Lie Groups in Quasi-Poisson Geometry and Braided Hopf Algebras

Pavol Ševera and Fridrich Valach[1]

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Abstract. We extend the notion of Poisson-Lie groups and Lie bialgebras from Poisson to $\mathfrak{g}$-quasi-Poisson geometry and provide a quantization to braided Hopf algebras in the corresponding Drinfeld category. The basic examples of these $\mathfrak{g}$-quasi-Poisson Lie groups are nilpotent radicals of parabolic subgroups. We also provide examples of moment maps in this new context coming from moduli spaces of flat connections on surfaces.

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1 Introduction

Quasi-Poisson geometry, introduced by Alekseev, Kosmann-Schwarzbach and Meinrenken [3] in their study of moduli spaces of flat connections, is an "infinitesimally braided" version of Poisson geometry. A $\mathfrak{g}$-quasi-Poisson manifold is by definition a manifold with an action of a Lie algebra $\mathfrak{g}$ and with an invariant bivector field π, satisfying

$$[\pi,\pi]/2=\rho(\phi),$$

where ρ is the action of $\mathfrak{g}$ and $\phi\in\bigwedge^3\mathfrak{g}$ is built from an invariant inner product on $\mathfrak{g}$ and from the structure constants of $\mathfrak{g}$.
There are two main reasons for studying quasi-Poisson geometry. The first one is the symplectic and Poisson geometry of moduli spaces of flat connections on compact oriented surfaces. As shown in [2, 3], there is a natural quasi-Poisson

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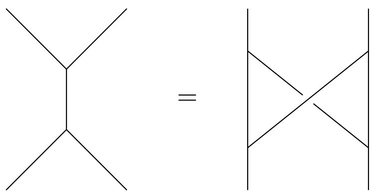
structure on such moduli spaces when we trivialize the principal bundle at a chosen point on the boundary of the surface. The Atiyah-Bott symplectic form [1] is then obtained by a quasi-Hamiltonian reduction. The main advantage of this method is that the quasi-Poisson moduli spaces are smooth and can be built from simple buiding blocks (obtained by cutting the surface to simple pieces), whereas the moduli space of a closed surface is not smooth and the Atiyah-Bott symplectic form requires a global definition.

The second main reason is deformation quantization. Quasi-Poisson geometry is "infinitesimally braided" in the following sense. The inner product on $\mathfrak{g}$ and a Drinfeld associator were used by Drinfeld [4] to deform the symmetric monoidal structure on the category of $U\mathfrak{g}$-modules to a braided monoidal structure. As observed by Enriquez and Etingof [5], the natural definition of deformation quantization of a $\mathfrak{g}$-quasi-Poisson manifold is a deformed product which is associative in Drinfeld's category. As it turns out, it is often easier to quantize a quasi-Poisson manifold in this sense and then perform a reduction and thus obtain a quantization of the reduced Poisson manifold, than to directly quantize the Poisson manifold. This method was used in [10] for quantization of Poisson-Lie groups and related Poisson manifolds.

Indeed, Poisson-Lie groups are among the most interesting and useful Poisson manifolds. In this work we extend the theory of Poisson-Lie groups and Lie bialgebras to the quasi-Poisson world, and we quantize these $\mathfrak{g}$-quasi-Poisson versions to *braided Hopf algebras* in the Drinfeld category. This means, in particular, that the deformed product and coproduct satisfy the relation



This quantization is a braided extension of the celebrate result of Etingof and Kazhdan [6] on quantization of Lie bialgebras to Hopf algebras. To achieve it we use a quantization method developed in [11].

We also show that many basic notions from the theory of Poisson-Lie groups / Lie bialgebras have a natural $\mathfrak{g}$-quasi-Poisson version. This is true in particular for Manin triples (that get replaced by quadruples) and for moment maps. These constructions are closely related to moduli spaces of flat connections on surfaces with decorated boundaries, as studied in [8]; in particular, these moduli spaces provide natural examples of moment maps.

The nilpotent radicals of parabolic subgroups are the simplest examples of $\mathfrak{g}$-quasi-Poisson Lie groups.

2 Quasi-Poisson manifolds

Let $\mathfrak{g}$ be a Lie algebra with an invariant element $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$. We shall suppose that t is non-degenerate and denote the corresponding bilinear pairing (the inverse of t) on $\mathfrak{g}$ by $\langle\, ,\, \rangle$. Let $\phi \in \bigwedge^3 \mathfrak{g}$ be given by

$$\phi(\alpha, \beta, \gamma) = \frac{1}{4}\alpha\big([t^\sharp\beta, t^\sharp\gamma]\big) \text{ for all } \alpha, \beta, \delta \in \mathfrak{g}^*.$$

Definition 2.1 ([3]). *A* $\mathfrak{g}$-quasi-Poisson manifold *is a manifold M with an action ρ of $\mathfrak{g}$ and with a $\mathfrak{g}$-invariant bivector field π, satisfying*

$$[\pi, \pi]/2 = \rho(\phi),$$

where $\rho : \mathfrak{g} \to \Gamma(TM)$ is extended to an algebra morphism $\rho : \bigwedge \mathfrak{g} \to \Gamma(\bigwedge TM)$. Equivalently, the bracket $\{f, g\} := \pi(df, dg)$ on $C^\infty(M)$ satisfies

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \rho(\phi)(df, dg, dh). \tag{1}$$

A map $f : M_1 \to M_2$ between two $\mathfrak{g}$-quasi-Poisson manifolds (M_1, ρ_1, π_1) and (M_2, ρ_2, π_2) is quasi-Poisson *if it is $\mathfrak{g}$-equivariant and if $f_*\pi_1 = \pi_2$.*

If M is $\mathfrak{g}$-quasi-Poisson and if the action $\mathfrak{g}$ on M integrates to an action of a connected Lie group G, we shall say that M is G-quasi-Poisson.
There is a natural $\mathfrak{g}$-quasi-Poisson structure on the product of two $\mathfrak{g}$-quasi-Poisson manifolds:

Definition 2.2 ([3]). *If (M_1, ρ_1, π_1) and (M_2, ρ_2, π_2) are $\mathfrak{g}$-quasi-Poisson manifolds then their* fusion product *$M_1 \circledast M_2$ is the $\mathfrak{g}$-quasi-Poisson manifold $M_1 \times M_2$ with the diagonal action of $\mathfrak{g}$ and with the bivector field*

$$\pi = \pi_1 + \pi_2 - \frac{1}{2}\, t^{ij}\rho_1(e_i) \wedge \rho_2(e_j),$$

where $t = t^{ij} e_i \otimes e_j$ in a basis e_i of $\mathfrak{g}$.

Let us note that the fusion product is associative, but not commutative. It makes the category of $\mathfrak{g}$-quasi-Poisson manifolds to a (non-symmetric) monoidal category.
A Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called *coisotropic* if $\mathfrak{p}^\perp \subset \mathfrak{p}$.

Example 2.1 ([8]). A particularly useful class of $\mathfrak{g}$-quasi-Poisson manifolds is given by triples $(M, \rho, 0)$, where ρ is an action of $\mathfrak{g}$ such that $\rho(t) = 0 \in \Gamma(T^{\otimes 2}M)$, or equivalently such that the stabilizers of the action are coisotropic Lie subalgebras of $\mathfrak{g}$. In this case the diagonal map $M \to M \circledast M$ is $\mathfrak{g}$-quasi-Poisson.

Another useful construction is the *coisotropic reduction*:

Proposition 2.1 ([9]). *Let* $\mathfrak{p} \subset \mathfrak{g}$ *be a coisotropic Lie subalgebra and let* $\mathfrak{g}' := \mathfrak{p}/\mathfrak{p}^\perp$. *If* $M' := M/\mathfrak{p}^\perp$ *is a manifold, i.e. if there is a manifold* M' *and a surjective submersion* $p : M \to M'$ *whose fibers are the* $\mathfrak{p}^\perp$*-orbits, then* (M', ρ', π') *is* $\mathfrak{g}'$*-quasi-Poisson, where* $\pi' = p_*\pi$ *and* ρ' *is induced from* ρ. *Similarly, if* $M'' = M/\mathfrak{p}$ *is a manifold then* (M'', π'') *is Poisson, where* π'' *is the push-forward of* π.

The projection $(M_1 \circledast_{\mathfrak{g}} M_2)/\mathfrak{p}^\perp \to (M_1/\mathfrak{p}^\perp) \circledast_{\mathfrak{g}'} (M_2/\mathfrak{p}^\perp)$ is $\mathfrak{g}'$-quasi-Poisson, i.e. reduction can be seen as a (colax) monoidal functor.

