On the Center-Valued Atiyah Conjecture for L^2-Betti Numbers

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Abstract. The so-called Atiyah conjecture states that the $\mathcal{N}(G)$-dimensions of the L^2-homology modules of finite free G-CW-complexes belong to a certain set of rational numbers, depending on the finite subgroups of G. In this article we extend this conjecture to a statement for the center-valued dimensions. We show that the conjecture is equivalent to a precise description of the structure as a semisimple Artinian ring of the division closure $D(\mathbb{Q}[G])$ of $\mathbb{Q}[G]$ in the ring of affiliated operators. We prove the conjecture for all groups in Linnell's class $\mathfrak{C}$, containing in particular free-by-elementary amenable groups.

The center-valued Atiyah conjecture states that the center-valued L^2-Betti numbers of finite free G-CW-complexes are contained in a certain discrete subset of the center of $\mathbb{C}[G]$, the one generated as an additive group by the center-valued traces of all projections in $\mathbb{C}[H]$, where H runs through the finite subgroups of G.

Finally, we use the approximation theorem of Knebusch [15] for the center-valued L^2-Betti numbers to extend the result to many groups which are residually in $\mathfrak{C}$, in particular for finite extensions of products of free groups and of pure braid groups.

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1 Introduction

In [3], Atiyah introduced L^2-Betti numbers for manifolds with cocompact free G-action for a discrete group G (later generalized to finite free G-CW-complexes). There, he asked [3, p. 72] about the possible values these can assume. This question was later popularized in precise form as the so-called "strong Atiyah conjecture". One easily sees that the possible values depend on G. For a finite subgroup of order n in G, a free cocompact G-manifold with L^2-Betti number $1/n$ can be constructed. For certain groups G which contain finite subgroups of arbitrarily large order, with quite some effort manifolds M with $\pi_1(M) = G$ and with transcendental L^2-Betti numbers have been constructed [4, 12, 26]. In the following, we will therefore concentrate on G with a bound on the orders of finite subgroups.
The L^2-Betti numbers are defined using the L^2-chain complex. The chain groups there are of the form $l^2(G)^d$, and the differentials are given by convolution multiplication with a matrix over $\mathbb{Z}[G]$. The strong Atiyah conjecture for free finite G-CW-complexes is equivalent to the following (with $K = \mathbb{Z}$):

1.1 Definition. Let G be a group with a bound on the orders of finite subgroups and let $\operatorname{lcm}(G) \in \mathbb{N}$ (the positive integers) denote the least common multiple of these orders. Let $K \subset \mathbb{C}$ be a subring.
We say that G satisfies the *strong Atiyah conjecture over K, or $K[G]$ satisfies the strong Atiyah conjecture* if for every $n \in \mathbb{N}$ and every $A \in \mathrm{M}_n(K[G])$

$$\dim_G(\ker(A)) := \operatorname{tr}_G(\operatorname{pr}_{\ker A}) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Here, as before, we consider $A\colon l^2(G)^n \to l^2(G)^n$ as a bounded operator, acting by left convolution multiplication — the continuous extension of the left multiplication action on the group ring to $l^2(G)$. tr_G is the canonical trace on $\mathrm{M}_n(\mathcal{N}(G))$, i.e. the extension (using the matrix trace) of $\operatorname{tr}_G\colon \mathcal{N}(G) \to \mathbb{C}$; $a \mapsto \langle a\delta_e, \delta_e\rangle_{l^2(G)}$, where $\mathcal{N}(G)$, the weak closure of $\mathbb{C}[G] \subset \mathcal{B}(\ell^2(G))$ is the group von Neumann algebra.
If G contains arbitrarily large finite subgroups, we set $\operatorname{lcm}(G) := +\infty$.

A projection P will always be a self adjoint idempotent, so $P = P^2 = P^*$, where * indicates the involution on $\mathcal{N}(G)$. If E is an idempotent, then E is similar to a projection P and then $\operatorname{tr}_G(E) = \operatorname{tr}_G(P)$. Also a central idempotent is always a projection. Note that if G is an infinite group, then the set $\{\operatorname{tr}_G(P)\}$, where P runs through the projectors in $\mathrm{M}_n(\mathcal{N}(G))$, $n \in \mathbb{N}$ consists of all non-negative real numbers. The strong Atiyah conjecture predicts, on the other hand, that the L^2-Betti numbers take values in the subgroup of $\mathbb{R}$ generated by traces of projectors defined already over $\mathbb{Q}[H]$ for the finite subgroups H of G: the projector $p_H = (\sum_{h\in H} h)/|H|$ satisfies $\operatorname{tr}_G(p_H) = 1/|H|$. And by the Chinese remainder theorem, the additive subgroup of $\mathbb{R}$ generated by the $|H|^{-1}$ is exactly $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z}$.

We now turn to the center-valued refinements of the above statements. The center-valued L^2-Betti numbers are obtained by replacing the canonical (complex-valued) trace tr_G by the center-valued trace tr^u_G (see Definition 2.1), taking values in the center of $\mathcal{N}(G)$. Note that by general theory [14, Chapter 8], as every finite von Neumann algebra has a unique normalized center-valued trace, this is a powerful invariant: two finitely generated projective Hilbert $\mathcal{N}(G)$-modules are isomorphic if and only if their center-valued dimensions coincide. The center of a ring R will be denoted $\mathcal{Z}(R)$.

1.2 Definition. Let G be a group with $\mathrm{lcm}(G) < \infty$, let K be a subring of $\mathbb{C}$, let F be the field of fractions of K, and assume that F is closed under complex conjugation. Let $L_K(G)$ be the additive subgroup of $\mathcal{Z}(\mathcal{N}(G))$ generated by $\mathrm{tr}^u_G(P) \in \mathcal{Z}(\mathbb{C}[G]) \subset \mathcal{Z}(\mathcal{N}(G))$ where P runs through projections $P \in F[H]$ with $H \le G$ a finite subgroup.
We say that G satisfies the *center-valued Atiyah conjecture over K, or $K[G]$ satisfies the center-valued conjecture* if for every $n \in \mathbb{N}$ and every $A \in \mathrm{M}_n(K[G])$ we have $\dim^u_G(\ker(A)) := \mathrm{tr}^u_G(\mathrm{pr}_{\ker A}) \in L_K(G)$.

Observe that G satisfies the center-valued Atiyah conjecture over K if and only if G satisfies the center-valued conjecture over its field of fractions F. Indeed the "only if" is obvious. On the other hand if $A \in \mathrm{M}_n(F[G])$, then ("clearing denominators") there exists $0 \neq k \in K$ such that $kA \in \mathrm{M}_n(K[G])$, and $\ker A = \ker kA$, which verifies the "if" part.

