Iwasawa Theory and F-Analytic Lubin-Tate (φ, Γ)-Modules

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Abstract. Let K be a finite extension of $\mathbf{Q}_p$. We use the theory of (φ, Γ)-modules in the Lubin-Tate setting to construct some corestriction-compatible families of classes in the cohomology of V, for certain representations V of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. If in addition V is crystalline, we describe these classes explicitly using Bloch-Kato's exponential maps. This allows us to generalize Perrin-Riou's period map to the Lubin-Tate setting.

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Contents

Introduction

Let K be a finite extension of $\mathbf{Q}_p$ and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. In this article, we use the theory of (φ, Γ)-modules in the Lubin-Tate setting to construct some classes in $\mathrm{H}^1(K, V)$, for "F-analytic" representations V of G_K. If in addition V is crystalline, we describe these classes explicitly using Bloch and Kato's exponential maps and generalize Perrin-Riou's period map to the Lubin-Tate setting.

We now describe our constructions in more detail, and introduce some notation which is used throughout this paper. Let F be a finite Galois extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F$, let π be a uniformizer of $\mathcal{O}_F$ and let $k_F = \mathcal{O}_F/\pi$ and $q = \mathrm{Card}(k_F)$. Let LT be the Lubin-Tate formal group [LT65] attached to π. We fix a coordinate T on LT, so that for each $a \in \mathcal{O}_F$ the multiplication-by-a map is given by a power series $[a](T) = aT + \mathrm{O}(T^2) \in \mathcal{O}_F[\![T]\!]$. Let $\log_{\mathrm{LT}}(T)$ denote the attached logarithm and $\exp_{\mathrm{LT}}(T)$ its inverse for the composition. Let $\chi_\pi : G_F \to \mathcal{O}_F^\times$ be the attached Lubin-Tate character. If K is a finite extension of F, let $K_n = K(\mathrm{LT}[\pi^n])$ and $K_\infty = \cup_{n \geqslant 1} K_n$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.

Let $\mathbf{A}_F$ denote the set of power series $\sum_{i \in \mathbf{Z}} a_i T^i$ with $a_i \in \mathcal{O}_F$ such that $a_i \to 0$ as $i \to -\infty$ and let $\mathbf{B}_F = \mathbf{A}_F[1/\pi]$, which is a field. It is endowed with a Frobenius map $\varphi_q : f(T) \mapsto f([\pi](T))$ and an action of Γ_F given by $g : f(T) \mapsto f([\chi_\pi(g)](T))$. If K is a finite extension of F, the theory of the field of norms ([FW79a, FW79b] and [Win83]) provides us with a finite unramified extension $\mathbf{B}_K$ of $\mathbf{B}_F$. Recall [Fon90] that a (φ, Γ)-module over $\mathbf{B}_K$ is a finite dimensional $\mathbf{B}_K$-vector space endowed with a compatible Frobenius map φ_q and action of Γ_K. We say that a (φ, Γ)-module over $\mathbf{B}_K$ is étale if it has a basis in which $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(\mathbf{A}_K)$. The relevance of these objects is explained by the result below (see [Fon90], [KR09]).

Theorem. *There is an equivalence of categories between the category of F-linear representations of G_K and the category of étale (φ, Γ)-modules over $\mathbf{B}_K$.*

Let $\mathbf{B}_F^\dagger$ denote the set of power series $f(T) \in \mathbf{B}_F$ that have a non-empty domain of convergence. The theory of the field of norms again provides us [Mat95] with a finite extension $\mathbf{B}_K^\dagger$ of $\mathbf{B}_F^\dagger$. We say that a (φ, Γ)-module over $\mathbf{B}_K$ is overconvergent if it has a basis in which $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(\mathbf{B}_K^\dagger)$ and $\mathrm{Mat}(g) \in \mathrm{GL}_d(\mathbf{B}_K^\dagger)$ for all $g \in \Gamma_K$. If $F = \mathbf{Q}_p$, every étale (φ, Γ)-module over $\mathbf{B}_K$ is overconvergent [CC98]. If $F \neq \mathbf{Q}_p$, this is no longer the case [FX13].

Let us say that an F-linear representation V of G_K is F-analytic if for all embeddings $\tau : F \to \overline{\mathbf{Q}}_p$, with $\tau \neq \mathrm{Id}$, the representation $\mathbf{C}_p \otimes_F^\tau V$ is trivial (as a semilinear $\mathbf{C}_p$-representation of G_K). The following result is known [Ber16].

Theorem. *If V is an F-analytic representation of G_K, it is overconvergent.*

Another source of overconvergent representations of G_K is the set of representations that factor through Γ_K (see §1.3). Our first result is the following (theorem 1.3.1).

Theorem A. *If V is an overconvergent representation of G_K, there exists an F-analytic representation X_{an} of G_K, a representation Y_Γ of G_K that factors through Γ_K, and a surjective G_K-equivariant map $X_{\mathrm{an}} \otimes_F Y_\Gamma \to V$.*

We next focus on F-analytic representations. Let $\mathbf{B}^\dagger_{\mathrm{rig},F}$ denote the Robba ring, which is the ring of power series $f(T) = \sum_{i \in \mathbf{Z}} a_i T^i$ with $a_i \in F$ such that there exists $\rho < 1$ such that $f(T)$ converges for $\rho < |T| < 1$. We have $\mathbf{B}^\dagger_F \subset \mathbf{B}^\dagger_{\mathrm{rig},F}$. The theory of the field of norms again provides us with a finite extension $\mathbf{B}^\dagger_{\mathrm{rig},K}$ of $\mathbf{B}^\dagger_{\mathrm{rig},F}$. If V is an F-linear representation of G_K, let $\mathrm{D}(V)$ denote the (φ, Γ)-module over $\mathbf{B}_K$ attached to V. If V is overconvergent, there is a well defined (φ, Γ)-module $\mathrm{D}^\dagger(V)$ over $\mathbf{B}^\dagger_K$ attached to V, such that $\mathrm{D}(V) = \mathbf{B}_K \otimes_{\mathbf{B}^\dagger_K} \mathrm{D}^\dagger(V)$. We call $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$ the (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ attached to V, given by $\mathrm{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_K} \mathrm{D}^\dagger(V)$.

The ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is a free $\varphi_q(\mathbf{B}^\dagger_{\mathrm{rig},K})$-module of degree q. This allows us to define [FX13] a map $\psi_q : \mathbf{B}^\dagger_{\mathrm{rig},K} \to \mathbf{B}^\dagger_{\mathrm{rig},K}$ that is a Γ_K-equivariant left inverse of φ_q, and likewise, if V is an overconvergent representation of G_K, a map $\psi_q : \mathrm{D}^\dagger_{\mathrm{rig}}(V) \to \mathrm{D}^\dagger_{\mathrm{rig}}(V)$ that is a Γ_K-equivariant left inverse of φ_q.

The main result of this article is the construction, for an F-analytic representation V of G_K, of a collection of maps

$$h^1_{K_n,V} : \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \to \mathrm{H}^1(K_n, V),$$

having a certain number of properties. For example, these maps are compatible with corestriction: $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$ if $n \geqslant 1$. Another property is that if $F = \mathbf{Q}_p$ and $\pi = p$ (the cyclotomic case), these maps coïncide with those constructed in [CC99] (and generalized in [Ber03]).

If now $K = F$ and V is a crystalline F-analytic representation of G_F, we give explicit formulas for $h^1_{F_n,V}$ using Bloch and Kato's exponential maps [BK90]. Let V be as above, let $\mathrm{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris},F} \otimes_F V)^{G_F}$ (note that because the $\otimes$ is over F, this is the identity component of the usual $\mathrm{D}_{\mathrm{cris}}$) and let $t_\pi = \log_{\mathrm{LT}}(T)$. Let $\{u_n\}_{n \geqslant 0}$ be a compatible sequence of primitive π^n-torsion points of LT. Let $\mathbf{B}^+_{\mathrm{rig},F}$ denote the positive part of the Robba ring, namely the ring of power series $f(T) = \sum_{i \geqslant 0} a_i T^i$ with $a_i \in F$ such that $f(T)$ converges for $0 \leqslant |T| < 1$. If $n \geqslant 0$, we have a map $\varphi_q^{-n} : \mathbf{B}^+_{\mathrm{rig},F} \to F_n[\![t_\pi]\!]$ given by $f(T) \mapsto f(u_n \oplus \exp_{\mathrm{LT}}(t_\pi/\pi^n))$. Using the results of [KR09], we prove that there is a natural (φ,Γ)-equivariant inclusion $\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \to \mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$. This provides us, by composition, with maps $\varphi_q^{-n} : \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \to F_n(\!(t_\pi)\!) \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ and $\partial_V \circ \varphi_q^{-n} : \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \to F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ where ∂_V is the "coefficient of t_π^0" map. Recall finally that we have two maps, Bloch and Kato's exponential $\exp_{F_n,V} : F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{H}^1(F_n, V)$ and its dual $\exp^*_{F_n,V^*(1)} \mathrm{H}^1(F_n,V) \to F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ (the subscript $V^*(1)$ denotes the dual of V twisted by the cyclotomic character, but is merely a notation here). The first result is as follows (theorem 3.3.1).

Theorem B. *If V is as above and $y \in \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1}$, then*

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) = \begin{cases} q^{-n}\partial_V(\varphi_q^{-n}(y)) & \text{if } n \geqslant 1 \\ (1-q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Let $\nabla = t_\pi \cdot d/dt_\pi$, let $\nabla_i = \nabla - i$ if $i \in \mathbf{Z}$ and let $h \geqslant 1$ be such that $\mathrm{Fil}^{-h}\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{cris}}(V)$. We prove that if $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$, then $\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \in \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1}$, and we have the following result (theorem 3.3.2).

Theorem C. *If V is as above and $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$, then*

$$h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) = \\ (-1)^{h-1}(h-1)! \begin{cases} \exp_{F_n,V}(q^{-n}\partial_V(\varphi_q^{-n}(y))) & \text{if } n \geqslant 1 \\ \exp_{F,V}((1-q^{-1}\varphi_q^{-1})\partial_V(y)) & \text{if } n = 0. \end{cases}$$

Using theorems B and C, we give in §3.5 a Lubin-Tate analogue of Perrin-Riou's "big exponential map" [PR94] using the same method as that of [Ber03] which treats the cyclotomic case. It will be interesting to compare this big exponential map with the "big logarithms" constructed in [Fou05] and [Fou08]. It is also instructive to specialize theorem C to the case $V = F(\chi_\pi)$, which corresponds to "Lubin-Tate" Kummer theory. Recall that if L is a finite extension of F, Kummer theory gives us a map $\delta : \mathrm{LT}(\mathfrak{m}_L) \to \mathrm{H}^1(L, F(\chi_\pi))$. When L varies among the F_n, these maps are compatible: the diagram

$$\begin{array}{ccc} \mathrm{LT}(\mathfrak{m}_{F_{n+1}}) & \xrightarrow{\delta} & \mathrm{H}^1(F_{n+1}, V) \\ {\scriptstyle \mathrm{Tr}^{\mathrm{LT}}_{F_{n+1}/F_n}}\downarrow & & \downarrow{\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \\ \mathrm{LT}(\mathfrak{m}_{F_n}) & \xrightarrow{\delta} & \mathrm{H}^1(F_n, V) \end{array}$$

commutes. Let S denote the set of sequences $\{x_n\}_{n\geqslant 1}$ with $x_n \in \mathfrak{m}_{F_n}$ and such that $\mathrm{Tr}^{\mathrm{LT}}_{F_{n+1}/F_n}(x_{n+1}) = [q/\pi](x_n)$ for $n \geqslant 1$. We prove that S is big, in the sense that (if $F \neq \mathbf{Q}_p$) the projection on the n-th coordinate map $S \otimes_{\mathcal{O}_F} F \to F_n$ is onto (this would not be the case if we did not have the factor q/π in the definition of S). Furthermore, we prove that if $x \in S$, there exists a power series $f(T) \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=1/\pi}$ such that $f(u_n) = \log_{\mathrm{LT}}(x_n)$ for $n \geqslant 1$. We have $d/dt_\pi(f(T)) \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=1}$ and the following holds (theorem 3.4.5), where u is the basis of $F(\chi_\pi)$ corresponding to the choice of $\{u_n\}_{n\geqslant 0}$.

THEOREM D. *We have* $h^1_{F_n,F(\chi_\pi)}(d/dt_\pi(f(T))\cdot u) = (q/\pi)^{-n}\cdot\delta(x_n)$ *for all* $n \geqslant 1$.

In the cyclotomic case, there is [Col79] a power series $\mathrm{Col}_x(T)$ such that $\mathrm{Col}_x(u_n) = x_n$ for $n \geqslant 1$. We then have $f(T) = \log \mathrm{Col}_x(T)$, and theorem D is proved in [CC99]. In the general Lubin-Tate case, we do not know whether there is a "Coleman power series" of which $f(T)$ would be the $\log_{\mathrm{LT}}$. This seems like a non-trivial question.

It would be interesting to compare our results with those of [SV17]. The authors of [SV17] also construct some classes in $\mathrm{H}^1(K,V)$, but start from the space $\mathrm{D}(V(\chi_\pi\cdot\chi_{\mathrm{cyc}}^{-1}))^{\psi_q=\pi/q}$. In another direction, is it possible to extend our constructions to representations of the form $V\otimes_F Y_\Gamma$ with V F-analytic and Y_Γ factoring through Γ_K, and in particular recover the explicit reciprocity law of [Tsu04]?

1 LUBIN-TATE (φ,Γ)-MODULES

In this chapter, we recall the theory of Lubin-Tate (φ,Γ)-modules and classify overconvergent representations.

1.1 NOTATION

Let F be a finite Galois extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_F$, and residue field k_F. Let π be a uniformizer of $\mathcal{O}_F$. Let $d = [F : \mathbf{Q}_p]$ and e be the ramification index of $F/\mathbf{Q}_p$. Let $q = p^f$ be the cardinality of k_F and let $F_0 = W(k_F)[1/p]$ be the maximal unramified extension of $\mathbf{Q}_p$ inside F. Let σ denote the absolute Frobenius map on F_0.

