Vector-Valued Modular Forms and the Gauss Map

Francesco Dalla Piazza, Alessio Fiorentino,
Samuel Grushevsky[1], Sara Perna,
Riccardo Salvati Manni[2].

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Abstract. We use the gradients of theta functions at odd two-torsion points — thought of as vector-valued modular forms — to construct holomorphic differential forms on the moduli space of principally polarized abelian varieties, and to characterize the locus of decomposable abelian varieties in terms of the Gauss images of two-torsion points.

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Introduction

The geometry of Siegel modular varieties — the quotients of the Siegel upper half-space $\mathcal{H}_g$ by discrete groups — has been under intense investigation for the last forty years, with various results obtained about their birational geometry, compactifications, and other properties. Some of the first results in this direction are due to Freitag, who in [Fre75a, Fre75b] showed that some Siegel modular varieties are not unirational, by constructing non-zero differential forms on them. This proof requires two ingredients: suitably compactifying the variety and arguing that the differential form extends, and actually constructing the differential forms. Freitag proved the appropriate general extension result for differential forms. Thanks to [AMRT10], [Tai82], and much subsequent work on the theory of compactifications of locally symmetric domains, the extension of differential forms is now well-known in full generality.

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In this paper, we focus on the original problem of constructing differential forms on Siegel modular varieties. We recall that differential forms on Siegel modular varieties can be constructed from suitable vector-valued modular forms. In general, vector-valued modular forms can be constructed using theta series with pluriharmonic coefficients, but the question of whether the series thus constructed are identically zero is very complicated. General results on the existence and non-vanishing of holomorphic differential forms can be found in [Wei83] and [Wei87]. In connection with the possibility of finding special divisors in the Siegel modular varieties in the sense of Weissauer [Wei87], we will restrict our attention to non-zero differential forms of degree one less than the top.

In [Fre78] Freitag constructed such forms on $\mathcal{A}_g$ for $g \equiv 1 \pmod 8$, for $g \geq 17$, while the fifth author, in [SM87], gave a completely different construction for $g \equiv 1 \pmod 4$, $g \neq 1, 5, 13$. In this paper, we present an easier and more natural method of constructing such differentials forms, providing also a natural bridge between methods of [Fre78] and [SM87]. Our tools will be the gradients of theta functions and expressions in terms of them considered by the third and fifth author in [GSM04, GSM06]. Our result is the following.

Denote by $\partial := \left(\frac{(1+\delta_{ij})}{2}\partial_{\tau_{ij}}\right)$ the matrix of partial derivatives with respect to τ. Let f, h be two scalar modular forms of the same weight, for some modular group Γ acting on $\mathcal{H}_g$. Then $A := h^2\partial(f/h)$ is a matrix-valued modular form. Denote by A^{ad} the adjoint matrix of A (the transpose of the matrix of cofactors), and denote by $d\check{\tau}_{ij}$ the wedge product of all $d\tau_{ab}$ for $1 \leq a \leq b \leq g$ except $d\tau_{ij}$, with the suitable sign. Denote by $d\check{\tau}$ the matrix of all $d\check{\tau}_{ij}$. Then

Theorem 1. *Let $g \geq 2$, let $f := \Theta[\varepsilon](\tau)$ and $h := \Theta[\delta](\tau)$ be second order theta constants. Then the modular form*

$$\omega := \mathrm{Tr}(A^{ad}_{\varepsilon,\delta} d\check{\tau}),$$

where A^{ad} and $d\check{\tau}$ are as defined above, is a non-zero holomorphic differential form on $\mathcal{A}_g(\Gamma) := \mathcal{H}_g/\Gamma$ of degree one less than the top (i.e. of degree $g(g+1)/2 - 1$). Here, for g odd we have $\Gamma = \Gamma_g(2,4)$, while for g even it is an index two subgroup $\Gamma_g^*(2,4) \subset \Gamma_g(2,4)$.

In what follows, we will discuss the relation of special cases of this construction to those of Freitag [Fre75b] and the fifth author [SM87]. In a related direction, we revisit the method of constructing vector-valued modular forms using gradients of odd theta functions with half integral characteristics. Recall that the gradients at $z = 0$ of odd theta functions with half integral characteristics can be thought of as the images of two-torsion points that are smooth points of the theta divisor under the Gauss map. In this direction, we obtain a proof of the following geometric statement.

Theorem 2. *A principally polarized abelian variety is decomposable (i.e. is a product of lower-dimensional ones) if and only if the images under the Gauss map of all smooth two-torsion points in the theta divisor lie on a quadric in $\mathbb{P}^{g-1}$.*

The structure of the paper is as follows. In section 1, we recall some basic facts about theta functions and vector-valued modular forms. In section 2, we collect several results about gradients of odd theta functions. In section 3, we prove Theorem 2. In section 4, we recall and improve results of Freitag and the fifth author about holomorphic differential forms on Siegel varieties. Finally, in section 5 we prove theorem 1 and explain the relation among the approaches to constructing differential forms on Siegel modular varieties.

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1. Definitions and notation

We use the standard definitions and notation in working with complex principally polarized abelian varieties (ppav), as used in [GSM04], which we now quickly summarize.

1.1. Siegel modular forms. Let $\mathcal{H}_g$ be the Siegel upper-half-space of degree g, namely the space of $g \times g$ complex symmetric matrices with positive definite imaginary part. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts transitively on $\mathcal{H}_g$ via

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1} \quad \text{where} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are the $g \times g$ blocks of the matrix γ. We will keep this block notation for a symplectic matrix throughout the paper.

The Siegel modular group is $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$. The principal congruence subgroup of level $n \in \mathbb{N}$ is defined as:

$$\Gamma_g(n) := \{\gamma \in \Gamma_g \,|\, \gamma \equiv 1_{2g} \bmod n\}.$$

A subgroup of finite index in Γ_g is called a congruence subgroup of level n if it contains $\Gamma_g(n)$. Notice that if $g > 1$, every subgroup of finite index is a congruence subgroup. The Siegel modular varieties obtained by taking the quotients with respect to the action of congruence subgroups are of central importance, as they define moduli spaces of ppav with suitable level structures.

More precisely, an element $\tau \in \mathcal{H}_g$ defines the complex abelian variety $X_\tau := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, hence τ is usually called a period matrix of the abelian variety X_τ. The quotient of $\mathcal{H}_g$ by the action of the Siegel modular group is classically known to be the moduli space of ppav: $\mathcal{A}_g := \mathcal{H}_g/\Gamma_g$.

