A More General Method to Classify up to Equivariant KK-Equivalence

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Abstract. Using a homological invariant together with an obstruction class in a certain Ext^2-group, we classify objects in triangulated categories that have projective resolutions of length two. This invariant gives strong classification results for actions of the circle group on C*-algebras, C*-algebras over finite unique path spaces, and graph C*-algebras with finitely many ideals.

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1. Introduction

The C*-algebra classification program aims at classifying certain C*-algebras up to isomorphism by suitable invariants. Such a classification usually has two steps. First, an isomorphism between the invariants is lifted to an equivalence in a suitable equivariant KK-theory; then the latter is lifted to an isomorphism. These two steps are quite different in nature. The first is mainly algebraic topology, the second mainly analysis. This article deals with the first step of getting equivariant KK-equivalences from isomorphisms on suitable invariants. The invariants used previously are homological functors – variants of K-theory. There are, however, many situations where there is no known homological invariant that is sufficiently fine to detect KK-equivalence. This article introduces a more complex invariant with two layers: the primary invariant is a homological functor as usual; the secondary is a certain obstruction class, which lives

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in an Ext^2-group constructed from the primary invariant. We took this idea from Wolbert [40]. It goes back further to Bousfield [5].

Our two-layer invariants are complete invariants up to KK-equivalence in several new cases and shed light on previous classification results for non-simple Cuntz–Krieger algebras and graph algebras. We explain how to classify arbitrary objects in the bootstrap class in $\mathrm{KK}^{\mathbb{T}}$, where $\mathbb{T}$ is the circle group, and in $\mathrm{KK}(X)$ for a finite unique path space X (see Definition 4.2). The latter result is far more general than previous ones in [4, 27]. Furthermore, we deduce a classification theorem up to stable isomorphism for purely infinite graph C^*-algebras with finitely many ideals; this contains the class of real-rank-zero Cuntz–Krieger algebras classified by Restorff [33]. Our approach has the additional advantage that the resulting classification result is strong, that is, every isomorphism on the level of invariants lifts to an isomorphism of C^*-algebras; this also leads to a classification theorem up to actual isomorphism for the class of unital graph C^*-algebras as above.

Our method is based on homological algebra in triangulated categories, see [22, 26]. This starts with a homological invariant on a triangulated category, which defines a homological ideal as its kernel on morphisms. The general theory gives projective resolutions, derived functors, and a Universal Coefficient Theorem for objects with a projective resolution of length 1. This implies that a certain universal homological invariant F – in practice, this is often the one we started from – is a complete invariant up to isomorphism in the triangulated category. Here we extend this method to also classify objects with a projective resolution of length 2: we find that objects x in the triangulated category with given invariant $F(x)$ are in bijection with the group $\mathrm{Ext}^2(F(x), F(x)[-1])$. Thus $F(x)$ together with a class in $\mathrm{Ext}^2(F(x), F(x)[-1])$ is a complete invariant.

We make this more concrete by examining the example of the triangulated category $\mathrm{KK}^{\mathbb{T}}$ of C^*-algebras with a circle action. Our theorem classifies objects of the ($\mathbb{T}$-equivariant) bootstrap class in $\mathrm{KK}^{\mathbb{T}}$ up to $\mathrm{KK}^{\mathbb{T}}$-equivalence. Here the homological invariant is $\mathbb{T}$-equivariant K-theory $\mathrm{K}_*^{\mathbb{T}}$. This is a functor from $\mathrm{KK}^{\mathbb{T}}$ to the category of $\mathbb{Z}/2$-graded, countable modules over the commutative ring $R = \mathbb{Z}[x, x^{-1}]$, the representation ring of $\mathbb{T}$. Generic R-modules have projective resolutions of length two, not one. Hence there is no Universal Coefficient Theorem in this case. Let M be a countable $\mathbb{Z}/2$-graded R-module. We show

(1) there is a $\mathbb{T}$-C^*-algebra A in the bootstrap class with $\mathrm{K}_*^{\mathbb{T}}(A) \cong M$;
(2) for such A, there is an invariant $\delta(A) \in \mathrm{Ext}^2_R(M, M)^-$ such that $\delta(A_1) \cong \delta(A_2)$ if and only if there is a $\mathrm{KK}^{\mathbb{T}}$-equivalence $A_1 \to A_2$ inducing the identity map on $M = \mathrm{K}_*^{\mathbb{T}}(A_i)$; here $\mathrm{Ext}^2_R(M, M)^-$ denotes the odd part of the $\mathbb{Z}/2$-graded group $\mathrm{Ext}^2_R(M, M)$.

In particular, if $\mathrm{Ext}^2_R(M, M)^- = 0$ then M lifts uniquely to a $\mathbb{T}$-C^*-algebra A in the bootstrap class.

The above result does not yet finish the classification of $\mathbb{T}$-C^*-algebras A in the bootstrap class because there may be isomorphisms $A_1 \to A_2$ that induce a non-identity isomorphism $M \to M$ on $M = \mathrm{K}_*^{\mathbb{T}}(A_1) = \mathrm{K}_*^{\mathbb{T}}(A_2)$. The complete invariant takes values in a category of pairs (M, δ), where M is a countable, $\mathbb{Z}/2$-graded R-module and $\delta \in \mathrm{Ext}^2_R(M, M)^-$ and where a morphism $(M_1, \delta_1) \to (M_2, \delta_2)$ is a grading-preserving R-module homomorphism $f\colon M_1 \to M_2$ with $\delta_2 f = f\delta_1$ in $\mathrm{Ext}^2_R(M_1, M_2)^-$. We show that isomorphism classes in the bootstrap class in $\mathrm{KK}^{\mathbb{T}}$ are in bijection with isomorphism classes in this category of pairs.

Many purely infinite C*-algebras carry a gauge action by $\mathbb{T}$ by construction. As examples of our classification, we consider Cuntz–Krieger algebras and some C*-algebras constructed by Nekrashevych in [29]. In these cases, $\mathrm{Ext}^2_R(M, M)^- = 0$, so that there is no obstruction class.

The above classification result is very efficient for *counting* isomorphism classes of objects in the bootstrap class with a given $\mathbb{Z}/2$-graded R-module M as its equivariant K-theory. It may be hard, however, to compute the obstruction class in $\mathrm{Ext}^2_R(M, M)^-$ for a given $\mathbb{T}$-C*-algebra A with $\mathrm{K}^{\mathbb{T}}_*(A) \cong M$. At the moment, we have no examples of non-equivalent $\mathbb{T}$-C*-algebras that are distinguished only by the obstruction class. The authors intend to provide adequate methods for computing obstruction classes in future work.

Our next application concerns the bootstrap class in $\mathrm{KK}(X)$ for a finite topological T_0-space X, see [25]. Kirchberg's Classification Theorem says that an equivalence in $\mathrm{KK}(X)$ between two strongly purely infinite, stable, separable, nuclear C*-algebras with primitive ideal space X lifts to a *-isomorphism, so classification up to $\mathrm{KK}(X)$-equivalence already implies classification theorems up to isomorphism for suitable C*-algebras. Previous classification results in $\mathrm{KK}(X)$ in [4, 27] only apply to very special X because projective resolutions of length 1 are rare.

Invariants with enough projective resolutions of length 2 are more common. If X is a unique path space, then the K-theories of the ideals corresponding to minimal neighbourhoods of points in X give an invariant with this property. This invariant is much smaller than filtrated K-theory. Since any object of the bootstrap class has a projective resolution of length 2, our new classification method applies to arbitrary objects in the bootstrap class of $\mathrm{KK}(X)$ for a unique path space X.

Even if X is not a unique path space, our classification theorem applies to objects in the bootstrap class in $\mathrm{KK}(X)$ that have projective resolutions of length 2. We show that this is the case for graph C*-algebras with finitely many ideals. Furthermore, we compute the obstruction class of a graph algebra from the Pimsner–Voiculescu type sequences that compute the K-theory groups of its ideals. Hence our complete invariant may be computed effectively in this case. We get a strong classification functor up to stable isomorphism for purely infinite graph C*-algebras with finitely many ideals; strong classification means that every isomorphism on the invariants lifts to a stable isomorphism. This is the first strong classification result – even for the class of purely infinite Cuntz–Krieger algebras – without the assumption of a specific ideal structure. The invariant and its computation are described in more detail in Section 5.

Our abstract setup should also work in many other situations. One of them is connective K-theory, regarded as an invariant on connective E-theory. We refer to Andreas Thom's thesis [39] for details. See [9] for applications of connective K-theory to C*-algebras. Another instance is Kasparov's KK-theory for C*-algebras over a zero-dimensional compact metrisable space X. Here the K-theory of the total algebra has a natural module structure over the ring of locally constant functions $C(X, \mathbb{Z})$. This ring has global dimension 2 by [13, Examples 2.5(b)].

For C*-algebras over the unit interval and filtrated K-theory as the invariant, the relevant Abelian category has dimension 2 once again. So far, we cannot treat this example, however, because there are not enough projective objects in this case.

1.1. Outline. The structure of this article is as follows. Section 2 develops the general theory of obstruction classes. Section 3 applies it to circle actions on C*-algebras, Section 4 to C*-algebras over unique path spaces, and Section 5 to graph C*-algebras; this includes a return to general triangulated categories in order to compute obstruction classes for objects of a specific type.

1.2. Notation. We use $\sqcup$ to denote the place for the input object of a functor. For example, $\mathfrak{T}(\sqcup, B)$ denotes the contravariant hom-functor represented by an object B in a category $\mathfrak{T}$, and $\sqcup \otimes V$ denotes the functor given by tensoring with the object V. We write $\in\in$ for objects of a category.

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2. Lifting two-dimensional objects

Throughout the article, we use the language of homological algebra in triangulated categories introduced in [26]. We recall what this means in the setting of KK^G to help readers with an operator algebraic background. We write standing assumptions in italics below to make them easy to spot.

Let $\mathfrak{T}$ be a triangulated category with countable coproducts, also called direct sums. For instance, let G be a compact Lie group. The G-equivariant Kasparov category KK^G has C*-algebras with continuous G-actions as objects and $\mathrm{KK}_0^G(A, B)$ as arrows from A to B. A triangulated category structure on KK^G is introduced in [23]. Countable direct sums in the C*-algebraic sense are coproducts in KK^G.

To do homological algebra in $\mathfrak{T}$, we assume a stable homological ideal $\mathfrak{I}$ in $\mathfrak{T}$ that is compatible with countable direct sums. Any such ideal is of the form

$$\mathfrak{I}(A, B) = \{\varphi \in \mathfrak{T}(A, B) \mid F(\varphi) = 0\} \qquad \text{for all } A, B \in\in \mathfrak{T} \tag{2.1}$$

for some stable homological functor F to a stable Abelian category $\mathfrak{A}$. "Stability" means that $\mathfrak{A}$ carries a suspension automorphism and that F intertwines the suspensions in $\mathfrak{T}$ and $\mathfrak{A}$ up to natural isomorphism.

If $\mathfrak{T} = \mathrm{KK}^G$, we choose F and $\mathfrak{J}$ as follows. The representation ring R of G is naturally isomorphic to $\mathrm{KK}^G_0(\mathbb{C},\mathbb{C})$. The G-equivariant K-theory

$$\mathrm{K}^G_*(A) = \mathrm{K}_*(A \rtimes G) \cong \mathrm{KK}^G_*(\mathbb{C}, A)$$

is a $\mathbb{Z}/2$-graded module over R by Kasparov's intersection product. It is countable if A is separable because it is the K-theory of a separable C^*-algebra. Let $\mathfrak{A}$ be the category of countable, $\mathbb{Z}/2$-graded R-modules, with the suspension automorphism shifting the grading. Let $F := \mathrm{K}^G_*\colon \mathfrak{T} \to \mathfrak{A}$ be the equivariant K-theory functor. Let $\mathfrak{J}$ be the kernel of F on morphisms as in (2.1). This is a stable homological ideal by definition. Its compatibility with countable direct sums follows from the "continuity" of topological K-theory.

We return to a general stable homological ideal $\mathfrak{J}$ in a triangulated category $\mathfrak{T}$. An object $P \in\in \mathfrak{T}$ is $\mathfrak{J}$-*projective* if the map $\mathfrak{T}(P, f)\colon \mathfrak{T}(P, A) \to \mathfrak{T}(P, B)$ is 0 for all $f \in \mathfrak{J}(A, B)$ and $A, B \in\in \mathfrak{T}$. For instance, $\mathbb{C}$ with the trivial G-action is a projective object in KK^G for the ideal defined above because $F(A) = \mathrm{KK}^G_*(\mathbb{C}, A)$.

We assume that $\mathfrak{T}$ has enough $\mathfrak{J}$-projective objects. That is, for any $A \in\in \mathfrak{T}$, there are an $\mathfrak{J}$-projective object P and an arrow $\pi\colon P \to A$ such that $F(\pi)\colon F(P) \to F(A)$ is surjective. The last property depends only on $\mathfrak{J}$, not on F: If

$$P \xrightarrow{\pi} A \xrightarrow{\gamma} C \xrightarrow{\iota} P[1]$$

is an exact triangle containing π, then $F(\pi)$ is surjective if and only if $\gamma \in \mathfrak{J}$ if and only if $F(\iota)$ is injective. We also call the maps π and ι as above $\mathfrak{J}$-*epic* and $\mathfrak{J}$-*monic*, respectively. If there are enough projective objects, then any object A has an $\mathfrak{J}$-*projective resolution*, that is, there is a chain complex

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \cdots \to P_0 \to A \to 0$$

in the additive category $\mathfrak{T}$ such that all P_n, $n \in \mathbb{N}$, are $\mathfrak{J}$-projective and such that F maps it to an *exact* chain complex in $\mathfrak{A}$. The length of this complex is a measure of the complexity of the object A of $\mathfrak{T}$. We shall classify objects of increasing complexity, where this length is 0, 1 and 2. It seems, however, that this cannot be continued for objects that only admit projective resolutions of length 3.

The ideal $\mathfrak{J}$ does not determine the functor F with (2.1) uniquely. But among the functors F defining $\mathfrak{J}$, there is a "universal" choice. *We assume that F is the universal $\mathfrak{J}$-exact stable homological functor.* For the ideal in KK^G considered above, the functor $F(A) := \mathrm{KK}^G_*(\mathbb{C}, A)$ is indeed universal, see [26, Theorem 72]; this theorem also asserts that $\mathfrak{J}$ has enough projective objects.

When $\mathfrak{J}$ has enough projective objects and $F\colon \mathfrak{T} \to \mathfrak{A}$ is the universal $\mathfrak{J}$-exact stable homological functor, then the following happens by [26, Theorem 57]. First the category $\mathfrak{A}$ has enough projective objects; secondly, the adjoint functor $F^\vdash$ of F is defined on all projective objects of $\mathfrak{A}$; and, thirdly, $F \circ F^\vdash(A) \cong A$

for all projective objects A of $\mathfrak{A}$. Let $\langle\mathfrak{P}_{\mathfrak{I}}\rangle \subseteq \mathfrak{T}$ denote the localising subcategory generated by the $\mathfrak{I}$-projective objects in $\mathfrak{T}$. [22, Theorem 3.22] implies that $\hat{A} \in\in \langle\mathfrak{P}_{\mathfrak{I}}\rangle$ if and only if $\mathfrak{T}(\hat{A}, B) = 0$ for all objects $B \in\in \mathfrak{T}$ with $\mathrm{id}_B \in \mathfrak{I}$ (such objects are also called $\mathfrak{I}$-*contractible*).