Let LT be the Lubin-Tate formal $\mathcal{O}_F$-module attached to π and choose a coordinate T for the formal group law, such that the action of π on LT is given by $[\pi](T) = T^q + \pi T$. If $a \in \mathcal{O}_F$, let $[a](T)$ denote the power series that gives the action of a on LT. Let $\log_{\mathrm{LT}}(T)$ denote the attached logarithm and $\exp_{\mathrm{LT}}(T)$ its inverse. If K is a finite extension of F, let $K_n = K(\mathrm{LT}[\pi^n])$ and let $K_\infty = \cup_{n\geqslant 1} K_n$. Let $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. By Lubin-Tate theory (see [LT65]), Γ_K is isomorphic to an open subgroup of $\mathcal{O}_F^\times$ via the Lubin-Tate character $\chi_\pi : \Gamma_K \to \mathcal{O}_F^\times$.

Let $n(K) \geqslant 1$ be such that if $n \geqslant n(K)$, then $\chi_\pi : \Gamma_{K_n} \to 1+\pi^n\mathcal{O}_F$ is an isomorphism, and $\log_p : 1+\pi^n\mathcal{O}_F \to \pi^n\mathcal{O}_F$ is also an isomorphism.

Since $\log_{\mathrm{LT}}(T)$ converges on the open unit disk, it can be seen as an element of $\mathbf{B}^+_{\mathrm{rig},F}$ and we denote it by t_π. Recall that $g(t_\pi) = \chi_\pi(g)\cdot t_\pi$ if $g \in G_K$ and that $\varphi_q(t_\pi) = \pi\cdot t_\pi$. Let $\partial = d/dt_\pi$ so that $\partial f(T) = a(T)\cdot df(T)/dT$, where $a(T) = (d\log_{\mathrm{LT}}(T)/dT)^{-1} \in \mathcal{O}_F[\![T]\!]^\times$. We have $\partial\circ g = \chi_\pi(g)\cdot g\circ\partial$ if $g \in \Gamma_K$ and $\partial\circ\varphi_q = \pi\cdot\varphi_q\circ\partial$.

Recall that $\mathbf{B}^\dagger_{\mathrm{rig},F}$ denotes the Robba ring, the ring of power series $f(T) = \sum_{i\in\mathbf{Z}} a_i T^i$ with $a_i \in F$ such that there exists $\rho < 1$ such that $f(T)$ converges for $\rho < |T| < 1$. We have $\mathbf{B}^\dagger_F \subset \mathbf{B}^\dagger_{\mathrm{rig},F}$ and by writing a power series as the sum of its plus part and its minus part, we get $\mathbf{B}^\dagger_{\mathrm{rig},F} = \mathbf{B}^+_{\mathrm{rig},F} + \mathbf{B}^\dagger_F$.

Each ring $R \in \{\mathbf{B}^\dagger_{\mathrm{rig},F}, \mathbf{B}^+_{\mathrm{rig},F}, \mathbf{B}^\dagger_F, \mathbf{B}_F\}$ is equipped with a Frobenius map $\varphi_q : f(T) \mapsto f([\pi](T))$ and an action of Γ_F given by $g : f(T) \mapsto f([\chi_\pi(g)](T))$. Moreover, the ring R is a free $\varphi_q(R)$-module of rank q, and we define $\psi_q : R \to R$ by the formula $\varphi_q(\psi_q(f)) = 1/q \cdot \mathrm{Tr}_{R/\varphi_q(R)}(f)$. The map ψ_q has the following properties (see for instance §2A of [FX13] and §1.2.3 of [Col16]): $\psi_q(x \cdot \varphi_q(y)) = \psi_q(x) \cdot y$, the map ψ_q commutes with the action of Γ_F, $\partial \circ \psi_q = \pi^{-1} \cdot \psi_q \circ \partial$ and if $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ then $\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \mathrm{LT}[\pi]} f(T \oplus z)$. If M is a free R-module with a semilinear Frobenius map φ_q such that $\mathrm{Mat}(\varphi_q)$ is invertible, then any $m \in M$ can be written as $m = \sum_i r_i \cdot \varphi_q(m_i)$ with $r_i \in R$ and $m_i \in M$ and the map $\psi_q : m \mapsto \sum_i \psi_q(r_i) \cdot m_i$ is then well-defined. This applies in particular to the rings $\mathbf{B}^\dagger_{\mathrm{rig},K}$, $\mathbf{B}^+_{\mathrm{rig},K}$, $\mathbf{B}^\dagger_K$, $\mathbf{B}_K$ and to the (φ,Γ)-modules over them.

1.2 Construction of Lubin-Tate (φ,Γ)-modules

A (φ,Γ)-module over $\mathbf{B}_K$ (or over $\mathbf{B}^\dagger_K$ or over $\mathbf{B}^\dagger_{\mathrm{rig},K}$) is a finite dimensional $\mathbf{B}_K$-vector space D (or a finite dimensional $\mathbf{B}^\dagger_K$-vector space or a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module of finite rank respectively), along with a semilinear Frobenius map φ_q whose matrix (in some basis) is invertible, and a continuous, semilinear action of Γ_K that commutes with φ_q.

We say that a (φ,Γ)-module D over $\mathbf{B}_K$ is étale if D has a basis in which $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(\mathbf{A}_K)$. Let $\mathbf{B}$ be the p-adic completion of $\cup_{M/F} \mathbf{B}_M$ where M runs through the finite extensions of F. By specializing the constructions of [Fon90], Kisin and Ren prove the following theorem (theorem 1.6 of [KR09]).

THEOREM 1.2.1. *The functors* $V \mapsto \mathrm{D}(V) = (\mathbf{B} \otimes_F V)^{H_K}$ *and* $\mathrm{D} \mapsto (\mathbf{B} \otimes_{\mathbf{B}_K} \mathrm{D})^{\varphi_q = 1}$ *give rise to mutually inverse equivalences of categories between the category of F-linear representations of G_K and the category of étale (φ,Γ)-modules over $\mathbf{B}_K$.*

We say that a (φ,Γ)-module D is overconvergent if there exists a basis of D in which the matrices of φ_q and of all $g \in \Gamma_K$ have entries in $\mathbf{B}^\dagger_K$. This basis then generates a $\mathbf{B}^\dagger_K$-vector space $\mathrm{D}^\dagger$ which is canonically attached to D. If V is a p-adic representation, we say that it is overconvergent if $\mathrm{D}(V)$ is overconvergent, and then $\mathrm{D}^\dagger(V)$ denotes the corresponding (φ,Γ)-module over $\mathbf{B}^\dagger_K$. The main result of [CC98] states that if $F = \mathbf{Q}_p$, then every étale (φ,Γ)-module over $\mathbf{B}_K$ is overconvergent (the proof is given for $\pi = p$, but it is easy to see that it works for any uniformizer). If $F \neq \mathbf{Q}_p$, some simple examples (see [FX13]) show that this is no longer the case.

Recall that an F-linear representation of G_K is F-analytic if $\mathbf{C}_p \otimes^\tau_F V$ is the trivial $\mathbf{C}_p$-semilinear representation of G_K for all embeddings $\tau \neq \mathrm{Id} \in \mathrm{Gal}(F/\mathbf{Q}_p)$.

This definition is the natural generalization of Kisin and Ren's notion of F-crystalline representation. Kisin and Ren then show that if $K \subset F_\infty$, and if V is a crystalline F-analytic representation of G_K, the (φ, Γ)-module attached to V is overconvergent (see §3.3 of [KR09]; they actually prove a stronger result, namely that the (φ, Γ)-module attached to such a V is of finite height).

If $\mathrm{D}^\dagger_{\mathrm{rig}}$ is a (φ, Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see §2.1 of [KR09] or §1C of [FX13]), the series $\log(g) = \log(1 + (g-1))$ gives rise to a differential operator $\nabla_g : \mathrm{D}^\dagger_{\mathrm{rig}} \to \mathrm{D}^\dagger_{\mathrm{rig}}$. The map $v \mapsto \exp(v)$ is defined on a neighborhood of 0 in $\mathrm{Lie}\,\Gamma_K$; the map $\mathrm{Lie}\,\Gamma_K \to \mathrm{End}(\mathrm{D}^\dagger_{\mathrm{rig}})$ arising from $v \mapsto \nabla_{\exp(v)}$ is $\mathbf{Q}_p$-linear, and we say that $\mathrm{D}^\dagger_{\mathrm{rig}}$ is F-analytic if this map is F-linear (see §2.1 of [KR09] and §1.3 of [FX13]).

If V is an overconvergent representation of G_K, we let $\mathrm{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_K} \mathrm{D}^\dagger(V)$. The following is theorem D of [Ber16].

Theorem 1.2.2. *The functor $V \mapsto \mathrm{D}^\dagger_{\mathrm{rig}}(V)$ gives rise to an equivalence of categories between the category of F-analytic representations of G_K and the category of étale F-analytic Lubin-Tate (φ, Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$.*

In general, representations of G_K that are not F-analytic are not overconvergent (see §1.3), and the analogue of theorem 1.2.2 without the F-analyticity condition on both sides does not hold.

1.3 Overconvergent Lubin-Tate (φ, Γ)-modules

By theorem 1.2.2, there is an equivalence of categories between the category of F-analytic representations of G_K and the category of étale F-analytic Lubin-Tate (φ, Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. The purpose of this section is to prove a conjecture of Colmez that describes *all* overconvergent representations of G_K. Any representation V of G_K that factors through Γ_K is overconvergent, since H_K acts trivially on V so that $\mathrm{D}(V) = \mathbf{B}_K \otimes_F V$ and therefore $\mathrm{D}(V)$ has a basis in which $\mathrm{Mat}(\varphi_q) = \mathrm{Id}$ and $\mathrm{Mat}(g) \in \mathrm{GL}_d(\mathcal{O}_F)$ if $g \in \Gamma_K$. If X is F-analytic and Y factors through Γ_K, $X \otimes_F Y$ is therefore overconvergent. We prove that any overconvergent representation of G_K is a quotient (and therefore also a subobject, by dualizing) of some representation of the form $X \otimes_F Y$ as above.

Theorem 1.3.1. *If V is an overconvergent representation of G_K, there exists an F-analytic representation X of G_K, a representation Y of G_K that factors through Γ_K, and a surjective G_K-equivariant map $X \otimes_F Y \to V$.*

Proof. Recall (see §3 of [Ber16]) that if $r > 0$, then inside $\mathbf{B}^\dagger_{\mathrm{rig},K}$ we have the subring $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ of elements defined on a fixed annulus whose inner radius depends on r and whose outer raidus is 1, and that (φ, Γ)-modules over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ can be defined over $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ if r is large enough, giving us a module $\mathrm{D}^{\dagger,r}_{\mathrm{rig}}(V)$. We also have rings $\mathbf{B}^{[r;s]}_K$ of elements defined on a closed annulus whose radii depend on $r \leqslant s$. One can think of an element of $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ as a compatible family

of elements of $\{\mathbf{B}_K^I\}_I$ where I runs over a set of closed intervals whose union is $[r;+\infty[$. In the rest of the proof, we use this principle of glueing objects defined on closed annuli to get an object on the annulus corresponding to $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$.
Choose $r > 0$ large enough such that $\mathrm{D}^{\dagger,r}_{\mathrm{rig}}(V)$ is defined, and $s \geqslant qr$. Let $\mathrm{D}^{[r;s]}(V) = \mathbf{B}_K^{[r;s]} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}^{\dagger,r}_{\mathrm{rig}}(V)$. If $a \in \mathcal{O}_F$, and if $\mathrm{val}_p(a) \geqslant n$ for $n = n(r,s)$ large enough, the series $\exp(a\cdot\nabla)$ converges in the operator norm to an operator on the Banach space $\mathrm{D}^{[r;s]}(V)$. This way, we can define a twisted action of Γ_{K_n} on $\mathrm{D}^{[r;s]}(V)$, by the formula $h \star x = \exp(\log_p(\chi_\pi(h)) \cdot \nabla)(x)$. This action is now F-analytic by construction.
Since $s \geqslant qr$, the modules $\mathrm{D}^{[q^m r;q^m s]}(V)$ for $m \geqslant 0$ are glued together (using the idea explained above) by φ_q and we get a new action of Γ_{K_n} on $\mathrm{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathrm{D}^{[r;+\infty[}(V)$ and hence on $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$. Since φ_q is unchanged, this new (φ,Γ)-module is étale, and therefore corresponds to a representation W of G_{K_n}. The representation W is F-analytic by theorem 1.2.2, and its restriction to H_K is isomorphic to V.
Let $X = \mathrm{ind}_{G_{K_n}}^{G_K} W$. By Mackey's formula, $X|_{H_K}$ contains $W|_{H_K} \simeq V|_{H_K}$ as a direct summand. The space $Y = \mathrm{Hom}(\mathrm{ind}_{G_{K_n}}^{G_K} W, V)^{H_K}$ is therefore a nonzero representation of Γ_K, and there is an element $y \in Y$ whose image is V. The natural map $X \otimes_F Y \to V$ is therefore surjective. Finally, X is F-analytic since W is F-analytic. □

By dualizing, we get the following variant of theorem 1.3.1.

COROLLARY 1.3.2. *If V is an overconvergent representation of G_K, there exists an F-analytic representation X of G_K, a representation Y of G_K that factors through Γ_K, and an injective G_K-equivariant map $V \to X \otimes_F Y$.*

1.4 EXTENSIONS OF (φ,Γ)-MODULES

In this section, we prove that there are no non-trivial extensions between an F-analytic (φ,Γ)-module and the twist of an F-analytic (φ,Γ)-module by a character that is not F-analytic. This is not used in the rest of the paper, but is of independent interest.
If $\delta\colon \Gamma_K \to \mathcal{O}_F^\times$ is a continuous character, and $g \in \Gamma_K$, let $w_\delta(g) = \log\delta(g)/\log\chi_\pi(g)$. Note that δ is F-analytic if and only if $w_\delta(g)$ is independent of $g \in \Gamma_K$.
We define the first cohomology group $\mathrm{H}^1(\mathrm{D})$ of a (φ,Γ)-module D as in §4 of [FX13]. Let D be a (φ,Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. Let G denote the semigroup $\varphi_q^{\mathbf{Z}_{\geqslant 0}} \times \Gamma_K$ and let $\mathrm{Z}^1(\mathrm{D})$ denote the set of continuous functions $f\colon G \to \mathrm{D}$ such that $(h-1)f(g) = (g-1)f(h)$ for all $g,h \in G$. Let $\mathrm{B}^1(\mathrm{D})$ be the subset of $\mathrm{Z}^1(\mathrm{D})$ consisting of functions of the form $g \mapsto (g-1)y$, $y \in D$ and let $\mathrm{H}^1(\mathrm{D}) = \mathrm{Z}^1(\mathrm{D})/\mathrm{B}^1(\mathrm{D})$. If $g \in G$ and $f \in \mathrm{Z}^1$, then $[h \mapsto (g-1)f(h)] = [h \mapsto (h-1)f(g)] \in \mathrm{B}^1$. The natural actions of Γ_K and φ_q on H^1 are therefore trivial.

