Free Actions of Compact Quantum Groups on Unital C*-Algebras

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Abstract. Let F be a field, Γ a finite group, and $\mathrm{Map}(\Gamma, F)$ the Hopf algebra of all set-theoretic maps $\Gamma \to F$. If E is a finite field extension of F and Γ is its Galois group, the extension is Galois if and only if the canonical map $E \otimes_F E \to E \otimes_F \mathrm{Map}(\Gamma, F)$ resulting from viewing E as a $\mathrm{Map}(\Gamma, F)$-comodule is an isomorphism. Similarly, a finite covering space is regular if and only if the analogous canonical map is an isomorphism. In this paper, we extend this point of view to actions of compact quantum groups on unital C^*-algebras. We prove that such an action is free if and only if the canonical map (obtained using the underlying Hopf algebra of the compact quantum group) is an isomorphism. In particular, we are able to express the freeness of a compact Hausdorff topological group action on a compact Hausdorff topological space in algebraic terms. As an application, we show that a field of free actions on unital C^*-algebras yields a global free action.

Contents

Introduction 826
1. Equivalence of freeness and the Peter-Weyl-Galois condition 830
2. Equivalence of principality and strong monoidality 833
3. The classical case 835
4. Vector-bundle interpretation 838
5. Application: fields of free actions 840
Appendix: Finite Galois coverings 843
References 846

Introduction

A *compact quantum group* [W-SL87, W-SL98] is a unital C^*-algebra H with a given unital injective $*$-homorphism Δ (referred to as comultiplication)

$$\Delta\colon H \longrightarrow H \underset{\min}{\otimes} H \tag{0.1}$$

that is coassociative, i.e. it renders the diagram

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"] & H \underset{\min}{\otimes} H \arrow[d, "\Delta\otimes\mathrm{id}"] \\ H \underset{\min}{\otimes} H \arrow[r, "\mathrm{id}\otimes\Delta"'] & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{tikzcd} \tag{0.2}$$

commutative, and such that the two-sided cancellation property holds:

$$\{(a\otimes 1)\Delta(b) \mid a,b\in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1\otimes b) \mid a,b\in H\}^{\mathrm{cls}}. \tag{0.3}$$

Here $\otimes_{\min}$ denotes the spatial tensor product of C^*-algebras and cls denotes the closed linear span of a subset of a Banach space.

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a *coaction* (or an action of the compact quantum group (H,Δ) on A, cf. [P-P95, Definition 1.4]) iff

1. $(\delta\otimes\mathrm{id})\circ\delta = (\mathrm{id}\otimes\Delta)\circ\delta$ (coassociativity),
2. $\{\delta(a)(1\otimes h) \mid a\in A,\ h\in H\}^{\mathrm{cls}} = A\otimes_{\min} H$ (counitality).

We shall consider three properties of coactions.

Definition 0.1 ([E-DA00]). *The coaction $\delta : A \to A\otimes_{\min} H$ is* free *iff*

$$\{(x\otimes 1)\delta(y) \mid x,y\in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H.$$

Given a compact quantum group (H,Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of its irreducible unitary representations [W-SL98, MV98]. This is Woronowicz's Peter-Weyl theory in the case of compact quantum groups. Moreover, denoting by $\otimes$ the purely algebraic tensor product over the field $\mathbb{C}$ of complex numbers, we define the *Peter-Weyl subalgebra* of A (cf. [P-P95, S-PM11]) as

$$\mathcal{P}_H(A) := \{\, a\in A \,|\, \delta(a)\in A\otimes\mathcal{O}(H)\,\}. \tag{0.4}$$

Using the coassociativity of the coaction δ, one can check that $\mathcal{P}_H(A)$ is a right $\mathcal{O}(H)$-comodule algebra. In particular, $\mathcal{P}_H(H) = \mathcal{O}(H)$. The assignment $A \mapsto \mathcal{P}_H(A)$ is functorial with respect to equivariant unital $*$-homomorphisms and comodule algebra maps. We call it the *Peter-Weyl functor.*

Definition 0.2. *The coaction* $\delta : A \to A \otimes_{\min} H$ *satisfies the* Peter-Weyl-Galois (PWG) condition *iff the canonical map*

$$can\colon \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A) \otimes \mathcal{O}(H),$$
$$can\colon x \otimes y \longmapsto (x \otimes 1)\delta(y), \tag{0.5}$$

is bijective. Here $B := A^{\mathrm{co}\,H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ *is the unital* C^**-subalgebra of coaction invariants (fixed-point subalgebra).*

Throughout this paper the tensor product over an algebra denotes the purely algebraic tensor product over that algebra. Note that $\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A)$ is not in general an algebra, and even if we lift the canonical map to

$$\widetilde{can}\colon \mathcal{P}_H(A) \otimes \mathcal{P}_H(A) \ni x \otimes y \longmapsto (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes \mathcal{O}(H), \tag{0.6}$$

it is not an algebra homomorphism, and cannot as such be completed into a continuous map between C*-algebras. However, it can be defined on the level of Hilbert modules (see [DY13]).

Definition 0.3. *The coaction* $\delta : A \to A \otimes_{\min} H$ *is* strongly monoidal *iff for all left* $\mathcal{O}(H)$*-comodules* V *and* W *the map*

$$\beta : (\mathcal{P}_H(A)\Box V) \underset{B}{\otimes} (\mathcal{P}_H(A)\Box W) \longrightarrow \mathcal{P}_H(A)\Box(V \otimes W),$$
$$\Big(\sum_i a_i \otimes v_i\Big) \otimes \Big(\sum_j b_j \otimes w_j\Big) \longmapsto \sum_{i,j} a_i b_j \otimes (v_i \otimes w_j),$$

is bijective.

In the above definition, we have used the cotensor product

$$\mathcal{P}_H(A)\Box V := \{t \in \mathcal{P}_H(A) \otimes V \mid (\delta \otimes \mathrm{id})(t) = (\mathrm{id} \otimes {}_V\Delta)(t)\}, \tag{0.7}$$

where ${}_V\Delta\colon V \to \mathcal{O}(H) \otimes V$ is the given left coaction of $\mathcal{O}(H)$ on V. The coaction of $\mathcal{O}(H)$ on $V \otimes W$ is the diagonal coaction.
The theorem of this paper is:

Theorem 0.4. *Let* A *be a unital* C^**-algebra equipped with an action of a compact quantum group* (H, Δ) *given by* $\delta\colon A \to A \otimes_{\min} H$*. Then the following are equivalent:*

(1) *The action of* (H, Δ) *on* A *is free.*
(2) *The action of* (H, Δ) *on* A *satisfies the Peter-Weyl-Galois condition.*
(3) *The action of* (H, Δ) *on* A *is strongly monoidal.*

Note that of the three equivalent conditions, the first uses functional analysis, the second is algebraic, and the third is categorical. The difficult implication, which is the core of the theorem, is (1) $\Longrightarrow$ (2). It proves that, for any free action, there exists a strong connection, a key technical device for index-pairing computations (e.g. [HMS03]). In the spirit of Woronowicz's Peter-Weyl theory, our result states that the original functional-analysis formulation of free action is equivalent to the much more algebraic PWG-condition.

We now proceed to explain our main result in the classical setting. Let G be a compact Hausdorff topological group acting on a compact Hausdorff topological space X by a continuous right action $X \times G \to X$. It is immediate that the action is free, i.e. $xg = x \implies g = e$ (where e is the identity element of G), if and only if

$$\begin{aligned} X \times G &\longrightarrow X \underset{X/G}{\times} X, \\ (x, g) &\longmapsto (x, xg), \end{aligned} \tag{0.8}$$

is a homeomorphism. Here $X \times_{X/G} X$ is the subset of $X \times X$ consisting of pairs (x_1, x_2) such that x_1 and x_2 are in the same G-orbit.

This is equivalent to the assertion that the $*$-homomorphism

$$C(X \underset{X/G}{\times} X) \longrightarrow C(X \times G) \tag{0.9}$$

obtained from the above map $(x, g) \mapsto (x, xg)$ is an isomorphism. Here, as usual, $C(Y)$ denotes the commutative C^*-algebra of all continuous complex-valued functions on a compact Hausdorff space Y.