In the example of KK^G, the adjoint functor $F^{\vdash}$ maps the free R-module R to $\mathbb{C}$ with trivial G-action because

$$\mathrm{KK}^G_*(\mathbb{C}, A) \cong \mathrm{K}^G_*(A) \cong \mathfrak{A}\bigl(R, \mathrm{K}^G_*(A)\bigr).$$

The functor $F^{\vdash}$ automatically commutes with countable direct sums. So its value on R determines what it does on all countable projective R-modules. An object B of KK^G is $\mathfrak{I}$-contractible if and only if $\mathrm{K}^G_*(B) = 0$. By [22, Theorem 3.22], we have $\mathrm{KK}^G_*(A, B) = 0$ for all $\mathfrak{I}$-contractible B if and only if A belongs to $\langle\mathbb{C}\rangle$, the localising subcategory of KK^G generated by $\mathbb{C}$. We denote this subcategory by $\mathcal{B}^G$ because it is the correct analogue of the bootstrap class in KK^G for nice enough compact groups G, that is, if G is a connected compact Lie group with torsion-free $\pi_1(G)$. (For general compact Lie groups, we should allow more generators to define the equivariant bootstrap class, see [10, Section 3.1].) From now on, we will develop some general theory in the abstract setting of triangulated categories. In Section 3, we shall return to the particular case of KK^G, for nice enough groups G such as $G = \mathbb{T}$.

Definition 2.2. A *lifting* of $A \in\in \mathfrak{A}$ is a pair $(\hat{A}, \alpha)$ consisting of $\hat{A} \in\in \mathfrak{T}$ with $\mathfrak{T}(\hat{A}, B) = 0$ for all $\mathfrak{I}$-contractible $B \in\in \mathfrak{T}$ and an isomorphism $\alpha\colon F(\hat{A}) \xrightarrow{\cong} A$. An *equivalence* between two liftings $(\hat{A}_1, \alpha_1)$, $(\hat{A}_2, \alpha_2)$ is an isomorphism $\varphi \in \mathfrak{T}(\hat{A}_1, \hat{A}_2)$ with $\alpha_1 = \alpha_2 \circ F(\varphi)$. We often drop α from the notation and call $\hat{A}$ a lifting of A.

If $A \in\in \mathfrak{A}$ is projective, then $F^{\vdash}(A)$ with the canonical isomorphism $F\bigl(F^{\vdash}(A)\bigr) \cong A$ is a *natural* lifting of A.

Proposition 2.3. *Let $A \in\in \mathfrak{A}$ have cohomological dimension* 1. *Then A has a lifting, and any two liftings are equivalent.*

Proof. Let

$$0 \to P_1 \xrightarrow{\partial} P_0 \to A \to 0$$

be a projective resolution in $\mathfrak{A}$. Then $F^{\vdash}(\partial)\colon F^{\vdash}(P_1) \to F^{\vdash}(P_0)$ is an arrow in $\mathfrak{T}$ with $F\bigl(F^{\vdash}(\partial)\bigr) \cong \partial$. Let $\hat{A}$ be the cone of $F^{\vdash}(\partial)$. Since ∂ is monic, $F^{\vdash}(\partial)$ is $\mathfrak{I}$-monic. Hence $F\bigl(F^{\vdash}(P_1)\bigr) \rightarrowtail F\bigl(F^{\vdash}(P_0)\bigr) \twoheadrightarrow F(\hat{A})$ is a short exact sequence, proving that $F(\hat{A}) \cong A$. If B is $\mathfrak{I}$-contractible, then $\mathfrak{T}(F^{\vdash}(P_j), B) \cong \mathfrak{T}\bigl(P_j, F(B)\bigr) = 0$ and hence $\mathfrak{T}(\hat{A}, B) = 0$ by the long exact sequence for $\mathfrak{T}(\sqcup, B)$. Hence $\hat{A}$ is a lifting of A.

Let $\hat{A}_1$ and $\hat{A}_2$ be liftings of A. This includes a choice of isomorphisms $F(\hat{A}_1) \cong A$ and $F(\hat{A}_2) \cong A$. The Universal Coefficient Theorem [26, Theorem 66] applies to $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$. Hence there is $f \in \mathfrak{T}(\hat{A}_1, \hat{A}_2)$ that lifts the identity map on A when we identify $F(\hat{A}_1) \cong A$ and $F(\hat{A}_2) \cong A$. Since f is an $\mathfrak{I}$-equivalence, its cone B is $\mathfrak{J}$-contractible. Thus $\mathfrak{T}_*(\hat{A}_i, B) = 0$ for $i = 1, 2$, and this implies $\mathfrak{T}_*(B, B) = 0$ and hence $B = 0$ by the long exact sequence. Thus f is invertible. $\square$

The equivalence between two liftings in Proposition 2.3 is not canonical, and the lifting is not natural, unlike for projective objects. The Universal Coefficient Theorem [26, Theorem 66] only shows that any arrow $A_1 \to A_2$ in $\mathfrak{A}$ between objects of cohomological dimension 1 lifts to an arrow in $\mathfrak{T}$. But this lifting is only unique up to $\operatorname{Ext}^1(A_1, A_2[-1])$. With parity assumptions as in Section 2.1, there is a canonical lifting for any arrow $A_1 \to A_2$ between objects of cohomological dimension 1: lift its even and odd parts separately (the corresponding Ext^1-terms in the Universal Coefficient Theorem vanish) and then take the direct sum. This shows that the UCT short exact sequence splits under parity assumptions. This splitting is not natural, however.
Proposition 2.3 implies that isomorphism classes of objects in $\mathfrak{A}$ of cohomological dimension 1 correspond bijectively to isomorphism classes of objects A in $\langle \mathfrak{P}_{\mathfrak{J}} \rangle$ with $F(A)$ of cohomological dimension 1. This is used in [4,20,27] and other classification results. It may, however, be very hard to find computable invariants F for which all objects in its image have cohomological dimension 1.

Lemma 2.4. *Any $A \in\in \mathfrak{A}$ of cohomological dimension 2 has a lifting in $\mathfrak{T}$.*

Proof. Let

$$0 \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0 \tag{2.5}$$

be an exact chain complex in $\mathfrak{A}$ with projective P_0, P_1 and P_2. Let

$$\Omega A := \partial_1(P_1) \cong \operatorname{coker} \partial_2,$$

so that we get short exact sequences

$$P_2 \rightarrowtail P_1 \twoheadrightarrow \Omega A, \qquad \Omega A \rightarrowtail P_0 \twoheadrightarrow A.$$

Since ΩA has a projective resolution of length 1, it has a lifting $D \in\in \mathfrak{T}$ by Proposition 2.3. Let $\hat{P}_0 := F^{\vdash}(P_0)$ be the canonical lifting of P_0.
The Universal Coefficient Theorem [26, Theorem 66] gives a short exact sequence

$$\operatorname{Ext}^1(\Omega A[1], P_0) \rightarrowtail \mathfrak{T}(D, \hat{P}_0) \twoheadrightarrow \operatorname{Hom}(\Omega A, P_0). \tag{2.6}$$

Hence the inclusion map $\Omega A \hookrightarrow P_0$ lifts to some $f \in \mathfrak{T}(D, \hat{P}_0)$, which is $\mathfrak{J}$-monic. The mapping cone $\hat{A}$ of f belongs to $\langle \mathfrak{P}_{\mathfrak{J}} \rangle$ by construction, and has $F(\hat{A}) \cong P_0/\Omega A \cong A$ by the short exact sequence (2.5), so it is a lifting of A. $\square$

[27, Theorem 4.10] shows that liftings of objects of cohomological dimension two cannot be unique in general. We may, however, classify liftings up to equivalence:

Theorem 2.7. *Let $A \in\in \mathfrak{A}$ have cohomological dimension 2. The set of equivalence classes of liftings of A is in bijection with $\operatorname{Ext}^2(A, A[-1])$.*

Proof. Fix a length-two projective resolution of A as in (2.5) and a lifting $\hat{A}$ of A, which exists by Lemma 2.4. Let $\hat{P}_0 := F^{\vdash}(P_0)$. The map $P_0 \to A$ in $\mathfrak{A}$ is adjoint to a map $\hat{P}_0 \to \hat{A}$ in $\mathfrak{T}$. We may complete this to an exact triangle

$$D \xrightarrow{\varphi} \hat{P}_0 \to \hat{A} \to D[1]. \tag{2.8}$$

Since $P_0 \to A$ is surjective, the map $\hat{P}_0 \to \hat{A}$ is $\mathfrak{I}$-epic. Hence F maps (2.8) to a short exact sequence $F(D) \rightarrowtail P_0 \twoheadrightarrow A$. Thus D is a lifting of $\Omega A := \ker \partial_0 \cong \operatorname{coker} \partial_2$. Since ΩA has cohomological dimension 1, its lifting D is unique up to isomorphism by Proposition 2.3. The exact triangle (2.8) shows that $\hat{A}$ is the cone of φ. So any other lifting $\hat{A}'$ must be the cone of some arrow $\varphi'\colon D \to \hat{P}_0$ that induces the inclusion map $F(D) \to P_0$. Conversely, if $\varphi'\colon D \to \hat{P}_0$ lifts the inclusion map $\Omega A \to P_0$, then its cone is a lifting of A by the proof of Lemma 2.4.

Let $\hat{A}$ and $\hat{A}'$ be the liftings associated to φ and φ'. We claim that an isomorphism $\alpha\colon \hat{A} \to \hat{A}'$ that induces the identity map on $F(\hat{A}') \cong A \cong F(\hat{A})$ may be embedded in a morphism of triangles

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \hat{P}_0 \arrow[r, "\pi_0"] \arrow[d, equal] & \hat{A} \arrow[d, "\alpha"] \\ D \arrow[r, "\varphi'"] & \hat{P}_0 \arrow[r, "\pi_0'"] & \hat{A}'. \end{tikzcd} \tag{2.9}$$

The assumption that α induces the identity map on A means that the right square commutes. Then by the third axiom of triangulated categories (see [28, §1.1]), there is an arrow $\psi\colon D \to D$ that gives a triangle morphism. The arrow ψ induces the identity map on $F(D) = \Omega A$ because the map $F(\varphi) = F(\varphi')\colon F(D) \to P_0$ is injective. Thus $\varphi' \circ \psi = \varphi$ for some $\psi\colon D \to D$ that induces the identity map on $F(D)$.

Conversely, let $\varphi' \circ \psi = \varphi$ for some $\psi\colon D \to D$ that induces the identity map on $F(D)$. This means that the left square in (2.9) commutes. We may embed this square in a triangle morphism to construct $\alpha\colon \hat{A} \to \hat{A}'$. Since ψ induces the identity map on $F(D)$, it is invertible. Hence α is also invertible by the Five Lemma for exact triangles. Summing up, $\hat{A}$ and $\hat{A}'$ are equivalent liftings if and only if there is $\psi\colon D \to D$ with $\varphi' \circ \psi = \varphi$ and $F(\psi) = \mathrm{id}_{\Omega A}$.

By the Universal Coefficient Theorem, the possible choices for $\psi - \mathrm{id}_D$ and $\varphi' - \varphi$ lie in $\operatorname{Ext}^1(\Omega A, \Omega A[-1])$ and $\operatorname{Ext}^1(\Omega A, P_0[-1])$, respectively. The short exact sequence $\Omega A \rightarrowtail P_0 \twoheadrightarrow A$ induces a long exact sequence

$$\cdots \to \operatorname{Ext}^1(\Omega A, \Omega A[-1]) \xrightarrow{j} \operatorname{Ext}^1(\Omega A, P_0[-1]) \xrightarrow{p} \operatorname{Ext}^1(\Omega A, A[-1]) \to 0$$

because $\operatorname{Ext}^2(\Omega A, \sqcup) = 0$. We claim that the two liftings $\hat{A}$ and $\hat{A}'$ are equivalent if and only if $\varphi' - \varphi$ belongs to the image of the map j.

If $\hat{A}$ and $\hat{A}'$ are equivalent, we can write $\varphi' \circ \psi = \varphi$ as above. Then $\varphi' - \varphi = \varphi' \circ (\mathrm{id}_D - \psi)$. Since $\mathrm{id}_D - \psi$ belongs to $\operatorname{Ext}^1(\Omega A, \Omega A[-1]) \subseteq \mathfrak{T}(D, D)$, the naturality of the Universal Coefficient Theorem allows to compute the element

$\varphi' \circ (\mathrm{id}_D - \psi)$ in $\mathrm{Ext}^1(\Omega A, P_0[-1]) \subseteq \mathfrak{T}(D, \hat{P}_0)$ as the Ext-product of $(\mathrm{id}_D - \psi) \in \mathrm{Ext}^1(\Omega A, \Omega A[-1])$ with the induced homomorphism $F(\varphi') \in \mathrm{Hom}(\Omega A, P_0)$. Since $F(\varphi')$ is the inclusion map $\Omega A \hookrightarrow P_0$, it takes $\mathrm{id}_D - \psi$ to $j(\mathrm{id}_D - \psi)$. In particular, $\varphi' - \varphi$ belongs to the image of j.

Conversely, if $\varphi' - \varphi = j(\alpha)$ for some $\alpha \in \mathrm{Ext}^1(\Omega A, \Omega A[-1])$, we may write $\varphi' - \varphi = j(\mathrm{id}_D - \psi)$ by setting $\psi = \mathrm{id}_D - \alpha \in \mathrm{Ext}^1(\Omega A, \Omega A[-1]) \subseteq \mathfrak{T}(D, D)$. Since $F(\alpha) = 0$ we have $F(\psi) = \mathrm{id}_{\Omega A}$. Moreover, $\varphi' \circ \psi = \varphi$ holds because $\varphi' - \varphi = j(\mathrm{id}_D - \psi) = \varphi' \circ (\mathrm{id}_D - \psi) = \varphi' - \varphi' \circ \psi$. Hence $\hat{A}$ and $\hat{A}'$ are equivalent. It follows that $\hat{A}$ and $\hat{A}'$ are equivalent if and only if $p(\varphi') = p)\varphi)$ in $\mathrm{Ext}^1(\Omega A, A[-1])$, and any element in $\mathrm{Ext}^1(\Omega A, A[-1])$ is of the form $p(\varphi')$ for some φ'. Since P_0 is projective, another long exact sequence implies

$$\mathrm{Ext}^1(\Omega A, A[-1]) \cong \mathrm{Ext}^2(A, A[-1]).$$

Thus $\hat{A}$ and $\hat{A}'$ are equivalent if and only if $\varphi' - \varphi$ is mapped to 0 in $\mathrm{Ext}^2(A, A[-1])$, and any element in $\mathrm{Ext}^2(A, A[-1])$ comes from some φ'.