If D_0 and D_1 are two (φ,Γ)-modules, then $\mathrm{Hom}(\mathrm{D}_1,\mathrm{D}_0) = \mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},K}\text{-mod}}(\mathrm{D}_1,\mathrm{D}_0)$ is a free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module of rank $\mathrm{rk}(\mathrm{D}_0)\,\mathrm{rk}(\mathrm{D}_1)$ which is easily seen to be itself a (φ,Γ)-module. The space $\mathrm{H}^1(\mathrm{Hom}(\mathrm{D}_1,\mathrm{D}_0))$ classifies the extensions of D_1 by D_0. More precisely, if D is such an extension and if $s\colon \mathrm{D}_1 \to \mathrm{D}$ is a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-linear map that is a section of the projection $\mathrm{D} \to \mathrm{D}_1$, then $g \mapsto s - g(s)$ is a cocycle on G with values in $\mathrm{Hom}(\mathrm{D}_1,\mathrm{D}_0)$ (the element $g(s) \in \mathrm{Hom}(\mathrm{D}_1,\mathrm{D})$ being defined by $g(s)(g(x)) = g(s(x))$ for all $g \in G$ and all $x \in \mathrm{D}_1$). The class of this cocycle in the quotient $\mathrm{H}^1(\mathrm{Hom}(\mathrm{D}_1,\mathrm{D}_0))$ does not depend on the choice of the section s, and every such class defines a unique extension of D_1 by D_0 up to isomorphism.

Theorem 1.4.1. *If* D *is an* F*-analytic* (φ,Γ)*-module, and if* $\delta\colon \Gamma_K \to \mathcal{O}_F^\times$ *is* not *locally* F*-analytic, then* $\mathrm{H}^1(\mathrm{D}(\delta)) = \{0\}$.

Proof. If $g \in \Gamma_K$ and $x(\delta) \in \mathrm{D}(\delta)$ with $x \in \mathrm{D}$, we have

$$\nabla_g(x(\delta)) = \nabla(x)(\delta) + w_\delta(g)\cdot x(\delta).$$

If $g,h \in \Gamma_K$, this implies that $\nabla_g(x(\delta)) - \nabla_h(x(\delta)) = (w_\delta(g) - w_\delta(h))\cdot x(\delta)$. If $\overline{f} \in \mathrm{H}^1(\mathrm{D}(\delta))$ and $g \in \Gamma_K$, then $g(\overline{f}) = \overline{f}$ and therefore $\nabla_g(\overline{f}) = 0$. The formula above shows that if $k \in \Gamma_K$, then $\nabla_g(f(k)) - \nabla_h(f(k)) = (w_\delta(g) - w_\delta(h))\cdot f(k)$, so that $0 = (\nabla_g - \nabla_h)(\overline{f}) = (w_\delta(g) - w_\delta(h))\cdot\overline{f}$, and therefore $\overline{f} = 0$ if δ is not locally analytic. ☐

2 Analytic cohomology and Iwasawa theory

In this chapter, we explain how to construct classes in the cohomology groups of F-analytic (φ,Γ)-modules. This allows us to define our maps $h^1_{K_n,V}$.

2.1 Analytic cohomology

Let G be an F-analytic semigroup and let M be a Fréchet or LF space with a pro-F-analytic (§2 of [Ber16]) action of G. Recall that this means that we can write $M = \varinjlim_i \varprojlim_j M_{ij}$ where M_{ij} is a Banach space with a locally analytic action of G. A function $f : G \to M$ is said to be pro-F-analytic if its image lies in $\varprojlim_j M_{ij}$ for some i and if the corresponding function $f : G \to M_{ij}$ is locally F-analytic for all j.

The analytic cohomology groups $\mathrm{H}^i_{\mathrm{an}}(G,M)$ are defined and studied in §4 of [FX13] and §5 of [Col16]. In particular, we have $\mathrm{H}^0_{\mathrm{an}}(G,M) = M^G$ and $\mathrm{H}^1_{\mathrm{an}}(G,M) = \mathrm{Z}^1_{\mathrm{an}}(G,M)/\mathrm{B}^1_{\mathrm{an}}(G,M)$ where $\mathrm{Z}^1_{\mathrm{an}}(G,M)$ is the set of pro-F-analytic functions $f : G \to M$ such that $(g-1)f(h) = (h-1)f(g)$ for all $g,h \in G$ and $\mathrm{B}^1_{\mathrm{an}}(G,M)$ is the set of functions of the form $g \mapsto (g-1)m$.

Let M be a Fréchet space, and write $M = \varprojlim_n M_n$ with M_n a Banach space such that the image of M_{n+j} in M_n is dense for all $j \geqslant 0$.

Proposition 2.1.1. *We have* $\mathrm{H}^1_{\mathrm{an}}(G,M) = \varprojlim_n \mathrm{H}^1_{\mathrm{an}}(G,M_n)$.

Proof. By definition, we have an exact sequence

$$0 \to \mathrm{B}^1_{\mathrm{an}}(G, M_n) \to \mathrm{Z}^1_{\mathrm{an}}(G, M_n) \to \mathrm{H}^1_{\mathrm{an}}(G, M_n) \to 0.$$

It is clear that $\mathrm{B}^1_{\mathrm{an}}(G, M) = \varprojlim_n \mathrm{B}^1_{\mathrm{an}}(G, M_n)$ and that $\mathrm{Z}^1_{\mathrm{an}}(G, M) = \varprojlim_n \mathrm{Z}^1_{\mathrm{an}}(G, M_n)$, since these spaces are spaces of functions on G satisfying certain compatible conditions. The Banach spaces $\mathrm{B}^1_{\mathrm{an}}(G, M_n)$ satisfy the Mittag-Leffler condition: $\mathrm{B}^1_{\mathrm{an}}(G, M_n) = M_n/M_n^G$ and the image of M_{n+j} in M_n is dense for all $j \geqslant 0$. This implies that the sequence

$$0 \to \varprojlim_n \mathrm{B}^1_{\mathrm{an}}(G, M_n) \to \varprojlim_n \mathrm{Z}^1_{\mathrm{an}}(G, M_n) \to \varprojlim_n \mathrm{H}^1_{\mathrm{an}}(G, M_n) \to 0$$

is exact, and the proposition follows. □

In this paper, we mainly use the semigroups Γ_K, $\Gamma_K \times \Phi$ where $\Phi = \{\varphi_q^n, n \in \mathbf{Z}_{\geqslant 0}\}$ and $\Gamma_K \times \Psi$ where $\Psi = \{\psi_q^n, n \in \mathbf{Z}_{\geqslant 0}\}$. The semigroups Φ and Ψ are discrete and the F-analytic structure comes from the one on Γ_K.

DEFINITION 2.1.2. Let G be a compact group and let H be an open subgroup of G. We have the *corestriction* map $\mathrm{cor} : \mathrm{H}^1_{\mathrm{an}}(H, M) \to \mathrm{H}^1_{\mathrm{an}}(G, M)$, which satisfies $\mathrm{cor} \circ \mathrm{res} = [G : H]$. This map has the following equivalent explicit descriptions (see §2.5 of [Ser94] and §II.2 of [CC99]). Let $X \subset G$ be a set of representatives of G/H and let $f \in \mathrm{Z}^1_{\mathrm{an}}(H, M)$ be a cocycle.

1. By Shapiro's lemma, $\mathrm{H}^1_{\mathrm{an}}(H, M) = \mathrm{H}^1_{\mathrm{an}}(G, \mathrm{ind}_H^G M)$ and cor is the map induced by $i \mapsto \sum_{x \in X} x \cdot i(x^{-1})$;
2. if $M \subset N$ where N is a G-module and if there exists $n \in N$ such that $f(h) = (h-1)(n)$, then $\mathrm{cor}(f)(g) = (g-1)(\sum_{x \in X} xn)$;
3. if $g \in G$, let $\tau_g : X \to X$ be the permutation defined by $\tau_g(x)H = gxH$. We have $\mathrm{cor}(f)(g) = \sum_{x \in X} \tau_g(x) \cdot f(\tau_g(x)^{-1} gx)$.

If $g \in \Gamma_K$, let $\ell(g) = \log_p \chi_\pi(g)$. If M is a Fréchet space with a pro-F-analytic action of Γ_K and if $g \in \Gamma_K$ is such that $\chi_\pi(g) \in 1 + 2p\mathcal{O}_F$, then $\lim_{n \to \infty}(g^{p^n} - 1)/(p^n \ell(g))$ converges to an operator ∇ on M, which is independent of g thanks to the F-analyticity assumption. If $c : \Gamma_K \to M$ is an F-analytic map, let $c'(1)$ denote its derivative at the identity.

PROPOSITION 2.1.3. *If M is a Fréchet space with a pro-F-analytic action of Γ_K, the map $c \mapsto c'(1)$ induces an isomorphism* $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M) = (M/\nabla M)^{\Gamma_K}$, *under which* $\mathrm{cor}_{L/K}$ *corresponds to* $\mathrm{Tr}_{L/K}$.

Proof. Assume for the time being that M is a Banach space. We first show that the map induced by $c \mapsto c'(1)$ is well-defined and lands in $(M/\nabla M)^{\Gamma_K}$. The map $c \mapsto c'(1)$ from $\mathrm{Z}^1_{\mathrm{an}}(\Gamma_K, M) \to M$ is well-defined, and if $c(g) = (g-1)m$, then $c'(1) = \nabla m$ so that there is a well-defined map $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M) \to M/\nabla M$. If

$h \in \Gamma_K$ then $(h-1)c'(1) = \lim_{g\to 1}(h-1)c(g)/\ell(g) = \lim_{g\to 1}(g-1)c(h)/\ell(g) = \nabla c(h)$ so that the image of $c \mapsto c'(1)$ lies in $(M/\nabla M)^{\Gamma_K}$.

The formula for the corestriction follows from the explicit descriptions above: if $h \in \Gamma_L$ then $\tau_h(x) = x$ so that $\operatorname{cor}(c)(h) = \sum_{x\in X} x \cdot c(h)$ and

$$\operatorname{cor}(c)'(1) = \lim_{h\to 1} \operatorname{cor}(c)(h)/\ell(h) = \sum_{x\in X} x \cdot c'(1) = \operatorname{Tr}_{L/K}(c'(1)).$$

We now show that the map is injective. If $c'(1) = \nabla m$, then the derivative of $g \mapsto c(g) - (g-1)m$ at $g = 1$ is zero and hence $c(g) = (g-1)m$ on some open subgroup Γ_L of Γ_K and $c = [L : K]^{-1}\operatorname{cor}_{L/K} \circ \operatorname{res}_{K/L}(c) = 0$.

We finally show that the map is surjective. Suppose now that $y \in (M/\nabla M)^{\Gamma_K}$. The formula $g \mapsto (\exp(\ell(g)\nabla)-1)/\nabla \cdot y$ defines an analytic cocycle c_L on some open subgroup Γ_L of Γ_K. The image of $[L : K]^{-1}c_L$ under $\operatorname{cor}_{L/K}$ gives a cocyle $c \in \mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M)$ such that $c'(1) = y$.

We now let $M = \varprojlim_n M_n$ be a Fréchet space. The map $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M) \to (M/\nabla M)^{\Gamma_K}$ induced by $c \mapsto c'(1)$ is well-defined, and in the other direction we have the map $y \mapsto c_y$:

$$(M/\nabla M)^{\Gamma_K} \to \varprojlim_n (M_n/\nabla M_n)^{\Gamma_K} \to \varprojlim_n \mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M_n) \to \mathrm{H}^1_{\mathrm{an}}(\Gamma_K, M).$$

These two maps are inverses of each other. □

Remark 2.1.4. Compare with the following theorem (see [Tam15], corollary 21): if G is a compact p-adic Lie group and if M is a locally analytic representation of G, then $\mathrm{H}^i_{\mathrm{an}}(G, M) = \mathrm{H}^i(\mathrm{Lie}(G), M)^G$.

2.2 Cohomology of F-analytic (φ,Γ)-modules

If V is an F-analytic representation, let $\mathrm{H}^1_{\mathrm{an}}(K, V) \subset \mathrm{H}^1(K, V)$ classify the F-analytic extensions of F by V. Let D denote an F-analytic (φ,Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, such as $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$.

Proposition 2.2.1. *If V is F-analytic, then* $\mathrm{H}^1_{\mathrm{an}}(K, V) = \mathrm{H}^1_{\mathrm{an}}(\Gamma_K \times \Phi, \mathrm{D}^\dagger_{\mathrm{rig}}(V))$.

Proof. The group $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K \times \Phi, \mathrm{D}^\dagger_{\mathrm{rig}}(V))$ classifies the F-analytic extensions of $\mathbf{B}^\dagger_{\mathrm{rig},K}$ by $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$, which correspond to F-analytic extensions of F by V by theorem 1.2.2. □

Theorem 2.2.2. *If* D *is an F-analytic (φ,Γ)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and $i = 0, 1$, then* $\mathrm{H}^i_{\mathrm{an}}(\Gamma_K, \mathrm{D}^{\psi_q=0}) = 0$.

Proof. Since $\mathbf{B}^\dagger_{\mathrm{rig},F} \subset \mathbf{B}^\dagger_{\mathrm{rig},K}$, the $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module D is a free $\mathbf{B}^\dagger_{\mathrm{rig},F}$-module of finite rank. Let $\mathcal{R}_F$ denote $\mathbf{B}^\dagger_{\mathrm{rig},F}$ and let $\mathcal{R}_{\mathbf{C}_p}$ denote $\mathbf{C}_p \widehat{\otimes}_F \mathbf{B}^\dagger_{\mathrm{rig},F}$ the Robba ring with coefficients in $\mathbf{C}_p$. There is an action of G_F on the coefficients of $\mathcal{R}_{\mathbf{C}_p}$ and $\mathcal{R}_{\mathbf{C}_p}^{G_F} = \mathcal{R}_F$.

