On the Invertibility of Motives of Affine Quadrics

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Abstract. We show that the reduced motive of a smooth affine quadric is invertible as an object of the triangulated category of motives $\mathbf{DM}(k, \mathbb{Z}[1/e])$ (where k is a perfect field of exponential characteristic e). We also establish a motivic version of the conjectures of Po Hu on products of certain affine Pfister quadrics. Both of these results are obtained by studying a novel conservative functor on (a subcategory of) $\mathbf{DM}(k, \mathbb{Z}[1/e])$, the construction of which constitutes the main part of this work.

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1 Introduction

For a perfect field k, Voevodsky has constructed a triangulated category $\mathbf{DM}(k)$ containing the classical category $Chow(k)$ of Chow motives [20]. Like $Chow(k)$, $\mathbf{DM}(k)$ is a tensor category. We denote the tensor product by $\otimes = \otimes_{\mathbf{DM}(k)}$ and the unit by $\mathbb{1} = \mathbb{1}_{\mathbf{DM}(k)}$. As in any tensor category, we have the notion of *invertible objects*: an object $E \in \mathbf{DM}(k)$ is called invertible if there exists an object $F \in \mathbf{DM}(k)$ and an isomorphism $E \otimes F \approx \mathbb{1}$. The set of isomorphism classes of invertible objects forms an abelian group under $\otimes$ and is called the *Picard group*. We denote it $Pic(\mathbf{DM}(k))$.
Po Hu [12] was the first to construct interesting elements in $Pic(\mathbf{DM}(k))$, related to certain low-dimensional quadrics. In this direction we prove the following result (see Theorem 33 in Section 5).

Theorem. *Let k be a perfect field of exponential characteristic e not two, $\phi(t_1, \ldots, t_n)$ a non-degenerate quadratic form over k and $a \in k^\times$. Write X_ϕ^a for the affine quadric defined by the equation $\phi(t_1, \ldots, t_n) = a$.*
Then the reduced motive $\tilde{M}(X_\phi^a) \in \mathbf{DM}(k, \mathbb{Z}[1/e])$ is invertible.

This result has a number of predecessors. Work of Voevodsky can be used to show that reduced versions of the Rost motives [19] are invertible. As observed by Hu-Kriz [13, Proposition 5.5], the reduced Rost motives are reduced motives of affine Pfister quadrics. They go further and explore analogies with the Hopf invariant one problem. In [12] this culminates in certain conjectures about smash products of affine Pfister quadrics implying their invertibility. Moreover the conjectures are proven in low dimensions.

The best method the author knows of attacking the study of Picard groups of tensor categories (to the extent that it even deserves the name "method") is to construct "realisation functors" $F : \mathbf{DM}(k) \to \mathcal{C}$. If F is a tensor functor, it induces a homomorphism $Pic(\mathbf{DM}(k)) \to Pic(\mathcal{C})$. If F is sufficiently nice, and $\mathcal{C}$ sufficiently simple, one may hope to compute $Pic(\mathcal{C})$ and relate it to $Pic(\mathbf{DM}(k))$. We mention in passing that a good test for the "niceness" of F seems to be *conservativity* (i.e. the property that F detects isomorphisms). This will be illustrated later.

There are well known realisation functors out of $\mathbf{DM}(k)$, but none of them seem helpful to our problem. If $k \subset \mathbb{C}$ there is the Hodge realisation, but this factors through the natural functor $\mathbf{DM}(k) \to \mathbf{DM}(\mathbb{C})$ and hence provides no interesting information about quadrics (which over $\mathbb{C}$ are distinguished by only their dimension). There is also étale realisation, but this factors through $\mathbf{DM}(k) \to \mathbf{DM}_{et}(k)$. In $\mathbf{DM}_{et}(k)$ our problem turns out to be very simple and not indicative of the complexity encountered in $\mathbf{DM}(k)$ (i.e. in the Nisnevich topology). What we propose in this work is to construct purpose-built realisation functors $\mathbf{DM}(k) \to \mathcal{C}$ into big but easy to understand categories. (Actually we do not quite achieve this; limitations will be explained later.) To motivate our constructions, we explain two analogous but simpler problems obtained by replacing $\mathbf{DM}(k)$ by another category.

First let G be a finite group. There exists the *stable G-equivariant homotopy category* $\mathbf{SH}(G)$ [15]. Its objects (called genuine G-spectra) are roughly pointed G-spaces, where maps inducing weak equivalences on all fixed point sets have been turned into isomorphisms, and all representation spheres are invertible objects. If $H \le G$ is a subgroup, the set of cosets G/H can naturally be turned into a pointed G-space (adding a base point $*$ with trivial action). We denote the associated spectrum by $\Sigma^\infty G/H_+$. The objects $\Sigma^\infty G/H_+$ generate $\mathbf{SH}(G)$. There is a functor, called *geometric fixed points functor*, and denoted $\Phi = \Phi^G : \mathbf{SH}(G) \to \mathbf{SH}$ (where $\mathbf{SH} = \mathbf{SH}(\{e\})$ is the classical stable homotopy category) which turns out to be very useful. It is a tensor functor with the property that $\Phi^G(\Sigma^\infty G/G_+) = S$ (the sphere spectrum), whereas $\Phi^G(\Sigma^\infty G/H_+) = 0$ for any proper subgroup $H < G$. There are also natural functors $\mathbf{SH}(G) \to \mathbf{SH}(H)$ (treating G-spaces as H-spaces) allowing us to construct the more general geometric fixed points functors $\Phi^H : \mathbf{SH}(G) \to \mathbf{SH}(H) \to \mathbf{SH}$. As it turns out the collection $\{\Phi^H\}_H$ (with H ranging over all subgroups of G) is as nice as one may ask (in particular conservative). Consequently these functors were used in [10] to study $Pic(\mathbf{SH}(G))$.

We now come to a second, more algebraic, example. Let R be a (commutative

unital) ring. Suppose we want to study $Pic(D(R))$, the Picard group of the derived category of R-modules. Let m be a maximal ideal of R. Recalling that $D(R)$ can be identified with a subcategory of $K(P(R))$, the homotopy category of chain complexes of projective R-modules, it is easy to construct a functor $\Phi^m : D(R) \to D(R/m)$ with the propetry that $\Phi^m(R[0]) = R/m[0]$. (This is just $\otimes^L_R R/m$.) It turns out that the collection $\{\Phi^m\}_m$ (where m ranges over all maximal ideals) is as nice as one needs (at least when restricted to subcategories of sufficiently small objects in $D(R)$). Moreover the categories $D(R/m)$ are easy to understand. Consequently, these functors have implicitly been used by Fausk in his study of the Picard group of derived categories [9].
Our construction for $\mathbf{DM}^{gm}(k)$ uses a conglomerate of these ideas. The technical notion of weight structures is the glue that holds our constructions together. We proceed roughly as follows. Recall that $\mathbf{DM}^{gm}(k)$ is generated as a triangulated category by the Chow motives. Let $\mathcal{S}$ be the triangulated subcategory generated by those Chow motives not affording a (non-vanishing) Tate summand. The basic idea is to consider the (Verdier Quotient) functor $\varphi_0^k : \mathbf{DM}^{gm}(k) \to \mathbf{DM}^{gm}(k)/\mathcal{S}$. The right hand side does not seem initially easier to understand, but it is at least clear that it is generated by the images of Tate motives. Using weight structure theory one obtains a functor $t : \mathbf{DM}^{gm}(k)/\mathcal{S} \to K^b(Tate)$, where $Tate$ is the category of (pure) Tate motives, and K^b means bounded chain homotopy category.[1] Combined with base change to arbitrary fields, we thus obtain a collection of functors $\Phi^l : \mathbf{DM}^{gm}(k) \to \mathbf{DM}^{gm}(l) \to \mathbf{DM}^{gm}(l)/\mathcal{S} \to K^b(Tate)$. We not that if $T \in Tate$ is a Tate motive then $\Phi^k(T) = T$. If instead $M \in Chow$ affords no (non-zero) Tate summands, then $\Phi^k(M) = 0$. This is rather similar to the geometric fixed points functor Φ^G from stable equivariant homotopy theory. Since the general Φ^l are obtained from Φ^k by base change, just as Φ^H is obtained from the Φ^G construction by base change (restriction to a subgroup), we will call the functors Φ^l "generalized geometric fixed points functors."
A natural question is when these functors have good properties. For our purposes we definitely need *tensor* functors, which is to say we need $\mathcal{S}$ to be a tensor ideal. This is just not true in general. However, if instead of looking at the full $\mathbf{DM}^{gm}(k)$ we look at the subcategory $\mathbf{DQM}^{gm}(k)$ generated by the (products of) smooth projective quadrics, and use coefficients modulo two, then we can show that $\mathcal{S}$ is a tensor ideal. Moreover, using more properties of weight structures, we prove the collection of generalized fixed points functors to be conservative and Pic-injective (i.e. inducing an injection on Picard groups; see Theorem 31 in Section 5):