We claim that the map sending $\hat{A}'$ to the image of $\varphi' - \varphi$ in $\mathrm{Ext}^2(A, A[-1])$ is a bijection from the set of equivalence classes of liftings of A to $\mathrm{Ext}^2(A, A[-1])$. Indeed, if $\hat{A}'_1$ and $\hat{A}'_2$ are two arbitrary liftings, they are cones of maps $\varphi'_1, \varphi'_2 \colon D \to \hat{P}_0$ both inducing the inclusion map $\Omega A \to P_0$ on F. The liftings $\hat{A}'_1$ and $\hat{A}'_2$ are equivalent if and only if the map

$$\mathfrak{T}(D, \hat{P}_0) \supseteq \mathrm{Ext}^1(\Omega A, P_0[-1]) \to \mathrm{Ext}^1(\Omega A, A[-1]) \to \mathrm{Ext}^2(A, A[-1]) \tag{2.10}$$

sends $\varphi'_1 - \varphi'_2$ to 0. Since (2.10) is a group homomorphism, this happens if and only if $\varphi'_1 - \varphi$ and $\varphi'_2 - \varphi$ have the same image in $\mathrm{Ext}^2(A, A[-1])$. □

Corollary 2.11. *If $A \in\in \mathfrak{A}$ has cohomological dimension 2 and $\mathrm{Ext}^2(A, A[-1]) = 0$, then A has a unique lifting up to equivalence.*

Our construction actually shows that the set of equivalence classes of liftings carries a free and transitive action of the Abelian group $\mathrm{Ext}^2(A, A[-1])$. Once we pick a single element, we thus get a bijection to $\mathrm{Ext}^2(A, A[-1])$; but this bijection depends on the choice of one lifting, namely, the one corresponding to $0 \in \mathrm{Ext}^2(A, A[-1])$. For two liftings $\hat{A}_1$ and $\hat{A}_2$ associated to classes $\delta_1, \delta_2 \in \mathrm{Ext}^2(A, A[-1])$, the difference $\delta_2 - \delta_1 \in \mathrm{Ext}^2(A, A[-1])$ is canonically defined. It is called the *relative obstruction class* $\delta(\hat{A}_2, \hat{A}_1)$.

We discuss another approach to the relative obstruction class, which relates it to the boundary map on the second page of the ABC spectral sequence for $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$ associated to the ideal $\mathfrak{I}$. This cohomological spectral sequence, named after Adams, Brinkmann and Christensen, is discussed in great detail in [22, Section 4]. Its E_2-term is

$$E_2^{p,q} = \mathrm{Ext}^p(A, A[q])$$

for $p \geq 0$, $q \in \mathbb{Z}$. By assumption, $E_2^{p,q} = 0$ for $p \neq 0, 1, 2$. Hence $E_k^{p,q} = 0$ for $p \neq 0, 1, 2$ and $k \geq 3$ as well. Since the boundary map d_k on E_k has bidegree

$(k, 1-k)$, we get $d_k = 0$ for $k \geq 3$, and the only part of d_2 that may be non-zero is $d_2^{0,q}\colon E_2^{0,q} \to E_2^{2,q-1}$. Hence

$$\begin{aligned}
E_\infty^{0,q} &= \ker(d_2^{0,q}\colon E_2^{0,q} \to E_2^{2,q-1}),\\
E_\infty^{1,q} &= E_2^{1,q},\\
E_\infty^{2,q} &= \operatorname{coker}(d_2^{0,q+1}\colon E_2^{0,q+1} \to E_2^{2,q}).
\end{aligned}$$

As a consequence, $\varphi \in \operatorname{Hom}(A, A)$ lifts to $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$ if and only if $d_2^{0,0}(\varphi) = 0$. In particular, $\hat{A}_1$ and $\hat{A}_2$ are equivalent liftings if and only if $\mathrm{id}_A \in \operatorname{Hom}(A, A)$ lifts to $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$, if and only if

$$d_2^{0,0}(\mathrm{id}_A) \in E_2^{2,-1} = \operatorname{Ext}^2(A, A[-1])$$

vanishes. Thus both conditions $d_2^{0,0}(\mathrm{id}_A) = 0$ and $\delta(\hat{A}_1, \hat{A}_2) = 0$ are necessary and sufficient for an equivalence of liftings. This suggests the following lemma:

Lemma 2.12. $d_2^{0,0}(\mathrm{id}_A) = \delta(\hat{A}_1, \hat{A}_2) = -\delta(\hat{A}_2, \hat{A}_1)$.

Proof. The cohomological spectral sequence for $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$ in [22, Section 4] is constructed using a phantom tower for $\hat{A}_1$. We implicitly already constructed such a phantom tower when lifting A to $\hat{A}_1$. In the notation above, it looks as follows (where circled arrows denote maps of degree -1):

$$\begin{tikzcd}
\hat{A}_1 \arrow[rr, "\iota_0", "\circ"{marking}] & & D \arrow[rr, "\iota_1", "\circ"{marking}] \arrow[dl, "\varphi_1"'] & & \hat{P}_2 \arrow[rr] \arrow[dl, "\hat{\partial}_2"'] \arrow[dr, equal] & & 0 \arrow[rr] \arrow[dl] & & \cdots \arrow[dl] & \\
& \hat{P}_0 \arrow[ul, "\hat{\partial}_0"] & & \hat{P}_1 \arrow[ll, "\hat{\partial}_1"] \arrow[ul, "\pi_1"] & & \hat{P}_2 \arrow[ll, "\hat{\partial}_2"] & & 0 \arrow[ll] \arrow[ul] & & \cdots \arrow[ll] \arrow[ul]
\end{tikzcd} \tag{2.13}$$

(The conventions about the degrees of the maps in the phantom tower are different in [22].) Here $\hat{P}_j$ and $\hat{\partial}_j$ are the unique liftings of the projective objects P_j and the boundary maps ∂_j in (2.5) for $j = 0, 1, 2$, and φ_1 is the map with cone $\hat{A}_1$ that was called φ in the arguments above. The triangles involving ι_n are $\mathfrak{I}$-exact, and the other triangles commute. This together with the $\mathfrak{I}$-projectivity of the objects $\hat{P}_n$ means that (2.13) is a phantom tower.

The relevant cohomological spectral sequence is constructed by applying the cohomological functor $\mathfrak{T}(\sqcup, \hat{A}_2)$ to the phantom tower for $\hat{A}_1$ in (2.13). The boundary map $d_2 := d_2^{0,0}$ on the second page maps $E_2^{0,0}$ to $E_2^{2,-1}$, where

$$\begin{aligned}
E_2^{0,0} &:= \{x \in \mathfrak{T}_0(\hat{P}_0, \hat{A}_2) \mid \hat{\partial}_1^*(x) = 0\} &&\cong \operatorname{Hom}(A, A),\\
E_2^{2,-1} &:= \mathfrak{T}_{-1}(\hat{P}_2, \hat{A}_2) \,/\, \hat{\partial}_2^*\bigl(\mathfrak{T}_{-1}(\hat{P}_1, \hat{A}_2)\bigr) &&\cong \operatorname{Ext}^2(A, A[-1]).
\end{aligned}$$

We describe how d_2 acts on $\mathrm{id}_A \in \operatorname{Hom}(A, A)$ (see [22, Section 4.1]). By [26, Theorem 59] and projectivity of P_0, there is a unique element $\hat{\partial}_0' \in \mathfrak{T}_0(\hat{P}_0, \hat{A}_2)$ such that $F(\hat{\partial}_0') = \partial_0\colon P_0 \to A$. We have $\hat{\partial}_0' \circ \hat{\partial}_1 = 0$ because $F(\hat{\partial}_0' \circ \hat{\partial}_1) = 0$. The isomorphism $\operatorname{Hom}(A, A) \to E_2^{0,0}$ above maps id_A to $\hat{\partial}_0' \in \mathfrak{T}_0(\hat{P}_0, \hat{A}_2)$. Since

$\hat{\partial}'_0 \circ \varphi_1 \circ \pi_1 = \hat{\partial}'_0 \circ \hat{\partial}_1 = 0$, there is $\rho \in \mathfrak{T}_{-1}(\hat{P}_2, \hat{A}_2)$ with $\iota_1^*(\rho) = \hat{\partial}'_0 \circ \varphi_1$. The image of ρ in $E_2^{2,-1}$ is $d_2(\mathrm{id}_A)$.

The UCT exact sequence for $\mathfrak{T}_*(D, \hat{A}_2)$ is the long exact sequence associated to the triangle $\hat{P}_2\hat{P}_1 D$ in (2.13). This exact sequence shows that ι_1^* is an isomorphism from $E_2^{2,-1}$, the cokernel of $\hat{\partial}_2^*$, onto

$$\mathrm{Ext}^1(\Omega A, A[-1]) \subseteq \mathfrak{T}_0(D, \hat{A}_2).$$

The map d_2 is constructed so that $\iota_1^*\bigl(d_2(\mathrm{id}_A)\bigr) = \hat{\partial}'_0 \circ \varphi_1$.

Embed $\hat{\partial}'_0$ in an exact triangle

$$D \xrightarrow{\varphi_2} \hat{P}_0 \xrightarrow{\hat{\partial}'_0} \hat{A}_2 \to D[1]$$

as in (2.8). Then $\hat{\partial}'_0 \circ \varphi_2 = 0$ and hence $\iota_1^*\bigl(d_2(\mathrm{id}_A)\bigr) = \hat{\partial}'_0 \circ (\varphi_1 - \varphi_2)$.

The map $\varphi_2 - \varphi_1$ induces the zero map $F(D) \to F(\hat{P}_0)$ and hence corresponds to an element x in $\mathrm{Ext}^1(\Omega A, P_0[-1])$ by the UCT sequence. By definition, the obstruction class $\delta(\hat{A}_2, \hat{A}_1)$ is the image of x under the map

$$\mathrm{Ext}^1(\Omega A, P_0[-1]) \to \mathrm{Ext}^1(\Omega A, A[-1]) \cong \mathrm{Ext}^2(A, A[-1]),$$

where the first map is induced by the projection $P_0 \to A$. By the naturality of the UCT sequence, this maps x to $\hat{\partial}'_0 \circ (\varphi_2 - \varphi_1)$. Comparing this with our computation of $d_2(\mathrm{id}_A)$ shows that $\delta(\hat{A}_2, \hat{A}_1) = -d_2(\mathrm{id}_A)$. □

Our description of equivalence classes of liftings is not yet a classification of objects in $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ up to isomorphism. Two objects $\hat{A}_1, \hat{A}_2 \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ are isomorphic if and only if there is an isomorphism $F(\hat{A}_1) \to F(\hat{A}_2)$ that lifts to $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$. If $F(\hat{A}_1) = F(\hat{A}_2)$ and $\delta(\hat{A}_1, \hat{A}_2) \neq 0$, then the identity map $F(\hat{A}_1) \to F(\hat{A}_2)$ does not lift; but there may be another isomorphism $F(\hat{A}_1) \cong F(\hat{A}_2)$ that lifts to $\mathfrak{T}(\hat{A}_1, \hat{A}_2)$. This seems hard to decide given only $F(\hat{A}_i)$ and $\delta(\hat{A}_1, \hat{A}_2) \neq 0$. The parity assumptions that we are about to make remedy this situation.

2.1. Parity assumptions. We are going to impose an extra assumption on $\mathfrak{A}$ that provides a *canonical* lifting for each object of $\mathfrak{A}$ of cohomological dimension 2. This allows us to understand the action of automorphisms on obstruction classes and to classify objects of $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ with length-2-projective resolutions up to isomorphism.

Definition 2.14. A stable Abelian category is called *paired* if $\mathfrak{A} = \mathfrak{A}_+ \times \mathfrak{A}_-$ with $\mathfrak{A}_+[-1] = \mathfrak{A}_-$ and $\mathfrak{A}_-[-1] = \mathfrak{A}_+$; that is, any object of $\mathfrak{A}$ is a direct sum of objects of even and odd parity, and the suspension automorphism on $\mathfrak{A}$ shifts parity.

Example 2.15. Let $\mathfrak{A}$ be the category of countable, $\mathbb{Z}/2$-graded modules over a ring R. Then $\mathfrak{A}$ is paired, with $\mathfrak{A}_\pm$ being the subcategories of countable R-modules concentrated in even or odd degree, respectively.

Let $\mathfrak{T} = \mathrm{KK}^G$ for a compact group G and $\mathfrak{I}$ is the kernel on morphisms of the functor K_*^G as in (2.1). Then $\mathfrak{A}$ is the category of countable, $\mathbb{Z}/2$-graded modules over the representation ring of G. Hence $\mathfrak{A}$ is paired in this example.

Assume that $\mathfrak{A}$ is paired. Since the two subcategories $\mathfrak{A}_\pm$ are orthogonal, we have $\operatorname{Ext}^2(A_+, A_-) = 0$ and $\operatorname{Ext}^2(A_-, A_+) = 0$ for $A_+ \in \mathfrak{A}_+$, $A_- \in \mathfrak{A}_-$. Now write $A \in\in \mathfrak{A}$ as $A \cong A_+ \oplus A_-$ with $A_\pm \in \mathfrak{A}_\pm$. Then $\operatorname{Ext}^2(A_+, A_+[-1]) = 0$ because $A_+[-1] \in \mathfrak{A}_-$ and $\operatorname{Ext}^2(A_-, A_-[-1]) = 0$ because $A_-[-1] \in \mathfrak{A}_+$. Corollary 2.11 shows that there are unique liftings $\hat{A}_+$ and $\hat{A}_-$ for A_+ and A_- (up to equivalence). We call $\hat{A}_0 := \hat{A}_+ \oplus \hat{A}_- \in\in \mathfrak{T}$ the *canonical* lifting of A and let $\delta(\hat{A}) := \delta(\hat{A}, \hat{A}_0)$ for any other lifting. This defines a canonical obstruction class in $\operatorname{Ext}^2(A, A[-1])$ for all liftings $\hat{A}$ of A. A simple computation as in [40, Proposition 9] shows that, for $f \in \operatorname{Hom}(A, B)$, the element $d_2^{0,0}(f) \in \operatorname{Ext}^2(A, B[-1])$ is given by the formula

$$d_2^{0,0}(f) = \delta(\hat{B})f - f\delta(\hat{A}). \tag{2.16}$$

To see this, let $\alpha\colon F(\hat{A}_0) \to A$ and $\beta\colon F(\hat{B}_0) \to B$ be isomorphisms. Then $f' := \beta^{-1} f \alpha \in \operatorname{Hom}\bigl(F(\hat{A}_0), F(\hat{B}_0)\bigr)$ can be lifted componentwise, so that $d_2^{0,0}(f') = 0$. Thus

$$\begin{aligned} &d_2^{0,0}(f) = d_2^{0,0}(\beta f' \alpha^{-1}) = d_2^{0,0}(\beta) f' \alpha^{-1} + \beta f' d_2^{0,0}(\alpha^{-1}) \\ &= d_2^{0,0}(\beta)\beta^{-1} f + f\alpha d_2^{0,0}(\alpha^{-1}) = \delta(\hat{B}, \hat{B}_0) f + f\delta(\hat{A}_0, \hat{A}) = \delta(\hat{B})f - f\delta(\hat{A}). \end{aligned}$$

Definition 2.17. Let $\mathfrak{A}\delta$ denote the additive category of pairs (A, δ) with $A \in\in \mathfrak{A}$ and $\delta \in \operatorname{Ext}^2(A, A[-1])$; morphisms from (A, δ) to (A', δ') in $\mathfrak{A}\delta$ are morphisms f from A to A' in $\mathfrak{A}$ which satisfy the compatibility condition $\delta' f = f\delta$.