Theorem 5.5 of [Col16] says that $\mathrm{H}^i_{\mathrm{an}}(\Gamma_K, (\mathcal{R}_{\mathbf{C}_p} \otimes_{\mathcal{R}_F} \mathrm{D})^{\psi_q=0}) = 0$. For $i=0$, this implies our claim. For $i=1$, it says that if $c : \Gamma_K \to \mathrm{D}^{\psi_q=0}$ is an F-analytic cocycle, there exists $m \in (\mathcal{R}_{\mathbf{C}_p} \otimes_{\mathcal{R}_F} \mathrm{D})^{\psi_q=0}$ such that $c(g) = (g-1)m$ for all $g \in \Gamma_K$. If $\alpha \in G_F$, then $c(g) = (g-1)\alpha(m)$ as well, so that $\alpha(m) - m \in ((\mathcal{R}_{\mathbf{C}_p} \otimes_{\mathcal{R}_F} \mathrm{D})^{\psi_q=0})^{\Gamma_K} = 0$. This shows that $m \in ((\mathcal{R}_{\mathbf{C}_p} \otimes_{\mathcal{R}_F} \mathrm{D})^{\psi_q=0})^{G_F} = \mathrm{D}^{\psi_q=0}$. □

Corollary 2.2.3. *The groups* $\mathrm{H}^i_{\mathrm{an}}(\Gamma_K \times \Phi, \mathrm{D})$ *and* $\mathrm{H}^i_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ *are isomorphic for* $i = 0, 1$.

Proof. If $i=0$, then we have an inclusion $\mathrm{D}^{\varphi_q=1,\Gamma_K} \subset \mathrm{D}^{\psi_q=1,\Gamma_K}$. If $x \in \mathrm{D}^{\psi_q=1,\Gamma_K}$, then $x - \varphi_q(x) \in \mathrm{D}^{\psi_q=0,\Gamma_K} = \{0\}$ by theorem 2.2.2, so that $x = \varphi_q(x)$ and the above inclusion is an equality.

Now let $i=1$. If $f \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Phi, \mathrm{D})$, let $Tf \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ be the function defined by $Tf(g) = f(g)$ if $g \in \Gamma_K$ and $Tf(\psi_q) = -\psi_q(f(\varphi_q))$.

If $f \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ and $g \in \Gamma_K$, then $(\varphi_q\psi_q - 1)f(g) \in \mathrm{D}^{\psi_q=0}$ and the map $g \mapsto (\varphi_q\psi_q - 1)f(g)$ is an element of $\mathrm{Z}^1_{\mathrm{an}}(\Gamma_K, \mathrm{D}^{\psi_q=0})$. By theorem 2.2.2, applied once for existence and once for unicity, there is a unique $m_f \in \mathrm{D}^{\psi_q=0}$ such that $(\varphi_q\psi_q - 1)f(g) = (g-1)m_f$. Let $Uf \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Phi, \mathrm{D})$ be the function defined by $Uf(g) = f(g)$ if $g \in \Gamma_K$ and $Uf(\varphi_q) = -\varphi_q(f(\psi_q)) + m_f$.

It is straightforward to check that U and T are inverses of each other (even at the level of the $\mathrm{Z}^1_{\mathrm{an}}$) and that they descend to the $\mathrm{H}^1_{\mathrm{an}}$. □

Theorem 2.2.4. *The map* $f \mapsto f(\psi_q)$ *from* $\mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ *to* D *gives rise to an exact sequence:*

$$0 \to \mathrm{H}^1_{\mathrm{an}}(\Gamma_K, \mathrm{D}^{\psi_q=1}) \to \mathrm{H}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D}) \to \left(\frac{\mathrm{D}}{\psi_q - 1}\right)^{\Gamma_K}$$

Proof. If $f \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ and $g \in \Gamma_K$, then $(g-1)f(\psi_q) = (\psi_q - 1)f(g) \in (\psi_q - 1)\mathrm{D}$ so that the image of f is in $(\mathrm{D}/(\psi_q-1))^{\Gamma_K}$. The other verifications are similar. □

2.3 The space $\mathrm{D}/(\psi_q - 1)$

By theorem 2.2.4 in the previous section, the cokernel of the map $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K, \mathrm{D}^{\psi_q=1}) \to \mathrm{H}^1_{\mathrm{an}}(\Gamma_K \times \Psi, \mathrm{D})$ injects into $(\mathrm{D}/(\psi_q-1))^{\Gamma_K}$. It can be useful to know that this cokernel is not too large. In this section, we bound $\mathrm{D}/(\psi_q - 1)$ when $\mathrm{D} = \mathbf{B}^\dagger_{\mathrm{rig},F}$, with the action of φ_q twisted by a^{-1}, for some $a \in F^\times$.

Theorem 2.3.1. *If* $a \in F^\times$, *then* $\psi_q - a : \mathbf{B}^\dagger_{\mathrm{rig},F} \to \mathbf{B}^\dagger_{\mathrm{rig},F}$ *is onto unless* $a = q^{-1}\pi^m$ *for some* $m \in \mathbf{Z}_{\geqslant 1}$, *in which case* $\mathbf{B}^\dagger_{\mathrm{rig},F}/(\psi_q - a)$ *is of dimension* 1.

In order to prove this theorem, we need some results about the action of ψ_q on $\mathbf{B}^\dagger_{\mathrm{rig},F}$. Recall that the map $\partial = d/dt_\pi$ was defined in §1.1.

Lemma 2.3.2. *If $a \in F^\times$, then $a\varphi_q - 1 : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}$ is an isomorphism, unless $a = \pi^{-m}$ for some $m \in \mathbf{Z}_{\geqslant 0}$, in which case*

$$\ker(a\varphi_q - 1 : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}) = Ft_\pi^m$$
$$\mathrm{im}(a\varphi_q - 1 : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}) = \{f(T) \in \mathbf{B}^+_{\mathrm{rig},F} \mid \partial^m(f)(0) = 0\}.$$

Proof. This is lemma 5.1 of [FX13]. □

Lemma 2.3.3. *If $m \in \mathbf{Z}_{\geqslant 0}$, there is an $h(T) \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}$ such that $\partial^m(h)(0) \neq 0$.*

Proof. We have $\psi_q(T) = 0$ by (the proof of) proposition 2.2 of [FX13]. If there was some m_0 such that $\partial^m(T)(0) = 0$ for all $m \geqslant m_0$, then T would be a polynomial in t_π, which it is not. This implies that there is a sequence $\{m_i\}_i$ of integers with $m_i \to +\infty$, such that $\partial^{m_i}(T)(0) \neq 0$, and we can take $h(T) = \partial^{m_i - m}(T)$ for any $m_i \geqslant m$. □

Corollary 2.3.4. *If $a \in F^\times$, then $\psi_q - a : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}$ is onto.*

Proof. If $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ and if we can write $f = (1 - a\varphi_q)g$, then $f = (\psi_q - a)(\varphi_q(g))$. If this is not possible, then by lemma 2.3.2 there exists $m \geqslant 0$ such that $a = \pi^{-m}$ and $\partial^m(f)(0) \neq 0$. Let h be the function provided by lemma 2.3.3. The function $f - (\partial^m(f)(0)/\partial^m(h)(0)) \cdot h$ is in the image of $1 - a\varphi_q$ by lemma 2.3.2, and $h = (\psi_q - a)(-a^{-1}h)$ since $\psi_q(h) = 0$. This implies that f is in the image of $\psi_q - a$. □

Lemma 2.3.5. *If $a^{-1} \in q \cdot \mathcal{O}_F$, then $\psi_q - a : \mathbf{B}^\dagger_{\mathrm{rig},F} \to \mathbf{B}^\dagger_{\mathrm{rig},F}$ is onto.*

Proof. We have $\mathbf{B}^\dagger_{\mathrm{rig},F} = \mathbf{B}^+_{\mathrm{rig},F} + \mathbf{B}^\dagger_F$ (by writing a power series as the sum of its plus part and of its minus part) and by corollary 2.3.4, $\psi_q - a : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}$ is onto. Take $f(T) \in \mathbf{B}^\dagger_F$, choose some $r > 0$ and let $\mathbf{B}^{(0,r]}_F$ be the set of $f(T) \in \mathbf{B}^\dagger_F$ that converge and are bounded on the annulus $0 < \mathrm{val}_p(x) \leqslant r$. It follows from proposition 1.4 of [Col16] that if $n \gg 0$, then $\psi_q^n(f) \in \mathbf{B}^{(0,r]}_F$ and by proposition 2.4(d) of [FX13], the sequence $(q/\pi \cdot \psi_q)^n(f)$ is bounded in $\mathbf{B}^{(0,r]}_F$. The series $\sum_{n\geqslant 0} a^{-1-n}\psi_q^n(f)$ therefore converges in $\mathbf{B}^{(0,r]}_F$, and we can write $f = (\psi_q - a)g$ where $g = a^{-1}(1 - a^{-1}\psi_q)^{-1} f = \sum_{n\geqslant 0} a^{-1-n}\psi_q^n(f)$. □

Let $\mathrm{Res} : \mathbf{B}^\dagger_{\mathrm{rig},F} \to F$ be defined by $\mathrm{Res}(f) = a_{-1}$ where $f(T)dt_\pi = \sum_{n\in\mathbf{Z}} a_n T^n dT$. The following lemma combines propositions 2.12 and 2.13 of [FX13].

Lemma 2.3.6. *The sequence $0 \to F \to \mathbf{B}^\dagger_{\mathrm{rig},F} \xrightarrow{\partial} \mathbf{B}^\dagger_{\mathrm{rig},F} \xrightarrow{\mathrm{Res}} F \to 0$ is exact, and $\mathrm{Res}(\psi_q(f)) = \pi/q \cdot \mathrm{Res}(f)$.*

Proof of theorem 2.3.1. Since $\partial \circ \psi_q = \pi^{-1}\psi_q \circ \partial$, the map ∂ induces a map:

$$\frac{\mathbf{B}^{\dagger}_{\mathrm{rig},F}}{\psi_q - a} \xrightarrow{\partial} \frac{\mathbf{B}^{\dagger}_{\mathrm{rig},F}}{\psi_q - a\pi}. \tag{Der}$$

Take $x \in \mathbf{B}^{\dagger}_{\mathrm{rig},F}$ such that $\mathrm{Res}(x) = 1$. We have $\mathrm{Res}((\psi_q - a\pi)x) = \pi/q - a\pi$. If $a \neq q^{-1}$, this is non-zero and if $f \in \mathbf{B}^{\dagger}_{\mathrm{rig},F}$, proposition 2.3.6 allows us to write $f = \partial g + \mathrm{Res}(f)/(\pi/q - a\pi) \cdot (\psi_q - a\pi)x$. This implies that (Der) is onto if $a \neq q^{-1}$.
Combined with lemma 2.3.5, this implies that $\mathbf{B}^{\dagger}_{\mathrm{rig},F}/(\psi_q - a) = 0$ if a is not of the form $q^{-1}\pi^m$ for some $m \in \mathbf{Z}_{\geqslant 1}$.
When $a = q^{-1}$, we have an exact sequence

$$\frac{\mathbf{B}^{\dagger}_{\mathrm{rig},F}}{\psi_q - q^{-1}} \xrightarrow{\partial} \frac{\mathbf{B}^{\dagger}_{\mathrm{rig},F}}{\psi_q - q^{-1}\pi} \xrightarrow{\mathrm{Res}} F \to 0,$$

which now implies that $\mathbf{B}^{\dagger}_{\mathrm{rig},F}/(\psi_q - q^{-1}\pi) = F$, generated by the class of x.
We now assume again that $a \neq q^{-1}$ and compute the kernel of (Der). If $f \in \mathbf{B}^{\dagger}_{\mathrm{rig},F}$ is such that $\partial f = (\psi_q - a\pi)g$, then $\mathrm{Res}\,\partial f = \mathrm{Res}(\psi_q - a\pi)g = (\pi/q - a\pi)\,\mathrm{Res}(g)$, so that $\mathrm{Res}(g) = 0$ and we can write $g = \partial h$. We have $\partial(f - (\psi_q - a)h) = 0$, so that $f = (\psi_q - a)h + c$, with $c \in F$. By corollary 2.3.4, there exists $b \in \mathbf{B}^{+}_{\mathrm{rig},F}$ such that $(\psi_q - a)(b) = c$, so that $f = (\psi_q - a)(h + b)$ and (Der) is bijective. We then have, by induction on $m \geqslant 1$, that $\mathbf{B}^{\dagger}_{\mathrm{rig},F}/(\psi_q - q^{-1}\pi^m) = F$, generated by the class of $\partial^m(x)$. $\square$

Remark 2.3.7. More generally, we expect that the following holds: if D is a (φ, Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, the F-vector space $\mathrm{D}/(\psi_q - 1)$ is finite dimensional.

2.4 The operator Θ_b

The power series $F(X) = X/(\exp(X) - 1)$ belongs to $\mathbf{Q}_p[\![X]\!]$ and has a nonzero radius of convergence. If M is a Banach space with a locally F-analytic action of Γ_K and $h \in \Gamma_K$ is close enough to 1, then

$$\frac{\nabla}{h-1} = \frac{\nabla}{\exp(\ell(h)\nabla) - 1} = \ell(h)^{-1} F(\ell(h)\nabla)$$

converges to a continuous operator on M. If $g \in \Gamma_K$, we then define

$$\frac{\nabla}{1-g} = \frac{\nabla}{1-g^n} \cdot \frac{1-g^n}{1-g}.$$

This operator is independent of the choice of n such that g^n is close enough to 1, and can be seen as an element of the locally F-analytic distribution algebra acting on M.

If M is a Fréchet space, write $M = \varprojlim_i M_i$ and define operators $\frac{\nabla}{1-g}$ on each M_i as above. These operators commute with the maps $M_j \to M_i$ (because n can be taken large enough for both M_i and M_j). This defines an operator $\frac{\nabla}{1-g}$ on M itself. The definition of $\frac{\nabla}{1-g}$ extends to an LF space with a pro-F-analytic action of Γ_K.