3 $\mathfrak{g}$-Quasi-Poisson Lie groups and Lie $\mathfrak{g}$-quasi-bialgebras

Definition 3.1. *A* $\mathfrak{g}$-quasi-Poisson Lie group *is a Lie group* H *with a* $\mathfrak{g}$*-quasi-Poisson structure* (ρ, π)*, such that the multiplication on* H *is a quasi-Poisson map*

$$H \circledast H \to H.$$

In more detail it means the following: the $\mathfrak{g}$-quasi-Poisson structure (ρ, π) is such that the action ρ of $\mathfrak{g}$ on H is by infinitesimal automorphisms of the Lie group H, and the bivector field π satisfies the relation (for any $h, h' \in H$)

$$\pi|_{hh'} = h\,\pi|_{h'} + \pi|_h\, h' - \frac{1}{2} t^{ij} (\rho(e_i)|_h\, h') \wedge (h\, \rho(e_j)|_{h'}), \tag{2}$$

where $\pi|_h \in \bigwedge^2 T_h H$ is the value of the bivector field π at h, and $h\,\pi|_{h'} \in \bigwedge^2 T_{hh'} H$ is $\pi|_{h'}$ left-translated by h to hh' (and similarly for $\pi|_h\, h'$ and right translation). Setting $h = h' = 1$ we get

$$\pi|_1 = 0.$$

Example 3.1. As observed already in [3], if G is a Lie group with the Lie algebra $\mathfrak{g}$ then G, with the action $\rho(X) = X^L - X^R$ and the bivector field $\pi = -\frac{1}{2} t^{ij}\, e_i^L \wedge e_j^R$, is a $\mathfrak{g}$-quasi-Poisson-group. Here, for $X \in \mathfrak{g}$, X^L and X^R denote the left/right-invariant vector field on G.

The corresponding infinitesimal notion is as follows.

Definition 3.2. *A* Lie $\mathfrak{g}$-quasi-bialgebra *is a triple* $(\mathfrak{h}, \dot\rho, \delta)$*, where*

- $\mathfrak{h}$ *is a Lie algebra*
- $\dot\rho$ *is an action of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *by Lie algebra derivations*
- $\delta : \mathfrak{h} \to \bigwedge^2 \mathfrak{h}$ *is a* $\mathfrak{g}$*-invariant Lie cobracket on* $\mathfrak{h}$

such that

$$\mathrm{ad}^{(2)}_X \delta(Y) - \mathrm{ad}^{(2)}_Y \delta(X) - \delta([X, Y]) - t^{ij}\, \dot\rho(e_i)(X) \wedge \dot\rho(e_j)(Y) = 0 \tag{3}$$

for all $X, Y \in \mathfrak{h}$*, where* $\mathrm{ad}^{(2)}_X := \mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X$.

Theorem 3.1. *If (H,ρ,π) is a $\mathfrak{g}$-quasi-Poisson Lie group then $(\mathfrak{h},\dot\rho,\delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra, with*

$$\big(\dot\rho(u)X\big)^L := [\rho(u),X^L] \quad \forall u\in\mathfrak{g}, X\in\mathfrak{h}$$

where, for $X\in\mathfrak{h}$, X^L denotes the corresponding left-invariant vector field on H, and

$$\delta(X)=[X^L,\pi]|_1,$$

i.e. δ is the linearization of π at the unit element $1\in H$.

Proof. Rewriting (2) as

$$\pi|_{hh'}\,h'^{-1}-\pi|_h = h\,\pi|_{h'}\,h'^{-1}-\frac{1}{2}\,t^{ij}\rho(e_i)|_h\wedge(h\,\rho(e_j)|_{h'}\,h'^{-1}),$$

setting $h'=\exp(\epsilon X)$ and differentiating w.r.t. ϵ at $\epsilon=0$ yields

$$\mathcal{L}_{X^L}\pi=\delta(X)^L-\frac{1}{2}\,t^{ij}\rho(e_i)\wedge[X^L,\rho(e_j)]. \tag{4}$$

The identity $\mathcal{L}_{X^L}\mathcal{L}_{Y^L}-\mathcal{L}_{Y^L}\mathcal{L}_{X^L}=\mathcal{L}_{[X,Y]^L}$ then gives

$$(\mathrm{ad}^{(2)}_X\,\delta(Y))^L-(\mathrm{ad}^{(2)}_Y\,\delta(X))^L-t^{ij}\,[X^L,\rho(e_i)]\wedge[Y^L,\rho(e_j)]=\delta([X,Y])^L,$$

i.e. δ satisfies (3).
It remains to prove that $\delta:\mathfrak{h}\to\mathfrak{h}\otimes\mathfrak{h}$ is a Lie cobracket, i.e. that its transpose $\delta^t:\mathfrak{h}^*\otimes\mathfrak{h}^*\to\mathfrak{h}^*$ is a Lie bracket. To see it, notice that δ^t is the linearization of the quasi-Poisson bracket $\{f,g\}:=\pi(df,dg)$ at $1\in H$. The Jacobi identity for δ^t then follows from the quasi-Jacobi identity (1) and from the fact that the linearization of the trivector field $\rho(\phi)$ at $1\in H$ vanishes. □

Example 3.2. The Lie $\mathfrak{g}$-quasi-bialgebra corresponding to Example 3.1 is $(\mathfrak{g},\mathrm{ad},[,]^t)$, where $[,]^t:\mathfrak{g}\to\bigwedge^2\mathfrak{g}$ is the transpose of $[,]:\bigwedge^2\mathfrak{g}\to\mathfrak{g}$ (here we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the bilinear pairing on $\mathfrak{g}$).

Remark. To avoid potential confusion, let us note that *quasi-Poisson Lie groups* introduced in [7] and our $\mathfrak{g}$-quasi-Poisson Lie groups are different notions. The former integrate quasi-Lie bialgebras and are semiclassical limits of quasi-Hopf algebras; the letter are semiclassical limits of braided Hopf algebras.

4 Manin quadruples

Lie bialgebras can be conveniently rephrased in terms of Manin triples. The corresponding notion for Lie $\mathfrak{g}$-quasi-bialgebras is as follows.

Definition 4.1. $(\mathfrak{d},\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *is a* Manin quadruple *if $\mathfrak{d}$ is a Lie algebra with an invariant nondegenerate symmetric pairing $\langle,\rangle$ and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are its Lie subalgebras such that*

- $\mathfrak{d} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{c}$ *(as vector spaces)*
- $\mathfrak{a}^\perp = \mathfrak{a} \oplus \mathfrak{b}, \quad \mathfrak{c}^\perp = \mathfrak{b} \oplus \mathfrak{c}$.

Example 4.1. Let $\mathfrak{d}$ be a semisimple Lie algebra and $\mathfrak{p}_+, \mathfrak{p}_- \subset \mathfrak{d}$ a pair of opposite parabolic Lie subalgebras. Let $\mathfrak{n}_\pm = \mathfrak{p}_\pm^\perp \subset \mathfrak{p}_\pm$ be the nilpotent radicals of $\mathfrak{p}_\pm$. Then

$$(\mathfrak{d}, \mathfrak{n}_+, \mathfrak{p}_+ \cap \mathfrak{p}_-, \mathfrak{n}_-)$$

is a Manin quadruple.