Theorem. *Let k be a perfect field of exponential characteristic e not two, and $\Phi^l : \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to Tate(\mathbb{F}_2)$ the functors constructed above.*
Then $\{\Phi^l\}_l$, as l ranges over finitely generated extensions of k, forms a conservative, Pic-injective family of tensor triangulated functors.

[1] Actually for this to be true we need to consider $\mathbf{DM}^{gm}(k, \mathbb{F})$ where $\mathbb{F}$ is a *field*. This is not really a problem.

It is then not hard to use general properties of base change and change of coefficients for **DM** to build a conservative and Pic-injective family for $\mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$. It turns out that one additional functor $\Psi : \mathbf{DQM}(k, \mathbb{Z}[1/e]) \to Tate(\mathbb{Z}[1/e])$ suffices. (It is related to geometric base change.)

In more detail, the paper is organised as follows. In Section 2 we introduce our notations regarding Chow motives and collect some results. The main idea is to use the absence of degree one zero-cycles in a variety to conclude that it is free of Tate summands in a strong sense. This observation is what will allow us in a later section to establish that our "geometric fixed points functors" Φ^l are tensor.

In Section 3 we review in some detail the categories $\mathbf{DM}(k, A)$ (triangulated motives over the perfect field k with coefficients in the commutative ring A) and their behaviour under change of coefficients and base. All the material is well known, but sometimes hard to source. We then construct a convenient conservative and Pic-injective collection on $\mathbf{DM}(k, A)$. The targets are always $\mathbf{DM}(k', A')$ with either k simplified (e.g. k' separably closed) or A simplified (e.g. A' a field).

Section 4 constitutes the technical heart of our work. We first rapidly review Bondarko's theory of weight structures. After that we carry out the programme outlined above, of constructing a conservative and Pic-injective family of functors $\{\Phi^l\}_l : \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to K^b(Tate(\mathbb{F}_2))$.

The remaining sections contain applications. In Section 5 we prove that all affine quadrics have invertible motives. This is rather satisfying, since affine quadrics are fairly natural "generalised spheres." Also the result has been known in the étale topology for a long time. Compare the beginning of this introduction for a history of this problem.

Section 6 contains the second set of applications. In [12, Conjecture 1.4] Po Hu has stated certain conjectures about the motivic spectra of affine Pfister quadrics, namely certain formulas they should satisfy under smash product. We establish the analogues (or "images") of these formulas in $\mathbf{DM}(k)$ by an easy computation involving our fixed points functors.

The list of applications of our methods does not end here, but the amount of material we want to stuff into one article does. As directions of future work, let us mention the following possibilities. The structure of $Pic(\mathbf{DQM}(k))$ can be investigated. One may replace the set of projective quadrics by projective homogeneous varieties for a fixed group G. Also using (almost) the same methods, it is possible to study $\mathbf{DATM}(k)$, the subcategory of $\mathbf{DM}(k)$ generated by $M(Spec(l))\{i\}$ for l/k finite separable and $i \in \mathbb{Z}$, i.e. Artin-Tate motives. This will be treated in forthcoming work.

We also note that our results for **DM** have applications to the study of the stable motivic homotopy category $\mathbf{SH}(k)$. In forthcoming work [1] we show that if k is a field of finite 2-étale cohomological dimension, then the functor $\mathbf{SH}(k) \to \mathbf{DM}(k)$ is conservative and Pic-injective, when restricted to compact spectra. Consequently the suspension spectral of affine quadrics are also

invertible, and the Hu conjectures hold for spectra (over such fields).
Whenever we talk about quadrics or quadratic forms, we shall assume that the base field has characteristic different from two. This will be restated with the most important theorems.
Our results are stated over perfect base fields, because this is when $\mathbf{DM}(k)$ is best understood. However actually everything goes through over arbitrary base fields, using [7].
Throughout this text, we will omit brackets around the arguments of functors whenever convenient. For example MX means the same thing as $M(X)$.
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2 Some Results about Chow Motives

We begin with some notation. We take for granted the notion of an *additive category*. An additive category $\mathcal{C}$ is called *Karoubi-closed* if every idempotent endomorphism of an object of $\mathcal{C}$ corresponds to a direct sum decomposition. By a *tensor category* we mean an additive category provided with a suitably compatible symmetric monoidal structure [8, Section 1]. In particular this means that the monoidal operation is bi-additive. We shall always denote the monoidal operation by $\otimes = \otimes_\mathcal{C}$ and call it tensor product. The tensor unit is generically denoted $\mathbb{1} = \mathbb{1}_\mathcal{C}$.
Now our conventions regarding Chow motives. By $SmProj(k)$ we denote the category of smooth projective varieties over the field k. It is a symmetric monoidal category using cartesian product as monoidal product. We shall assume understood the existence and functoriality properties of the *Chow ring* $A^*(X)$. Grading is by codimension and the equivalence relation we use is rational equivalence. Lower index means grading by dimension. For convenience if $\mathbb{F}$ is any coefficient ring, we put $A^*(X, \mathbb{F}) = A^*(X) \otimes_\mathbb{Z} \mathbb{F}$. It is then possible to construct a Karoubi-closed tensor category $Chow(k, \mathbb{F})$ together with a covariant symmetric monoidal functor $M = M_\mathbb{F} : SmProj(k) \to Chow(k, \mathbb{F})$ which has the following properties. The unit object is $\mathbb{1}_{Chow(k,\mathbb{F})} = \mathbb{1} = M(Spec(k))$. There exists an object $\mathbb{1}\{1\}$ such that $M(\mathbb{P}^1) \approx \mathbb{1} \oplus \mathbb{1}\{1\}$. We call $\mathbb{1}\{1\}$ the Lefschetz motive. It is invertible. For any $n \in \mathbb{Z}$ and $M \in Chow(k, \mathbb{F})$ we write $M\{n\} := M \otimes \mathbb{1}\{1\}^{\otimes n}$. For any $X, Y \in SmProj(k)$ and $i, j \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{Chow(k,\mathbb{F})}(M(X)\{i\}, M(Y)\{j\}) = A_{\dim X+i-j}(X \times Y).$$

In particular we have $\mathrm{Hom}(MX, \mathbb{1}\{i\}) = A^i(X, \mathbb{F})$ and $\mathrm{Hom}(\mathbb{1}\{i\}, MX) = A_i(X, \mathbb{F})$. Composition is by the usual push-pull convolution.
In the remainder of this section we collect some results about Chow motives which we will need throughout the article. None of them are hard, so probably most of this is well known.

Recall first that if l/k is a field extension then the functor $SmProj(k) \to SmProj(l), X \mapsto X_l$ induces a functor $Chow(k, \mathbb{F}) \to Chow(l, \mathbb{F})$ called *base change* and denoted $M \mapsto M_l$. We need to know something about this in the inseparable case.