In turn, the assertion that the $*$-homomorphism (0.9) is an isomorphism is readily proved equivalent to

$$\{(x \otimes 1)\delta(y) \mid x, y \in C(X)\}^{\mathrm{cls}} = C(X) \underset{\min}{\otimes} C(G), \tag{0.10}$$

where

$$\delta \colon C(X) \longrightarrow C(X) \underset{\min}{\otimes} C(G), \quad (\delta(f)(g))(x) := f(xg), \tag{0.11}$$

is the $*$-homomorphism obtained from the action map $X \times G \to X$. Hence, in the case of a compact Hausdorff group acting on a compact Hausdorff space, freeness in the usual sense agrees with freeness as defined in the setting of a compact quantum group acting on a unital C^*-algebra. Thus Theorem 0.4 provides the following characterization of free actions in the classical case.

Theorem 0.5. *Let G be a compact Hausdorff group acting continuously on a compact Hausdorff space X. Then the action is free if and only if the canonical map*

$$\mathit{can} \colon \mathcal{P}_{C(G)}(C(X)) \underset{C(X/G)}{\otimes} \mathcal{P}_{C(G)}(C(X)) \longrightarrow \mathcal{P}_{C(G)}(C(X)) \otimes \mathcal{O}(C(G)) \tag{0.12}$$

is an isomorphism.

Observe that even in the above special case of a compact Hausdorff group acting on a compact Hausdorff space, a proof is required for the equivalence of freeness of the action and the bijectivity of the canonical map (PWG-condition). Theorem 0.5 brings a new algebraic tool (strong connection) to the realm of compact Hausdorff principal bundles.

In this classical setting, the Peter-Weyl algebra $\mathcal{P}_{C(G)}(C(X))$ is the algebra of continuous global sections of the associated bundle of algebras $X \times_G \mathcal{O}(C(G))$:

$$\mathcal{P}_{C(G)}(C(X)) = \Gamma\Big(X \underset{G}{\times} \mathcal{O}(C(G))\Big). \tag{0.13}$$

Here $\mathcal{O}(C(G))$ is the subalgebra of $C(G)$ spanned by the matrix coefficients of irreducible unitary representations of G. We view $\mathcal{O}(C(G))$ as a representation space of G via the formula

$$\big(\varrho(g)(f)\big)(h) := f(g^{-1}h). \tag{0.14}$$

The algebra $\mathcal{O}(C(G))$ is topologized as the direct limit of its finite-dimensional subspaces. Multiplication and addition of sections is pointwise.

Note that, since $\mathcal{O}(C(G))$ is cosemisimple, it belongs to the category of representations of G that are purely algebraic direct sums of finite-dimensional representations of G. We denote this category by $\mathrm{FRep}^{\oplus}(G)$. Due to the cosemisimplicity of $\mathcal{O}(C(G))$, the following formula for the left coaction of $\mathcal{O}(C(G))$ on V

$$({}_V\Delta(v))(g) := \varrho(g^{-1})(v), \text{ where } \varrho : G \longrightarrow GL(V) \text{ is a representation,} \tag{0.15}$$

establishes an equivalence of $\mathrm{FRep}^{\oplus}(G)$ with the category of all left $\mathcal{O}(C(G))$-comodules. As with the special case $V = \mathcal{O}(C(G))$, all vector spaces in this category are topologized as the direct limits of their finite-dimensional subspaces.

Theorem 0.5 unifies continuous free actions of compact Hausdorff groups on compact Hausdorff spaces and principal actions of affine algebraic groups on affine schemes [DG70, S-P04]. Thus the main result of our paper might be viewed as continuing the Atiyah-Hirzebruch program of transferring ideas (e.g. K-theory) from algebraic geometry to topology [AH59, AH61]. In the same spirit, our main theorem (Theorem 0.4) unifies the C^*-algebraic concept of free actions of compact quantum groups [E-DA00] with the Hopf-algebraic concept of principal coactions [HKMZ11]. Theorem 0.4 implies the existence of strong connections [H-PM96] for free actions of compact quantum groups on unital C^*-algebras (connections on compact quantum principal bundles) thus providing a theoretical foundation for the plethora of concrete constructions studied over the past two decades within the general framework noncommutative geometry [C-A94]. In this paper, we apply Theorem 0.4 to fields of C^*-algebras (Corollary 5.3).

The paper is organized as follows. In Section 1, we prove the key part of our main theorem, that is the equivalence of freeness and the Peter-Weyl-Galois condition. In Section 2, we consider the general algebraic setting of principal coactions. Following Ulbrich [U-KH89] and Schauenburg [S-P04], we prove that the principality of a comodule algebra $\mathcal{P}$ over a Hopf algebra $\mathcal{H}$ is equivalent to the exactness and strong monoidality of the cotensor product functor $\mathcal{P}\Box^{\mathcal{H}}$. In particular, this proves the equivalence of the Peter-Weyl-Galois condition and strong monoidality for actions of compact quantum groups, thus completing the proof of the main theorem.

Although Theorem 0.5 is a special case of Theorem 0.4, the proof we give of Theorem 0.5 is not a special case of the proof of Theorem 0.4. Therefore, we treat Theorem 0.5 separately, and prove it in Section 3. The proof uses the strong monoidality (i.e. the preservation of tensor products) of the Serre-Swan equivalence and a general algebraic argument (Corollary 2.4) of Section 2. In Section 4, we give a vector-bundle interpretation of the aforementioned general algebraic argument. This provides a much desired translation between the algebraic and topological settings.

In Section 5, as an application of our main result, we prove that if a unital C^*-algebra A equipped with an action of a compact quantum group can be fibred over a compact Hausdorff space X with the PWG-condition valid on each fibre, then the PWG-condition is valid for the action on A. We end with an appendix discussing the well-known fact that regularity of a finite covering is equivalent to bijectivity of the canonical map (0.12).

1. Equivalence of freeness and the Peter-Weyl-Galois condition

The implication "PWG-condition $\Longrightarrow$ freeness" is proved as follows. The PWG-condition immediately implies that

$$(\mathcal{P}_H(A)\otimes\mathbb{C})\delta(\mathcal{P}_H(A)) = \mathcal{P}_H(A)\otimes\mathcal{O}(H). \tag{1.1}$$

As the right-hand side is a dense subspace of $A\otimes_{\min} H$ (see [P-P95, Theorem 1.5.1] and [S-PM11, Proposition 2.2]), we obtain the density condition defining freeness.

For the converse implication "PWG-condition $\Longleftarrow$ freeness" we need some preparations. If (V,δ_V) is a finite-dimensional right H-comodule, we write H_V for the smallest vector subspace of H such that $\delta_V(V)\subseteq V\otimes H_V$. We write

$$A_V := \{a\in A \mid \delta(a)\in A\otimes H_V\}. \tag{1.2}$$

Note that in the case $(A,\delta)=(H,\Delta)$, we have $A_V = H_V$. Thus H_V is a coalgebra.

One can define a continuous projection map E_V from A onto A_V as follows [P-P95, Theorem 1.5.1]. Let us call two finite-dimensional comodules of H *disjoint* if the set of morphisms between them only contains the zero map. Then E_V is the unique endomorphism of A which is the identity on A_V and which vanishes on A_W for any finite-dimensional comodule W that is disjoint from V. In the special case of $(A,\delta)=(H,\Delta)$, we use the notation e_V instead of E_V. The equivariance property

$$\delta\circ E_V = (\mathrm{id}\otimes e_V)\circ\delta \tag{1.3}$$

is proved by a straightforward verification. When V is the trivial representation, we write $E_V = E_B$ and $e_V = \varphi_H$, where $B := A^{\mathrm{co}\,H}$ is the algebra of coaction invariants and φ_H is the invariant state on H. Then the formula (1.3) specializes to

$$E_B = (\mathrm{id} \otimes \varphi_H) \circ \delta. \tag{1.4}$$

The key lemma in the proof of Theorem 0.4 is:

Lemma 1.1 (Theorem 1.2 in [DY13]). *Let $\delta\colon A \to A \otimes_{\min} H$ be a free coaction, and let V be a finite-dimensional H-comodule. Then A_V is finitely generated projective as a right B-module.*

Note that in the classical case $X \times G \to X$, we have $H = C(G)$ and $B = C(X/G)$. The B-module A_V is then $\Gamma(X \times_G H_V)$, and thus it is finitely generated projective.