We will use the so-called theta groups, which are congruence subgroups of level $2n$ defined as

$$\Gamma_g(n, 2n) := \left\{\gamma \in \Gamma_g(n) \,|\, \mathrm{diag}(A^tB) \equiv \mathrm{diag}(C^tD) \equiv 0 \bmod 2n\right\}.$$

We will also need the level 4 congruence subgroup

$$\Gamma^*_g(2,4) := \{\gamma \in \Gamma_g(2,4) \,|\, \mathrm{Tr}(A - 1_g) \equiv 0 \bmod 4\}, \tag{1}$$

which is of index 2 within $\Gamma_g(2,4)$. From now on, we will assume $g > 1$ and denote by Γ an arbitrary congruence subgroup of Γ_g. We denote $N := g(g+1)/2$, so that $\mathcal{A}_g(\Gamma) := \mathcal{H}_g/\Gamma$ is a complex N-dimensional orbifold.

Let $\rho : \mathrm{GL}(g,\mathbb{C}) \to \mathrm{End}(V)$ be an irreducible finite-dimensional rational representation; such representations are characterized by their highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_g) \in \mathbb{Z}^g$, with $\lambda_1 \geq \cdots \geq \lambda_g$. It will also be convenient for us to allow half-integer weights, which means to consider also $\det^{1/2} \otimes \rho'$ for a representation ρ' with integer weight. Let then $[\Gamma, \rho]$ be the space of holomorphic functions $f : \mathcal{H}_g \to V_\rho$ defined by the following property:

$$[\Gamma,\rho] := \{f : \mathcal{H}_g \to V_\rho \mid f(\gamma\cdot\tau) = \rho(C\tau + D)f(\tau), \quad \forall\gamma \in \Gamma, \forall\tau \in \mathcal{H}_g\}.$$

Such a function f is called a vector-valued modular form or ρ-valued modular form with respect to the representation $\rho = (\lambda_1, \lambda_2, \ldots, \lambda_g)$ and the group Γ. We call λ_g the *weight* of the vector-valued modular form f.

Since $\mathcal{H}_g$ is contractible, a ρ-valued modular form is a holomorphic section of a corresponding vector bundle on $\mathcal{A}_g(\Gamma)$. Denoting by $\mathbb{E}$ the rank g vector bundle over $\mathcal{A}_g$ whose fiber over A is the space $H^{1,0}(A,\mathbb{C})$, sections of $\mathbb{E}$ are modular forms for the standard representation of $\mathrm{GL}(g,\mathbb{C})$ on $\mathbb{C}^g$ and the group Γ_g.

More generally, it is possible to define a vector-valued modular form with a multiplier system for this kind of representation, see [Fre91] for details. We will make use of them when necessary.

1.2. Theta functions. Many examples of modular forms can be constructed by means of theta functions. Denote by $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. For $\varepsilon, \delta \in \mathbb{F}_2^g$, the theta function with characteristic $m = [\varepsilon, \delta]$ is the holomorphic function $\theta_m : \mathcal{H}_g \times \mathbb{C}^g \to \mathbb{C}$ defined by the series:

$$\theta_m(\tau, z) := \sum_{p\in\mathbb{Z}^g} e^{\pi i\left[(p+\varepsilon/2)^t\tau(p+\varepsilon/2)+2(p+\varepsilon/2)^t(z+\delta/2)\right]}.$$

We shall write $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z)$ for $\theta_m(\tau,z)$ if we need to emphasize the dependence on the characteristics. The characteristic m is called even or odd depending on whether the scalar product $\varepsilon\cdot\delta \in \mathbb{F}_2$ is zero or one, and the corresponding theta function is then even or odd as a function of z, respectively. The number of even (resp. odd) theta characteristics is $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$). Furthermore, theta functions with characteristics are solutions of the heat equation:

$$\frac{\partial^2}{\partial z_i \partial z_j}\theta_m(\tau,z) = 2\pi i(1+\delta_{ij})\frac{\partial}{\partial\tau_{ij}}\theta_m(\tau,z), \quad 1 \leq i,j \leq g. \tag{2}$$

For $\sigma \in \mathbb{F}_2^g$, the corresponding theta function of second order is defined as

$$\Theta[\sigma](\tau,z) := \theta\begin{bmatrix}\sigma\\ 0\end{bmatrix}(2\tau, 2z).$$

A theta constant is the evaluation at $z = 0$ of a theta function. Throughout the paper we will drop the argument $z = 0$ in the notation for theta constants. All odd theta constants with characteristics vanish identically in τ, as the corresponding theta functions are odd functions of z, and thus there are $2^{g-1}(2^g+1)$ non-trivial theta constants. All the 2^g second order theta functions are even in z, so there are 2^g theta constants of the second order.

As far as we are concerned, we will focus on the behavior of the theta constants under the action of subgroups of $\Gamma_g(2)$. By [Igu72], we have the following transformation formula:

$$\theta_m(\gamma \cdot \tau) = \kappa(\gamma) e^{2\pi i \phi_m(\gamma)} \det (C\tau + D)^{1/2} \theta_m(\tau), \qquad \forall \gamma \in \Gamma_g(2), \tag{3}$$

where

$$\phi_m(\gamma) = -\frac{1}{8}(\varepsilon^t B^t D\varepsilon + \delta^t A^t C\delta - 2\varepsilon^t B^t C\delta) + \frac{1}{4}\operatorname{diag}(A^t B)^t (D\varepsilon - C\delta),$$

and $\kappa(\gamma)$ is an 8^{th} root of unity, with the same sign ambiguity as $\det (C\tau + D)^{\frac{1}{2}}$.

Regarding second order theta constants, we will focus on the action of subgroups of $\Gamma_g(2,4)$. For every $\gamma \in \Gamma_g(2,4)$, let $\tilde{\gamma} \in \Gamma_g$ be such that $2(\gamma \cdot \tau) = \tilde{\gamma} \cdot (2\tau)$, that is $\tilde{\gamma} = \left(\begin{smallmatrix} A & 2B \\ C/2 & D \end{smallmatrix}\right)$. Hence, applying the transformation formula (3) to the second order theta constants, we get:

$$\Theta[\sigma](\gamma \cdot \tau) = \kappa(\tilde{\gamma}) \det(C\tau + D)^{1/2} \Theta[\sigma](\tau), \qquad \forall \gamma \in \Gamma_g(2,4). \tag{4}$$