There is an additive functor

$$F\delta\colon \mathfrak{T} \to \mathfrak{A}\delta, \qquad \hat{A} \mapsto \bigl(F(\hat{A}), \delta(\hat{A})\bigr).$$

The following classification result generalises [5, Theorem 9.1] and [40, Theorem 11].

Theorem 2.18. *Assume that $\mathfrak{A}$ is paired and has global dimension 2. Then the functor $F\delta$ is full and induces a bijection between isomorphism classes of objects $\hat{A}$ in $\langle \mathfrak{P}_\mathfrak{I} \rangle$ and isomorphism classes of objects in the category $\mathfrak{A}\delta$. Furthermore, every lift of an isomorphism in $\mathfrak{A}\delta$ is an isomorphism in $\mathfrak{T}$.*

Proof. The last claim in the theorem follows from a standard argument: if $\hat{A}_1$ and $\hat{A}_2$ belong to $\langle \mathfrak{P}_\mathfrak{I} \rangle$ and if $f \in \mathfrak{T}(\hat{A}_1, \hat{A}_2)$ is an $\mathfrak{I}$-equivalence, then the mapping cone C_f of f is both $\mathfrak{I}$-contractible and in $\langle \mathfrak{P}_\mathfrak{I} \rangle$; hence $\mathfrak{T}(C_f, B) = 0$ for all $\mathfrak{I}$-contractible $B \in\in \mathfrak{T}$ and in particular $\mathfrak{T}(C_f, C_f) = 0$, so that $C_f \cong 0$, that is, f is invertible.
Theorem 2.7 shows that every class in $\operatorname{Ext}^2(A, A[-1])$ appears as $\delta(\hat{A})$ for some lifting $\hat{A}$ of A. Hence $F\delta$ is essentially surjective. Since a morphism $f \in \operatorname{Hom}(A, B)$ lifts to a morphism $\hat{A} \to \hat{B}$ if (and only if) $d_2^{0,0}(f) = 0$, (2.16) shows that the functor $F\delta$ is full. Hence $F\delta$ distinguishes isomorphism classes. □

3. Kasparov theory for circle actions

Let G be a connected compact Lie group with torsion-free fundamental group. We will soon specialise to the circle group $G = \mathbb{T}$, but some results hold more generally. As in Section 2, let $\mathfrak{T} := \mathrm{KK}^G$ be the G-equivariant Kasparov theory. Let $\mathfrak{A}$ be the category of countable, $\mathbb{Z}/2$-graded R-modules, where R is the representation ring of G, and let $F := \mathrm{K}^G_*\colon \mathfrak{T} \to \mathfrak{A}$ be the equivariant K-theory functor. Let $\mathfrak{I}$ be the kernel of F on morphisms. We have already seen that $\mathbb{C}$ with the trivial action is $\mathfrak{I}$-projective, and we identified the localising subcategory generated by the projective objects with $\langle\mathbb{C}\rangle = \mathcal{B}^G$. The following result is implicit in [24].

Proposition 3.1. *Let G be a connected compact Lie group such that $\pi_1(G)$ is torsion-free. Let T be a maximal torus in G. A G-C^*-algebra A belongs to $\mathcal{B}^G$ if and only if $A \rtimes G$ belongs to the usual bootstrap class $\mathcal{B}$ in* KK*, if and only if $A \rtimes T$ belongs to $\mathcal{B}$.*

Proof. If $A \in\in \mathcal{B}^G$, then $A\rtimes G \in\in \langle\mathbb{C}\rtimes G\rangle = \langle \mathrm{C}^*(G)\rangle = \langle\mathbb{C}\rangle = \mathcal{B}$ because $\mathrm{C}^*(G)$ is a direct sum of matrix algebras; similarly, $A \rtimes T \in\in \mathcal{B}$. Conversely, assume that $A\rtimes T \in \mathcal{B} \subseteq \mathrm{KK}$. The assumptions on G imply that $H^2(G, \mathbb{T}) = 0$. Hence [24, Proposition 3.3] says that any G-C^*-algebra A belongs to the localising subcategory of $\mathfrak{T}$ generated by $A\rtimes T$ equipped with the trivial G-action. Taking the trivial G-action is a triangulated functor $\mathfrak{t}\colon \mathrm{KK} \to \mathrm{KK}^G$, so $A \in\in \langle\mathfrak{t}(A\rtimes T)\rangle$ and $A \rtimes T \in\in \mathcal{B} = \langle\mathbb{C}\rangle$ give $A \in\in \langle\mathfrak{t}(\mathbb{C})\rangle = \mathcal{B}^G$ as asserted.
There is a Morita–Rieffel equivalence $A\rtimes T \sim \bigl(A\otimes \mathrm{C}(G/T)\bigr)\rtimes G$. [24, Proposition 2.1] says that $\mathrm{C}(G/T)$ is KK^G-equivalent to $\mathbb{C}^w$, where w is the size of the Weyl group of G. Hence $A\rtimes T$ is KK-equivalent to $(A\rtimes G)^w$. Thus $A\rtimes T \in\in \mathcal{B}$ if and only if $A \rtimes G \in\in \mathcal{B}$. □

Let n be the rank of the maximal torus in G and let W be the Weyl group of G. Then $R \cong \mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]^W$, where the action of W comes from the canonical action on T. Even more, we have

$$R \cong \mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_l^{-1}]$$

for some l with $0 \le l \le n$; see for instance [38]. This ring has cohomological dimension $n+1$ because $\mathbb{Z}$ has cohomological dimension 1 and each independent variable adds 1 to the length of resolutions (see [35, Theorem 3.2.3 and Corollary 3.2.4]).
The cohomological dimension of R is 2 if and only if $n = 1$. The two groups G with $n = 1$ are the circle group $\mathbb{T}$ and $\mathrm{SU}(2)$. (The group $\mathrm{SO}(3)$ has torsion in π_1 and therefore is not covered by Proposition 3.1.) If $n = 1$, then Theorem 2.7 applies to all objects of $\mathfrak{A}$. That is, any $M \in\in \mathfrak{A}$ has a lifting, and equivalence classes of liftings are in bijection with $\mathrm{Ext}^2_{\mathfrak{A}}(M, M[-1])$; this is the even part of $\mathrm{Ext}^2_R(M, M[-1])$ with its usual $\mathbb{Z}/2$-grading, so that we may also denote it by $\mathrm{Ext}^2_R(M, M)^-$. The category $\mathfrak{A}$ is even, so that the results of Section 2.1 apply as well. That is, there is a canonical lifting of any $M \in\in \mathfrak{A}$, namely, the direct sum $\hat{M}_+ \oplus \hat{M}_-$, where $\hat{M}_+$ and $\hat{M}_-$ are the unique liftings of the

even and odd parts of M, respectively. Every object $A \in\in \mathcal{B}^G$ has an invariant $(M,\delta) \in\in \mathfrak{A}\delta$ with $M := \mathrm{K}_*^G(A)$ and $\delta \in \mathrm{Ext}_R^2(M,M)^-$; Theorem 2.18 says that $A_1, A_2 \in\in \mathcal{B}^G$ corresponding to (M_1,δ_1) and (M_2,δ_2) in $\mathfrak{A}\delta$ are isomorphic if and only if there is a grading preserving R-module isomorphism $f\colon M_1 \to M_2$ with $f\delta_1 = \delta_2 f$ in $\mathrm{Ext}_R^2(M_1,M_2)^-$.

If $n = 2$, then Theorem 2.7 still applies, among others, to objects of $\mathfrak{A}$ that are free as Abelian groups. Groups G for which this happens are $\mathbb{T}^2$, $\mathbb{T}\times\mathrm{SU}(2)$, $\mathrm{SU}(2)\times\mathrm{SU}(2)$, $\mathrm{SU}(3)$, $\mathrm{Spin}(5)$, and the simply connected compact Lie group with Dynkin diagram of type G_2. For even higher rank, we know no useful sufficient criterion for an R-module to have a projective resolution of length 2.

We now consider some natural examples of circle actions on C*-algebras. Thus $G = \mathbb{T}$ and $R = \mathbb{Z}[x,x^{-1}]$ from now on.

Example 3.2. Consider the Cuntz algebra $\mathcal{O}_n$ for $n < \infty$ with its usual gauge action, defined by multiplying each generator by $z \in \mathbb{T}$. Then $\mathcal{O}_n \rtimes \mathbb{T}$ is Morita–Rieffel equivalent to the fixed-point algebra $\mathcal{O}_n^{\mathbb{T}}$. This is the UHF-algebra of type n^∞. It belongs to the bootstrap class, so that $\mathcal{O}_n \in \mathcal{B}^{\mathbb{T}}$ by Proposition 3.1. And it has K-theory $\mathbb{Z}[1/n]$ concentrated in degree 0. The generator of the representation ring x acts on this by multiplication by n. Thus

$$M := \mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_n) \cong \mathbb{Z}[x,x^{-1}]/(x-n),$$

where $(x-n)$ means the principal ideal generated by $x-n$. This is concentrated in degree 0 and has a length-1-projective resolution

$$0 \to \mathbb{Z}[x,x^{-1}] \xrightarrow{x-n} \mathbb{Z}[x,x^{-1}] \twoheadrightarrow \mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_n). \tag{3.3}$$

Either of these two facts shows that $\mathrm{Ext}^2_{\mathbb{Z}[x,x^{-1}]}(M,M)^- = 0$. Hence $\mathcal{O}_n$ is the unique object of $\mathcal{B}^{\mathbb{T}}$ with $\mathrm{K}_*^{\mathbb{T}}(A) \cong \mathbb{Z}[x,x^{-1}]/(x-n)$.

Similarly, $C(S^1)$ with the translation action is the unique object of $\mathcal{B}^{\mathbb{T}}$ up to KK^T-equivalence with $\mathrm{K}_i^{\mathbb{T}}(A) \cong \mathbb{Z}$ for $i = 0,1$ and x acting trivially.

3.1. Cuntz–Krieger algebras. Now consider the Cuntz–Krieger algebra $\mathcal{O}_A$ with its usual gauge action; it is defined by an $n\times n$-matrix A with entries in $\{0,1\}$ or more generally in the non-negative integers, such that no row or column vanishes identically. The crossed product $\mathcal{O}_A \rtimes \mathbb{T}$ is Morita–Rieffel equivalent to the fixed-point algebra $\mathcal{O}_A^{\mathbb{T}}$ by [30, Theorem 3.2.2 and Lemma 4.1.1]. The fixed-point algebra is an AF-algebra, and its K_0-group is isomorphic to the direct limit of the iteration sequence

$$\mathbb{Z}^n \xrightarrow{A^{\mathrm{t}}} \mathbb{Z}^n \xrightarrow{A^{\mathrm{t}}} \mathbb{Z}^n \xrightarrow{A^{\mathrm{t}}} \mathbb{Z}^n \xrightarrow{A^{\mathrm{t}}} \cdots;$$

the action of the generator x is induced by multiplication with A^{t} (see [8, Proof of Proposition 3.1]). In particular, given two Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$, for degree reasons we have $\mathrm{Ext}^2_{\mathbb{Z}[x,x^{-1}]}\bigl(\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_A), \mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_B)\bigr)^- = 0$. Alternatively, we may write down a projective resolution of $\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_A)$ of length 1 as in (3.3). Hence every grading-preserving $\mathbb{Z}[x,x^{-1}]$-module isomorphism $\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_A) \to \mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_B)$ lifts to a $\mathrm{KK}^{\mathbb{T}}$-equivalence. We get the following characterisation of $\mathrm{KK}^{\mathbb{T}}$-equivalence for Cuntz–Krieger algebras:

Theorem 3.4. *Let A and B be finite square matrices with non-negative integral entries such that no row or column vanishes identically. The following are equivalent:*

- *The gauge actions on $\mathcal{O}_A$ and $\mathcal{O}_B$ are $\mathrm{KK}^{\mathbb{T}}$-equivalent.*
- *The $\mathbb{Z}[x,x^{-1}]$-modules $\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_A)$ and $\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_B)$ are isomorphic.*
- *The matrices A and B are shift equivalent over the integers.*

Proof. The equivalence of the first two statements follows from the argument above. For the equivalence of the second and third statement, see [21, Theorem 7.5.7]. □

[6, Example 2.13] gives two irreducible non-negative 4×4-matrices A and B that are shift equivalent over the integers but not over the *non-negative* integers. Then the gauge actions on the purely infinite simple Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are $\mathrm{KK}^{\mathbb{T}}$-equivalent by the previous theorem; but the *ordered* $\mathbb{Z}[x,x^{-1}]$-modules $\mathrm{K}_0(\mathcal{O}_A^{\mathbb{T}})$ and $\mathrm{K}_0(\mathcal{O}_B^{\mathbb{T}})$ are not isomorphic by [21, Theorem 7.5.8]. Hence the shift automorphisms on the gauge fixed-point algebras $\mathcal{O}_A^{\mathbb{T}}$ and $\mathcal{O}_B^{\mathbb{T}}$ cannot be stably conjugate. By Takai duality, the gauge actions on $\mathcal{O}_A$ and $\mathcal{O}_B$ cannot be stably conjugate. We cannot expect a Kirchberg–Phillips type classification result for circle actions unless the fixed-point algebra is also purely infinite and simple. For the most useful gauge actions, the fixed-point algebra is AF, however. So we cannot expect isomorphisms in $\mathrm{KK}^{\mathbb{T}}$ to lift to *-isomorphisms.

Remark 3.5. It was already observed in [36, Proposition 10.4] that the $\mathrm{KK}^{\mathbb{T}}$-equivalence class of an object in the $\mathbb{T}$-equivariant bootstrap class with equivariant K-theory concentrated in one degree is determined by its equivariant K-theory. [36, §10] contains some more results establishing KK^G-equivalence in special cases for Hodgkin–Lie groups G.

3.2. Computing Ext^2. We now describe $\mathrm{Ext}^*_R(V,W)$ for two general R-modules V and W, where $R = \mathbb{Z}[x,x^{-1}]$. We view an R-module V as an Abelian group with an automorphism x_V, namely, the action of the generator $x \in R$.