1.3 Proposition. *If a group G satisfies the center-vlaued Atiyah conjecture over K of Definition 1.2, then G also satisfies the (classical) strong Atiyah conjecture over K of Definition 1.1.*

Proof. By the universal property of the center-valued trace [14, Chapter 8], $\mathrm{tr}_G = \mathrm{tr}_G \circ \mathrm{tr}^u_G$. We therefore only have to check that $\mathrm{tr}_G(a) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$ for all $a \in L_K(G)$. By the definition of $L_K(G)$, we just have to show that $\mathrm{tr}_G(P) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$ for each projector $P \in F[H]$, where $H \le G$ is an arbitrary finite subgroup. This is of course well known to be true, it follows e.g. from the fact that finite groups satisfy the strong Atiyah conjecture over K. ☐

1.4 Proposition (compare Corollary 3.5). *If $\mathrm{lcm}(G) < \infty$ then $L_K(G) \subset \mathcal{Z}(\mathcal{N}(G))$ is discrete. In particular, the center-valued Atiyah conjecture predicts a "quantization" of the center-valued L^2-Betti numbers.*

1.5 Remark. As for the ordinary strong Atiyah conjecture, the center-valued Atiyah conjecture over $\mathbb{Z}[G]$ is equivalent to the statement that the center-valued L^2-Betti numbers for finite free G-CW-complexes take values in $L_{\mathbb{Z}}(G)$. The center-valued L^2-Betti numbers have been introduced and used in [21].

The strong Atiyah conjecture has many applications. Most interesting are those for a torsion-free group G, i.e. if $\mathrm{lcm}(G) = 1$. This is exemplified by the following surprising result of Linnell [17]. We first recall the notion of "the" division closure of $K[G]$.

1.6 Definition. Let G be a discrete group and let $K \subset \mathbb{C}$ be a subring. Let $\mathcal{U}(G)$ denote the ring of unbounded operators on $l^2(G)$ affiliated to $\mathcal{N}(G)$ (algebraically, $\mathcal{U}(G)$ is the Ore localization of $\mathcal{N}(G)$ at the set of all non-zero-divisors).
Define the *division closure* $D(K[G])$ to be the smallest subring of $\mathcal{U}(G)$ containing $K[G]$ which is closed under taking inverses in $\mathcal{U}(G)$.

1.7 Theorem. *Let G be a discrete group with $\mathrm{lcm}(G) = 1$ and let K be a subring of $\mathbb{C}$. Then $K[G]$ satisfies the strong Atiyah conjecture if and only if $D(K[G])$ is a skew field.*

The appealing feature of this theorem is that it provides a canonical over-ring, namely $D(K[G])$ of $K[G]$ which should be a skew field, provided G is torsion free. Observe that this implies in particular that $K[G]$ has no non-trivial zero-divisors. For more information on this, see [23, Remark 4.11].
Part of the motivation for the work at hand was the question of how to generalize Theorem 1.7 if $\mathrm{lcm}(G) > 1$. It turns out that one expects that $D(K[G])$ is semisimple Artinian. In the situation at hand this means that $D(K[G])$ is a finite direct sum of matrix rings over skew fields. This is proved in many cases e.g. in [17].
The present paper gives a very precise (conjectural) description of $D(K[G])$, and if it is satisfied we call $D(K[G])$ *Atiyah-expected Artinian*: the lattice of finite subgroups and their K-linear representations give a precise prediction into which matrix summands $D(K[G])$ decomposes and the size of the corresponding matrices. The precise formula is a bit cumbersome, so we don't give it here but refer to Definition 3.6.
One of our main theorems is the precise generalization of Theorem 1.7.

1.8 Theorem. *Let G be a discrete group with $\mathrm{lcm}(G) < \infty$ and let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Then $K[G]$ satisfies the center-valued Atiyah conjecture if and only if $D(K[G])$ is Atiyah-expected Artinian.*

Indeed, we show in Theorem 3.7 that these two properties are also equivalent to the property that $K_0(D(K[G]))$ is generated by the images of $K_0(K[H])$ as H runs over the finite subgroups of G.

1.9 Definition. Given a discrete group G with $\mathrm{lcm}(G) < \infty$, let $\Delta^+(G)$ denote the maximal finite normal subgroup, and let $\Delta(G)$ denote the *finite conjugacy center*, i.e. the set of those elements of G which have only a finite number of conjugates.
Indeed, by [25, §1], $\Delta(G)$ is a normal subgroup of G. Recall that the product of two normal subgroups is a normal subgroup, therefore, as $\mathrm{lcm}(G) < \infty$, $\Delta^+(G)$ makes sense. Note that $\Delta^+(G) \subset \Delta(G)$, indeed, using [25, Lemma 19.3] it is exactly the subset of all elements of finite order in $\Delta(G)$.

In the special case $\Delta^+(G) = \{1\}$, we have that $D(K[G])$ is Atiyah-expected Artinian if and only if it is an $\mathrm{lcm}(G) \times \mathrm{lcm}(G)$-matrix ring over a skew field,

and by Theorem 3.7 this is equivalent to the center-valued Atiyah conjecture (which in this case is implied by the usual Atiyah conjecture, as the relevant part of $\mathcal{Z}(\mathcal{N}(G))$ is $\mathbb{C}[\Delta^+(G)]$). This special case (and slightly more general situations) have already been covered in [20], but without the use of the center-valued trace. It turns out that the general case requires this more refined dimension function. However, much of our arguments for Theorem 3.7 follow closely the arguments of [20].
In [20], a variant of the division closure, namely the ring $\mathcal{E}(K[G])$ is introduced and used (compare Definition 2.2). It is closed under adding central idempotents in $\mathcal{U}(G)$ which generated the same submodules as elements already in the ring. We expect that this actually coincides with $D(K[G])$.

1.10 Theorem. *If* $\mathrm{lcm}(G) < \infty$, *K is a subfield of $\mathbb{C}$ which is closed under complex conjugation and $K[G]$ satisfies the center-valued Atiyah conjecture, then $\mathcal{E}(K[G]) = D(K[G])$.*

As the second main result of the paper we establish the center-valued Atiyah conjecture for certain classes of groups (namely almost all for which the original Atiyah conjecture is known). The algebraic closure of $\mathbb{Q}$ will be denoted $\overline{\mathbb{Q}}$.

1.11 Theorem. *Let K be a subfield of $\mathbb{C}$ which is closed under complex conjugation. The center-valued Atiyah conjecture over K is true for the following groups G:*

1. *all groups G which belong to Linnell's class of groups $\mathfrak{C}$ of Definition 2.7, in particular all free by elementary amenable groups G.*
2. *if K is contained in $\overline{\mathbb{Q}}$, all elementary amenable extensions of*
    - *pure braid groups*
    - *right-angled Artin groups*
    - *primitive link groups*
    - *virtually cocompact special groups, where a "cocompact special groups" is a fundamental group of a compact special cube complex —this class of groups contains Gromov hyperbolic groups which act cocompactly and properly on CAT(0) cube complexes, fundamental groups of compact hyperbolic 3-manifolds with empty or toroidal boundary, and Coxeter groups without a Euclidean triangle Coxeter subgroup,*
    - *or of products of the above.*

1.12 Question. Missing in the above list are congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ and finite extensions thereof. Note that the usual Atiyah conjecture for these groups, as long as they are torsion free, is proved in [11]. For torsion-free groups, the center-valued Atiyah conjecture is not stronger than the usual Atiyah conjecture. However, it would be interesting to generalize the work of [11] to certain extensions which are not torsion free, and then (or along the way) to deal with the center-valued Atiyah conjecture for these.