The second order theta constants are thus modular forms of weight one half with respect to the congruence subgroup $\Gamma_g(2,4)$, and $v_\Theta(\gamma) := \kappa(\tilde{\gamma})$ is a fourth root of unity. For a fixed $\tau \in \mathcal{H}_g$, the abelian variety X_τ comes with a principal polarization given by its theta divisor Θ_τ, namely the zero locus of the holomorphic function $\theta_0(\tau, z)$. One can identify, even though in a non-canonical way, the characteristic $m = [\varepsilon, \delta] \in \mathbb{F}_2^g$ with the two-torsion point $x_m = (\varepsilon\tau + \delta)/2$ on the ppav X_τ. To this divisor we associate the symmetric line bundle $\mathcal{L} = \mathcal{O}_{X_\tau}(\Theta_\tau)$. Then the theta functions with characteristic m is, up to a constant factor, the unique section of the line bundle $t_{x_m}^* \mathcal{L}$. A two-torsion point x_m is called even/odd depending on whether the characteristic m is even or odd. Denoting by $X_\tau[2]$ the set of two-torsion points, note that for any $x_m \in X_\tau[2]$ we have $\mathcal{O}_{X_\tau}(2\Theta_\tau) \simeq \mathcal{L}^{\otimes 2} \simeq (t_{x_m}^* \mathcal{L})^{\otimes 2}$. Thus squares of theta functions with characteristics can be expressed in terms of a basis of sections of $\mathcal{L}^{\otimes 2}$, and such a basis is given by theta functions of the second order. The explicit formula is Riemann's bilinear relation:

$$\theta \left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, z)^2 = \sum_{\sigma \in \mathbb{F}_2^g} (-1)^{\sigma \cdot \delta} \Theta[\sigma + \varepsilon](\tau, z) \Theta[\sigma](\tau, 0). \tag{5}$$

Similarly, for every α, $\varepsilon \in \mathbb{F}_2^g$ the following relation holds:

$$\Theta[\alpha](\tau)\Theta[\alpha + \varepsilon](\tau) = \frac{1}{2^g} \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\alpha \cdot \sigma} \theta \left[\begin{smallmatrix} \varepsilon \\ \sigma \end{smallmatrix}\right](\tau)^2. \tag{6}$$

It is easily seen that the character v_Θ^2 is trivial precisely on the subgroup $\Gamma_g^*(2,4) \subset \Gamma_g(2,4)$.

As we are interested in the characterization of the locus of decomposable abelian varieties, we need to recall the following analytic characterization:

Theorem 3 ([Sas83],[SM94]). *A ppav is indecomposable (that is, is not equal to a product of lower-dimensional ppav) if and only if the matrix*

$$M(\tau) := \begin{pmatrix} \dots & \Theta[\varepsilon] & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \partial_{\tau_{ij}}\Theta[\varepsilon] & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

(with entries taken for all $\varepsilon \in \mathbb{F}_2^g$ and for all $1 \le i \le j \le g$) has maximal rank, i.e. rank $\frac{g(g+1)}{2} + 1$.

We recall also that taking the gradient with respect to z of the holomorphic function $\theta_0(\tau, z)$, we get the Gauss map

$$G : \Theta_\tau \dashrightarrow \mathbb{P}^{g-1}$$

defined on the smooth locus of the theta divisor $\Theta_\tau \subset X_\tau$. The Gauss map is dominant if and only if the ppav (X_τ, Θ_τ) is indecomposable.

We will also have to deal with indexing by subsets of the coordinates, and fix notation for this now. For any set X, we denote by $P(X)$ the collection of all its subsets, and by $P_k(X)$ the collection of all its subsets of cardinality k. If $X \subset \mathbb{Z}$, we can view it as an order (i.e. as a set ordered increasingly), and denote by $P_k^*(X) \subset P^*(X)$ respectively the collection of its sub-orders (i.e. increasingly ordered subsets). If $I \in P_k^*(X)$, we denote by I^c its complementary set thought of as an ordered set. Finally, we denote $X_g := \{1, \dots, g\}$, thought of as an ordered set.

2. Gradients of theta functions

In [GSM04], gradients of theta functions are used to study the geometry of the moduli space of principally polarized abelian varieties — this study was further continued in [GSM05, GSM06, GSM09, GH12, GH11]. Indeed, for any odd m the gradient

$$v_m(\tau) := \operatorname{grad}_z \theta_m(\tau, z)|_{z=0} \tag{7}$$

is a not identically zero vector-valued modular form for the group $\Gamma_g(4, 8)$ for the representation $\det^{\otimes 1/2} \otimes \mathrm{std}$, where std is the standard representation of $\mathrm{GL}(g, \mathbb{C})$ on $\mathbb{C}^g$. We have

$$v_m \in H^0(\mathcal{A}_g(4, 8), \det \mathbb{E}^{\otimes 1/2} \otimes \mathbb{E}).$$

In [GSM04], it is shown that in fact the set of gradients of theta functions for all odd m defines a generically injective map of $\mathcal{A}_g(4, 8)$ to the set of $g \times 2^{g-1}(2^g - 1)$ complex matrices (and in fact to the corresponding Grassmannian), providing a weaker analog for ppav of the results of Caporaso and Sernesi [CS03b, CS03a] characterizing a generic curve by its bitangents or its theta hyperplanes.

For $\varepsilon, \delta \in \mathbb{F}_2^g$, define the $g \times g$ symmetric matrix $C_{\varepsilon\,\delta}(\tau)$ with entries

$$C_{\varepsilon\,\delta,ij}(\tau) := 2\partial_{z_i}\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)\,\partial_{z_j}\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0), \tag{8}$$

where $\partial_{z_i} := \frac{\partial}{\partial z_i}$. Notice that $C_{\varepsilon\,\delta} = 2\,v_{\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]} v^t_{\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]}$. Moreover, define the $g \times g$ symmetric matrix $A_{\varepsilon\,\delta}$ with entries

$$A_{\varepsilon\,\delta,ij}(\tau) := \Theta[\varepsilon](\tau)\partial_{z_i}\partial_{z_j}\Theta[\delta](\tau) - \Theta[\delta](\tau)\partial_{z_i}\partial_{z_j}\Theta[\varepsilon](\tau). \tag{9}$$

In the current paper, it will be convenient also to write $C_{\varepsilon\,\delta}$ and $A_{\varepsilon\,\delta}$ as column vectors of size $N = g(g+1)/2$, which we will denote $\mathbf{C}_{\varepsilon\delta}$ and $\mathbf{A}_{\varepsilon\,\delta}$ respectively.

Because of the modularity of the gradients of odd theta functions, both $C_{\varepsilon\,\delta}$ and $A_{\varepsilon\,\delta}$ are vector-valued modular forms with respect to the group $\Gamma_g(4,8)$ (a more careful analysis of the transformation formula shows that it is in fact modular with respect to $\Gamma_g^*(2,4)$) for the representation $\det\otimes\mathrm{Sym}^2(\mathrm{std})$ — that is, with highest weight $(3,1,\ldots,1)$.

Using the fact that both theta functions with characteristic and theta functions of the second order satisfy the heat equation (2), one can express $C_{\varepsilon\delta}$ in terms of derivatives of second order theta constants, and vice versa.

Lemma 4 ([GSM04]). *We have the following identities of vector-valued modular forms:*

$$C_{\varepsilon\delta} = \frac{1}{2}\sum_{\alpha\in\mathbb{F}_2^g}(-1)^{\alpha\cdot\delta}A_{\varepsilon+\alpha\,\alpha}; \tag{10}$$

$$A_{\varepsilon+\alpha\,\alpha} = \frac{1}{2^{g-1}}\sum_{\{\delta\in\mathbb{F}_2^g\mid[\varepsilon,\delta]\,\mathrm{odd}\}}(-1)^{\alpha\cdot\delta}C_{\varepsilon\delta}. \tag{11}$$

Of course, we have the same identities relating $\mathbf{A}_{\varepsilon+\alpha\,\alpha}$ and $\mathbf{C}_{\varepsilon\delta}$.