The ring R has a very short R-bimodule resolution

$$0 \to R \otimes R \xrightarrow{x\otimes 1 - 1\otimes x} R \otimes R \xrightarrow{\text{mult}} R \to 0.$$

This remains exact when we apply the functor $\sqcup \otimes_R V$ for a left R-module V. This gives a short exact sequence of R-modules

$$0 \to R \otimes V \xrightarrow{x\otimes 1 - 1\otimes x_V} R \otimes V \xrightarrow{\text{mult}} V \to 0$$

for any R-module V. Given another R-module W, the long exact cohomology sequence for this short exact sequence becomes

$$\begin{aligned} 0 \to \mathrm{Hom}_R(V,W) \to \mathrm{Hom}_R(R\otimes V,W) \to \mathrm{Hom}_R(R\otimes V,W) \\ \to \mathrm{Ext}^1_R(V,W) \to \mathrm{Ext}^1_R(R\otimes V,W) \to \mathrm{Ext}^1_R(R\otimes V,W) \\ \to \mathrm{Ext}^2_R(V,W) \to \mathrm{Ext}^2_R(R\otimes V,W) \to \mathrm{Ext}^2_R(R\otimes V,W) \to 0. \end{aligned}$$

This simplifies considerably because

$$\mathrm{Ext}^n_R(R \otimes V, W) \cong \mathrm{Ext}^n_{\mathbb{Z}}(V, W)$$

by adjoint associativity. Thus we get a long exact sequence

$$\begin{aligned}(3.6)\quad 0 &\to \mathrm{Hom}_R(V,W) \to \mathrm{Hom}_{\mathbb{Z}}(V,W) \to \mathrm{Hom}_{\mathbb{Z}}(V,W)\\ &\to \mathrm{Ext}^1_R(V,W) \to \mathrm{Ext}^1_{\mathbb{Z}}(V,W) \to \mathrm{Ext}^1_{\mathbb{Z}}(V,W) \to \mathrm{Ext}^2_R(V,W) \to 0.\end{aligned}$$

Here the maps $\mathrm{Hom}_{\mathbb{Z}}(V,W) \to \mathrm{Hom}_{\mathbb{Z}}(V,W)$ and $\mathrm{Ext}^1_{\mathbb{Z}}(V,W) \to \mathrm{Ext}^1_{\mathbb{Z}}(V,W)$ are $f \mapsto (x_W)_*f - (x_V)^*f$, using the automorphisms x_V and x_W of V and W. Hence

$$(3.7)\qquad \mathrm{Ext}^2_R(V,W) \cong \mathrm{coker}\big((x_W)_* - (x_V)^*\big)\colon \mathrm{Ext}^1_{\mathbb{Z}}(V,W) \to \mathrm{Ext}^1_{\mathbb{Z}}(V,W).$$

Remark 3.8. We may view this cokernel as the first Hochschild cohomology for R with coefficients in $\mathrm{Ext}^1_{\mathbb{Z}}(V,W)$ with the induced R-bimodule structure. The kernel of this map is the zeroth Hochschild cohomology. The above long exact sequence is equivalent to a spectral sequence

$$\mathrm{HH}^p\big(R, \mathrm{Ext}^q_{\mathbb{Z}}(V,W)\big) \Rightarrow \mathrm{Ext}^{p+q}_R(V,W).$$

Eusebio Gardella shows in [14, 15] that for circle actions on unital Kirchberg algebras A with the Rokhlin property, such that A satisfies the Universal Coefficient Theorem and has finitely generated K-theory groups, the action of the generator of R on equivariant K-theory is the identity (this is analogous to the situation of finite group actions with the Rokhlin property, see [31]), and equivariant K-theory together with the unit class is a complete invariant. Moreover, $\mathrm{K}_0(A) \cong \mathrm{K}_1(A) \cong \mathrm{K}_0(A^{\mathbb{T}}) \oplus \mathrm{K}_1(A^{\mathbb{T}})$, and every pair (G_0, G_1) of finitely generated Abelian groups with any unit class in G_0 may be realised as $\big(\mathrm{K}_0(A^{\mathbb{T}}), \mathrm{K}_1(A^{\mathbb{T}})\big)$. Another classification result for Rokhlin actions of finite groups on Kirchberg algebras was proved by Masaki Izumi [16].

If x acts identically on V and W, then $\mathrm{Ext}^2_R(V,W) \cong \mathrm{Ext}^1_{\mathbb{Z}}(V,W)$ by (3.7). Therefore, there must be a unique obstruction class in $\mathrm{Ext}^1_{\mathbb{Z}}\big(\mathrm{K}_*(A^{\mathbb{T}}), \mathrm{K}_{*+1}(A^{\mathbb{T}})\big)$ that comes from a Rokhlin action on a unital Kirchberg algebra. We do not know, however, which obstruction class this is.

3.3. Nekrashevych's C*-algebras of self-similar groups. Nekrashevych [29] constructs purely infinite simple C*-algebras with a gauge action of $\mathbb{T}$ from self-similar groups. He proves that the conjugacy class of this gauge action essentially determines the underlying self-similar group and hence is a very fine invariant. This is, however, far from true for the $\mathrm{KK}^{\mathbb{T}}$-equivalence class.

We consider only the particular case considered in [29, Theorem 4.8] to use Nekrashevych's K-theory computation. The self-similar group G in question is the iterated monodromy group of a post-critically finite, hyperbolic, rational function f on $\hat{\mathbb{C}}$. Let n be the (mapping) degree of this rational function, that is, each non-critical point has precisely n preimages. The function f has at most finitely many attracting cycles; let their lengths be $\ell_1, \dots, \ell_c$, listed with repetitions. Thus f has c attracting cycles.

It is asserted in [29, Theorem 4.8] that the K-theory of the gauge fixed-point algebra $\mathcal{O}_G^{\mathbb{T}}$ of the C*-algebra $\mathcal{O}_G$ associated to f is $\mathbb{Z}[1/n]$ in even degrees and $\mathbb{Z}^{k-1}$ in odd degrees, where $k = \sum_{i=1}^{c} \ell_i$. We can be more precise: the proof of [29, Theorem 4.8] also gives the $\mathbb{T}$-equivariant K-theory of $\mathcal{O}_G$.

First, $\mathcal{O}_G^{\mathbb{T}}$ is Morita–Rieffel equivalent to the crossed product in this case, so that the K-theory of $\mathcal{O}_G^{\mathbb{T}}$ is isomorphic to the $\mathbb{T}$-equivariant K-theory and carries a $\mathbb{Z}[x, x^{-1}]$-module structure. The action of x on this module is given by multiplication by n on the even part, as for the Cuntz algebra $\mathcal{O}_n$. Thus $\mathrm{K}_0^{\mathbb{T}}(\mathcal{O}_G) \cong R/(x-n)$ has a projective resolution of length 1.

The odd part is the quotient of $H = \mathbb{Z}^{\ell_1} \oplus \dots \oplus \mathbb{Z}^{\ell_c}$ by the diagonally embedded copy of $\mathbb{Z}$. We may view H as the space of functions from the union of the attracting cycles of f to $\mathbb{Z}$. The generator x acts like f on these functions, that is, it is a cyclic permutation in each copy of $\mathbb{Z}^{\ell_i}$. Thus we get the quotient of the module

$$V := \bigoplus_{i=1}^{c} \mathbb{Z}[x_i, x_i^{-1}]/(x_i^{\ell_i} - 1)$$

by the copy of $\mathbb{Z}$ generated by $N_i := 1 + x_i + \dots + x_i^{\ell_i - 1}$ in each component.

Lemma 3.9. $\mathrm{Ext}^2_R\big(\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_G), \mathrm{K}_{*+1}^{\mathbb{T}}(\mathcal{O}_G)\big) = 0$.

Proof. Let $\mathrm{K}_* := \mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_G)$. Since K_1 is free as an Abelian group, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1, \mathrm{K}_0) = 0$ and hence $\mathrm{Ext}^2_R(\mathrm{K}_1, \mathrm{K}_0) = 0$ by the long exact sequence (3.6). Since K_0 has a projective R-module resolution of length 1 by (3.3), $\mathrm{Ext}^2_R(\mathrm{K}_0, \mathrm{K}_1) = 0$ as well. □

Since $\mathcal{O}_G^{\mathbb{T}}$ is the C*-algebra of an amenable groupoid, it belongs to the bootstrap class. Hence so does the Morita–Rieffel equivalent C*-algebra $\mathcal{O}_G \rtimes \mathbb{T}$. So our classification results apply by Proposition 3.1. Lemma 3.9 shows that, up to circle-equivariant KK-equivalence, the C*-algebra $\mathcal{O}_G$ is classified completely by the $\mathbb{Z}/2$-graded $\mathbb{Z}[x, x^{-1}]$-module $\mathrm{K}_*^{\mathbb{T}}(\mathcal{O}_G)$. This module only remembers the degree n of f and the multiset of lengths ℓ_i, so it is a rather coarse invariant.

It would be very interesting to refine our invariant to detect the conjugacy class of the gauge action because this determines the action of f on its Julia set up to topological conjugacy by Nekrashevych's main result ([29, Section 4.2]). Unfortunately, we know no useful refinements for our invariant. Both for $\mathcal{O}_G$ and for Cuntz–Krieger algebras, the fixed-point algebra of the gauge action has a unique trace. For Cuntz–Krieger algebras, the order structure on $\mathrm{K}_0^{\mathbb{T}}$ gives a finer invariant (see Section 3.1), but for $\mathcal{O}_G$, the group $\mathrm{K}_0^{\mathbb{T}}(\mathcal{O}_G) \cong \mathbb{Z}[1/n]$ carries no interesting order structure (an order on $\mathbb{Z}[1/n]$ is determined by its restrictions to the subgroups $\mathbb{Z} \cdot 1/n^k \cong \mathbb{Z}$, $k \in \mathbb{N}$).

4. Kasparov theory for C*-algebras over unique path spaces

Let X be a finite T_0-space. In this section, we consider Kirchberg's ideal-related KK-theory $\mathfrak{T} := \mathfrak{KK}(X)$, following [25,27]. Let $i_x\mathbb{C} \in\in \mathfrak{T}$ denote the C*-algebra of complex numbers $\mathbb{C}$ equipped with the continuous map $\mathrm{Prim}(\mathbb{C}) \to X$ taking the unique element of $\mathrm{Prim}(\mathbb{C})$ to $x \in X$. The bootstrap class $\mathcal{B}(X)$ in $\mathfrak{T}$ is the localising subcategory generated by the collection $\{i_x\mathbb{C} \mid x \in X\}$ of one-dimensional C*-algebras over X (see [25, Definition 4.11]).

We apply the homological machinery from [26] to the family of functors represented by the objects $i_x\mathbb{C}$, respectively. Let $A(U_x)$ be the distinguished ideal of A corresponding to the smallest open neighbourhood U_x of x in X. The adjointness relations in [25, Proposition 3.13] specialise to

$$\mathrm{KK}_*(X; i_x\mathbb{C}, A) \cong \mathrm{KK}_*\big(\mathbb{C}, A(U_x)\big) \cong \mathrm{K}_*\big(A(U_x)\big). \tag{4.1}$$

To make this relation plausible, observe that a *-homomorphism over X from $i_x\mathbb{C}$ to A is just a *-homomorphism from $\mathbb{C}$ to A such that the distinguished ideal $\mathbb{C}(U_x) = \mathbb{C}$ is mapped into the distinguished ideal $A(U_x)$.

For $x \in X$, consider the stable homological functor

$$F_x : \mathfrak{T} \to \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}, \quad A \mapsto \mathrm{KK}_*(X; i_x\mathbb{C}, A)$$

and the homological ideal $\mathfrak{I}_x := \ker F_x$. Since

$$\mathrm{KK}_*(X; i_x\mathbb{C}, A) \cong \mathrm{Hom}_{\mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}}\big(\mathbb{Z}[0], F_x(A)\big),$$

the adjoint functor $F_x^\vdash$ takes the free rank-one Abelian group in even degree to the object $i_x\mathbb{C}$. [26, Theorem 57] implies that F_x is the universal $\mathfrak{I}_x$-exact functor and that $\mathfrak{I}_x$ has enough projective objects.

Now we consider the homological ideal $\mathfrak{I} := \bigcap_{x\in X} \mathfrak{I}_x$. [25, Theorem 4.17] gives $\mathrm{KK}_*(X; A, B) = 0$ for all $\mathfrak{I}$-contractible B if and only if A belongs to $\mathcal{B}(X)$. By [26, Proposition 55], the ideal $\mathfrak{I}$ has enough projective objects. An argument as in [27, Section 4.3] shows that the universal $\mathfrak{I}$-exact stable homological functor is

$$\mathrm{XK} := \mathrm{KK}_*(X; \mathcal{R}, \sqcup) : \mathfrak{T} \to \mathfrak{Mod}\big(\mathrm{KK}_*(X; \mathcal{R}, \mathcal{R})^{\mathrm{op}}\big)_{\mathrm{c}},$$

where $\mathcal{R} := \bigoplus_{x\in X} i_x\mathbb{C}$ and $\mathfrak{Mod}\big(\mathrm{KK}_*(X; \mathcal{R}, \mathcal{R})^{\mathrm{op}}\big)_{\mathrm{c}}$ denotes the category of countable $\mathbb{Z}/2$-graded *right* modules over the $\mathbb{Z}/2$-graded ring $\mathrm{KK}_*(X; \mathcal{R}, \mathcal{R})$.

Partially order X by $x \preceq y$ if and only if $x \in \overline{\{y\}}$, if and only if $y \in U_x$. Equation (4.1) implies

$$\mathrm{KK}_*(X; i_x\mathbb{C}, i_y\mathbb{C}) \cong \begin{cases} \mathbb{Z}[0] & \text{if } x \preceq y, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x, y \in X$. The proof of (4.1) shows that the generator of $\mathrm{KK}_0(X; i_x\mathbb{C}, i_y\mathbb{C}) = \mathbb{Z}$ for $x \preceq y$ is the class i_x^y of the identity map on $\mathbb{C}$, viewed as a *-homomorphism over X from $i_x\mathbb{C}$ to $i_y\mathbb{C}$. Since $i_y^z \circ i_x^y = i_x^z$, the $\mathbb{Z}/2$-graded ring $\mathrm{KK}_*(X; \mathcal{R}, \mathcal{R})^{\mathrm{op}}$ is isomorphic to the integral incidence algebra $\mathbb{Z}[X]$ of the poset $(X, \preceq)$ in even degree and vanishes in odd degree; here we use the convention that $\mathbb{Z}[X]$ is the free Abelian group generated by

elements $f_{x\preceq y}$ for all pairs (x,y) with $x \preceq y$; the multiplication is defined by $f_{x\preceq y} f_{y\preceq z} = f_{x\preceq z}$. An early reference for the incidence algebra is [37, §3].
We write $x \to y$ for $x, y \in X$ if $x \succ y$ and there is no $z \in X$ with $x \succ z \succ y$. Then $x \succ y$ if and only if there is a path $x = x_0 \to x_1 \to \dots \to x_\ell = y$ with some $x_1, \dots, x_{\ell-1} \in X$.

Definition 4.2. A finite T_0-space X is called a *unique path space* if the path $x = x_0 \to x_1 \to \dots \to x_\ell = y$ is unique for all $x, y \in X$ with $x \succ y$.