Lemma 1. *Let l/k be a purely inseparable extension of fields of characteristic p and $\mathbb{F}$ a coefficient ring in which (the image of) p is invertible. Then the base change $Chow(k, \mathbb{F}) \to Chow(l, \mathbb{F})$ is fully faithful.*

Proof. It suffices to prove that for $X \in SmProj(k)$ we have $A_*(X, \mathbb{F}) = A_*(X_l, \mathbb{F})$. By the definition of rational equivalence as in [11, Section 1.6] it is enough to show that $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is an isomorphism for all X.
Let $Z \subset X$ be a reduced closed subscheme and $|Z_l|$ the reduced closed subscheme underlying Z_l. Then the image of $[Z]$ under $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is $n[|Z_l|]$, where n is the multiplicity of Z_l. This is easily seen to be a power of p, whence $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is injective. It is also surjective since $X_l \to X$ is a homeomerphism on underlying spaces. This concludes the proof. $\square$

We now investigate "Tate summands". Denote by $Tate(k, \mathbb{F}) \subset Chow(k, \mathbb{F})$ the smallest (strictly) full Karoubi-closed additive subcategory containing $\mathbb{1}\{i\}$ for all i. This is independent up to equivalence of k and we will just write $Tate(\mathbb{F})$ if no confusion can arise. (It is a tensor category.)
We say $M \in Chow(k, \mathbb{F})$ is *Tate-free* if whenever $M \approx T \oplus M'$ with $T \in Tate(k, \mathbb{F})$, then $T \approx 0$. The next proposition holds in much greater generality, but this version is all we need.

Proposition 2. *Let $\mathbb{F}$ be a finite ring and $M \in Chow(k, \mathbb{F})$. Then there exist $T \in Tate(\mathbb{F})$ and $M' \in Chow(k, \mathbb{F})$ with M' Tate-free and $M \approx T \oplus M'$.*

Proof. Splitting off Tate summands inductively, the only problem which could occur is that M might afford arbitrarily large Tate summands. The impossibility of this follows (for example) from the finiteness of étale cohomology of complete varieties [17, Corollary VI.2.8]. $\square$

Lemma 3. *Let $\mathbb{F}$ be a field. Then if $M, N \in Chow(k, \mathbb{F})$ are Tate-free so is $M \oplus N$.*

Proof. A motive with $\mathbb{F}$-coefficients is Tate-free if and only if it affords no summand of the form $\mathbb{1}\{n\}$ for any n.
Let $i : \mathbb{1}\{n\} \to M \oplus N$ and $p : M \oplus N \to \mathbb{1}\{n\}$ be inclusion of and projection to a summand, for M, N arbitrary. Write $i = (i_M, i_N)^T$ and $p = (p_M, p_N)$. Then $\mathrm{id} = pi = p_M i_M + p_N i_N$. Since $\mathrm{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{n\}) = \mathbb{F} \neq 0$ we must have $p_M i_M \neq 0$ or $p_N i_N \neq 0$. Suppose the former holds. Then since $\mathbb{F}$ is a field we may replace i_M by a multiple ci_M such that $p_M(ci_M) = 1$. Thus $\mathbb{1}\{n\}$ is a summand of M. Similarly in the other case. This establishes the contrapositive of the lemma. $\square$

Lemma 4. *Let $\mathbb{F}$ be a field. Then any morphism in $Tate(k,\mathbb{F})$ factoring through a Tate-free object is zero.*

Proof. Since $\mathbb{F}$ is a field any Tate motive is a sum of $\mathbb{1}\{n\}$ for various n, so it suffices to consider a morphism $\mathbb{1}\{n\} \to \mathbb{1}\{m\}$ factoring through a Tate-free object. Since $\mathrm{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{m\}) = 0$ for $n \neq m$ we may assume $n = m$. Consider $a \in \mathrm{Hom}(\mathbb{1}\{n\}, M)$ and $b \in \mathrm{Hom}(M, \mathbb{1}\{n\})$. If $ba \neq 0$ then there exists $c \in \mathbb{F}$ such that $(cb)a = \mathrm{id}$. It follows that $(cb), a$ present $\mathbb{1}\{n\}$ as a summand of M. This establishes the contrapositive. $\square$

We need tools to recognise Tate-free motives. To do so, we introduce some more notation. For $X \in SmProj(k)$ there exists the degree map $deg : A_0(X,\mathbb{F}) \to \mathbb{F}$ (corresponding to pushforward along the structure map $\mathrm{Hom}(\mathbb{1}, MX) \to \mathrm{Hom}(\mathbb{1}, \mathbb{1})$). Write $I_{\mathbb{F}}(X) = deg(A_0(X,\mathbb{F}))$ for the image of the degree map. This is the ideal inside $\mathbb{F}$ generated by the degrees of closed points. The utility of this notion is as follows.

Lemma 5. *Let $\mathbb{F}$ be a field and suppose $I_{\mathbb{F}}(X) \neq \mathbb{F}$. Then MX is Tate-free.*

Proof. As before MX is Tate-free if and only if it affords no summand $\mathbb{1}\{N\}$ for any N. Given $i \in \mathrm{Hom}(\mathbb{1}\{N\}, MX) = A_N(X,\mathbb{F})$ and $p \in \mathrm{Hom}(MX, \mathbb{1}\{N\}) = A^N(X,\mathbb{F})$, the composite $pi \in \mathrm{Hom}(\mathbb{1}\{N\}, \mathbb{1}\{N\}) = \mathbb{F}$ is obtained by push-pull convolution. In this case it is just $deg(p \cap i)$ and so is contained in $I_{\mathbb{F}}(X)$. Thus $pi \neq 1$ and (p, i) is *not* a presentation of $\mathbb{1}\{N\}$ as a summand of X. $\square$

Lemma 6. *Let $X, Y \in SmProj(k)$. Then $I_{\mathbb{F}}(X \times Y) \subset I_{\mathbb{F}}(X) \cap I_{\mathbb{F}}(Y)$.*

Proof. We recall that $I_{\mathbb{F}}(X \times Y)$ is just the ideal generated by degrees of closed points. So let $z \in X \times Y$ be a closed point. Then $z \to X \times Y$ corresponds to morphisms $z \to X$ and $z \to Y$. It follows that $deg(z) \in I_{\mathbb{F}}(X)$ and similarly $deg(z) \in I_{\mathbb{F}}(Y)$. This implies the result. $\square$

Suppose $S \subset SmProj(k)$ is a set of smooth projective varieties. We write $\langle S \rangle^{\otimes,T}_{Chow(k,\mathbb{F})}$ for the smallest strictly full, additive, Karoubi-closed, tensor subcategory of $Chow(k,\mathbb{F})$ containing all Tate motives and also MX for each $X \in S$. Assuming $\mathbb{F}$ is a field, this means that a general object of $\langle S \rangle^{\otimes,T}_{Chow(k,\mathbb{F})}$ is (isomorphic to) a summand of

$$T \oplus M(X_1^{(1)} \times \cdots \times X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \times \cdots \times X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$ and $X_i^{(j)} \in S, i_r \in \mathbb{Z}$.
The following proposition (or rather its failure to generalise) is the basic reason why in the construction of fixed point functors we will need to restrict to subcategories.

