Equality of Two Non-Logarithmic Ramification Filtrations of Abelianized Galois Group in Positive Characteristic

Yuri Yatagawa

Received: Septeber 7, 2016
Revised: April 17, 2017

Communicated by Thomas Geisser

Abstract. We prove the equality of two non-logarithmic ramification filtrations defined by Matsuda and Abbes-Saito of the abelianized absolute Galois group of a complete discrete valuation field in positive characteristic. We compute the refined Swan conductor and the characteristic form of a character of the fundamental group of a smooth separated scheme over a perfect field of positive characteristic by using sheaves of Witt vectors.

2010 Mathematics Subject Classification: primary 11S15, secondary 14G22
Keywords and Phrases: local field, ramification filtration, characteristic form, Witt vector.

Introduction

Let K be a complete discrete valuation field with residue field F_K and $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the absolute Galois group of K. In [Se], the definition of (upper numbering) ramification filtration of G_K is given in the case where F_K is perfect. In the general residue field case, Abbes-Saito ([AS1]) have given definitions of two ramification filtrations of G_K geometrically, one is logarithmic and the other is non-logarithmic. In Saito's recent work ([Sa1], [Sa2]) on characteristic cycle of a constructible sheaf, the non-logarithmic filtration in equal characteristic plays important roles to give an example of characteristic cycle.

Assume that K is of positive characteristic. Let $H^1(K, \mathbf{Q}/\mathbf{Z})$ be the character group of G_K. In this case, Matsuda ([M]) has defined a non-logarithmic ramification filtration of $H^1(K, \mathbf{Q}/\mathbf{Z})$ as a non-logarithmic variant of Brylinski-Kato's logarithmic filtration ([B], [K1]) using Witt vectors. In this paper, we prove that the abelianization of Abbes-Saito's non-logarithmic filtration $\{G_K^r\}_{r\in\mathbf{Q}_{\geq 1}}$ is the same as Matsuda's filtration $\{\mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})\}_{m\in\mathbf{Z}_{\geq 1}}$ by taking dual, which enable us to compute abelianized Abbes-Saito's filtration by using Witt vectors. This is stated as follows and proved in Section 3:

Theorem 0.1. *Let $m \geq 1$ be an integer and r a rational number such that $m \leq r < m+1$. For $\chi \in H^1(K, \mathbf{Q}/\mathbf{Z})$, the following are equivalent:*

(i) $\chi \in \mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})$.

(ii) $\chi(G_K^{m+}) = 0$.

(iii) $\chi(G_K^{r+}) = 0$.

For $m > 2$, Theorem 0.1 has been proved by Abbes-Saito ([AS3]). The proof goes similarly as the proof by Abbes-Saito (loc. cit.). The proof in this paper relies on the characteristic form defined by Saito ([Sa1]) even in the exceptional case where $p = 2$ and an explicit computation of the characteristic form.
Let X be a smooth separated scheme over a perfect field of positive characteristic and $U = X - D$ the complement of a divisor D on X with simple normal crossings. The characteristic form of a character of the abelianized fundamental group $\pi_1^{\mathrm{ab}}(U)$ is an element of the restriction to a radicial covering of a sub divisor Z of D of a differential module of X. We compute the characteristic form using sheaves of Witt vectors. By taking X and D so that the local field at a generic point of D is K and using the injections defined by the characteristic form from the graded quotients of $\{\mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})\}_{m\in\mathbf{Z}_{\geq 1}}$ and the modules of characters of the graded quotients of $\{G_K^r\}_{r\in\mathbf{Q}_{\geq 1}}$, we obtain the proof of Theorem 0.1.
This paper consists of three sections. In Section 1, we recall Kato and Matsuda's ramification theories in positive characteristic. We give some complements to these theories to compute the refined Swan conductor ([K1]) and the characteristic form for a character of the fundamental group of a smooth separated scheme over a perfect field of positive characteristic in terms of sheaves of Witt vectors. In Section 2, we recall Abbes-Saito's non-logarithmic ramification theory in positive characteristic in terms of schemes over a perfect field. We recall the definition of the characteristic form defined by Saito and show that this characteristic form is computed with sheaves of Witt vectors. Section 3 is devoted to prove Theorem 0.1.
This paper is a refinement of a part of the author's thesis at University of Tokyo. The author would like to express her sincere gratitude to her supervisor Takeshi Saito for suggesting her to refine the computation of characteristic form using sheaves of Witt vectors, reading the manuscript carefully, and giving a lot of advice on the manuscript. The research was partially supported by the

Program for Leading Graduate Schools, MEXT, Japan and JSPS KAKENHI Grant Number 15J03851.

Contents

1 Kato and Matsuda's ramification theories and complements

1.1 Local theory: logarithmic case

We recall Kato's ramification theory ([K1], [K2]) and prove some properties of graded quotients of some filtrations for the proof of Proposition 1.29 in Subsection 1.3.

Let K be a complete discrete valuation field of characteristic $p > 0$. We regard $H^1_{\text{ét}}(K, \mathbf{Z}/n\mathbf{Z})$ as a subgroup of $H^1_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}) = \varinjlim_n H^1_{\text{ét}}(K, \mathbf{Z}/n\mathbf{Z})$. Let $W_s(K)$ be the Witt ring of K of length $s \geq 0$. By definition, $W_0(K) = 0$ and $W_1(K) = K$. We write

$$F : W_s(K) \to W_s(K);\ (a_{s-1}, \cdots, a_0) \mapsto (a_{s-1}^p, \cdots, a_0^p)$$

for the Frobenius. By the Artin-Schreier-Witt theory, we have the exact sequence

$$0 \to W_s(\mathbf{F}_p) \to W_s(K) \xrightarrow{F-1} W_s(K) \to H^1(K, \mathbf{Z}/p^s\mathbf{Z}) \to 0. \tag{1.1}$$

We define

$$\delta_s : W_s(K) \to H^1(K, \mathbf{Q}/\mathbf{Z}) \tag{1.2}$$

to be the composition

$$W_s(K) \to H^1(K, \mathbf{Z}/p^s\mathbf{Z}) \to H^1(K, \mathbf{Q}/\mathbf{Z}),$$

where the first arrow is the fourth morphism in (1.1).

Let $\mathcal{O}_K$ be the valuation ring of K and F_K the residue field of K. We write G_K for the absolute Galois group of K.

Definition 1.1 ([K1, Definition (3.1)]). Let $s \geq 0$ be an integer.

(i) Let $a = (a_{s-1}, \ldots, a_0)$ be an element of $W_s(K)$. We define $\operatorname{ord}_K(a)$ by $\operatorname{ord}_K(a) = \min_{0 \leq i \leq s-1}\{p^i \operatorname{ord}_K(a_i)\}$.

(ii) We define an increasing filtration $\{\mathrm{fil}_n W_s(K)\}_{n \in \mathbf{Z}}$ of $W_s(K)$ by

$$\mathrm{fil}_n W_s(K) = \{a \in W_s(K) \mid \operatorname{ord}_K(a) \geq -n\}. \tag{1.3}$$

The filtration $\{\mathrm{fil}_n W_s(K)\}_{n \in \mathbf{Z}}$ in Definition 1.1 is first defined by Brylinski ([B, Proposition 1]) and $\mathrm{fil}_n W_s(K)$ is a submodule of $W_s(K)$ for $n \in \mathbf{Z}$ (loc. cit.). Let $n \geq 0$ be an integer and put $s' = \operatorname{ord}_p(n)$. Suppose that $s' < s$. Let V denote the Verschiebung

$$V : W_s(K) \to W_{s+1}(K);\ (a_{s-1}, \cdots, a_0) \mapsto (0, a_{s-1}, \cdots, a_0).$$

Since $(a_{s-1}, \ldots, a_0) = (a_{s-1}, \ldots, a_{s'+1}, 0, \ldots, 0) + V^{s-s'-1}(a_{s'}, \ldots, a_0)$, we have

$$\mathrm{fil}_n W_s(K) = \mathrm{fil}_{n-1} W_s(K) + V^{s-s'-1} \mathrm{fil}_n W_{s'+1}(K). \tag{1.4}$$

Definition 1.2 ([K1, Corollary (2.5), Theorem (3.2) (1)]). Let δ_s be as in (1.2).

(i) We define an increasing filtration $\{\mathrm{fil}_n H^1(K, \mathbf{Z}/p^s\mathbf{Z})\}_{n \in \mathbf{Z}_{\geq 0}}$ of $H^1(K, \mathbf{Z}/p^s\mathbf{Z})$ by

$$\mathrm{fil}_n H^1(K, \mathbf{Z}/p^s\mathbf{Z}) = \delta_s(\mathrm{fil}_n W_s(K)).$$

(ii) We define an increasing filtration $\{\mathrm{fil}_n H^1(K, \mathbf{Q}/\mathbf{Z})\}_{n \in \mathbf{Z}_{\geq 0}}$ of $H^1(K, \mathbf{Q}/\mathbf{Z})$ by

$$\mathrm{fil}_n H^1(K, \mathbf{Q}/\mathbf{Z}) = H^1(K, \mathbf{Q}/\mathbf{Z})\{p'\} + \bigcup_{s \geq 1} \delta_s(\mathrm{fil}_n W_s(K)), \tag{1.5}$$

where $H^1(K, \mathbf{Q}/\mathbf{Z})\{p'\}$ denotes the prime-to-p part of $H^1(K, \mathbf{Q}/\mathbf{Z})$.

Definition 1.3 ([K1, Definition (2.2)]). Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. We define the *Swan conductor* $\mathrm{sw}(\chi)$ of χ by $\mathrm{sw}(\chi) = \min\{n \in \mathbf{Z}_{\geq 0} \mid \chi \in \mathrm{fil}_n H^1(K, \mathbf{Q}/\mathbf{Z})\}$.

We recall the definition of refined Swan conductor of $\chi \in H^1(K, \mathbf{Q}/\mathbf{Z})$ given by Kato ([K2, (3.4.2)]). Let Ω_K^1 be the differential module of K over $K^p \subset K$.

Definition 1.4. We define an increasing filtration $\{\mathrm{fil}_n \Omega_K^1\}_{n \in \mathbf{Z}_{\geq 0}}$ of Ω_K^1 by

$$\mathrm{fil}_n \Omega_K^1 = \{(\alpha d\pi/\pi + \beta)/\pi^n \mid \alpha \in \mathcal{O}_K, \beta \in \Omega_{\mathcal{O}_K}^1\} = \mathfrak{m}^{-n} \Omega_{\mathcal{O}_K}^1(\log), \tag{1.6}$$

where π is a uniformizer of K and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_K$.

We consider the morphism

$$-F^{s-1}d : W_s(K) \to \Omega^1_K;\ (a_{s-1}, \cdots, a_0) \mapsto -\sum_{i=0}^{s-1} a_i^{p^i-1} da_i. \tag{1.7}$$

The morphism $-F^{s-1}d$ (1.7) satisfies $-F^{s-1}d(\mathrm{fil}_n W_s(K)) \subset \mathrm{fil}_n\Omega^1_K$. We put $\mathrm{gr}_n = \mathrm{fil}_n/\mathrm{fil}_{n-1}$ for $n \in \mathbf{Z}_{\geq 1}$. Then, for $n \in \mathbf{Z}_{\geq 1}$, the morphism (1.7) induces

$$\varphi_s{}^{(n)} \colon \mathrm{gr}_n W_s(K) \to \mathrm{gr}_n\Omega^1_K.$$

Let $\delta_s^{(n)} \colon \mathrm{gr}_n W_s(K) \to \mathrm{gr}_n H^1(K, \mathbf{Q}/\mathbf{Z})$ denote the morphism induced by δ_s (1.2) for $n \in \mathbf{Z}_{\geq 1}$. For $n \in \mathbf{Z}_{\geq 1}$, there exists a unique injection $\phi^{(n)} \colon \mathrm{gr}_n H^1(K, \mathbf{Q}/\mathbf{Z}) \to \mathrm{gr}_n\Omega^1_K$ such that the diagram

$$\begin{tikzcd} \mathrm{gr}_n W_s(K) \arrow[rr, "\varphi_s^{(n)}"] \arrow[dr, "\delta_s^{(n)}"'] & & \mathrm{gr}_n\Omega^1_K \\ & \mathrm{gr}_n H^1(K, \mathbf{Q}/\mathbf{Z}) \arrow[ur, "\phi^{(n)}"'] & \end{tikzcd} \tag{1.8}$$

is commutative for any $s \in \mathbf{Z}_{\geq 0}$ by [M, Remark 3.2.12], or [AS3, §10] for more detail. We note that $\mathrm{gr}_n\Omega^1_K \simeq \mathfrak{m}^{-n}\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} F_K$ is a vector space over F_K.

Definition 1.5 ([K2, (3.4.2)], [M, Remark 3.2.12], see also [AS3, Définition 10.16]). Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. We put $n = \mathrm{sw}(\chi)$. If $n \geq 1$, then we define the *refined Swan conductor* $\mathrm{rsw}(\chi)$ of χ to be the image of χ by $\phi^{(n)}$ in (1.8).

In the rest of this subsection, we prove some properties of graded quotients of filtrations.

For $q \in \mathbf{R}$, let $[q]$ denote the integer n such that $q - 1 < n \leq q$.

Lemma 1.6. *Let m and $r \geq 0$ be integers.*

(i) $[m/p^r] = [(m-1)/p^r] + 1$ *if* $m \in p^r\mathbf{Z}$ *and* $[m/p^r] = [(m-1)/p^r]$ *if* $m \notin p^r\mathbf{Z}$.

(ii) $[[m/p^r]/p] = [m/p^{r+1}] = [[m/p]/p^r]$.

Proof. (i) We put $m = p^r q + a$, where $q, a \in \mathbf{Z}$ and $0 \leq a < p^r$. Then $[m/p^r] = q$. Further $[(m-1)/p^r] = q + [(a-1)/p^r]$. Since $[(a-1)/p^r] = -1$ if $a = 0$ and $[(a-1)/p^r] = 0$ if $0 < a < p^r$, the assertion holds.

(ii) We put $m = p^{r+1}q' + a'$, where $q', a' \in \mathbf{Z}$ and $0 \leq a' < p^{r+1}$. Then $[m/p^r] = pq' + [a'/p^r]$ and $0 \leq [a'/p^r] < p$. Further $[m/p] = p^r q' + [a'/p]$ and $0 \leq [a'/p] < p^r$. Hence we have $[[m/p^r]/p] = q' = [m/p^{r+1}]$ and $[[m/p]/p^r] = q' = [m/p^{r+1}]$. □

LEMMA 1.7. *Let a be an element of $W_s(K)$.*

(i) $\mathrm{ord}_K(F(a)) = p \cdot \mathrm{ord}_K(a)$.

(ii) $\mathrm{ord}_K((F-1)(a)) = p \cdot \mathrm{ord}_K(a)$ *if* $\mathrm{ord}_K(a) < 0$ *and* $\mathrm{ord}_K((F-1)(a)) \geq 0$ *if* $\mathrm{ord}_K(a) \geq 0$.

(iii) For an integer $n \geq 0$, we have $F^{-1}(\mathrm{fil}_n W_s(K)) = (F-1)^{-1}(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_{[n/p]} W_s(K)$.

Proof. (i) We put $a = (a_{s-1}, \ldots, a_0)$. Since $F(a) = (a_{s-1}^p, \ldots, a_0^p)$, the assertion holds.
(ii) Suppose that $\mathrm{ord}_K(a) \geq 0$. Then, since both a and $F(a)$ belong to $\mathrm{fil}_0 W_s(K)$, we have $(F-1)(a) \in \mathrm{fil}_0 W_s(K)$. Hence we have $\mathrm{ord}_K((F-1)(a)) \geq 0$ by (1.3).
Suppose that $\mathrm{ord}_K(a) < 0$. We put $\mathrm{ord}_K(a) = -n$. Since both a and $F(a)$ belong to $\mathrm{fil}_{pn} W_s(K)$, we have $(F-1)(a) \in \mathrm{fil}_{pn} W_s(K)$. Since $\mathrm{ord}_K(F(a)) = -pn < \mathrm{ord}_K(a) = -n$, we have $(F-1)(a) \notin \mathrm{fil}_{pn-1} W_s(K)$. Hence we have $\mathrm{ord}_K((F-1)(a)) = -pn$.
(iii) By (i), we have $F(a) \in \mathrm{fil}_n W_s(K)$ if and only if $\mathrm{ord}_K(a) \geq -n/p$ for $a \in W_s(K)$. Hence we have $F^{-1}(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_{[n/p]} W_s(K)$. By (ii), we have $(F-1)^{-1}(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_{[n/p]} W_s(K)$ similarly. □

Let $n \geq 1$ be an integer. By Lemma 1.7 (iii), the Frobenius $F\colon W_s(K) \to W_s(K)$ induces the injection

$$\bar{F}\colon \mathrm{fil}_{[n/p]} W_s(K)/\mathrm{fil}_{[(n-1)/p]} W_s(K) \to \mathrm{gr}_n W_s(K). \tag{1.9}$$

By Lemma 1.6 (i), the domain of (1.9) is equal to $\mathrm{gr}_{n/p} W_s(K)$ if $n \in p\mathbf{Z}$ and it is 0 if $n \notin p\mathbf{Z}$.
By Lemma 1.7 (iii), the morphism $F-1\colon W_s(K) \to W_s(K)$ induces the injection

$$\overline{F-1}\colon \mathrm{fil}_{[n/p]} W_s(K)/\mathrm{fil}_{[(n-1)/p]} W_s(K) \to \mathrm{gr}_n W_s(K). \tag{1.10}$$

Since $[n/p] < n$ if $n \geq 1$, the morphisms (1.9) and (1.10) are the same.

LEMMA 1.8 (cf. [K1, Theorem (3.2), Corollary (3.3)]). *Let $n \geq 1$ be an integer. Then we have the exact sequence*

$$0 \to \mathrm{fil}_{[n/p]} W_s(K)/\mathrm{fil}_{[(n-1)/p]} W_s(K) \xrightarrow{\bar{F}} \mathrm{gr}_n W_s(K) \xrightarrow{\varphi_s^{(n)}} \mathrm{gr}_n \Omega_K^1,$$

where $\mathrm{fil}_{[n/p]} W_s(K)/\mathrm{fil}_{[(n-1)/p]} W_s(K)$ *is* $\mathrm{gr}_{n/p} W_s(K)$ *if* $n \in p\mathbf{Z}$ *and* 0 *if* $n \notin p\mathbf{Z}$.

Proof. As in the proof of [AS3, Proposition 10.7], the morphism $\varphi_s^{(n)}$ factors through

$$\mathrm{gr}_n H^1(K, \mathbf{Z}/p^s\mathbf{Z}) \simeq \mathrm{fil}_n W_s(K)/((F-1)(W_s(K)) \cap \mathrm{fil}_n W_s(K) + \mathrm{fil}_{n-1} W_s(K)).$$

Since this factorization defines the injection $\phi^{(n)}$ in (1.8) by [AS3, Proposition 10.14] and since the morphism $\bar{F}$ (1.9) is equal to the morphism $\overline{F-1}$ (1.10), the assertion holds. □

DEFINITION 1.9. Let $s \geq 0$ and $r \geq 0$ be integers. We define an increasing filtration $\{\mathrm{fil}_n^{(r)} W_s(K)\}_{n \in \mathbf{Z}_{\geq 0}}$ of $W_s(K)$ by

$$\mathrm{fil}_n^{(r)} W_s(K) = \{a \in W_s(K) \mid \mathrm{ord}_K(a) \geq -n/p^r\} = \mathrm{fil}_{[n/p^r]} W_s(K). \tag{1.11}$$

By (1.11), we have $\mathrm{fil}_n^{(0)} W_s(K) = \mathrm{fil}_n W_s(K)$ for $n \in \mathbf{Z}_{\geq 0}$.
For integers $0 \leq t \leq s$, let pr_t denote the projection

$$\mathrm{pr}_t \colon W_s(K) \to W_t(K) \ ; \ (a_{s-1}, \ldots, a_0) \mapsto (a_{s-1}, \ldots, a_{s-t}). \tag{1.12}$$

We put $\mathrm{gr}_n^{(r)} = \mathrm{fil}_n^{(r)} / \mathrm{fil}_{n-1}^{(r)}$ for $r \in \mathbf{Z}_{\geq 0}$ and $n \in \mathbf{Z}_{\geq 1}$.