Assume that K contains F_1 and let $r(K) = f + \mathrm{val}_p([K : F_1])$. For example, $p^{r(F_n)} = q^n$ if $n \geqslant 1$. Assume further that K contains $F_{n(K)}$, so that $\chi_\pi : \Gamma_K \to \mathcal{O}_F^\times$ is injective and its image is a free $\mathbf{Z}_p$-module of rank d. If $b = (b_1, \dots, b_d)$ is a basis of Γ_K (that is, $\Gamma_K = b_1^{\mathbf{Z}_p} \cdots b_d^{\mathbf{Z}_p}$), then let $\ell^*(b) = \ell(b_1) \cdots \ell(b_d)/p^{r(K)}$ and

$$\Theta_b = \ell^*(b) \cdot \frac{\nabla^d}{(b_1 - 1) \cdots (b_d - 1)}.$$

Lemma 2.4.1. *If $K = F_n$ and $m \geqslant 0$ and $x \in F_{m+n}$, then*

$$\Theta_b(x) = q^{-m-n} \cdot \mathrm{Tr}_{F_{m+n}/F_n}(x).$$

Proof. Since $\nabla = \lim_{k\to\infty} (b^{p^k} - 1)/p^k \ell(b)$, we have

$$\Theta_b = \lim_{k\to\infty} \frac{1}{q^n p^{kd}} \cdot \frac{(b_1^{p^k} - 1) \cdots (b_d^{p^k} - 1)}{(b_1 - 1) \cdots (b_d - 1)}.$$

The set $\{b_1^{a_1} \cdots b_d^{a_d}\}$ with $0 \leqslant a_i \leqslant p^k - 1$ runs through a set of representatives of $\Gamma_n/\Gamma_n^{p^k} = \Gamma_n/\Gamma_{n+ek}$ so that

$$\frac{1}{q^n p^{kd}} \cdot \frac{(b_1^{p^k} - 1) \cdots (b_d^{p^k} - 1)}{(b_1 - 1) \cdots (b_d - 1)} = \frac{1}{q^n p^{kd}} \mathrm{Tr}_{F_{n+ek}/F_n} = \frac{1}{q^{n+ek}} \cdot \mathrm{Tr}_{F_{n+ek}/F_n}.$$

The lemma follows from taking k large enough so that $ek \geqslant m$. ☐

For $i \in \mathbf{Z}$, let $\nabla_i = \nabla - i$.

Lemma 2.4.2. *If b is a basis of Γ_{F_n} and if $f(T) \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}$, then $\Theta_b(f(T)) \in (t_\pi/\varphi_q^n(T)) \cdot \mathbf{B}^+_{\mathrm{rig},F}$, and if $h \geqslant 2$ then $\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b(f(T)) \in (t_\pi/\varphi_q^n(T))^h \cdot \mathbf{B}^+_{\mathrm{rig},F}$.*

Proof. If $m \geqslant 1$, then by lemma 2.4.1 and using repeatedly the fact (see §1.1) that $\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \mathrm{LT}[\pi]} f(T \oplus z)$,

$$\Theta_b(f(u_{n+m})) = 1/q^{m+n} \cdot \mathrm{Tr}_{F_{m+n}/F_n} f(u_{m+n}) = \psi_q^m(f)(u_n) = 0.$$

This proves the first claim, since an element $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ is divisible by $t_\pi/\varphi_q^n(T)$ if and only if $f(u_{n+m}) = 0$ for all $m \geqslant 1$. The second claim follows easily. ☐

Let D be a φ_q-module over F. Let $\varphi_q^{-n}\colon \mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F D \to F_n(\!(t_\pi)\!) \otimes_F D$ be the map

$$\varphi_q^{-n}\colon t_\pi^{-h} f(T) \otimes x \mapsto \pi^{nh} t_\pi^{-h} f(u_n \oplus \exp_{\mathrm{LT}}(t_\pi/\pi^n)) \otimes \varphi_q^{-n}(x).$$

If $f(t_\pi) \in F_n(\!(t_\pi)\!) \otimes_F D$, let $\partial_D(f) \in F_n \otimes_F D$ denote the coefficient of t_π^0.

LEMMA 2.4.3. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F D)^{\psi_q=1}$ and if $m \geqslant n$, then*

$$q^{-m}\mathrm{Tr}_{F_m/F_n}\partial_D(\varphi_q^{-m}(y)) = \begin{cases} q^{-n}\partial_D(\varphi_q^{-n}(y)) & \text{if } n \geqslant 1 \\ (1-q^{-1}\varphi_q^{-1})\partial_D(y) & \text{if } n = 0. \end{cases}$$

Proof. If $y = t_\pi^{-\ell}\sum_{k=0}^{+\infty} a_k T^k \in \mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F D$, then (by definition of φ_q^{-m})

$$\varphi_q^{-m}(y) = \pi^{m\ell} t_\pi^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(u_m \oplus \exp_{\mathrm{LT}}(t_\pi/\pi^m))^k,$$

and $\psi_q(y) = y$ means that:

$$\varphi_q(y)(T) = \frac{1}{q}\sum_{[\pi](\omega)=0} y(T \oplus \omega).$$

If $m \geqslant 2$, the conjugates of u_m under $\mathrm{Gal}(F_m/F_{m-1})$ are the $\{\omega \oplus u_m\}_{[\pi](\omega)=0}$ so that:

$$\begin{aligned}
&\mathrm{Tr}_{F_m/F_{m-1}}\partial_D(\varphi_q^{-m}(y)) \\
&\qquad = \partial_D\left(\sum_{[\pi](\omega)=0} \pi^{m\ell} t_\pi^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(\omega \oplus u_m \oplus \exp_{\mathrm{LT}}(t_\pi/\pi^m))^k\right) \\
&\qquad = \partial_D\left(\varphi_q^{-m}\left(\sum_{[\pi](\omega)=0} y(T \oplus \omega)\right)\right) \\
&\qquad = q\partial_D(\varphi_q^{-(m-1)}(y)).
\end{aligned}$$

For $m = 1$, the computation is similar, except that the conjugates of u_1 under $\mathrm{Gal}(F_1/F)$ are the ω, where $[\pi](\omega) = 0$ but $\omega \neq 0$, which results in:

$$\mathrm{Tr}_{F_1/F}\partial_D(\varphi_q^{-1}(y)) = \partial_D\left(\varphi_q^{-1}\left(\sum_{\substack{[\pi](\omega)=0 \\ \omega\neq 0}} y(T \oplus \omega)\right)\right) = \partial_D(qy - \varphi_q^{-1}(y)).$$

$\square$

2.5 Construction of extensions

Let D be an F-analytic (φ, Γ)-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. The space $\mathrm{D}^{\psi_q=1}$ is a closed subspace of D and therefore an LF space. Take K such that K contains $F_{n(K)}$ and let b be a basis of Γ_K.

Proposition 2.5.1. *If $y \in \mathrm{D}^{\psi_q=1}$, there is a unique cocycle $c_b(y) \in \mathrm{Z}^1_{\mathrm{an}}(\Gamma_K, \mathrm{D}^{\psi_q=1})$ such that for all $1 \leqslant j \leqslant d$ and $k \geqslant 0$, we have*

$$c_b(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(y).$$

We then have $c_b(y)'(1) = \Theta_b(y)$.

Proof. There is obviously one and only one continuous cocycle satisfying the conditions of the proposition. It is $\mathbf{Q}_p$-analytic, and in order to prove that it is F-analytic, we need to check that the directional derivatives are independent of j. We have

$$\lim_{k \to 0} \frac{c_b(y)(b_j^k)}{\ell(b_j^k)} = \ell^*(b) \cdot \frac{\nabla^d}{\prod_i (b_i - 1)}(y) = \Theta_b(y),$$

which is indeed independent of j, and thus $c_b(y)'(1) = \Theta_b(y)$. □

Lemma 2.5.2. *If $n \geqslant n(K)$ and $L = K_n$ and $M = K_{n+e}$ and b is a basis of Γ_L, then b^p is a basis of Γ_M and $\mathrm{cor}_{M/L} c_{b^p}(y) = c_b(y)$.*

Proof. The Lubin-Tate character maps Γ_L to $1 + \pi^n \mathcal{O}_F$, and $\Gamma_M = \Gamma_L^p$ because $(1 + \pi^n \mathcal{O}_F)^p = 1 + \pi^{n+e} \mathcal{O}_F$. Since $\{b_1^{k_1} \cdots b_d^{k_d}\}$ with $0 \leqslant k_i \leqslant p - 1$ is a set of representatives for Γ_L / Γ_M, and since $[M : L] = q^e = p^d$, the explicit formula for the corestriction (definition 2.1.2) implies (here and elsewhere $\lceil x \rceil$ is the smallest integer $\geqslant x$)

$$\begin{aligned}
&\mathrm{cor}_{M/L}(c_{b^p}(y))(b_j^k) \\
&\quad = \sum_{0 \leqslant k_1, \dots, k_d \leqslant p-1} b_1^{k_1} \dots b_d^{k_d} \cdot \ell^*(b^p) \cdot \frac{b_j^{p \lceil \frac{k - k_j}{p} \rceil} - 1}{b_j^p - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i^p - 1)}(y) \\
&\quad = \ell^*(b) \left(\sum_{k_j = 0}^{p-1} b_j^{k_j} \frac{b_j^{p \lceil \frac{k - k_j}{p} \rceil} - 1}{b_j^p - 1} \right) \cdot \left(\prod_{i \neq j} \frac{b_i^p - 1}{b_i - 1} \right) \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i^p - 1)}(y) \\
&\quad = \ell^*(b) \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(y) \\
&\quad = c_b(y)(b_j^k).
\end{aligned}$$

This proves the lemma. □

Lemma 2.5.3. *If a and b are two bases of Γ_K, then $c_a(y)$ and $c_b(y)$ are cohomologous.*

Proof. If $\alpha_1,\dots,\alpha_d$ and $\beta_1,\dots,\beta_d$ are in $F^\times$, the Laurent series

$$\frac{\alpha_1\cdots\alpha_d\cdot T^{d-1}}{(\exp(\alpha_1 T)-1)\cdots(\exp(\alpha_d T)-1)}-\frac{\beta_1\cdots\beta_d\cdot T^{d-1}}{(\exp(\beta_1 T)-1)\cdots(\exp(\beta_d T)-1)}$$

is the difference of two Laurent series, each having a simple pole at 0 with equal residues, and therefore belongs to $F[\![T]\!]$. Let a and b be two bases of Γ_K and take $y\in \mathrm{D}^{\psi_q=1}$.

Let N be a Γ_K-stable Fréchet subspace of D that contains y and write $N=\varprojlim M_j$. Since $M=M_j$ is F-analytic, we have $g=\exp(\ell(g)\nabla)$ on M for g in some open subgroup of Γ_K. Let $k\gg 0$ be large enough such that $a_i^{p^k}$ and $b_i^{p^k}$ are in this subgroup, and let $\alpha_i=p^k\ell(a_i)$ and $\beta_i=p^k\ell(b_i)$. Taking k large enough (depending on M), we can assume moreover that the power series $T/(\exp(T)-1)$ applied to the operators $\alpha_i\nabla$ and $\beta_i\nabla$ converges on M. The element

$$w=\left(\frac{\alpha_1\cdots\alpha_d\cdot\nabla^{d-1}}{(\exp(\alpha_1\nabla)-1)\cdots(\exp(\alpha_d\nabla)-1)}\right.\\ \left.-\frac{\beta_1\cdots\beta_d\cdot\nabla^{d-1}}{(\exp(\beta_1\nabla)-1)\cdots(\exp(\beta_d\nabla)-1)}\right)(y)$$

of M is well defined. By proposition 2.5.1, we have

$$c_{a^{p^k}}(y)'(1)-c_{b^{p^k}}(y)'(1)=\Theta_{a^{p^k}}(y)-\Theta_{b^{p^k}}(y)=p^{-r(L)}\nabla(w)$$

where L is the extension of K such that $\Gamma_L=\Gamma_K^{p^k}$. Thus, for g close enough to 1, we have $c_{a^{p^k}}(y)(g)-c_{b^{p^k}}(y)(g)=(g-1)(p^{-r(L)}w)$. Lemma 2.5.2 now implies by corestricting that this holds for all g, and, by corestricting again, that $c_a(y)$ and $c_b(y)$ are cohomologous in M. By varying M, we get the same result in N, which implies the proposition. □

Lemma 2.5.4. *If L/K is a finite extension contained in K_∞, and if b is a basis of Γ_K and a is a basis of Γ_L, then $\mathrm{cor}_{L/K}c_a(y)=c_b(y)$.*

Proof. The groups Γ_K and Γ_L are both free $\mathbf{Z}_p$-modules of rank d, so that by the elementary divisors theorem, we can change the bases a and b in such a way that there exists $e_1,\dots,e_d$ with $a_i=b_i^{p^{e_i}}$.

Since $\{b_1^{k_1}\cdots b_d^{k_d}\}$ with $0\leqslant k_i\leqslant p^{e_i}-1$ is a set of representatives for Γ_K/Γ_L, and since $[L:K]=p^{e_1+\cdots+e_d}$, the explicit formula for the corestriction implies

$$\begin{aligned}
&\operatorname{cor}_{L/K}(c_a(y))(b_j^k) \\
&= \sum_{\substack{0\leqslant k_1\leqslant p^{e_1}-1 \\ \vdots \\ 0\leqslant k_d\leqslant p^{e_d}-1}} b_1^{k_1}\dots b_d^{k_d}\cdot \ell^*(a)\cdot \frac{a_j^{\left\lceil \frac{k-k_j}{p^{e_j}}\right\rceil}-1}{a_j-1}\cdot \frac{\nabla^{d-1}}{\prod_{i\neq j}(a_i-1)}(y) \\
&= \ell^*(b)\cdot\left(\sum_{k_j=0}^{p^{e_j}-1}\frac{a_j^{\left\lceil \frac{k-k_j}{p^{e_j}}\right\rceil}-1}{a_j-1}\right)\cdot\left(\prod_{i\neq j}\frac{a_i-1}{b_i-1}\right)\cdot\frac{\nabla^{d-1}}{\prod_{i\neq j}(a_i-1)}(y) \\
&= \ell^*(b)\cdot\frac{b_j^k-1}{b_j-1}\cdot\frac{\nabla^{d-1}}{\prod_{i\neq j}(b_i-1)}(y) \\
&= c_b(y)(b_j^k).
\end{aligned}$$

□

Definition 2.5.5. Let $h^1_{K,V} : \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \to \mathrm{H}^1_{\mathrm{an}}(K,V)$ denote the map obtained by composing $y\mapsto \overline{c}_b(y)$ with $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K,\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1})\to \mathrm{H}^1_{\mathrm{an}}(\Gamma_K\times\Psi,\mathrm{D}^\dagger_{\mathrm{rig}}(V))$ (theorem 2.2.4) and with $\mathrm{H}^1_{\mathrm{an}}(\Gamma_K\times\Psi,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\simeq \mathrm{H}^1_{\mathrm{an}}(K,V)$ (proposition 2.2.1 and corollary 2.2.3).