Proposition 4.1. *If $(\mathfrak{d}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ is a Manin quadruple then $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}$. As a result, $\mathfrak{a} \oplus \mathfrak{b} \subset \mathfrak{d}$ is a Lie subalgebra which is a semidirect sum of $\mathfrak{a}$ and $\mathfrak{b}$. The restriction of $\langle , \rangle$ to $\mathfrak{b}$ is non-degenerate, and $\mathfrak{a}, \mathfrak{c} \subset \mathfrak{d}$ are isotropic subspaces. The pairing $\langle , \rangle$ provides an isomorphism $\mathfrak{a}^* \cong \mathfrak{c}$.*

Proof. We have $\langle [\mathfrak{b}, \mathfrak{a}], \mathfrak{a} \rangle = \langle \mathfrak{b}, [\mathfrak{a}, \mathfrak{a}] \rangle \subset \langle \mathfrak{b}, \mathfrak{a} \rangle = 0$, and similarly $\langle [\mathfrak{b}, \mathfrak{a}], \mathfrak{b} \rangle = \langle \mathfrak{a}, [\mathfrak{b}, \mathfrak{b}] \rangle \subset \langle \mathfrak{a}, \mathfrak{b} \rangle = 0$, which implies $[\mathfrak{b}, \mathfrak{a}] \subset (\mathfrak{a} \oplus \mathfrak{b})^\perp = \mathfrak{a}$, as we wanted to show. The rest of the proposition follows easily. $\square$

Theorem 4.1. *There is an equivalence between Lie $\mathfrak{g}$-quasi-bialgebras and Manin quadruples, given as follows: if $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$ is a Manin quadruple then $(\mathfrak{h}, \dot\rho, \delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra, where*

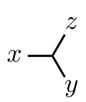
$$\dot\rho(v)X = [v, X] \quad (v \in \mathfrak{g}, X \in \mathfrak{h})$$

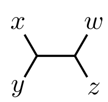
$$\langle \delta(X), A \otimes B \rangle = \langle X, [B, A] \rangle \quad (X \in \mathfrak{h}, A, B \in \mathfrak{h}^*).$$

Proof. If $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$ is a Manin quadruple then δ is clearly a Lie cobracket and the action $\dot\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$ preserves both the bracket and the cobracket on $\mathfrak{h}$. To show that $(\mathfrak{h}, \dot\rho, \delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra we thus need to verify the relation (3). In fact, this relation is simply the Jacobi identity

$$\langle [A, X], [Y, B] \rangle - \langle [B, X], [Y, A] \rangle - \langle [X, Y], [B, A] \rangle = 0 \tag{5}$$

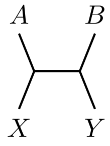
for $X, Y \in \mathfrak{h}$, $A, B \in \mathfrak{h}^*$.
To see it, it is convenient to use the graphical notation

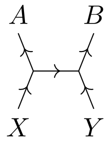
$\langle [x, y], z \rangle =$ 

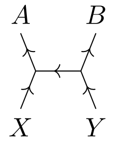
$\langle [x, y], [z, w] \rangle =$ 

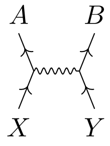
$(x, y, z, w \in \mathfrak{d})$.

In particular, if $X, Y \in \mathfrak{h}$ and $A, B \in \mathfrak{h}^*$, we get

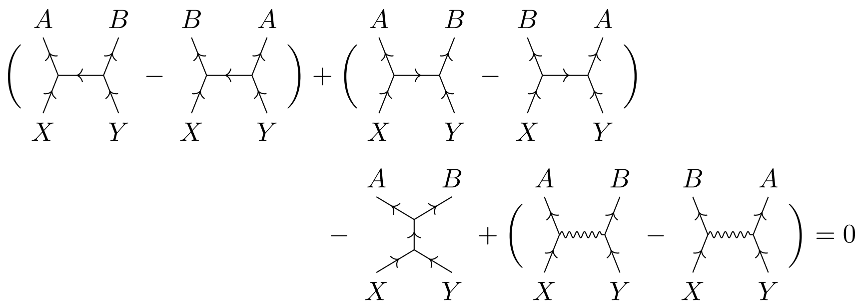
$\langle [A, X], [Y, B] \rangle =$ 

$=$ 

$+$ 

$+$ 

where directed edges go from $\mathfrak{h}$ to $\mathfrak{h}^*$ and the wiggly edge connects $\mathfrak{g}$ with $\mathfrak{g}$. The identity (5) thus gives us



which is the relation (3) contracted with $A \otimes B$ (the terms are parenthesized to correspond to the 4 terms of (3)).
This construction is easily seen to be reversible: if $(\mathfrak{h}, \dot{\rho}, \delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra then we get a Lie bracket on $\mathfrak{d}$, as (5) is the non-trivial part of the Jacobi identity in $\mathfrak{d}$. $\square$

Example 4.2. The Manin quadruple corresponding to Example 3.2 is

$$(\mathfrak{g} \oplus \bar{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}^{12}, \mathfrak{g}^{123}, \mathfrak{g}^{23}).$$

Here $\bar{\mathfrak{g}}$ denotes $\mathfrak{g}$ with t replaced by $-t$, $\mathfrak{g}\oplus\bar{\mathfrak{g}}\oplus\mathfrak{g}$ denotes the Lie algebra $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$ with the pairing given by $(t, -t, t)$, and finally

$$\mathfrak{g}^{12} = \{(X, X, 0), X \in \mathfrak{g}\}, \quad \mathfrak{g}^{23} = \{(0, X, X), X \in \mathfrak{g}\}$$
$$\mathfrak{g}^{123} = \{(X, X, X), X \in \mathfrak{g}\}$$

5 Construction of $\mathfrak{g}$-quasi-Poisson groups

Let $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$ be a Manin quadruple and let H be the 1-connected group integrating $\mathfrak{h}$ (or at least such that the adjoint action of H on $\mathfrak{d}$ is well-defined). In this section we shall describe how to make H to a $\mathfrak{g}$-quasi-Poisson group.
The adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$ extends to an action ρ of $\mathfrak{g}$ on H by infinitesimal automorphisms. More generally, for any $v \in \mathfrak{d}$ let us define a vector field $\hat{\rho}(v)$ on H via

$$\hat{\rho}(v)|_h := p(\mathrm{Ad}_h\, v)\, h \qquad (h \in H)$$

where $p : \mathfrak{d} \to \mathfrak{h}$ is the projection w.r.t. $\mathfrak{g} \oplus \mathfrak{h}^*$. By construction

$$\hat{\rho}(X) = X^L \ \text{ for } X \in \mathfrak{h}, \quad \hat{\rho}(v) = \rho(v) \ \text{ for } v \in \mathfrak{g}.$$

Proposition 5.1. *$\hat{\rho}$ is an action of $\mathfrak{d}$ on H.*

Proof. Let D be the 1-connected Lie group integrating $\mathfrak{d}$ and let $P_- \subset D$ be the connected Lie subgroup integrating the Lie subalgebra $\mathfrak{p}_- := \mathfrak{g} \oplus \mathfrak{h}^* \subset \mathfrak{d}$. Let us first suppose that P_- is closed, so that the quotient $P_- \backslash D$ is a manifold. The action of $\mathfrak{d}$ on D by the left-invariant vector fields projects to an action of $\mathfrak{d}$ on $P_- \backslash D$. Moreover the composition $H \to D \to P_- \backslash D$ is a local diffeomorphism, so the action of $\mathfrak{d}$ on $P_- \backslash D$ pulls back to an action on H. By construction, this action is $\hat{\rho}$.

This argument is easily refined to the case when $P_- \subset D$ is not closed. The cosets $P_- \, d$, $d \in D$, form a foliation $\mathcal{F}$ of D. For any $h \in H$ let us choose an open subset $h \in U_h \subset D$ with the property that $H \cap U_h$ intersects every leaf of $\mathcal{F}$ once, so that we have a diffeomorphism $H \cap U_h \cong U_h/\mathcal{F}$. The Lie algebra $\mathfrak{d}$ acts on $U_h/\mathcal{F}$ and under the diffeomorphism $H \cap U_h \cong U_h/\mathcal{F}$ this action is equal to $\hat{\rho}$. $\square$

Theorem 5.1. *Let $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$ be a Manin quadruple, $(\mathfrak{h}, \dot{\rho}, \delta)$ the corresponding Lie $\mathfrak{g}$-quasi-bialgebra, and H the 1-connected Lie group integrating $\mathfrak{h}$. There is a unique $\mathfrak{g}$-quasi-Poisson structure (ρ, π) on H making H to a $\mathfrak{g}$-quasi-Poisson group, such that the corresponding Lie $\mathfrak{g}$-quasi-bialgebra is $(\mathfrak{h}, \dot{\rho}, \delta)$. Explicitly,*

$$\pi = \frac{1}{2}\, \hat{\rho}(E^i) \wedge (E_i)^L, \tag{6}$$

where E_i is a basis of $\mathfrak{h}$ and E^i the dual basis of $\mathfrak{h}^*$.

Proof. Uniqueness follows from $\pi|_1 = 0$ and from (4).

Let us note that $\hat{H} := (H, \hat{\rho}, 0)$ is a $\mathfrak{d}$-quasi-Poisson manifold, as the stabilizer of $\hat{\rho}$ at any point $h \in H$ is coisotropic, namely $\mathrm{Ad}_{h^{-1}} \mathfrak{p}_-$, where $\mathfrak{p}_- = \mathfrak{g} \oplus \mathfrak{h}^* \subset \mathfrak{d}$. The diagonal map $\hat{H} \to \hat{H} \circledast_{\mathfrak{d}} \hat{H}$ is $\mathfrak{d}$-quasi-Poisson.