If X is a unique path space, then $\mathbb{Z}[X]$ is the integral path algebra of the quiver $(X, \to)$, where paths are concatenated such that arrows point to the left.
There is a canonical family of orthogonal idempotent elements $e_x \in \mathbb{Z}[X]$ for $x \in X$ with $\sum_{x\in X} e_x = 1$. Viewing $\mathbb{Z}[X]$-bimodules as modules over the ring $\mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{Z}[X^{\mathrm{op}}]$, we see that $P_{x\otimes y} := \mathbb{Z}[X]e_x \otimes e_y\mathbb{Z}[X]$ is a projective $\mathbb{Z}[X]$-bimodule (corresponding to the idempotent $e_x \otimes e_y$). There are canonical $\mathbb{Z}[X]$-bimodule maps

$$\begin{aligned} \psi_x\colon P_{x\otimes x} &\to \mathbb{Z}[X], && a\otimes b \mapsto a\cdot b,\\ \varphi_{x,y}\colon P_{x\otimes y} &\to P_{x\otimes x} \oplus P_{y\otimes y}, && a\otimes b \mapsto (a\otimes f_{x\preceq y}b, -af_{x\preceq y}\otimes b), \end{aligned}$$

for $x, y \in X$ with $x \preceq y$.

Lemma 4.3. *Let X be a unique path space. The sequence*

$$0 \to \bigoplus_{y\to x} P_{x\otimes y} \xrightarrow{\bigoplus \varphi_{x,y}} \bigoplus_{x\in X} P_{x\otimes x} \xrightarrow{\bigoplus \psi_x} \mathbb{Z}[X] \to 0$$

is a length-one projective bimodule resolution.

Proof. We clearly have a complex of bimodule maps. The underlying Abelian groups of the three modules are free. To verify exactness, we choose the following canonical bases. A basis for $\mathbb{Z}[X]$ is given by paths $p = (x_0 \leftarrow \dots \leftarrow x_k)$, $k \geq 0$, in X. The standard basis for $P_{x\otimes x}$ consists of pairs of paths p_1, p_2 with p_1 beginning and p_2 ending at x. Thus pairs of composable paths give a basis for $\bigoplus_x P_{x\otimes x}$. We write such a pair as a single path $p_l = (x_0 \leftarrow \dots \leftarrow \overline{x_l} \leftarrow \dots \leftarrow x_k)$ with a distinguished time $l \in \{0, \dots, k\}$, namely, the place where to cut the path into two; so $p_l \in P_{x_l\otimes x_l}$. Similarly, a basis for $\bigoplus_{y\to x} P_{x\otimes y}$ is given by paths $p_{l,l+1} = (x_1 \leftarrow \dots \leftarrow \overline{x_l \leftarrow x_{l+1}} \leftarrow \dots \leftarrow x_k)$ with two consecutive distinguished times x_l and x_{l+1}, and $p_{l,l+1} \in P_{x_l\otimes x_{l+1}}$. In this picture, $\psi := \bigoplus \psi_x$ simply forgets the distinguished time of a basis vector. Hence it is surjective. The map $\varphi := \bigoplus \varphi_{x,y}$ takes a path $p_{l,l+1}$ with two distinguished times to the linear combination $p_l - p_{l+1}$ of paths with only one distinguished time. This map does not change the underlying path. So it suffices to check injectivity of φ on elements of the form $\sum_{l=0}^{k-1} n_l p_{l,l+1}$ with a fixed path p of length k. Applying φ yields $n_0p_0 + \left(\sum_{l=1}^{k-1}(n_l - n_{l-1})p_l\right) - n_{k-1}p_k$. If this sum vanishes, then $n_0 = 0$, and then $n_1 = 0$, and so on. So φ is injective. Finally, we show exactness in the middle. The kernel of ψ is generated by elements of the form $\sum_{l=0}^{k} n_l p_l$ with $\sum_{l=0}^{k} n_l = 0$. We rewrite such an element as

$\sum_{l=0}^{k-1}\left(\sum_{j=1}^{l} n_j\right)(p_l - p_{l+1})$ using $-\sum_{l=0}^{k-1} n_l = n_k$. This shows that it belongs to the image of φ. $\square$

Proposition 4.4. *If X is a unique path space, then $\mathrm{KK}_*(X;\mathcal{R},\mathcal{R})^{\mathrm{op}}$ has cohomological dimension at most 2.*

In fact, it is easy to see that the cohomological dimension of $\mathrm{KK}_*(X;\mathcal{R},\mathcal{R})^{\mathrm{op}}$ is equal to 2 unless the space X is discrete (in which case it is 1).

Proof. Tensoring the above short exact bimodule sequence over $\mathbb{Z}[X]$ with a left $\mathbb{Z}[X]$-module V gives the short exact sequence

$$0 \to \bigoplus_{y\to x} P_{x\otimes y} \otimes V \to \bigoplus_{x\in X} P_{x\otimes x} \otimes V \to V \to 0.$$

We have $P_{x\otimes y} \otimes_{\mathbb{Z}[X]} V \cong \mathbb{Z}[X]e_x \otimes_{\mathbb{Z}} V_y$, where $V_y := e_y \cdot V$ is the entry group of the module V at y. It follows that $\mathrm{Ext}^n_{\mathbb{Z}[X]}(P_{x\otimes y} \otimes V, W) \cong \mathrm{Ext}^n_{\mathbb{Z}}(V_y, W_x)$. Hence the long exact cohomology sequence for the functor $\mathrm{Hom}_{\mathbb{Z}[X]}(\textvisiblespace, W)$ applied to the above short exact sequence is of the form

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathbb{Z}[X]}(V,W) \to \bigoplus_{x\in X} \mathrm{Hom}_{\mathbb{Z}}(V_x, W_x) \to \bigoplus_{y\to x} \mathrm{Hom}_{\mathbb{Z}}(V_y, W_x)\\ \to \mathrm{Ext}^1_{\mathbb{Z}[X]}(V,W) \to \bigoplus_{x\in X} \mathrm{Ext}^1_{\mathbb{Z}}(V_x, W_x) \to \bigoplus_{y\to x} \mathrm{Ext}^1_{\mathbb{Z}}(V_y, W_x)\\ \to \mathrm{Ext}^2_{\mathbb{Z}[X]}(V,W) \to 0,\end{aligned}$$

and $\mathrm{Ext}^n_{\mathbb{Z}[X]}$ vanishes for $n \ge 3$ because $\mathrm{Ext}^n_{\mathbb{Z}}$ vanishes for $n \ge 2$. $\square$

The maps $\bigoplus_{x\in X} \mathrm{Ext}^n_{\mathbb{Z}}(V_x, W_x) \to \bigoplus_{y\to x} \mathrm{Ext}^n_{\mathbb{Z}}(V_y, W_x)$ in the exact sequence above are the sum of the maps

$$\mathrm{Ext}^n_{\mathbb{Z}}(V_x, W_x) \oplus \mathrm{Ext}^n_{\mathbb{Z}}(V_y, W_y) \xrightarrow{\left((i_W)_*, -(i_V)^*\right)} \mathrm{Ext}^n_{\mathbb{Z}}(V_y, W_x),$$

induced by the arrows $i\colon y \to x$ in X. This gives a scheme for computing the groups $\mathrm{Ext}^n_{\mathbb{Z}[X]}(V,W)$. As in Section 3.2, the above long exact sequence is equivalent to a spectral sequence

$$\mathrm{HH}^p\big(\mathbb{Z}[X], \mathrm{Ext}^q_{\mathbb{Z}}(V,W)\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Z}[X]}(V,W).$$

Definition 4.5. A C^*-algebra over X is called a *Kirchberg X-algebra* if it is separable, tight (see [25, Definition 5.1]), $\mathcal{O}_\infty$-absorbing and nuclear.

Combining Theorem 2.18 and Proposition 4.4 with Kirchberg's Classification Theorem in [18], we get the following purely algebraic complete classification of Kirchberg X-algebras in the bootstrap class $\mathcal{B}(X)$:

Corollary 4.6. *Let X be a unique path space. Then the functor $\mathrm{XK}\delta$ induces a bijection between the sets of *-isomorphism classes over X of stable Kirchberg X-algebras in the bootstrap class $\mathcal{B}(X)$ and of isomorphism classes in the category $\mathfrak{Mod}\big(\mathbb{Z}[X]\big)^{\mathbb{Z}/2}_{\mathrm{c}}\delta$. Every isomorphism in $\mathfrak{Mod}\big(\mathbb{Z}[X]\big)^{\mathbb{Z}/2}_{\mathrm{c}}\delta$ lifts to a *-isomorphism over X.*

Example 4.7. If $X = \bullet$ is the one-point space, then $\mathbb{Z}[X]$ is simply the ring of integers, which has global dimension 1. Hence, in this case, the functor $\mathrm{XK}\delta$ reduces to plain $\mathbb{Z}/2$-graded K-theory. A *Kirchberg X-algebra* is just a (simple) Kirchberg algebra. Corollary 4.6 reduces to the Kirchberg–Phillips Classification Theorem.

Example 4.8. Let $X = \bullet \to \bullet$ be the two-point Sierpiński space. Stable Kirchberg X-algebras in $\mathcal{B}(X)$ are essentially the same as stable extensions of UCT Kirchberg algebras. Rørdam [34] classified these by their six-term exact sequences in K-theory. Moreover, every six-term exact sequence of countable Abelian groups arises as the K-theory sequence of a stable Kirchberg X-algebra in $\mathcal{B}(X)$. The relation between Rørdam's invariant and ours becomes apparent by the following direct computation: given two objects $G_1 \xrightarrow{\varphi} G_2$ and $H_1 \xrightarrow{\psi} H_2$ in $\mathfrak{Mod}\big(\mathbb{Z}[X]\big)$, there are natural isomorphisms

$$\begin{aligned}\mathrm{Ext}^2_{\mathbb{Z}[X]}(G_1 \xrightarrow{\varphi} G_2, H_1 \xrightarrow{\psi} H_2) &\cong \mathrm{Ext}^2_{\mathbb{Z}[X]}\big(\ker(\varphi) \to 0, 0 \to \operatorname{coker}(\psi)\big)\\ &\cong \mathrm{Ext}^1_{\mathbb{Z}[X]}\big(0 \to \ker(\varphi), 0 \to \operatorname{coker}(\psi)\big) \cong \mathrm{Ext}^1_{\mathbb{Z}}\big(\ker(\varphi), \operatorname{coker}(\psi)\big).\end{aligned}$$

The group $\mathrm{Ext}^1_{\mathbb{Z}}\big(\ker(\varphi), \operatorname{coker}(\psi)\big)$ is in natural bijection to the set of equivalence classes of exact sequences of the form

$$H_1 \xrightarrow{\psi} H_2 \to E \to G_1 \xrightarrow{\varphi} G_2.$$

In fact, our invariant factors through Rørdam's; it remembers isomorphism classes but forgets certain morphisms.

Example 4.9. Let X be totally ordered (for two points, this is Example 4.8). Then filtrated K-theory is a complete invariant for objects in $\mathcal{B}(X)$ by the main result of [27]. Since totally ordered spaces are unique path spaces, we now have two seemingly different complete invariants for objects in $\mathcal{B}(X)$. Both invariants must contain exactly the same information. The authors, however, do not understand the relationship between these two invariants. If, for instance, $X = \bullet \to \bullet \to \bullet$, then the issue is to relate elements in $\mathrm{Ext}^2_{\mathbb{Z}[X]}(G_1 \to G_2 \to G_3, H_1 \to H_2 \to H_3)$ to commuting diagrams of the form

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & & \\
 & & H_3 \arrow[ur] \arrow[dr] & & G_1 \arrow[dr] & & \\
 & H_2 \arrow[ur] \arrow[dr] & & L \arrow[ur] \arrow[dr] & & G_2 \arrow[dr] & \\
H_1 \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & G_3 \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] &
\end{tikzcd}$$

with certain exactness conditions.

5. Graph C*-algebras

If the finite T_0-space X is not a unique path space, then we may still classify those objects A of $\mathcal{B}(X)$ for which $\mathrm{XK}(A)$ has a projective resolution of length 2. We are going to show that graph C*-algebras with finitely many ideals have this property. Even better, we may compute their obstruction classes in terms of the Pimsner–Voiculescu type sequence that computes their K-theory.

5.1. A computation of obstruction classes. First we prove a general result in the abstract setting of a triangulated category $\mathfrak{T}$ with a universal $\mathfrak{J}$-exact stable homological functor $F\colon \mathfrak{T} \to \mathfrak{A}$; we also impose the parity assumptions of Section 2.1. For certain objects in $\mathfrak{T}$ that are constructed from a length-2 projective resolution in $\mathfrak{A}$, we compute the obstruction class explicitly. Let

$$0 \to M_1 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\varepsilon} M_0 \to 0 \tag{5.1}$$

be an exact chain complex in $\mathfrak{A}_+$ with projective objects M_1, Q_1 and Q_0. The adjoint functor $F^\vdash$ on projective objects of $\mathfrak{A}$ gives objects $\hat{M}_1$, $\hat{Q}_1$ and $\hat{Q}_0$ of $\mathfrak{T}$ lifting M_1, Q_1 and Q_0, and maps $\hat{\partial}_2 \in \mathfrak{T}(\hat{M}_1, \hat{Q}_1)$ and $\hat{\partial}_1 \in \mathfrak{T}(\hat{Q}_1, \hat{Q}_0)$ lifting ∂_2 and ∂_1. Embed $\hat{\partial}_1$ into an exact triangle

$$\hat{Q}_1 \xrightarrow{\hat{\partial}_1} \hat{Q}_0 \xrightarrow{p} A \xrightarrow{r} \hat{Q}_1[1].$$

The long exact sequence for F applied to this triangle has the form

$$\cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{F(p)} F(A) \xrightarrow{F(r)} Q_1[1] \xrightarrow{\partial_1[1]} Q_0[1] \to \cdots$$

Since the cokernel $M_0 \in\in \mathfrak{A}_+$ of ∂_1 and the kernel $M_1[1] \in\in \mathfrak{A}_-$ of $\partial_1[1]$ have different parity, we get

$$F(A) \cong M_0 \oplus M_1[1].$$

This has the following projective resolution of length 2:

$$0 \to M_1 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \oplus M_1[1] \xrightarrow{(\varepsilon, \mathrm{id}_{M_1[1]})} M_0 \oplus M_1[1] \to 0. \tag{5.2}$$

Theorem 5.3. *The obstruction class of A is the class of the 2-step extension* (5.1) *in $\mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1)$, which we embed as a direct summand into*

$$\begin{aligned}\mathrm{Ext}^2_{\mathfrak{A}}(F(A), F(A)[-1]) \cong \mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_0[-1]) \oplus \mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1) \\ \oplus \mathrm{Ext}^2_{\mathfrak{A}}(M_1[1], M_0[-1]) \oplus \mathrm{Ext}^2_{\mathfrak{A}}(M_1[1], M_1).\end{aligned}$$

Let

$$0 \to M_1' \xrightarrow{\partial_2'} Q_1' \xrightarrow{\partial_1'} Q_0' \xrightarrow{\varepsilon'} M_0' \to 0 \tag{5.4}$$

be another exact chain complex in $\mathfrak{A}$ with even projective objects M_1', Q_1' and Q_0', and let A' be the cone of the lifting $\hat{\partial}_1'$ of ∂_1. Then $A \cong A'$ if and *only if there is a commutative diagram*

$$\begin{tikzcd}[ampersand replacement=\&]
M_1 \arrow[r, tail, "(\partial_2, 0)"] \arrow[d, "\cong"', "\varphi_1"] \& Q_1 \oplus Q_0' \arrow[r, "(\partial_1, \mathrm{id}_{Q_0'})"] \arrow[d, "\cong"', "\varphi_2"] \& Q_0 \oplus Q_0' \arrow[r, two heads, "(\varepsilon, 0)"] \arrow[d, "\cong"', "\varphi_3"] \& M_0 \arrow[d, "\cong"', "\varphi_4"] \\
M_1' \arrow[r, tail, "(0, \partial_2')"] \& Q_0 \oplus Q_1' \arrow[r, "(\mathrm{id}_{Q_0}, \partial_1')"] \& Q_0 \oplus Q_0' \arrow[r, two heads, "(0, \varepsilon')"] \& M_0'
\end{tikzcd} \tag{5.5}$$

in $\mathfrak{A}$, where the maps φ_i for $i = 1, 2, 3, 4$ are isomorphisms.