Proposition 7. *Let $\mathbb{F}$ be a finite field and $S \subset SmProj(k)$ be such that $I_{\mathbb{F}}(X) = 0$ for all $X \in S$ (i.e. such that all closed points of X have degree* *divisible by the characteristic of* $\mathbb{F}$*). Then any object* $M \in \langle S \rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$ *can be written as* $T \oplus M'$*, where* $T \in Tate(\mathbb{F})$ *and* M' *is (isomorphic to) a summand of*

$$M(X_1^{(1)} \times \cdots \times X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \times \cdots \times X_{n_m}^{(m)})\{i_m\},$$

for some $X_i^{(j)} \in S, i_r \in \mathbb{Z}$*. Moreover any such* M' *is Tate-free.*

Proof. By Lemma 6 we know that $I_{\mathbb{F}}(X_1^{(j)} \times \ldots X_{n_j}^{(j)}) = 0$ and thus by Lemmas 5 and 3 we conclude that any M' as displayed is indeed Tate-free. So it suffices to establish the first part.
By definition we may write

$$M \oplus M'' \approx T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$ and $X_i^{(j)} \in S$. Using Proposition 2 we write $M \oplus M'' \approx M' \oplus M''' \oplus T'$, where M', M'' are maximal Tate-free summands in M, M'' respectively and T' is Tate. Writing out the inverse isomorphisms $M' \oplus M''' \oplus T' \leftrightarrows T \oplus M(X_1^{(1)} \ldots) \oplus \ldots$ in matrix form and using Lemma 4 we conclude that $T' \approx T$ via the induced map. The Lemma below yields that $M' \oplus M'' \approx M(X_1^{(1)} \ldots) \oplus \ldots$. This finishes the proof. $\square$

Lemma 8. *Let* $\mathcal{C}$ *be an additive category and let* $U, T, X, T' \in \mathcal{C}$ *be four objects. Suppose we are given an isomorphism* $\phi : U \oplus T \to X \oplus T'$ *such that the component* $T \to T'$ *is also an isomorphism. Then there is an isomorphism* $\tilde{\phi} : U \to X$*.*

Proof. Let us write

$$\phi = \begin{pmatrix} \alpha & a \\ b & f \end{pmatrix} \quad \psi = \begin{pmatrix} \beta & a' \\ b' & g \end{pmatrix},$$

where ψ is the inverse of ϕ. By assumption f is an isomorphism. Writing out $\phi\psi = \mathrm{id}_{X \oplus T}$ and $\psi\phi = \mathrm{id}_{U \oplus T'}$ one obtains

$$b\beta = -fb' \quad \beta a = -a' f \quad \alpha\beta + ab' = \mathrm{id}_U \quad \beta\alpha + a'b = \mathrm{id}_X .$$

Put $\tilde{\alpha} = \alpha - af^{-1}b : U \to X$. Then the above relations imply that $\tilde{\alpha}$ is an isomorphism with inverse β. $\square$

3 Review of Voevodsky Motives

In this section we collect some facts about $\mathbf{DM}(k, A)$ that we will need for the applications. Most of this is well-known, but in some cases we were unable to locate adequate references. We will assume throughout that k is a perfect field, and re-state this assumption with each theorem.
Fix a ring A. We follow the construction of $\mathbf{DM}(k, A)$ and $\mathbf{DM}^{\mathrm{eff}}(k, A)$ in [5]. Write $Sm(k)$ for the symmetric monoidal category of smooth schemes over k (monoidal operation being cartesian product) and $Cor(k)$ for the symmetric

monoidal category with same objects as $Sm(k)$ but morphisms given by finite correspondences. There is a natural monoidal functor $Sm(k) \to Cor(k)$. Write $Shv^{tr}(k, A)$ for the abelian category of Nisnevich sheaves of A-modules on $Cor(k)$, i.e. (additive) presheaves $Cor(k)^{op} \to A$-Mod such that the restriction $Sm(k)^{op} \to Cor(k) \to A$-Mod is a sheaf in the Nisnevich topology. There is a functor $A_{tr}\bullet : Sm(k) \to Shv^{tr}(k, A)$ sending $X \in Sm(k)$ to the presheaf with transfers it represents (which turns out to be a sheaf). The category $Shv^{tr}(k, A)$ carries a right exact tensor structure making A_{tr} a monoidal functor.
The category $\mathbf{DM}^{\text{eff}}(k, A)$ is then the $\mathbb{A}^1$-local derived category of $Shv^{tr}(k, A)$ [5, Example 3.15]. We write $L_{\mathbb{A}^1} : D(Shv^{tr}(k, A)) \to \mathbf{DM}^{\text{eff}}(k, A) \subset D(Shv^{tr}(k, A))$ for the localisation functor and denote the composite $Sm(k) \xrightarrow{A_{tr}} Shv^{tr}(k, A) \to \mathbf{DM}^{\text{eff}}(k, A)$ by M^{eff}_A or M^{eff} if no confusion can arise. The category $\mathbf{DM}^{\text{eff}}(k, A)$ is compactly generated, and the subcategory of compact objects $\mathbf{DM}^{\text{eff},gm}(k, A)$ is the thick subcategory generated by $M^{\text{eff}}_A(X)$ for $X \in Sm(k)$ [5, Example 5.5]. The category $\mathbf{DM}^{\text{eff}}(k, A)$ carries a symmetric monoidal structure making $M^{\text{eff}}_A : Sm(k) \to \mathbf{DM}^{\text{eff}}(k, A)$ a monoidal functor [5, Example 2.4].
If $X \in Sm(k)$ then we write $\tilde{M}^{\text{eff}}X$ for the homotopy fibre of $M^{\text{eff}}X \to M^{\text{eff}}Spec(k) = \mathbb{1}$. This is the *reduced* (effective) motive of X. If (X, x) is a pointed scheme then there is a canonical isomorphism $M^{\text{eff}}X \simeq \tilde{M}^{\text{eff}}X \oplus M^{\text{eff}}x = \tilde{M}^{\text{eff}}X \oplus \mathbb{1}$. In this situation we will write $M^{\text{eff}}(X, x)$ for $\tilde{M}^{\text{eff}}X$, with this direct sum decomposition understood.
Throughout this text, we write $\mathbb{G}_m$ for the pointed scheme $(\mathbb{A}^1 \setminus 0, 1)$.
The next step is to stabilise $\mathbf{DM}^{\text{eff}}(k, A)$ by inverting $M^{\text{eff}}\mathbb{G}_m$ in the monoidal structure. Here we depart slightly from the notation of [5] and write $Sp(Shv^{tr}(k, A))$ for the abelian category of symmetric spectra in $Shv^{tr}(k, A)$ [5, Section 6]. This is in keeping with the notation in [6, Section 5.3]. Then $\mathbf{DM}(k, A)$ is the $\Omega - \mathbb{A}^1$-local derived category of $Sp(Shv^{tr}(k, A))$ [5, Example 6.25]. There is an adjunction $\Sigma^\infty : Shv^{tr}(k, A) \leftrightarrows Sp(Shv^{tr}(k, A))$ extending to an adjunction

$$\Sigma^\infty : \mathbf{DM}^{\text{eff}}(k, A) \leftrightarrows \mathbf{DM}(k, A) : \Omega^\infty.$$

We write $M = M_A : Sm(k) \to \mathbf{DM}(k, A)$ for the evident composite, and call MX for $X \in Sm(k)$ the motive of X. Similarly for the reduced motive $\tilde{M}X$.
As before, $\mathbf{DM}(k, A)$ is a compactly generated tensor triangulated category. The subcategory $\mathbf{DM}^{gm}(k, A)$ of compact objects is the thick triangulated subcategory generated by the objects of the form $MX \otimes (M\mathbb{G}_m)^{\otimes i}$ for $X \in Sm(k)$ and $i \in \mathbb{Z}$.
So far all of this is very formal and the base k did not really enter. Since k is perfect, we have Voevodsky's remarkable results at hand. Firstly, a bounded above complex in $D(Shv^{tr}(k, A))$ is $\mathbb{A}^1$-local if and only if its homology sheaves are [16, Proposition 14.8]. Moreover a model for $L_{\mathbb{A}^1}$ is given by the $\mathbb{A}^1$-chain complex C_* [16, Corollary 14.9].
Next there is the cancellation theorem: for $E, F \in \mathbf{DM}^{\text{eff}}(k, A)$ we have