Let $\mathfrak{p}_+ := \mathfrak{h} \oplus \mathfrak{g} \subset \mathfrak{d}$; we have $\mathfrak{h} = (\mathfrak{p}_+)^\perp \subset \mathfrak{p}_+$ and $\mathfrak{p}_+/\mathfrak{h} = \mathfrak{g}$. Using the coisotropic reduction by $\mathfrak{h} = (\mathfrak{p}_+)^\perp \subset \mathfrak{p}_+$ we turn

$$(\hat{H} \circledast_{\mathfrak{d}} \hat{H})/H$$

to a $\mathfrak{g}$-quasi-Poisson manifold.

Let us identify $(\hat{H} \circledast_{\mathfrak{d}} \hat{H})/H$ with H via $(h_1, h_2) \mapsto h_1 h_2^{-1}$. We thus constructed a $\mathfrak{g}$-quasi-Poisson structure (ρ, π) on H. Let us check that it makes H to a $\mathfrak{g}$-quasi-Poisson group, i.e. that the group product $H \times H \to H$ is a $\mathfrak{g}$-quasi-Poisson map. To see it, consider the $\mathfrak{g}$-quasi-Poisson maps

$$\begin{tikzcd}
(\hat{H} \circledast_{\mathfrak{d}} \hat{H} \circledast_{\mathfrak{d}} \hat{H})/H \arrow[r, "\Delta"] \arrow[d, "\epsilon"] & (\hat{H} \circledast_{\mathfrak{d}} \hat{H} \circledast_{\mathfrak{d}} \hat{H} \circledast_{\mathfrak{d}} \hat{H})/H \arrow[d, "\kappa"] \\
(\hat{H} \circledast_{\mathfrak{d}} \hat{H})/H & \big((\hat{H} \circledast_{\mathfrak{d}} \hat{H})/H\big) \circledast_{\mathfrak{g}} \big((\hat{H} \circledast_{\mathfrak{d}} \hat{H})/H\big) \arrow[l, dashed, "\epsilon\circ(\kappa\circ\Delta)^{-1}"']
\end{tikzcd}$$

given by

$$\epsilon([h_1, h_2, h_3]) = [h_1, h_3]$$

$$\Delta([h_1,h_2,h_3]) = [h_1,h_2,h_2,h_3]$$
$$\kappa([h_1,h_2,h_3,h_4]) = ([h_1,h_2],[h_3,h_4])$$

where $h_1,h_2,h_3,h_4 \in H$ and the square bracket denotes the respective cosets. It can be easily seen that $\kappa\circ\Delta$ is in fact a diffeomorphism. Moreover, since all the maps at the diagram are $\mathfrak{g}$-quasi-Poisson maps, we have that $\epsilon\circ(\kappa\circ\Delta)^{-1}$, which coincides with the multiplication $H\times H\to H$ under our identification $H\cong(\hat H\circledast_{\mathfrak{d}}\hat H)/H$, is also $\mathfrak{g}$-quasi-Poisson.
Let us now prove that the bivector field π on $H=(\hat H\circledast_{\mathfrak{d}}\hat H)/H$ satisfies (6). Let e_α be a basis of $\mathfrak{g}$ and e^α the dual basis of $\mathfrak{g}$ (w.r.t. $\langle,\rangle$). We have

$$t_{\mathfrak{d}} = E^i\otimes E_i + E_i\otimes E^i + e_\alpha\otimes e^\alpha$$

and thus

$$\pi = -\frac12\tau_*\big(\hat\rho_1(E^i)\wedge\hat\rho_2(E_i)+\hat\rho_1(E_i)\wedge\hat\rho_2(E^i)+\hat\rho_1(e_\alpha)\wedge\hat\rho_2(e^\alpha)\big)$$

where $\tau: H\times H\to H$, $\tau(h_1,h_2)=h_1h_2^{-1}$ is our identification of $H=(\hat H\circledast_{\mathfrak{d}}\hat H)/H$ with H. Since $\tau(h,1)=h$ and $\hat\rho(\mathfrak{h}^*)|_1=\hat\rho(\mathfrak{g})|_1=0$ and $\hat\rho(X)=X^L$ for $X\in\mathfrak{h}$, we get

$$\pi|_h = -\frac12\tau_*\big(\hat\rho_1(E^i)|_h\wedge\hat\rho_2(E_i)|_1\big)=\frac12\,\hat\rho(E^i)|_h\wedge h\,E_i$$

as we wanted to show.
To finish the proof, we need to show that $[X^L,\pi]|_1=\delta(X)$, i.e.

$$\langle [X^L,\pi]|_1, A\otimes B\rangle = \langle X,[B,A]\rangle \qquad \forall X\in\mathfrak{h},\ A,B\in\mathfrak{h}^*.$$

Using (6) we get

$$\begin{aligned}\langle [X^L,\pi]|_1, A\otimes B\rangle &= \frac12\big(\langle [X,E^i],A\rangle\langle E_i,B\rangle - \langle [X,E^i],B\rangle\langle E_i,A\rangle\big)\\ &= \frac12\big(\langle [X,B],A\rangle - \langle [X,A],B\rangle\big) = \langle X,[B,A]\rangle \qquad (7)\end{aligned}$$

as we needed. □

Remark. The proof above is not the shortest possible, but it contains a conceptual construction of the $\mathfrak{g}$-quasi-Poisson group H as $(\hat H\circledast_{\mathfrak{d}}\hat H)/H$ which will be useful when we consider moment maps and deformation quantization. This construction is of interest also in the special case of $\mathfrak{g}=0$, i.e. of Poisson-Lie groups.

Example 5.1. Let us consider the Manin quadruple of Example 4.1 in the case of $\mathfrak{d}=\mathfrak{sl}(N)$, with the inner product $\langle A,B\rangle=\operatorname{Tr}AB$. In this case $\mathfrak{g}$ consists of the traceless block-diagonal $N\times N$-matrices (for some partition of $(1,2,\dots,N)$ into blocks), $\mathfrak{h}$ of matrices with blocks above the diagonal, and $\mathfrak{h}^*$ of matrices with blocks below the diagonal. We have $H = 1 + \mathfrak{h} \subset SL(N)$. For an element $X \in H$ let x^k_l denote the matrix elements of X; x^k_l's are functions on H. Equation (6) gives the following expression for the $\mathfrak{g}$-quasi-Poisson bracket on H:

$$\{x^k_l, x^m_n\} = \frac{1}{2}\Big(x^k_n x^m_l \big(\theta^{\dot l}_{\dot n} - \theta^{\dot n}_{\dot l} + \theta^{\dot k}_{\dot m} - \theta^{\dot m}_{\dot k}\big) + \delta^k_n \sum_r \delta^{\dot r}_{\dot n} x^m_r x^r_l - \delta^m_l \sum_r \delta^{\dot r}_{\dot l} x^k_r x^r_n\Big).$$

where

$$\theta^m_n = \begin{cases} 1 & m < n \\ 0 & m \geq n \end{cases}$$

and $\dot n$, for $1 \leq n \leq N$, denotes the number of the block to which n belongs.

6 Moment maps

In this section we shall see that the theory of moment maps, as known from the case of Lie bialgebras and Poisson-Lie groups, can be developed also for Lie $\mathfrak{g}$-quasi-bialgebras and $\mathfrak{g}$-quasi-Poisson Lie groups.

Definition 6.1. *If (H, ρ_H, π_H) is a $\mathfrak{g}$-quasi-Poisson group and (M, ρ_M, π_M) a $\mathfrak{g}$-quasi-Poisson manifold, an action $H \times M \to M$ is* $\mathfrak{g}$-quasi-Poisson *if it is a $\mathfrak{g}$-quasi-Poisson map*

$$H \circledast M \to M.$$

If $(\mathfrak{h}, \dot\rho, \delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra, an action ψ of $\mathfrak{h}$ on M is $\mathfrak{g}$-quasi-Poisson *if it is $\mathfrak{g}$-equivariant and if*

$$[\psi(X), \pi_M] = -\psi(\delta(X)) + \frac{1}{2} t^{ij}\, \psi(\dot\rho(e_i)X) \wedge \rho_M(e_j) \tag{8}$$

for every $X \in \mathfrak{h}$.