LEMMA 1.10. *Let $r \geq 0$ and $0 \leq t \leq s$ be integers. Let $\mathrm{pr}_t \colon W_s(K) \to W_t(K)$ be as in (1.12). Let $n \geq 0$ be an integer.*

(i) $\mathrm{pr}_t(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_n^{(s-t)} W_t(K)$.

(ii) $(F-1)^{-1}(\mathrm{fil}_n^{(r)} W_s(K)) = \mathrm{fil}_{[n/p]}^{(r)} W_s(K)$.

Proof. (i) By (1.3), we have $\mathrm{pr}_t(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_{[n/p^{s-t}]} W_t(K)$. Hence the assertion holds by (1.11).
(ii) By Lemma 1.7 (iii) and (1.11), we have $(F-1)^{-1}(\mathrm{fil}_n^{(r)} W_s(K)) = \mathrm{fil}_{[[n/p^r]/p]} W_s(K)$. By Lemma 1.6 (ii) and (1.11), the assertion holds. $\square$

Let $n \geq 0$ and $0 \leq t \leq s$ be integers. Since $\mathrm{pr}_t(\mathrm{fil}_n W_s(K)) = \mathrm{fil}_n^{(s-t)} W_t(K)$ by Lemma 1.10 (i), we have the exact sequence

$$0 \to \mathrm{fil}_n W_{s-t}(K) \xrightarrow{V^t} \mathrm{fil}_n W_s(K) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}_n^{(s-t)} W_t(K) \to 0. \tag{1.13}$$

LEMMA 1.11. *Let $n \geq 1$ be an integer. Then the exact sequence (1.13) induces the exact sequence*

$$0 \to \mathrm{gr}_n W_{s-t}(K) \xrightarrow{\bar{V}^t} \mathrm{gr}_n W_s(K) \xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}_n^{(s-t)} W_t(K) \to 0,$$

where $\mathrm{gr}_n^{(s-t)} W_t(K)$ is equal to $\mathrm{gr}_{n/p^{s-t}} W_t(K)$ if $n \in p^{s-t}\mathbf{Z}$ and 0 if $n \notin p^{s-t}\mathbf{Z}$.

Proof. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{fil}_{n-1} W_{s-t}(K) \arrow[r, "V^t"] \arrow[d] & \mathrm{fil}_{n-1} W_s(K) \arrow[r, "\mathrm{pr}_t"] \arrow[d] & \mathrm{fil}_{n-1}^{(s-t)} W_t(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{fil}_n W_{s-t}(K) \arrow[r, "V^t"] & \mathrm{fil}_n W_s(K) \arrow[r, "\mathrm{pr}_t"] & \mathrm{fil}_n^{(s-t)} W_t(K) \arrow[r] & 0,
\end{tikzcd} \tag{1.14}$$

where the horizontal lines are exact and the vertical arrows are inclusions. By applying the snake lemma to (1.14), we obtain the exact sequence which we have desired. The last supplement to $\mathrm{gr}_n^{(s-t)} W_t(K)$ follows by Lemma 1.6 (i) and (1.11). $\square$

1.2 Local theory: non-logarithmic case

We recall a non-logarithmic variant, given by Matsuda ([M]), of Kato's logarithmic ramification theory recalled in Subsection 1.1, and we consider the exceptional case of Matsuda's theory. We also consider the graded quotients of filtrations. We keep the notation in Subsection 1.1.

Definition 1.12 (cf. [M, 3.1]). We define an increasing filtration $\{\mathrm{fil}'_m W_s(K)\}_{m\in\mathbf{Z}_{\geq 1}}$ of $W_s(K)$ by

$$\mathrm{fil}'_m W_s(K) = \mathrm{fil}_{m-1} W_s(K) + V^{s-s'}\mathrm{fil}_m W_{s'}(K). \tag{1.15}$$

Here $s' = \min\{\mathrm{ord}_p(m), s\}$.

The definition of $\{\mathrm{fil}'_m W_s(K)\}_{m\in\mathbf{Z}_{\geq 1}}$ in Definition 1.12 is shifted by 1 from Matsuda's definition ([M, 3.1]). Since $\mathrm{fil}_n W_s(K)$ is a submodule of $W_s(K)$ for $n \in \mathbf{Z}$, the subset $\mathrm{fil}'_m W_s(K)$ is a submodule of $W_s(K)$ for $m \in \mathbf{Z}_{\geq 1}$.
By (1.15), we have

$$\mathrm{fil}_{m-1} W_s(K) \subset \mathrm{fil}'_m W_s(K) \subset \mathrm{fil}_m W_s(K) \tag{1.16}$$

for $m \in \mathbf{Z}_{\geq 1}$. Since $\min\{\mathrm{ord}_p(1), s\} = 0$ for $s \in \mathbf{Z}_{\geq 0}$, we have

$$\mathrm{fil}_0 W_s(K) = \mathrm{fil}'_1 W_s(K). \tag{1.17}$$

Definition 1.13 (cf. [M, Definition 3.1.1]). Let δ_s be as in (1.2).

(i) We define an increasing filtration $\{\mathrm{fil}'_m H^1(K, \mathbf{Z}/p^s\mathbf{Z})\}_{m\in\mathbf{Z}_{\geq 1}}$ of $H^1(K, \mathbf{Z}/p^s\mathbf{Z})$ by

$$\mathrm{fil}'_m H^1(K, \mathbf{Z}/p^s\mathbf{Z}) = \delta_s(\mathrm{fil}'_m W_s(K)).$$

(ii) We define an increasing filtration $\{\mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})\}_{m\in\mathbf{Z}_{\geq 1}}$ of $H^1(K, \mathbf{Q}/\mathbf{Z})$ by

$$\mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z}) = H^1(K, \mathbf{Q}/\mathbf{Z})\{p'\} + \bigcup_{s\geq 1} \delta_s(\mathrm{fil}'_m W_s(K)), \tag{1.18}$$

where $H^1(K, \mathbf{Q}/\mathbf{Z})\{p'\}$ denotes the prime-to-p part of $H^1(K, \mathbf{Q}/\mathbf{Z})$.

By (1.16), we have

$$\mathrm{fil}_{m-1} H^1(K, \mathbf{Q}/\mathbf{Z}) \subset \mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z}) \subset \mathrm{fil}_m H^1(K, \mathbf{Q}/\mathbf{Z}) \tag{1.19}$$

for $m \in \mathbf{Z}_{\geq 1}$. By (1.17), we have $\mathrm{fil}_0 H^1(K, \mathbf{Q}/\mathbf{Z}) = \mathrm{fil}'_1 H^1(K, \mathbf{Q}/\mathbf{Z})$.

Definition 1.14 (cf. [M, Definition 3.2.5]). Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. We define the *total dimension* $\mathrm{dt}(\chi)$ of χ by $\mathrm{dt}(\chi) = \min\{m \in \mathbf{Z}_{\geq 1} \mid \chi \in \mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})\}$.

DEFINITION 1.15. We define an increasing filtration $\{\mathrm{fil}'_m\Omega^1_K\}_{m\in\mathbf{Z}_{\geq 1}}$ of Ω^1_K by

$$\mathrm{fil}'_m\Omega^1_K = \{\gamma/\pi^m \mid \gamma \in \Omega^1_{\mathcal{O}_K}\} = \mathfrak{m}^{-m}\Omega^1_{\mathcal{O}_K},$$

where π is a uniformizer of K and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_K$.

Since $\mathfrak{m}\Omega^1_{\mathcal{O}_K}(\log) \subset \Omega^1_{\mathcal{O}_K} \subset \Omega^1_{\mathcal{O}_K}(\log)$, we have

$$\mathrm{fil}_{m-1}\Omega^1_K \subset \mathrm{fil}'_m\Omega^1_K \subset \mathrm{fil}_m\Omega^1_K \tag{1.20}$$

for $m \in \mathbf{Z}_{\geq 1}$.
We consider the morphism (1.7). The morphism (1.7) satisfies $-F^{s-1}d(\mathrm{fil}'_mW_s(K)) \subset \mathrm{fil}'_m\Omega^1_K$ for $m \in \mathbf{Z}_{\geq 1}$. We put $\mathrm{gr}'_m = \mathrm{fil}'_m/\mathrm{fil}'_{m-1}$ for $m \in \mathbf{Z}_{\geq 2}$. Then, for $m \in \mathbf{Z}_{\geq 2}$, the morphism (1.7) induces

$$\varphi'^{(m)}_s \colon \mathrm{gr}'_mW_s(K) \to \mathrm{gr}'_m\Omega^1_K. \tag{1.21}$$

Let $\delta'^{(m)}_s \colon \mathrm{gr}'_mW_s(K) \to \mathrm{gr}'_mH^1(K,\mathbf{Q}/\mathbf{Z})$ denote the morphism induced by δ_s (1.2) for $m \in \mathbf{Z}_{\geq 2}$. If $(p,m) \neq (2,2)$, there exists a unique injection $\phi'^{(m)} \colon \mathrm{gr}'_mH^1(K,\mathbf{Q}/\mathbf{Z}) \to \mathrm{gr}'_m\Omega^1_K$ such that the diagram

$$\begin{tikzcd} \mathrm{gr}'_mW_s(K) \arrow[rr, "\varphi'^{(m)}_s"] \arrow[dr, "\delta'^{(m)}_s"'] & & \mathrm{gr}'_m\Omega^1_K \\ & \mathrm{gr}'_mH^1(K,\mathbf{Q}/\mathbf{Z}) \arrow[ur, "\phi'^{(m)}"'] & \end{tikzcd} \tag{1.22}$$

is commutative for any $s \in \mathbf{Z}_{\geq 0}$ by [M, Proposition 3.2.3]. We note that $\mathrm{gr}'_m\Omega^1_K \simeq \mathfrak{m}^{-m}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} F_K$ is a vector space over F_K.
We consider the exceptional case where $(p,m) = (2,2)$.

LEMMA 1.16. *Let $s \geq 1$ be an integer. Assume that $p = 2$. Then $V^{s-1} \colon K \to W_s(K)$ induces an isomorphism $\mathrm{gr}'_2K \to \mathrm{gr}'_2W_s(K)$.*

Proof. Since $p = 2$, we have $s' = \min\{\mathrm{ord}_p(2), s\} = 1$. Hence we have

$$\begin{aligned} \mathrm{fil}'_2W_s(K) &= \mathrm{fil}_1W_s(K) + V^{s-1}\mathrm{fil}_2K \\ &= \mathrm{fil}'_1W_s(K) + V^{s-1}\mathrm{fil}_2K \end{aligned}$$

by applying (1.15) for the first equality and (1.4) and (1.17) for the second equality. Since $\mathrm{fil}_2K = \mathrm{fil}'_2K$ by (1.15), the assertion holds. □

PROPOSITION 1.17. *Assume that $p = 2$. Let $F_K^{1/2} \subset \bar{F}_K$ denote the subfield of an algebraic closure $\bar{F}_K$ of F_K consisting of the square roots of F_K.*

(i) There exists a unique morphism

$$\tilde{\varphi}'^{(2)}_s \colon \mathrm{gr}'_2W_s(K) \to \mathrm{gr}'_2\Omega^1_K \otimes_{F_K} F_K^{1/2}$$

such that $\tilde{\varphi}_s'^{(2)}(\bar{a}) = -da_0 + \sqrt{\overline{\pi^2 a_0}}d\pi/\pi^2$ *for every* $\bar{a} \in \mathrm{gr}_2' W_s(K)$ *whose lift in* $\mathrm{fil}_2' W_s(K)$ *is* $a = (0, \ldots, 0, a_0)$ *and for every uniformizer* $\pi \in K$. *Here* $\sqrt{\overline{\pi^2 a_0}} \in F_K^{1/2}$ *denotes the square root of the image* $\overline{\pi^2 a_0}$ *of* $\pi^2 a_0$ *in* F_K.

(ii) *There exists a unique injection* $\tilde{\phi}'^{(2)} \colon \mathrm{gr}_2' H^1(K, \mathbf{Q}/\mathbf{Z}) \to \mathrm{gr}_2' \Omega_K^1 \otimes_{F_K} F_K^{1/2}$ *such that the following diagram is commutative for every* $s \geq 0$*:*

$$\begin{tikzcd} \mathrm{gr}_2' W_s(K) \arrow[rr, "\tilde{\varphi}_s'^{(2)}"] \arrow[dr, "\delta_s'^{(2)}"'] & & \mathrm{gr}_2' \Omega_K^1 \otimes_{F_K} F_K^{1/2} \\ & \mathrm{gr}_2' H^1(K, \mathbf{Q}/\mathbf{Z}). \arrow[ur, "\tilde{\phi}'^{(2)}"'] & \end{tikzcd} \tag{1.23}$$

Proof. By Lemma 1.16, we may assume that $s = 1$.
(i) Let a be an element of $\mathrm{fil}_2' K$ and π a uniformizer of K. Since $p = 2$, we have $\mathrm{fil}_2' K = \mathrm{fil}_2 K$ by (1.15). Hence we have $\pi^2 a \in \mathcal{O}_K$ by (1.3). Since $-d(\mathrm{fil}_2' K) \subset \mathrm{fil}_2' \Omega_K^1$, we have $-da + \sqrt{\overline{\pi^2 a}}d\pi/\pi^2 \in \mathrm{gr}_2' \Omega_K^1 \otimes_{F_K} F_K^{1/2}$. If $a \in \mathrm{fil}_1' K$, we have $a \in \mathcal{O}_K$ by (1.3) and (1.17). Since $-d(\mathrm{fil}_1' K) \subset \mathrm{fil}_1' \Omega_K^1$, we have $-da + \sqrt{\overline{\pi^2 a}}d\pi/\pi^2 = 0$ in $\mathrm{gr}_2' \Omega_K^1 \otimes_{F_K} F_K^{1/2}$. For $a, b \in \mathrm{fil}_2' K$, we have $\sqrt{\overline{\pi^2(a+b)}} = \sqrt{\overline{\pi^2 a}} + \sqrt{\overline{\pi^2 b}}$, since $p = 2$.
We prove that $\sqrt{\overline{\pi^2 a}}d\pi/\pi^2$ is independent of the choice of a uniformizer π of K. Let $u \in \mathcal{O}_K^\times$ be a unit. Then, in $\mathrm{gr}_2' \Omega_K^1 \otimes_{F_K} F_K^{1/2}$, we have

$$\sqrt{\overline{(u\pi)^2 a}}d(u\pi)/(u\pi)^2 = u\sqrt{\overline{\pi^2 a}}ud\pi/(u\pi)^2 = \sqrt{\overline{\pi^2 a}}d\pi/\pi^2.$$

Hence the assertion holds.
(ii) Since $p = 2$ and $\mathrm{fil}_2' K = \mathrm{fil}_2 K$, we have $\mathrm{fil}_2' K \cap (F-1)(K) = (F-1)(\mathrm{fil}_1 K)$ by Lemma 1.7 (iii). Hence it is sufficient to prove that $\mathrm{Ker}\, \tilde{\varphi}_1'^{(2)}$ is the image of $(F-1)(\mathrm{fil}_1 K)$ in $\mathrm{gr}_2' K$.
Let a be an element of $\mathrm{fil}_1 K$. By (1.3), we may put $a = a'/\pi$, where $a' \in \mathcal{O}_K$. Then we have

$$\tilde{\varphi}_1'^{(2)}(\bar{a}^2 - \bar{a}) = -\bar{a}' d\pi/\pi^2 + \sqrt{\bar{a}'^2}d\pi/\pi^2 = 0. \tag{1.24}$$

Conversely, let $a \in \mathrm{fil}_2' K$ be a lift of an element of $\mathrm{Ker}\, \tilde{\varphi}_1'^{(2)}$. Since $\mathrm{fil}_2' K = \mathrm{fil}_2 K$, we can put $a = a'/\pi^2$, where $a' \in \mathcal{O}_K$, by (1.3). Suppose that $\mathrm{ord}_K(a') > 0$, that is $a \in \mathrm{fil}_1 W_s(K)$. Since $\tilde{\varphi}_1'^{(2)}(\bar{a}) = -(a'\pi^{-1})d\pi/\pi^2 = 0$, we have $a'\pi^{-1} = 0$ in F_K. Hence $a \in \mathrm{fil}_0 K = \mathrm{fil}_1' K$, that is $\bar{a} = 0$ in $\mathrm{gr}_2' K$.
Assume that $a' \in \mathcal{O}_K^\times$ is a unit. Since we have

$$\tilde{\varphi}_1'^{(2)}(\bar{a}) = -da + \sqrt{\bar{a}'}d\pi/\pi^2 = 0, \tag{1.25}$$

we have $\sqrt{a'} \in F_K$. Hence there exist a unit $a'' \in \mathcal{O}_K^\times$ and an element $b \in \mathrm{fil}_1 K$ such that $a = (F-1)(a''/\pi) + b$. By (1.24) and (1.25), we have $\tilde{\varphi}_1'^{(2)}(\bar{b}) = 0$. Hence we have $b \in \mathrm{fil}'_1 K$ by the case where $\mathrm{ord}_K(a') > 0$, which is proved above. Therefore $\bar{a} \in \mathrm{gr}'_2 K$ is the image of an element of $(F-1)(\mathrm{fil}_1 K)$. $\square$

Let $m \geq 2$ be an integer. By abuse of notation, we write

$$\phi'^{(m)} \colon \mathrm{gr}'_m H^1(K, \mathbf{Q}/\mathbf{Z}) \to \mathrm{gr}'_m \Omega^1_K \otimes_{F_K} F_K^{1/p} \tag{1.26}$$

for the composition of $\phi'^{(m)}$ in (1.22) and the inclusion $\mathrm{gr}'_m \Omega^1_K \to \mathrm{gr}'_m \Omega^1_K \otimes_{F_K} F_K^{1/p}$ if $(p,m) \neq (2,2)$ and $\tilde{\phi}'^{(2)}$ in Proposition 1.17 (ii) if $(p,m) = (2,2)$.

DEFINITION 1.18. Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. We put $m = \mathrm{dt}(\chi)$ and assume that $m \geq 2$. We define the *characteristic form* $\mathrm{char}(\chi) \in \mathrm{gr}'_m \Omega^1_K \otimes_{F_K} F_K^{1/p}$ of χ to be the image of χ by $\phi'^{(m)}$ (1.26).

By (1.22) and Proposition 1.17, we need $F_K^{1/p}$ only in the case where $p = 2$ and $\chi \in \mathrm{fil}'_2 H^1(K, \mathbf{Q}/\mathbf{Z}) - \mathrm{fil}_1 H^1(K, \mathbf{Q}/\mathbf{Z})$.
In the rest of this subsection, we prepare some lemmas for the proof of Proposition 1.29.