$\mathrm{Hom}(E \otimes M\mathbb{G}_m, F \otimes M\mathbb{G}_m) = \mathrm{Hom}(E, F)$ [21, Corollary 4.10]. (Voevodsky only states this for E, F bounded above, but the general case follows using compact generation and taking limits: Let $\mathcal{C}_1 \subset \mathbf{DM}^{\mathrm{eff}}(k, A)$ be the class of objects F such that $\mathrm{Hom}(MX[i], F) = \mathrm{Hom}(MX[i] \otimes M\mathbb{G}_m, F \otimes M\mathbb{G}_m)$ for all $X \in Sm(k)$. Then $\mathcal{C}_1$ is closed under cones (by the five lemma), shifts, isomorphisms and arbitrary sums (because the MX are compact) and contains all bounded above complexes, hence $\mathcal{C}_1 = \mathbf{DM}^{\mathrm{eff}}(k, A)$. Next let $\mathcal{C}_2 \subset \mathbf{DM}^{\mathrm{eff}}(k, A)$ be the class of objects E such that $\mathrm{Hom}(E, F) = \mathrm{Hom}(E \otimes M\mathbb{G}_m, F \otimes M\mathbb{G}_m)$ for all $F \in \mathbf{DM}^{\mathrm{eff}}(k, A)$. Then $\mathcal{C}_2$ is closed under shifts, cones (by the five lemma) isomorphisms and arbitrary sums, and contains $\mathbf{DM}^{gm}(k, A)$ (since $\mathcal{C}_1 = \mathbf{DM}^{\mathrm{eff}}(k, A)$) and hence $\mathcal{C}_2 = \mathbf{DM}^{\mathrm{eff}}(k, A)$.) This implies that $\Sigma^\infty : \mathbf{DM}^{\mathrm{eff}}(k, A) \to \mathbf{DM}(k, A)$ is fully faithful. Indeed if $E \in D(Shv^{tr}(k, A))$ is $\mathbb{A}^1$-local then $\Sigma^\infty E \in D(Sp(Shv^{tr}(k, A)))$ is an Ω-spectrum by cancellation (i.e. $E \simeq R\underline{Hom}(M\mathbb{G}_m, E \otimes M\mathbb{G}_m)$).

Finally there are the homotopy t-structures: the category $\mathbf{DM}^{\mathrm{eff}}(k, A)$ has a non-degenerate t-structure with heart the category of homotopy invariant sheaves with transfers, and $\mathbf{DM}(k, A)$ affords a non-degenerate t-structure with heart the category of homotopy modules with transfers. The functor Ω^∞ is t-exact and hence Σ^∞ is right-t-exact.

We will now discuss functoriality of $\mathbf{DM}(k, A)$ in k and A. The tool to do this is [5, Proposition 3.11], which implies that if $f^* : Shv^{tr}(k, A) \to Shv^{tr}(l, B)$ is a functor which preserves colimits, coverings, and multiplication by $\mathbb{A}^1$ and $\mathbb{G}_m$, then there are induced adjunctions

$$Lf^* : \mathbf{DM}^{\mathrm{eff}}(k, A) \leftrightarrows \mathbf{DM}^{\mathrm{eff}}(l, B) : Rf_*$$

$$Lf^* : \mathbf{DM}(k, A) \leftrightarrows \mathbf{DM}(l, B) : Rf_*.$$

Here Lf^* commutes with Σ^∞ and Rf_* commutes with Ω^∞. If the functor f^* we started with was monoidal then so is the extended functor Lf^*.

We now specialise to base change. For this, let $f : Spec(l) \to Spec(k)$ be an extension of (perfect) fields. Then $f^* : Shv^{tr}(k, A) \to Shv^{tr}(l, A)$ satisfies the requirements outlined above and so we get $Lf^* : \mathbf{DM}(k, A) \leftrightarrows \mathbf{DM}(l, A) : Rf_*$, and similarly for $\mathbf{DM}^{\mathrm{eff}}$. If f is finite separable then $f_\# : Sm(l) \to Sm(k)$ induces $f_\# : Shv^{tr}(l, A) \to Shv^{tr}(k, A)$ and then $Lf_\# : \mathbf{DM}(l, A) \leftrightarrows \mathbf{DM}(k, A) : Rf^*$. See also [5, Example 6.25] again. In this situation we have $Rf^* = f^* = Lf^*$.

The following result is surely well-known, but we could not find a reference, so include the easy proof.

Proposition 9. *Let $f : Spec(l) \to Spec(k)$ be an algebraic (separable) extension of the perfect field k. Then $Lf^* : \mathbf{DM}(k, A) \to \mathbf{DM}(l, A)$ is t-exact.*
Suppose that A is a ring such that for each finite subextension $l/l'/k$, the (image of the) integer $[l' : k]$ is a unit in A.
Then $Lf^* : \mathbf{DM}(k, A) \to \mathbf{DM}(l, A)$ is conservative and t-exact.

Thus we shall write $Lf^* = f^*$ also in this situation.

Proof. Since $f^* : Shv^{tr}(k, A) \to Shv^{tr}(l, A)$ preserves homotopy invariant sheaves, it follows that $Lf^* : D(Sp(Shv^{tr}(k, A))) \to D(Sp(Shv^{tr}(k, A)))$ preserves $\mathbb{A}^1$-local objects. Since $f^* : Shv^{tr}(k, A) \to Shv^{tr}(l, A)$ preserves contractions, $Lf^* : D(Sp(Shv^{tr}(k, A))) \to D(Sp(Shv^{tr}(l, A)))$ preserves $\mathbb{A}^1 - \Omega$-local objects. Thus t-exactness of $Lf^* : D(Sp(Shv^{tr}(k, A))) \to D(Sp(Shv^{tr}(l, A)))$ implies t-exactness of $Lf^* : \mathbf{DM}(k, A) \to \mathbf{DM}(l, A)$.
It thus remains to show: if $F \in Shv^{tr}(k, A)$ and $f^*F = 0$, then $F = 0$. Let $X \in Sm(k)$, $x \in F(X)$. It suffices to show that $x = 0$. By [17, Proposition II.2.2 and Lemma II.3.3] we have $0 = f^*x \in F(X \otimes_k l) = \operatorname{colim}_{l/l'/k} F(X \otimes_k l')$, where the colimit is over finite subextensions. Thus there exists a finite subextension $l/l'/k$ with $(l'/k)^*(x) = 0$. But then by a transfer argument one finds that $[l' : k]x = 0$, whence $x = 0$ since $[l' : k]$ is a unit in A by assumption. □

Next we consider change of coefficients. The construction and basic properties must be well known, but again we could not find convenient references. Let $\alpha : A \to B$ be a ring homomorphism. There is a natural adjoint functor pair

$$\alpha_\# : Shv^{tr}(k, A) \rightleftarrows Shv^{tr}(k, B) : \alpha^*.$$

Here $\alpha_\# F$ is the sheaf associated to $X \mapsto F(X) \otimes_A B$ and $\alpha^* F(X) = F(X)$, viewed as an A-module. In particular $\alpha_\#$ is monoidal and preserves colimits, so [5, Proposition 3.11] applies to give us adjunctions

$$L\alpha_\# : \mathbf{DM}^{\mathrm{eff}}(k, A) \to \mathbf{DM}^{\mathrm{eff}}(k, B) : R\alpha_\#$$

$$L\alpha_\# : \mathbf{DM}(k, A) \to \mathbf{DM}(k, B) : R\alpha_\#.$$

In order to manipulate these functors efficiently, we need a standard result.