Proposition 6.1. *If (H, ρ_H, π_H) is a connected $\mathfrak{g}$-quasi-Poisson group and (M, ρ_M, π_M) a $\mathfrak{g}$-quasi-Poisson manifold, and $\cdot : H \times M \to M$ is a $\mathfrak{g}$-equivariant action of H on M, then this action is $\mathfrak{g}$-quasi-Poisson iff the corresponding action ψ of $\mathfrak{h}$, $\psi(X)|_m := \frac{d}{dt}\big|_0 e^{-tX} \cdot m$ ($m \in M$), is $\mathfrak{g}$-quasi-Poisson.*

Proof. Let us define an action $*$ of H on $\bigwedge^2 TM$: for $h \in H$, $m \in M$, $b_m \in \bigwedge^2 T_m M$

$$h * b_m := h \cdot b_m + \pi_H|_h \cdot m - \frac{1}{2} t^{ij} (\rho_H(e_i)|_h \cdot m) \wedge (h \cdot \rho_M(e_j)|_m) \in \bigwedge\nolimits^2 T_{h\cdot m} M.$$

The fact that $*$ is an action follows easily from the multiplicativity property (2) of π_H and from the $\mathfrak{g}$-equivariance of the action $\cdot : H \times M \to M$ and of the product $H \times H \to H$.

The action $*$ has the property that the map $\cdot : H \circledast M \to M$ is $\mathfrak{g}$-quasi-Poisson iff the bivector field π_M is H-invariant under the $*$-action, i.e. iff π_M is invariant under the corresponding action of the Lie algebra $\mathfrak{h}$. The latter invariance is Equation (8). ∎

As observed in [8], if (M,ρ,π) is a $\mathfrak{g}$-quasi-Poisson manifold then the operators

$$d_\pm := \pm[\pi,\cdot] + \frac{1}{2}t^{ij}\,\rho(e_i)\wedge[\rho(e_j),\cdot]$$

are differentials on the graded algebra $\Gamma(\bigwedge TM)$ of polyvector fields on M. Comparing with (8) we see that a $\mathfrak{g}$-equivariant action ψ of $\mathfrak{h}$ on M is $\mathfrak{g}$-quasi-Poisson iff

$$d_-\psi(X) = -\psi(\delta(X)).$$

Let us recall that if $A \to M$ is a vector bundle and if d_A is a differential on the graded algebra $\Gamma(\bigwedge A^*)$ then A is a Lie algebroid with the bracket

$$\langle [u,v]_A,\omega\rangle = [d_A i_u + i_u d_A, i_v]\omega \qquad u,v\in\Gamma(A),\omega\in\Gamma(A^*)$$

and with the anchor $\mathsf{a}: A\to TM$

$$\mathsf{a}(u)f = \langle u, d_A f\rangle \qquad u\in\Gamma(A), f\in C^\infty(M).$$

In particular, $d_\pm$ give us two Lie algebroid structures $([,]_+,\mathsf{a}_+)$ and $([,]_-,\mathsf{a}_-)$ on T^*M. The anchors are

$$\mathsf{a}_+(\alpha) = \sigma(\alpha,\cdot), \quad \mathsf{a}_-(\alpha) = \sigma(\cdot,\alpha)$$

where

$$\sigma := \pi + \frac{1}{2}\rho(t) \in \Gamma(T^{\otimes 2}M).$$

Proposition 6.2. *Let (H,ρ,π) be a $\mathfrak{g}$-quasi-Poisson Lie group. Then for any $\alpha,\beta\in\mathfrak{h}^*$*

$$[\alpha^L,\beta^L]_+ = -[\alpha,\beta]^L.$$

Proof. For any $X\in\mathfrak{h}$ we have, by (4),

$$d_+X^L = -\delta(X)^L$$

and thus

$$\begin{aligned}\langle[\alpha^L,\beta^L]_+, X^L\rangle &= [d_+ i_{\alpha^L} + i_{\alpha^L}d_+, i_{\beta^L}]X^L \\ &= -i_{\beta^L}i_{\alpha^L}d_+X^L = \langle \alpha\otimes\beta,\delta(X)\rangle = -\langle[\alpha,\beta],X\rangle.\end{aligned}$$

Since this equality holds for every $X\in\mathfrak{h}$, we have indeed $[\alpha^L,\beta^L]_+ = -[\alpha,\beta]^L$. □

If $(\mathfrak{d},\mathfrak{h},\mathfrak{g},\mathfrak{h}^*)$ is a Manin quadruple then both $\mathfrak{h}$ and $\mathfrak{h}^*$ are $\mathfrak{g}$-quasi-bialgebras; we shall call $\mathfrak{h}^*$ *the dual Lie $\mathfrak{g}$-quasi-bialgebra of* $\mathfrak{h}$. We can now formulate and prove the main result of this section.

Theorem 6.1 (Moment map). *Let $(\mathfrak{h}, \dot\rho, \delta)$ be a Lie $\mathfrak{g}$-quasi-bialgebra and let H^* be a connected $\mathfrak{g}$-quasi-Poisson Lie group integrating its dual. If*

$$\mu : M \to H^*$$

is a $\mathfrak{g}$-quasi-Poisson map then the map

$$\psi : \mathfrak{h} \to \Gamma(TM), \quad \psi(X) = -\mathsf{a}_+(\mu^* X^L)$$

is a $\mathfrak{g}$-quasi-Poisson action of $\mathfrak{h}$ on M.

Proof. Let us extend the Lie brackets $[,]_\pm$ to Gerstenhaber brackets $[,]_\pm$ on the graded algebras $\Omega(M)$ and $\Omega(H^*)$. The morphisms of graded algebras

$$(\Omega(H^*), d, [,]_+) \xrightarrow{\mu^*} (\Omega(M), d, [,]_+) \xrightarrow{\bigwedge \mathsf{a}_+} (\Gamma(\textstyle\bigwedge TM), d_-, [,]) \tag{9}$$

preserve the indicated differentials and Gerstenhaber brackets: for the first map it follows immediately from the fact that μ^* is $\mathfrak{g}$-quasi-Poisson. For the second map it follows from the more general fact that if $(A, [,]_A, \mathsf{a})$ is a Lie algebroid over M then $\bigwedge \mathsf{a} : (\Gamma(\bigwedge A), [,]_A) \to (\Gamma(\bigwedge TM), [,])$ is a morphism of Gerstenhaber algebras, and $\bigwedge \mathsf{a}^t : (\Omega(M), d) \to (\Gamma(\bigwedge A^*), d_A)$ is a morphism of differential graded algebras, where $\mathsf{a}^t : T^*M \to A^*$ is the transpose of a, together with the fact that $\mathsf{a}_-^t = \mathsf{a}_+$.
Let now $\Psi : (\Omega(H^*), d, [,]_+) \to (\Gamma(\bigwedge TM), d_-, [,])$ be the composition of (9). By definition we have, for $X \in \mathfrak{h}$, that $\psi(X) = -\Psi(X^L)$. The fact that ψ is an action of $\mathfrak{h}$ now follows from Proposition 6.2, and the fact that this action is $\mathfrak{g}$-quasi-Poisson, i.e. that $d_-\psi(X) = -\psi(\delta(X))$, follows from $d(X^L) = (\delta(X))^L$. □

7 Examples from moduli spaces of flat connections

One important class of $\mathfrak{g}$-quasi-Poisson manifolds arises from moduli spaces of flat connections on surfaces with marked points on the boundary. It is also this place where one can find many examples of momentum maps, which will be our focus in this section.
Firstly, let us mention a more general notion of fusion, which we will need in this section.

Definition 7.1 ([3]). *Let $(\mathfrak{g}, t)$, $(\mathfrak{g}', t')$ be two Lie algebras with invariant inner products. If (M, ρ, π) is a $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}'$-quasi-Poisson manifold (where $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}'$ is equipped with the inner product $t \oplus t \oplus t'$) then the (internal) fusion of M is the $\mathfrak{g} \oplus \mathfrak{g}'$-quasi-Poisson manifold $(M, \rho_\circledast, \pi_\circledast)$ given by*

$$\rho_\circledast(X, Y) = \rho(X, X, Y) \quad \forall X \in \mathfrak{g},\, Y \in \mathfrak{g}',$$

$$\pi_\circledast = \pi - \frac{1}{2}(\rho_1 \wedge \rho_2)(t),$$

where ρ_1 and ρ_2 are the actions of the first and of the second copy of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}'$, respectively.