Proposition 2.5.6. *We have* $\operatorname{cor}_{M/L}\circ h^1_{M,V}=h^1_{L,V}$ *if* M/L *is a finite extension contained in* $K_\infty/K_{n(K)}$. *In particular,* $\operatorname{cor}_{K_{n+1}/K_n}\circ h^1_{K_{n+1},V}=h^1_{K_n,V}$ *if* $n\geqslant n(K)$.

Proof. This follows from the definition and from lemma 2.5.4 above. □

Remark 2.5.7. Proposition 2.5.6 allows us to extend the definition of $h^1_{K,V}$ to all K, without assuming that K contains $F_{n(K)}$, by corestricting.

Some of the constructions of this section are summarized in the following theorem. Recall (see §3 of [Ber16]) that there is a ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ that contains $\mathbf{B}^\dagger_{\mathrm{rig},F}$, is equipped with a Frobenius map φ_q and an action of G_F and such that $V=(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbf{B}^\dagger_{\mathrm{rig},F}}\mathrm{D}^\dagger_{\mathrm{rig}}(V))^{\varphi_q=1}$.

Theorem 2.5.8. *If* $y\in\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1}$ *and* K *contains* $K_{n(K)}$ *and* b *is a basis of* Γ_K*, then*

1. *there is a unique* $c_b(y)\in\mathrm{Z}^1_{\mathrm{an}}(\Gamma_K,\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1})$ *such that for* $k\in\mathbf{Z}_p$,

$$c_b(y)(b_j^k)=\ell^*(b)\cdot\frac{b_j^k-1}{b_j-1}\cdot\frac{\nabla^{d-1}}{\prod_{i\neq j}(b_i-1)}(y);$$

2. *there is a unique* $m_c \in \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=0}$ *such that* $(\varphi_q-1)c_b(y)(g) = (g-1)m_c$ *for all* $g \in \Gamma_K$*;*

3. *the* (φ,Γ)*-module corresponding to this extension has a basis in which*

$$\mathrm{Mat}(g) = \begin{pmatrix} * & c_b(y)(g) \\ 0 & 1 \end{pmatrix} \text{ if } g \in \Gamma_K, \qquad \text{and} \qquad \mathrm{Mat}(\varphi_q) = \begin{pmatrix} * & m_c \\ 0 & 1 \end{pmatrix};$$

4. *if* $z \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_F V$ *is such that* $(\varphi_q - 1)z = m_c$*, then the cocycle*

$$g \mapsto c_b(y)(g) - (g-1)z$$

defined on G_K *has values in* V *and represents* $h^1_{K,V}(y)$ *in* $\mathrm{H}^1_{\mathrm{an}}(K,V)$*.*

Proof. Items (1), (2) and (3) are reformulations of the constructions of this chapter. Let us prove (4). Let us write the (φ,Γ)-module corresponding to the extension in (3) as $\mathrm{D}' = \mathrm{D}^\dagger_{\mathrm{rig}}(V) \oplus \mathbf{B}^\dagger_{\mathrm{rig},F} \cdot e$. It is an étale (φ,Γ)-module that comes from the p-adic representation $V' = (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},F}} \mathrm{D}')^{\varphi_q=1}$. We have $V' = V \oplus F \cdot (e-z)$ as F-vector spaces since $\varphi_q(e-z) = e-z$. If $g \in G_K$, then

$$g(e-z) = e + c_b(y)(g) - g(z) = e - z + c_b(y)(g) - (g-1)z.$$

This proves (4). □

Let $F = \mathbf{Q}_p$ and $\pi = p = q$, and let V be a representation of G_K. In §II.1 of [CC99], Cherbonnier and Colmez define a map $\mathrm{Log}^*_{V^*(1)} : \mathrm{D}^\dagger(V)^{\psi=1} \to \mathrm{H}^1_{\mathrm{Iw}}(K,V)$, which is an isomorphism (theorem II.1.3 and proposition III.3.2 of [CC99]).

Proposition 2.5.9. *If* $F = \mathbf{Q}_p$ *and* $\pi = p$*, then the map*

$$\mathrm{D}^\dagger(V)^{\psi=1} \to \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} \xrightarrow{\{h^1_{K_n,V}\}_{n\geqslant 1}} \varprojlim_n \mathrm{H}^1_{\mathrm{an}}(K_n,V) \to \varprojlim_n \mathrm{H}^1(K_n,V)$$

coincides with the map $\mathrm{Log}^*_{V^*(1)} : \mathrm{D}^\dagger(V)^{\psi=1} \to \mathrm{H}^1_{\mathrm{Iw}}(K,V) \subset \varprojlim_n \mathrm{H}^1(K_n,V)$*.*

Proof. The map $\mathrm{Log}^*_{V^*(1)}$ is contructed by mapping $x \in \mathrm{D}^\dagger(V)^{\psi=1}$ to the sequence $(\dots, \iota_{\psi,n}(x), \dots) \in \varprojlim_n \mathrm{H}^1(K_n,V)$ (see theorem II.1.3 in [CC99] and the paragraph preceding it), where

$$\iota_{\psi,n}(x) = \left[\sigma \mapsto \ell_{K_n}(\gamma_n)\left(\frac{\sigma-1}{\gamma_n-1}x - (\sigma-1)b\right)\right]$$

on G_{K_n} and where (see proposition I.4.1, lemma I.5.2 and lemma I.5.5 of ibid.)

1. $\gamma_n = \gamma_1^{[K_n:K_1]}$ and γ_1 is a fixed generator of Γ_{K_1};

2. $\ell_{K_n}(\gamma_n) = \frac{\log \chi(\gamma_n)}{p^{r(K_n)}}$ where $r(K_n)$ is the integer such that $\log \chi(\Gamma_{K_n}) = p^{r(K_n)}\mathbf{Z}_p$;

3. $b \in \widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{Q}_p} V$ is such that $(\varphi-1)b = a$ and $a \in \mathrm{D}^\dagger(V)^{\psi=1}$ is such that $(\gamma_n - 1)a = (\varphi-1)x$ (using the fact that $\gamma_n - 1$ is bijective on $\mathrm{D}^\dagger(V)^{\psi=0}$).

The theorem follows from comparing this with the explicit formula of theorem 2.5.8. □

3 Explicit formulas for crystalline representations

In this chapter, we explain how the constructions of the previous chapter are related to p-adic Hodge theory, via Bloch and Kato's exponential maps. Let $\mathbf{B}_{\mathrm{dR}}$ be Fontaine's ring of periods [Fon94] and let $\mathbf{B}^+_{\max,F}$ be the subring of $\mathbf{B}^+_{\mathrm{dR}}$ that is constructed in §8.5 of [Col02] (recall that $\mathbf{B}^+_{\max,F} = F \otimes_{F_0} \mathbf{B}^+_{\max}$ where $F_0 = F \cap \mathbf{Q}_p^{\mathrm{unr}}$ and $\mathbf{B}^+_{\max}$ is a ring that is similar to Fontaine's $\mathbf{B}_{\mathrm{cris}}$). We assume throughout this chapter that $K = F$ and that the representation V is crystalline and F-analytic.

3.1 Crystalline F-analytic representations

If V is an F-analytic crystalline representation of G_F, let $\mathrm{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\max,F} \otimes_F V)^{G_F}$ (this is the "component at identity" of the usual $\mathrm{D}_{\mathrm{cris}}$). By corollary 3.3.8 of [KR09], F-analytic crystalline representations of G_F are overconvergent. Moreover, if $\mathcal{M}(D) \subset \mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F D$ is the object constructed in §2.2 of ibid., then by §2.4 of ibid., $\mathcal{M}(\mathrm{D}_{\mathrm{cris}}(V))$ contains a basis of $\mathrm{D}^\dagger(V)$ and $\mathrm{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{B}^\dagger_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_{\mathrm{rig},F}} \mathcal{M}(\mathrm{D}_{\mathrm{cris}}(V))$. This implies that $\mathrm{D}^\dagger_{\mathrm{rig}}(V) \subset \mathbf{B}^\dagger_{\mathrm{rig},F}[1/t_\pi] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$.

Theorem 3.1.1. *We have* $\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \subset \mathbf{B}^+_{\mathrm{rig},F}[1/t_\pi] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$.

Proof. Take $h \geqslant 0$ such that the slopes of $\pi^{-h}\varphi_q$ on $\mathrm{D}_{\mathrm{cris}}(V)$ are $\leqslant -d$. Let E be an extension of F such that E contains the eigenvalues of φ_q on $\mathrm{D}_{\mathrm{cris}}(V)$. We show that $\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \subset t_\pi^{-h} E \otimes_F \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$. Let $e_1, \dots, e_n$ be a basis of $t_\pi^{-h} E \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ in which the matrix $(p_{i,j})$ of φ_q is upper triangular. If $y = \sum_{i=1}^d y_i \otimes \varphi_q(e_i)$ with $y_i \in E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},F}$, then $\psi_q(y) = y$ if and only if $\psi_q(y_k) = p_{k,k} y_k + \sum_{j>k} p_{k,j} y_j$ for all k. The theorem follows from applying lemma 3.1.2 below to $k = n, n-1, \dots, 1$. □

Lemma 3.1.2. *Take* $y \in E \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},F}$ *and* $\alpha \in F$ *such that* $\mathrm{val}_\pi(\alpha) \leqslant -d$. *If* $\psi_q(y) - \alpha y \in E \otimes_F \mathbf{B}^+_{\mathrm{rig},F}$, *then* $y \in E \otimes_F \mathbf{B}^+_{\mathrm{rig},F}$.

Proof. This is lemma 5.4 of [FX13]. □

3.2 Bloch-Kato's exponentials for analytic representations

We now recall the definition of Bloch-Kato's exponential map and its dual, and give a similar definition for F-analytic representations.

Lemma 3.2.1. *We have an exact sequence*

$$0 \to F \to (\mathbf{B}^+_{\max,F}[1/t_\pi])^{\varphi_q=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \to 0.$$

Proof. This is lemma 9.25 of [Col02]. □

If V is a de Rham F-linear representation of G_K, we can $\otimes_F$ the above sequence with V and we get a connecting homomorphism $\exp_{K,V} : (\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K} \to \mathrm{H}^1(K,V)$. Recall that if W is an F-vector space, there is a natural injective map $W \otimes_F V \to W \otimes_{\mathbf{Q}_p} V$.

Lemma 3.2.2. *If V is F-analytic, the map $\exp_{K,V} : (\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K} \to \mathrm{H}^1(K,V)$ defined above coincides with Bloch-Kato's exponential via the inclusion $(\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K} \subset (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$, and its image is in $\mathrm{H}^1_{\mathrm{an}}(K,V)$.*

Proof. Bloch and Kato's exponential is defined as follows (definition 3.10 of [BK90]): if φ_p denotes the Frobenius map that lifts $x \mapsto x^p$ and if $x \in (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$, there exists $\tilde{x} \in \mathbf{B}^{\varphi_p=1}_{\max,\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} V$ such that $\tilde{x} - x \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$, and $\exp(x)$ is represented by the cocyle $g \mapsto (g-1)\tilde{x}$.
Lemma 3.2.1 says that we can lift $x \in (\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K}$ to some $\tilde{x} \in (\mathbf{B}^+_{\max,F}[1/t_\pi])^{\varphi_q=1} \otimes_F V$ such that $\tilde{x} - x \in \mathbf{B}^+_{\mathrm{dR}} \otimes_F V \subset \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. In addition, $\mathbf{B}^{\varphi_q=1}_{\max,\mathbf{Q}_p} = F_0 \otimes_{\mathbf{Q}_p} \mathbf{B}^{\varphi_p=1}_{\max,\mathbf{Q}_p}$ (see lemma 1.1.11 of [Ber08]) so that $(\mathbf{B}^+_{\max,F}[1/t_\pi])^{\varphi_q=1} \subset F \otimes_{\mathbf{Q}_p} \mathbf{B}^{\varphi_p=1}_{\max,\mathbf{Q}_p}$. We can therefore view $\tilde{x}$ as an element of $\mathbf{B}^{\varphi_p=1}_{\max,\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} V$, and $\exp_{K,V}(x) = [g \mapsto (g-1)\tilde{x}] = \exp(x)$.
The construction of $\exp_{K,V}(x)$ shows that the cocycle $\exp_{K,V}(x)$ is de Rham. At each embedding $\tau \neq \mathrm{Id}$ of F, the extension of F by V given by $\exp_{K,V}(x)$ is therefore Hodge-Tate with weights 0. This finishes the proof of the lemma. □

Recall the following theorem of Kato (see §II.1 of [Kat93]).

Theorem 3.2.3. *If V is a de Rham representation, the map from $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ to $\mathrm{H}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ defined by $x \mapsto [g \mapsto \log(\chi_{\mathrm{cyc}}(\overline{g}))x]$ is an isomorphism, and the dual exponential map $\exp^*_{K,V^*(1)} : \mathrm{H}^1(K,V) \to (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is equal to the composition of the map $\mathrm{H}^1(K,V) \to \mathrm{H}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ with the inverse of this isomorphism.*

Concretely, if $c \in \mathrm{Z}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ is some cocycle, there exists $w \in \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ such that $c(g) = \log(\chi_{\mathrm{cyc}}(\overline{g})) \cdot \exp^*_{K,V^*(1)}(c) + (g-1)(w)$.

