$

if $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_4$ and $q_1 - q_i \notin \mathrm{Sdec}(G)$ for all $2 \leq i \leq m$ if $\boldsymbol{\mu} \simeq \boldsymbol{\mu}_2$, which completes the proof. □

Remark 9.3. *(1) Let* $G' = G \times (\prod_{i=1}^{m'} \mathbf{Spin}_{2n'_i})$*, where* G *is the group from Corollary 9.2. Then, similar to the proof of Proposition 9.1 one can show that* $\mathrm{Inv}^3_{sd}(G') = \mathrm{Inv}^3_{sd}(G)$*. Moreover, by [4, Theorem 15.4]* $\mathrm{Inv}^3_{ind}(G') = \mathrm{Inv}^3_{ind}(G) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus m'}$.

(2) Let $G' = G \times (\prod_{i=1}^{m'} \mathbf{Spin}_{2n'_i})$*, where* G *is either* $\mathbf{SO}_{2m}$ *or* $\mathbf{HSpin}_{2m}$*. If* $G = \mathbf{SO}_{2m}$*, then by [4, §15]* $\mathrm{Dec}(G) = \mathrm{Dec}(\mathbf{Spin}_{2m})$*, thus* $\mathrm{Dec}(G') = \mathrm{Sdec}(G')$*. Similarly, if* $G = \mathbf{HSpin}_{2m}$*, then it follows from [2, §5] and [15, §3d] that* $\mathrm{Dec}(G') = \mathrm{Sdec}(G')$*. One can also easily compute the indecomposable groups.*

10 The $\mathbf{PGO}_8$-case

In the present section we use the techniques developed in section 3 to give a direct proof of the main result of [15, Appendix].

In this section, $G = \mathbf{PGO}_8$ that is an adjoint group of Dynkin type D_4. The weight lattice of type D_4 can be constructed as follows. We first take a $\mathbb{Q}$-vector space with basis $e_1, \ldots, e_4$. Then Λ has the following $\mathbb{Z}$-basis consisting of fundamental weights:

$$\omega_1 = e_1,\ \omega_2 = e_1 + e_2,\ \omega_3 = (e_1 + e_2 + e_3 - e_4)/2,\ \omega_4 = (e_1 + e_2 + e_3 + e_4)/2.$$

So the coordinates of elements of Λ are either all integers or half-integers.

The group T^* consists of all points such that all coordinates are integers, and the sum of coordinates is divisible by 2. We have $\Lambda/T^* = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with elements $\bar{0} = (0,0)$, $\bar{\omega}_1 = (0,1)$, $\bar{\omega}_3 = (1,0)$, $\bar{\omega}_4 = (1,1)$. The quotient map $\Lambda \to \Lambda/T^*$ induces a grading on Λ and, hence, on $\mathbb{Z}[\Lambda]$. We denote by $\Lambda^{(0,0)}, \Lambda^{(0,1)}, \Lambda^{(1,0)}, \Lambda^{(1,1)}$ the respective homogeneous components. Each polynomial $f \in \mathbb{Z}[\Lambda]$ can be split into a sum of its homogeneous components, which we will denote by $f^{(i,j)}$. Denote the orbits in $\mathbb{Z}[\Lambda]$ by $\rho(\omega_1), \ldots, \rho(\omega_4)$ and the augmented orbits by $\rho_1, \ldots, \rho_4$ as in section 3.

Lemma 10.1. *Let* $f_1, \ldots, f_4 \in \mathbb{Z}[\Lambda]$ *be such that* $f_1\rho_1 + \ldots + f_4\rho_4 \in \mathbb{Z}[T^*]$*. Then, for each element* $i \in \Lambda/T^*$ *except for* $i = (0,1)$ *the sum of coefficients of* $f_1^{(i)}$ *is even.*

Proof. Consider a subgroup

$$\Lambda' = \{x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in T^* \mid x_2 + x_3 + x_4 \text{ and } x_1 \text{ is even}\}.$$

We have $\Lambda/\Lambda' = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ with a generator $\bar{\omega}_1$ of order 2 and $\bar{\omega}_4$ of order 4.