Let now Σ be an compact oriented surface with boundary (not necessarily connected) and let $V \subset \partial\Sigma$ be a finite set meeting every component of Σ. Let, as above, $\mathfrak{d}$ be a Lie algebra with invariant inner product, and D a Lie group integrating $\mathfrak{d}$. The moduli space

$$M_D(\Sigma, V)$$

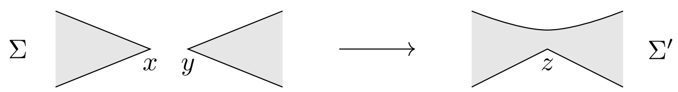
of flat principal D-bundles over Σ trivialized over V is naturally identified with the space of groupoid morphisms

$$\mathrm{Hom}(\Pi_1(\Sigma, V), D).$$

There is a natural action of the group D^V on $M_D(\Sigma, V)$ given by changing the trivialization of a principal D-bundle $P \to \Sigma$ over V.

Theorem 7.1 ([9]). *There is a natural D^V-quasi-Poisson structure $\pi_{\Sigma,V}$ on $M_D(\Sigma, V)$ characterized by these properties:*

1. *If Σ is a disk and V consists of two points then $\pi_{\Sigma,V} = 0$.*
2. *If $\Sigma = \Sigma_1 \sqcup \Sigma_2$ (and so $M_D(\Sigma, V) = M_D(\Sigma_1, V_1) \times M_D(\Sigma_1, V_1)$) then $\pi_{\Sigma,V} = \pi_{\Sigma_1,V_1} + \pi_{\Sigma_2,V_2}$.*
3. *Let $x, y \in V$ and let Σ' be obtained from Σ by a "corner connected sum":*



Let V' be the image of V in Σ', i.e. with x and y identified (denoted z on the picture). In this case $M_D(\Sigma', V') \cong M_D(\Sigma, V)$ (the isomorphism is induced by the gluing map $\Sigma \to \Sigma'$). Then the $D^{V'}$-quasi-Poisson structure on $M_D(\Sigma', V')$ is obtained from the D^V-quasi-Poisson structure on $M_D(\Sigma, V)$ by fusion of the $\mathfrak{d}$'s acting at x and y.

This quasi-Poisson structure is functorial under embeddings of surfaces: if $i : \Sigma' \to \Sigma$ is an embedding then the induced map $i^* : M_D(\Sigma, V) \to M_D(\Sigma', V' = V \cap i(\Sigma'))$ is D^V-quasi-Poisson (where $D^{V \setminus V'}$ acts trivially on $M_D(\Sigma', V')$).

Let now $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$ be a Manin quadruple, D a connected Lie group integrating $\mathfrak{d}$, and $H, G, H^* \subset D$ the connected subgroups integrating $\mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*$. We shall suppose that the map

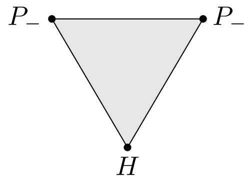
$$H \times G \times H^* \to D, \quad (h, g, h^*) \mapsto hgh^*$$

is a diffeomorphism. (If we drop this assumption then we need to replace the moduli spaces considered below with appropriate locally diffeomorphic manifolds.) Let $P_- \subset D$ be the connected Lie subgroup integrating the Lie subalgebra $\mathfrak{p}_- := \mathfrak{g} \oplus \mathfrak{h}^* \subset \mathfrak{d}$; it is a semi-direct product of G with H^*.

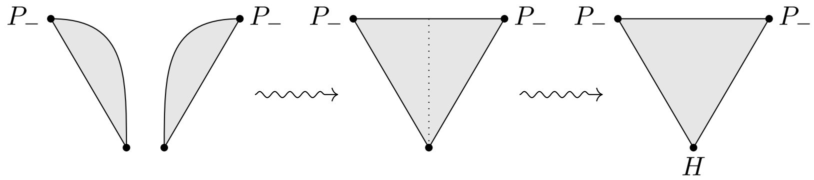
Example 7.1 ($\mathfrak{g}$-quasi-Poisson Lie groups as moduli spaces). Let us consider the moduli space $M_D(\triangle, V)$ for the triangle (i.e. for the disk with 3 marked points on the boundary). It is D^3-quasi-Poisson; after we take its quotient by

$$P_- \times P_- \times H \subset D \times D \times D,$$

we get a G-quasi-Poisson manifold. For every marked point we specify on the picture the subgroup of D by which we mod out:



By Theorem 7.1, this $\mathfrak{g}$-quasi-Poisson manifold $M_D(\triangle, V)/P_- \times P_- \times H$ can be obtained by a fusion followed by a reduction



$$M_D(\triangle, V)/P_- \times P_- \times H \cong ((D/P_-) \circledast_{\mathfrak{d}} (D/P_-))/H.$$

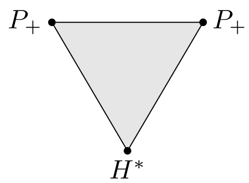
Here D, the moduli space for a bigon (which is D^2-quasi-Poisson), has $\pi = 0$ (see Theorem 7.1), and thus the D-quasi-Poisson space D/P_- has also $\pi = 0$. We can identify D/P_- with H. We thus have an isomorphism

$$M_D(\triangle, V)/P_- \times P_- \times H \cong (H \circledast_{\mathfrak{d}} H)/H$$

and thus, as we noticed in the proof of Theorem 5.1 (where we denoted the $\mathfrak{d}$-quasi-Poisson manifold H with $\pi = 0$ by $\hat{H}$), we have an isomorphism of $\mathfrak{g}$-quasi-Poisson manifolds

$$M_D(\triangle, V)/P_- \times P_- \times H \cong H$$

where H on the RHS is the $\mathfrak{g}$-quasi-Poisson Lie group given by the Manin quadruple $(\mathfrak{d}, \mathfrak{h}, \mathfrak{g}, \mathfrak{h}^*)$. Similarly, the moduli space given by



is isomorphic, as a $\mathfrak{g}$-quasi-Poisson manifold, to the dual $\mathfrak{g}$-quasi-Poisson Lie group

$$M_D(\triangle, V)/P_+ \times P_+ \times H^* \cong H^*. \tag{10}$$

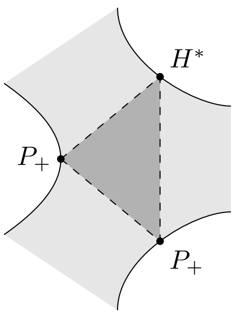
Example 7.2 (Moment map from 3 marked points). Let Σ be a connected compact oriented surface and $V_\Sigma \subset \partial\Sigma$ a set of 3 marked points. Let $i : \triangle \to \Sigma$ be an embedding of a triangle sending the vertices $V_\triangle$ of $\triangle$ to V_Σ. Then, by Theorem 7.1, the induced map

$$i^* : M_D(\Sigma, V_\Sigma) \to M_D(\triangle, V_\triangle)$$

is D^3-quasi-Poisson, and thus, in view of (10), it induces a $\mathfrak{g}$-quasi-Poisson map

$$M_D(\Sigma, V_\Sigma)/P_+ \times P_+ \times H^* \to H^*,$$

i.e. a moment map.



Example 7.3 (Moment map from 2 marked points). Let now Σ be a connected compact oriented surface and $V_\Sigma = \{A, B\} \subset \partial\Sigma$ a set of 2 marked points. Let $i : \triangle \to \Sigma$ be a map which sends one vertex of $\triangle$ to A, the two remaining vertices to B, and is an embedding of $\triangle$ with two vertices identified to Σ. By Theorem 7.1 the map

$$i^* : M_D(\Sigma, V_\Sigma) \to M_D(\triangle, V_\triangle)^{\circledast}$$

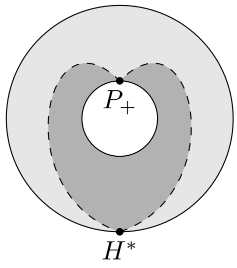
is D^2-quasi-Poisson, where $M_D(\triangle, V_\triangle)^{\circledast}$ is $M_D(\triangle, V_\triangle)$ with fusion applied at the two identified vertices. (Geometrically, $M_D(\triangle, V_\triangle)^{\circledast}$ is the moduli space for an annulus with two marked points; the two points can be either on the same boundary circle, or on both of them, depending on the order of the fusion, giving two different quasi-Poisson structures.)
The projection

$$p : M_D(\triangle, V_\triangle)^{\circledast}/P_+ \times H^* \to M_D(\triangle, V_\triangle)/P_+ \times P_+ \times H^*$$

is $\mathfrak{g}$-quasi-Poisson. As a result,

$$p \circ i^* : M_D(\Sigma, V_\Sigma)/P_+ \times H^* \to M_D(\triangle, V_\triangle)/P_+ \times P_+ \times H^* \cong H^*$$

is a $\mathfrak{g}$-quasi-Poisson map to the dual $\mathfrak{g}$-quasi-Poisson group H^*, i.e. it is a moment map.
As a simple example, let Σ be an annulus and let V_Σ consist of two points, one on each boundary circle of Σ.