DEFINITION 1.19. Let $s \geq 0$ and $r \geq 0$ be integers. We put $r' = \min\{\mathrm{ord}_p(m), s+r\}$ and $s'' = \max\{0, r'-r\}$. We define increasing filtrations $\{\mathrm{fil}'^{(r)}_m W_s(K)\}_{m \in \mathbf{Z}_{\geq 1}}$ and $\{\mathrm{fil}''^{(r)}_m W_s(K)\}_{m \in \mathbf{Z}_{\geq 1}}$ of $W_s(K)$ by

$$\mathrm{fil}'^{(r)}_m W_s(K) = \mathrm{fil}^{(r)}_{m-1} W_s(K) + V^{s-s''} \mathrm{fil}^{(r)}_m W_{s''}(K), \tag{1.27}$$

$$\mathrm{fil}''^{(r)}_m W_s(K) = \mathrm{fil}^{(r)}_{[(m-1)/p]} W_s(K) + V^{s-s''} \mathrm{fil}^{(r)}_{[m/p]} W_{s''}(K). \tag{1.28}$$

If $r = 0$, then we simply write $\mathrm{fil}''_m W_s(K)$ for $\mathrm{fil}''^{(0)}_m W_s(K)$.

If $r = 0$, since $s'' = s' = \min\{\mathrm{ord}_p(m), s\}$, we have $\mathrm{fil}'^{(0)}_m W_s(K) = \mathrm{fil}'_m W_s(K)$. Further we have

$$\mathrm{fil}''_m W_s(K) = \mathrm{fil}_{[(m-1)/p]} W_s(K) + V^{s-s'} \mathrm{fil}_{[m/p]} W_{s'}(K). \tag{1.29}$$

LEMMA 1.20. *Let $r \geq 0$ and $0 \leq t \leq s$ be integers. Let $\mathrm{pr}_t \colon W_s(K) \to W_t(K)$ be as in (1.12). Let $m \geq 1$ be an integer.*

(i) $\mathrm{pr}_t(\mathrm{fil}'_m W_s(K)) = \mathrm{fil}'^{(s-t)}_m W_t(K)$.

(ii) We have the exact sequence

$$0 \to \mathrm{fil}'_m W_{s-t}(K) \xrightarrow{V^t} \mathrm{fil}'_m W_s(K) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}'^{(s-t)}_m W_t(K) \to 0. \tag{1.30}$$

(iii) $\mathrm{pr}_t(\mathrm{fil}''_m W_s(K)) = \mathrm{fil}''^{(s-t)}_m W_t(K)$.

(iv) We have the exact sequence

$$0 \to \mathrm{fil}''_m W_{s-t}(K) \xrightarrow{V^t} \mathrm{fil}''_m W_s(K) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}''^{(s-t)}_m W_t(K) \to 0. \tag{1.31}$$

(v) $\mathrm{fil}''^{(r)}_m W_s(K) = (F-1)^{-1}(\mathrm{fil}'^{(r)}_m W_s(K))$. *Especially,* $\mathrm{fil}''_m W_s(K) = (F-1)^{-1}(\mathrm{fil}'_m W_s(K))$.

Proof. We put $s' = \min\{\mathrm{ord}_p(m), s\}$, $r' = \min\{\mathrm{ord}_p(m), s+r\}$, and $s'' = \max\{0, r'-r\}$.

(i) By (1.27), we have $\mathrm{fil}'^{(s-t)}_m W_t(K) = \mathrm{fil}^{(s-t)}_{m-1} W_t(K)$ if $t \leq s-s'$ and $\mathrm{fil}'^{(s-t)}_m W_t(K) = \mathrm{fil}^{(s-t)}_{m-1} W_t(K) + V^{s-s'}\mathrm{fil}^{(s-t)}_m W_{t-s+s'}(K)$ if $t > s-s'$. By Lemma 1.10 (i), we have $\mathrm{pr}_t(\mathrm{fil}_{m-1} W_s(K)) = \mathrm{fil}^{(s-t)}_{m-1} W_t(K)$ and, if $t > s-s'$, we have $\mathrm{pr}_t(V^{s-s'}\mathrm{fil}_m W_{s'}(K)) = V^{s-s'}\mathrm{fil}^{(s-t)}_m W_{t-s+s'}(K)$. Hence the assertion holds by (1.15).

(ii) The assertion holds by (1.15) and (i).

(iii) The assertion holds similarly as the proof of (i) by (1.28) and (1.29).

(iv) The assertion holds by (1.29) and (iii).

(v) Since $V^{s-s''}$ and $\mathrm{pr}_{s-s''}$ commute with $F-1$, the morphisms $V^{s-s''}\colon W_{s''}(K) \to W_s(K)$ and $\mathrm{pr}_{s-s''}\colon W_s(K) \to W_{s-s''}(K)$ induce $V^{s-s''}\colon (F-1)^{-1}(\mathrm{fil}^{(r)}_m W_{s''}(K)) \to (F-1)^{-1}(\mathrm{fil}'^{(r)}_m W_s(K))$ and $\mathrm{pr}_{s-s''}\colon (F-1)^{-1}(\mathrm{fil}'^{(r)}_m W_s(K)) \to (F-1)^{-1}(\mathrm{fil}^{(r+s'')}_{m-1} W_{s-s''}(K))$ respectively.

We prove that $\mathrm{fil}''^{(r)}_m W_s(K) \subset (F-1)^{-1}(\mathrm{fil}'^{(r)}_m W_s(K))$. By (1.11) and (1.28), we have $\mathrm{fil}''^{(r)}_m W_s(K) = \mathrm{fil}_{[[(m-1)/p]/p^r]} W_s(K) + V^{s-s''}\mathrm{fil}_{[[m/p]/p^r]} W_{s''}(K)$. By (1.11) and (1.27), we have $\mathrm{fil}'^{(r)}_m W_s(K) = \mathrm{fil}_{[(m-1)/p^r]} W_s(K) + V^{s-s''}\mathrm{fil}_{[m/p^r]} W_{s''}(K)$. Hence, by Lemma 1.6 (ii) and Lemma 1.7 (iii), we have $\mathrm{fil}''^{(r)}_m W_s(K) \subset (F-1)^{-1}(\mathrm{fil}'^{(r)}_m W_s(K))$.

We put $A_n = \mathrm{fil}^{(r)}_n W_{s''}(K)$ and $B_n = \mathrm{fil}^{(r+s'')}_n W_{s-s''}(K)$ for $n \in \mathbf{Z}_{\geq 0}$. We also put $C_n = \mathrm{fil}'^{(r)}_n W_s(K)$ and $D_n = \mathrm{fil}''^{(r)}_n W_s(K)$ for $n \in \mathbf{Z}_{\geq 1}$. We consider the commutative diagram

$$\begin{tikzcd}
A_{[m/p]} \arrow[r, "V^{s-s''}"] \arrow[d] & D_m \arrow[r, "\mathrm{pr}_{s-s''}"] \arrow[d] & B_{[(m-1)/p]} \arrow[r] \arrow[d] & 0 \\
(F-1)^{-1}(A_m) \arrow[r, "V^{s-s''}"] & (F-1)^{-1}(C_m) \arrow[r, "\mathrm{pr}_{s-s''}"] & (F-1)^{-1}(B_{m-1}), &
\end{tikzcd}$$

where the left and right vertical arrows are the identities by Lemma 1.10 (ii), the middle vertical arrow is the inclusion, and the lower horizontal line is exact. Since the upper horizontal line is exact by Lemma 1.10 (i) and (1.28), the assertion holds by applying the snake lemma. □

COROLLARY 1.21. *Let $m \geq 2$ and $0 \leq t \leq s$ be integers.*

(i) The exact sequence (1.30) induces the exact sequence

$$0 \to \mathrm{gr}'_m W_{s-t}(K) \xrightarrow{\bar{V}^t} \mathrm{gr}'_m W_s(K) \xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}'^{(s-t)}_m W_t(K) \to 0.$$

(ii) The exact sequence (1.31) induces the exact sequence

$$0 \to \mathrm{gr}''_m W_{s-t}(K) \xrightarrow{\bar{V}^t} \mathrm{gr}''_m W_s(K) \xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}''^{(s-t)}_m W_t(K) \to 0.$$

Proof. The assertion holds similarly as the proof of Lemma 1.11. □

Let $m \geq 2$ be an integer. By abuse of notation, let

$$\varphi'^{(m)}_s \colon \mathrm{gr}'_m W_s(K) \to \mathrm{gr}'_m \Omega^1_K \otimes_{F_K} F_K^{1/p}$$

be the composition of $\varphi'^{(m)}_s$ (1.21) and the inclusion $\mathrm{gr}'_m\Omega^1_K \to \mathrm{gr}'_m\Omega^1_K \otimes_{F_K} F_K^{1/p}$ if $(p,m) \neq (2,2)$ and $\tilde{\varphi}'^{(2)}_s$ in Proposition 1.17 (i) if $(p,m) = (2,2)$.
Let $r \geq 0$ be an integer. By Lemma 1.20 (v), the morphism $F-1 \colon W_s(K) \to W_s(K)$ induces the injection

$$\overline{F-1} \colon \mathrm{gr}''^{(r)}_m W_s(K) \to \mathrm{gr}'^{(r)}_m W_s(K).$$

Especially, the morphism $F-1$ induces the injection

$$\overline{F-1} \colon \mathrm{gr}''_m W_s(K) \to \mathrm{gr}'_m W_s(K).$$

Lemma 1.22 (cf. [M, Proposition 3.2.1, Proposition 3.2.3]). *Let $m \geq 2$ be an integer. Then we have the exact sequence*

$$0 \to \mathrm{gr}''_m W_s(K) \xrightarrow{\overline{F-1}} \mathrm{gr}'_m W_s(K) \xrightarrow{\varphi'^{(m)}_s} \mathrm{gr}'_m \Omega^1_K \otimes_F F^{1/p}.$$

Proof. As in the proof of [M, Proposition 3.2.1] and Proposition 1.17 (ii), the morphism $\varphi'^{(m)}_s$ factors through

$$\mathrm{gr}'_m H^1(K, \mathbf{Z}/p^s\mathbf{Z}) \simeq \mathrm{fil}'_m W_s(K)/((F-1)(W_s(K)) \cap \mathrm{fil}'_m W_s(K) + \mathrm{fil}'_{m-1} W_s(K)).$$

Since this factorization defines the injection $\phi'^{(m)}$ by [M, Proposition 3.2.3] and Proposition 1.17 (ii), the assertion holds. □

Lemma 1.23. *Let $m \geq 1$ and $r \geq 0$ be integers.*

(i) $\mathrm{fil}'^{(r)}_m K = \mathrm{fil}_{m/p^r} K$ *if* $m \in p^{r+1}\mathbf{Z}$ *and* $\mathrm{fil}'^{(r)}_m K = \mathrm{fil}_{[(m-1)/p^r]} K$ *if* $m \notin p^{r+1}\mathbf{Z}$.

(ii) $\mathrm{fil}''^{(r)}_m K = \mathrm{fil}_{[m/p^{r+1}]} K$.

Proof. (i) By (1.27), we have $\mathrm{fil}'^{(r)}_m K = \mathrm{fil}^{(r)}_m K$ if $m \in p^{r+1}\mathbf{Z}$ and $\mathrm{fil}'^{(r)}_m K = \mathrm{fil}^{(r)}_{m-1} K$ if $m \notin p^{r+1}\mathbf{Z}$. Hence the assertion holds by (1.11).
(ii) By Lemma 1.20 (v), we have $\mathrm{fil}''^{(r)}_m K = (F-1)^{-1}(\mathrm{fil}'^{(r)}_m K)$. By (i) and Lemma 1.7 (iii), we have $\mathrm{fil}''^{(r)}_m K = \mathrm{fil}_{m/p^{r+1}} W_s(K)$ if $m \in p^{r+1}\mathbf{Z}$ and $\mathrm{fil}''^{(r)}_m K = \mathrm{fil}_{[[(m-1)/p^r]/p]} W_s(K)$ if $m \notin p^{r+1}\mathbf{Z}$. Hence the assertion holds by Lemma 1.6. □

Corollary 1.24. *Let $m \geq 2$ and $r \geq 0$ be integers.*

(i) *Assume that $r \geq 1$. Then $\mathrm{gr}'^{(r)}_m K = \mathrm{gr}_{[m/p^r]} K$ if $m \in p^{r+1}\mathbf{Z}$ or $\mathrm{ord}_p(m-1) = r$, and $\mathrm{gr}'^{(r)}_m K = 0$ otherwise.*

(ii) *$\mathrm{gr}''^{(r)}_m K = \mathrm{gr}_{m/p^{r+1}} K$ if $m \in p^{r+1}\mathbf{Z}$, and $\mathrm{gr}''^{(r)}_m K = 0$ if $m \notin p^{r+1}\mathbf{Z}$.*

Proof. (i) Assume that $m \in p^{r+1}\mathbf{Z}$. Since $r \geq 1$, we have $m-1 \notin p^r\mathbf{Z}$. Hence $\mathrm{gr}'^{(r)}_m K = \mathrm{fil}_{[m/p^r]} K/\mathrm{fil}_{[(m-2)/p^r]} K$ by Lemma 1.23 (i). By Lemma 1.6 (i), the assertion holds in this case.
Assume that $m \notin p^{r+1}\mathbf{Z}$. By Lemma 1.23 (i), we have $\mathrm{gr}'^{(r)}_m K = \mathrm{fil}_{[(m-1)/p^r]} K/\mathrm{fil}_{[(m-2)/p^r]} K$ if $m-1 \notin p^{r+1}\mathbf{Z}$ and $\mathrm{gr}'^{(r)}_m K = 0$ if $m-1 \in p^{r+1}\mathbf{Z}$. Suppose that $m-1 \notin p^{r+1}\mathbf{Z}$. By Lemma 1.6 (i), we have $\mathrm{gr}'^{(r)}_m K = \mathrm{gr}_{[(m-1)/p^r]} K$ if $m-1 \in p^r\mathbf{Z}$ and $\mathrm{gr}'^{(r)}_m K = 0$ if $m-1 \notin p^r\mathbf{Z}$. If $m-1 \in p^r\mathbf{Z}$, then we have $m \notin p^r\mathbf{Z}$, since $r \geq 1$. Hence the assertion holds by Lemma 1.6 (i).
(ii) By Lemma 1.23 (ii), we have $\mathrm{gr}''^{(r)}_m K = \mathrm{fil}_{[m/p^{r+1}]} K/\mathrm{fil}_{[(m-1)/p^{r+1}]} K$. Hence the assertion holds by Lemma 1.6 (i). □

We note that if $r = 0$ and if $m \in p\mathbf{Z}$ then $\mathrm{gr}'^{(r)}_m K = \mathrm{gr}'_m K = \mathrm{fil}_m K/\mathrm{fil}_{m-2} K$.

1.3 Sheafification: logarithmic case

Let X be a smooth separated scheme over a perfect field k of characteristic $p > 0$. Let D be a divisor on X with simple normal crossings and $\{D_i\}_{i \in I}$ the irreducible components of D. The generic point of D_i is denoted by $\mathfrak{p}_i$ for $i \in I$. We put $U = X - D$ and let $j \colon U \to X$ be the canonical open immersion. For $i \in I$, let $\mathcal{O}_{K_i}$ denote the completion $\hat{\mathcal{O}}_{X,\mathfrak{p}_i}$ of the local ring $\mathcal{O}_{X,\mathfrak{p}_i}$ at $\mathfrak{p}_i$ and K_i the fractional field of $\mathcal{O}_{K_i}$ called *local field* at $\mathfrak{p}_i$.
Let $\epsilon \colon X_{\text{ét}} \to X_{\text{Zar}}$ be the canonical mapping from the étale site of X to the Zariski site of X. We use the same notation j_* for the push-forward of both étale sheaves and Zariski sheaves. We consider the exact sequence

$$0 \to W_s(\mathbf{F}_p) \to W_s(\mathcal{O}_{U_{\text{ét}}}) \xrightarrow{F-1} W_s(\mathcal{O}_{U_{\text{ét}}}) \to 0$$

of étale sheaves on U for $s \in \mathbf{Z}_{\geq 0}$. Since $R^1(\epsilon \circ j)_* W_s(\mathcal{O}_{U_{\text{ét}}}) = 0$, we have an exact sequence

$$0 \to j_* W_s(\mathbf{F}_p) \to j_* W_s(\mathcal{O}_U) \xrightarrow{F-1} j_* W_s(\mathcal{O}_U) \to R^1(\epsilon \circ j)_* \mathbf{Z}/p^s\mathbf{Z} \to 0 \quad (1.32)$$

We write

$$\delta_s\colon j_*W_s(\mathcal{O}_U) \to R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z} \tag{1.33}$$

for the fourth morphism in (1.32).

Let V be an open subset of X. Since we have the spectral sequence $E_2^{p,q} = H^p_{\mathrm{Zar}}(V, R^q(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}) \Rightarrow H^{p+q}_{\text{ét}}(U\cap V, \mathbf{Z}/p^s\mathbf{Z})$ and $E_2^{1,0} = E_2^{2,0} = 0$, the canonical morphism

$$H^1_{\text{ét}}(U\cap V, \mathbf{Z}/p^s\mathbf{Z}) \to \Gamma(V, R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z})$$

is an isomorphism. By the exact sequence (1.32), the morphism δ_s (1.33) induces an isomorphism

$$j_*W_s(\mathcal{O}_U)/(F-1)j_*W_s(\mathcal{O}_U) \to R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}.$$

If $D_i\cap V\neq\emptyset$ and if $a\in\Gamma(U\cap V, W_s(\mathcal{O}_U))$, let $a|_{K_i}$ denote the image of a by

$$\Gamma(U\cap V, W_s(\mathcal{O}_U)) \to W_s(K_i).$$

Similarly, if $D_i\cap V\neq\emptyset$ and if $\chi\in H^1_{\text{ét}}(U\cap V, \mathbf{Z}/p^s\mathbf{Z})$, let $\chi|_{K_i}$ denote the image of χ by

$$H^1_{\text{ét}}(U\cap V, \mathbf{Z}/p^s\mathbf{Z}) \to H^1(K_i, \mathbf{Z}/p^s\mathbf{Z}).$$

Definition 1.25. Let $R = \sum_{i\in I} n_iD_i$, where $n_i\in\mathbf{Z}_{\geq 0}$ for $i\in I$, and let $j_i\colon \operatorname{Spec} K_i\to X$ denote the canonical morphism for $i\in I$.

(i) We define a subsheaf $\mathrm{fil}_R j_*W_s(\mathcal{O}_U)$ of Zariski sheaf $j_*W_s(\mathcal{O}_U)$ to be the pull-back of $\bigoplus_{i\in I} j_{i*}\mathrm{fil}_{n_i}W_s(K_i)$ by the morphism $j_*W_s(\mathcal{O}_U)\to \bigoplus_{i\in I} j_{i*}W_s(K_i)$.

(ii) We define a subsheaf $\mathrm{fil}_R R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}$ of $R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}$ to be the image of $\mathrm{fil}_R j_*W_s(\mathcal{O}_U)$ by δ_s (1.33).

(iii) We define a subsheaf $\mathrm{fil}_R j_*\Omega^1_U$ of $j_*\Omega^1_U$ to be $\Omega^1_X(\log D)(R)$.