3. Characterization of decomposable ppav

We are now ready to prove our first result on the characterization of decomposable ppav. Indeed, recall that if $\tau = \left(\begin{smallmatrix}\tau_1 & 0\\ 0 & \tau_2\end{smallmatrix}\right)$, with $\tau_i \in \mathcal{H}_{g_i}$, for $g_1+g_2=g$, then the theta function with characteristic splits as a product

$$\theta_m(\tau,z) = \theta_{m_1}(\tau_1,z_1)\cdot\theta_{m_2}(\tau_2,z_2),$$

where $z_i \in \mathbb{C}^{g_i}$, and we have written m as $m_1\,m_2$, with $m_i \in \mathbb{F}_2^{2g_i}$. Computing the partial derivatives and evaluating at zero, we get

$$v_m(\tau) = \Big(v_{m_1}(\tau_1)\cdot\theta_{m_2}(\tau_2,0),\ \theta_{m_1}(\tau_1,0)\cdot v_{m_2}(\tau_2)\Big).$$

Since m is odd, it follows that precisely one of m_1 and m_2 is odd, and thus only the corresponding g_i entries of the gradient vector are non-zero. Thus, if we arrange the gradients for all odd m in a matrix, it will have a block form, with the two non-zero blocks of sizes $g_i \times 2^{g_i-1}(2^{g_i}-1)$, and two "off-diagonal" zero blocks. This is simply to say that the set of gradients of all odd theta functions, at a point τ as above, lies in the union of coordinate linear spaces $\mathbb{C}^{g_1}\cup\mathbb{C}^{g_2}\subset\mathbb{C}^g$. Since $\mathrm{grad}_z\,\theta_m(\tau,z)|_{z=0}$ and $\mathrm{grad}_z\,\theta_0(\tau,z)|_{z=m}$ differ by a constant factor, and thus give the same point in $\mathbb{P}^{g-1}$, this implies that the images of all the smooth two-torsion points of Θ_τ under the Gauss map lie on g_1g_2 reducible quadrics in $\mathbb{P}^{g-1}$ written explicitly as

$$X_iX_j = 0, \qquad \forall\, 1 \le i \le g_1 < j \le g.$$

This is equivalent to these Gauss images all lying on a union of two hyperplanes, and a weaker condition is that they all lie on some quadric (not necessarily a reducible one). We now show that this weak condition is enough to characterize the locus of decomposable ppav, proving one of our two main results.

PROOF OF THEOREM 2. The discussion above proves that for a decomposable ppav with a period matrix $\tau = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$ the images of all the odd two-torsion points lie on a quadric. In general, if a ppav is decomposable its period matrix does not need to have this block shape, and would rather be conjugate to it under Γ_g. Since $v_m(\tau)$ are vector-valued modular forms for the representation $\det^{1/2} \otimes \mathrm{std}$, they transform linearly under the group action, and hence the condition that the images of the odd two-torsion points under the Gauss map lie on a quadric is preserved under the action of Γ_g. Thus, for any decomposable ppav the images of all smooth two-torsion points lying on Θ_τ are contained in (many) quadrics.

For the other direction of the theorem, we manipulate the gradients to reduce to the characterization of the locus of decomposable ppav given by Theorem 3. Indeed, suppose all images of the odd two-torsion points m lie on a quadric with homogenous equation $Q(x_1, \ldots, x_g)$: this is to say that

$$Q(v_m) = v_m^t B v_m = 0$$

for all odd $m \in X_\tau[2]$ that are smooth points of Θ_τ (where we have denoted by B the matrix of coefficients of Q). We thus have

$$\mathrm{Tr}(v_m^t B v_m) = \mathrm{Tr}(B v_m v_m^t) = \mathrm{Tr}(BC_m) = 0$$

for all odd m (if $m \in \mathrm{Sing}\, X_\tau$, then $v_m = 0$, so $C_m = 0$, and this still holds). Since by (11) each $A_{\alpha\,\beta}$ is a linear combination of the C_m's, it follows that we also have

$$\mathrm{Tr}(BA_{\alpha\,\beta}) = 0$$

for all α, β, and in particular this implies that the matrix

$$\mathbf{A} := (\mathbf{A}_{\alpha\,\beta})_{\alpha \ne \beta \in \mathbb{F}_2^g}, \tag{12}$$

where each $\mathbf{A}_{\alpha\,\beta}$ is a column-vector in $\mathbb{C}^{g(g+1)/2}$, is degenerate. The following lemma in linear algebra shows that this implies that the matrix $M(\tau)$ in Theorem 3 is degenerate, and thus that X_τ is decomposable — completing the proof of the theorem. $\square$

LEMMA 5. *The* $\frac{g(g+1)}{2} \times 2^{g-1}(2^g - 1)$ *matrix* $\mathbf{A}(\tau)$ *in* (12) *has rank less than* $\frac{g(g+1)}{2}$ *(i.e. non-maximal) if and only if the matrix* $M(\tau)$ *has non-maximal rank.*

Proof. For $1 \le i \le j \le g$, we denote M_{ij} and $\mathbf{A}_{ij}$, correspondingly, the (i,j) rows of the matrices $M(\tau)$ and $\mathbf{A}(\tau)$, and denote M_0 the first row of $M(\tau)$ (the vector of second order theta constants). We then have

$$M_0 \wedge M_{ij} = \mathbf{A}_{ij}\ ,$$

where by the wedge we mean taking the row vector whose entries are all two by two minors of the matrix formed by two row vectors M_0 and M_{ij}. If the vectors $\mathbf{A}_{\alpha\beta}$ are linearly dependent, this means we have some linear relation $0 = \sum a_{ij}\mathbf{A}_{ij}$ among the rows of $\mathbf{A}(\tau)$, which is equivalent to

$$0 = \sum_{i,j} a_{ij}(M_0 \wedge M_{ij}) = M_0 \wedge \left(\sum_{i,j} a_{ij} M_{ij}\right),$$

and thus M_0 must be proportional to $\sum a_{ij}M_{ij}$, so that the matrix M does not have maximal rank. □

Remark 6. The proof above shows that in fact a quadric in $\mathbb{P}^{g-1}$ contains the Gauss images of all the two-torsion points lying on the theta divisor if and only if it contains the entire image of the Gauss map.