Define a B-valued inner product on A_V by

$$\langle a, b\rangle_B := E_B(a^* b). \tag{1.5}$$

Lemma 1.2 (Corollary 2.6 in [DY13]). *The B-valued inner product* (1.5) *makes A_V a right Hilbert B-module* [L-EC95]. *The Hilbert module norm $\|a\|_B := \|\langle a, a\rangle_B\|^{1/2}$ is equivalent to the C^*-norm of A restricted to A_V.*

We will need the following lemma concerning the interior tensor product of Hilbert modules.

Lemma 1.3 (cf. Proposition 4.5 in [L-EC95]). *Let C and D be unital C^*-algebras, and let $(\mathscr{E}, \langle\,\cdot\,,\,\cdot\,\rangle_C)$ be a right Hilbert C-module that is finitely generated projective as a right C-module. Let $(\mathscr{F}, \langle\,\cdot\,,\,\cdot\,\rangle_D)$ be an arbitrary right Hilbert D-module, and $\pi : C \to \mathcal{L}(\mathscr{F})$ be a unital *-homomorphism of C into the C^*-algebra of adjointable operators on $\mathscr{F}$. Then the algebraic tensor product $\mathscr{E} \otimes_C \mathscr{F}$ is a right Hilbert D-module with respect to the inner product given by*

$$\langle x \otimes y, z \otimes w\rangle := \langle y, \pi(\langle x, z\rangle_C) w\rangle_D.$$

Proof. We need to prove that the semi-norm $\|z\| = \|\langle z, z\rangle_D\|^{1/2}$ on $\mathscr{E} \otimes_C \mathscr{F}$ is in fact a norm with respect to which $\mathscr{E} \otimes_C \mathscr{F}$ is complete. The statement obviously holds for $\mathscr{E} = C^n$, the n-fold direct sum of the standard right C-module C. Since $\mathscr{E}$ is finitely generated projective, $\mathscr{E}$ can be realized as a direct summand of C^n, so that the conclusion also applies in this case. □

We are now ready to prove the implication "PWG-condition $\Longleftarrow$ freeness". By the freeness assumption, the image of *can* is dense in $A \otimes H$. In particular, for a given finite-dimensional comodule V and any $h \in H_V$, we can find a sequence $k_n \in \mathbb{N}$ and elements $p_{n,i}$ and $q_{n,i}$ in $\mathcal{P}_H(A)$ with $1 \leq i \leq k_n$ such that

$$\sum_{i=1}^{k_n} (p_{n,i} \otimes 1)\delta(q_{n,i}) \underset{n\to\infty}{\longrightarrow} 1 \otimes h \tag{1.6}$$

in the C^*-norm. Applying $\mathrm{id} \otimes e_V$ to this expression, and using (1.3), we see that we can take $q_{n,i} \in A_V$.

Applying δ to the first leg of (1.6) and using coassociativity, we obtain

$$\sum_{i=1}^{k_n} (\delta(p_{n,i}) \otimes 1)(\mathrm{id} \otimes \Delta)(\delta(q_{n,i})) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes h. \tag{1.7}$$

Observe now that, since $q_{n,i} \in A_V$, by (1.2) we obtain $(\mathrm{id} \otimes \Delta)(\delta(q_{n,i})) \in A_V \otimes H_V \otimes H_V$. Hence the left-hand side of (1.7) belongs to the tensor product $(A \otimes_{\min} H) \otimes H_V$. As H_V is finite dimensional, the restriction of the antipode S of $\mathcal{O}(H)$ to H_V is continuous. Therefore, we can apply S to the third leg of (1.7) to conclude

$$\sum_{i=1}^{k_n} (\delta(p_{n,i}) \otimes 1)(\mathrm{id} \otimes (\mathrm{id} \otimes S) \circ \Delta)(\delta(q_{n,i})) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes S(h). \tag{1.8}$$

Again by the finite dimensionality of H_V, multiplying the second and third legs is a continuous operation, so that

$$\sum_{i=1}^{k_n} \delta(p_{n,i})(q_{n,i} \otimes 1) \underset{n\to\infty}{\longrightarrow} 1 \otimes S(h). \tag{1.9}$$

Since $S(h) \in H_{\bar{V}}$, where $\bar{V}$ is the contragredient of V, applying $\mathrm{id} \otimes e_{\bar{V}}$ to the above limit and using the equivariance property (1.3) shows that in the above limit we can choose $p_{n,i} \in A_{\bar{V}}$.

Consider now the right B-module map

$$G_V : A_{\bar{V}} \underset{B}{\otimes} A_V \longrightarrow A_{\bar{V}\otimes V} \otimes H_{\bar{V}}, \qquad a \otimes b \longmapsto \delta(a)(b \otimes 1). \tag{1.10}$$

By Lemma 1.1 and Lemma 1.3, the algebraic tensor product on the left-hand side becomes an interior tensor product of right Hilbert B-modules. The inner product for $A_{\bar{V}} \otimes_B A_V$ is

$$\langle c \otimes d, a \otimes b\rangle_B = E_B(d^* E_B(c^* a) b). \tag{1.11}$$

On the other hand, equipping $H_{\bar{V}}$ with the Hilbert-space structure $\langle h, k\rangle = \varphi_H(h^* k)$, the right-hand side is a right Hilbert B-module with inner product

$$\langle b \otimes h, a \otimes g\rangle_B = \varphi_H(h^* g) E_B(b^* a). \tag{1.12}$$

It follows from these formulas and (1.4) that G_V is an isometry between these Hilbert modules. Hence the range of G_V is closed.

From (1.9) and the equivalence of C^*-module and Hilbert C^*-module norms in Lemma 1.2, we infer that the range of G_V contains $1 \otimes S(h)$. Therefore, as the domain of G_V is an algebraic tensor product, we can find a finite number of elements $p_i, q_i \in \mathcal{P}_H(A)$ such that

$$\sum_i \delta(p_i)(q_i \otimes 1) = 1 \otimes S(h). \tag{1.13}$$

Now applying the map $a \otimes g \mapsto (1 \otimes S^{-1}(g))\delta(a)$ to both sides yields

$$\sum_i (p_i \otimes 1)\delta(q_i) = 1 \otimes h. \tag{1.14}$$

As h was arbitrary in $\mathcal{O}(H)$, it follows that *can* is surjective.

Finally, as the Hopf algebra $\mathcal{O}(H)$ is cosemisimple, according to [S-HJ90, Remark 3.9], bijectivity of the canonical map *can* follows from its surjectivity. This completes the proof of the implication "PWG-condition $\Longleftarrow$ freeness".

2. Equivalence of principality and strong monoidality

The framework of principal comodule algebras unifies in one category many algebraically constructed noncommutative examples and classical compact Hausdorff principal bundles.

Definition 2.1 ([BH04]). *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode, and let $\Delta_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$ be a coaction making $\mathcal{P}$ an $\mathcal{H}$-comodule algebra. We call $\mathcal{P}$* principal *if and only if:*

1. *$\mathcal{P}\otimes_{\mathcal{B}}\mathcal{P} \ni p \otimes q \mapsto can(p \otimes q) := (p \otimes 1)\Delta_{\mathcal{P}}(q) \in \mathcal{P} \otimes \mathcal{H}$ is bijective, where $\mathcal{B} := \mathcal{P}^{\mathrm{co}\,\mathcal{H}} := \{p \in \mathcal{P} \mid \Delta_{\mathcal{P}}(p) = p \otimes 1\}$;*
2. *there exists a left $\mathcal{B}$-linear right $\mathcal{H}$-colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{P} \to \mathcal{P}$.*

Here (1) is the Hopf-Galois condition and (2) is the right equivariant left projectivity of $\mathcal{P}$.