Set $R = \mathbb{Z}/4\mathbb{Z}$. Consider a natural map $\mathbb{Z}[\Lambda] \to R[\Lambda/\Lambda']$ given by $f \mapsto \bar{f}$. Since $\Lambda' \subset T^*$, $R[\Lambda/\Lambda']$ is also a Λ/T^*-graded algebra, and this map preserves the grading. By definition, the sum of coefficients of f modulo 4 is the sum of coefficients of $\bar{f}$. So, it is sufficient to prove that for each element $i \in \Lambda/T^*$ except for $i = (0,1)$, the sum of coefficients of $\bar{f}_1^{(i)} = \bar{f}_1^{(i)}$ is even.

Since $\bar\rho_1 = 2\overline{e^{e_1}} + 2\overline{e^{e_2}}$, $\bar\rho_2 = \bar\rho_3 = \bar\rho_4 = 0$ in $R[\Lambda/\Lambda']$ and $\bar f_1\bar\rho_1 + \ldots + \bar f_4\bar\rho_4 \in R[T^*/\Lambda']$, we obtain that $\bar f_1\bar\rho_1 \in R[T^*/\Lambda']$. Since $\bar\rho_1 \in \mathbb{Z}[\Lambda^{(0,1)}]$, for each $i \in \Lambda/T^*$, $i \neq 0$, we have

$$0 = (\bar f_1\bar\rho_1)^{(i)} = \bar f_1^{(i-(0,1))}\bar\rho_1 = 2\overline{e^{e_1}}(1 + \overline{e^{e_1+e_2}})\bar f_1^{(i-(0,1))}.$$

Therefore, for each $j \in \Lambda/T^*$, $j \neq (0,1)$, all coefficients of $(1 + \overline{e^{e_1+e_2}})\bar f_1^{(j)}$ are divisible by 2.
Observe that the classes of 0 and of ω_2 in Λ/Λ' are all elements of Λ/Λ' that belong to T^*/Λ'. So, if $\lambda \in \Lambda/\Lambda'$, and $f \in R[\Lambda/\Lambda']$, then the coefficient of $(1+\overline{e^{e_1+e_2}})f$ at e^λ is the sum of coefficients of f in front of $e^{\lambda'}$ for all $\lambda' \in \Lambda/\Lambda'$ such that $\lambda = \lambda' \bmod T^*$ (there are exactly two such λ', one of them is λ, the other is $\lambda + e_1 + e_2$). If f is a homogeneous polynomial of degree $j \in \Lambda/T^*$, and $\lambda \in \Lambda/\Lambda'$ is mapped to j by the natural projection $\Lambda/\Lambda' \to \Lambda/T^*$, then the coefficient of $(1 + \overline{e^{e_1+e_2}})f$ in front of e^λ is the sum of all coefficients of f. Therefore, the sum of all coefficients of $\bar f_1^{(j)}$ is divisible by 2 for each $j \in \Lambda/T^*$, $j \neq (0,1)$. □

Lemma 10.2. *Let $f_1, \ldots, f_4 \in \mathbb{Z}[\Lambda]$ be such that $f_1\rho_1 + \ldots + f_4\rho_4 \in \mathbb{Z}[T^*]$. Then,*
(1) for each element $i \in \Lambda/T^*$ except for $i = (1,0)$ the sum of coefficients of $f_3^{(i)}$ is even;
(2) for each element $i \in \Lambda/T^*$ except for $i = (1,1)$ the sum of coefficients of $f_4^{(i)}$ is even.