4. A review of constructions of holomorphic differential forms on Siegel modular varieties

For a finite index subgroup $\Gamma \subset \Gamma_g$, we denote, as before, $\mathcal{A}_g(\Gamma) := \mathcal{H}_g/\Gamma$, and we are then interested in constructing non-zero degree k differential forms on it, that is elements of $\Omega^k(\mathcal{A}_g(\Gamma))$. It is known that for $g \ge 2$

$$\Omega^k(\mathcal{A}_g(\Gamma)) \cong \Omega^k(\mathcal{H}_g)^\Gamma,$$

where $\Omega^k(\mathcal{H}_g)^\Gamma$ is the vector space of elements of $\Omega^k(\mathcal{H}_g)$ equivariant under the action of Γ. Whenever $k < N = g(g+1)/2$ and $g \ge 2$, such holomorphic differential forms always extend to a compactification. More precisely, if $\mathcal{H}_g^0/\Gamma$ is the set of regular points of $\mathcal{H}_g/\Gamma$, and $\tilde{X}$ denotes the desingularization of the Satake compactification of $\mathcal{H}_g/\Gamma$, which contains $\mathcal{H}_g^0/\Gamma$ as a dense open subset, then every holomorphic differential form $\omega \in \Omega^k(\mathcal{H}_g^0/\Gamma)$ of degree $k < N$ extends to $\tilde{X}$ (see [FP82]).

Holomorphic differential forms can thus also be thought of as vector-valued modular forms for a suitable representation. We have the following fundamental result of Weissauer:

Theorem 7 ([Wei83]). *The space $\Omega^k(\mathcal{A}_g(\Gamma))$ is zero unless $k = g\alpha - \alpha(\alpha-1)/2$ for some $0 \le \alpha \le g$, in which case*

$$\Omega^{\alpha g - \frac{1}{2}\alpha(\alpha-1)}(\mathcal{A}_g(\Gamma)) = [\Gamma, \rho_\alpha] \tag{13}$$

is the space of vector-valued modular forms for the representation of $\mathrm{GL}(g,\mathbb{C})$ *with highest weight* $(g+1,\dots,g+1,\alpha,\dots,\alpha)$, *with* α *appearing* $g-\alpha$ *times.*

The case $k = N-1$, corresponding to the representation ρ_{g-1} with highest weight $(g+1,\dots,g+1,g-1)$, turns out to be of great interest, as it is related to the construction of special divisors on the Satake compactification of Siegel modular varieties. Indeed, Weissauer [Wei87] proved that the zero locus D_h of a modular form h on the Satake compactification of $\mathcal{A}_g(\Gamma)$ is a special divisor if and only if there exists a non-vanishing $\omega \in \Omega^{N-1}(\mathcal{H}_g)^\Gamma$ such that $\operatorname{Tr}(\omega(\tau)\partial_\tau h(\tau))$ is identically zero on D_h. Moreover, using theta series with pluriharmonic coefficients, Weissauer [Wei87] proved that for any g the space $\Omega^{N-1}(\mathcal{A}_g(\Gamma))$ is non-zero for a suitable Γ. Such differential forms can be constructed as follows. Let

$$d\check{\tau}_{ij} = \pm \bigwedge_{1\le h\le k\le g,\, (h,k)\neq(i,j)} d\tau_{hk},$$

where the sign is chosen in such a way that $d\check{\tau}_{ij} \wedge d\tau_{ij} = \bigwedge_{1\le i<j\le g} d\tau_{ij}$, see [Fre78]. Then we have

$$\omega = \operatorname{Tr}(A(\tau)d\check{\tau}) = \sum_{1\le i,j\le g} A_{ij}(\tau)d\check{\tau}_{ij}, \tag{14}$$

with

$$A(\gamma\cdot\tau) = \det(C\tau+D)^{g+1}\,(C\tau+D)^{-t}A(\tau)(C\tau+D)^{-1}. \tag{15}$$

In [Fre75a], Freitag provides a method to construct holomorphic differential $(N-1)$-forms in genus g, invariant with respect to any subgroup Γ of finite index of the symplectic group Γ_g, starting from two scalar valued modular forms in genus g, both of weight $\frac{g-1}{2}$. We briefly recall this construction and slightly improve his result. To simplify the notation, we set

$$\partial_{ij} = \frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\tau_{ij}}; \qquad \partial := (\partial_{ij}). \tag{16}$$

For any $I, J \in P_k(X_g)$ with $0 \le k \le g$, we denote by ∂^I_J the submatrix of ∂ obtained by taking the rows corresponding to the elements in I and the columns corresponding to the elements in J:

$$\partial^I_J = (\partial_{ij})_{\substack{i\in I\\ j\in J}}$$

and denote by $|\partial^I_J|$ the determinant $|\partial^I_J| = \det(\partial^I_J)$. For $k=0$, we set both ∂^I_J and $|\partial^I_J|$ to be the identity operator.

For any congruence subgroup Γ, Freitag [Fre75a] then defines the linear pairing $\{\ ,\ \}$ by

$$\begin{aligned}\{\ ,\ \} : [\Gamma,(g-1)/2] \times [\Gamma,(g-1)/2] &\to \Omega^{N-1}(\mathcal{A}_g(\Gamma))\\ (f,h) &\mapsto \{f,h\} := \operatorname{Tr}(B(\tau)d\check{\tau}),\end{aligned}$$

where

$$B(\tau)_{ij} := (-1)^{i+j}\sum_{k=0}^{g-1}\frac{(-1)^k}{\binom{g-1}{k}}\sum_{\substack{I\in P^*_k(X_g\setminus\{i\})\\ J\in P^*_k(X_g\setminus\{j\})}} s(I)s(J)\left|\partial^I_J\right| f(\tau)\left|\partial^{I^c}_{J^c}\right| h(\tau),$$

where $s(I)$ (resp. $s(J)$) denotes the sign of the permutation of the elements of $X_g \setminus \{i\}$ (resp. $X_g \setminus \{j\}$) that turns the set $I \cup I^c$ (resp. $J \cup J^c$) into an increasing ordered set. One then easily checks that the parity of the pairing is $\{f,h\} = (-1)^{g+1}\{h,f\}$.

In [Fre78], Freitag then proved that the holomorphic differential form

$$F^{(g)} := \left\{ \sum_m \theta_m^{g-1}(\tau), \sum_m \theta_m^{g-1}(\tau) \right\} \tag{17}$$

does not vanish identically when $g \equiv 1 \pmod 8$, for $g \geq 17$. We extend this result to $g = 9$:

Proposition 8. *The vector-valued modular form $F^{(9)}$ does not vanish identically, and thus gives a non-zero differential form in $\Omega^{44}(\mathcal{A}_9)$.*

Proof. Since the set of all $d\check{\tau}_{ij}$ for $1 \leq i \leq j \leq g$ is a basis of $\Omega^{N-1}(\mathcal{H}_g)$, it suffices to prove that at least one $B(\tau)_{ij}$ is not identically zero. By Freitag's computation [Fre78, eg. 61], the Fourier coefficient of the pairing $\{f,h\}$ with respect to a matrix T is given by

$$a_{\{f,h\}}(T)_{gg} = \sum_{k=1}^{g} \frac{(-1)^k}{\binom{g-1}{k-1}} \sum_{\substack{I,J \in P^*_{k-1}(X_{g-1}) \\ T_1+T_2=T}} s(I)s(J)|T_1|_J^I |T_2|_{J^c}^{I^c} a_f(T_1) a_h(T_2), \tag{18}$$

where $I^c = X_{g-1} \setminus I$ denotes the complement, and $a_f(T_1)$ and $a_h(T_2)$ are the Fourier coefficients of f and h corresponding to the matrices T_1 and T_2 respectively.