Alternately, one can approach principality through strong connections:

Definition 2.2 ([BH04]). *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode S, and $\Delta_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$ be a coaction making $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra. A* strong connection *ℓ on $\mathcal{P}$ is a unital linear map $\ell : \mathcal{H} \to \mathcal{P} \otimes \mathcal{P}$ satisfying:*

1. $(\mathrm{id} \otimes \Delta_{\mathcal{P}}) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta$*;*
2. $({}_{\mathcal{P}}\Delta \otimes \mathrm{id}) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta$*, where* ${}_{\mathcal{P}}\Delta := (S^{-1} \otimes \mathrm{id}) \circ \mathrm{flip} \circ \Delta_{\mathcal{P}}$*;*
3. $\widetilde{can} \circ \ell = 1 \otimes \mathrm{id}$*, where* $\widetilde{can}\colon \mathcal{P} \otimes \mathcal{P} \ni p \otimes q \mapsto (p \otimes 1)\Delta_{\mathcal{P}}(q) \in \mathcal{P} \otimes \mathcal{H}$*.*

One can prove (see [BH] and references therein) that a comodule algebra is principal if and only if it admits a strong connection.

If $\Delta_M : M \to M \otimes \mathcal{C}$ is a coaction making M a right comodule over a coalgebra $\mathcal{C}$ and N is a left $\mathcal{C}$-comodule via a coaction ${}_N\Delta\colon N \to \mathcal{C} \otimes N$, then we define their *cotensor product* as

$$M \overset{\mathcal{C}}{\square} N := \big\{ t \in M \otimes N \mid (\Delta_M \otimes \mathrm{id})(t) = (\mathrm{id} \otimes {}_N\Delta)(t) \big\}. \tag{2.1}$$

In particular, for a right $\mathcal{H}$-comodule algebra $\mathcal{P}$ and a left $\mathcal{H}$-comodule V, we observe that $\mathcal{P}\square^{\mathcal{H}} V$ is a left $\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$- module in a natural way. One of the key properties of principal comodule algebras is that, for any finite-dimensional left $\mathcal{H}$-comodule V, the left $\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$-module $\mathcal{P}\square^{\mathcal{H}} V$ is finitely generated projective [BH04]. Here $\mathcal{P}$ plays the role of a principal bundle and $\mathcal{P}\square^{\mathcal{H}} V$ plays the role of an associated vector bundle. Therefore, we call $\mathcal{P}\square^{\mathcal{H}} V$ an *associated module.*

Principality can also be characterized by the exactness and strong monoidality of the cotensor functor. This characterisation uses the notion of coflatness of a comodule: a right comodule is *coflat* if and only if cotensoring it with left comodules preserves exact sequences.

Theorem 2.3. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode and $\mathcal{P}$ be a right $\mathcal{H}$-comodule algebra. Then $\mathcal{P}$ is principal if and only if $\mathcal{P}$ is right $\mathcal{H}$-coflat and for all left $\mathcal{H}$-comodules V and W the map*

$$\beta : (\mathcal{P}\square V) \underset{\mathcal{B}}{\otimes} (\mathcal{P}\square W) \longrightarrow \mathcal{P}\square(V \otimes W),$$
$$\Big(\sum_i a_i \otimes v_i\Big) \otimes \Big(\sum_j b_j \otimes w_j\Big) \longmapsto \sum_{i,j} a_i b_j \otimes (v_i \otimes w_j),$$

is bijective. In other words, $\mathcal{P}$ is principal if and only if the cotensor product functor is exact and strongly monoidal with respect to the above map β.

Proof. The proof relies on putting together [S-HJ90, Theorem I], [S-P98, Theorem 6.15], [BH04, Theorem 2.5] and [SS05, Theorem 5.6]. First assume that $\mathcal{P}$ is principal. Then $\mathcal{P}$ is right equivariantly projective, and it follows from [BH04, Theorem 2.5] that $\mathcal{P}$ is faithfully flat. Now we can apply [S-P98, Theorem 6.15] to conclude that β is bijective. Furthermore, by [S-HJ90, Theorem I], the faithful flatness of $\mathcal{P}$ implies the coflatness of $\mathcal{P}$. Conversely, assume that cotensoring with $\mathcal{P}$ is exact and strongly monoidal with respect to β. Then substituting $\mathcal{H}$ for V and W yields the Hopf-Galois condition. Now [SS05, Theorem 5.6] implies the equivariant projectivity of $\mathcal{P}$. □

Corollary 2.4. *Let A be a unital C^*-algebra equipped with an action of a compact quantum group (H, Δ) given by $\delta\colon A \to A \otimes_{\min} H$. Then the following are equivalent:*

- *The action of (H, Δ) on A satisfies the Peter-Weyl-Galois condition.*
- *The action of (H, Δ) on A is strongly monoidal.*

Proof. The Hopf algebra $\mathcal{O}(H)$ always has bijective antipode. It follows from [W-SL87, Theorem 4.2] and [BH04, Lemma 2.4] that any comodule over this Hopf algebra is coflat. Hence [SS05, Theorem 5.6] implies that the equivariant projectivity condition (i.e. Condition (2) of Definition 2.1) is valid for any $\mathcal{O}(H)$-comodule algebra such that the canonical map is bijective. The corollary now follows from Theorem 2.3. (As an alternative to [SS05, Theorem 5.6], one can use the combination of [BB08, Theorem 4] and [BH04, Lemma 2.2].) $\square$

3. The classical case

In this section, we prove our main result in the classical case, i.e. we prove Theorem 0.5. As in the proof of the general noncommutative case, we rely on the fact that the module of continuous sections of an associated vector bundle is finitely generated projective. However, unlike in the proof in Section 1, herein we first prove strong monoidality, and then conclude the PWG-condition. An entirely different proof of Theorem 0.5, using local triviality, can be found in [BH14].

To be consistent with general notation, we should only use C^*-algebras $C(G)$, $C(X)$, etc., rather than spaces themselves. However, this would make formulas too cluttered, so that throughout this section we consistently omit writing $C(\ )$ in the subscript and the argument of the Peter-Weyl functor.

The implication "PWG-condition $\Longrightarrow$ freeness" is proved as follows. The PWG-condition immediately implies that

$$(\mathcal{P}_G(X) \otimes \mathbb{C})\delta(\mathcal{P}_G(X)) = \mathcal{P}_G(X) \otimes \mathcal{O}(G). \tag{3.1}$$

As the right-hand side is a dense subspace of $C(X) \otimes_{\min} C(G)$, we obtain the density condition (0.10). The latter is equivalent to freeness, as explained in the introduction.

For the converse implication "PWG-condition $\Longleftarrow$ freeness", we shall use the Serre-Swan theorem.

Theorem 3.1 ([S-R62]). *Let Y be a compact Hausdorff topological space. Then a $C(Y)$-module is finitely generated and projective if and only if it is isomorphic to the module of continuous global sections of a vector bundle over Y.*

For a compact Hausdorff topological space Y, we denote by $\mathrm{Vect}(Y)$ the category of $\mathbb{C}$ vector bundles on Y. An object in $\mathrm{Vect}(Y)$ is a $\mathbb{C}$ vector bundle E with base space Y. The projection of E onto Y is denoted by $\pi_E \colon E \to Y$. A section of E is a continuous map

$$s\colon Y \longrightarrow E \quad \text{with} \quad \pi_E \circ s = \mathrm{id}_Y\,. \tag{3.2}$$

A morphism in $\mathrm{Vect}(Y)$ is a vector bundle map, i.e. a continuous map

$$\varphi\colon E \longrightarrow F \quad \text{such that} \quad \pi_F \circ \varphi = \pi_E \tag{3.3}$$

and, for all $y \in Y$, the restriction-corestriction map $\varphi_y \colon \pi_E^{-1}(y) \to \pi_F^{-1}(y)$ is a linear map between finite-dimensional vector spaces.

View the commutative C^*-algebra $C(Y)$ as a commutative ring with unit. Denote by $\mathrm{FProj}(C(Y))$ the category of finitely generated projective $C(Y)$-modules. An object in the category $\mathrm{FProj}(C(Y))$ is a finitely generated projective $C(Y)$-module. A morphism in $\mathrm{FProj}(C(Y))$ is a map of $C(Y)$-modules $\psi \colon M \to N$.