Proof. (1) Consider an automorphism ψ of Λ (induced by an outer automorphism of $\mathbf{PGO}_8$) that interchanges ω_1 and ω_3 and keeps ω_2 and ω_4 invariant. It preserves T^*. So, it also acts on $\Lambda/T^* = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ by interchanging $(0,1)$ and $(1,0)$ (and keeping $(0,0)$ and $(1,1)$).
By definition ψ maps the graded components of a polynomial $f \in \mathbb{Z}[T^*]$ to graded components of $\psi(f)$, more precisely, $\psi(f^{(i)}) = \psi(f)^{\psi(i)}$. In particular, $\psi(f)^{(0,1)} = \psi(f^{(1,0)})$.
Since ψ interchanges ρ_1 and ρ_3, and keeps ρ_2 and ρ_4 unchanged,

$$\psi(f_1\rho_1 + \ldots + f_4\rho_4) = \psi(f_1)\rho_3 + \psi(f_2)\rho_2 + \psi(f_3)\rho_1 + \psi(f_4)\rho_4.$$

Finally, observe that the sum of coefficients of $\psi(f)$ is the same as the sum of coefficients of f. We then apply Lemma 10.1.
(2) The proof is completely similar to the proof of the previous case, the only difference is that now ψ should interchange ω_1 and ω_4 and keep ω_2 and ω_3 unchanged. □

Proposition 10.3. *Let $f_1, \ldots, f_4 \in \mathbb{Z}[\Lambda]$ be such that $f_1\rho_1 + \ldots + f_4\rho_4 \in \mathbb{Z}[T^*]$. Then the sum of all coefficients of f_i is even for $i = 1, 3, 4$.*

Proof. Set $\omega_{(0,1)} = \omega_1$, $\omega_{(1,0)} = \omega_3$, $\omega_{(1,1)} = \omega_4$, and $\omega_{(0,0)} = \omega_2$. Then the class of $\omega_{(i,j)}$ in Λ/T^* is precisely (i,j).
Set $R = \mathbb{Z}/16\mathbb{Z}$ and consider the natural map $\mathbb{Z}[\Lambda] \to R[\Lambda/T^*]$ given by $f \mapsto \bar{f}$. Since $f_1\rho_1 + \ldots + f_4\rho_4 \in \mathbb{Z}[T^*]$, $\bar{\rho}_i = 8\overline{e^{\omega_i}} - 8$, $i = 1,3,4$ and $\bar{\rho}_2 = 24\overline{e^{\omega_2}} - 24 = 0$,

$$\bar{f}_1\bar{\rho}_1 + \bar{f}_3\bar{\rho}_3 + \bar{f}_4\bar{\rho}_4 \text{ is a constant in } R[\Lambda/T^*].$$

For $i = 1,3,4$ and $(j,k) \in \Lambda/T^*$, denote by $c_i^{(j,k)}$ the sum of coefficients of $f_i^{(j,k)}$ modulo 16. Then $\bar{f}_i^{(j,k)} = c_i^{(j,k)}\overline{e^{\omega_{(j,k)}}}$. By Lemma 10.1 and 10.2, all numbers $c_i^{(j,k)}$ are even, except for, possibly, $c_1^{(0,1)}$, $c_3^{(1,0)}$, and $c_4^{(1,1)}$. Observe that if $c_i^{(j,k)}$ is even, then $c_i^{(j,k)}\overline{e^{\omega_{(j,k)}}}\bar{\rho}_i = 0$ in $R[\Lambda/T^*]$ since $\bar{\rho}_i$ is divisible by 8. Therefore, both expressions

$$c_1^{(0,1)}\overline{e^{\omega_1}}\bar{\rho}_1 + c_3^{(1,0)}\overline{e^{\omega_3}}\bar{\rho}_3 + c_4^{(1,1)}\overline{e^{\omega_4}}\bar{\rho}_4 \quad \text{and}$$

$$8c_1^{(0,1)}\overline{e^{\omega_1}} + 8c_3^{(1,0)}\overline{e^{\omega_3}} + 8c_4^{(1,1)}\overline{e^{\omega_4}} + 8(c_1^{(0,1)} + c_3^{(1,0)} + c_4^{(1,1)})$$

are constants in $R[\Lambda/T^*]$.
So, the coefficients $c_1^{(0,1)}$, $c_3^{(1,0)}$, and $c_4^{(1,1)}$ are even which means that for all $i = 1,3,4$ and $(j,k) \in \Lambda/T^*$, $c_i^{(j,k)}$ is even. But then the sum of all coefficients of f_i is even for $i = 1,3,4$. □