Corollary 3.2.4. *If $c \in \mathrm{Z}^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_F V)$, and if there exist $x \in (\mathbf{B}_{\mathrm{dR}} \otimes_F V)^{G_K}$ and $w \in \mathbf{B}_{\mathrm{dR}} \otimes_F V$ such that $c(g) = \ell(\overline{g}) \cdot x + (g-1)(w)$, then $\exp^*_{K,V^*(1)}(c) = x$.*

Proof. This follows from theorem 3.2.3 and from the fact that $g \mapsto \log(\chi_\pi(\overline{g})/\chi_{\mathrm{cyc}}(\overline{g}))$ is $\mathbf{B}_{\mathrm{dR}}$-admissible, since $t_\pi/t \in (\mathbf{B}_{\mathrm{dR}}^+)^\times$ so that $\log(t_\pi/t) \in \mathbf{B}_{\mathrm{dR}}^+$ is well-defined. □

3.3 Interpolating exponentials and their duals

Let V be an F-analytic crystalline representation. By theorem 3.1.1, we have $\mathrm{D}_{\mathrm{rig}}^\dagger(V)^{\psi_q=1} \subset \mathbf{B}_{\mathrm{rig},F}^+[1/t_\pi] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$. Let ∂_V denote the map ∂_D of §2.4 for $D = \mathrm{D}_{\mathrm{cris}}(V)$.

Theorem 3.3.1. *If $y \in \mathrm{D}_{\mathrm{rig}}^\dagger(V)^{\psi_q=1}$, then*

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) = \begin{cases} q^{-n}\partial_V(\varphi_q^{-n}(y)) & \text{if } n \geqslant 1 \\ (1-q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Proof. Since the diagram

$$\begin{array}{ccc} \mathrm{H}^1(F_{n+1},V) & \xrightarrow{\exp^*_{F_{n+1},V^*(1)}} & F_{n+1} \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \\ {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}}\downarrow & & {\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}}\downarrow \\ \mathrm{H}^1(F_n,V) & \xrightarrow{\exp^*_{F_n,V^*(1)}} & F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \end{array}$$

is commutative, we only need to prove the theorem when $n \geqslant n(F)$ by lemma 2.4.3 and proposition 2.5.6. By theorem 2.5.8, we have

$$h^1_{F_n,V}(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(y) - (b_j^k - 1)z,$$

with $z \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_F V$ so that if $m \gg 0$, then $\varphi_q^{-m}(z) \in \mathbf{B}_{\mathrm{dR}}^+ \otimes_F V$ (see §3 of [Ber16] and §2.2 of [Ber02]). Moreover, $\varphi_q^{-m}(y) \in F_m(\!(t_\pi)\!) \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$. Let $W = \{w \in F_m(\!(t_\pi)\!) \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ such that $\partial_V(w) = 0\}$. The operator ∇ is bijective on W, and $F_m(\!(t_\pi)\!) \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ injects into $\mathbf{B}_{\mathrm{dR}} \otimes_F V$, hence there exists $u \in \mathbf{B}_{\mathrm{dR}} \otimes_F V$ such that

$$\begin{aligned} h^1_{F_n,V}(y)(b_j^k) &= \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j}(b_i - 1)}(\partial_V(\varphi_q^{-m}(y))) - (b_j^k - 1)u \\ &= \ell(b_j^k) \cdot \Theta_b(\partial_V(\varphi_q^{-m}(y))) - (b_j^k - 1)u \\ &= \ell(b_j^k) \cdot q^{-n}\partial_V(\varphi_q^{-n}(y))) - (b_j^k - 1)u, \end{aligned}$$

by lemmas 2.4.1 and 2.4.3. This proves the theorem by corollary 3.2.4. □

We now give explicit formulas for $\exp_{F_n,V}$. Take $h \geqslant 0$ such that $\mathrm{Fil}^{-h}\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{cris}}(V)$, so that $t_\pi^h(\mathbf{B}_{\mathrm{rig},F}^+ \otimes_F \mathrm{D}_{\mathrm{cris}}(V)) \subset \mathrm{D}_{\mathrm{rig}}^\dagger(V)$ (in the notation of §2.2 of [KR09], we have $t_\pi^h(\mathbf{B}_{\mathrm{rig},F}^+ \otimes_F \mathrm{D}_{\mathrm{cris}}(V)) \subset \mathcal{M}(\mathrm{D}_{\mathrm{cris}}(V))$). In particular, if $y \in (\mathbf{B}_{\mathrm{rig},F}^+ \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$, then $\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \in \mathrm{D}_{\mathrm{rig}}^\dagger(V)^{\psi_q=1}$.

Theorem 3.3.2. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$, then*

$$h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) = \\ (-1)^{h-1}(h-1)! \begin{cases} \exp_{F_n,V}(q^{-n}\partial_V(\varphi_q^{-n}(y))) & \text{if } n \geqslant 1 \\ \exp_{F,V}((1-q^{-1}\varphi_q^{-1})\partial_V(y)) & \text{if } n=0. \end{cases}$$

Proof. Since the diagram

$$\begin{array}{ccc} F_{n+1} \otimes_F \mathrm{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_{n+1},V}} & \mathrm{H}^1(F_{n+1},V) \\ {\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}}\downarrow & & {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}}\downarrow \\ F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_n,V}} & \mathrm{H}^1(F_n,V) \end{array}$$

is commutative, we only need to prove the theorem when $n \geqslant n(F)$ by lemma 2.4.3 and proposition 2.5.6. By theorem 2.5.8, we have

$$\begin{aligned} &h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(b_j^k) \\ &\quad = \ell^*(b) \cdot \frac{b_j^k-1}{b_j-1} \cdot \frac{\nabla^{d-1}}{\prod_{i\neq j}(b_i-1)}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) - (b_j^k-1)z \\ &\qquad\qquad = (b_j^k-1) \cdot (\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - (b_j^k-1)z, \end{aligned}$$

so that $h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(g) = (g-1)(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - (g-1)z$ if $g \in \Gamma_K$. By lemma 2.4.2, we have

$$\begin{aligned} &(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)((\varphi_q-1)y) \\ &\qquad \in (t_\pi/\varphi_q^n(T))^h(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=0} \subset \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=0}, \end{aligned}$$

so that (in the notation of theorem 2.5.8) $m_c = (\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)((\varphi_q-1)y)$. Since $(\varphi_q-1)z = m_c$, we have $(\varphi_q-1)((\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z) = 0$, and therefore

$$(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z \in (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t_\pi])^{\varphi_q=1} \otimes_F V$$

The ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ contains $\mathbf{B}^+_{\max,F}$ and the inclusion $(\mathbf{B}^+_{\max,F}[1/t_\pi])^{\varphi_q=1} \subset (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t_\pi])^{\varphi_q=1}$ is an equality (proposition 3.2 of [Ber02]). This implies that

$$(\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)(y) - z \subset (\mathbf{B}^+_{\max,F}[1/t_\pi])^{\varphi_q=1} \otimes_F V.$$

Moreover, we have $z \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_F V$ so that if $m \gg 0$, then $\varphi_q^{-m}(z) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_F V$. In addition, $\varphi_q^{-m}(y)$ belongs to $F_m[\![t_\pi]\!] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$, so that $\varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y))$ belongs to $t_\pi F_m[\![t_\pi]\!] \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ and therefore

$$\begin{aligned} (\nabla_{h-1} \circ \cdots \circ \nabla_1 \circ \Theta_b)\left(\varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y))\right) &\in t_\pi^h F_m[\![t_\pi]\!] \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \\ &\subset \mathbf{B}^+_{\mathrm{dR}} \otimes_F V. \end{aligned}$$

We can hence write

$$h^1_{F_n,V}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y))(g) = (g-1)(\nabla_{h-1}\circ\cdots\circ\nabla_1\circ\Theta_b\circ\partial_V(\varphi_q^{-m}(y)) - (g-1)u,$$

with $u \in \mathbf{B}_{\mathrm{dR}}^+ \otimes_F V$. The theorem now follows from the fact that

$$\Theta_b \circ \partial_V(\varphi_q^{-m}(y)) = q^{-n}\partial_V(\varphi_q^{-n}(y)) \in F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$$

by lemmas 2.4.2 and 2.4.3, that $\nabla_{h-1}\circ\cdots\circ\nabla_1 = (-1)^{h-1}(h-1)!$ on $F_n \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$, and from the reminders given in §3.2, in particular the fact that $\exp_{K,V}$ is the connecting homomorphism when tensoring the exact sequence of lemma 3.2.1 with V and taking Galois invariants. □

3.4 Kummer theory and the representation $F(\chi_\pi)$

Throughout this section, $V = F(\chi_\pi)$. Let $L \subset \overline{\mathbf{Q}}_p$ be an extension of K. The Kummer map $\delta : \mathrm{LT}(\mathfrak{m}_L) \to \mathrm{H}^1(L,V)$ is defined as follows. Choose a generator $u = (u_k)_{k\geqslant 0}$ of $T_\pi \mathrm{LT} = \varprojlim_k \mathrm{LT}[\pi^k]$. If $x \in \mathrm{LT}(\mathfrak{m}_L)$, let $x_k \in \mathrm{LT}(\mathfrak{m}_{\overline{\mathbf{Q}}_p})$ be such that $[\pi^k](x_k) = x$. If $g \in G_L$, then $g(x_k) - x_k \in \mathrm{LT}[\pi^k]$ so that we can write $g(x_k) - x_k = [c_k(g)](u_k)$ for some $c_k(g) \in \mathcal{O}_F/\pi^k$. If $c(g) = (c_k(g))_{k\geqslant 0} \in \mathcal{O}_F$ then $\delta(x) = [g \mapsto c(g)] \in \mathrm{H}^1(L,V)$.
If $x \in \mathrm{LT}(\mathfrak{m}_L)$, and L/K is finite Galois, let $\mathrm{Tr}^{\mathrm{LT}}_{L/K}$ be the map defined by $\mathrm{Tr}^{\mathrm{LT}}_{L/K}(x) = \sum^{\mathrm{LT}}_{g\in\mathrm{Gal}(L/K)} g(x)$ where the superscript LT means that the summation is carried out using the Lubin-Tate addition. If $F = \mathbf{Q}_p$ and $\mathrm{LT} = \mathbf{G}_{\mathrm{m}}$, we recover the classical Kummer map, and $\mathrm{Tr}^{\mathrm{LT}}_{L/K}(x) = \mathrm{N}_{L/K}(1+x) - 1$.

Lemma 3.4.1. *We have the following commutative diagram:*

$$\begin{array}{ccc}
\mathrm{LT}(\mathfrak{m}_{K_{n+1}}) & \xrightarrow{\delta} & \mathrm{H}^1(K_{n+1},V) \\
\Big\downarrow{\scriptstyle \mathrm{Tr}^{\mathrm{LT}}_{K_{n+1}/K_n}} & & \Big\downarrow{\scriptstyle \mathrm{cor}_{K_{n+1}/K_n}} \\
\mathrm{LT}(\mathfrak{m}_{K_n}) & \xrightarrow{\delta} & \mathrm{H}^1(K_n,V).
\end{array}$$

Proof. This is a straightforward consequence of the explicit description of the corestriction map. □

Recall that $\varphi_q \circ \psi_q(f) = \frac{1}{q}\sum_{\omega\in\mathrm{LT}[\pi]} f(T \oplus \omega)$, so that for $n \geqslant 1$:

$$\psi_q(f)(u_n) = \frac{1}{q}\sum_{\omega\in\mathrm{LT}[\pi]} f(u_{n+1}\oplus\omega) = \frac{1}{q}\mathrm{Tr}_{F_{n+1}/F_n} f(u_{n+1}).$$

In particular, if $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ is such that $\psi_q(f(T)) = 1/\pi \cdot f(T)$ and $y_n = f(u_n)$, then $\mathrm{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n$.

Proposition 3.4.2. *Assume that $F \neq \mathbf{Q}_p$. If $\{y_n\}_{n\geqslant 1}$ is a sequence with $y_n \in F_n$ and $\mathrm{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n$, there exists $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ such that $\psi_q(f(T)) = 1/\pi \cdot f(T)$ and $y_n = f(u_n)$ for all $n \geqslant 1$.*

Proof. By [Laz62], there exists a power series $g(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ such that $g(u_n) = y_n$ for all $n \geqslant 1$. We also have

$$\psi_q g(0) = \frac{1}{q} g(0) + \frac{1}{q} \mathrm{Tr}_{F_1/F_0} g(u_1),$$

and since $q \neq \pi$ (because $F \neq \mathbf{Q}_p$), we can choose $g(0)$ such that

$$\frac{1}{\pi} g(0) = \frac{1}{q} g(0) + \frac{1}{q} \mathrm{Tr}_{F_1/F_0} y_1.$$

This implies that $(\psi_q(g) - 1/\pi \cdot g)(u_n) = 0$ for all $n \geqslant 0$, so that $\psi_q(g) - 1/\pi \cdot g \in t_\pi \cdot \mathbf{B}^+_{\mathrm{rig},F}$. It is therefore enough to prove that $\psi_q - 1/\pi : t_\pi \cdot \mathbf{B}^+_{\mathrm{rig},F} \to t_\pi \cdot \mathbf{B}^+_{\mathrm{rig},F}$ is onto. Since $\psi_q(t_\pi f) = 1/\pi \cdot t_\pi \psi_q(f)$, this amounts to proving that $\psi_q - 1 : \mathbf{B}^+_{\mathrm{rig},F} \to \mathbf{B}^+_{\mathrm{rig},F}$ is onto, which follows from corollary 2.3.4. ☐

Definition 3.4.3. Let S denote the set of sequences $\{x_n\}_{n \geqslant 1}$ with $x_n \in \mathfrak{m}_{F_n}$ and $\mathrm{Tr}^{\mathrm{LT}}_{F_{n+1}/F_n}(x_{n+1}) = [q/\pi](x_n)$ for $n \geqslant 1$.

The following proposition says that if $F \neq \mathbf{Q}_p$, then S is quite large: for any $k \geqslant 1$, the "k-th component" map $F \otimes_{\mathcal{O}_F} S \to F_k$ is surjective (if $F = \mathbf{Q}_p$, there are restrictions on "universal norms").

Proposition 3.4.4. *Assume that $F \neq \mathbf{Q}_p$. If $z \in \mathfrak{m}_{F_k}$, there exists $\ell \geqslant 0$ and $x \in S$ such that $x_k = [\pi^\ell](z)$.*

Proof. We claim that $\mathrm{Tr}_{F_{n+1}/F_n}(\mathcal{O}_{F_{n+1}}) = \pi \mathcal{O}_{F_n}$. Indeed, let $\mathcal{D}$ denote the different. We have (see for instance proposition 7.11 of [Iwa86])

$$\mathrm{val}_p(\mathcal{D}_{F_{n+1}/F_n}) = \frac{1}{e}\left(n + 1 - \frac{1}{q-1}\right) - \frac{1}{e}\left(n - \frac{1}{q-1}\right) = \mathrm{val}_p(\pi).$$

This implies that $\mathrm{Tr}_{F_{n+1}/F_n}(\mathcal{O}_{F_{n+1}}) = \pi \mathcal{O}_{F_n}$ by proposition 7 of Chapter III of [Ser68].