In this case $M_D(\Sigma, V_\Sigma) \cong D\times D$, with the identification given by the holonomies along the dashed curves. Moreover $M_D(\Sigma, V_\Sigma)/P_+ \times H^* \cong D$, as the projection

$$D \times D \cong M_D(\Sigma, V_\Sigma) \to M_D(\Sigma, V_\Sigma)/P_+ \times H^*$$

restricts to a diffeomophism on $\{1\} \times D \subset D \times D$.
The $\mathfrak{g}$-quasi-Poisson structure on $D \cong M_D(\Sigma, V_\Sigma)/P_+ \times H^*$ is

$$\pi = \frac{1}{2}\left[(E^i \wedge E_i)^L - (E^i \wedge E_i)^R - (e^\alpha)^L \wedge (e_\alpha)^R\right]$$

with $\mathfrak{g}$ acting on D by conjugation. One can check that it makes D to a $\mathfrak{g}$-quasi-Poisson group and that $H \subset D$ (though not $H^* \subset D$) is a $\mathfrak{g}$-quasi-Poisson inclusion. The moment map $D \to H^*$ is the projection $D \to D/P_+ \cong H^*$.

8 Quantization to braided Hopf algebras

One of the main reason for introducing $\mathfrak{g}$-quasi-Poisson Lie groups and Lie $\mathfrak{g}$-quasi-bialgebras is that they can be quantized to braided Hopf algebras. We shall describe this quantization procedure in this section.
Let us briefly describe Drinfeld's construction of braided monoidal categories (BMCs) using a Drinfeld associator [4]. If U and V are two $U\mathfrak{g}$-modules, let $t^{U,V} : U \otimes V \to U \otimes V$ be given by $t^{U,V} := (\rho_U \otimes \rho_V)(t)$ (where, as above, $t \in (S^2\mathfrak{g})^\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$), and let $\sigma^{U,V} : U \otimes V \to V \otimes U$ be the flip. Let $U\mathfrak{g}\text{-mod}_\hbar$ be the category of $U\mathfrak{g}$-modules, where the morphisms are allowed to be formal power series in $\hbar$. If $\Phi \in K\langle\langle x, y\rangle\rangle$ is a Drinfeld associator, where K is our base field of characteristic 0, then the braiding

$$\beta^{U,V} = \sigma^{U,V} \circ \exp\Big(\frac{\hbar}{2}\, t^{U,V}\Big) : U \otimes V \to V \otimes U$$

and the associativity constraint

$$\gamma^{U,V,W} = \gamma_0^{U,V,W} \circ \Phi(\hbar\, t^{U,V}, \hbar\, t^{V,W}) : (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$

where

$$\gamma_0^{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

is the standard associativity constraint, make $U\mathfrak{g}\text{-mod}_\hbar$ to a BMC. We shall denote this braided monoidal category by $U\mathfrak{g}\text{-mod}_\hbar^\Phi$.

If M is a $\mathfrak{g}$-quasi-Poisson manifold then its *deformation quantization* is a $\mathfrak{g}$-equivariant bilinear product $*$ on $C^\infty(M)$

$$f_1 * f_2 = f_1 f_2 + \hbar\, B_1(f_1, f_2) + \hbar^2 B_2(f_1, f_2) + \dots$$

where B_k are bidifferential operators, such that $(C^\infty(M), *, 1)$ is an associative algebra in the BMC $U\mathfrak{g}\text{-mod}_\hbar^\Phi$, and such that

$$\{f_1, f_2\} = B_1(f_1, f_2) - B_1(f_2, f_1).$$

It is not expected to exist in general. However, if H is a $\mathfrak{g}$-quasi-Poisson Lie group, such a $*$ does exist, and, moreover, is a part of a Hopf algebra structure. In order to describe this quantization it is convenient to introduce the following category (a version of $U\mathfrak{g}\text{-mod}_\hbar$).

Definition 8.1. *The category $\mathfrak{g}\text{-man}_\hbar$ has $\mathfrak{g}$-manifolds as objects, and a morphism $F : M \to N$ is a $\mathfrak{g}$-equivariant linear map*

$$F^* : C^\infty(N) \to C^\infty(M)[[\hbar]], \quad F^* = F_0^* + \hbar\, F_1^* + \hbar^2 F_2^* + \dots$$

such that $F_0^* = f^*$ for some smooth map $f : M \to N$ and such that F_k^*, $k \geq 1$, are differential operators w.r.t. the algebra map F_0^*. The composition of morphisms is given by $(F \circ G)^* := G^* \circ F^*$.

The category $\mathfrak{g}\text{-man}_\hbar$ is symmetric monoidal, with the product $M_1 \otimes M_2 := M_1 \times M_2$. Its deformation to a BMC via a Drinfeld associator $\Phi \in \mathbb{R}\langle\langle x, y\rangle\rangle$ will be denoted by $\mathfrak{g}\text{-man}_\hbar^\Phi$.

Let $\mathfrak{g}\text{-man}$ denote the symmetric monoidal category of $\mathfrak{g}$-manifolds. Whenever M is a $\mathfrak{g}$-manifold then (M, Δ, ϵ) is a comonoid in $\mathfrak{g}$-man, where $\Delta : M \to M \times M$ is the diagonal map, and $\epsilon : M \to$ point is the unique map. A comonoid structure $(M, \Delta_\hbar, \epsilon)$ in $\mathfrak{g}\text{-man}_\hbar^\Phi$, such that $(\Delta_\hbar^*)_0 = \Delta^*$, is equivalent to a $*$-product on M via

$$f * g = \Delta_\hbar^*(f \otimes g) \tag{11}$$

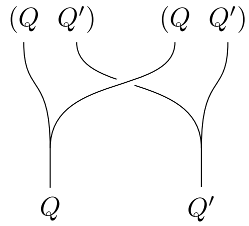
making $C^\infty(M)$ to an associative algebra in $U\mathfrak{g}\text{-mod}_\hbar^\Phi$.

We can now formulate the main result of this section.

Theorem 8.1. *If H is a $\mathfrak{g}$-quasi-Poisson Lie group then there is a Hopf algebra structure on H in the BMC $\mathfrak{g}\text{-man}_\hbar^\Phi$ deforming the Hopf algebra (i.e. Lie group) structure of H in $\mathfrak{g}$-man, such that the $*$-product on H given by (11) is a deformation quantization of the $\mathfrak{g}$-quasi-Poisson manifold H.*

Similarly, if $(\mathfrak{h}, \dot\rho, \delta)$ is a Lie $\mathfrak{g}$-quasi-bialgebra, there is a deformation of the Hopf algebra structure on $U\mathfrak{h}$ to a Hopf algebra structure in the BMC $U\mathfrak{g}\text{-mod}_\hbar^\Phi$, such that, for $v \in \mathfrak{g} \subset U\mathfrak{g}$, $\Delta_\hbar(v) - \Delta_\hbar^{op}(v) = \hbar\, \delta(v) + O(\hbar^2)$.

The proof of the theorem is an application of the method of quantization of Lie bialgebras presented in [11] (one can probably also use the original approach of Etingof and Kazhdan [6] in combination with the $\mathfrak{g}$-quasi-Poisson group D of Example 7.3, but it seems more complicated). For further details about the relevant definitions etc. we refer the reader to [11].
Let us first observe that if $(Q, \Delta_Q, \epsilon_Q)$ and $(Q', \Delta_{Q'}, \epsilon_{Q'})$ are comonoids in a BMC $\mathcal{D}$ then $Q \otimes Q'$ is a comonoid as well, with the counit $\epsilon_Q \otimes \epsilon_{Q'}$ and the coproduct