If E is a $\mathbb{C}$ vector bundle on Y, then $\Gamma(E)$ denotes the $C(Y)$-module consisting of all continuous sections of E. The module structure is pointwise. According to the Serre-Swan theorem, the functor Γ

$$\mathrm{Vect}(Y) \longrightarrow \mathrm{FProj}(C(Y)), \qquad E \longmapsto \Gamma(E), \tag{3.4}$$

is an equivalence of categories and preserves all the basic properties of the two categories. In particular, $E \longmapsto \Gamma(E)$ preserves $\oplus$ and $\otimes$:

$$\begin{aligned} \Gamma(E \oplus F) &= \Gamma(E) \oplus \Gamma(F), \\ \Gamma(E \otimes F) &= \Gamma(E) \underset{C(Y)}{\otimes} \Gamma(F). \end{aligned} \tag{3.5}$$

Let X be a compact Hausdorff space equipped with a continuous free action of a compact Hausdorff group G. Next, let $\mathrm{FRep}(G)$ denote the category of representations of G on finite-dimensional complex vector spaces. Due to the freeness asumption, we can define the functor

$$\mathrm{FRep}(G) \longrightarrow \mathrm{Vect}(X/G), \quad V \longmapsto X \underset{G}{\times} V, \tag{3.6}$$

preserving $\oplus$ and $\otimes$:

$$\begin{aligned} X \underset{G}{\times} (V \oplus W) &= (X \underset{G}{\times} V) \,\oplus\, (X \underset{G}{\times} W), \\ X \underset{G}{\times} (V \otimes W) &= (X \underset{G}{\times} V) \,\otimes\, (X \underset{G}{\times} W). \end{aligned} \tag{3.7}$$

Combining the functor Γ with the functor $X \times_G$ yields the functor

$$\mathrm{FRep}(G) \longrightarrow \mathrm{FProj}(C(X/G)), \quad V \longmapsto \Gamma(X \underset{G}{\times} V). \tag{3.8}$$

Furthermore, note that the $C(X/G)$-module $C_G(X, V)$ of all continuous G-equivariant functions from X to V is naturally isomorphic with $\Gamma(X \times_G V)$. Here G-equivariance means

$$\forall\, x \in X,\, g \in G: \ f(xg) = \varrho(g^{-1})(f(x)), \qquad \varrho: \ G \longrightarrow GL(V). \tag{3.9}$$

Hence we can replace the above $\otimes$-preserving functor with the $\otimes$-preserving functor

$$\mathrm{FRep}(G) \longrightarrow \mathrm{FProj}(C(X/G)), \quad V \longmapsto C_G(X, V). \tag{3.10}$$

The following elementary observation is key in translating from the topological to the algebraic setting.

Lemma 3.2. *Let X be a compact Hausdorff space equipped with a continuous action of a compact Hausdorff group G, and let V be a finite-dimensional representation of G. Then the evident identification $C(X,V) = C(X) \otimes V$ determines an equivalence of tensor functors:*

$$C_G(X,V) = \mathcal{P}_G(X)\Box V.$$

Proof. Let $\{e_i\}_{i=1}^n$ be a basis of V and $\{e^i\}_{i=1}^n$ be the basis of V^* dual to $\{e_i\}_{i=1}^n$. Given $f \in C(X,V)$, we note that

$$\begin{gathered}\sum_{i=1}^n (e^i \circ f) \otimes e_i \in \mathcal{P}_G(X)\Box V \\ \Updownarrow \\ \sum_{i=1}^n \delta(e^i \circ f) \otimes e_i = \sum_{i=1}^n (e^i \circ f) \otimes_V \Delta(e_i) \\ \Updownarrow \\ \forall\, x \in X,\ g \in G:\ f(xg) = \varrho(g^{-1})(f(x)). \end{gathered} \tag{3.11}$$

The second equivalence is an immediate consquence of the definitions of δ and ϱ (see (0.11) and (0.15)). The first equivalence follows directly from the definition of cotensor product (see (0.7)) and the fact that

$$\sum_{i=1}^n (e^i \circ f) \otimes_V \Delta(e_i) \in C(X) \otimes \mathcal{O}(G) \otimes V. \tag{3.12}$$

Thus the evident identification yields $C_G(X,V) = \mathcal{P}_G(X)\Box V$.

Finally, let β be the map defined in Theorem 2.3, and let

$$\begin{gathered}\mathrm{diag}\colon C_G(X,V) \underset{C(X/G)}{\otimes} C_G(X,W) \longrightarrow C_G(X, V \otimes W), \\ \mathrm{diag}\colon f_1 \otimes f_2 \longmapsto \big(x \mapsto f_1(x) \otimes f_2(x)\big). \end{gathered} \tag{3.13}$$

The commutativity of the diagram

$$\begin{tikzcd} C_G(X,V) \underset{C(X/G)}{\otimes} C_G(X,W) \arrow[r, "\mathrm{diag}"] \arrow[d] & C_G(X, V\otimes W) \arrow[d] \\ (\mathcal{P}_G(X)\Box V) \underset{C(X/G)}{\otimes} (\mathcal{P}_G(X)\Box W) \arrow[r, "\beta"] & \mathcal{P}_G(X)\Box(V \otimes W) \end{tikzcd} \tag{3.14}$$

proves that the identification $C_G(X,V) = \mathcal{P}_G(X)\Box V$ defines an equivalence of tensor functors. $\square$

Assume now that the action of G on X is free. Then, by the Serre-Swan theorem, the functor $\Gamma(X\times_G\ )$ is strongly monoidal. Since it is equivalent as a tensor functor to $C_G(X,\ )$, we conclude from Lemma 3.2 that the cotensor product functor

$$\mathrm{FRep}(G) \longrightarrow \mathrm{FProj}(C(X/G)), \quad V \longmapsto \mathcal{P}_G(X)\Box V, \tag{3.15}$$

is also strongly monoidal.

Next, since $\mathcal{O}(G)$ is cosemisimple, any $\mathcal{O}(G)$-comodule is a purely algebraic direct sum of finite-dimensional comodules. Furthermore, as the cotensor product is defined as the kernel of a linear map, it commutes with such direct sums. As it is also clear that the map β commutes with such direct sums, we infer that the extended cotensor product functor

$$\mathrm{FRep}^{\oplus}(G) \longrightarrow \mathrm{FProj}^{\oplus}(C(X/G)), \quad V \longmapsto \mathcal{P}_G(X)\Box V, \tag{3.16}$$

is strongly monoidal. Here $\mathrm{FProj}^{\oplus}(C(X/G))$ is the category of projective modules over $C(X/G)$ that are purely algebraic direct sums of finitely generated projective $C(X/G)$-modules, and $\mathrm{FRep}^{\oplus}(G)$ is the category of representations of G defined above (0.15). (One can think of these categories as the ind-completions in the sense of [AGV72, Expose I, Section 8.2].) Combining this with Corollary 2.4 allows us to conclude the proof of the implication "PWG-condition $\Longleftarrow$ freeness".

4. Vector-bundle interpretation

We now give a vector-bundle interpretation of the proof of the preceding section. To this end, we need to extend the functor $C_G(X,\ )$ to the category $\mathrm{FRep}^{\oplus}(G)$, which includes the representation $\mathcal{O}(G)$. Let V be a purely algebraic direct sum of finite-dimensional representations of G. We topologize V as the direct limit of its finite-dimensional subspaces, and denote by $C(X,V)$ the space of all continuous maps from X to V. An elementary topological argument shows that the image of any continuous map from X to V is contained in a finite-dimensional subspace of V. Therefore, Lemma 3.2 generalizes to:

Corollary 4.1. *Let V be an object in the category $\mathrm{FRep}^{\oplus}(G)$. Then the evident identification $C(X,V)=C(X)\otimes V$ determines an equivalence of tensor functors:*

$$C_G(X,V)=\mathcal{P}_G(X)\Box V.$$

Taking $V=\mathcal{O}(G)$ topologized with the direct limit topology, we immediately obtain the following presentation of the Peter-Weyl algebra:

$$C_G(X,\mathcal{O}(G))=\mathcal{P}_G(X)\Box\mathcal{O}(G)=\mathcal{P}_G(X). \tag{4.17}$$