We consider the morphism

$$-F^{s-1}d\colon j_*W_s(\mathcal{O}_U)\to j_*\Omega^1_U\ ;\ (a_{s-1},\ldots,a_0)\mapsto -\sum_{i=0}^{s-1} a_i^{p^i-1}da_i. \tag{1.34}$$

Let $R = \sum_{i\in I} n_iD_i$, where $n_i\in\mathbf{Z}_{\geq 0}$ for $i\in I$. Then (1.34) induces the morphism

$$\mathrm{fil}_R j_*W_s(\mathcal{O}_U)\to \mathrm{fil}_R j_*\Omega^1_U.$$

Let $R' = \sum_{i\in I} n_i'D_i$, where $n_i'\in\mathbf{Z}_{\geq 0}$ such that $n_i'\leq n_i$ for $i\in I$. Then we have $\mathrm{fil}_R\supset\mathrm{fil}_{R'}$ and put $\mathrm{gr}_{R/R'} = \mathrm{fil}_R/\mathrm{fil}_{R'}$. Then the morphism (1.34) induces the morphism

$$\varphi_s^{(R/R')}\colon \mathrm{gr}_{R/R'} j_*W_s(\mathcal{O}_U)\to \mathrm{gr}_{R/R'} j_*\Omega^1_U. \tag{1.35}$$

If $R = R' + D_i$ for some $i \in I$, then we simply write $\varphi_s^{(R,i)}$ for $\varphi_s^{(R,R')}$ and $\mathrm{gr}_{R,i}$ for $\mathrm{gr}_{R/R'}$.

Let $0 \leq t \leq s$ be integers. We put $[R/p^j] = \sum_{i\in I}[n_i/p^j]D_i$. We consider the projection

$$\mathrm{pr}_t \colon j_*W_s(\mathcal{O}_U) \to j_*W_t(\mathcal{O}_U)\ ;\ (a_{s-1}, \dots, a_0) \mapsto (a_{s-1}, \dots, a_{s-t}). \tag{1.36}$$

Since we have $\mathrm{pr}_t(\mathrm{fil}_R j_*W_s(\mathcal{O}_U)) = \mathrm{fil}_{[R/p^{s-t}]} j_*W_t(\mathcal{O}_U)$ by (1.11) and Lemma 1.10 (i), we have the exact sequence

$$0 \to \mathrm{fil}_R j_*W_{s-t}(\mathcal{O}_U) \xrightarrow{V^t} \mathrm{fil}_R j_*W_s(\mathcal{O}_U) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}_{[R/p^{s-t}]} j_*W_t(\mathcal{O}_U) \to 0. \tag{1.37}$$

Lemma 1.26. *Let $R = \sum_{i\in I} n_iD_i$ and $R' = \sum_{i\in I} n_i'D_i$, where $n_i, n_i' \in \mathbf{Z}_{\geq 0}$ and $n_i' \leq n_i$ for every $i \in I$. Then the exact sequence (1.37) induces the exact sequence*

$$\begin{aligned} 0 \to \mathrm{gr}_{R/R'} j_*W_{s-t}(\mathcal{O}_U) &\xrightarrow{\bar{V}^t} \mathrm{gr}_{R/R'} j_*W_s(\mathcal{O}_U) \\ &\xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}_{[R/p^{s-t}]/[R'/p^{s-t}]} j_*W_t(\mathcal{O}_U) \to 0. \end{aligned} \tag{1.38}$$

Especially, if $R = R' + D_i$ for some $i \in I$, we have the exact sequence

$$\begin{aligned} 0 \to \mathrm{gr}_{R,i} j_*W_{s-t}(\mathcal{O}_U) &\xrightarrow{\bar{V}^t} \mathrm{gr}_{R,i} j_*W_s(\mathcal{O}_U) \\ &\xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}_{[R/p^{s-t}]/[(R-D_i)/p^{s-t}]} j_*W_t(\mathcal{O}_U) \to 0. \end{aligned}$$

Proof. The assertion holds similarly as the proof of Lemma 1.11. In fact, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{fil}_{R'} j_*W_{s-t}(\mathcal{O}_U) \arrow[r, "V^t"] \arrow[d] & \mathrm{fil}_{R'} j_*W_s(\mathcal{O}_U) \arrow[r, "\mathrm{pr}_t"] \arrow[d] & \mathrm{fil}_{[R'/p^{s-t}]} j_*W_t(\mathcal{O}_U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{fil}_R j_*W_{s-t}(\mathcal{O}_U) \arrow[r, "V^t"] & \mathrm{fil}_R j_*W_s(\mathcal{O}_U) \arrow[r, "\mathrm{pr}_t"] & \mathrm{fil}_{[R/p^{s-t}]} j_*W_t(\mathcal{O}_U) \arrow[r] & 0, \end{tikzcd} \tag{1.39}$$

where the horizontal lines are exact and the vertical arrows are inclusions. Then this diagram induces the sequence (1.38). By taking stalks of (1.39), the exactness of (1.38) follows. □

Let $R = \sum_{i\in I} n_iD_i$ and $R' = \sum_{i\in I} n_i'D_i$, where $n_i, n_i' \in \mathbf{Z}_{\geq 0}$ and $n_i' \leq n_i$ for every $i \in I$. We consider the morphism

$$\bar{F} \colon \mathrm{gr}_{[R/p]/[R'/p]} j_*W_s(\mathcal{O}_U) \to \mathrm{gr}_{R/R'} j_*W_s(\mathcal{O}_U) \tag{1.40}$$

induced by the Frobenius $F \colon j_*W_s(\mathcal{O}_U) \to j_*W_s(\mathcal{O}_U)$. Since $F^{-1}(\mathrm{fil}_R j_*W_s(\mathcal{O}_U)) = \mathrm{fil}_{[R/p]} j_*W_s(\mathcal{O}_U)$ by Lemma 1.7 (iii) and similarly for R', the morphism (1.40) is injective.

We consider the morphism

$$\overline{F-1}\colon \mathrm{gr}_{[R/p]/[R'/p]} j_*W_s(\mathcal{O}_U) \to \mathrm{gr}_{R/R'} j_*W_s(\mathcal{O}_U) \tag{1.41}$$

induced by $F-1\colon j_*W_s(\mathcal{O}_U) \to j_*W_s(\mathcal{O}_U)$. If $R = R' + D_i$ for some $i \in I$, then the morphisms (1.40) and (1.41) are the same, since $[R/p] \leq R'$ with respect to product order.

Lemma 1.27. *Let A be a smooth ring over k. Let $t_1, \ldots, t_r$ be elements of A such that $(t_1 \cdots t_r = 0)$ is a divisor on $\operatorname{Spec} A$ with simple normal crossings whose irreducible components are $\{(t_i = 0)\}_{i=1}^r$. Let a be an element of $\operatorname{Frac} A$. Assume that $a^p t_1^{n_1} \cdots t_r^{n_r} \in A$, where $n_1, \ldots, n_r$ are integers such that $0 \leq n_i < p$ for $i = 1, \ldots, r$. Then we have $a \in A$.*

Proof. Since $a^p t_1^{n_1} \cdots t_r^{n_r} \in A$, the valuation of $a^p t_1^{n_1} \cdots t_r^{n_r}$ in $A_{(t_i)}$ is non-negative for $i = 1, \ldots, r$. Since the normalized valuation of a^p in $\operatorname{Frac} A_{(t_i)}$ for $i = 1, \ldots, r$ is divided by p and $0 \leq n_i < p$ for $i = 1, \ldots, r$, the valuation of a in $\operatorname{Frac} A_{(t_i)}$ for $i = 1, \ldots, r$ is non-negative. Since A is factorial, we have $A[1/t_1 \cdots t_r] \cap \bigcap_{i=1}^r A_{(t_i)} = A$. Hence the assertion holds. □

Lemma 1.28. *Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be sheaves of abelian groups on X and let $\mathcal{F}_i$, $\mathcal{G}_i$, and $\mathcal{H}_i$ be subsheaves of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ respectively for $i = 1, 2, 3$. Assume that $\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2$, $\mathcal{H}_3 = \mathcal{H}_1 \cap \mathcal{H}_2$, and that $\mathcal{G}_3 \subset \mathcal{G}_1 \cap \mathcal{G}_2$. If we have an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ and if this exact sequence induces the exact sequence $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ for $i = 1, 2, 3$, then we have $\mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_3 \arrow[r] \arrow[d] & \mathcal{G}_3 \arrow[r] \arrow[d] & \mathcal{H}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \oplus \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{G}_1 \oplus \mathcal{G}_2 \arrow[r] \arrow[d] & \mathcal{H}_1 \oplus \mathcal{H}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0,
\end{tikzcd} \tag{1.42}$$

where the bottom vertical arrows are defined by the difference. Since $\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{H}_3 = \mathcal{H}_1 \cap \mathcal{H}_2$, the left and right vertical columns are exact. By applying the snake lemma to the lower two lines, we have $\mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$. □

Proposition 1.29. *Let $R = \sum_{i \in I} n_i D_i$, where $n_i \in \mathbf{Z}_{\geq 0}$ for $i \in I$. Let $s \geq 0$ be an integer and let i be an element of I such that $n_i \geq 1$. We put $R' = R - D_i$. Then we have the exact sequence*

$$0 \to \mathrm{gr}_{[R/p]/[R'/p]} j_*W_s(\mathcal{O}_U) \xrightarrow{\bar{F}} \mathrm{gr}_{R,i} j_*W_s(\mathcal{O}_U) \xrightarrow{\varphi_s^{(R,i)}} \mathrm{gr}_{R,i} j_*\Omega_U^1,$$

where $\mathrm{gr}_{[R/p]/[R'/p]}j_*W_s(\mathcal{O}_U)$ *is* $\mathrm{gr}_{[R/p],i}j_*W_s(\mathcal{O}_U)$ *if* $n_i \in p\mathbf{Z}$ *and* 0 *if* $n_i \notin p\mathbf{Z}$.

Proof. We may assume that $s \geq 1$, $I = \{1, \ldots, r\}$, and that $i = 1$. Let $j_1 \colon \operatorname{Spec} K_1 \to X$ be the canonical morphism. We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{gr}_{[R/p]/[R'/p]}j_*W_s(\mathcal{O}_U) \arrow[r, "\bar{F}"] \arrow[d] & \mathrm{gr}_{R,1}j_*W_s(\mathcal{O}_U) \arrow[r, "\varphi_s^{(R,1)}"] \arrow[d] & \mathrm{gr}_{R,1}j_*\Omega^1_U \arrow[d] \\ 0 \arrow[r] & j_{1*}(\mathrm{gr}_{[n_1/p]/[(n_1-1)/p]}W_s(K_1)) \arrow[r, "\bar{F}"] & j_{1*}\mathrm{gr}_{n_1}W_s(K_1) \arrow[r, "\varphi_s^{(n_1)}"] & j_{1*}\mathrm{gr}_{n_1}\Omega^1_{K_1}, \end{tikzcd} \tag{1.43}$$

where we put $\mathrm{gr}_{[n_1/p]/[(n_1-1)/p]} = \mathrm{fil}_{[n_1/p]}/\mathrm{fil}_{[(n_1-1)/p]}$ and the vertical arrows are inclusions. Since the lower line is exact by Lemma 1.8, it is sufficient to prove that the left square in (1.43) is cartesian.

If $n_1 \notin p\mathbf{Z}$, then the assertion holds since $\mathrm{gr}_{[R/p]/[R'/p]}j_*W_s(\mathcal{O}_U) = 0$ and $\mathrm{gr}_{[n_1/p]/[(n_1-1)/p]}W_s(K_1) = 0$ by Lemma 1.6 (i).

Assume that $n_1 \in p\mathbf{Z}$. Then we have $\mathrm{gr}_{[R/p]/[R'/p]}j_*W_s(\mathcal{O}_U) = \mathrm{gr}_{[R/p],1}j_*W_s(\mathcal{O}_U)$ and $\mathrm{gr}_{[n_1/p]/[(n_1-1)/p]}W_s(K_1) = \mathrm{gr}_{n_1/p}W_s(K_1)$ by Lemma 1.6 (i).

We prove the assertion by induction on s. Suppose that $s = 1$. Since the assertion is local, we may assume that $X = \operatorname{Spec} A$ is affine and that $D_i = (t_i = 0)$ for $i \in I$, where $t_i \in A$ for $i \in I$. Further we may assume that the invertible $\mathcal{O}_{D_1}$-modules $\mathrm{gr}_{R,1}j_*\mathcal{O}_U$ and $\mathrm{gr}_{[R/p],1}j_*\mathcal{O}_U$ are gererated by $c_0 = 1/t_1^{n_1}\cdots t_r^{n_r}$ and $c_1 = 1/t_1^{n_1/p}t_2^{m'_2}\cdots t_r^{m'_r}$ respectively, where $m'_i = [n_i/p]$ for $i \in I - \{1\}$. Let $k(D_1)$ denote the functional field of D_1. We identify $\mathrm{gr}_{n_1}K_1$ with $k(D_1)\cdot c_0$ and $\mathrm{gr}_{n_1/p}K_1$ with $k(D_1)\cdot c_1$.

Let $\bar{a}$ be an element of $k(D_1)$ such that $\bar{F}(\bar{a}c_1) = \bar{a}^p c_1^p \in \mathrm{gr}_{R,1}j_*\mathcal{O}_U$. Since $(\bar{a}^p c_1^p/c_0)\cdot c_0 \in \mathrm{gr}_{R,1}j_*\mathcal{O}_U = \mathcal{O}_{D_1}\cdot c_0$, we have $\bar{a}^p c_1^p/c_0 \in \mathcal{O}_{D_1}$. Since $c_1^p/c_0 = t_2^{n_2-pm'_2}\cdots t_r^{n_r-pm'_r}$ and $0 \leq n_i - pm'_i < p$ for $i \in I - \{1\}$, we have $\bar{a} \in \mathcal{O}_{D_1}$ by Lemma 1.27. Hence we have $\bar{a}c_1 \in \mathcal{O}_{D_1}\cdot c_1 = \mathrm{gr}_{[R/p],1}j_*\mathcal{O}_U$. Hence the assertion holds if $s = 1$.

If $s > 1$, we put $\mathcal{F} = j_{1*}\mathrm{gr}_{n_1}W_{s-1}(K_1)$, $\mathcal{F}_1 = \mathrm{gr}_{R,1}j_*W_{s-1}(\mathcal{O}_U)$, $\mathcal{F}_2 = j_{1*}\mathrm{gr}_{n_1/p}W_{s-1}(K_1)$, and $\mathcal{F}_3 = \mathrm{gr}_{[R/p],1}j_*W_{s-1}(\mathcal{O}_U)$. Since the canonical morphisms $\mathcal{F}_1 \to \mathcal{F}$ and $\mathcal{F}_3 \to \mathcal{F}_2$ are injective and both $\bar{F} \colon \mathcal{F}_3 \to \mathcal{F}_1$ and $\bar{F} \colon \mathcal{F}_2 \to \mathcal{F}$ are injective, we may identify $\mathcal{F}_i$ with a subsheaf of $\mathcal{F}$ for $i = 1, 2, 3$. We also put $\mathcal{G} = j_{1*}\mathrm{gr}_{n_1}W_s(K_1)$, $\mathcal{G}_1 = \mathrm{gr}_{R,1}j_*W_s(\mathcal{O}_U)$, $\mathcal{G}_2 = j_{1*}\mathrm{gr}_{n_1/p}W_s(K_1)$, and $\mathcal{G}_3 = \mathrm{gr}_{[R/p],1}j_*W_s(\mathcal{O}_U)$. We further put $\mathcal{H} = j_{1*}(\mathrm{gr}_n^{(s-1)}K_1)$, $\mathcal{H}_1 = \mathrm{gr}_{[R/p^{s-1}]/[R'/p^{s-1}]}j_*\mathcal{O}_U$, $\mathcal{H}_2 = j_{1*}(\mathrm{gr}_{n_1/p}^{(s-1)}K_1)$, and $\mathcal{H}_3 = \mathrm{gr}_{[R/p^s]/[R'/p^s]}j_*\mathcal{O}_U$. Similarly as $\mathcal{F}_i$, we may identify $\mathcal{G}_i$ and $\mathcal{H}_i$ with subsheaves of $\mathcal{G}$ and $\mathcal{H}$ respectively for $i = 1, 2, 3$.

By the induction hypothesis, we have $\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2$. If $n_1 \notin p^s\mathbf{Z}$, then $\mathcal{H}_2 = \mathcal{H}_3 = 0$ by Lemma 1.6 (i) and (1.11). If $n_1 \in p^s\mathbf{Z}$, then we have $\mathcal{H}_3 = \mathcal{H}_1 \cap \mathcal{H}_2$ by Lemma 1.6 (i), (1.11), and the induction hypothesis. By the commutativity of (1.43), we have $\mathcal{G}_3 \subset \mathcal{G}_1 \cap \mathcal{G}_2$. Since exact sequences in Lemma 1.11 and Lemma 1.26 in the case where $t = 1$ are compatible with the inclusions of sheaves above, the assertion holds by Lemma 1.28. ☐

LEMMA 1.30. *Let $f \colon \mathcal{F} \to \mathcal{G}$ be a surjection of sheaves of abelian groups on X. Let $g \colon \mathcal{G} \to \mathcal{H}$ be a morphism of sheaves of abelian groups on X. We put $\Gamma = (\mathbf{Z}_{\geq 0})^r$, where $r > 0$ is an integer, and let $1_i \in \Gamma$ be the element whose i-th component is 1 and the others are 0 for $i = 1, \ldots, r$. Let $\{\mathrm{fil}_n \mathcal{F}\}_{n \in \Gamma}$ and $\{\mathrm{fil}_n \mathcal{H}\}_{n \in \Gamma}$ be increasing filtrations of $\mathcal{F}$ and $\mathcal{H}$ respectively with respect to product order. Assume that $\bigcup_{n \in \Gamma} \mathrm{fil}_n \mathcal{F} = \mathcal{F}$ and $\bigcup_{n \in \Gamma} \mathrm{fil}_n \mathcal{H} = \mathcal{H}$. We put $\mathrm{fil}_n \mathcal{G} = f(\mathrm{fil}_n \mathcal{F})$ for $n \in \Gamma$, which define an increasing filtration of $\mathcal{G}$. If $g(\mathrm{fil}_n \mathcal{G}) \subset \mathrm{fil}_n \mathcal{H}$ for every $n \in \Gamma$ and if the morphism $\mathrm{fil}_{n+1_i} \mathcal{G}/\mathrm{fil}_n \mathcal{G} \to \mathrm{fil}_{n+1_i} \mathcal{H}/\mathrm{fil}_n \mathcal{H}$ induced by g is injective for every $n \in \Gamma$ and $i = 1, \ldots, r$, then we have $\mathrm{fil}_n \mathcal{G} = g^{-1}(\mathrm{fil}_n \mathcal{H})$ for every $n \in \Gamma$.*