Recall that the center-valued Atiyah conjecture for a group G only makes an assertion when $\operatorname{lcm}(G) < \infty$. For the proof of 1 of Theorem 1.11 we closely follow the method of [17], making use of the equivalent algebraic formulations of the Atiyah conjecture of Theorem 3.7. Indeed, we show that the conjecture is stable under extensions by torsion-free elementary amenable groups. We actually show (and use) slightly more refined stability properties.

For 2 of Theorem 1.11 we use the approximation theorem for the center-valued L^2-Betti numbers, [15, Theorem 3.2]. Because of the discreteness of the possible center-valued L^2-Betti numbers, the Atiyah conjecture for a suitable sequence of quotients implies the Atiyah conjecture for the group itself. We follow here the general idea as already applied in [29] and for more general coefficient rings in [9]. That this idea can be used for the class of groups listed in 2 was shown for the pure braid groups in [19], for primitive link groups in [8] and for right-angled Coxeter and Artin groups in [18], and for cocompact special groups by Schreve in [30] (who uses fundamentally the geometric insights of Haglund-Wise [13], and develops further the methods of [18]). Agol [1] shows in breakthrough work that Gromov hyperbolic cocompact CAT(0) cube groups are virtually cocompact special; with Bergeron-Wise' construction of a cocompact action of a hyperbolic 3-manifold group on a CAT(0) cube complex [5] this implies that hyperbolic 3-manifold groups are virtually cocompact special.

2 Preliminaries on rings associated to groups

$\mathcal{U}(G)$, $D(K[G])$ and traces on these

2.1 Definition. Let G be a discrete group. The center-valued trace is the uniquely defined $\mathbb{C}$-linear map

$$\operatorname{tr}_G^u : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G))$$

such that for $a, b \in \mathcal{N}(G)$ and $c \in \mathcal{Z}(\mathcal{N}(G))$, we have

- $\operatorname{tr}_G^u(ab) = \operatorname{tr}_G^u(ba)$;
- $\operatorname{tr}_G^u(c) = c$;
- $\operatorname{tr}_G^u(a) \in (\mathcal{Z}(\mathcal{N}(G)))^+$ if $a \in (\mathcal{N}(G))^+$.

The trace can be extended to $\mathrm{M}_d(\mathcal{N}(G))$ by taking $\operatorname{tr}_G^u := \operatorname{tr}_G^u \otimes \operatorname{tr}_{\mathrm{M}_d(\mathbb{C})}$ (by abuse of notation), with $\operatorname{tr}_{\mathrm{M}_d(\mathbb{C})}$ the non-normalized trace on $\mathrm{M}_d(\mathbb{C})$.
If $P \in \mathrm{M}_d(\mathcal{N}(G))$ is a projector with image the (Hilbert $\mathcal{N}(G)$-module) V, set $\dim_G^u(V) := \operatorname{tr}_G^u(P)$.

That a unique such trace exists is established e.g. in [14, Chapter 8].

Later, we want to apply the trace also for the division closure. Recall that we have (by definition) the following diagram of inclusions of rings

$$\begin{array}{ccc} K[G] & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ D(K[G]) & \longrightarrow & \mathcal{U}(G). \end{array}$$

Given a finitely presented $K[G]$-module M, represented by $A \in \mathrm{M}_{k\times l}(K[G])$, i.e. with exact sequence $K[G]^l \xrightarrow{A} K[G]^k \to M \to 0$, the induced modules $M \otimes_{K[G]} \mathcal{N}(G)$, $M \otimes_{K[G]} \mathcal{U}(G)$, $M \otimes_{K[G]} D(K[G])$ are also finitely presented with the same presenting matrix A. The standard theory of Hilbert $\mathcal{N}(G)$-modules gives a center-valued dimension for each finitely presented $\mathcal{N}(G)$-module, in particular for $M \otimes_{K[G]} \mathcal{N}(G)$, and $\dim_G^u(M \otimes_{K[G]} \mathcal{N}(G)) = k - \dim_G^u(\ker(A))$ in the above situation (compare [21]). In [27], this dimension is extended to finitely presented $\mathcal{U}(G)$-modules, of course in such a way that the value is unchanged if we induce up from $\mathcal{N}(G)$ to $\mathcal{U}(G)$. More precisely, [27] describes the extension of dimensions based on arbitrary $\mathbb{C}$-valued traces on $\mathcal{N}(G)$, this implies easily the corresponding extension for $\dim_G^u$.

The central idempotent division closure $\mathcal{E}(K[G])$

2.2 Definition. Let R be a subring of the ring S and let $C = \{e \in S \mid e \text{ is a central idempotent of } S \text{ and } eS = rS \text{ for some } r \in R\}$. Then we define

$$\mathcal{C}(R,S) = \sum_{e\in C} eR,$$

a subring of S. In the case $S = \mathcal{U}(G)$, we write $\mathcal{C}(R)$ for $\mathcal{C}(R,\mathcal{U}(G))$. For each ordinal α, define $\mathcal{E}_\alpha(R,S)$ as follows:

- $\mathcal{E}_0(R,S) = R$;
- $\mathcal{E}_{\alpha+1}(R,S) = \mathcal{D}(\mathcal{C}(\mathcal{E}_\alpha(R,S),S),S)$;
- $\mathcal{E}_\alpha(R,S) = \bigcup_{\beta<\alpha} \mathcal{E}_\beta(R,S)$ if α is a limit ordinal.

Then $\mathcal{E}(R,S) = \bigcup_\alpha \mathcal{E}_\alpha(R,S)$. Also in the case $R = K[G]$ where G is a group and K is a subfield of $\mathbb{C}$, we write $\mathcal{E}(K[G])$ for $\mathcal{E}(K[G],\mathcal{U}(G))$.

2.3 Conjecture. *Let G be a discrete group and $K \subset \mathbb{C}$ a subfield. Then $D(K[G]) = \mathcal{E}(K[G])$, at least if* $\mathrm{lcm}(G) < \infty$.

We cite some properties of $\mathcal{E}(K[G])$ from [20] which will be useful later. Indeed, we generalize from the canonical trace to the center-valued trace, but the proofs literally also cover this more general situation.