Lemma 10. *Any object $E \in \mathbf{DM}(k, A)$ is a filtered homotopy colimit of objects of the form $E' \otimes \mathbb{G}_m^{\otimes i}$ with $E' \in \mathbf{DM}^{eff}(k, A)$ bounded above and $i \in \mathbb{Z}$.*

Proof. Let $\mathcal{DM}^{\mathrm{eff}}(k, A)$ be a model for $\mathbf{DM}(k, A)$. Then we may alternatively model $\mathbf{DM}(k, A)$ via $Spt^\Sigma(\mathcal{DM}^{\mathrm{eff}}(k, A), M\mathbb{G}_m)$, i.e. via (symmetric) $M\mathbb{G}_m$-spectra in $\mathcal{DM}^{\mathrm{eff}}(k, A)$.
Let $E \in \mathbf{DM}(k, A) = Ho(Spt^\Sigma(\mathcal{DM}^{\mathrm{eff}}(k, A), M\mathbb{G}_m))$ have fibrant replacement $(E_1, E_2, \dots)$. Then $E \simeq \operatorname{hocolim}_i (E_i)_{\ge -i} \otimes \mathbb{G}_m^{\otimes -i}$, as one sees immediately by computing the homotopy sheaves on both sides. □

In order to use this, recall that any left adjoint functor of triangulated categories (assumed to have all countable coproducts) commutes with filtered homotopy colimits (by filtered we always mean ω-filtered, i.e. with a countable indexing set) and if it additionally preserves a compact generating set, then its right adjoint also commutes with filtered homotopy colimits. Thus essentially all our functors commute with filtered homotopy colimits. In particular $R\alpha^*$ commutes with filtered homotopy colimits.

Lemma 11. *Let k be perfect and $\alpha : A \to B$ a ring homomorphism. Then the functors $R\alpha^*$ commute with Σ^∞. In fact for $E \in \mathbf{DM}^{eff}(k, B)$ we have $R\alpha^*(E \otimes \mathbb{G}_m) \simeq R\alpha^*(E) \otimes \mathbb{G}_m$.*

Of course $L\alpha_\#$ always commutes with Σ^∞, for formal reasons.

Proof. Let $E \in \mathbf{DM}^{\text{eff}}(k, B)$. Then $\Sigma^\infty E \simeq (E, E \otimes \mathbb{G}_m, E \otimes \mathbb{G}_m^{\otimes 2}, \dots)$. (Here by $E \otimes \mathbb{G}_m$ we mean the derived tensor product in $\mathbf{DM}^{\text{eff}}$, in particular this notation implies an $\mathbb{A}^1$-local object.) By the cancellation theorem, this is an Ω-spectrum. It follows that $R\alpha^*\Sigma^\infty E = (R\alpha^*E, R\alpha^*(E \otimes \mathbb{G}_m), \dots)$. It is thus enough to show that $R\alpha^*(E \otimes \mathbb{G}_m) \simeq R\alpha^*(E) \otimes \mathbb{G}_m$.
Since $L\alpha_\#$ is symmetric monoidal $R\alpha^*$ is lax symmetric monoidal and there is a natural comparison map. Since $\mathbf{DM}^{\text{eff}}(k, B)$ is generated as a localising subcategory by MX for $X \in Sm(k)$ and $\otimes, R\alpha^*$ commute with arbitrary sums, we may assume $E = MX$. In this case a fibrant model of $E \otimes \mathbb{G}_m$ is given by $C_*B_{tr}(X_+ \wedge \mathbb{G}_m)$. Resolving B freely as an A-module, it follows that $R\alpha^*E \in \mathbf{DM}^{\text{eff}}(k, A) \otimes \mathbb{G}_m$. A calculation using adjunction and the cancellation theorem allows us to conclude by the Yoneda lemma. $\square$

Proposition 12. *Let k be perfect, $\alpha : A \to B$ be flat, $E \in \mathbf{DM}^{gm}(k, A)$ and $F \in \mathbf{DM}(k, A)$. Then*

$$\mathrm{Hom}(E, F) \otimes_A B \approx \mathrm{Hom}(L\alpha_\# E, L\alpha_\# F).$$

Proof. By Lemma 10 (and twisting E), we may assume that $F \in \mathbf{DM}^{\text{eff}}(k, A)$ and is bounded above. Then by the cancellation theorem we may assume that $E \in \mathbf{DM}^{gm,\text{eff}}(k, A)$ as well.
There is a natural map from the left hand side to the right hand side. Using the 5-lemma and the fact that $\mathbf{DM}^{gm,\text{eff}}(k, A)$ is generated by MX for $X \in Sm(k)$, we may reduce to $E = MX$ (shifting F if necessary). In this case $\mathrm{Hom}(MX, F)$ is given by the hypercohomology $H^0(X, F^\bullet)$. Since $\otimes_A B$ is exact it commutes with hypercohomology and preserves $\mathbb{A}^1$-invariance of cohomology sheaves, so we have $H^0(X, F^\bullet) \otimes_A B = H^0(X, F^\bullet \otimes_A B) = H^0(X, (L\alpha_\# F)^\bullet)$. $\square$

Proposition 13. *Let k be perfect, A a ring, $a \in A$ a non zero divisor and $\alpha : A \to A/(a)$ the natural map. Then for $E \in \mathbf{DM}(k, A)$ there is a natural distinguished triangle*

$$E \xrightarrow{\cdot a} E \to R\alpha^* L\alpha_\# E.$$

This triangle yields the typical *Bockstein sequences* one expects for reduction of coefficients.

Proof. By Lemmas 10 and 11 we may assume that $E \in \mathbf{DM}^{\text{eff}}(k, A)$ and is bounded above.
In this case $R\alpha^* L\alpha_\# E$ is computed by resolving E by a complex of representable sheaves $C^\bullet$ and then $C^\bullet/(a)$ is a model for $R\alpha^* L\alpha_\# E$. (Note that since $C^\bullet$ has homotopy invariant cohomology, so does $\alpha_\# C^\bullet = C^\bullet/(a)$, by

considering the (ordinary) Bockstein sequence. Hence we may apply α^* immediately to $\alpha_\# C^\bullet$ instead of having to $\mathbb{A}^1$-localise first.) Since a is not a zero divisor the sequence $0 \to C^\bullet \to C^\bullet \to C^\bullet/(a) \to 0$ is exact and yields the desired triangle. □

With this preparation out of the way, we can prove our conservativity and Pic-injectivity theorem. Recall that $\mathrm{Hom}_{\mathbf{DM}(k,A)}(\mathbb{1}, \mathbb{1}[i]) = A$ if $i = 0$ and $= 0$ else.

Theorem 14. *Let k be a perfect field and A a PID of characteristic zero. Let $f : Spec(k^s) \to Spec(k)$ be a separable closure.*
The collection of functors $\{f^*\} \cup \{L\alpha_{\pi\#}\}_\pi$ is conservative. If A has primes of arbitrary large characteristic, the collection is also Pic-injective (both on $\mathbf{DM}(k, A)$). Here $\alpha_\pi : A \to A/(\pi)$ runs through the primes of A.

We could prove essentially the same theorem with A replaced by a Dedekind domain (of characteristic zero) with only slightly more work.