Proof. Let $n \in \Gamma$ be an element. We prove that the morphism $\mathcal{G}/\mathrm{fil}_n \mathcal{G} \to \mathcal{H}/\mathrm{fil}_n \mathcal{H}$ is injective. Since $\mathcal{F} = \bigcup_{n \in \Gamma} \mathrm{fil}_n \mathcal{F}$ and f is surjective, we have $\mathcal{G} = \bigcup_{n \in \Gamma} \mathrm{fil}_n \mathcal{G}$ and hence $\mathcal{G}/\mathrm{fil}_n \mathcal{G} = \varinjlim_{n'} \mathrm{fil}_{n'} \mathcal{G}/\mathrm{fil}_n \mathcal{G}$, where n' runs through the elements of Γ greater than n with respect to product order. Since $\mathcal{H} = \bigcup_{n \in \Gamma} \mathrm{fil}_n \mathcal{H}$, we have $\mathcal{H}/\mathrm{fil}_n \mathcal{H} = \varinjlim_{n'} \mathrm{fil}_{n'} \mathcal{H}/\mathrm{fil}_n \mathcal{H}$, where n' runs through the elements of Γ greater than n. Hence it is sufficient to prove that $\mathrm{fil}_{n'} \mathcal{G}/\mathrm{fil}_n \mathcal{G} \to \mathrm{fil}_{n'} \mathcal{H}/\mathrm{fil}_n \mathcal{H}$ is injective for every $n' \in \Gamma$ such that $n' \geq n$. We prove this assertion by induction on n'.
If $n' = n$, the assertion holds since $\mathrm{fil}_{n'} \mathcal{G}/\mathrm{fil}_n \mathcal{G} = 0$ and $\mathrm{fil}_{n'} \mathcal{H}/\mathrm{fil}_n \mathcal{H} = 0$. For $n' > n$, take i such that $n' - 1_i \geq n$. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{fil}_{n'-1_i} \mathcal{G}/\mathrm{fil}_n \mathcal{G} \arrow[r] \arrow[d] & \mathrm{fil}_{n'} \mathcal{G}/\mathrm{fil}_n \mathcal{G} \arrow[r] \arrow[d] & \mathrm{fil}_{n'} \mathcal{G}/\mathrm{fil}_{n'-1_i} \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{fil}_{n'-1_i} \mathcal{H}/\mathrm{fil}_n \mathcal{H} \arrow[r] & \mathrm{fil}_{n'} \mathcal{H}/\mathrm{fil}_n \mathcal{H} \arrow[r] & \mathrm{fil}_{n'} \mathcal{H}/\mathrm{fil}_{n'-1_i} \mathcal{H} \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal lines are exact. By the induction hypothesis, the left vertical arrow is injective. Since the right vertical arrow is injective, the middle vertical arrow is injective. Hence the assertion holds. ☐

PROPOSITION 1.31. *Let $R = \sum_{i \in I} n_i D_i$, where $n_i \in \mathbf{Z}_{\geq 0}$ for $i \in I$. Let $j_i \colon \operatorname{Spec} K_i \to X$ be the canonical morphism for $i \in I$.*

(i) *The subsheaf $\mathrm{fil}_R R^1(\epsilon \circ j)_* \mathbf{Z}/p^s \mathbf{Z}$ is equal to the pull-back of $\bigoplus_{i \in I} j_{i*} \mathrm{fil}_{n_i} H^1(K_i, \mathbf{Q}/\mathbf{Z})$ by the morphism $R^1(\epsilon \circ j)_* \mathbf{Z}/p^s \mathbf{Z} \to \bigoplus_{i \in I} j_{i*} H^1(K_i, \mathbf{Q}/\mathbf{Z})$.*

(ii) *Let $R' = \sum_{i \in I} n'_i D_i$, where $n'_i \in \mathbf{Z}_{\geq 0}$ and $n_i - 1 \leq n'_i \leq n_i$ for $i \in I$. Then there exists a unique injection $\phi_s^{(R/R')} \colon \mathrm{gr}_{R/R'} R^1(\epsilon \circ j)_* \mathbf{Z}/p^s \mathbf{Z} \to$*

$\mathrm{gr}_{R/R'}j_*\Omega^1_U$ *such that the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{gr}_{R/R'}j_*W_s(\mathcal{O}_U) \arrow[rr, "\varphi_s^{(R/R')}"] \arrow[dr, "\delta_s^{(R/R')}"'] & & \mathrm{gr}_{R/R'}j_*\Omega^1_U \\ & \mathrm{gr}_{R/R'}R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}. \arrow[ur, "\phi_s^{(R/R')}"'] & \end{tikzcd} \tag{1.44}$$

Proof. Let i be an element of I such that $n_i \geq 1$. Since the kernel of $\delta_s^{(R,i)}$ is the image of $\overline{F-1}$ (1.41) and the morphisms $\bar{F}$ (1.40) and $\overline{F-1}$ (1.41) are the same, the kernel of $\delta_s^{(R,i)}$ is equal to the kernel of $\varphi_s^{(R,i)}$ by Proposition 1.29. Since $\delta_s^{(R,i)}$ is surjective, there exists a unique injection $\phi_s^{(R,i)}\colon \mathrm{gr}_{R,i}R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z} \to \mathrm{gr}_{R,i}j_*\Omega^1_U$ such that the diagram (1.44) for $R' = R - D_i$ is commutative.
(i) Let i be an element of I such that $n_i \geq 1$. We consider the commutative diagram

$$\begin{tikzcd} \mathrm{gr}_{R,i}R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z} \arrow[r] \arrow[d, "\phi_s^{(R,i)}"'] & j_{i*}\mathrm{gr}_{n_i}H^1(K_i,\mathbf{Q}/\mathbf{Z}) \arrow[d, "\phi^{(n_i)}"] \\ \mathrm{gr}_{R,i}j_*\Omega^1_U \arrow[r] & j_{i*}\mathrm{gr}_{n_i}\Omega^1_{K_i}, \end{tikzcd}$$

where the lower horizontal arrow is the inclusion and $\phi^{(n_i)}$ is as in (1.8). Since the left vertical arrow is injective as proved above, the upper horizontal arrow is injective. Hence the assertion holds by applying Lemma 1.30 to the case where $\mathcal{F} = j_*W_s(\mathcal{O}_U)$, $\mathcal{G} = R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}$, and $\mathcal{H} = \bigoplus_{i\in I} j_{i*}H^1(K_i,\mathbf{Q}/\mathbf{Z})$.
(ii) Let J be the subset of I consisting of $i \in I$ such that $n_i' \neq n_i$. We consider the commutative diagram

$$\begin{tikzcd} \mathrm{gr}_{R/R'}j_*W_s(\mathcal{O}_U) \arrow[rr, "\varphi_s^{(R/R')}"] \arrow[d, "\delta_s^{(R/R')}"'] & & \mathrm{gr}_{R/R'}j_*\Omega^1_U \arrow[d] \\ \mathrm{gr}_{R/R'}R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z} \arrow[r] & \bigoplus_{i\in J} j_{i*}\mathrm{gr}_{n_i}H^1(K_i,\mathbf{Q}/\mathbf{Z}) \arrow[r, "\oplus\phi^{(n_i)}"] & \bigoplus_{i\in J} j_{i*}\mathrm{gr}_{n_i}\Omega^1_{K_i}, \end{tikzcd}$$

where $\phi^{(n_i)}$ is as in (1.8) for $i \in J$. By (i), the left lower horizontal arrow is injective. Since $\mathrm{gr}_{n_i}\Omega^1_{K_i}$ is the stalk of $\mathrm{gr}_{R/R'}j_*\Omega^1_U$ at the generic point of D_i for $i \in J$, the kernel of the canonical morphism $\mathrm{fil}_R j_*\Omega^1_U \to \bigoplus_{i\in J} j_{i*}\mathrm{gr}_{n_i}\Omega^1_{K_i}$ is the intersection of $\mathrm{fil}_{R-D_i}j_*\Omega^1_U$ for $i \in J$. Hence the right vertical arrow is injective. Since the right lower horizontal arrow is injective, the kernel of $\varphi_s^{(R/R')}$ is equal to that of $\delta_s^{(R/R')}$. Since $\delta_s^{(R/R')}$ is surjective, the assertion holds. ☐

Definition 1.32. Let χ be an element of $H^1_{\text{ét}}(U,\mathbf{Q}/\mathbf{Z})$. We define the *Swan conductor divisor* R_χ of χ by $R_\chi = \sum_{i\in I}\mathrm{sw}(\chi|_{K_i})D_i$.

Definition 1.33. Let χ be an element of $H^1_{\text{ét}}(U, \mathbf{Q}/\mathbf{Z})$. Assume that $\mathrm{sw}(\chi|_{K_i}) > 0$ for some $i \in I$. Let p^s be the order of the p-part of χ. We put $Z = \mathrm{Supp}(R_\chi)$. We define the *refined Swan conductor* $\mathrm{rsw}(\chi)$ of χ to be the image of the p-part of χ by the composition

$$\begin{aligned}\Gamma(X, \mathrm{fil}_{R_\chi} R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}) &\to \Gamma(X, \mathrm{gr}_{R_\chi/(R_\chi - Z)} R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}) \\ &\xrightarrow{\phi_s^{(R_\chi/(R_\chi-Z))}(X)} \Gamma(X, \mathrm{gr}_{R_\chi/(R_\chi-Z)} j_*\Omega^1_U) = \Gamma(Z, \Omega^1_X(\log D)(R_\chi)\otimes_{\mathcal{O}_X}\mathcal{O}_Z).\end{aligned}$$

By the construction of $\phi_s^{(R_\chi/(R_\chi-Z))}$, the germ $\mathrm{rsw}(\chi)_{\mathfrak{p}_i}$ of $\mathrm{rsw}(\chi)$ at the generic point $\mathfrak{p}_i$ of D_i contained in Z is equal to $\mathrm{rsw}(\chi|_{K_i})$. This implies that $\mathrm{rsw}(\chi)$ in Definition 1.33 is none other than the refined Swan conductor of χ in the sense of [K2, (3.4.2)].

1.4 Sheafification: non-logarithmic case

We recall the definition of the radicial covering $S^{1/p}$ of a scheme S over a perfect field k of characteristic $p > 0$. We consider the commutative diagram

$$\begin{tikzcd} S^{1/p} \arrow[r] \arrow[d] & S \arrow[r, "F_S"] \arrow[d] & S \arrow[d] \\ \mathrm{Spec}\,k \arrow[r, "F_k^{-1}"'] & \mathrm{Spec}\,k \arrow[r, "F_k"'] & \mathrm{Spec}\,k, \end{tikzcd}$$

where the left square is the base change over k by the inverse F_k^{-1} of F_k. The symbols F_S and F_k denote the absolute Frobenius of S and $\mathrm{Spec}\,k$ respectively. We define the *radicial covering* $S^{1/p} \to S$ as the composition of morphisms in the upper line.

We keep the notation in Subsection 1.3.

Definition 1.34. Let $R = \sum_{i\in I} n_i D_i$, where $n_i \in \mathbf{Z}_{\geq 1}$ for $i \in I$, and let $j_i\colon \mathrm{Spec}\,K_i \to X$ denote the canonical morphism for $i \in I$. Let $r \geq 0$ be an integer.

(i) We define subsheaves $\mathrm{fil}'^{(r)}_R j_*W_s(\mathcal{O}_U)$ and $\mathrm{fil}''^{(r)}_R j_*W_s(\mathcal{O}_U)$ of Zariski sheaf $j_*W_s(\mathcal{O}_U)$ to be the pull-back of $\bigoplus_{i\in I} j_{i*}\mathrm{fil}'^{(r)}_{n_i} W_s(K_i)$ and $\bigoplus_{i\in I} j_{i*}\mathrm{fil}''^{(r)}_{n_i} W_s(K_i)$ by the morphism $j_*W_s(\mathcal{O}_U) \to \bigoplus_{i\in I} j_{i*}W_s(K_i)$ respectively.

If $r = 0$, then we simply write $\mathrm{fil}'_R j_*W_s(\mathcal{O}_U)$ and $\mathrm{fil}''_R j_*W_s(\mathcal{O}_U)$ for $\mathrm{fil}'^{(0)}_R j_*W_s(\mathcal{O}_U)$ and $\mathrm{fil}''^{(0)}_R j_*W_s(\mathcal{O}_U)$ respectively.

(ii) We define a subsheaf $\mathrm{fil}'_R R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}$ of $R^1(\epsilon\circ j)_*\mathbf{Z}/p^s\mathbf{Z}$ to be the image of $\mathrm{fil}'_R j_*W_s(\mathcal{O}_U)$ by δ_s (1.33).

(iii) We define a subsheaf $\mathrm{fil}'_R j_*\Omega^1_U$ of $j_*\Omega^1_U$ to be $\Omega^1_X(R)$.

Similarly as in the logarithmic case, we consider the morphism $-F^{s-1}d$ (1.34). Let $R=\sum_{i\in I}n_iD_i$, where $n_i\in\mathbf{Z}_{\geq 1}$ for $i\in I$. Then $-F^{s-1}d$ (1.34) induces the morphism

$$\mathrm{fil}'_R j_*W_s(\mathcal{O}_U)\to \mathrm{fil}'_R j_*\Omega^1_U.$$

For $R'=\sum_{i\in I}n'_iD_i$, where $n'_i\in\mathbf{Z}_{\geq 1}$ such that $n'_i\leq n_i$ for $i\in I$, we put $\mathrm{gr}'_{R/R'}=\mathrm{fil}'_R/\mathrm{fil}'_{R'}$. Then the morphism $-F^{s-1}d$ (1.34) induces the morphism

$$\varphi'^{(R/R')}_s\colon \mathrm{gr}'_{R/R'}j_*W_s(\mathcal{O}_U)\to \mathrm{gr}'_{R/R'}j_*\Omega^1_U. \tag{1.45}$$

For $R'=\sum_{i\in I}n'_iD_i$, where $n'_i\in\mathbf{Z}_{\geq 1}$ such that $n_i-1\leq n'_i\leq n_i$ for $i\in I$, we put $D^{(R/R')}=R-R'\subset D$. If $p\neq 2$ or there is no $i\in I$ such that $(n_i,n'_i)=(2,1)$, let $\tilde{\varphi}'^{(R/R')}_s\colon \mathrm{gr}'_{R/R'}j_*W_s(\mathcal{O}_U)\to \mathrm{gr}'_{R/R'}j_*\Omega^1_U\otimes_{\mathcal{O}_{D^{(R/R')}}}\mathcal{O}_{D^{(R/R')1/2}}$ be the composition

$$\mathrm{gr}'_{R/R'}j_*W_s(\mathcal{O}_U)\xrightarrow{\varphi'^{(R/R')}_s}\mathrm{gr}'_{R/R'}j_*\Omega^1_U\to \mathrm{gr}'_{R/R'}j_*\Omega^1_U\otimes_{\mathcal{O}_{D^{(R/R')}}}\mathcal{O}_{D^{(R/R')1/2}}.$$

Otherwise, as in the proof of Proposition 1.17 (i), there exists a unique morphism

$$\tilde{\varphi}'^{(R/R')}_s\colon \mathrm{gr}'_{R/R'}j_*W_s(\mathcal{O}_U)\to \mathrm{gr}'_{R/R'}j_*\Omega^1_U\otimes_{\mathcal{O}_{D^{(R/R')}}}\mathcal{O}_{D^{(R/R')1/p}} \tag{1.46}$$

such that locally $\tilde{\varphi}'^{(R/R')}_s(\bar{a})=-\sum_{i=0}^{s-1}a_i^{p^i-1}da_i+\sum_{(n_i,n'_i)=(2,1)}\sqrt{t_i^2a_0}\,dt_i/t_i^2$ for every $\bar{a}\in\mathrm{gr}'_{R/R'}j_*W_s(\mathcal{O}_U)$ whose lift is $a=(a_{s-1},\dots,a_0)\in\mathrm{fil}'_Rj_*W_s(\mathcal{O}_U)$ and for every local equation t_i of D_i for $i\in I$ such that $(n_i,n'_i)=(2,1)$.
If $R=R'+D_i$ for some $i\in I$, then we simply write $\mathrm{gr}'_{R,i}$ for $\mathrm{gr}'_{R/R'}$, $\tilde{\varphi}'^{(R,i)}$ for $\tilde{\varphi}'^{(R/R')}$, and similarly for $\mathrm{gr}''_{R/R'}$, $\mathrm{gr}'^{(r)}_{R/R'}$, and $\mathrm{gr}''^{(r)}_{R/R'}$.

Lemma 1.35. *Let $R=\sum_{i\in I}n_iD_i$, where $n_i\in\mathbf{Z}_{\geq 1}$ for $i\in I$, and let $r\geq 0$ be an integer. Then we have $\mathrm{fil}''^{(r)}_Rj_*W_s(\mathcal{O}_U)=(F-1)^{-1}(\mathrm{fil}'^{(r)}_Rj_*W_s(\mathcal{O}_U))$. Especially, we have $\mathrm{fil}''_Rj_*W_s(\mathcal{O}_U)=(F-1)^{-1}(\mathrm{fil}'_Rj_*W_s(\mathcal{O}_U))$.*

Proof. Let $j_i\colon \operatorname{Spec}K_i\to X$ be the canonical morphism for $i\in I$. Since $F-1$ is compatible with the canonical morphism $j_*W_s(\mathcal{O}_U)\to\bigoplus_{i\in I}j_{i*}W_s(K_i)$, the assertions hold by Lemma 1.20 (v). □

Lemma 1.36. *Let $r\geq 0$ be an integer. Let $R=\sum_{i\in I}n_iD_i$ and $R'=\sum_{i\in I}n'_iD_i$, where $n_i,n'_i\in\mathbf{Z}_{\geq 1}$ such that $n'_i=n_i/p^r$ if $n_i\in p^{r+1}\mathbf{Z}$ and $n'_i=[(n_i-1)/p^r]$ if $n_i\notin p^{r+1}\mathbf{Z}$ for every $i\in I$.*

(i) $\mathrm{fil}'^{(r)}_Rj_*\mathcal{O}_U=\mathrm{fil}_{R'}j_*\mathcal{O}_U$.

(ii) $\mathrm{fil}''^{(r)}_Rj_*\mathcal{O}_U=\mathrm{fil}_{[R/p^{r+1}]}j_*\mathcal{O}_U$.

Proof. The assertions hold by Lemma 1.23. □

COROLLARY 1.37. *Let the notation be as in Lemma 1.36. Let i be an element of I such that $n_i \geq 2$.*

(i) Assume that $r \geq 1$. Then $\mathrm{gr}'^{(r)}_{R,i} j_*\mathcal{O}_U = \mathrm{gr}_{R',i} j_*\mathcal{O}_U$ if $n_i \in p^{r+1}\mathbf{Z}$ or $\mathrm{ord}_p(n_i - 1) = r$, and $\mathrm{gr}'^{(r)}_{R,i} j_*\mathcal{O}_U = 0$ otherwise.

(ii) $\mathrm{gr}''^{(r)}_{R,i} j_*\mathcal{O}_U = \mathrm{gr}_{[R/p^{r+1}],i} j_*\mathcal{O}_U = \mathrm{gr}_{[R'/p],i} j_*\mathcal{O}_U$ if $n_i \in p^{r+1}\mathbf{Z}$, and $\mathrm{gr}''^{(r)}_{R,i} j_*\mathcal{O}_U = 0$ if $n_i \notin p^{r+1}\mathbf{Z}$.