Proof. We first compute the obstruction class of A. For this, we compare A to the canonical lifting of $F(A)$. The latter is the direct sum of the canonical lifting of M_0 with $\hat{M}_1[1]$. To lift M_0 canonically, we first embed $\hat{\partial}_2 \colon \hat{M}_1 \to \hat{Q}_1$ in an exact triangle

$$\hat{M}_1 \xrightarrow{\hat{\partial}_2} \hat{Q}_1 \xrightarrow{u} D \xrightarrow{v} \hat{M}_1[1].$$

Then $F(D) \cong \operatorname{coker} \partial_2 \cong \ker \varepsilon$. The UCT gives $\mathfrak{T}_0(D, \hat{Q}_0) \cong \mathfrak{A}(\ker \varepsilon, Q_0)$ for parity reasons. Hence there is a unique $x \in \mathfrak{T}_0(D, \hat{Q}_0)$ for which $F(x)$ is the inclusion of $\ker \varepsilon$ into Q_0. The cone of x is the canonical lifting of M_0. Since direct sums of exact triangles remain exact, the canonical lifting of $F(A)$ is the cone of the map $(x, 0) \colon D \to \hat{Q}_0 \oplus \hat{M}_1[1]$.

The map $\hat{\partial}_2 \circ \hat{\partial}_1 = 0$ is part of an exact triangle

$$\hat{M}_1 \xrightarrow{0} \hat{Q}_0 \xrightarrow{i_1} \hat{Q}_0 \oplus \hat{M}_1[1] \xrightarrow{p_2} \hat{M}_1[1],$$

where i_1 is the inclusion of the first summand and p_2 the projection onto the second summand. The octahedral axiom applied to $\hat{\partial}_1$ and $\hat{\partial}_2$ gives maps $\bar{x} \colon D \to \hat{Q}_0$, $y \colon D \to \hat{M}_1[1]$ and $t \colon \hat{M}_1[1] \to A$ such that the diagram

$$\begin{tikzcd}[ampersand replacement=\&]
\hat{M}_1 \arrow[r, "\hat{\partial}_2"] \arrow[d, equal] \& \hat{Q}_1 \arrow[r, "u"] \arrow[d, "\hat{\partial}_1"] \& D \arrow[r, "v"] \arrow[d, dotted, "\binom{\bar{x}}{v}"] \& \hat{M}_1[1] \arrow[d, equal] \\
\hat{M}_1 \arrow[r, "0"] \arrow[d] \& \hat{Q}_0 \arrow[r, "i_1"] \arrow[d, "p"] \& \hat{Q}_0 \oplus \hat{M}_1[1] \arrow[r, "p_2"] \arrow[d, dotted, "\binom{p}{t}"] \& \hat{M}_1[1] \arrow[d] \\
0 \arrow[r] \arrow[d] \& A \arrow[r, equal] \arrow[d, "r"] \& A \arrow[r] \arrow[d, "u[1] \circ r"] \& 0 \arrow[d] \\
\hat{M}_1[1] \arrow[r, "\hat{\partial}_2[1]"] \& \hat{Q}_1[1] \arrow[r, "u[1]"] \& D[1] \arrow[r, "v[1]"] \& \hat{M}_1[2]
\end{tikzcd}$$

commutes and has exact rows and columns.

We claim that $\bar{x} = x$. Recall that $F(u)$ is surjective. So $F(\bar{x})$ is determined by its composition with $F(u)$, which is equal to $F(\hat{\partial}_1) = \partial_1$ by the commuting diagram. Hence $F(\bar{x}) = F(x)$, which gives $\bar{x} = x$ by the uniqueness of x. The exactness of the third column means that A is the cone of the map $(x, v) \colon D \to \hat{Q}_0 \oplus \hat{M}_1[1]$. The canonical lifting of $F(A)$ is the cone of the map $(x, 0) \colon D \to$

$\hat{Q}_0 \oplus \hat{M}_1[1]$. Hence the obstruction class of A is the image of $(x, v) - (x, 0) = (0, v)$ under the map from $\mathrm{Ext}^1_{\mathfrak{A}}\bigl(F(D), Q_0[-1] \oplus M_1\bigr) \subseteq \mathfrak{T}(D, \hat{Q}_0 \oplus \hat{M}_1[1])$ to $\mathrm{Ext}^2_{\mathfrak{A}}\bigl(F(A), F(A)[-1]\bigr)$ constructed in the proof of Theorem 2.7.

We may describe the element in $\mathrm{Ext}^1_{\mathfrak{A}}\bigl(F(D), Q_0[-1] \oplus M_1\bigr)$ induced by $(0, v)$ because v also appears in the first row: it is represented by the extension

$$0 \to Q_0[-1] \oplus M_1 \xrightarrow{(\mathrm{id}_{Q_0[-1]}, \partial_2)} Q_0[-1] \oplus Q_1 \xrightarrow{(0,u)} F(D) \to 0.$$

The next step is to push forward along the map

$$(\varepsilon[-1], \mathrm{id}_{M_1})\colon Q_0[-1] \oplus M_1 \to M_0[-1] \oplus M_1 \cong F(A)[-1].$$

The resulting element in $\mathrm{Ext}^1_{\mathfrak{A}}\bigl(F(D), F(A)[-1]\bigr)$ is represented by the extension

$$0 \to M_0[-1] \oplus M_1 \xrightarrow{(\mathrm{id}_{M_0[-1]}, \partial_2)} M_0[-1] \oplus Q_1 \xrightarrow{(0,u)} F(D) \to 0.$$

To get the obstruction class for A in $\mathrm{Ext}^2_{\mathfrak{A}}\bigl(F(A), F(A)[-1]\bigr)$, we need to splice the extension above with the extension

$$0 \to F(D) \xrightarrow{(\iota,0)} Q_0 \oplus M_1[1] \xrightarrow{(\varepsilon, \mathrm{id}_{M_1[1]})} M_0 \oplus M_1[1] \to 0,$$

where $\iota\colon F(D) \to Q_0$ denotes the inclusion map $F(D) \cong \ker \varepsilon \subseteq Q_0$. Up to the identification stated in the theorem, this yields indeed the class of the 2-step extension (5.1).

Now we establish the isomorphism criterion. First assume that there are invertible maps φ_i as in (5.5). Since the cone of the identity map is the zero object and since cones are additive for direct sums, the cone of $\hat{\partial}_1 \oplus \mathrm{id}_{\hat{Q}'_0}$ is again A, and the cone of $\hat{\partial}'_1 \oplus \mathrm{id}_{\hat{Q}_0}$ is again A'. Since the maps $\hat{\partial}_1 \oplus \mathrm{id}_{\hat{Q}'_0}$ and $\hat{\partial}'_1 \oplus \mathrm{id}_{\hat{Q}_0}$ are isomorphic by (5.5), they have isomorphic cones. Thus $A \cong A'$.

Conversely, assume that $A \cong A'$. Then $F(A) \cong F(A')$, so that we get isomorphisms $\varphi_1\colon M_1 \to M'_1$ and $\varphi_4\colon M_0 \to M'_0$. To simplify notation, we assume without loss of generality that φ_1 and φ_4 are identity maps. Then the isomorphism $A \cong A'$ is an equivalence of liftings, so that A and A' have the same obstruction class in $\mathrm{Ext}^2_{\mathfrak{A}}\bigl(F(A), F(A)[-1]\bigr)$. By our computation of the obstruction class, this means that the 2-step extensions (5.1) and (5.4) (with $M'_i = M_i$) have the same class in $\mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1)$. The classes in $\mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1)$ are not changed by adding $\mathrm{id}_{Q'_0}\colon Q'_0 \to Q'_0$ in (5.1) and $\mathrm{id}_{Q_0}\colon Q_0 \to Q_0$ in (5.4). Thus the two rows in (5.5) have the same class in $\mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1)$.

Since Q_0 and Q'_0 are projective, there are maps $\psi\colon Q_0 \to Q'_0$ and $\psi'\colon Q'_0 \to Q_0$ with $\varepsilon' \circ \psi = \varepsilon\colon Q_0 \to M$ and $\varepsilon \circ \psi' = \varepsilon'\colon Q'_0 \to M$. Then

$$\varphi_3 := \begin{pmatrix} \mathrm{id}_{Q_0} & 0 \\ \psi & \mathrm{id}_{Q'_0} \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_{Q_0} & -\psi' \\ 0 & \mathrm{id}_{Q'_0} \end{pmatrix} \colon Q_0 \oplus Q'_0 \to Q_0 \oplus Q'_0$$

is an isomorphism with $(0, \varepsilon') \circ \varphi_3 = (\varepsilon, 0)$. Hence φ_3 makes the third square in (5.5) commute.

Let K be the kernel of $(\varepsilon, 0)\colon Q_0 \oplus Q'_0 \to M_0$. This is isomorphic to the kernel of $(0, \varepsilon')$ via φ_3. Since $Q_0 \oplus Q'_0$ is projective, composition with the class of the extension $K \rightarrowtail Q_0 \oplus Q'_0 \twoheadrightarrow M_0$ gives an isomorphism $\mathrm{Ext}^2_{\mathfrak{A}}(M_0, M_1) \cong$

$\mathrm{Ext}^1_{\mathfrak{A}}(K, M_1)$. Hence the equality of the obstruction classes shows that the extensions $M_1 \rightarrowtail Q_1 \oplus Q_0' \twoheadrightarrow K$ and $M_1 \rightarrowtail Q_0 \oplus Q_1' \twoheadrightarrow K$ that we get from the two rows in (5.5) and the isomorphism φ_3 have the same class in $\mathrm{Ext}^1_{\mathfrak{A}}(K, M_1)$. Equality in $\mathrm{Ext}^1_{\mathfrak{A}}(K, M_1)$ means that the extensions really are isomorphic in the strongest possible sense, that is, there is an isomorphism $\varphi_2 \colon Q_1 \oplus Q_0' \to Q_0 \oplus Q_1'$ that induces an isomorphism of extensions. This means that it makes the remaining two squares in (5.5) commute. Thus $A \cong A'$ implies that there are isomorphisms φ_i making (5.5) commute. □

Remark 5.6. The same argument works if the objects in (5.1) all belong to $\mathfrak{A}_-$. If the objects in (5.1) belong to $\mathfrak{A}$, then we may split (5.1) into its even and odd parts. Thus the obvious adaptation of Theorem 5.3 still holds without any parity assumptions on the objects M_j and Q_j.

Remark 5.7. There are several variants of the criterion (5.5). Since (5.1) and (5.4) are exact, isomorphisms φ_2 and φ_3 making the middle square in (5.5) commute give φ_1 and φ_4 making all squares in (5.5) commute. Furthermore, if φ_i are isomorphisms for $i = 1, 4$ and for $i = 2$ or $i = 3$, then the remaining one is an isomorphism as well by the Five Lemma.
If there are maps φ_i making (5.5) commute, and such that φ_1 and φ_4 are invertible, then it already follows that $A \cong A'$. This is because φ_1 and φ_4 induce an isomorphism $F(A) \cong F(A')$, and (5.5) shows that the obstruction classes also agree, no matter whether φ_2 or φ_3 are invertible.

5.2. Crossed products for C*-algebras over topological spaces. In this subsection, we generalise some basic results about crossed products to C*-algebras over topological spaces. Let G be a locally compact group. Let X be a second countable topological space.

Definition 5.8. A *G-C*-algebra over X* is a C*-algebra over X whose underlying C*-algebra is a G-C*-algebra such that all distinguished ideals are G-invariant.

If (A, α) is a G-C*-algebra over X then the crossed product $A \rtimes_\alpha G$ is a C*-algebra over X via $(A \rtimes_\alpha G)(U) := A(U) \rtimes_{\alpha|_{A(U)}} G$ for all $U \in \mathbb{O}(X)$. If G is Abelian, then $(A \rtimes_\alpha G, \hat{\alpha})$ is a $\hat{G}$-C*-algebra over X.

Proposition 5.9 (Takai Duality). *Let G be Abelian. Let (A, α) be a G-C*-algebra over X. Then there is a natural isomorphism of G-C*-algebras over X from $\big((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}, \hat{\hat{\alpha}}\big)$ to $\big(A \otimes \mathbb{K}(L^2 G), \alpha \otimes \mathrm{ad}_\lambda\big)$. In particular, there is a natural* $\mathrm{KK}(X)$*-equivalence between $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ and A.*

Proof. We have only added X-equivariance to the classical statement. This follows immediately from the naturality of the classical version (see [7, Theorem 6 in Appendix C of Chapter 2]). □

In the following, we assume for convenience that X is finite and A belongs to the category $\mathfrak{KK}(X)_{\mathrm{loc}}$ defined in [25, Definition 4.8]. It should be possible to remove these assumptions by carefully checking the naturality of the homotopies implementing the respective equivalences.

Proposition 5.10 (Green Imprimitivity). *Let H be a closed subgroup of G and let (A,α) be an H-C^*-algebra over X. Assume that X is finite and $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. There is a natural* $\mathrm{KK}(X)$*-equivalence between $A \rtimes_\alpha H$ and $\mathrm{Ind}_H^G(A,\alpha) \rtimes_{\mathrm{Ind}\,\alpha} G$.*

Proof. By naturality, the imprimitivity bimodule constructed in [11, Theorem 4.1] induces a $\mathrm{KK}(X)$-element which is a pointwise KK-equivalence. By [25, Proposition 4.9], it is a $\mathrm{KK}(X)$-equivalence. □

Proposition 5.11 (Connes–Thom Isomorphism). *Let (A,α) be an $\mathbb{R}$-C^*-algebra over X. Assume that X is finite and $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. Then $A \rtimes_\alpha \mathbb{R}$ is naturally* $\mathrm{KK}(X)$*-equivalent to* $A[-1]$.