2.4 Lemma. *(cf.* [20, Lemma 2.4]*) The following additive subgroups of $\mathcal{Z}(\mathcal{N}(G))$ coincide:*

$$\begin{aligned}&\langle \dim_G^u(x\mathcal{U}(G)^n) \mid x \in \mathrm{M}_n(K[G]), n \in \mathbb{N}\rangle \\ &\qquad\qquad = \langle \dim_G^u(x\mathcal{U}(G)^n) \mid x \in \mathrm{M}_n(\mathcal{E}(K[G])), n \in \mathbb{N}\rangle\end{aligned}$$

This has as an immediate corollary that $\mathcal{E}(K[G]) = D(K[G])$ if $K[G]$ satisfies the center-valued Atiyah conjecture:

Proof of Theorem 1.10. Let $e \in \mathcal{E}(K[G])$ be a central idempotent of $\mathcal{U}(G)$. Then all the spectral projections of e lie in $\mathcal{Z}(\mathcal{N}(G))$, therefore e is affiliated to $\mathcal{Z}(\mathcal{N}(G))$. Being an idempotent, even $e \in \mathcal{Z}(\mathcal{N}(G))$. Therefore, on the one hand, $\mathrm{tr}_G^u(e) = e$ while, on the other hand by Lemma 2.4, $\mathrm{tr}_G^u(e) = \dim_G^u(e\mathcal{U}(G)) \in L_K(G)$, in particular $e \in \mathcal{Z}(K[\Delta^+]) \subset K[\Delta^+]$. $\square$

2.5 Remark. The proof just given didn't need the full force of the center-valued Atiyah conjecture, only the statement that $\dim_G^u(x\mathcal{U}(G)^n) \in \mathcal{Z}(\mathcal{N}(G))$ is supported only on elements of finite order, i.e. lies in $\mathcal{Z}(K[\Delta^+])$.

Approximation of the center-valued trace

The following is a special case of [15, Theorem 3.2] which will be used in the next section.

2.6 Theorem. *Let G be a discrete group with a sequence $G = G_0 \geq G_1 \geq \cdots$ of normal subgroups with $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$.*
Let $A \in \mathrm{M}_d(\overline{\mathbb{Q}}[G])$ and $g \in \Delta(G)$. Let $A[i] \in \mathrm{M}_d(\overline{\mathbb{Q}}[G/G_i])$ be the image of A under the map induced by the projection $\mathrm{pr}_i \colon G \to G/G_i$.
Assume that all G/G_i satisfy the determinant bound property [9, Definition 3.1]*, e.g. are elementary amenable (or more generally belong to the class $\mathcal{G}$ of groups introduced in* [9, Definition 1.8] *and corrected in the errata to* [28] *at arXiv:math/9807032, or are sofic, compare* [10] *and* [15, Theorem 4.1]*). Then*

$$\lim_{i\to\infty} \langle \dim^u_{\mathcal{N}(G/G_i)}(\ker(A[i])), \mathrm{pr}_i(g)\rangle_{l^2(G/G_i)} = \langle \dim_G^u \ker(A), g\rangle_{l^2(G)}.$$

Linnell's class $\mathfrak{C}$

2.7 Definition. Let $\mathfrak{C}$ denote the smallest class of groups which

1. contains all free groups,
2. is closed under directed unions,
3. satisfies $G \in \mathfrak{C}$ whenever $H \lhd G$, $H \in \mathfrak{C}$ and G/H is elementary amenable.

3 Reformulation of the center-valued Atiyah conjecture

Let G be a group with $\operatorname{lcm}(G) < \infty$. We shall assume that K is a subfield of $\mathbb{C}$ which is closed under complex conjugation. Many of the arguments given below don't require this assumption; however if K is a subfield closed under complex conjugation and e is a central idempotent in $K[G]$, then e is a projection [6, Lemma 9.2(i)]. Furthermore if G is a finite group and $A \in \mathrm{M}_n(K[G])$, then $\mathrm{pr}_{\ker A} \in \mathrm{M}_n(K[G])$ (use [6, Proposition 9.3]); it is here where we are using the property that K is closed under complex conjugation.
Recall that Δ^+ is the (finite) normal subgroup consisting of all elements of finite order and having only finitely many conjugates.

3.1 Lemma. *Let $K \subset \mathbb{C}$ be a subfield which contains all $|\Delta^+|$-th roots of 1, and let c_G denote the number of finite conjugacy classes of elements of finite order in G, i.e. the dimension of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K[\Delta^+])$. There is a finite set of primitive central projections $\{U^1, \dots, U^{c_G}\}$ of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K[\Delta^+]) \subset \mathcal{Z}(K[G])$, given by*

$$U^i := \sum_{k \text{ s.t. } \exists g \in G: g u_i g^{-1} = u_k} u_k,$$

where u_i are the primitive central idempotents of the semisimple Artinian ring $K[\Delta^+]$. Furthermore $u_i = \frac{n_i}{|\Delta^+|} \sum_{s \in G} \chi_i(s^{-1}) s$, with n_i the dimensions of the irreducible representations (over $\mathbb{C}$) of Δ^+ and χ_i the corresponding characters (extended by 0 to all of G). Moreover the U^j form an orthogonal basis of the vector space $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(\mathbb{C}[\Delta^+])$.

Proof. By Maschke's theorem of standard representation theory, the algebra $K[\Delta^+]$ is semisimple Artinian, compare [16, XVIII, Theorem 1.2]. Therefore it has finitely many primitive central idempotents u_i.
Any algebra automorphism must permute the u_i, in particular the conjugation action of G. An element of $\mathcal{Z}(K[\Delta^+])$ belongs to the center of $K[G]$ (and then also of $\mathcal{N}(G)$) if and only if it is invariant under conjugation by elements of G. It follows immediately that the U^i are the primitive central idempotents of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K[\Delta^+])$, and furthermore they form an orthogonal basis for $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(\mathbb{C}[\Delta^+])$.
The formula for the u_i is also standard, [16, XVIII, Proposition 4.4 and Theorem 11.4]. ∎

3.2 Lemma. *Let K be a subfield of $\mathbb{C}$ and let L/K be a finite Galois extension of K with Galois group F. Let G be a finite group, let $\{e_1, \dots, e_n\}$ denote the primitive central idempotents of $K[G]$, and let $\{u_1, \dots, u_m\}$ denote the primitive central idempotents of $L[G]$. Then F acts as automorphisms on $L[G]$ according to the rule $\theta \sum_{g \in G} a_g g = \sum_{g \in G} \theta(a_g) g$ for $\theta \in F$. The u_i form an orthogonal set and $\langle u_i, 1 \rangle = \langle \theta u_i, 1 \rangle$ for all i. For each i, define $N_i = \{j \in \mathbb{N} \mid e_i u_j = u_j\}$. Then F acts transitively on $\{u_j \mid j \in N_i\}$ and $e_i = \sum_{j \in N_i} u_j$.*

Proof. This is well-known, and follows from Galois descent. Note that $u_i e_j$ is a central idempotent in $L[G]$ and $u_i = u_i e_j + (1-e_j)u_i$. It follows for all i, j, either $u_i e_j = 0$ or $u_i e_j = u_i$, because u_i is primitive. It follows easily that $e_i = \sum_{j\in N_i} u_j$. Also F acts on $\{u_j \mid j \in N_i\}$, and the sum of the u_j in an orbit is fixed by F and is therefore in $K[G]$. Since e_i is primitive, it follows that this orbit must be the whole of N_i. Finally if $e = \sum_{g\in G} e_g g \in L[G]$ is an idempotent, then $e_1 \in \mathbb{Q}$ (by the character formula of Lemma 3.1) and we see that $\langle u_i, 1\rangle = \langle \theta u_i, 1\rangle$ for all i. ☐