Proof. Since $[R/p^{r+1}] = [R'/p]$ by Lemma 1.6, the assertions hold by Corollary 1.24 and Lemma 1.36. □

Let $R = \sum_{i \in I} n_i D_i$ and $R' = \sum_{i \in I} n'_i D_i$, where $n_i, n'_i \in \mathbf{Z}_{\geq 1}$ and $n'_i \leq n_i$ for every $i \in I$. Let $0 \leq t \leq s$ be integers. Since we have $\mathrm{pr}_t(\mathrm{fil}'_R j_* W_s(\mathcal{O}_U)) = \mathrm{fil}'^{(s-t)}_R j_* W_t(\mathcal{O}_U)$ by Lemma 1.20 (i), we have the exact sequence

$$0 \to \mathrm{fil}'_R j_* W_{s-t}(\mathcal{O}_U) \xrightarrow{V^t} \mathrm{fil}'_R j_* W_s(\mathcal{O}_U) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}'^{(s-t)}_R j_* W_t(\mathcal{O}_U) \to 0. \quad (1.47)$$

Similarly, since $\mathrm{pr}_t(\mathrm{fil}''_R j_* W_s(\mathcal{O}_U))$ is $\mathrm{fil}''^{(s-t)}_R j_* W_t(\mathcal{O}_U)$ by Lemma 1.20 (iii), we have the exact sequence

$$0 \to \mathrm{fil}''_R j_* W_{s-t}(\mathcal{O}_U) \xrightarrow{V^t} \mathrm{fil}''_R j_* W_s(\mathcal{O}_U) \xrightarrow{\mathrm{pr}_t} \mathrm{fil}''^{(s-t)}_R j_* W_t(\mathcal{O}_U) \to 0. \quad (1.48)$$

LEMMA 1.38. *Let $R = \sum_{i \in I} n_i D_i$ and $R' = \sum_{i \in I} n'_i D_i$, where $n_i, n'_i \in \mathbf{Z}_{\geq 1}$ and $n_i - 1 \leq n'_i \leq n_i$ for every $i \in I$. Let $0 \leq t \leq s$ be integers.*

(i) The exact sequence (1.47) induces the exact sequence

$$0 \to \mathrm{gr}'_{R/R'} j_* W_{s-t}(\mathcal{O}_U) \xrightarrow{\bar{V}^t} \mathrm{gr}'_{R/R'} j_* W_s(\mathcal{O}_U) \xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}'^{(s-t)}_{R/R'} j_* W_t(\mathcal{O}_U) \to 0.$$

(ii) The exact sequence (1.48) induces the exact sequence

$$0 \to \mathrm{gr}''_{R/R'} j_* W_{s-t}(\mathcal{O}_U) \xrightarrow{\bar{V}^t} \mathrm{gr}''_{R/R'} j_* W_s(\mathcal{O}_U) \xrightarrow{\overline{\mathrm{pr}}_t} \mathrm{gr}''^{(s-t)}_{R/R'} j_* W_t(\mathcal{O}_U) \to 0.$$

Proof. The assertions hold similarly as the proof of Lemma 1.26. □

Let $r \geq 0$ be an integer. By Lemma 1.35, the morphism $F - 1 \colon j_* W_s(\mathcal{O}_U) \to j_* W_s(\mathcal{O}_U)$ induces the injection

$$\overline{F-1} \colon \mathrm{gr}''^{(r)}_{R/R'} j_* W_s(\mathcal{O}_U) \to \mathrm{gr}'^{(r)}_{R/R'} j_* W_s(\mathcal{O}_U).$$

Especially, the morphism $F - 1$ induces the injection

$$\overline{F-1} \colon \mathrm{gr}''_{R/R'} j_* W_s(\mathcal{O}_U) \to \mathrm{gr}'_{R/R'} j_* W_s(\mathcal{O}_U).$$

Lemma 1.39. *Let $R = \sum_{i \in I} n_i D_i$, where $n_i \in \mathbf{Z}_{\geq 1}$ for $i \in I$. Let $s \geq 0$ be an integer and let i be an element of I such that $n_i \geq 2$. Then we have the exact sequence*

$$0 \to \mathrm{gr}''_{R,i} j_* W_s(\mathcal{O}_U) \xrightarrow{\overline{F-1}} \mathrm{gr}'_{R,i} j_* W_s(\mathcal{O}_U) \xrightarrow{\tilde{\varphi}_s'^{(R,i)}} \mathrm{gr}'_{R,i} j_* \Omega^1_U \otimes_{\mathcal{O}_{D_i}} \mathcal{O}_{D_i^{1/p}}.$$

Proof. We may assume that $s \geq 1$, $I = \{1, \dots, r\}$, and that $i = 1$. Let $j_1 \colon \operatorname{Spec} K_1 \to X$ be the canonical morphism. We consider the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathrm{gr}''_{R,1} j_* W_s(\mathcal{O}_U) & \xrightarrow{\overline{F-1}} & \mathrm{gr}'_{R,1} j_* W_s(\mathcal{O}_U) & \xrightarrow{\varphi_s'^{(R,1)}} & \mathrm{gr}'_{R,1} j_* \Omega^1_U \otimes_{\mathcal{O}_{D_1}} \mathcal{O}_{D_1^{1/p}} \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & j_{1*}\mathrm{gr}''_{n_1} W_s(K_1) & \xrightarrow{\overline{F-1}} & j_{1*}\mathrm{gr}'_{n_1} W_s(K_1) & \xrightarrow{\varphi_s'^{(n_1)}} & j_{1*}(\mathrm{gr}'_{n_1} \Omega^1_{K_1} \otimes_{F_{K_1}} F_{K_1}^{1/p}), \end{array} \tag{1.49}$$

where F_{K_1} denotes the residue field of K_1 and the vertical arrows are canonical injections. By Lemma 1.22, the lower horizontal line is exact. Hence it is sufficient to prove that the left square in (1.49) is cartesian.

We prove the assertion by induction on s. Suppose that $s = 1$. If $n_1 \notin p\mathbf{Z}$, then we have $\mathrm{gr}''_{n_1} W_s(K_1) = 0$ and $\mathrm{gr}''_{R,1} j_* \mathcal{O}_U = 0$ by Corollary 1.24 (ii) and Corollary 1.37 (ii). Hence the assertion holds in this case.

Assume that $n_1 \in p\mathbf{Z}$. By (1.15), we have $\mathrm{gr}'_{n_1} K_1 = \mathrm{fil}_{n_1} K_1 / \mathrm{fil}_{n_1 - 2} K_1$. By Corollary 1.24 (ii), we have $\mathrm{gr}''_{n_1} K_1 = \mathrm{gr}_{n_1/p} K_1$. Since the assertion is a local property, we may assume that $X = \operatorname{Spec} A$ is affine and that $D_i = (t_i = 0)$ for $i \in I$, where $t_i \in A$ for $i \in I$. Further we may assume that the invertible $\mathcal{O}_{2D_1}$-module $\mathrm{gr}'_{R,1} j_* \mathcal{O}_U$ is generated by $c_0 = 1/t_1^{n_1} \cdots t_r^{n_r}$, and that the invertible $\mathcal{O}_{D_1}$-module $\mathrm{gr}''_{R,1} j_* \mathcal{O}_U$ is generated by $c_1 = 1/t_1^{n_1/p} t_2^{m'_2} \cdots t_r^{m'_r}$, where $m'_i = [n_i/p]$ for $i \in I - \{1\}$. Let $R(2D_1)$ denote the stalk of $\mathcal{O}_{2D_1}$ at the generic point of $2D_1$ and let $k(D_1)$ denote the functional field of D_1. Then we may identify $\mathrm{gr}'_{n_1} K_1$ with $R(2D_1) \cdot c_0$ and $\mathrm{gr}''_{n_1} K_1$ with $k(D_1) \cdot c_1$.

Let $\bar{a}$ be an element of $k(D_1)$ such that $\overline{(F-1)}(\bar{a} c_1) \in \mathrm{gr}'_{R,1} j_* \mathcal{O}_U$. Since we have $\overline{(F-1)}(\bar{a} c_1) = ((\bar{a}^p c_1^p - \bar{a} c_1)/c_0) \cdot c_0 \in \mathrm{gr}'_{R,1} j_* \mathcal{O}_U = \mathcal{O}_{2D_1} \cdot c_0$, we have $(\bar{a}^p c_1^p - \bar{a} c_1)/c_0 \in \mathcal{O}_{2D_1}$. Since $c_1/c_0 = t_1^{n_1 - n_1/p} t_2^{n_2 - m'_2} \cdots t_r^{n_r - m'_r}$ and $n_1 - n_1/p \geq 1$, we have $(\bar{a}^p c_1^p - \bar{a} c_1)/c_0 = \bar{a}^p c_1^p / c_0$ in $\mathcal{O}_{D_1}$. Since $c_1^p / c_0 = t_2^{n_2 - pm'_2} \cdots t_r^{n_r - pm'_r}$ and $0 \leq n_i - pm'_i < p$ for $i \in I - \{1\}$, we have $\bar{a} \in \mathcal{O}_{D_1}$ by Lemma 1.27. Hence we have $\bar{a} c_1 \in \mathcal{O}_{D_1} \cdot c_1 = \mathrm{gr}''_{R,1} j_* \mathcal{O}_U$. Thus the assertion holds if $s = 1$.

If $s > 1$, we put $\mathcal{F} = j_{1*}\mathrm{gr}'_{n_1} W_{s-1}(K_1)$, $\mathcal{F}_1 = \mathrm{gr}'_{R,1} j_* W_{s-1}(\mathcal{O}_U)$, $\mathcal{F}_2 = j_{1*}\mathrm{gr}''_{n_1} W_{s-1}(K_1)$, and $\mathcal{F}_3 = \mathrm{gr}''_{R,1} j_* W_{s-1}(\mathcal{O}_U)$. Since the canonical morphisms $\mathcal{F}_1 \to \mathcal{F}$ and $\mathcal{F}_3 \to \mathcal{F}_2$ are injective and both $\overline{F-1} \colon \mathcal{F}_3 \to \mathcal{F}_1$ and $\overline{F-1} \colon \mathcal{F}_2 \to \mathcal{F}$ are injective, we may identify $\mathcal{F}_i$ with a subsheaf of $\mathcal{F}$ for $i = 1, 2, 3$. We also put $\mathcal{G} = j_{1*}\mathrm{gr}'_{n_1} W_s(K_1)$, $\mathcal{G}_1 = \mathrm{gr}'_{R,1} j_* W_s(\mathcal{O}_U)$, $\mathcal{G}_2 = j_{1*}\mathrm{gr}''_{n_1} W_s(K_1)$, and

$\mathcal{G}_3 = \mathrm{gr}''_{R,1} j_* W_s(\mathcal{O}_U)$. Further we put $\mathcal{H} = j_{1*}\mathrm{gr}'^{(s-1)}_{n_1} K_1$, $\mathcal{H}_1 = \mathrm{gr}'^{(s-1)}_{R,1} j_*\mathcal{O}_U$, $\mathcal{H}_2 = j_{1*}\mathrm{gr}''^{(s-1)}_{n_1} K_1$, and $\mathcal{H}_3 = \mathrm{gr}''^{(s-1)}_{R,1} j_*\mathcal{O}_U$. Similarly as $\mathcal{F}_i$, we may identify $\mathcal{G}_i$ and $\mathcal{H}_i$ with subsheaves of $\mathcal{G}$ and $\mathcal{H}$ respectively for $i = 1, 2, 3$.
By the induction hypothesis, we have $\mathcal{F}_3 = \mathcal{F}_1 \cap \mathcal{F}_2$. If $n_1 \notin p^s\mathbf{Z}$, then we have $\mathcal{H}_2 = \mathcal{H}_3 = 0$ by Corollary 1.24 (ii) and Corollary 1.37 (ii). If $n_1 \in p^s\mathbf{Z}$, then we have $\mathcal{H}_3 = \mathcal{H}_1 \cap \mathcal{H}_2$ by Corollary 1.24, Corollary 1.37, and the case where $s = 1$ in the proof of Proposition 1.29. By the commutativity of (1.49), we have $\mathcal{G}_3 \subset \mathcal{G}_1 \cap \mathcal{G}_2$. Since exact sequences in Corollary 1.21 and Lemma 1.38 in the case where $t = 1$ are compatible with the inclusions of sheaves above, the assertion holds by Lemma 1.28. □

Proposition 1.40. *Let $R = \sum_{i\in I} n_i D_i$, where $n_i \in \mathbf{Z}_{\geq 1}$ for $i \in I$. Let $j_i \colon \operatorname{Spec} K_i \to X$ be the canonical morphism for $i \in I$.*

(i) The subsheaf $\mathrm{fil}'_R R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z}$ is equal to the pull-back of $\bigoplus_{i\in I} j_{i*}\mathrm{fil}'_{n_i} H^1(K_i, \mathbf{Q}/\mathbf{Z})$ by the morphism $R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z} \to \bigoplus_{i\in I} j_{i*}H^1(K_i, \mathbf{Q}/\mathbf{Z})$.

(ii) Let $R' = \sum_{i\in I} n'_i D_i$, where $n'_i \in \mathbf{Z}_{\geq 1}$ such that $n_i - 1 \leq n'_i \leq n_i$ for $i \in I$. Then there exists a unique injection $\phi'^{(R/R')}_s \colon \mathrm{gr}'_{R/R'} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z} \to (\mathrm{gr}'_{R/R'} j_*\Omega^1_U)|_{D^{(R/R')1/p}} = \mathrm{gr}'_{R/R'} j_*\Omega^1_U \otimes_{\mathcal{O}_{D^{(R/R')}}} \mathcal{O}_{D^{(R/R')1/p}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{gr}'_{R/R'} j_* W_s(\mathcal{O}_U) \arrow[rr, "\tilde{\varphi}'^{(R/R')}_s"] \arrow[dr, "\delta'^{(R/R')}_s"'] & & (\mathrm{gr}'_{R/R'} j_*\Omega^1_U)|_{D^{(R/R')1/p}} \\
& \mathrm{gr}'_{R/R'} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z}. \arrow[ur, "\phi'^{(R/R')}_s"'] &
\end{tikzcd} \tag{1.50}$$

Proof. Let i be an element of I such that $n_i \geq 2$. By Lemma 1.39, the kernel of $\delta'^{(R,i)}_s$ is equal to the kernel of $\tilde{\varphi}'^{(R,i)}_s$. Since $\delta'^{(R,i)}_s$ is surjective, there exists a unique injection $\phi'^{(R,i)}_s \colon \mathrm{gr}'_{R,i} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z} \to \mathrm{gr}'_{R,i} j_*\Omega^1_U \otimes_{\mathcal{O}_{D_i}} \mathcal{O}_{D_i^{1/p}}$ such that the diagram (1.50) for $R' = R - D_i$ is commutative.
(i) Let i be an element of I such that $n_i \geq 2$. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{gr}'_{R,i} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z} \arrow[r] \arrow[d, "\phi'^{(R,i)}_s"'] & j_{i*}\mathrm{gr}_{n_i} H^1(K_i, \mathbf{Q}/\mathbf{Z}) \arrow[d, "\phi'^{(n_i)}"] \\
\mathrm{gr}'_{R,i} j_*\Omega^1_U \otimes_{\mathcal{O}_{D_i}} \mathcal{O}_{D_i^{1/p}} \arrow[r] & j_{i*}(\mathrm{gr}'_{n_i}\Omega^1_{K_i} \otimes_{F_{K_i}} F_{K_i}^{1/p}),
\end{tikzcd}$$

where F_{K_i} is the residue field of K_i, the lower horizontal arrow is the inclusion, and $\phi'^{(n_i)}$ is as in (1.26). Since the left vertical arrow is injective as proved

above, the upper horizontal arrow is injective. Hence the assertion holds by Lemma 1.30 similarly as the proof of Proposition 1.31 (i).
(ii) Let J be the subset of I consisting of $i \in I$ such that $n'_i \neq n_i$. Since $\mathrm{gr}'_{n_i} j_{i*}\Omega^1_{K_i} \otimes_{F_{K_i}} F^{1/p}_{K_i}$ is the stalk of $\mathrm{gr}'_{R/R'} j_*\Omega^1_U \otimes_{\mathcal{O}_{D(R/R')}} \mathcal{O}_{D^{(R/R')1/p}}$ at the generic point of $D^{1/p}_i$ for $i \in J$, the assertion holds similarly as the proof of Proposition 1.31 (ii). $\square$

Definition 1.41. Let χ be an element of $H^1_{\text{ét}}(U, \mathbf{Q}/\mathbf{Z})$. We define the *total dimension divisor* R'_χ of χ by $R'_\chi = \sum_{i \in I} \mathrm{dt}(\chi|_{K_i}) D_i$.

We note that we have $\mathrm{Supp}(R'_\chi - D) = \mathrm{Supp}(R_\chi)$ by (1.17).

Definition 1.42. Let χ be an element of $H^1_{\text{ét}}(U, \mathbf{Q}/\mathbf{Z})$. Assume that $\mathrm{dt}(\chi|_{K_i}) > 1$ for some $i \in I$. Let p^s be the order of the p-part of χ. We put $Z = \mathrm{Supp}(R'_\chi - D)$. We define the *characteristic form* $\mathrm{char}(\chi)$ of χ to be the image of the p-part of χ by the composition

$$\begin{aligned}\Gamma(X, \mathrm{fil}'_{R_\chi} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z}) &\to \Gamma(X, \mathrm{gr}'_{R'_\chi/(R'_\chi - Z)} R^1(\epsilon \circ j)_*\mathbf{Z}/p^s\mathbf{Z}) \\ &\xrightarrow{\phi_s'^{(R'_\chi/(R'_\chi - Z))}(X)} \Gamma(X, \mathrm{gr}'_{R'_\chi/(R'_\chi - Z)} j_*\Omega^1_U \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z^{1/p}}) \\ &= \Gamma(Z^{1/p}, \Omega^1(R'_\chi) \otimes_{\mathcal{O}_X} \mathcal{O}_{Z^{1/p}}).\end{aligned}$$

2 Abbes-Saito's ramification theory and Witt vectors

2.1 Abbes-Saito's ramification theory

We briefly recall Abbes-Saito's non-logarithmic ramification theory ([Sa1, Section 2, Subsection 3.1]).

Definition 2.1 ([Sa1, Definition 1.12]). Let P be a scheme. Let D be a Cartier divisor on P and X a closed subscheme of P. We define the *dilatation* $P^{(D \cdot X)}$ of P with respect to (D, X) to be the complement of the proper transform of D in the blow-up of X along $D \cap X$.

Let X be a smooth separated scheme over a perfect field k of characteristic $p > 0$. Let D be a divisor on X with simple normal crossings and $\{D_i\}_{i \in I}$ the irreducible components of D. We put $U = X - D$. Let $R = \sum_{i \in I} r_i D_i$ be a linear combination with integral coefficients $r_i \geq 1$ for every $i \in I$. Let Z be the support of $R - D$.
We put $P = X \times_k X$. Let $\Delta \colon X \to P$ be the diagonal and $\mathrm{pr}_i \colon P \to X$ the i-th projection for $i = 1, 2$. We identify $D \subset X$ with closed subschemes of P by the diagonal. We put $P^{(D)} = \bigcap_{i=1}^2 P^{(\mathrm{pr}_i^* D \cdot X)}$, where the intersection is taken in the blow-up of P along $D \subset P$.
Let $D^{(D)}_i$ be the inverse image of D_i by $P^{(D)} \to P$. Then $D^{(D)} = \sum_{i \in I} D^{(D)}_i$ is a divisor on $P^{(D)}$ with simple normal crossings. The diagonal Δ is canonically lifted to the closed immersion $X \to P^{(D)}$ and we identify X with a closed

subscheme of $P^{(D)}$ by the lift. We define $P^{(R)}$ to be the dilatation of $P^{(D)}$ with respect to $(\sum_{i\in I}(r_i-1)D_i^{(D)}, X)$. Let $T^{(R)}\subset D^{(R)}$ be the inverse image of $Z\subset D$ by $P^{(R)}\to P$. Then the complement $P^{(R)}-D^{(R)}$ is $U\times_k U$ ([Sa1, Lemma 2.4.3]) and $T^{(R)}$ is $TX(-R)\times_X Z$ ([Sa1, Corollary 2.9]), where $TX = \operatorname{Spec} S^\bullet\Omega^1_X$ denotes the tangent bundle of X and $TX(-R)=\operatorname{Spec} S^\bullet\Omega^1_X(R)$. Let G be a finite group and $V\to U$ a G-torsor. We consider the open immersion $U\times_k U = P^{(R)}-D^{(R)}\to P^{(R)}$. The quotient $(V\times_k V)/\Delta G$ of $V\times_k V$ by the diagonal action of G is finite étale over $U\times_k U$. Let $Q^{(R)}$ be the normalization of $P^{(R)}$ in the finite étale covering $(V\times_k V)/\Delta G\to U\times_k U$. Then the canonical lift $X\to P^{(R)}$ of the diagonal is canonically lifted to $X\to Q^{(R)}$.