We now give a direct proof of the result obtained in [15, Appendix] using a computer algorithm

Corollary 10.4. *If $G = \mathbf{PGO}_8$, then any semi-decomposable invariant of G is decomposable.*

Proof. Let $x \in \mathbb{Z}[T^*] \cap I_{sc}^W$. Similar to [15, §3c] we write

$$x = \sum_{i=1}^{4}(d_i + \delta_i)\bar{\rho}(\omega_i) \text{ for some } d_i \in \mathbb{Z},\ \delta_i \in I_{sc}.$$

Then $c_2(x) = \sum_{i=1}^4 d_i c_2(\bar{\rho}_i)$. By Proposition 10.3 we have $d_1 \equiv d_3 \equiv d_4 \equiv 0 \bmod 2$. Since $c_2(\bar{\rho}_i) = 2q$ for $i = 1,3,4$ and $c_2(\bar{\rho}_2) = 12q$ by [15, §3d], we obtain that $c_2(x) \in 4\mathbb{Z}q = \mathrm{Dec}(G)$. □

11 Type E

We now treat the exceptional cases. In the following we show that any semi-decomposable invariant is decomposable for semisimple groups of type E_6 and E_7.

Lemma 11.1. *Let G be a split semisimple group of type E_6 or E_7. Then, $\mathrm{Sdec}(G) = \mathrm{Dec}(G)$, i.e., each semi-decomposable invariant is decomposable.*

Proof. We denote by E_6^{ad} (resp. E_7^{ad}) a split simple adjoint group of type E_6 (resp. E_7) and by E_6^{sc} (resp. E_7^{sc}) a split simple simply connected group of type E_6 (resp. E_7). We first consider the case where G is a semisimple group of type E_6, i.e., $G = (E_6^{sc} \times \cdots \times E_6^{sc})/\boldsymbol{\mu}$ (n copies of E_6^{sc}) for some central subgroup $\boldsymbol{\mu}$. Let $q_1, \cdots, q_n$ be the corresponding normalized Killing forms for each copy of E_6^{sc} in G. Since $\mathrm{Dec}(E_6^{sc}) = \mathrm{Dec}(E_6^{ad}) = 6q_i$ by [14, §4b], we have $\mathrm{Dec}(E_6^{sc} \times \cdots \times E_6^{sc}) = \mathrm{Dec}(E_6^{ad} \times \cdots \times E_6^{ad}) = 6\mathbb{Z}q_1 \oplus \cdots \oplus 6\mathbb{Z}q_n$. As $\mathrm{Dec}(E_6^{ad} \times E_6^{ad}) \subseteq \mathrm{Dec}(G) \subseteq \mathrm{Dec}(E_6^{sc} \times E_6^{sc})$, we conclude that $\mathrm{Dec}(G) = 6\mathbb{Z}q_1 \oplus \cdots \oplus 6\mathbb{Z}q_n$.
Now we show $\mathrm{Sdec}(G) \subseteq \mathrm{Dec}(G)$. Similar to the proof of Proposition 7.1, we consider an arbitrary element $x \in I_{sc}^W$:

$$x = \sum_{j=1}^{n} \sum_{i=1}^{6} (d_{ij} + \delta_{ij})\bar{\rho}(\omega_{ij})$$

for some $d_{ij} \in \mathbb{Z}$ and $\delta_{ij} \in I_{sc}$, where $\{\omega_{1j}, \ldots, \omega_{6j}\}$ is the fundamental weights of each copy of E_6^{sc}. Since $c_2(I_{sc}^3) = 0$, we obtain

$$c_2(x) = \sum_{j=1}^{n} \sum_{i=1}^{6} d_{ij} c_2(\bar{\rho}(\omega_{ij})). \tag{20}$$