Proof. We first show conservativity. Let $E \in \mathbf{DM}(k, A)$ with $L\alpha_{\pi\#}E = 0$ for all π and $f^*E = 0$. We must show that $E = 0$. Let $T \in \mathbf{DM}^{gm}(k, A)$. It suffices to prove that $\mathrm{Hom}(T, E) = 0$. Now by Proposition 13 we have the triangle $E \xrightarrow{\pi} E \to R\alpha_\pi^* L\alpha_{\pi\#}E = 0$. Thus multiplication by π is an isomorphism on $\mathrm{Hom}(T, E)$. Let $K = Frac(A)$. Since π was arbitrary it follows that $\mathrm{Hom}(T, E)$ is a K-vector space. Since $K \otimes_A K \neq 0$ we concude that $\mathrm{Hom}(T, E) = 0$ provided that $\mathrm{Hom}(T, E) \otimes_A K = 0$. Let $\alpha_0 : A \to K$ be the (flat) localisation. By proposition 12 we know that $\mathrm{Hom}(T, E) \otimes_A K = \mathrm{Hom}(L\alpha_{0\#}T, L\alpha_{0\#}E)$, so it suffices to show that $L\alpha_{0\#}E = 0$. But K is of characteristic zero, so by proposition 9 it is enough to show that $f^*L\alpha_{0\#}E = 0$. Since $L\alpha_{0\#}$ and f^* "commute", this follows from the assumption that $f^*E = 0$.
Now we prove Pic-injectivity. Let $E \in \mathbf{DM}(k, A)$ be such that $f^*E \approx \mathbb{1}_{k^s}$ and $L\alpha_{\pi\#}E \approx \mathbb{1}_{A/(\pi)}$. As a first step, I claim that there exists a finite extension $k \subset l \subset k^s$ such that $g^*E \approx \mathbb{1}_l$, where $g : Spec(l) \to Spec(k)$. As in the proof of Proposition 9 we find that $\mathrm{Hom}(\mathbb{1}_{k^s}, f^*E) = \mathrm{colim}_{k \subset l \subset k^s} \mathrm{Hom}(\mathbb{1}_l, (l/k)^*E)$, where the colimit is over finite subextensions. Hence there exist l and an element $t \in \mathrm{Hom}(\mathbb{1}_l, g^*E)$ such that $(k^s/l)^*(t)$ is an isomorphism. The commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathbb{1}_l, g^*E) & \longrightarrow & \mathrm{Hom}(\mathbb{1}_{k^s}, f^*E) \approx A \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathbb{1}_{l,A/(\pi)}, L\alpha_{\pi\#}g^*E) & \xrightarrow{\approx} & \mathrm{Hom}(\mathbb{1}_{k^s,A/(\pi)}, L\alpha_{\pi\#}f^*E) \approx A/(\pi)
\end{array}$$

shows that $L\alpha_{\pi\#}(t)$ is an isomorphism. Thus by the first part (conservativity), t is an isomorphism.

Now we consider $\mathrm{Hom}(\mathbb{1}_k, E)$. From the Bockstein triangles and the assumption $L\alpha_{\pi\#} \approx \mathbb{1}_{A/(\pi)}$ we get the exact sequences

$$\begin{aligned}\mathrm{Hom}(\mathbb{1}_{A/(\pi)}, L\alpha_{\pi\#}E[-1]) = 0 \to \mathrm{Hom}(\mathbb{1}_k, E) \xrightarrow{\pi} \mathrm{Hom}(\mathbb{1}_k, E)\\ \to \mathrm{Hom}(\mathbb{1}_{A/(\pi)}, L\alpha_{\pi\#}E) \approx A/(\pi) \to \mathrm{Hom}(\mathbb{1}_k, E[1])\end{aligned}$$

It follows that $\mathrm{Hom}(\mathbb{1}_k, E)$ is a torsion-free A-module (hence abelian group). Thus by transfer it follows that $\mathrm{Hom}(\mathbb{1}_k, E) \to \mathrm{Hom}(\mathbb{1}_l, g^*E) \approx A$ is injective. Let us denote the image by $I \subset A$. This is a free A-module (of rank zero or one).
Since $\mathrm{Hom}(\mathbb{1}_l, g^*(E)[1]) = 0$ it follows by transfer that $\mathrm{Hom}(\mathbb{1}_k, E[1])$ is $[l:k]$-torsion. Choosing π of sufficiently large characteristic, we find that $A/(\pi) \to \mathrm{Hom}(\mathbb{1}_k, E[1])$ is the zero map. Thus $I = \mathrm{Hom}(\mathbb{1}_k, E) \neq 0$, i.e. $I \approx A$. It follows that $\mathrm{Hom}(\mathbb{1}_k, E) \to \mathrm{Hom}(\mathbb{1}_{A/(\pi)}, L\alpha_{\pi\#}E) \approx A/(\pi)$ is surjective for each π.
Consider the commutative diagram

$$\begin{tikzcd}\mathrm{Hom}(\mathbb{1}_k, E) \arrow[r] \arrow[d, "(*)"'] & \mathrm{Hom}(\mathbb{1}_l, g^*E) \approx A \arrow[d, "(**)"'] \\ \mathrm{Hom}(\mathbb{1}_{A/(\pi)}, L\alpha_{\pi\#}E) \arrow[r, "\approx"] & \mathrm{Hom}(\mathbb{1}_{l,A/(\pi)}, L\alpha_{\pi\#}g^*E) \approx A/(\pi)\end{tikzcd}$$

The map (**) is the natural surjection and (*) is surjective as we just proved. It follows that $I + (\pi) = A$ for each π and so $I = A$. Thus there exists $t' \in \mathrm{Hom}(\mathbb{1}_k, E)$ with $g^*(t') = t$ an isomorphism. Considering the diagram again one finds that $L\alpha_{\pi\#}(t')$ is also an isomorphism. Thus t' is an isomorphism (by the first part, again) and we are done. □

We need two more auxiliary results. For the first, let $f : Spec(l) \to Spec(k)$ be a Galois extension with group G. If $M \in \mathbf{DM}^{gm}(k, A)$ then the A-module $\mathrm{Hom}(\mathbb{1}, f^*M) \approx \mathrm{Hom}(M(Spec(l)), M)$ has a natural action by G (coming from automorphisms of $Spec(l)$). We denote this action by $\kappa_M : G \to Aut(\mathrm{Hom}(\mathbb{1}, f^*M))$.

PROPOSITION 15. *Let $f : Spec(l) \to Spec(k)$ be (finite) Galois and $[l : k]$ invertible in A. Then the above construction yields an injective homomorphism*

$$\kappa : Ker(f^* : Pic(\mathbf{DM}^{gm}(k, A)) \to Pic(\mathbf{DM}^{gm}(l, A))) \to \mathrm{Hom}(Gal(l/k), A^\times).$$

Proof. Suppose that $M \in Pic(\mathbf{DM}^{gm}(k, A))$, $f^*M \simeq \mathbb{1}$ and let us show that $M \simeq \mathbb{1}$ if and only if the action is trivial. Necessity is clear, we show sufficiency. Independent of the assumptions on $[l : k]$ and M I claim we have the following: if $t : \mathbb{1}_L \to f^*M$ is any morphism, then $f^*(tr(t)) : \mathbb{1}_L \to f^*M$ is the sum of the conjugates under the G-action. Indeed the action on $\mathrm{Hom}_L(\mathbb{1}_L, f^*M) \approx \mathrm{Hom}_k(\mathbb{1}_L, M)$ comes from premultiplication by elements

of $\mathrm{Hom}_k(\mathbb{1}_L, \mathbb{1}_L)$, whereas transfer comes from premultiplication with the adjunction morphism. Thus to prove the claim we may assume that $M = \mathbb{1}$ and $t = \mathrm{id}$, in which case the result follows from [16, Exercise 1.11].
Thus reinstating our assumptions, let $t : \mathbb{1}_L \to f^*M$ be an isomorphism and assume that the G-action is trivial. Then $tr(t/[l : k]) : \mathbb{1} \to M$ is an isomorphism since $f^*(tr(t/[l : k])) = t$ is, by Proposition 9.
Finally we have to prove that κ is a homomorphism, i.e. that $\kappa_{M\otimes N} = \kappa_M \kappa_N$. For this let us denote the adjunction isomorphism $\mathrm{Hom}_{\mathbf{DM}(k,A)}(M(l), T) \to \mathrm{Hom}_{\mathbf{DM}(l,A)}(\mathbb{1}, f^*T)$ generically by ad. One checks that given $f \in \mathrm{Hom}(M(l), M), g \in \mathrm{Hom}(M(l), N)$ then $ad(f) \otimes ad(g) = ad((f \otimes g) \circ \alpha)$, where $\alpha : M(l) \to M(l) \otimes M(l)$ is the map corresponding to $l \otimes l \to l, a \otimes b \to ab$. Next observe that α is G-equivariant if G acts diagonally on $M(l) \otimes M(l)$. The result follows. ☐

For the statement of the next result, we need $\mathbf{DM}(l, A)$ even if l is not perfect. It is explained in the next section what we mean by that. Under our assumptions on A, it is equivalent to $\mathbf{DM}(l^p, A)$, where l^p is the perfect closure of l.