Definition 2.2 ([Sa1, Definition 2.12]). Let $V\to U$ be a G-torsor for a finite group G and $R=\sum_{i\in I} r_iD_i$ a linear combination with integral coefficients $r_i\geq 1$ for every $i\in I$.

(i) We say that the ramification of V over U at a point x on D is *bounded by $R+$* if the finite morphism $Q^{(R)}\to P^{(R)}$ is étale on a neighborhood of the image of x by the lift $X\to Q^{(R)}$.

(ii) We say that the ramification of V over U along D is *bounded by $R+$* if the finite morphism $Q^{(R)}\to P^{(R)}$ is étale on a neighborhood of the image of the lift $X\to Q^{(R)}$.

Lemma 2.3. *Let $V\to U$ be a G-torsor for a finite group G and $R=\sum_{i\in I} r_iD_i$ a linear combination with integral coefficients $r_i\geq 1$ for every $i\in I$. Let $\mathfrak{p}_i$ be the generic point of D_i for $i\in I$. Then the following are equivalent:*

(i) The ramification of V over U at $\mathfrak{p}_i$ is bounded by $R+$ for every $i\in I$.

(ii) The ramification of V over U along D is bounded by $R+$.

Proof. Since $Q^{(R)}\to P^{(R)}$ is an isomorphism outside of the inverse image of D, the assertion holds by the purity of Zariski-Nagata. ☐

In [Sa1], the notion of the bound of ramification of V over U is defined for $R=\sum_{i\in I} r_iD_i$ of rational coefficients $r_i\geq 1$. The next proposition relates the ramification of G-torsor to the ramification of local field.

Proposition 2.4 ([Sa1, Proposition 2.27]). *Assume that D is irreducible. Let K be the local field at the generic point $\mathfrak{p}$ of D. Let $\{G^r_K\}_{r\in\mathbf{Q}_{>0}}$ be the ramification filtration of the absolute Galois group G_K of K ([AS1, Definition 3.4]). Let $r\geq 1$ be a rational number and let $G^{r+}_K=\overline{\bigcup_{s>r} G^s_K}$ denote the closure of the union of G^s_K for $s>r$. For a G-torsor $V\to U$ for a finite group G, the following are equivalent:*

(i) The ramification of V over U at $\mathfrak{p}$ is bounded by $rD+$.

(ii) G^{r+}_K acts trivially on the finite étale K-algebra $L=\Gamma(V\times_U K,\mathcal{O}_{V\times_U K})$.

We note that the filtration $\{G_K^r\}_{r\in\mathbf{Q}_{>0}}$ is decreasing.
We recall the characteristic form defined in [Sa1, Subsection 2.4]. Let $W^{(R)}$ be the largest open subscheme of $Q^{(R)}$ étale over $P^{(R)}$. We define a scheme $E^{(R)}$ over $T^{(R)}$ to be the fiber product $T^{(R)} \times_{P^{(R)}} W^{(R)}$. Then there is a unique open sub group scheme $E^{(R)0}$ of a smooth group scheme $E^{(R)}$ over Z such that for every $x \in Z$ the fiber $E^{(R)0} \times_Z x$ is the connected component of $E^{(R)} \times_Z x$ containing the unit section ([Sa1, Proposition 2.16]). Further $E^{(R)0}$ is étale over $T^{(R)}$.
Assume that the ramification of V over U along D is bounded by $R+$. Then, we say that the ramification of V over U along D is *non-degenerate* at the multiplicity R if the étale morphism $E^{(R)0} \to T^{(R)}$ is finite. We note that this condition is satisfied if we remove a sufficiently large closed subscheme of X of codimension ≥ 2. Assume that the ramification of V over U along D is non-degenerate at the multiplicity R. Then the exact sequence $0 \to \tilde{G}^{(R)} \to E^{(R)0} \to T^{(R)} \to 0$ defines a closed immersion $\tilde{G}^{(R)\vee} \to T^{(R)\vee}$ of commutative group schemes to the dual vector bundle defined over Z^{1/p^n}, where $n \geq 0$ is an integer.

Definition 2.5 ([Sa1, Definition 2.19]). Let $V \to U$ be a G-torsor for a finite group G. Assume that the ramification of V over U along D is bounded by $R+$ and non-degenerate at the multiplicity R. We define the *characteristic form* $\mathrm{Char}_R(V/U)$ to be the morphism $\tilde{G}^{(R)\vee} \to T^{(R)\vee} = (T^*X \times_X Z)(R)$ over Z^{1/p^n} for a sufficiently large integer $n \geq 0$.

Proposition 2.6 (cf. [Sa1, Corollary 2.28.2]). *Let the notation be as in Proposition 2.4. Let $\mathcal{O}_K$ be the valuation ring of K and F_K the residue field of K. We put $N^{(r)} = \mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}$, where $\mathfrak{m}_{\bar{K}}^r = \{a \in \bar{K} \mid \mathrm{ord}_K(a) \geq r\}$ and $\mathfrak{m}_{\bar{K}}^{r+} = \{a \in \bar{K} \mid \mathrm{ord}_K(a) > r\}$. Let $r > 1$ be a rational number. Assume that the ramification of V over U along D is bounded by $R+$ and non-degenerate at the multiplicity rD. Then the following are equivalent:*

(i) The characteristic form $\mathrm{Char}_{rD}(V/U)$ defines the non-zero mapping by taking the stalk at the generic point of D.

(ii) G_K^r acts non-trivially on L.

Proof. The assertion holds by [Sa1, Corollary 2.28.2] and its proof. □

2.2 Valuation of Witt vectors

We keep the notation in Subsection 2.1. In this subsection, we assume that X is a smooth affine scheme $\mathrm{Spec}\, A$ over k and that D is an irreducible divisor defined by $\pi \in A$. We put $U = \mathrm{Spec}\, B$ and $R = rD$, where $r \geq 1$ is an integer. Let $J \subset A$ be the kernel of the multiplication $A \otimes_k A \to A$. Following the construction of $P^{(R)}$ recalled in the previous section, we have

$$P^{(R)} = \mathrm{Spec}(A \otimes_k A)[J/(\pi^r \otimes 1), ((1 \otimes \pi)/(\pi \otimes 1))^{-1}].$$

The divisor $D^{(R)}$ is defined by $\pi \otimes 1$.
We put $P^{(R)} = \operatorname{Spec} A^{(r)}$. Let $\hat{A}$ denote the completion of the local ring $\mathcal{O}_{X,\mathfrak{p}}$ at the generic point $\mathfrak{p}$ of D and $\hat{A}^{(r)}$ the completion of the local ring $\mathcal{O}_{P^{(R)},\mathfrak{q}}$ at the generic point $\mathfrak{q}$ of $D^{(R)}$ respectively. Let $u\colon \hat{A} \to \hat{A}^{(r)}$ and $v\colon \hat{A} \to \hat{A}^{(r)}$ be the morphisms induced by the first and second projections $P \to X$ respectively. We put $K = \operatorname{Frac} \hat{A}$ and $L^{(r)} = \operatorname{Frac} \hat{A}^{(r)}$.

Lemma 2.7. *Let F_K be the residue field of K. Let $a = a'\pi^n \in K$ be an element, where n is an integer and $a' \in \hat{A}^\times$ is a unit. Let $r \geq 1$ be an integer.*

(i) If $n = 0$ and if $r = 1$, then we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a)) = 0$.

(ii) If $n \notin p\mathbf{Z}$ or $r = 1$, then $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) = r - 1$.

(iii) If $n \in p\mathbf{Z}$ and if $r > 1$, then $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) \geq r$. *Further if a' is not a p-power in F_K, the equality holds.*

Proof. We put $w = (v(\pi) - u(\pi))/u(\pi)^r$ and $w' = (v(a') - u(a'))/u(\pi)^r$. Then we have $v(\pi)/u(\pi) = 1 + u(\pi)^{r-1}w$ and $v(a')/u(a') = 1 + u(a')^{-1}u(\pi)^r w'$. Hence we have

$$v(a)/u(a) - 1 \tag{2.1}$$
$$= \begin{cases} (1 + u(a')^{-1}u(\pi)^r w')(1 + u(\pi)^{r-1}w)^n - 1 & (n \geq 0) \\ (1 + u(\pi)^{r-1}w)^n \left(\left(1 + u(a')^{-1}u(\pi)^r w'\right) - \left(1 + u(\pi)^{r-1}w\right)^{-n} \right) & (n < 0). \end{cases}$$

Suppose that $n = 0$ and $r = 1$. Then we have $v(a)/u(a) = 1 + u(a')^{-1}u(\pi)w'$. Since $u(\pi) = \pi \otimes 1$ is a uniformizer of $\hat{A}^{(r)}$, the assertion (i) holds.
Suppose that $n \notin p\mathbf{Z}$. Then we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) = r - 1$.
Assume that $n \in p\mathbf{Z}$. Suppose that $n = 0$. Then we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) \geq r$, and the equality holds if w' is a unit in $\hat{A}^{(r)}$.
Suppose that $n \neq 0$. We put $n = p^{s'}n'$, where $s' = \operatorname{ord}_p(n) \geq 1$. Then we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) \geq \min\{r, p^{s'}(r-1)\}$. If $r = 1$, then we have $r > p^{s'}(r-1) = 0 = r - 1$. Since $w \in \hat{A}^{(r)\times}$ is a unit, the assertion holds if $r = 1$.
If $r > 1$, then $p^{s'}(r-1) \geq r$. Further the equality holds only if $(p, r, s') = (2, 2, 1)$. Hence we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) \geq r$. Further, if $(p, r, s') \neq (2, 2, 1)$ and if w' is a unit in $\hat{A}^{(r)}$, the equality holds. If $(p, r, s') = (2, 2, 1)$, then we have $\operatorname{ord}_{L^{(r)}}(v(a)/u(a) - 1) = r$ if and only if $u(a')^{-1}w' \neq n'w^p$.
Assume that a is not a p-power in F_K. Then the elements π and a' are p-independent over K^p. Hence the images in $\hat{A}^{(r)}/u(\pi)\hat{A}^{(r)}$ of w and w' form a part of a basis of the F_K-vector space $\pi^{-r}\Omega^1_A \otimes_A F_K$, since $T^{(R)}$ is $TX(-R) \times_X D$. Hence w' is a unit in $\hat{A}^{(r)}$ and $u(a')^{-1}w' \neq n'w^p$. Thus the assertions (ii) and (iii) follow. □

Let $s \geq 0$ be an integer and put $\mathbf{Z}[T, S]_d = \mathbf{Z}[T_d, \ldots, T_{s-1}, S_d, \ldots, S_{s-1}]$ for an integer d such that $0 \leq d \leq s - 1$. We define polynomials $Q_d(T, S) \in$

$\mathbf{Z}[T,S]_d[1/p]$ for $0 \leq d \leq s-1$ inductively by the relation

$$\sum_{i=d}^{s-1} p^{s-1-i}(T_i(1+S_i))^{p^{i-d}} = \sum_{i=d}^{s-1} p^{s-1-i}T_i^{p^{i-d}} + \sum_{i=d}^{s-1} p^{s-1-i}Q_i^{p^{i-d}}. \tag{2.2}$$

It is well-known in the theory of Witt vectors that Q_d is an element of $\mathbf{Z}[T,S]_d$. For elements $x = (x_{s-1}, \dots x_0)$ and $y = (y_{s-1}, \dots, y_0)$ of $W_s(A)$ for a ring A, we put $x' = (x'_{s-1}, \dots, x'_0)$, where $x'_i = x_i(1+y_i)$ for $i = 0, \dots, s-1$. Then we have

$$x' - x = (Q_{s-1}(x,y), Q_{s-2}(x,y), \dots, Q_0(x,y)). \tag{2.3}$$

Lemma 2.8 (cf. [AS3, Lemma 12.2]). *Let the notation be as above.*

(i) *$Q_d(T,S)$ belongs to the ideal of $\mathbf{Z}[T,S]_d$ generated by $(S_i)_{d \leq i \leq s-1}$ for $d = 0, \dots, s-1$.*

(ii) *$Q_d(T,S) - \sum_{i=d}^{s-1} T_i^{p^{i-d}} S_i$ belongs to the ideal of $\mathbf{Z}[T,S]_d$ generated by $(S_iS_j)_{d \leq i,j \leq s-1}$ for $d = 0, \dots, s-1$.*

(iii) *If we replace T_i by $T_i^{p^{s-1-i}}$ in $Q_d(T,S)$, the polynomial $Q_d(T,S)$ is homogeneous of degree p^{s-1-d} as a polynomial of multi-value T for $0 \leq d \leq s-1$.*

Proof. The assertions (i) and (ii) are the same as (i) and (ii) in [AS3, Lemma 12.2] respectively.
We prove (iii) by induction on d. If $d = s-1$, we have $Q_{s-1} = T_{s-1}S_{s-1}$. Hence the assertion holds.
If $d < s-1$, we have

$$Q_d = p^{d-s+1}\left(\sum_{i=d}^{s-1} p^{s-1-i}T_i^{p^{i-d}}\left((1+S_i)^{p^{i-d}} - 1\right) - \sum_{i=d+1}^{s-1} p^{s-1-i}Q_i^{p^{i-d}}\right).$$

By the induction hypothesis, the polynomial $Q_i(T,S)$ is homogeneous of degree p^{s-1-i} for T for $d+1 \leq i \leq s-1$ with T_j replaced by $T_j^{p^{s-1-j}}$ for $i \leq j \leq s-1$. Hence $Q_i(T,S)^{p^{i-d}}$ is homogeneous of degree p^{s-1-d} for T for $d+1 \leq i \leq s-1$ with the same replacement of T_j for $i \leq j \leq s-1$. Hence the assertion holds. $\square$

Lemma 2.9. *Let $a = (a_{s-1}, \dots, a_0)$ be an element of $W_s(K)$ and put $b = (b_{s-1}, \dots, b_0) \in W_s(L^{(r)})$, where $b_i = v(a_i)/u(a_i) - 1$ if $a_i \neq 0$ and $b_i = 0$ if $a_i = 0$ for $0 \leq i \leq s-1$. Let $m \geq 1$ be an integer and assume that $a \in \mathrm{fil}'_m W_s(K)$. Let $r \geq 1$ be an integer.*

(i) *If $(m,r) = (1,1)$, then $p^i \operatorname{ord}_{L^{(r)}}(Q_d(u(a),b)) \geq -m+1$ for every $0 \leq d \leq s-1$.*

(ii) If $r > 1$, then $p^i \operatorname{ord}_{L^{(r)}}(Q_d(u(a), b)) > -m + r$ for every $0 < d \leq s - 1$, and $\operatorname{ord}_{L^{(r)}}(Q_0(u(a), b)) \geq -m + r$.

Proof. We put $s' = \min\{\operatorname{ord}_p(m), s\}$. Let $a' = (a'_{s-1}, \dots, a'_0)$ be an element of $W_s(K)$ such that $a'_i = 0$ if $p^i \operatorname{ord}_K(a_i) = -m$ and $a'_i = a_i$ if $p^i \operatorname{ord}_K(a_i) \geq -(m-1)$ for $0 \leq i \leq s-1$. We note that if $s' \leq i \leq s-1$ then $a'_i = a_i$ by (1.15). Let $a'' = (a''_{s'-1}, \dots, a''_0)$ be an element of $W_{s'}(K)$ such that $a''_i = 0$ if $p^i \operatorname{ord}_K(a_i) \geq -(m-1)$ and $a''_i = a_i$ if $p^i \operatorname{ord}_K(a_i) = -m$ for $0 \leq i \leq s'-1$. Then we have $a = a' + V^{s-s'}(a'')$. Let $b' \in W_s(L^{(r)})$ and $b'' \in W_{s'}(L^{(r)})$ be the elements defined from a' and a'' respectively similarly as b defined from a. Since we have $Q(u(a), b) = (Q_{s-1}(u(a), b), \dots, Q_0(u(a), b)) = v(a) - u(a)$ and similarly for a' and a'' by (2.3), we have $Q(u(a), b) = Q(u(a'), b') + V^{s-s'}(Q(u(a''), b''))$. Since $\operatorname{fil}_n W_s(L^{(r)})$ is a submodule of $W_s(L^{(r)})$ for $n \in \mathbf{Z}$, the assertions for a hold if the assertions for a' and a'' hold. Hence we prove the assertions for a' and a''.

By the definitions of a' and a'', we have $\operatorname{ord}_{L^{(r)}}(u(a'_i)) \geq -(m-1)/p^i$ for $0 \leq i \leq s-1$ and $\operatorname{ord}_{L^{(r)}}(u(a''_i)) \geq -m/p^i$ for $0 \leq i \leq s'-1$. If $r > 1$, then we have $\operatorname{ord}_{L^{(r)}}(b'_i) \geq r-1$ for $0 \leq i \leq s-1$ and $\operatorname{ord}_{L^{(r)}}(b''_i) \geq r$ for $0 \leq i \leq s'-1$ by Lemma 2.7 (ii) and (iii). If $(m, r) = (1, 1)$, then we have $s' = 0$ and $\operatorname{ord}_{L^{(r)}}(b'_i) \geq r-1$ for $0 \leq i \leq s-1$ by Lemma 2.7 (ii). Hence, by Lemma 2.8 (i) and (iii), we have

$$p^d \operatorname{ord}_{L^{(r)}}(Q_d(u(a'), b')) \geq -(m-1) + p^d(r-1) \geq -m + r. \tag{2.4}$$

Further we have

$$p^d \operatorname{ord}_{L^{(r)}}(Q_d(u(a''), b'')) \geq -m + p^d r \geq -m + r. \tag{2.5}$$

If $r > 1$, then the equality holds in the right inequality in (2.4) only if $d = 0$ and so in (2.5). Hence the assertions hold. $\square$

LEMMA 2.10. *Let the notation be as in Lemma 2.9. Let $m \geq 2$ be an integer and assume that $a \in \operatorname{fil}'_m W_s(K)$. Then we have $\operatorname{ord}_{L^{(m)}}(Q_0(u(a), b) - \sum_{i=0}^{s-1} u(a_i)^{p^i} b_i) > 0$.*

Proof. We put $s' = \min\{\operatorname{ord}_p(m), s\}$. Let $a' = (a'_{s-1}, \dots, a'_0)$ and $a'' = (a''_{s'-1}, \dots, a''_0)$ be as in the proof of Lemma 2.9. We have $a = a' + V^{s-s'}(a'')$. Let $b' \in W_s(L^{(m)})$ and $b'' \in W_{s'}(L^{(m)})$ be the elements defined from a' and a'' respectively similarly as b defined from a. Since $Q(u(a), b) = Q(u(a'), b') + V^{s-s'} Q(u(a''), b'')$ as in the proof of Lemma 2.9 and $\sum_{i=0}^{s-1} u(a_i)^{p^i} b_i = \sum_{i=0}^{s-1} u(a'_i)^{p^i} b'_i + \sum_{i=0}^{s'-1} u(a''_i)^{p^i} b''_i$, it is sufficient to prove the assertion for a' and a''.

As in the proof of Lemma 2.9, we have $\operatorname{ord}_{L^{(m)}}(u(a'_i)) \geq -(m-1)/p^i$ for $0 \leq i \leq s-1$ and $\operatorname{ord}_{L^{(m)}}(u(a''_i)) \geq -m/p^i$ for $0 \leq i \leq s'-1$. Further we have $\operatorname{ord}_{L^{(m)}}(b'_i) \geq m-1$ for $0 \leq i \leq s-1$ and $\operatorname{ord}_{L^{(m)}}(b''_i) \geq m$ for $0 \leq i \leq s'-1$.