Assume now that the action of G on X is free. Then $X \times_G \mathcal{O}(G)$ is a vector bundle in the sense that it is a direct sum of ordinary (i.e. with finite-dimensional fibers) vector bundles, and

$$\Gamma(X \times_G \mathcal{O}(G)) = C_G(X, \mathcal{O}(G)) = \mathcal{P}_G(X). \tag{4.18}$$

Moreover, arguing as for the cotensor product functor, we conclude that the functor

$$\mathrm{FRep}^{\oplus}(G) \longrightarrow \mathrm{FProj}^{\oplus}(C(X/G)), \quad V \longmapsto C_G(X, V), \tag{4.19}$$

is strongly monoidal. Hence, taking advantage of (4.17), we obtain

$$C_G(X, \mathcal{O}(G) \otimes \mathcal{O}(G)) = \mathcal{P}_G(X) \underset{C(X/G)}{\otimes} \mathcal{P}_G(X). \tag{4.20}$$

Next, denote by $\mathcal{O}(G)^{\mathrm{trivial}}$ the vector space $\mathcal{O}(G)$ with the trivial action of G, i.e. every $g \in G$ is acting by the identity map of $\mathcal{O}(G)$. Then, as before, we obtain

$$\begin{aligned} C_G(X, \mathcal{O}(G) \otimes \mathcal{O}(G)^{\mathrm{trivial}}) &= \mathcal{P}_G(X) \underset{C(X/G)}{\otimes} C(X/G) \otimes \mathcal{O}(G) \\ &= \mathcal{P}_G(X) \otimes \mathcal{O}(G). \end{aligned} \tag{4.21}$$

Lemma 4.2. *The G-equivariant homeomorphism*

$$W \colon G \times G^{\mathrm{trivial}} \longrightarrow G \times G, \quad W((g, g')) := (g, gg'),$$

gives an isomorphism of representations of G

$$\mathcal{O}(G) \otimes \mathcal{O}(G)^{\mathrm{trivial}} \cong \mathcal{O}(G) \otimes \mathcal{O}(G).$$

Here $G \times G^{\mathrm{trivial}}$ and $G \times G$ are right G-spaces via the formulas

$$(g, g')h := (h^{-1}g, g') \qquad \text{and} \qquad (g, g')h := (h^{-1}g, h^{-1}g'),$$

respectively.

Proof. Since $\mathcal{O}(G)$ is a Hopf algebra, the pullback of W restricts and corestricts to

$$W^* \colon \mathcal{O}(G) \otimes \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)^{\mathrm{trivial}}. \tag{4.22}$$

Taking into account (0.14) and (0.15), we infer that W^* is the required intertwining operator. □

Combining Lemma 4.2 with (4.20) and (4.21) gives

$$\mathcal{P}_G(X) \underset{C(X/G)}{\otimes} \mathcal{P}_G(X) \cong \mathcal{P}_G(X) \otimes \mathcal{O}(G). \tag{4.23}$$

Finally, to see that this isomorphism is indeed the canonical map, we explicitly put together all identifications used on the way. First, we observe that, since the isomorphism

$$\mathcal{P}_G(X) \longrightarrow \mathcal{P}_G(X) \square \mathcal{O}(G) \tag{4.24}$$

is given by the coaction δ, the identification (4.17) is implemented by the maps

$$\mathcal{P}_G(X) \underset{F}{\overset{E}{\rightleftarrows}} C_G(X, \mathcal{O}(G)),$$

$$\big(E(f)(x)\big)(g) := f(xg),\ F(\alpha)(x) := \alpha(x)(e),\ E \circ F = \mathrm{id},\ F \circ E = \mathrm{id}. \tag{4.25}$$

We can now easily check that the following composition of isomorphisms

$$\mathcal{P}_G(X) \underset{C(X/G)}{\otimes} \mathcal{P}_G(X) \xrightarrow{E \otimes E} C_G(X, \mathcal{O}(G)) \underset{C(X/G)}{\otimes} C_G(X, \mathcal{O}(G)) \xrightarrow{\mathrm{diag}}$$

$$C_G\big(X, \mathcal{O}(G) \otimes \mathcal{O}(G)\big) \xrightarrow{W^* \circ} C_G\big(X, \mathcal{O}(G) \otimes \mathcal{O}(G)^{\mathrm{trivial}}\big) \xrightarrow{\sum_i (\mathrm{id} \otimes e^i) \otimes e_i}$$

$$C_G(X, \mathcal{O}(G)) \otimes \mathcal{O}(G) \xrightarrow{F \otimes \mathrm{id}} \mathcal{P}_G(X) \otimes \mathcal{O}(G)$$

is the canonical map, as desired.

5. Application: fields of free actions

Let A be a unital C^*-algebra with center $Z(A)$, let X be a compact Hausdorff space and let $\theta : C(X) \to Z(A)$ be a unital inclusion. The triple $(A, C(X), \theta)$ is called a unital $C(X)$-*algebra* ([K-G88, p. 154]). In the following, we simply consider $C(X)$ as a subalgebra of A. For $x \in X$, let J_x be the closed two-sided ideal in A generated by the functions $f \in C(X)$ that vanish at x. Then we have quotient C^*-algebras $A_x := A/J_x$ with natural projection maps $\pi_x : A \to A_x$, and the triple (X, A, π_x) is a *field of* C^**-algebras*. For any $a \in A$, the map $n_x : X \to \mathbb{R}, x \mapsto \|\pi_x(a)\|$ is upper semi-continuous [DG83, Theorem 2.4] (see also [R-MA89, Proposition 1.2]). If the latter map is continuous, the field is called continuous, but this property will not be necessary to assume for our purposes.

LEMMA 5.1. *Let X be a compact Hausdorff space, A a unital $C(X)$-algebra, and (H, Δ) a compact quantum group acting on A via $\delta : A \to A \otimes_{\min} H$. Assume that $C(X) \subseteq A^{\mathrm{co}\, H}$. Then for each $x \in X$ there exists a unique coaction $\delta_x : A_x \to A_x \otimes_{\min} H$ such that for all $a \in A$*

$$\delta_x(\pi_x(a)) = (\pi_x \otimes \mathrm{id})(\delta(a)). \tag{5.1}$$

Proof. Let $x \in X$ and $f \in C(X)$ with $f(x) = 0$. As $\delta(f) = f \otimes 1$ by assumption, it follows that $(\pi_x \otimes \mathrm{id})(\delta(f)) = 0$. Hence $(\pi_x \otimes \mathrm{id})(\delta(a)) = 0$ for $a \in J_x$, so that δ_x can be defined by (5.1). It is straightforward to check that each δ_x satisfies the coassociativity and counitality conditions.

Finally, to see that δ_x is injective, assume that $\delta_x(\pi_x(a)) = 0$. Then

$$(\pi_x \otimes \mathrm{id})(\delta(a)) = 0, \tag{5.2}$$

whence $(\mathrm{id} \otimes \omega)(\delta(a)) \in J_x$ for all $\omega \in A^*$. In particular, if $(g_\alpha)_\alpha$ is a bounded positive approximate unit for $C_0(X \setminus \{x\})$, then

$$g_\alpha(\mathrm{id} \otimes \omega)(\delta(a)) \underset{\alpha}{\overset{\mathrm{norm}}{\longrightarrow}} (\mathrm{id} \otimes \omega)(\delta(a)). \tag{5.3}$$

Hence we obtain

$$(g_\alpha \otimes 1)\delta(a) \xrightarrow[\alpha]{\text{weakly}} \delta(a). \tag{5.4}$$

However, as $(g_\alpha \otimes 1)\delta(a) = \delta(g_\alpha a)$ and δ is injective, we find that

$$g_\alpha a \xrightarrow[\alpha]{\text{weakly}} a. \tag{5.5}$$

Consequently, $\pi_x(a) = 0$, and we conclude that δ_x is injective. □

THEOREM 5.2. *Let X be a compact Hausdorff space, A a unital $C(X)$-algebra, and (H, Δ) a compact quantum group acting on A via $\delta : A \to A \otimes_{\min} H$. Assume that $C(X) \subseteq A^{\operatorname{co} H}$. Then, the coaction δ is free if and only if the coactions δ_x are free for each $x \in X$.*

Proof. First note that $A \otimes_{\min} H$ is again a $C(X)$-algebra in a natural way. We will denote the quotient $(A \otimes_{\min} H)/(J_x \otimes_{\min} H)$ by $A_x \otimes_x H$. This will be a C^*-completion of the algebraic tensor product algebra $A_x \otimes H$ (not necessarily the minimal one). We will denote the quotient map at x by $\pi_x \otimes_x \mathrm{id} : A \otimes_{\min} H \to A_x \otimes_x H$.