Proof. We adopt the approach from [20, Proposition 8.3]. Let $\tilde{A}$ denote the C^*-algebra A over X with the trivial $\mathbb{R}$-action. Then $\mathrm{C}_0(\mathbb{R},A)$ and $\mathrm{C}_0(\mathbb{R},\tilde{A})$ with the diagonal actions are naturally *-isomorphic as $\mathbb{R}$-C^*-algebras, where the action on $\mathbb{R}$ is given by translation; explicitly, the isomorphism takes a function $f \in \mathrm{C}_0(\mathbb{R},\tilde{A})$ to the function $t \mapsto \alpha_t\bigl(f(t)\bigr)$ in $\mathrm{C}_0(\mathbb{R},A)$. By [17, Theorem 5.9] we may fix a $\mathrm{KK}^{\mathbb{R}}$-equivalence between $\mathbb{C}[-1]$ and $\mathrm{C}_0(\mathbb{R})$, where $\mathrm{C}_0(\mathbb{R})$ carries the translation action. In combination, this gives a natural $\mathrm{KK}^{\mathbb{R}}$-equivalence between $A[-1]$ and $\tilde{A}[-1]$, and consequently also between A and $\tilde{A}$. Taking crossed products gives a natural KK-equivalence between $A \rtimes_\alpha \mathbb{R}$ and $A[-1]$. As in the previous proof, the naturality of the constructed cycle and [25, Proposition 4.9] show that this is a $\mathrm{KK}(X)$-equivalence. □

Proposition 5.12 (Pimsner–Voiculescu Triangle). *Let (A,α) be a $\mathbb{Z}$-C^*-algebra over X. Assume that X is finite and $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. Then there is a natural exact triangle in $\mathfrak{KK}(X)$ of the form*

$$A[-1] \rtimes_\alpha \mathbb{Z} \to A \xrightarrow{\alpha_1-\mathrm{id}} A \to A \rtimes_\alpha \mathbb{Z}.$$

Proof. We abbreviate $\alpha = \alpha_1$ and let $T_\alpha = \bigl\{f\colon \mathrm{C}([0,1],A) \mid f(1) = \alpha\bigl(f(0)\bigr)\bigr\}$ be the mapping torus of α. The extension

$$A[-1] \rightarrowtail T_\alpha \overset{\mathrm{ev}_0}{\twoheadrightarrow} A$$

has a completely positive X-equivariant section taking $a \in A$ to the affine function $(1-t)\cdot a + t\cdot\alpha(a)$. We get a natural exact triangle in $\mathfrak{KK}(X)$ of the form

$$A[-1] \to A[-1] \to T_\alpha \to A. \tag{5.13}$$

The $\mathbb{R}$-C^*-algebra T_α is naturally *-isomorphic over X to $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A,\alpha)$. By Green's imprimitivity theorem and the Connes–Thom isomorphism, we have natural $\mathrm{KK}(X)$-equivalences

$$A \rtimes_\alpha \mathbb{Z} \simeq \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R} \simeq T_\alpha \rtimes \mathbb{R} \simeq T_\alpha[-1].$$

Plugging this into (5.13) and rotating as appropriate gives an exact triangle of the desired form. The formula for the map from A to A is a consequence of the naturality of the boundary map in the KK-theoretic six-term sequence applied to the morphisms of extensions

$$\begin{tikzcd}
A[-1] \arrow[r, tail] \arrow[d, equal] & T_\alpha \arrow[r, two heads] \arrow[d, hook] & A \arrow[d, "(\mathrm{id}_A, \alpha)"] \\
A[-1] \arrow[r, tail] & \mathrm{C}\bigl([0,1], A\bigr) \arrow[r, two heads] & A \oplus A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A[-1] \arrow[r, tail] \arrow[d, equal] & \mathrm{C}_0\bigl([0,1), A\bigr) \arrow[r, two heads] \arrow[d, hook] & A \arrow[d, "\iota_0"] \\
A[-1] \arrow[r, tail] & \mathrm{C}\bigl([0,1], A\bigr) \arrow[r, two heads] & A \oplus A \\
A[-1] \arrow[r, tail] \arrow[u, equal] & \mathrm{C}_0\bigl((0,1], A\bigr) \arrow[r, two heads] \arrow[u, hook] & A, \arrow[u, "\iota_1"']
\end{tikzcd}$$

together with the elementary fact that the extensions $\mathbb{C}[-1] \rightarrowtail \mathrm{C}_0\bigl([0,1)\bigr) \twoheadrightarrow \mathbb{C}$ and $\mathbb{C}[-1] \rightarrowtail \mathrm{C}_0\bigl((0,1]\bigr) \twoheadrightarrow \mathbb{C}$ correspond to the classes $-\mathrm{id}_{\mathbb{C}[-1]}$ and $\mathrm{id}_{\mathbb{C}[-1]}$ in $\mathrm{KK}_1(\mathbb{C}, \mathbb{C}[-1]) \cong \mathrm{KK}_0(\mathbb{C}[-1], \mathbb{C}[-1])$, respectively. □

Corollary 5.14 (Dual Pimsner–Voiculescu Triangle). *Let (A, α) be a $\mathbb{T}$-C^*-algebra over X. Assume that X is finite and $A \rtimes_\alpha \mathbb{T} \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. Then there is a natural exact triangle in $\mathfrak{KK}(X)$ of the form*

$$A[-1] \to A \rtimes_\alpha \mathbb{T} \xrightarrow{\hat{\alpha}_1 - \mathrm{id}} A \rtimes_\alpha \mathbb{T} \to A.$$

Proof. This follows from the Pimsner–Voiculescu Triangle and Takai Duality. □

5.3. Application to graph algebras. Let $A = \mathrm{C}^*(E)$ be the C^*-algebra of a countable graph E. We assume that A has only finitely many ideals or, equivalently, that its primitive ideal space is finite; this is necessary for our machinery to work. We set $X = \mathrm{Prim}(A)$. The gauge action $\gamma\colon \mathbb{T} \curvearrowright A$ turns A into a $\mathbb{T}$-C^*-algebra over X. Corollary 5.14 provides the following natural exact triangle in $\mathfrak{KK}(X)$:

$$A[-1] \to A \rtimes_\gamma \mathbb{T} \xrightarrow{\mathrm{id} - \hat{\gamma}_1^{-1}} A \rtimes_\gamma \mathbb{T} \to A.$$

We have composed with the automorphism $\hat{\gamma}_1^{-1}$ to replace $\hat{\gamma}_1 - \mathrm{id}$ by $\mathrm{id} - \hat{\gamma}_1^{-1}$ to prepare for computations below.
The C^*-algebra $A \rtimes_\gamma \mathbb{T}$ is AF. Hence the odd part of $\mathrm{XK}(A \rtimes_\gamma \mathbb{T})$ vanishes. Applying the functor XK to the dual Pimsner–Voiculescu triangle, we get the following dual Pimsner–Voiculescu exact sequence:

$$0 \to \mathrm{XK}_1(A) \to \mathrm{XK}_0(A \rtimes_\gamma \mathbb{T}) \xrightarrow{\mathrm{id} - (\hat{\gamma}_1^{-1})_*} \mathrm{XK}_0(A \rtimes_\gamma \mathbb{T}) \to \mathrm{XK}_0(A) \to 0. \tag{5.15}$$

The module $\mathrm{XK}(A \rtimes_\gamma \mathbb{T})$ is usually not projective, so we cannot directly apply Theorem 5.3. For this purpose, we replace $A \rtimes_\gamma \mathbb{T}$ by a suitable C^*-subalgebra. This construction is based on the ingredients of the computation of the K-theory of graph C*-algebras in [32, Section 3] and [2, Section 6]; we shall use the notation and a number of results proved in these articles.

We may identify $\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}$ with the C*-algebra of the so-called skew-product graph $E \times_1 \mathbb{Z}$. This becomes an isomorphism of C*-algebras over X via the canonical definitions $\bigl(\mathrm{C}^*(E) \rtimes_\gamma \mathbb{T}\bigr)(U) := \bigl(\mathrm{C}^*(E)(U)\bigr) \rtimes_\gamma \mathbb{T}$ and $\mathrm{C}^*(E \times_1 \mathbb{Z})(U) := J_{H_U\times\mathbb{Z}, B_U\times\mathbb{Z}}$ where (H_U, B_U) is the admissible pair such that $\mathrm{C}^*(E)(U) = J_{H_U,B_U}$.

We let N denote the set $\{n \in \mathbb{Z} \mid n \le 0\}$ and $N^* = N \setminus \{0\}$. We let $\hat{Q}_0 = \hat{Q}_1$ be the C*-subalgebra of $\mathrm{C}^*(E \times_1 \mathbb{Z})$ associated to the subgraph $E \times_1 N$, the restriction of the graph $E \times_1 \mathbb{Z}$ to the subset of vertices $E \times N$. This is a C*-algebra over X via $\mathrm{C}^*(E \times_1 N)(U) = J_{H_U\times N, B_U\times N^*}$, and the inclusion map $\mathrm{C}^*(E \times_1 N) \hookrightarrow \mathrm{C}^*(E \times_1 \mathbb{Z})$ is a *-homomorphism over X. The dual action on $A \rtimes_\gamma \mathbb{T}$ corresponds to the shift automorphism β on $\mathrm{C}^*(E \times_1 \mathbb{Z})$. Its inverse preserves the subalgebra $\hat{Q}_0 = \mathrm{C}^*(E \times_1 N)$. This is why we need $\hat{\gamma}_1^{-1}$ in (5.15).

Lemma 5.16. *The module* $\mathrm{XK}(\hat{Q}_0)$ *is projective and concentrated in even degree.*

Proof. The C*-algebra $\hat{Q}_0$ is an AF-algebra because the graph $E \times_1 N$ has no cycles. We claim that all distinguished subquotients of $\hat{Q}_0$ have free K_0-groups and vanishing K_1-groups. Since ideals and quotients of AF-algebras are again AF and since AF-algebras have vanishing K_1-groups, it suffices to show that the K_0-group of every distinguished quotient of $\mathrm{C}^*(E \times_1 N)$ is free. By [2, Corollary 3.5], the quotient of $\mathrm{C}^*(E \times_1 N)$ by the ideal $J_{H_U\times_1 N, B_U\times_1 N^*}$ is isomorphic to the C*-algebra of the graph $\bigl((E \times_1 N)/(H \times_1 N)\bigr) \setminus \beta(B \times_1 N^*)$; its K-theory is free by [2, Lemma 6.2] and continuity of K-theory. Now it follows from [3, Lemma 4.10] that the module $\mathrm{XK}(\hat{Q}_0)$ is projective. □

We write $s\colon \hat{Q}_1 \to \hat{Q}_0$ for the restricted morphism $1 - \beta^{-1}$ and form an exact triangle

$$C_s \to \hat{Q}_1 \xrightarrow{s} \hat{Q}_0 \to C_s[1],$$

where C_s denotes the generalised mapping cone of s. By the third axiom of triangulated categories (weak functoriality of the generalised mapping cone; see [28, §1.1]), there is a morphism $f\colon C_s \to A[-1]$ such that the diagram

$$\begin{array}{ccccccc} C_s & \longrightarrow & \hat{Q}_1 & \xrightarrow{s} & \hat{Q}_0 & \longrightarrow & C_s[1] \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ A[-1] & \longrightarrow & A \rtimes_\gamma \mathbb{T} & \xrightarrow{\mathrm{id} - \hat{\gamma}^{-1}} & A \rtimes_\gamma \mathbb{T} & \longrightarrow & A \end{array} \tag{5.17}$$

commutes. As in [32, Lemma 3.3], it follows that the morphism $f_*\colon \mathrm{K}_*\bigl(C_s(Y)\bigr) \to \mathrm{K}_*\bigl(A(Y)\bigr)$ induced by f is bijective for every closed subset $Y \subseteq X$. Hence f is a $\mathrm{KK}(X)$-equivalence by [25, Proposition 4.15] and the Five Lemma.

Lemma 5.18. *The module* $\ker\bigl(\mathrm{XK}(s)\bigr)$ *is projective.*

Proof. This module is concentrated in even degree and isomorphic to $\mathrm{XK}_1\bigl(\mathrm{C}^*(E)\bigr)$. Since $\mathrm{C}^*(E)$ has vanishing exponential maps and all its subquotients have free K_1-groups, it follows as in [3, Lemma 4.10] that this module is projective. □

Since the map f in (5.17) is a $\mathrm{KK}(X)$-equivalence, we get an exact triangle

$$A[-1] \to \hat{Q}_1 \to \hat{Q}_0 \to A.$$

Lemmas 5.16 and 5.18 show that Theorem 5.3 applies. The obstruction class is the image of the top row in (5.17) under the functor XK. The vertical maps in this diagram show that XK applied to the bottom row also represents the same class in Ext^2. The bottom row is exactly the dual Pimsner–Voiculescu sequence (5.15), as asserted. Combining these computations with Kirchberg's Classification Theorem gives the following theorem:

Theorem 5.19. *Let A_1 and A_2 be purely infinite graph* C^**-algebras such that*

$$\mathrm{Prim}(A_1) \cong \mathrm{Prim}(A_2) \cong X$$

is finite. Then any isomorphism $\mathrm{XK}\delta(A_1) \cong \mathrm{XK}\delta(A_2)$ *lifts to a stable isomorphism between A_1 and A_2. The obstruction classes $\delta(A_i)$ in* $\mathrm{Ext}^2\bigl(\mathrm{XK}(A_i), \mathrm{XK}(A_i)[-1]\bigr)$ *are determined by the dual Pimsner–Voiculescu sequences* (5.15) *for the gauge actions $\gamma\colon \mathbb{T} \curvearrowright A_i$.*

Proof. By [19, Corollary 9.4], a purely infinite, separable, nuclear C^*-algebra with real rank zero absorbs the infinite Cuntz algebra $\mathcal{O}_\infty$ tensorially. Hence Kirchberg's Classification Theorem applies. It gives the result together with Theorem 5.3. □

Roughly speaking, stable, purely infinite graph C^*-algebras with finitely many ideals are strongly classified by their dual Pimsner–Voiculescu sequence in XK (up to the correct notion of equivalence). These algebras always have real rank zero.

Corollary 5.20. *Let A_1 and A_2 be unital, purely infinite graph* C^**-algebras such that* $\mathrm{Prim}(A_1) \cong \mathrm{Prim}(A_2) \cong X$. *Then any isomorphism* $\mathrm{XK}\delta(A_1) \cong \mathrm{XK}\delta(A_2)$ *taking the unit class in* $\mathrm{K}_0(A_1)$ *to the unit class in* $\mathrm{K}_0(A_2)$ *lifts to a* **-isomorphism between A_1 and A_2.*

Proof. This follows from [12, Theorem 3.3] and our strong classification theorem up to stable isomorphism. Here we use that, when A has real rank zero, the group $\mathrm{K}_0(A)$ can be naturally recovered from the module $\mathrm{XK}_0(A)$ as a certain cokernel, see [1, Lemma 8.3]. □

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