For our case this formula can be greatly simplified. Indeed, we recall the result of Igusa [Igu81] that $\sum_m \theta_m^8(\tau) = 2^g \Theta_{E_8}^{(g)}$. We then choose $T := \begin{pmatrix} \zeta_{E_8} & 0 \\ 0 & 0 \end{pmatrix}$, where ζ_{E_8} is the matrix associated with the quadratic form corresponding to the E_8 lattice, given in a suitable basis by

$$\zeta_{E_8} := \begin{pmatrix} 2&0&0&1&0&0&0&0 \\ 0&2&1&0&0&0&0&0 \\ 0&1&2&1&0&0&0&0 \\ 1&0&1&2&1&0&0&0 \\ 0&0&0&1&2&1&0&0 \\ 0&0&0&0&1&2&1&0 \\ 0&0&0&0&0&1&2&1 \\ 0&0&0&0&0&0&1&2 \end{pmatrix}. \tag{19}$$

By Köcher principle, the Fourier coefficients $a_f(S)$ or $a_h(S)$ with respect to a non-semidefinite positive matrix S are zero, and thus only the terms with even semidefinite positive T_1 and T_2 produce non-zero summands in (18). Whenever the chosen T is written as $T = T_1 + T_2$ with T_1, T_2 positive semidefinite matrices, one of T_i must be zero. Finally, recall that for $g = 9$ we have

$$\Theta_{E_8}(\tau) = \sum_{x_1,\dots,x_9 \in \Lambda_{E_8}} e^{\pi i \mathrm{Tr}(x \cdot x)} = \sum_{p \in \mathbb{Z}^{g=9,8}} e^{\pi i \mathrm{Tr}(p \zeta_{E_8} p^t \tau)} = \sum_M N_M \prod_{i \leq j} e^{\pi i m_{ij} \tau_{ij}},$$

where, for $M = (m_{ij})$ a symmetric $g \times g$ integer matrix, $N_M \in \mathbb{N}$ is the number of integral matrix solutions of the Diophantine system $p\zeta_{E_8}p^t = M$. Setting $M = T$ and writing $p = \binom{p_1}{p_2}$, where p_1 and p_2 are respectively 8×8 and

1×8 integer matrices, it follows that for all solutions $p_2 = 0$, while p_1 satisfies $p_1 \zeta_{E_8} p_1^t = \zeta_{E_8}$.

The number of solutions of the previous equations equals the order of the group $U(\zeta_{E_8})$ of automorphisms of the E_8 lattice, i.e. $a(\zeta_{E_8}) = \#(U(\zeta_{E_8})) = 4!6!8!$, see [CS99, page 121]. Thus, we finally have $N_T = a_{F^{(9)}}(T)_{99} = 4!6!8!$, hence there is a non-empty set of summands in (18), all of them positive, so it follows that $A(T)_{99}$ is non-zero. $\square$

REMARK 9. The argument above generalizes to give an alternative proof of Freitag's result for $g = 8k+1$, for any $k \geq 1$, using the modular form $\Theta_{E_8}(\tau)^k$.

We now recall another construction of holomorphic differentials forms, due to the fifth author [SM87]. For $M = (m_1, \dots, m_{g-1})$ a set of distinct odd characteristics, define

$$F(m_1, \dots, m_{g-1})(\tau) := v_{m_1}(\tau) \wedge \dots \wedge v_{m_{g-1}}(\tau).$$

One can then use these wedge products of gradients of theta functions to construct further vector-valued modular forms. We set

$$W(M)(\tau) := \pi^{-2g+2} F(m_1, \dots, m_{g-1})(\tau)^t \, F(m_1, \dots, m_{g-1})(\tau), \tag{20}$$

and then have

PROPOSITION 10 ([SM87]). *For g odd, for any matrix of distinct odd characteristics $M = (m_1, \dots, m_{g-1}) \in M_{2g \times (g-1)}(\mathbb{F}_2)$*

$$\omega(M)(\tau) := \mathrm{Tr}\left(W(m_1, \dots, m_{g-1})(\tau) d\check{\tau}\right)$$

is a non-zero holomorphic differential form in $\Omega^{N-1}(\mathcal{A}_g(2,4))$. If g is even, it is a non-zero holomorphic differential form in $\Omega^{N-1}(\mathcal{A}_g^*(2,4))$

REMARK 11. Symmetrizing the $\omega(M)$ constructed above using the action of the entire modular group, differential forms for the entire modular group were obtained in [SM87], thus showing that $\Omega^{N-1}(\mathcal{A}_g)$ is non-zero for any $g \equiv 1 \pmod 4$, $g \neq 1, 5, 13$.

5. A NEW CONSTRUCTION OF DIFFERENTIAL FORMS

Our first main theorem, Theorem 1, gives an easy new method to construct non-zero holomorphic differential forms on Siegel modular varieties, using the modular forms $A_{\varepsilon\delta}$. We prove that it works, and then relate this new construction to the two constructions discussed above.

PROOF OF THEOREM 1. Recall that for fixed ε, δ the matrix $A_{\varepsilon\delta}$ can be written as

$$A_{\varepsilon\,\delta}(\tau) := 4\pi i \,\Theta[\delta]^2 \partial \left(\frac{\Theta[\varepsilon]}{\Theta[\delta]}\right),$$

and thus its entries are vector-valued modular forms for the representation of highest weight $(3, 1, \dots, 1)$.

We denote by $A^{ad}_{\varepsilon\,\delta}$ the adjoint matrix — the transpose of the matrix of cofactors of A. This matrix is then clearly a vector-valued modular form $A^{ad}_{\varepsilon\,\delta} \in$

$[\Gamma, (g+1, \ldots, g+1, g-1)]$ with $\Gamma = \Gamma_g(2,4)$ for g odd, and $\Gamma = \Gamma_g^*(2,4)$ for g even, and thus $\mathrm{Tr}(A^{ad}_{\varepsilon\,\delta}\, d\check{\tau})$ defines a differential form of degree $N-1$ as claimed. It remains to prove that this differential form is not identically zero. Recalling that the product of a matrix and the matrix of its cofactors is the determinant times the identity matrix, if we prove that $\det A_{\varepsilon\,\delta}$ is not identically zero, it would follow that $A^{ad}_{\varepsilon\,\delta}$ is not identically zero, and thus that $\mathrm{Tr}(A^{ad}_{\varepsilon\,\delta}\, d\check{\tau})$ is not identically zero. The proof is thus completed by the following proposition. $\square$

Proposition 12. *The determinant* $\det A_{\varepsilon\,\delta}$ *is a not identically zero scalar modular form of weight* $g+2$.