The implication "δ is free $\Longrightarrow$ the coactions δ_x are free for each $x \in X$" follows immediately from the commutativity of the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "can"] \arrow[d, "\pi_x \otimes \pi_x"'] & A \underset{\min}{\otimes} H \arrow[d, "\pi_x \underset{x}{\otimes} \mathrm{id}"] \\ A_x \otimes A_x \arrow[r] & A_x \underset{x}{\otimes} H . \end{tikzcd} \tag{5.6}$$

Here the upper horizontal arrow is given by the formula $a \otimes a' \mapsto (a \otimes 1)\delta(a')$, and the lower horizontal arrow is given by $a \otimes a' \mapsto (a \otimes 1)\delta_x(a')$.

Assume now that each δ_x is free. Fix $\varepsilon > 0$, and choose $h \in \mathcal{O}(H)$. By Theorem 0.4, for each $x \in X$ we can find an element $z_x \in (A \otimes \mathbb{C})\delta(A)$ such that $(\pi_x \otimes_x \mathrm{id})(z_x) = 1 \otimes h$ in $A_x \otimes_x H$. Consider the function

$$f_x : X \ni y \longmapsto \|(\pi_y \otimes_y \mathrm{id})(z_x - 1 \otimes h)\| = \|(\pi_y \otimes_y \mathrm{id})(z_x) - 1 \otimes h\| \in \mathbb{R}. \tag{5.7}$$

As the norm on the field $y \mapsto A_y \otimes_y H$ is upper semi-continuous, the function $y \mapsto f_x(y)$ is upper semi-continuous. Since $f_x(x) = 0$, we can find an open neighborhood U_x of x such that for all $y \in U_x$

$$f_x(y) = \|(\pi_y \otimes_y \mathrm{id})(z_x) - 1 \otimes h\|_{A_y \otimes_y H} < \varepsilon. \tag{5.8}$$

Let $\{f_i\}_i$ be a partition of unity subordinate to a finite subcover $\{U_{x_i}\}_i$. An easy estimate shows that for $z := \sum_i (f_i \otimes 1) z_{x_i}$ and all $y \in X$

$$\|(\pi_y \otimes_y \mathrm{id})(z - 1 \otimes h)\|_{A_y \otimes_y H} < \varepsilon. \tag{5.9}$$

Taking the supremum over all y, we conclude by [DG83, Theorem 2.4] and the compactness of X that $\|z - 1 \otimes h\| < \varepsilon$. Hence $(A \otimes \mathbb{C})\delta(A)$ is dense in $A \otimes H$, i.e. the coaction δ is free. $\square$

Combining Theorem 0.4 and Theorem 5.2, we obtain:

Corollary 5.3. *Let X be a compact Hausdorff space, A a unital $C(X)$-algebra, and (H, Δ) a compact quantum group acting on A via $\delta : A \to A \otimes_{\min} H$. Assume that $C(X) \subseteq A^{\mathrm{co}\, H}$. Then, the coaction δ satisfies the PWG-condition if and only if the coactions δ_x satisfy the PWG-condition for each $x \in X$.*

As a particular case we consider:

Definition 5.4 (cf. [DHH15]). *Let (H, Δ) be a compact quantum group acting on a unital C^*-algebra A via $\delta : A \to A \otimes_{\min} H$. We call the unital C^*-algebra*

$$A \overset{\delta}{\circledast} H := \left\{ f \in C\big([0,1], A \underset{\min}{\otimes} H\big) \,\middle|\, f(0) \in \mathbb{C} \otimes H,\ f(1) \in \delta(A) \right\}$$

the equivariant noncommutative join *of A and H.*

The C^*-algebra $A \circledast^\delta H$ is obviously a $C([0,1])$-algebra with $(A \circledast^\delta H)_x = A \otimes_{\min} H$ for $x \in (0,1)$, $(A \circledast^\delta H)_0 = H$ and $(A \circledast^\delta H)_1 \cong A$. The following lemma shows that $A \circledast^\delta H$ carries a natural action of (H, Δ).

Lemma 5.5. *The compact quantum group (H, Δ) acts on the unital C^*-algebra $A \circledast^\delta H$ via*

$$\delta_{A \circledast^\delta H} \colon A \overset{\delta}{\circledast} H \ni f \longmapsto (\mathrm{id} \otimes \Delta) \circ f \in (A \overset{\delta}{\circledast} H) \underset{\min}{\otimes} H.$$

Proof. We first show that the range of $\delta_{A \circledast^\delta H}$ is contained in $(A \circledast^\delta H) \otimes_{\min} H$. To this end, we take any function $f \in A \circledast^\delta H$ and identify $(A \circledast^\delta H) \otimes_{\min} H$ as a subalgebra of $C\big([0,1], A \otimes_{\min} H \otimes_{\min} H\big)$. Since f is uniformly continuous and $\mathcal{P}_H(A)$ is dense in A by [P-P95, Theorem 1.5.1] and [S-PM11, Proposition 2.2], an elementary partition of unity argument shows that f can be approximated by finite sums of functions of three kinds:

(1) $F_1 : [0,1] \ni t \mapsto \xi_0(t)(1 \otimes h) \in \mathbb{C} \otimes \mathcal{O}(H)$, where $\xi_0 \in C([0,1],[0,1])$, $\xi_0(1) = 0$, and h is a fixed element of $\mathcal{O}(H)$;
(2) $F_2 : [0,1] \ni t \mapsto \xi(t)(a \otimes h) \in \mathcal{P}_H(A) \otimes \mathcal{O}(H)$, where $\xi \in C([0,1],[0,1])$ with $\xi(0) = 0 = \xi(1)$, and a and h are respectively fixed elements of $\mathcal{P}_H(A)$ and $\mathcal{O}(H)$;
(3) $F_3 : [0,1] \ni t \mapsto \xi_1(t)\delta(a) \in \delta(\mathcal{P}_H(A))$, where $\xi_1 \in C([0,1],[0,1])$, $\xi_1(0) = 0$, and a is a fixed element of $\mathcal{P}_H(A)$.

It is clear that $(\mathrm{id} \otimes \Delta) \circ F_i \in C([0,1], A \otimes_{\min} H) \otimes H$ for all i. As the rightmost tensor product is algebraic, evaluations commute with $\mathrm{id} \otimes \Delta$, and δ is coassociative, we infer that $(\mathrm{id} \otimes \Delta) \circ F_i \in (A \circledast^\delta H) \otimes H$ for all i (cf. [DHH15, Lemma 5.2]). Furthermore, since $\delta_{A \circledast^\delta H}$ viewed as a map into $C\big([0,1], A \otimes_{\min} H \otimes_{\min} H\big)$ is a $*$-homomorphism, it is continuous, so that $(\mathrm{id} \otimes \Delta) \circ f \in (A \circledast^\delta H) \otimes_{\min} H$. Hence $\delta_{A \circledast^\delta H}$ has range in $(A \circledast^\delta H) \otimes_{\min} H$.

The injectivity and coassociativity of $\delta_{A\circledast^\delta H}$ are immediate respectively from the injectivity and coassociativity of Δ. The counitality condition follows from the same approximation argument as above. $\square$

Corollary 5.6. *If the coaction $\delta : A \to A \otimes_{\min} H$ is free, then so is the coaction $\delta_{A\circledast^\delta H} : A \circledast^\delta H \to (A \circledast^\delta H) \otimes_{\min} H$.*

Proof. The C^*-algebra $A \circledast^\delta H$ is a unital $C([0,1])$-algebra with $C([0,1]) \subseteq (A \circledast^\delta H)^{\operatorname{co} H}$. With the notation of Lemma 5.1, we have:

(1) $((A \circledast^\delta H)_0, \delta_0) = (H, \Delta)$,
(2) $((A \circledast^\delta H)_x, \delta_x) = (A \otimes_{\min} H, \mathrm{id} \otimes \Delta)$ for $x \in (0,1)$,
(3) $((A \circledast^\delta H)_1, \delta_1) \cong (A, \delta)$.