Hence, by Lemma 2.8 (ii) and (iii), we have

$$\mathrm{ord}_{L^{(m)}}(Q_0(u(a'), b') - \sum_{i=0}^{s-1} u(a'_i)^{p^i} b'_i) \geq -(m-1) + 2(m-1) = m-1 > 0.$$

Further we have

$$\mathrm{ord}_{L^{(m)}}(Q_0(u(a''), b'') - \sum_{i=0}^{s'-1} u(a''_i)^{p^i} b''_i) \geq -m + 2m = m > 0.$$

Hence the assertion holds. □

2.3 Calculation of characteristic forms

Let X be a smooth separated scheme over a perfect field k of characteristic $p > 0$. Let D be a divisor on X with simple normal crossings and $\{D_i\}_{i\in I}$ the irreducible components of D. We put $U = X - D$ and let $j\colon U \to X$ denote the canonical open immersion. Let K_i be the local field at the generic point of D_i for $i \in I$ and let $\mathcal{O}_{K_i}$ be the valuation ring of K_i for $i \in I$.
Let χ be an element of $H^1_{\text{ét}}(U, \mathbf{Q}/\mathbf{Z})$. In this subsection, we prove the equality of the characteristic form $\mathrm{char}(\chi)$ of χ and the characteristic form $\mathrm{Char}_R(V/U)$ of the Galois torsor $V \to U$ corresponding to χ.
Let $p_i\colon P^{(R)} \to X$ be the morphism induced by the i-th projection for $i = 1, 2$. Let $u\colon p_1^{-1}\mathcal{O}_X \to \mathcal{O}_{P^{(R)}}$ and $v\colon p_2^{-1}\mathcal{O}_X \to \mathcal{O}_{P^{(R)}}$ be the canonical morphisms of sheaves on $P^{(R)}$ by abuse of notation. Let $L_i^{(R)}$ be the fractional field of the completion of the local ring $\mathcal{O}_{P^{(R)},\mathfrak{q}_i}$, where $R = \sum_{i\in I} r_i D_i$ is a linear combination with integer coefficients $r_i \geq 1$ for every $i \in I$ and $\mathfrak{q}_i$ is the generic point of the pull-back $D_i^{(R)}$ of $D_i^{(D)}$ by $P^{(R)} \to P^{(D)}$. If $D = D_1$ is irreducible, then we simply write $L^{(r_1)}$ for $L_1^{(R)}$ as in the previous section.
We first consider the tamely ramified case.

Lemma 2.11. *Assume that the order n of χ is prime to p and regard χ as an element of $H^1_{\text{ét}}(U, \mathbf{Z}/n\mathbf{Z})$. We put $G = \mathbf{Z}/n\mathbf{Z}$. Let $V \to U$ be the G-torsor corresponding to χ. Let $R = \sum_{i\in I} r_i D_i$ be a linear combination with integral coefficients $r_i \geq 1$ for every $i \in I$.*

(i) The ramification of V over U along D is bounded by $D+$.

(ii) The characteristic form $\mathrm{Char}_R(V/U)$ is the zero mapping.

Proof. (i) By Lemma 2.3, we may assume that $D = D_1$ is irreducible. Since the assertion is local, we may assume that $X = \mathrm{Spec}\, A$ is affine and D is defined by an element of A. Since $G^1_{K_1}$ is the inertia group of G_{K_1} ([AS1, Proposition 3.7 (1)]), we may assume that k contains a primitive n-th root of unity by Lemma 2.3 and Proposition 2.4. Since $\mathrm{ord}_{L^{(r_1)}}(v(a)/u(a)) = 0$ for every unit $a \in \mathcal{O}_{K_1}^\times$ by Lemma 2.7 (i), the assertion holds.

(ii) Let Z be the support of $R-D$. By (i) and Proposition 2.4, the ramification group $G_{K_i}^{r_i+}$ acts trivially on $L_i = \Gamma(V \times_U K_i, \mathcal{O}_{V\times_U K_i})$ for D_i contained in Z. By Proposition 2.6, the stalk of the characteristic form $\mathrm{Char}_R(V/U)$ at the generic point of D_i defines the zero mapping for D_i contained in Z. Hence the assertion holds. □

By Lemma 2.11, the bound of the ramification of the Galois torsor $V \to U$ corresponding to χ and its characteristic form $\mathrm{Char}_R(V/U)$ does not depend on the prime-to-p-part of χ, that is, they are dependent only on the p-part of χ.

PROPOSITION 2.12. *Assume that the order of χ is p^s and regard χ as an element of $H^1_{\text{ét}}(U, \mathbf{Z}/p^s\mathbf{Z})$. We put $G = \mathbf{Z}/p^s\mathbf{Z}$. Let $V \to U$ be the G-torsor corresponding to χ.*

(i) The ramification of V over U along D is bounded by $R'_\chi+$, where R'_χ is the total dimension divisor of χ (Definition 1.41).

(ii) Assume that $R'_\chi \neq D$ and put $Z = \mathrm{Supp}(R'_\chi - D)$. Then the scheme $E^{(R'_\chi)} \to T^{(R'_\chi)} = TX(-R'_\chi)\times_X Z$ is defined by the Artin-Schreier equation $t^p - t = \mathrm{char}(\chi)$.

Proof. We put $m_i = \mathrm{dt}(\chi|_{K_i})$ for $i \in I$. Let $a = (a_{s-1}, \ldots, a_0) \in \mathrm{fil}'_{R'_\chi} j_* W_s(\mathcal{O}_U)$ be an element whose image by δ_s (1.33) is χ. Then $V \times_k V/\Delta G \to U \times_k U$ is the G-torsor defined by the Artin-Schreier-Witt equation $(F-1)(t) = v(a) - u(a)$.
(i) By Lemma 2.3, we may assume that D is irreducible. Since the assertion is local, we may assume that $X = \mathrm{Spec}\, A$ is affine and that D is defined by an element of A. By (2.3) and Lemma 2.9, the difference $v(a) - u(a)$ is a regular function on $P^{(R'_\chi)}$. Hence the assertion holds.
(ii) By (i), (2.3), Lemma 2.9 (ii), and Lemma 2.10, the scheme $E^{(R'_\chi)} \to T^{(R'_\chi)}$ is the G-torsor defined by the Artin-Schreier equation $t^p - t = \sum_{j=0}^{s-1} u(a_j)^{p^j-1}(v(a_j) - u(a_j))$. We put $n_{ij} = \mathrm{ord}_{K_i}(a_j)$ for $i \in I$ and $0 \leq j \leq s-1$. As calculating in the proof of Lemma 2.7, we have the following on a neighborhood of the generic point of $D_i^{(R'_\chi)}$ for $i \in I$ such that $m_i > 1$:

(a) If $n_{ij} \notin p\mathbf{Z}$, we have $u(a_j)^{p^j-1}(v(a_j) - u(a_j)) = n_{ij} u(a_j)^{p^j} u(t_i)^{m_i - 1} w_i$;

(b) If $n_{ij} \in p\mathbf{Z}$ and if $(p, m_i, \mathrm{ord}_p(n_{ij})) \neq (2,2,1)$, we have $u(a_j)^{p^j-1}(v(a_j) - u(a_j)) = u(a_j)^{p^j} u(a'_j)^{-1} u(t_i)^{m_i} w'_{ij}$;

(c) If $(p, m_i, \mathrm{ord}_p(n_{ij})) = (2,2,1)$, we have $u(a_j)^{p^j-1}(v(a_j) - u(a_j)) = u(a_j)^{p^j}(u(a'_j)^{-1} u(t_i)^2 w'_{ij} + (n_{ij}/2) u(t_i)^2 w_i^2)$,

where t_i is a local equation of D_i, $a'_j = a_j / t_i^{n_{ij}}$, $w_i = (v(t_i) - u(t_i))/u(t_i)^{m_i}$, and $w'_{ij} = (v(a'_j) - u(a'_j))/u(t_i)^{m_i}$ for every $j = 0, \ldots, s-1$. Since

$a \in \mathrm{fil}'_{R'_\chi} j_* W_s(\mathcal{O}_U)$, we have $p^j \operatorname{ord}_{L_i^{(m_i)}}(a_j) \geq -(m_i-1)$ if $n_{ij} \notin p\mathbf{Z}$ and $p^j \operatorname{ord}_{L_i^{(m_i)}}(a_j) \geq -m_i$ if $n_{ij} \in p\mathbf{Z}$. If $(p, m_i, \operatorname{ord}_p(n_{ij}), p^j n_{ij}) = (2,2,1,-2)$, we have $(p, j, n_{ij}) = (2, 0, -2)$. Hence the assertion holds by identifying w_i and w'_{ij} with $dt_i/t_i^{m_i}$ and $da'_j/t_i^{m_i}$ respectively. □

Corollary 2.13. *Let $V \to U$ be the Galois torsor corresponding to χ. Assume that the ramification of V over U along D is non-degenerate at the multiplicity R'_χ.*

(i) The image of the generator $1 \in \tilde{G}^{(R'_\chi)\vee} = \mathbf{F}_p$ by $\operatorname{Char}_{R'_\chi}(V/U)$ is equal to $\operatorname{char}(\chi)$.

(ii) Assume that $D = D_1$ is irreducible and that $\operatorname{dt}(\chi|_{K_1}) > 1$. Then the ramification of V over U at the generic point of D is not bounded by $rD+$ for any rational number r such that $1 \leq r < \operatorname{dt}(\chi|_{K_1})$.

Proof. (i) The assertion holds by Lemma 2.11 and Proposition 2.12 (ii).
(ii) We put $K = K_1$. Assume that G_K^{r+} acts trivially on $L = \Gamma(V \times_U K, \mathcal{O}_{V \times_U K})$ for a rational number r such that $1 \leq r < \operatorname{dt}(\chi|_K)$. Then, by (i) and Proposition 2.6, the stalk $\operatorname{char}(\chi|_K)$ of $\operatorname{char}(\chi)$ at the generic point of D must be 0. However $\operatorname{char}(\chi)$ is non-zero. Hence the assertion holds by Proposition 2.4. □

3 Equality of ramification filtrations

Let K be a complete discrete valuation field of characteristic $p > 0$ and F_K the residue field. Let G_K be the absolute Galois group of K. We show that the abelianization of Abbes-Saito's filtration $\{G_K^r\}_{r \in \mathbf{Q}>0}$ ([AS1, Definition 3.4]) is the same as $\{\mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})\}_{m \in \mathbf{Z}_{\geq 1}}$ (Definition 1.2) by taking dual. If $m > 2$, then it has been proved by Abbes-Saito ([AS3, Théorème 9.10]).

Theorem 3.1. *Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. Let $m \geq 1$ be an integer. Let r be a rational number such that $m \leq r < m+1$. If F_K is finitely generated over a perfect subfield $k \subset F_K$, then the following are equivalent:*

(i) $\chi \in \mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})$.

(ii) $\chi(G_K^{m+}) = 0$.

(iii) $\chi(G_K^{r+}) = 0$.

Proof. Since G_K^{1+} is a pro-p-subgroup of G_K ([AS1, Proposition 3.7.1]), we may assume that the order of χ is a power of p. Let p^s be the order of χ and regard χ as an element of $H^1_{\text{ét}}(U, \mathbf{Z}/p^s\mathbf{Z})$. We put $G = \mathbf{Z}/p^s\mathbf{Z}$. As in [AS3, 6.1], we take a smooth affine connected scheme X over k and a smooth irreducible divisor D on X such that the completion $\hat{\mathcal{O}}_{X,\mathfrak{p}}$ of the local ring $\mathcal{O}_{X,\mathfrak{p}}$ at the generic point $\mathfrak{p}$ of D is isomorphic to $\mathcal{O}_K$. By shrinking X if necessary, we take a G-torsor $V \to U = X - D$ corresponding to a character of $\pi_1^{\mathrm{ab}}(U)$ whose restriction to G_K is χ.

By Proposition 2.12 (i) and Corollary 2.13 (ii), the ramification of V over U at the generic point of D is bounded by $rD+$ for a rational number $r \geq 1$ if and only if $r \geq \mathrm{dt}(\chi)$. Further, by Proposition 2.4, the former condition is equivalent to that G_K^{r+} acts trivially on $L = \Gamma(V \times_U K, \mathcal{O}_{V \times_U K})$. Hence $\chi(G_K^{r+}) = 0$ if and only if $r \geq \mathrm{dt}(\chi)$.
Since the condition (i) holds if and only if $m \geq \mathrm{dt}(\chi)$, the equivalence of (i) and (ii) follows. Since $m \leq r$, the condition (ii) deduces the condition (iii). Suppose that the condition (iii) holds. Since $r \geq \mathrm{dt}(\chi)$, we have $m = [r] \geq \mathrm{dt}(\chi)$. Hence the condition (ii) holds. □

Proof of Theorem 0.1. We may identify K with $F_K((\pi))$ by taking a uniformizer of K. Let $K_h = \mathrm{Frac}(F_K[\pi]^h_{(\pi)})$ be the fractional field of the henselization of the localization $F_K[\pi]_{(\pi)}$ of $F_K[\pi]$ at the prime ideal (π). Since the completion of K_h is K, the canonical morphisms $G_K \to G_{K_h}$ and $H^1(K_h, \mathbf{Q}/\mathbf{Z}) \to H^1(K, \mathbf{Q}/\mathbf{Z})$ are isomorphisms.
Let k be a perfect subfield of F_K and take a separating transcendental basis S of F_K over k. For a finite subextension E of F_K over $k(S')$, where S' is a finite set of S, let $K_{E,h}$ denote the fractional field of the henselization of the local ring $E[\pi]_{(\pi)}$. Since $F_K = \varinjlim E$, we may identify K_h with the inductive limit $\varinjlim K_{E,h}$ and $H^1(K_h, \mathbf{Q}/\mathbf{Z})$ with $\varinjlim H^1(K_{E,h}, \mathbf{Q}/\mathbf{Z})$, where E runs through such subfields of F_K.
Let χ be an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$. Take a subfield E of F_K such that E is a subextension of F_K over $k(S')$ for a finite subset $S' \subset S$ and that $\chi \in H^1(K_{E,h}, \mathbf{Q}/\mathbf{Z})$. Let K_E denote the completion of $K_{E,h}$. We identify $H^1(K_E, \mathbf{Q}/\mathbf{Z})$ with $H^1(K_{E,h}, \mathbf{Q}/\mathbf{Z})$ and $\chi \in H^1(K_{E,h}, \mathbf{Q}/\mathbf{Z})$ with an element of $H^1(K_E, \mathbf{Q}/\mathbf{Z})$. We prove that each condition in Theorem 3.1 holds for K if and only if it holds for K_E.
Let $\mathrm{dt}_K(\chi)$ and $\mathrm{dt}_{K_E}(\chi)$ denote the total dimension of χ as an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$ and $H^1(K_E, \mathbf{Q}/\mathbf{Z})$ respectively. We put $\mathrm{dt}_K(\chi) = n$ and $\mathrm{dt}_{K_E}(\chi) = n'$ and prove that $n = n'$. Since $\mathrm{fil}'_m W_s(K_E) \subset \mathrm{fil}'_m W_s(K)$ for every integer $m \geq 1$, we have $\mathrm{fil}'_m H_1(K_E, \mathbf{Q}/\mathbf{Z}) \subset \mathrm{fil}'_m H^1(K, \mathbf{Q}/\mathbf{Z})$. Hence we have $1 \leq n \leq n'$, which proves that $n = 1$ if $n' = 1$.
Assume that $n' > 1$. Take an element $\bar{a}$ of $\mathrm{gr}'_{n'} W_s(E(\pi))$ whose image in $\mathrm{gr}'_{n'} H^1(K_E, \mathbf{Q}/\mathbf{Z})$ is χ. Let $\mathrm{char}_K(\chi)$ and $\mathrm{char}_{K_E}(\chi)$ denote the characteristic form of χ as an element of $H^1(K, \mathbf{Q}/\mathbf{Z})$ and $H^1(K_E, \mathbf{Q}/\mathbf{Z})$ respectively. Let $\mathcal{O}_K$ and $\mathcal{O}_{K_E}$ denote the valuation rings of K and K_E respectively. Since F_K is separable over E, we have an injection $\Omega^1_{E[\pi]_{(\pi)}} \to \Omega^1_{F_K[\pi]_{(\pi)}}$. This injection induces the injection $\Omega^1_{\mathcal{O}_{K_E}} \to \Omega^1_{\mathcal{O}_K}$, and further the injection $\mathrm{gr}'_{n'} \Omega^1_{K_E} \to \mathrm{gr}'_{n'} \Omega^1_K$. Hence the canonical morphism $\mathrm{gr}'_{n'} \Omega^1_{K_E} \otimes_{F_K} F_K^{1/p} \to \mathrm{gr}'_{n'} \Omega^1_K \otimes_{F_K} F_K^{1/p}$ is injective. Since $\mathrm{char}_{K_E}(\chi) \neq 0$, the image of $\mathrm{char}_{K_E}(\chi)$ in $\mathrm{gr}'_{n'} \Omega^1_K \otimes_{F_K} F_K^{1/p}$ is not 0. This implies that $\mathrm{char}_K(\chi)$ is the image of $\mathrm{char}_{K_E}(\chi)$ in $\mathrm{gr}'_{n'} \Omega^1_K \otimes_{F_K} F_K^{1/p}$. Hence we have $n = n'$. Since the condition (i) in Theorem 3.1 holds for K if and only if $m \geq n$ and similarly for K_E, the condition (i) in Theorem 3.1 for K is equivalent to that for K_E.

Let $r \geq 1$ be a rational number. Since K is an extension of K_E of ramification index 1 and the extension of residue fields is separable, by applying [AS2, Lemma 2.2], the canonical morphism $G_K \to G_{K_E}$ induces the surjection $G_K^s \to G_{K_E}^s$ for every $s \in \mathbf{Q}_{\geq 1}$. Hence we have $\chi(G_K^{r+}) = 0$ if and only if $\chi(G_{K_E}^{r+}) = 0$, which proves the assertions for conditions (ii) and (iii) in Theorem 3.1. $\square$

References

[AS1] A. Abbes and T. Saito, Ramification of local fields with imperfect residue fields, *Am. J. Math.* 124 (5) (2002), 879–920.

[AS2] A. Abbes and T. Saito, Ramification of local fields with imperfect residue fields, II, *Documenta Math.* Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), 3–70.

[AS3] A. Abbes and T. Saito, Analyse micro-locale ℓ-adique en caractéristique $p > 0$: le cas d'un trait, *Publ. Res. Inst. Math. Sci.* 45 (2009), no. 1, 25–74.

[B] J. L. Brylinski, Théorie du corps de classes de Kato et revêtements abéliens de surfaces, *Ann. Inst. Fourier* 33 (1983), no. 3, 23–38.

[K1] K. Kato, Swan conductors for characters of degree one in the imperfect residue field case, Algebraic K-theory and algebraic number theory, *Contemp. Math.* 83 (1989), 101–131.

[K2] K. Kato, Class field theory, $\mathcal{D}$-modules, and ramification on higher dimensional schemes, part I, *Am. J. of Math.* Vol. 116, No. 4 (1994), 757–784.

[M] S. Matsuda, On the Swan conductor in positive characteristic, *Am. J. of Math.* Vol. 119, No. 4 (1997), 705–739.

[Sa1] T. Saito, Wild Ramification and the Cotangent Bundle, J. of Alg. Geom. published online.

[Sa2] T. Saito, The characteristic cycle and the singular support of a constructible sheaf, *Inventiones Math.* 207(2) (2017), 597–695.

[Se] J.-P. Serre, *Local fields*, *Graduate Texts in Mathematics* 67, Springer, 1979.

Yuri Yatagawa
Fakultät für Mathematik
Universität Regensburg
Universitätsstraße 31
93040 Regensburg
Germany
yuri.yatagawa@mathematik.uni-regensburg.de