3.3 Lemma. *Let $K \subset \mathbb{C}$ be a subfield, let ω be a primitive $|\Delta^+|$-root of 1 and set $L = K(\omega)$. Let F denote the Galois group of L over K, and let $U^1, \dots, U^{c_{L[G]}}$ be the primitive central projections of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(L[\Delta^+]) \subset \mathcal{Z}(L[G])$ as described above in Lemma 3.1. There is a finite set of primitive central projections $\{P^1, \dots, P^{C_{K[G]}}\}$ of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K[\Delta^+]) \subset \mathcal{Z}(K[G])$, given by*

$$P^i := \sum_{k\ s.t.\ \exists g\in G: gp_ig^{-1}=p_k} p_k,$$

where p_i are the primitive central idempotents of the semisimple Artinian ring $K[\Delta^+]$. Set $N_i = \{j \in \mathbb{N} \mid P^iU^j = U^j\}$. Then

$$P^i = \sum_{j\in N_i} U^j$$

and F acts transitively on $\{U^j \mid j \in N_i\}$.

Proof. This follows from Lemmas 3.1 and 3.2. ☐

3.4 Lemma. *Let H be a finite subgroup of G which contains Δ^+. For an irreducible projection $Q \in K[H]$ (in the sense that if $Q = Q_1 + Q_2$ with projections in $Q_1, Q_2 \in K[H]$ satisfying $Q_1Q_2 = 0$ then either $Q_1 = 0$ or $Q_2 = 0$) we have $\mathrm{tr}^u_G(Q) \in \mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K[\Delta^+]) \subset \mathcal{Z}(\mathcal{N}(G))$. More precisely, using the central projections P^i of Lemma 3.3 we have*

$$\mathrm{tr}^u_G(Q) = \frac{\dim_{\mathbb{C}}(Q\cdot\mathbb{C}[H])\cdot|\Delta^+|}{|H|\cdot\dim_{\mathbb{C}}(P^i\cdot\mathbb{C}[\Delta^+])}P^i = \frac{\dim_{\mathcal{N}(G)}(Q\cdot l^2(G))}{\dim_{\mathcal{N}(G)}(P^i\cdot l^2(G))}P^i \qquad (1)$$

where P^i is characterized by the property $QP^i = Q$.

Proof. Let ω be a primitive $|\Delta^+|$-th root of 1, let $L = K(\omega)$ and let F denote the Galois group of L/K. The center-valued trace is obtained by orthogonal projection from $l^2(G)$ to the subspace of $l^2(\Delta)$ spanned by functions which are constant on G-conjugacy classes, using the standard embedding of $\mathcal{N}(G)$ into $l^2(G)$. For Q, which is supported on group elements of finite order, therefore $\mathrm{tr}^u_G(Q) \in \mathbb{C}[\Delta^+]$. Let $U^1, \dots, U^{c_G}$ and $P^1, \dots, P^{C_{K[G]}}$ be the primitive central projections as described in Lemma 3.3. Using the standard inner product on

$\mathbb{C}[H]$ we obtain, using that $(U^1,\dots,U^{c_G})$ is an orthogonal basis of $\mathcal{Z}(\mathcal{N}(G))\cap\mathcal{Z}(L[H])=\mathcal{Z}(\mathcal{N}(G))\cap\mathcal{Z}(L[\Delta^+])$

$$\operatorname{tr}^u_G(Q)=\sum_j\frac{\langle Q,U^j\rangle}{\langle U^j,U^j\rangle}U^j. \tag{2}$$

Moreover, we have for each j that $QP^j+Q(1-P^j)=Q$ and $QP^jQ(1-P^j)=0$, the latter because P^j is central. Since Q is irreducible, we get either $QP^j=Q$ or $QP^j=0$. If $QP^i=Q$ we have $Q\sum_{j\in N_i}U^j=Q$ and $QU^j=0$ for $j\notin N_i$. Also if $j_1,j_2\in N_i$, then $\theta(QU^{j_1})=QU^{j_2}$ for some $\theta\in F$ and we see that $\langle QU^{j_1},1\rangle=\langle QU^{j_2},1\rangle$, consequently $\langle Q,U^j\rangle$ is independent of j for $j\in N_i$. Similarly $\langle U^j,U^j\rangle$ is independent of j for $j\in N_i$. Thus $\langle Q,P^i\rangle=|N_i|\langle Q,U^j\rangle$, $\langle P^i,P^i\rangle=|N_i|\langle U^j,U^j\rangle$ for $j\in N_i$, hence

$$\frac{\langle Q,U^j\rangle}{\langle U^j,U^j\rangle}=\frac{\langle Q,P^i\rangle}{\langle P^i,P^i\rangle}.$$

Substitute this in equation (2) together with

$$\langle Q,P^j\rangle=\langle QP^j,1\rangle=\langle Q,1\rangle=\frac{\dim_{\mathbb{C}}(Q\cdot\mathbb{C}[H])}{|H|}$$
$$\langle P^j,P^j\rangle=\langle P^j,1\rangle=\frac{\dim_{\mathbb{C}}(P^j\cdot\mathbb{C}[\Delta^+])}{|\Delta^+|}.$$

These formulas follow from the character formula for projections or are directly obtained as follows: for a projection $P\in\mathbb{C}[E]$ and a finite group E we have $\langle P,1\rangle_{l^2(E)}=\langle Ph,h\rangle_{l^2(E)}$ for all $h\in E$, therefore $\dim_{\mathbb{C}}(P\cdot\mathbb{C}[E])=\operatorname{tr}(P)=\sum_{h\in E}\langle Ph,h\rangle=|E|\cdot\langle P,1\rangle$.
Note, finally, that $\frac{\dim_{\mathbb{C}}(Q\cdot\mathbb{C}[H])}{|H|}=\dim_{\mathcal{N}(H)}(Q\cdot l^2(H))=\dim_{\mathcal{N}(G)}(Q\cdot l^2(G))$ by the induction rule for von Neumann dimensions. $\square$

3.5 Corollary. *The additive subgroup $L_K(G)$ of $\mathcal{Z}(\mathcal{N}(G))$ of Definition 1.2 is discrete.*

Proof. Recall that F denotes the relevant subfield of $\mathbb{C}$ in the setup of Definition 1.2, namely F is the field of fractions of K. Given a finite subgroup H of G and a projection $P\in F[H]$, $\operatorname{tr}^u_G(P)$ is a positive integral linear combination of $\operatorname{tr}^u_G(Q_\alpha)$ where $Q_\alpha\in F[H]$ are irreducible projections, corresponding to the decomposition of $\operatorname{im}(P)$ into irreducible $F[H]$-modules.
It therefore suffices to check that the additive subgroup of $\mathcal{Z}(\mathcal{N}(G))$ generated by $\operatorname{tr}^u_G(Q)$ is discrete, where Q runs through the irreducible projections in $F[H]$ and H runs through the finite subgroups of G. Increasing the field and increasing the finite subgroup has the only potential effect that an irreducible projection breaks up as a sum of new irreducible projections and therefore the subgroup generated by their center-valued traces increases. Therefore we may assume that these subgroups contain Δ^+ and that $F=\mathbb{C}$. By Lemma