As each of the above actions is free, we infer from Theorem 5.2 that $\delta_{A\circledast^\delta H}$ is free. Alternatively, one can use a direct approximation argument as in Lemma 5.5. $\square$

Appendix: Finite Galois coverings

Let $\pi\colon X \to Y$ be a *covering map* of topological spaces. As usual, this means that given any $y \in Y$ there exists an open set U in Y with $y \in U$ such that $\pi^{-1}(U)$ is a disjoint union of open sets each of which π maps homeomorphically onto U. A *deck transformation* is a homeomorphism $h\colon X \to X$ with $\pi \circ h = \pi$.

Proposition A.7. *Let X and Y be compact Hausdorff topological spaces. Let $\pi\colon X \to Y$ be a covering map, and let Γ be the group of deck transformations of this covering map. Assume that Γ is finite. Then X is a principal Γ-bundle over Y if and only if the canonical map*

$$\begin{aligned} can\colon C(X) \underset{C(Y)}{\otimes} C(X) &\longrightarrow C(X) \otimes C(\Gamma), \\ can\colon f_1 \otimes f_2 &\longmapsto (f_1 \otimes 1)\delta(f_2), \end{aligned}$$

is an isomorphism. Here δ is given by (0.11).

Proof. If X is a principal Γ-bundle over Y, then $C(Y) = C(X/\Gamma) = C(X)^{\operatorname{co} C(\Gamma)}$ and, by (0.10), *can* is surjective. Furthermore, since $C(\Gamma)$ is cosemisimple, by the result of H.-J. Schneider [S-HJ90, Theorem I], the surjectivity of *can* implies its bijectivity.

Assume now that *can* is bijective. The local triviality assumption in the definition of a covering map implies that for any continuous function f on X one has a continuous function $\Theta(f)$ on Y given by the formula

$$(\Theta(f))(y) := \frac{1}{\#\pi^{-1}(y)} \sum_{x\in\pi^{-1}(y)} f(x)\,. \tag{A.10}$$

Note that the fibres are finite due to the compactness of X. Also, one immediately sees that Θ is a unital $C(Y)$-linear map from $C(X)$ to $C(Y)$.

Now it follows from the bijectivity of *can* and [DHS99, Lemma 1.7] that $C(Y) = C(X)^{\mathrm{co}\, C(\Gamma)} = C(X/\Gamma)$. Hence the fibres of the covering map $\pi\colon X \to Y$ are the orbits of Γ. Finally, the freeness of the action of Γ on X follows from the surjectivity of *can* and (0.10). ☐

If X is connected, then it is always the case that the group of deck transformations Γ is finite and that the action of Γ on X is free. The issue is then whether or not the action of Γ on each fiber of π is transitive. Thus we conclude from Proposition A.7:

Corollary A.8. *Let X and Y be connected compact Hausdorff topological spaces, and let $\pi\colon X \to Y$ be a covering map. Denote by Γ the group of deck transformations. Then the action of Γ on each fiber of π is transitive if and only if the canonical map*

$$can\colon C(X) \underset{C(Y)}{\otimes} C(X) \longrightarrow C(X) \otimes C(\Gamma)$$

is an isomorphism.

Remark A.9. To make the proof of Proposition A.7 more self-contained, let us unravel the crux of the argument proving [DHS99, Lemma 1.7]. We know that $C(Y) \subseteq C(X/\Gamma)$, and we need to prove the equality. To this end, let us take any $f \in C(X/\Gamma)$. Then, since $can(1 \otimes f) = can(f \otimes 1)$, it follows from the bijectivity of *can* that $1 \otimes f = f \otimes 1 \in C(X) \otimes_{C(Y)} C(X)$. Applying $\Theta \otimes \mathrm{id}$ to this equality yields $f = \Theta(f) \in C(Y)$.

Remark A.10. An alternative proof of Proposition A.7 is as follows. Consider the commutative diagram

$$\begin{tikzcd} C(X) \underset{C(Y)}{\otimes} C(X) \arrow[r, "can"] \arrow[d] & C(X) \otimes C(\Gamma) \arrow[d] \\ C(X \underset{Y}{\times} X) \arrow[r] & C(X \times \Gamma) \end{tikzcd} \tag{A.11}$$

in which each vertical arrow is the evident map and the lower horizontal arrow is the $*$-homomorphism resulting from the map of topological spaces

$$X \times \Gamma \longrightarrow X \underset{Y}{\times} X, \qquad (x, \gamma) \mapsto (x, x\gamma). \tag{A.12}$$

Note that X is a (locally trivial) principal Γ bundle on Y if and only if this map of topological spaces is a homeomorphism, and the latter is equivalent to bijectivity of the lower horizontal arrow.

Hence to prove Proposition A.7, it will suffice to prove that the two vertical arrows are isomorphisms. The right vertical arrow is an isomorphism because Γ is a finite group, so $C(\Gamma)$ is a finite-dimensional vector space over the complex numbers $\mathbb{C}$.

For the left vertical arrow, let E be the vector bundle on Y whose fiber at $y \in Y$ is $\mathrm{Map}(\pi^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from $\pi^{-1}(y)$ to $\mathbb{C}$. As $\pi^{-1}(y)$ is a discrete subset of the compact Hausdorff space X, it is finite. Let $\mathcal{S}(E)$ be the algebra consisting of all the continuous sections of E. Then $\mathcal{S}(E) = C(X)$.

Similarly, define

$$\pi^{(2)} \colon X \underset{Y}{\times} X \longrightarrow Y \quad \text{by} \quad \pi^{(2)} \colon (x_1, x_2) \longmapsto \pi(x_1) = \pi(x_2). \tag{A.13}$$

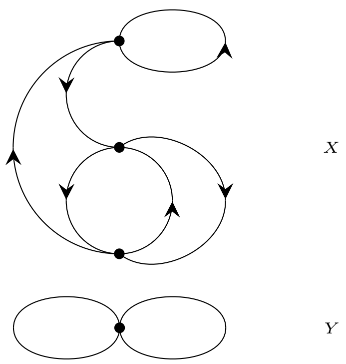
Let F be the vector bundle on Y whose fiber at $y \in Y$ is $\mathrm{Map}((\pi^{(2)})^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from $(\pi^{(2)})^{-1}(y)$ to $\mathbb{C}$. Then $\mathcal{S}(F) = C(X \times_Y X)$, where $\mathcal{S}(F)$ is the algebra consisting of all the continuous sections of F. Since $F = E \otimes E$ as vector bundles on Y, we conclude $\mathcal{S}(F) = \mathcal{S}(E) \otimes_{C(Y)} \mathcal{S}(E)$, which proves bijectivity for the left vertical arrow.

Example A.11. Without connectivity, the group of deck transformations can be infinite. For example, let Y be the Cantor set and let $\pi \colon Y \times \{0,1\} \to Y$ be the trivial twofold covering. Let U be a subset of Y which is both open and closed. Define $\gamma_U \colon Y \times \{0,1\} \to Y \times \{0,1\}$ by

$$\gamma_U(y,t) := \begin{cases} (y,t) & \text{for } y \notin U \\ \\ (y, 1-t) & \text{for } y \in U. \end{cases} \tag{A.14}$$

Then γ_U is a deck transformation and there are infinitely many closed and open subsets U.

Example A.12. The following example is a threefold covering X of the one-point union of two circles Y. Here the preimage of the left circle of the base space is the usual threefold covering of the circle. The preimage of the right circle of the base space is the disjoint union of the usual twofold covering of the circle and the onefold covering of the circle.



In this example, the group of deck transformations is trivial. Indeed, let γ be a deck transformation. Consider γ restricted to the preimage of the right circle of the base space. This preimage has two connected components. Since γ is a deck transformation of this preimage, it must map each connected compenent to itself. This implies that γ has a fixed point. Hence, as X is connected, $\gamma = \mathrm{id}$. In particular, this shows that the group of deck transformations need not act transitively on fibers of a covering map. The canonical map is surjective but not injective.

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