Proof. Since $\Theta[\varepsilon]$ and $\Theta[\delta]$ are different forms, there exist τ such that $\Theta[\varepsilon](\tau) = 0 \neq \Theta[\delta](\tau)$. We then denote $Z := 2\tau$, and work on the abelian variety X_Z, where $Z\varepsilon/2 \in \Theta_Z$ and $Z\delta/2 \notin \Theta_Z$ are thus two-torsion points. Since the characteristics are even, the point $Z\varepsilon/2$ is then an even two-torsion point lying on Θ_Z, and thus is a singular point of Θ_Z. From [GSM09], it follows that generically the singularity of Θ_Z at $Z\varepsilon/2$ is an ordinary double point. This is equivalent, via the heat equations, to the matrix $\partial\theta_m(Z,0)$, with $m = [\varepsilon, 0]$, having rank g. Moreover, we choose Z such that $\theta_n(Z) \neq 0$, with $n = [\delta, 0]$ and thus see that $\det A_{\varepsilon\,\delta}$ is not identically zero. $\square$

We will now compare the different constructions of modular forms. In Freitag's construction, let us consider Freitag's pairing when f and h are suitable powers of second order theta constants. For any $\varepsilon \neq \delta \in \mathbb{F}_2^g$ let

$$\omega_{\varepsilon\,\delta} := \{\Theta[\varepsilon]^{g-1}, \Theta[\delta]^{g-1}\}. \tag{21}$$

A simple computation on the characters shows that for g odd $\omega_{\varepsilon\,\delta} \in \Omega^{N-1}(\mathcal{A}_g(2,4))$, while for g even we only get $\omega_{\varepsilon\,\delta} \in \Omega^{N-1}(\mathcal{A}_g^*(2,4))$.

To relate this to the current construction, we first prove the following

Proposition 13. *For any* $\varepsilon \neq \delta$ *we have*

$$A^{ad}_{\varepsilon\,\delta}(\tau) = \left(\frac{\pi^2}{2^{g-2}}\right)^{g-1} \sum_{\substack{\alpha_1,\ldots,\alpha_{g-1}\in\mathbb{F}_2^g \\ \text{s.t. } [\varepsilon+\delta,\, \alpha_j]\text{ odd}}} (-1)^{\delta\cdot(\alpha_1+\cdots+\alpha_{g-1})} W([\varepsilon+\delta,\, \alpha_1], \ldots, [\varepsilon+\delta,\, \alpha_{g-1}]),$$

where W *is defined in* (20).

Proof. We will need some basic facts from linear algebra. First, we note that if A and B are an $m\times n$ and an $n \times m$ matrix, respectively. Then

$$AB = \sum_{i=1}^n A_i B^i, \tag{22}$$

where A_i is the i-th column of A and B^i is the i-th row of B. Furthermore, we will need that for $I, J \in P_k^*(X_m)$, the following holds:

$$(AB)^I_J = A^I B_J, \tag{23}$$

where A^I is the submatrix obtained from A by taking rows corresponding to the elements of I, and B_J is the submatrix obtained from B by taking columns corresponding to the elements of J. The last identity we need is the following generalization of the Binet formula:

$$\det(AB) = \sum_{S\in P_m^*(X_n)} \det(A_S)\,\det(B^S). \tag{24}$$

Notice that if $m > n$, $P_m^*(X_n)$ is empty and the right-hamd side of the previous identity is zero, as should be the case, since the rank of AB is bounded by the ranks of A and B. Defining the $g \times 2^g$ matrix

$$V_{\varepsilon+\delta} = \left(v_{\left[\begin{smallmatrix}\varepsilon+\delta\\ \alpha\end{smallmatrix}\right]} \right)_{\alpha\in\mathbb{F}_2^g},$$

whose columns are the gradients $v_{\left[\begin{smallmatrix}\varepsilon+\delta\\ \alpha\end{smallmatrix}\right]}$ indexed by $\alpha \in \mathbb{F}_2^g$, and defining the $2^g \times g$ matrix

$$V^-_{\varepsilon+\delta} = \left((-1)^{\delta\cdot\alpha}\, v^t_{\left[\begin{smallmatrix}\varepsilon+\delta\\ \alpha\end{smallmatrix}\right]} \right)_{\alpha\in\mathbb{F}_2^g},$$

relations (11) and (22) imply

$$A_{\varepsilon,\delta} = \frac{1}{2^{g-2}}\, V_{\varepsilon+\delta}\, V^-_{\varepsilon+\delta}.$$

Hence, by a straightforward computation from (23) and (24) the proposition follows. $\square$

We now compare our construction to that of Freitag, thus also linking the two previously known methods.

THEOREM 14. *For $\varepsilon \neq \delta$, denote by $B_{\varepsilon\,\delta}$ the vector-valued modular form such that $\{\Theta[\varepsilon]^{g-1}, \Theta[\delta]^{g-1}\} = \mathrm{Tr}(B_{\varepsilon\,\delta}(\tau)d\check{\tau})$. Then we have*

$$A^{ad}_{\varepsilon\,\delta} = \frac{(4\pi i)^{g-1}}{(g-1)!} B_{\varepsilon\,\delta}. \tag{25}$$

We note that of course the above is an identity of vector-valued modular forms, which also implies that the holomorphic differential forms constructed from them are equal in $\Omega^{N-1}(\mathcal{A}_g(2,4))$ and $\Omega^{N-1}(\mathcal{A}^*_g(2,4))$, for g odd and even respectively.

The proof of Theorem 14 relies on the following

LEMMA 15. *Let $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$ be elements of $P_k^*(X_g)$ with $k \leq n$. As a consequence of the heat equations, for every $\varepsilon \in \mathbb{F}_2^g$ the second order theta constant $\Theta[\varepsilon]$ satisfies the relation*

$$|\partial^I_J|\,\Theta[\varepsilon]^n = n(n-1)\dots(n-k+1)\Theta[\varepsilon]^{n-k}\,|(\partial\Theta[\varepsilon])^I_J|.$$

REMARK 16. We emphasize that the left-hand-side of the lemma means the determinant of the matrix of partial derivatives, considered as a degree k differential operator, applied to the power of the theta constant, while the right-hand-side is a different power of the theta constant multiplied by the determinant of the matrix of partial derivatives of the theta constants. When differentiating on the left, one would a priori expect terms involving higher order derivatives of the theta constant to appear, and the content of the lemma is that such cancel out.