By [11, §2], each element $c_2(\bar{\rho}(\omega_{ij}))$ in (20) iz contained in $6\mathbb{Z}q_j$. Hence, $\mathrm{Sdec}(G) \subseteq \mathrm{Dec}(G)$, so the equality holds.
Let $G = (E_7^{sc} \times \cdots \times E_7^{sc})/\boldsymbol{\mu}$ (n copies of E_7^{sc}) for some central subgroup $\boldsymbol{\mu}$. Then, the same argument together with $\mathrm{Dec}(E_6^{sc}) = \mathrm{Dec}(E_6^{ad}) = 12q_i$ ([14, §4b]) shows that $\mathrm{Dec}(G) = \mathrm{Sdec}(G) = 12\mathbb{Z}q_1 \oplus \cdots \oplus 12\mathbb{Z}q_n$. □

We determine the indecomposable groups for an arbitrary product of split simply-connected simple groups of type E_6 (resp. E_7) modulo the diagonal subgroups $\boldsymbol{\mu}_3$ (resp. $\boldsymbol{\mu}_2$).

Proposition 11.2. *(1) Let $G = (E_6^{sc} \times \cdots \times E_6^{sc})/\boldsymbol{\mu}_3$ with $n\,(\geq 2)$ copies of a split simple simply connected group E_6^{sc} of type E_6 and the diagonal subgroup $\boldsymbol{\mu}_3$. Then*

$$\mathrm{Inv}^3_{ind}(G) = \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^{\oplus n-1} \text{ and } \mathrm{Sdec}(G) = \mathrm{Dec}(G).$$

(2) Let $G = (E_7^{sc} \times \cdots \times E_7^{sc})/\boldsymbol{\mu}_2$ with $n\,(\geq 2)$ copies of a split simple simply connected group E_7^{sc} of type E_7 and the diagonal subgroup $\boldsymbol{\mu}_2$. Then

$$\mathrm{Inv}^3_{ind}(G) = \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/12\mathbb{Z})^{\oplus n-1} \text{ and } \mathrm{Sdec}(G) = \mathrm{Dec}(G).$$

Proof. By Lemma 11.1, it suffices to compute the indecomposable groups.
(1) Assume that $n = 2$. Then, by 5.1 the character group of the split maximal torus T of G is given by

$$T^* = \{\sum_{i=1}^{6} a_i\omega_i + \sum_{i=1}^{6} a_i'\omega_i' \,|\, a_1 + a_1' + a_5 + a_5' \equiv a_3 + a_3' + a_6 + a_6' \bmod 3\}. \tag{21}$$

Choose a basis $\{x_1,\dots x_6,x'_1,\dots,x'_6\}$ of (21) as follows

$$x_1=\omega_1+\omega'_6, x_2=\omega_2, x_3=\omega_3+2\omega'_6, x_4=\omega_4, x_5=\omega_5+\omega'_6, x_6=\omega_6+2\omega'_6,$$
$$x'_1=\omega'_1+\omega_6, x'_2=\omega'_2, x'_3=\omega'_3+2\omega'_6, x'_4=\omega'_4, x'_5=\omega'_5+\omega'_6, x'_6=3\omega'_6.$$

Let ϕ be a quadratic form on x_i, x'_i over $\mathbb{Z}$. Since the group $S^2(T^*_{sc})^W$ is generated by the normalized Killing forms

$$\begin{aligned} q &:= \omega_1^2-\omega_1\omega_3+\omega_2^2-\omega_2\omega_4+\omega_3^2-\omega_3\omega_4+\omega_4^2-\omega_4\omega_5+\omega_5^2-\omega_5\omega_6+\omega_6^2 \text{ and}\\ q' &:= \omega_1'^2-\omega'_1\omega'_3+\omega_2'^2-\omega'_2\omega'_4+\omega_3'^2-\omega'_3\omega'_4+\omega_4'^2-\omega'_4\omega'_5+\omega_5'^2-\omega'_5\omega'_6+\omega_6'^2, \end{aligned} \tag{22}$$

from the equation $\phi=dq+d'q'$, we obtain

$$\phi=(\tfrac{2d+2d'}{3})x_6'^2+\psi,$$

where ψ is a quadratic form with integer coefficients. Therefore,

$$Q(G)=\{dq+d'q' \mid d+d'\equiv 0 \bmod 3\}. \tag{23}$$

Therefore, by (23) and the proof of the previous lemma, we obtain

$$\mathrm{Inv}^3_{ind}(G)=\mathbb{Z}/2\mathbb{Z}(3q+3q')\oplus\mathbb{Z}/6\mathbb{Z}(q+2q').$$