Theorem 8.2 ([11]). *Let $\mathcal{D}$ and $\mathcal{C}$ be BMCs, $(Q, \Delta_Q, \epsilon_Q)$ a cocommutative comonoid in $\mathcal{D}$, and $F : \mathcal{D} \to \mathcal{C}$ a braided colax monoidal functor with these invertibility properties: the composition*

$$F(Q) \xrightarrow{F(\epsilon_Q)} F(1_{\mathcal{D}}) \to 1_{\mathcal{C}}$$

is an isomorphism, and for every objects $X, Y \in \mathcal{D}$ the morphism

$$\tau^{(Q)}_{X,Y} : F((X \otimes Q) \otimes Y) \to F(X \otimes Q) \otimes F(Q \otimes Y),$$

defined as the composition

$$\begin{aligned} F((X \otimes Q) \otimes Y) &\xrightarrow{F((\mathrm{id}_X \otimes \Delta_Q) \otimes \mathrm{id}_Y)} F\big((X \otimes (Q \otimes Q)) \otimes Y\big) \cong \\ &\cong F\big((X \otimes Q) \otimes (Q \otimes Y)\big) \to F(X \otimes Q) \otimes F(Q \otimes Y), \end{aligned}$$

is an isomorphism.
Then $F(Q \otimes Q)$ is a Hopf algebra in $\mathcal{C}$, with the structure defined as follows:

- *The coalgebra structure on $F(Q \otimes Q)$ is inherited from the coalgebra structure on $Q \otimes Q$.*
- *The product on $F(Q \otimes Q)$ is the composition*

$$\begin{aligned} F(Q \otimes Q) \otimes F(Q \otimes Q) &\xrightarrow{\tau^{(Q)\,-1}_{Q,Q}} F((Q \otimes Q) \otimes Q) \\ &\xrightarrow{F(\mathrm{id}_Q \otimes \epsilon_Q \otimes \mathrm{id}_Q)} F(Q \otimes Q). \end{aligned}$$

- *The unit is*

$$1_{\mathcal{C}} \cong F(Q) \xrightarrow{F(\Delta_Q)} F(Q \otimes Q).$$

- *The antipode is*
$$F(Q\otimes Q)\xrightarrow{F(\beta^{Q,Q})}F(Q\otimes Q)$$
where $\beta^{Q,Q}:Q\otimes Q\to Q\otimes Q$ *is the braiding in* $\mathcal{D}$.

Proof of Theorem 8.1. Let $(\mathfrak{d},\mathfrak{h},\mathfrak{g},\mathfrak{h}^*)$ be the Manin quadruple corresponding to the Lie $\mathfrak{g}$-quasi-bialgebra $\mathfrak{h}$. Let $\mathcal{C}=\mathfrak{g}\text{-man}^{\Phi}_{\hbar}$ and let $\mathcal{D}$ be the full subcategory of $\mathfrak{d}\text{-man}^{\Phi}_{\hbar}$ consisting of those $\mathfrak{d}$-manifolds for which the action of $\mathfrak{h}\subset\mathfrak{d}$ integrates to a free and proper action of the group H. Let $F:\mathcal{D}\to\mathcal{C}$ be given by
$$F(M)=M/H$$
and let $Q=H$ with the action $\hat\rho$ of $\mathfrak{d}$ (see Proposition 5.1) and with the undeformed coproduct Δ_Q and undeformed counit ϵ_Q.
By [11, Proposition 2] the functor F, with the coherence maps
$$(M\times N)/H\to M/H\times N/H$$
given by the natural projections, is a braided colax monoidal functor (here it is important to use the same associator $\Phi\in\mathbb{R}\langle\langle x,y\rangle\rangle$ in the definitions of $\mathcal{C}$ and $\mathcal{D}$) and (Q,Δ_Q,ϵ_Q) is a cocommutative comonoid in $\mathcal{D}$. The hypotheses of Theorem 8.2 are satisfied, hence
$$F(Q\otimes Q)=(H\times H)/H\cong H$$
is a Hopf algebra in $\mathfrak{g}\text{-man}^{\Phi}_{\hbar}$. The identification $(H\times H)/H\cong H$ we use is $(h_1,h_2)\mapsto h_1h_2^{-1}$, as in the proof of Theorem 5.1.
This Hopf algebra structure (in $\mathfrak{g}\text{-man}^{\Phi}_{\hbar}$) on H is clearly a deformation of the Lie group structure on H. To finish the proof we need to check that the $*$-product on H we obtained is indeed a deformation quantization of the $\mathfrak{g}$-quasi-Poisson structure on H. It follows from the fact [10] that if $(M^{(1)},\Delta^{(1)}_{\hbar})$ and $(M^{(2)},\Delta^{(2)}_{\hbar})$ are deformation quantizations of two $\mathfrak{d}$-quasi-Poisson structures on manifolds $M^{(1)}$ and $M^{(2)}$ then their tensor product in $\mathfrak{d}\text{-man}^{\Phi}_{\hbar}$ is a deformation quantization of the fusion $M^{(1)}\circledast M^{(2)}$. As a result the comonoid $Q\otimes Q$ is a deformation quantization of $\hat H\circledast_{\mathfrak{d}}\hat H$, where $\hat H$ denotes, as in the proof of Theorem 5.1, H with the action $\hat\rho$ and with $\pi=0$, and thus $F(Q\otimes Q)$ is a quantization of the $\mathfrak{g}$-quasi-Poisson manifold $(\hat H\circledast_{\mathfrak{d}}\hat H)/H$, which is H with its $\mathfrak{g}$-quasi-Poisson structure.
The deformation of the Hopf algebra structure on $U\mathfrak{h}$ is constructed similarly: we set $\mathcal{D}=U\mathfrak{d}\text{-mod}^{\Phi}_{\hbar}$, $\mathcal{C}=U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}$, $F(V)=V/(\mathfrak{h}\cdot V)$, and $Q=U\mathfrak{h}=U\mathfrak{d}/(\mathfrak{p}_-U\mathfrak{d})$ with its original cocommutative coalgebra structure. Finally we identify $F(U\mathfrak{h}\otimes U\mathfrak{h})$ with $U\mathfrak{h}$ via $x\otimes y\mapsto x\,S_0(y)$, where $S_0:U\mathfrak{h}\to U\mathfrak{h}$ is the original (non-deformed) antipode. The coproduct in the coalgebra $U\mathfrak{h}\otimes U\mathfrak{h}$ (in the category $\mathcal{D}$) applied to $x\otimes 1$, $x\in\mathfrak{h}$, is
$$\Delta_{\hbar}(x\otimes 1)=x\otimes 1\otimes 1\otimes 1+1\otimes 1\otimes x\otimes 1-\frac{\hbar}{2}1\otimes\delta(x)\otimes 1+O(\hbar^2)\in U\mathfrak{h}^{\otimes 4}$$

and thus the coproduct in $U\mathfrak{h} \cong F(U\mathfrak{h} \otimes U\mathfrak{h})$ applied to $x \in \mathfrak{h}$ is

$$\Delta_\hbar(x) = x \otimes 1 + 1 \otimes x + \frac{\hbar}{2}\,\delta(x) + O(\hbar^2),$$

so indeed

$$(\Delta_\hbar - \Delta_\hbar^{op})(x) = \hbar\,\delta(x) + O(\hbar^2)$$

as we wanted to show. □

References

[1] M.F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond. A 308 (1982), 523-615.

[2] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps. J. Differential Geom. 48 (1998), no. 3, 445495.

[3] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, Quasi-Poisson manifolds, Canad. J. Math. 54 (2002), no. 1, 3–29.

[4] V. Drinfed, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Algebra i Analiz 2 (1990), no. 4, 149–181.

[5] B. Enriquez, P. Etingof, Quantization of Alekseev-Meinrenken dynamical r-matrices. Lie groups and symmetric spaces, 81–98, Amer. Math. Soc. Transl. Ser. 2, 210, Amer. Math. Soc., Providence, RI, 2003.

[6] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras. I. Selecta Math. (N.S.) 2 (1996), no. 1, 1–41.

[7] Y. Kosmann-Schwarzbach, Quasi-bigébres de Lie et groupes de Lie quasi-Poisson, C.R. Acad. Sci. Paris 312, Serie I (1991) 391–394.

[8] D. Li-Bland, P. Ševera, Quasi-Hamiltonian groupoids and multiplicative Manin pairs, IMRN 10 (2011), 2295–2350.

[9] D. Li-Bland, P. Ševera, Moduli spaces for quilted surfaces and Poisson structures, Documenta Math. 20 (2015), 1071–1135.

[10] D. Li-Bland, P. Ševera, On Deformation Quantization of Poisson-Lie Groups and Moduli Spaces of Flat Connections, IMRN 15 (2015), 6734–6751.

[11] P. Ševera, Quantization of Lie bialgebras revisited, Selecta Math. (N.S.) 22 (2016) no. 3, 1563–1581.

Pavol Ševera
Section of Mathematics
University of Geneva
Switzerland
pavol.severa@gmail.com

Fridrich Valach
Section of Mathematics
University of Geneva
Switzerland
fridrich.valach@gmail.com