3.4, these are all integer multiples of $\operatorname{lcm}(G)^{-1}P^i$ with the orthogonal basis $(P^1,\dots,P^{c_G})$, therefore span a discrete subgroup of $\mathcal{Z}(\mathcal{N}(G))$. $\square$

3.6 Definition. Assume that G is a discrete group with $\operatorname{lcm}(G)<\infty$ and that K is a subfield of $\mathbb{C}$ which is closed under complex conjugation.
We say that $D(K[G])$ is *Atiyah-expected Artinian* if it is a semisimple Artinian ring such that its primitive central idempotents are the central idempotents $P^1,\dots,P^{C_{K[G]}}\in K[\mathcal{Z}(K[\Delta^+])]$ of Lemma 3.3, and if each direct summand $P^jD(K[G])P^j$ is an $L_j\times L_j$ matrix ring over a skew field.
Here, L_j is determined as follows: consider all irreducible sub-projections $Q_\alpha\in K[H_\alpha]$ of P^j (i.e. those satisfying $Q_\alpha P^j=Q_\alpha$), where H_α runs through all finite subgroups of G containing Δ^+. By Lemma 3.4, $\operatorname{tr}_G^u(Q_\alpha)=q_\alpha P^j$ for some rational number q_α. Because there are only finitely many isomorphism classes of finite subgroups of G, formula (1) shows that the collection of these rational numbers is finite. L_j is the smallest integer such that each q_α is an integer multiple of $\frac{1}{L_j}$. Explicitly,

$$L_j=\frac{\dim_{\mathbb{C}}(P^j\cdot\mathbb{C}[\Delta^+])\operatorname{lcm}(G)}{\gcd\Big(\dim_{\mathbb{C}}(P^j\cdot\mathbb{C}[\Delta^+])\operatorname{lcm}(G),\dim_{\mathbb{C}}(Q_\alpha\cdot\mathbb{C}[H_\alpha])\frac{\operatorname{lcm}(G)}{|H_\alpha|}\,|\Delta^+|\mid\alpha\Big)}\in\mathbb{Z}.$$

Proof. We have to show that the two descriptions of L_j coincide, using Equation (1), i.e. we have to find the smallest common denominator of all these fractions. We expand the denominators to the common value $\operatorname{lcm}(G)\cdot\dim_{\mathbb{C}}(P^j\cdot\mathbb{C}[\Delta^+])$, then we have to divide this by the greatest common divisor of this number and of all the new numerators. $\square$

3.7 Theorem. *Let G be a discrete group, with $\operatorname{lcm}(G)<\infty$ and let $K\subset\mathbb{C}$ be a subfield closed under complex conjugation. The following statements are equivalent.*

1. *$D(K[G])$ is Atiyah-expected Artinian as in Definition 3.6.*
2. *$\phi\colon\bigoplus_{E\le G\,:\,|E|<\infty}K_0(K[E])\to K_0(D(K[G]))$ is surjective and $D(K[G])$ is semisimple Artinian.*
3. *$\phi\colon\bigoplus_{E\le G\,;\,|E|<\infty}G_0(K[E])\to G_0(D(K[G]))$ is surjective.*
4. *KG satisfies the center-valued Atiyah conjecture.*

Recall here that, for a ring R, $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules, whereas $G_0(R)$ is the Grothendieck group of arbitrary finitely generated R-modules.

Proof of Theorem 3.7. 1 $\implies$ 2: We use the notation of Definition 3.6. Using the row projectors of matrix rings, there are projections $x_1,\dots,x_{C_{K[G]}}\in D(K[G])$ which represent a $\mathbb{Z}$-basis of the free abelian group $K_0(D(K[G]))$, and $[P^i]=L_i[x_i]$ in $K_0(D(K[G]))$. We have to show that each x_i is an integer linear combination of images of elements of $K_0(K[H_\alpha])$ with H_α finite. If

$Q_\alpha \in K[H_\alpha]$ is an irreducible sub-projection of P^i, then $\phi([Q_\alpha])$ is a multiple of $[x_i]$ in $K_0(D(K[G]))$, namely (comparing the center-valued dimensions which are defined for finitely generated projective $D(K[G])$-modules by the discussion of Section 2) $\phi([Q_\alpha]) = q_\alpha[P^i]$ if $\mathrm{tr}_G^u(Q_\alpha) = q_\alpha P^i$. By the Chinese remainder theorem and the definition of L_i as the smallest integers such that all the q_α are integer multiples of L_i^{-1}, also $[x_i] = L_i^{-1}[P^i]$ belongs to the image of ϕ.
$2 \implies 3$: For a semisimple Artinian ring every finitely generated module is projective, therefore $G_0 = K_0$ under the assumptions we make.
$3 \implies 4$: Let M be a finitely presented $K[G]$-module with presentation $K[G]^l \xrightarrow{A} K[G]^n \to M \to 0$, $A \in M_{n\times l}(K[G])$. Then $M \otimes_{K[G]} D(K[G])$ is finitely generated, therefore by the assumption stably isomorphic to an integer linear combination $\bigoplus a_i x_i D(K[G])$ with x_i projectors defined over finite subgroups E of G — note that $G_0(K[E]) = K_0(K[E])$ for any finite group E, as $K[E]$ is semisimple Artinian. Inducing further to $\mathcal{U}(G)$ and using that the dimension function extends to finitely presented $\mathcal{U}(G)$-modules (which is additive, so that we can leave out the stabilization summands), we read off that

$$\dim_G^u(M) = \dim_G^u(\bigoplus a_i x_i \mathcal{U}(G)) = \sum a_i \dim_G^u(x_i\mathcal{U}(G)) \in L_K(G)$$

by definition of $L_K(G)$. Finally, by additivity of the von Neumann dimension $\dim_G^u(\ker(A)) = n - \dim_G^u(M) \in L_K(G)$.
$4 \implies 1$: Here, we follow closely the argument of the proof of [20, Proposition 2.14]. Our assumption implies by Theorem 1.10 that $\mathcal{E}(K[G]) = D(K[G])$. Because the center-valued Atiyah conjecture implies that the ordinary L^2-Betti numbers are contained in a finitely generated subgroup of $\mathbb{Q}$ (generated by $\mathrm{tr}_G(P^j)/L_j$), by [20, Theorem 2.7] $D(K[G])$ is a semisimple Artinian ring.
The P^j are central idempotents in $D(K[G])$. We have to show that they are primitive central idempotents, and that each is the sum of exactly L_j orthogonal sub-idempotents which are themselves irreducible. The structure theory of rings then implies that each $P^jD(K[G])P^j$ is simple Artinian and an $L_j \times L_j$-matrix ring over a skew field.
Fix, as in Definition 3.6, the (finite) collection of sub-projections Q_α of P^j, where the Q_α are irreducible projections supported on $K[H_\alpha]$ and H_α runs through the (isomorphism classes of) finite extensions of $\Delta^+(G)$ inside G. Then $\mathrm{tr}_G^u(Q_\alpha) = \frac{n_\alpha}{L_j}P^j$ with integers n_α, and by definition of L_j we have $\gcd_\alpha(n_\alpha) = 1$. Set $d := \mathrm{lcm}_\alpha(n_\alpha)$.
Consider now $P^j\mathcal{U}(G)^d$. Because