For $n\geq 3$, the same argument shows that

$$Q(G)=\{d_1q+\cdots d_nq_n \mid \sum_{i=1}^n d_i\equiv 0 \bmod 3\},$$

where $q_1,\dots,q_n$ are the corresponding normalized Killing forms. Hence, the result for the indecomposable group follows from Lemma 11.1.
(2) Assume that $n=2$. Then, by 5.1 the character group of the split maximal torus T of G is given by

$$T^*=\{\sum_{i=1}^7 a_i\omega_i+\sum_{i=1}^7 a'_i\omega'_i \mid a_2+a_5+a_7\equiv a'_2+a'_5+a'_7 \bmod 2\}.$$

Since the group $S^2(T^*_{sc})^W$ is generated by the normalized Killing forms

$$q:=q_6-\omega_6\omega_7+\omega_7^2 \text{ and } q':=q'_6-\omega'_6\omega'_7+\omega_7'^2,$$

where q_6 and q'_6 are the normalized Killing forms of E_6 in (22), the same argument as in (1) shows

$$Q(G)=\{dq+d'q' \mid d+d'\equiv 0 \bmod 4\}.$$

Hence, by Lemma 11.1 $\mathrm{Inv}^3_{ind}(G)=\mathbb{Z}/3\mathbb{Z}(4q')\oplus\mathbb{Z}/12\mathbb{Z}(q-q')$. For $n\geq 3$, the same argument shows that $Q(G)=\{d_1q+\cdots d_nq_n \mid \sum_{i=1}^n d_i\equiv 0 \bmod 4\}$, where $q_1,\dots,q_n$ are the corresponding normalized Killing forms, which comptues the indecomposable group together with Lemma 11.1. □

REMARK 11.3. *(1) Let* $G = (E_6^{ad}\times\cdots\times E_6^{ad})\times(E_6^{sc}\times\cdots\times E_6^{sc})$, $n(\geq 0)$ *copies of* E_6^{ad} *and* $m(\geq 0)$ *copies of* E_6^{sc}. *It follows by* (2), *(23) and the proof of Lemma 11.1 that* $\mathrm{Inv}^3_{ind}(G) = (\mathbb{Z}/2\mathbb{Z})^{\oplus n}\oplus(\mathbb{Z}/6\mathbb{Z})^{\oplus m}$. *Similarly, for the same group* G *replacing* E_6 *by* E_7, *we have* $\mathrm{Inv}^3_{ind}(G) = (\mathbb{Z}/3\mathbb{Z})^{\oplus n}\oplus(\mathbb{Z}/12\mathbb{Z})^{\oplus m}$.
(2) Note that the center $\boldsymbol{\mu}_3\times\boldsymbol{\mu}_3$ *of* $E_6^{sc}\times E_6^{sc}$ *contains two nontrivial (*$\neq \boldsymbol{\mu}_3\times 1$, $1\times\boldsymbol{\mu}_3$*) central subgroups which is isomorphic to* $\boldsymbol{\mu}_3$*: a diagonal subgroup and a non-diagonal subgroup. Assume that* $\boldsymbol{\mu}$ *is non-diagonal. Then, the character group of* T *becomes*

$$T^* = \{\sum_{i=1}^{6} a_i\omega_i + \sum_{i=1}^{6} a_i'\omega_i' \,|\, a_1 + a_3' + a_5 + a_6' \equiv a_1' + a_3 + a_5' + a_6 \bmod 3\}.$$

In this case, we have the same $Q(G)$ *as in (23), thus have the same indecomposable group.*

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