Proof of lemma 15. The proof will be by induction in k. Clearly, for $k = 1$

$$\frac{(1+\delta_{i_1j_1})}{2}\partial_{\tau_{i_1j_1}}\Theta[\varepsilon]^n = n\Theta[\varepsilon]^{n-1}\frac{(1+\delta_{i_1j_1})}{2}\partial_{\tau_{i_1j_1}}\Theta[\varepsilon].$$

The first interesting case is $k = 2$, where $I = \{i_1, i_2\}$ and $J = \{j_1, j_2\}$. In this case we have

$$|\partial^I_J|\,\Theta[\varepsilon]^n = n(n-1)\Theta[\varepsilon]^{n-2}\,|(\partial\Theta[\varepsilon])^I_J| + n\Theta[\varepsilon]^{n-1}(|\partial^I_J|\,\Theta[\varepsilon]).$$

From the heat equation, it easily follows that for every $\varepsilon \in \mathbb{F}_2^g$

$$(1+\delta_{i_1j_1})(1+\delta_{i_2j_2})\partial_{\tau_{i_1j_1}}\partial_{\tau_{i_2j_2}}\Theta[\varepsilon] = (1+\delta_{i_2j_1})(1+\delta_{i_1j_2})\partial_{\tau_{i_2j_1}}\partial_{\tau_{i_1j_2}}\Theta[\varepsilon],$$

and hence

$$|\partial^I_J|\,\Theta[\varepsilon] = \begin{vmatrix} \frac{(1+\delta_{i_1j_1})}{2}\partial_{\tau_{i_1j_1}} & \frac{(1+\delta_{i_1j_2})}{2}\partial_{\tau_{i_1j_2}} \\ \frac{(1+\delta_{i_2j_1})}{2}\partial_{\tau_{i_2j_1}} & \frac{(1+\delta_{i_2j_2})}{2}\partial_{\tau_{i_2j_2}} \end{vmatrix}\Theta[\varepsilon] = 0. \tag{26}$$

Computing $|\partial^I_J|$ by the Laplace expansion along the first column for $k > 2$, we have

$$\begin{aligned}
|\partial^I_J|\,\Theta[\varepsilon]^n &= \Big(\sum_{h=1}^{k}(-1)^{h+1}\partial_{i_hj_1}\left|\partial^{I\setminus\{i_h\}}_{J\setminus\{j_1\}}\right|\Big)\Theta[\varepsilon]^n = \\
&= \sum_{h=1}^{k}(-1)^{h+1}\partial_{i_hj_1}\Big[n(n-1)\dots(n-k+2)\Theta[\varepsilon]^{n-k+1}\left|(\partial\Theta[\varepsilon])^{I\setminus\{i_h\}}_{J\setminus\{j_1\}}\right|\Big] = \\
&= n(n-1)\dots(n-k+1)\Theta[\varepsilon]^{n-k}\,|(\partial\Theta[\varepsilon])^I_J| + \\
&\quad + n(n-1)\dots(n-k+2)\Theta[\varepsilon]^{n-k+1}\sum_{h=1}^{k}(-1)^{h+1}\partial_{i_hj_1}\left|(\partial\Theta[\varepsilon])^{I\setminus\{i_h\}}_{J\setminus\{j_1\}}\right|.
\end{aligned}$$

The extra terms cancel out because of the heat equation, so the lemma is proved. ☐

We are now ready to prove the above theorem.

Proof of theorem 14. By [Wei83, lemma 4], to prove the identity of such vector-valued modular forms, it is enough to prove that, for example, the gg-th entries of the corresponding matrices agree.

We first recall that the determinant of a matrix can be expanded in its block submatrices as follows: for an $n \times n$ matrix M, and for any fixed $J \in P^*_k(X_n)$, we have

$$\det(M) = \sum_{I\in P^*_k(X_n)}(-1)^{I+J}\cdot|M^I_J|\cdot|M^{I^c}_{J^c}|\,,$$

where on the right we take the determinants of the corresponding submatrices, and $(-1)^I$ means $(-1)^{i_1+\dots+i_k}$, where $I = \{i_1, \dots, i_k\}$. Applying this to the

gg-th entry of the cofactor matrix, we get

$$(A^{ad}_{\varepsilon\,\delta})_{gg} = (4\pi i)^{g-1}\sum_{k=0}^{g-1}(-1)^k\Theta[\varepsilon]^{g-k-1}\Theta[\delta]^k \cdot \\ \cdot \sum_{I,J\in P^*_k(X_{g-1})}(-1)^{I+J}|(\partial\Theta[\varepsilon])^I_J|\cdot|(\partial\Theta[\delta])^{I^c}_{J^c}|.$$

By Lemma 15, it follows that

$$(B_{\varepsilon\,\delta})_{gg} = (g-1)!\sum_{k=0}^{g-1}(-1)^k\Theta[\varepsilon]^{g-k-1}\Theta[\delta]^k \cdot \\ \cdot \sum_{I,J\in P^*_k(X_{g-1})}s(I)s(J)|(\partial\Theta[\varepsilon])^I_J|\cdot|(\partial\Theta[\delta])^{I^c}_{J^c}|.$$

To complete the proof it is enough to check that $s(I)\,s(J) = (-1)^{I+J}$. This can be easily verified by induction in k, noting that for $I = \{i\}$ it holds that $s(I) = (-1)^{i-1}$, since it is the sign of the permutation that turns the set $\{i, 1, \ldots, i-1, i+1, \ldots, g-1\}$ into the set $\{1, \ldots, g-1\}$. □

Remark 17. In all of the constructions above, instead of starting from $A_{\varepsilon\delta}$, one can perform the same construction starting from theta constants of arbitrary level, or from two theta constants with characteristic. As a result, one gets vector-valued modular forms for suitable subgroups, which can be used to construct holomorphic differential forms on suitable Siegel modular varieties.

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Francesco Dalla Piazza
Dipartimento di Matematica
Università di Roma “La Sapienza”
Piazzale Aldo Moro, 2
I-00185 Roma
Italy
f.dallapiazza@gmail.com

Alessio Fiorentino
Dipartimento di Matematica
Università di Roma “La Sapienza”
Piazzale Aldo Moro, 2
I-00185 Roma
Italy
fiorentinoalessio@alice.it

Samuel Grushevsky
Mathematics Department
Stony Brook University,
Stony Brook
NY 11794-3651
USA
sam@math.stonybrook.edu

Sara Perna
Dipartimento di Matematica
Università di Roma “La Sapienza”
Piazzale Aldo Moro, 2
I-00185 Roma
Italy
sara.perna89@gmail.com

Riccardo Salvati Mannia
Dipartimento di Matematica
Università di Roma “La Sapienza”
Piazzale Aldo Moro, 2
I-00185 Roma
Italy
salvati@mat.uniroma1.it