$$\dim_G^u(P^j\mathcal{U}(G)^d) = dP^j = \dim_G^u(Q_\alpha\mathcal{U}(G)^{L_jd/n_\alpha})$$

by [22, Theorem 9.13(1)] then $P^j\mathcal{U}(G)^d \cong Q_\alpha\mathcal{U}(G)^{L_jd/n_\alpha}$, so we find L_jd/n_α mutually orthogonal projections in $\mathrm{M}_d(\mathcal{U}(G))$ corresponding to the copies of Q_α. Because the center-valued trace of each of those equals $\frac{n_\alpha}{L_j}P^j = \mathrm{tr}_G^u(Q_\alpha)$, by [7, Exercise 13.15A], there exist L_jd/n_α similarities (i.e. self-adjoint unitaries) $u_i \in \mathcal{U}(G)$ with $u_1 = 1$ such that these projections can be written

as $u_i P'_\alpha u_i$ (where P'_α is the diagonal matrix with first entry P_α and all other entries 0).
Then, exactly as in the proof of [20, Proposition 2.14] we can replace the u_i by $\tilde{u}_i \in \mathrm{M}_d(D(K[G]))$ which are invertible and such that we still have a direct sum decomposition

$$P^j D(K[G])^d = \bigoplus_{i=1}^{L_j d/n_\alpha} \tilde{u}_i P'_\alpha D(K[G])^d. \tag{3}$$

This uses the Kaplansky density theorem, the quantization of the center-valued trace and [20, Lemma 2.12].
Let us now take a central idempotent ϵ in $D(K[G])$ which is a sub-projection of P^j (i.e. $\epsilon P^j = \epsilon$). We have to show that $\epsilon = 0$ or $\epsilon = P^j$. To do this, we compute $\mathrm{tr}^u_G(\epsilon)$. Note that all the modules $\epsilon \tilde{u}_i P'_\alpha \mathcal{U}(G)^d$ are isomorphic, therefore by Equation (3)

$$d\,\mathrm{tr}^u_G(\epsilon) = \dim^u_G(\epsilon\mathcal{U}(G)^d) = \frac{L_j d}{n_\alpha} \dim^u_G(\epsilon P'_\alpha \mathcal{U}(G)^d). \tag{4}$$

By Lemma 2.4 and the assumption 4, $L_j \cdot \dim^u_G(\epsilon P'_\alpha \mathcal{U}(G)^d)$ is an integer multiple of P^j. Therefore, rearranging Equation (4)

$$n_\alpha\,\mathrm{tr}^u_G(\epsilon) \in \mathbb{Z}P^j.$$

As this holds for all α, even

$$\epsilon = \mathrm{tr}^u_G(\epsilon) = \mathrm{lcm}_\alpha(n_\alpha)\,\mathrm{tr}^u_G(\epsilon) \in \mathbb{Z}P^j.$$

So we can indeed conclude that P^j is a primitive central idempotent and therefore $P^j D(K[G])$ is an $l \times l$ matrix ring over a skew field. It follows that $P^j D(K[G])^{n_\alpha d}$ is the direct sum of $n_\alpha dl$ copies of an irreducible submodule. On the other hand, $P^j D(K[G])^{n_\alpha d}$ is the direct sum of $L_j d$ isomorphic summands for every α. As $\mathrm{lcm}_\alpha(n_\alpha) = 1$ we conclude that $L_j \mid l$. On the other hand, by the assumption 4 and Lemma 2.4, the center-valued dimension of the irreducible submodule (which is generated by one projector as $P^j D(K[G])$ is Artinian) is an integer multiple of $L_j^{-1} P^j$ and therefore $L_j \mid l$. It follows that $l = L_j$ as claimed. ☐

4 Special cases and inheritance properties of the center-valued Atiyah conjecture

Throughout this section, we assume that K is a subfield of $\mathbb{C}$ which is closed under complex conjugation.

4.1 Lemma. *The center-valued Atiyah conjecture is true for finitely generated virtually free groups.*

Proof. This follows from the proof of [17, Proposition 5.1(i) and Lemma 5.2(ii)] (in which $\mathbb{C}$ can be replaced by any subfield of $\mathbb{C}$) and Theorem 3.72. $\square$

4.2 Lemma. *If G is a directed union of groups G_i and the center-valued Atiyah conjecture over K is true for all groups G_i, then it is also true for G.*

Proof. By [17, Lemma 5.3], $D(K[G])$ is the directed union of the $D(K[G_i])$. Any matrix A over $D(K[G])$ is therefore already a matrix over $D(K[G_i])$ for some i, with $\dim^u_{G_i}(\ker(A)) \in L_K(G_i)$. Composition with the center-valued trace for G gives (by the induction formula for von Neumann dimensions) $\dim^u_G(\ker(A)) \in \mathrm{tr}^u_G(L_K(G_i)) \subset L_K(G)$. $\square$

4.3 Proposition. *Assume that we have an extension $1 \to H \to G \xrightarrow{\pi} E \to 1$ where E is elementary amenable and for each finite subgroup $F \leq E$, $\pi^{-1}(F) \leq G$ satisfies the center-valued Atiyah conjecture over K . Then also $K[G]$ satisfies the center-valued Atiyah conjecture.*

Proof. By transfinite induction, the statement is a formal consequence of the same assertion where E is finitely generated virtually abelian, as explained e.g. in the proof of [29, Proposition 3.1] or in [17].
If E is finitely generated virtually abelian then in the proof of [17, Lemma 5.3] it is shown that

$$\bigoplus_{F\leq E \text{ finite}} G_0(D(K[\pi^{-1}(F)])) \to G_0(D(K[G]))$$

is onto, using Moody's induction theorem [24, Theorem 1]. Since by assumption $\bigoplus_{U\leq \pi^{-1}(F) \text{ finite}} G_0(K[U]) \to G_0(D(K[\pi^{-1}(F)]))$ is onto for each such F and the composition of surjective maps is surjective we conclude that

$$\bigoplus_{F\in \mathcal{F}(G)} G_0(K[F]) \to G_0(D(K[G]))$$

is onto and 3 of Theorem 3.7 is established. $\square$

4.4 Proposition. *Let K be a subfield of $\overline{\mathbb{Q}}$ which is closed under complex conjugation. Assume that G is a group with a sequence $G \geq G_1 \geq \cdots$ of normal subgroups such that $\bigcap_{i\in\mathbb{N}} G_i = \{1\}$. Assume moreover that for each $i \in \mathbb{N}$ and each finite subgroup $F \leq G/G_i$ there is a finite subgroup $F' \leq G$ which is mapped isomorphically to F by the projection $G \to G/G_i$.*
Finally, assume that each G/G_i satisfies the determinant bound conjecture and the center-valued Atiyah conjecture over K. Then $K[G]$ satisfies the center-valued Atiyah conjecture.