Since π divides q/π, this shows that given $y \in \mathcal{O}_{F_k}$, there exists a sequence $\{y_n\}_{n \geqslant 1}$ with $x_n \in \mathcal{O}_{F_n}$ such that $y_k = y$, and $\mathrm{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n$ for $n \geqslant 1$. Take $\ell_1, \ell_2 \geqslant 0$ such that $\pi^{\ell_1} \mathcal{O}_{\mathbf{C}_p}$ is in the domain of $\exp_{\mathrm{LT}}$ and such that $\pi^{\ell_2} \log_{\mathrm{LT}}(z) \in \mathcal{O}_{F_k}$. Let $y = \pi^{\ell_2} \log_{\mathrm{LT}}(z)$. Let $\{y_n\}_{n \geqslant 1}$ be a sequence as above, let $x_n = \exp_{\mathrm{LT}}(\pi^{\ell_1} y_n)$ and $\ell = \ell_1 + \ell_2$. The elements $x_k \ominus [\pi^\ell](z)$, as well as $\mathrm{Tr}^{\mathrm{LT}}_{F_{n+1}/F_n}(x_{n+1}) \ominus [q/\pi](x_n)$ for all n, have their $\log_{\mathrm{LT}}$ equal to zero and are in a domain in which $\log_{\mathrm{LT}}$ is injective. This proves the proposition. ☐

If $x \in S$ and $y_n = \log_{\mathrm{LT}}(x_n)$, then $y_n \in F_n$ and $\mathrm{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n$, so that by proposition 3.4.2, there exists $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ such that $\psi_q(f(T)) = \pi^{-1} \cdot f(T)$ and $y_n = f(u_n)$ for all $n \geqslant 1$. If $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ is such that $\psi_q(f(T)) = \pi^{-1} \cdot f(T)$, then $\partial f \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q = 1}$ and $\partial f \cdot u$ can be seen as an element of $\mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q = 1}$.

THEOREM 3.4.5. *If* $x \in S$*, and if* $f(T) \in \mathbf{B}^+_{\mathrm{rig},F}$ *is such that* $f(u_n) = \log_{\mathrm{LT}}(x_n)$ *and* $\psi_q(f(T)) = \pi^{-1} \cdot f(T)$*, then* $h^1_{F_n,V}(\partial f(T) \cdot u) = (q/\pi)^{-n} \cdot \delta(x_n)$ *for all* $n \geqslant 1$.

Proof. Let $y = f(T) \otimes t_\pi^{-1} u$, so that $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$. By theorem 3.3.2 applied to y with $h=1$, we have $h^1_{F_n,V}(\nabla(y)) = \exp_{F_n,V}(q^{-n}\partial_V(\varphi_q^{-n}(y)))$ if $n \geqslant 1$. Since $\varphi_q^{-n} \circ \partial = \pi^n \cdot \partial \circ \varphi_q^{-n}$, this implies that

$$h^1_{F_n,V}(\partial f(T) \cdot u) = \exp_{F_n,V}(q^{-n}\partial_V(\varphi_q^{-n}(y))) = (q/\pi)^{-n} \cdot \exp_{F_n,V}(\log_{\mathrm{LT}}(x_n) \cdot u).$$

By example 3.10.1 of [BK90] and lemma 3.2.2, we have $\delta(x_n) = \exp_{F_n,V}(\log_{\mathrm{LT}}(x_n) \cdot u)$. This proves the theorem. □

Remark 3.4.6. If $F = \mathbf{Q}_p$ and $\pi = q = p$ and $x = \{x_n\}_{n \geqslant 1}$, this theorem says that $\mathrm{Exp}^*_{\mathbf{Q}_p}(\delta(x)) = \partial \log \mathrm{Col}_x(T)$, which is (iii) of proposition V.3.2 of [CC99] (see theorem II.1.3 of ibid for the definition of the map $\mathrm{Exp}^*_{\mathbf{Q}_p} : \mathrm{H}^1_{\mathrm{Iw}}(F, \mathbf{Q}_p(1)) \to \mathrm{D}^\dagger_{\mathrm{rig}}(\mathbf{Q}_p(1))^{\psi_q=1}$).

Remark 3.4.7. If $x \in S$, then by proposition 3.4.2, there is a power series $f(T)$ such that $f(u_n) = \log_{\mathrm{LT}}(x_n)$ for $n \geqslant 1$. Is there a power series $g(T) \in \mathcal{O}_F[\![T]\!]$ such that $g(u_n) = x_n$, so that $f(T) = \log g(T)$?
If $F = \mathbf{Q}_p$, such a power series is the classical Coleman power series [Col79]. If $F \neq \mathbf{Q}_p$ and $x \in S$ and z is a $[q/\pi]$-torsion point, and $k \geqslant d-1$ so that $z \in F_k$, then the sequence $x' = \{x'_n\}_{n \geqslant 1}$ defined by $x'_n = x_n$ if $n \neq k$ and $x'_k = x_k \oplus z$ also belongs to S. This means that we cannot naïvely interpolate x.

3.5 PERRIN-RIOU'S BIG EXPONENTIAL MAP

In this last section, we explain how the explicit formulas of the previous sections can be used to give a Lubin-Tate analogue of Perrin-Riou's "big exponential map" [PR94]. Take $h \geqslant 1$ such that $\mathrm{Fil}^{-h}\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{cris}}(V)$. If $f \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$, let $\Delta(f)$ be the image of $\oplus_{k=0}^h \partial^k(f)(0)$ in $\oplus_{k=0}^h \mathrm{D}_{\mathrm{cris}}(V)/(1-\pi^k\varphi_q)$.

LEMMA 3.5.1. *There is an exact sequence:*

$$0 \to \oplus_{k=0}^h t_\pi^k \mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-k}} \to \left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)\right)^{\psi_q=1} \xrightarrow{1-\varphi_q}$$
$$(\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0} \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \xrightarrow{\Delta} \oplus_{k=0}^h \frac{\mathrm{D}_{\mathrm{cris}}(V)}{1-\pi^k\varphi_q} \to 0.$$

Proof. Note that the map φ_q acts diagonally on tensor products. It is easy to see that $\ker(1-\varphi_q) = \oplus_{k=0}^h t_\pi^k \mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-k}}$, that Δ is surjective, and that $\mathrm{im}(1-\varphi_q) \subset \ker \Delta$, so we now prove that $\mathrm{im}(1-\varphi_q) = \ker \Delta$.
If $f, g \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ and $f = (1-\varphi_q)g$, then $\psi_q(f) = 0$ if and only if $\psi_q(g) = g$. It is therefore enough to show that if $f \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ is such that $\Delta(f) = 0$, then $f = (1-\varphi_q)g$ for some $g \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathrm{D}_{\mathrm{cris}}(V)$.

The map $1-\varphi_q : T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V) \to T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ is bijective because the slopes of φ_q on $T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F D$ are >0. This implies that $1-\varphi_q$ induces a sequence

$$0 \to \oplus_{k=0}^h t_\pi^k \mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-k}} \to \frac{\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)}{T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)} \xrightarrow{\overline{1-\varphi_q}} \frac{\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)}{T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)} \xrightarrow{\Delta} \oplus_{k=0}^h \frac{\mathrm{D}_{\mathrm{cris}}(V)}{1-\pi^k\varphi_q}.$$

We have $\ker(\overline{1-\varphi_q}) = \oplus_{k=0}^h t_\pi^k \mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-k}}$ and by comparing dimensions, we see that $\mathrm{coker}(\overline{1-\varphi_q}) = \oplus_{k=0}^h \mathrm{D}_{\mathrm{cris}}(V)/(1-\pi^k\varphi_q)$. This and the bijectivity of $1-\varphi_q$ on $T^{h+1}\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)$ imply the claim. ☐

If $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\Delta=0}$, then by lemma 3.5.1 there exists $y \in (\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$ such that $f=(1-\varphi_q)y$. Since $\nabla_{h-1}\circ\cdots\circ\nabla_0$ kills $\oplus_{k=0}^{h-1} t_\pi^k \mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-k}}$ we see that $\nabla_{h-1}\circ\cdots\circ\nabla_0(y)$ does not depend upon the choice of such a y (unless $\mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-h}}\neq 0$).

Definition 3.5.2. Let $h\geqslant 1$ be such that $\mathrm{Fil}^{-h}\mathrm{D}_{\mathrm{cris}}(V)=\mathrm{D}_{\mathrm{cris}}(V)$ and such that $\mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-h}}=0$. We deduce from the above construction a well-defined map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1},$$

given by $\Omega_{V,h}(f)=\nabla_{h-1}\circ\cdots\circ\nabla_0(y)$ where the element $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$ is such that $f=(1-\varphi_q)y$ and is provided by lemma 3.5.1.
If $\mathrm{D}_{\mathrm{cris}}(V)^{\varphi_q=\pi^{-h}}\neq 0$, we get a map

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1}/V^{G_F=\chi_\pi^h}.$$

Let u be a basis of $F(\chi_\pi)$ as above, and let $e_j=u^{\otimes j}$ if $j\in\mathbf{Z}$.

Theorem 3.5.3. *Take $y\in(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\psi_q=1}$ and let $h\geqslant 1$ be such that $\mathrm{Fil}^{-h}\mathrm{D}_{\mathrm{cris}}(V)=\mathrm{D}_{\mathrm{cris}}(V)$. Let $f=(1-\varphi_q)y$ so that $f\in((\mathbf{B}^+_{\mathrm{rig},F})^{\psi_q=0}\otimes_F \mathrm{D}_{\mathrm{cris}}(V))^{\Delta=0}$.*
If $j\in\mathbf{Z}$ and $h+j\geqslant 1$, then

$$h^1_{F_n,V(\chi_\pi^j)}(\Omega_{V,h}(f)\otimes e_j) = (-1)^{h+j-1}(h+j-1)!\times \begin{cases} \exp_{F_n,V(\chi_\pi^j)}(q^{-n}\partial_{V(\chi_\pi^j)}(\varphi_q^{-n}(\partial^{-j}y\otimes t_\pi^{-j}e_j))) & \text{if } n\geqslant 1 \\ \exp_{F,V(\chi_\pi^j)}((1-q^{-1}\varphi_q^{-1})\partial_{V(\chi_\pi^j)}(\partial^{-j}y\otimes t_\pi^{-j}e_j)) & \text{if } n=0. \end{cases}$$

If $j \in \mathbf{Z}$ and $h+j \leqslant 0$, then

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(\chi_\pi^j)}(\Omega_{V,h}(f)\otimes e_j)) = \frac{1}{(-h-j)!}\begin{cases} q^{-n}\partial_{V(\chi_\pi^j)}(\varphi_q^{-n}(\partial^{-j}y\otimes t_\pi^{-j}e_j)) & \text{if } n\geqslant 1\\ (1-q^{-1}\varphi_q^{-1})\partial_{V(\chi_\pi^j)}(\partial^{-j}y\otimes t_\pi^{-j}e_j) & \text{if } n=0.\end{cases}$$

Proof. If $h+j\geqslant 1$, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{D}^\dagger_{\mathrm{rig}}(V)^{\psi_q=1} \arrow[r, "\otimes e_j"] & \mathrm{D}^\dagger_{\mathrm{rig}}(V(\chi_\pi^j))^{\psi_q=1} \\ \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V)\right)^{\psi_q=1} \arrow[u, "\nabla_{h-1}\circ\cdots\circ\nabla_0"] \arrow[r, "\partial^{-j}\otimes t^{-j}e_j"] & \left(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathrm{D}_{\mathrm{cris}}(V(\chi_\pi^j))\right)^{\psi_q=1} \arrow[u, "\nabla_{h+j-1}\circ\cdots\circ\nabla_0"'] \end{tikzcd},$$

and the theorem is a straightforward consequence of theorem 3.3.2 applied to $\partial^{-j}y\otimes t^{-j}e_j$, $h+j$ and $V(\chi_\pi^j)$ (which are the j-th twists of y, h and V).
If $h+j\leqslant 0$, and Γ_{F_n} is torsion free, then theorem 3.3.1 shows that

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(\chi_\pi^j)}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = q^{-n}\partial_{V(\chi_\pi^j)}(\varphi_q^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j))$$

in $\mathrm{D}_{\mathrm{cris}}(V(\chi_\pi^j))$, and a short computation involving Taylor series shows that

$$\partial_{V(\chi_\pi^j)}(\varphi_q^{-n}(\nabla_{h-1}\circ\cdots\circ\nabla_0(y)\otimes e_j)) = (-h-j)!^{-1}\partial_{V(\chi_\pi^j)}(\varphi_q^{-n}(\partial^{-j}y\otimes t_\pi^{-j}e_j)).$$

To get the other n, we corestrict. □

Corollary 3.5.4. *We have $\Omega_{V,h}(x)\otimes e_j = \Omega_{V(\chi_\pi^j),h+j}(\partial^{-j}x\otimes t_\pi^{-j}e_j)$ and $\nabla_h\circ\Omega_{V,h}(x) = \Omega_{V,h+1}(x)$.*

Remark 3.5.5. The notation ∂^{-j} is somewhat abusive if $j\geqslant 1$ as ∂ is not injective on $\mathbf{B}^+_{\mathrm{rig},F}$ (it is surjective as can be seen by "integrating" directly a power series) but the reader can check that this leads to no ambiguity in the formulas of theorem 3.5.3 above.

If $F=\mathbf{Q}_p$ and $\pi=p$, definition 3.5.2 and theorem 3.5.3 are given in §II.5 of [Ber03]. They imply that $\Omega_{V,h}$ coïncides with Perrin-Riou's exponential map (see theorem 3.2.3 of [PR94]) after making suitable identifications (theorem II.13 of [Ber03]).
Our definition therefore generalizes Perrin-Riou's exponential map to the F-analytic setting. We hope to use the results of [Fou05] and [Fou08] to relate our constructions to suitable Iwasawa algebras as in the cyclotomic case.

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