Proof. As the statement is empty if $\mathrm{lcm}(G) = \infty$, we assume that $\mathrm{lcm}(G) < \infty$. We first show that, if i is large enough, π_i induces an isomorphism $\pi_i\colon \Delta^+(G) \to \Delta^+(G/G_i)$. Dropping finitely many terms in the sequence we can then assume that this is the case for all $i \in \mathbb{N}$. To prove the assertion, choose a finite subgroup M of G with maximal order (possible since $\mathrm{lcm}(G) < \infty$). Note that the product $\Delta^+ M$ is also a finite subgroup, therefore by maximality equal to M, consequently $\Delta^+ \le M$. Then choose finitely many $g_1, \dots, g_n \in G$ such that $\Delta^+(G) = \bigcap_{k=1}^n M^{g_k}$ (where M^g denotes the conjugate gMg^{-1}), which is possible by the descending chain condition for finite sets.

Finally, choose $r > 0$ such that $\pi_r \colon G \to G/G_r$ is injective when restricted to $\bigcup_{k=1}^n M^{g_k}$, which is possible because $\bigcap_i G_i = \{1\}$.

Because π_r is surjective, $\pi_r(\Delta^+(G))$ is a finite normal subgroup of G/G_i and therefore $\pi_r(\Delta^+(G)) \le \Delta^+(G/G_r)$. On the other hand, $\pi_r(M)$ is a finite subgroup with maximal order in G/G_r (because $\pi_r|_M$ is injective and every finite subgroup of G/G_r is an isomorphic image of a finite subgroup of G), therefore $\Delta^+(G/G_r) \le \pi_r(M)$, by normality even $\Delta^+(G/G_r) \le \bigcap_{k=1}^n \pi_r(M)^{\pi_r(g)}$. As $\bigcap_{k=1}^n M^g = \Delta^+(G)$ and by injectivity of π_r on $\bigcup_{k=1}^n M^g$ we finally get

$$\Delta^+(G/G_r) \le \bigcap_{k=1}^n \pi_r(M)^{\pi_r(g)} = \pi_r(\Delta^+(G)) \le \Delta^+(G/G_r).$$

This implies the statement for all $i \ge r$.

Secondly, given $g \in G$ of infinite order, for all sufficiently large i, the restriction of π_i to $\{1, g, g^2, \dots, g^{\mathrm{lcm}(G)}\}$ is injective and therefore, as by assumption the orders of finite subgroups of G/G_i are bounded by $\mathrm{lcm}(G)$, $\pi_i(g)$ also has infinite order.

Fix now $A \in \mathrm{M}_d(K[G])$ and denote by Q_i the projection onto the kernel of $A[i] := p_i(A)$. Recall that

$$\mathrm{tr}_G^u(Q_i) = \dim_G^u(\ker(A)) = \sum_{g \in G} \langle \dim_G^u(\ker(A)), g\rangle_{l^2(G)} g,$$

and we denote by $\langle \dim_G^u(\ker(A)), g\rangle$ the *coefficient of* g in $\dim_G^u(\ker(A))$, and correspondingly for $\mathrm{tr}_G^u(Q_i)$.

The center-valued Atiyah conjecture for $K[G/G_i]$ implies in particular that $\mathrm{tr}_G^u(Q_i)$ is contained in $K[\Delta^+(G/G_i)]$, therefore supported only on elements of finite order. Consequently, if $g \in G$ has infinite order, then $\langle \mathrm{tr}_G^u(Q_i), \mathrm{pr}_i(g)\rangle = 0$ for sufficiently large i and, by Theorem 2.6, $\langle \dim_G^u(\ker(A)), g\rangle = 0$. This implies that $\dim_G^u(\ker(A))$ is supported on elements of finite order, i.e. is contained in $\mathcal{Z}(\mathcal{N}(G)) \cap K[\Delta^+(G)]$.

As explained above, we can use π_i to identify $\Delta^+(G)$ and $\Delta^+(G/G_i)$ and consider $\mathrm{tr}_G^u(Q_i)$ as an element of $K[\Delta^+(G)]$. By Theorem 2.6, for each $g \in \Delta^+(G)$,

$$\langle \dim_G^u(\ker(A)), g\rangle = \lim_{i\to\infty} \langle \mathrm{tr}_G^u(Q_i), g\rangle.$$

Since all the (finitely many) coefficients converge, we even have

$$\lim_{i\to\infty} \mathrm{tr}_G^u(Q_i) = \dim_G^u(\ker(A)) \in \mathcal{Z}(\mathcal{N}(G)) \cap K[\Delta^+(G)].$$

Because the sets of isomorphism classes of finite subgroups of G/G_i and of G are identified by π_i, we get exactly the same relevant irreducible projections defined over finite subgroups and the same central idempotents in the formulas of Lemma 3.1 and Lemma 3.4 for $L_K(G)$ and $L_K(G/G_i)$. Consequently, π_i identifies $L_K(G)$ and $L_K(G/G_i)$. Finally, observe that by assumption about the Atiyah conjecture for G/G_i we have $\mathrm{tr}^u_G(Q_i) \in L_K(G)$. As the latter is a discrete subset of $\mathcal{Z}(\mathcal{N}(G))$, we finally observe that $\dim^u_G(\ker(A)) \in L_K(G)$, i.e. $K[G]$ satisfies the center-valued Atiyah conjecture. □

4.5 Theorem. *The center-valued Atiyah conjecture is true for all groups $G \in \mathfrak{C}$.*

Proof. In the proof of [17, Lemma 4.9] it is shown that the assertion follows (by transfinite induction) directly from Lemma 4.1, Lemma 4.2 and Proposition 4.3. □

4.6 Corollary. *Let K be a subfield of $\overline{\mathbb{Q}}$ which is closed under complex conjugation. Then the center-valued Atiyah conjecture is true for all elementary amenable extensions of pure braid groups, of right-angled Artin groups, of primitive link groups, of cocompact special groups, or of products of such.*

Proof. Each of the groups in the list has a sequence of normal subgroups with trivial intersection and with elementary amenable quotients such that in addition the condition of Proposition 4.4 is met. This is shown for the extensions of pure braid groups in [19], for primitive link groups in [8] and for right-angled Coxeter and Artin groups in [18], and combining [30] with [18] it also follows for special cocompact groups. Combining Theorem 4.5 and Proposition 4.4, the assertion follows. □

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