Lemma 16. *Let k be a perfect field, X/k a smooth variety, A a ring in which the exponential characteristic of k is invertible, and $M \in \mathbf{DM}(k, A)$.*
If for all $n \in \mathbb{Z}$ and all $x \in X$ (not necessarily closed) we have that $\mathrm{Hom}_{\mathbf{DM}(x,A)}(\mathbb{1}\{n\}, M_x) = 0$, then also for all $n \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathbf{DM}(k,A)}(MX\{n\}, M) = 0$.

Proof. We will prove the result by induction on $\dim X$. Thus in order to prove it for X we may assume that $\mathrm{Hom}_{\mathbf{DM}(k,A)}(MX'\{n'\}, M) = 0$ for every smooth, locally closed $X' \subset X$ with $\dim X' < \dim X$, and every $n' \in \mathbb{Z}$ (because the residue fields of X' form a subset of those of X). If $\dim X = 0$ then X is a disjoint union of spectra of fields, and the result is clear.
To prove the general case, we may assume that X is connected. Let $n \in \mathbb{Z}$ and $\alpha \in \mathrm{Hom}(MX\{n\}, M)$. It suffices to show that $\alpha = 0$. By considering the generic point and using continuity [7, Example 2.6(2)] we conclude that there exists a non-empty open subvariety $U \subset X$ such that $\alpha|_U = 0$. Let $Z = X \setminus U$. If Z is empty there is nothing to do. Otherwise there exists a non-empty, smooth, connected open subvariety $U_1 \subset Z$, since k is perfect.
Let $Z' = Z \setminus U_1$, $U' = U \cup U_1 = X \setminus Z'$. Then U' is smooth open in X and we have $X \setminus U' = Z'$, which is strictly smaller than Z. We shall prove that $\alpha|_{U'} = 0$. By repeating this argument with U replaced by U' (i.e. Noetherian induction on Z) it will follow that $\alpha = 0$.
Note that $U_1 = U' \setminus U$ is closed in U', say of codimension c. Thus we get the distinguished Gysin triangle

$$MU\{n\} \to MU'\{n\} \to MU_1\{n - c\}.$$

Now $\mathrm{Hom}(MU_1\{n - c\}, M) = 0$ by the induction on dimension. Thus $\mathrm{Hom}(MU'\{n\}, M) \to \mathrm{Hom}(MU\{n\}, M)$ is injective. But $(\alpha|_{U'})|_U = \alpha|_U = 0$ by assumption, so $\alpha|_{U'} = 0$. ☐

4 Weight Structures and the Geometric Fixed Points Functors

In this section, we will use Bondarko's theory of weight structures to construct "generalised geometric fixed points functors" and prove that they have good properties. We shall fix a coefficient ring $\mathbb{F}$ on which an integer e is invertible, and only work with fields of exponential characteristic e.

We shall have to deal with $\mathbf{DM}(k, \mathbb{F})$ for k an imperfect field. There is now a fairly complete theory of $\mathbf{DM}(X, \mathbb{F})$ for Noetherian schemes over a field of exponential characteristic e (assumed invertible in $\mathbb{F}$) [7]. It satisfies the six functors formalism, in particular *continuity*. We recall that if k is an imperfect field with perfect closure k^p, then the pullback $\mathbf{DM}(k, \mathbb{F}) \to \mathbf{DM}(k^p, \mathbb{F})$ is an equivalence of categories [7, Proposition 8.1 (d)]. This means that essentially all properties known over perfect fields hold over imperfect fields as well. We also mention that all of the categories $\mathbf{DM}(X, \mathbb{F})$ afford DG-enhancements. (This is well known if k is a perfect field and hence holds for k any field by the previous remark, and this is all we need. But it is actually clear that the constructions in [7] all yield DG categories.)

We shall work extensively in this section with weight structures [2], which we now review rapidly. Recall that given a category $\mathcal{C}$ and a full subcategory $\mathcal{D} \subset \mathcal{C}$, we call $\mathcal{D}$ *Karoubi-closed in* $\mathcal{C}$ if $\mathcal{D}$ is closed under retracts [2, p. 11]. In other words whenever $X \in \mathcal{C}$ and id_X factorises through an object of $\mathcal{D}$, then $X \in \mathcal{D}$. For example, if $\mathcal{C}$ is Karoubi-closed itself, then $\mathcal{D}$ is Karoubi-closed in $\mathcal{C}$ if and only if $\mathcal{D}$ is a Karoubi-closed category, and strictly full in $\mathcal{C}$.

Similarly, given a category $\mathcal{C}$ and a full subcategory $\mathcal{D} \subset \mathcal{C}$, by the *Karoubi-closure of* $\mathcal{D}$ *in* $\mathcal{C}$ we mean the full subcategory of $\mathcal{C}$ spanned by all the objects which are retracts of objects of $\mathcal{D}$. For example, if $\mathcal{D} \subset \mathcal{C}$ is strictly full and $\mathcal{D}$ is a Karoubi-closed category, then $\mathcal{D}$ is Karoubi-closed in $\mathcal{C}$.

Definition. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0} \subset \mathcal{C}$ two classes of objects. We call this a weight structure if the following hold:*

(i) $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0}$ *are additive and Karoubi-closed in* $\mathcal{C}$.

(ii) $\mathcal{C}^{w\geq 0} \subset \mathcal{C}^{w\geq 0}[1]$, $\mathcal{C}^{w\leq 0}[1] \subset \mathcal{C}^{w\leq 0}$

(iii) For $X \in \mathcal{C}^{w\geq 0}, Y \in \mathcal{C}^{w\leq 0}$ *we have* $\mathrm{Hom}(X, Y[1]) = 0$.

(iv) For each $X \in \mathcal{C}$ *there is a distinguished triangle*

$$B[-1] \to X \to A$$

with $B \in \mathcal{C}^{w\geq 0}$ *and* $A \in \mathcal{C}^{w\leq 0}$.

These axioms look quite similar to those of a t-structure, but in practice weight structures behave rather differently. We call a decomposition as in (iv) a *weight decomposition*. It is usually far from being unique. We put $\mathcal{C}^{w\geq n} = \mathcal{C}^{w\geq 0}[-n]$ and $\mathcal{C}^{w\leq n} = \mathcal{C}^{w\leq 0}[-n]$. We also write $\mathcal{C}^{w>n} = \mathcal{C}^{w\geq n+1}$ etc. The intersection $\mathcal{C}^{w=0} := \mathcal{C}^{w\geq 0} \cap \mathcal{C}^{w\leq 0}$ is called the *heart* of the weight structure.

A weight structure is called *non-degenerate* if $\cap_n \mathcal{C}^{w\geq n} = 0 = \cap_n \mathcal{C}^{w\leq n}$. It is called *bounded* if $\cup_n \mathcal{C}^{w\geq n} = \mathcal{C} = \cup_n \mathcal{C}^{w\leq n}$

A functor $F : \mathcal{C} \to \mathcal{D}$ between categories with weight structures is called *w-exact* if $F(\mathcal{C}^{w\leq 0}) \subset \mathcal{D}^{w\leq 0}$ and $F(\mathcal{C}^{w\geq 0}) \subset \mathcal{D}^{w\geq 0}$. It is called *$w$-conservative* if given $X \in \mathcal{C}$ with $F(X) \in \mathcal{D}^{w\leq 0}$ we have $X \in \mathcal{C}^{w\leq 0}$, and similarly for $w \geq 0$. Note that a w-conservative functor on a non-degenerate weight structure is conservative.

In the following proposition we summarise properties of weight structures we use.

Proposition 17. *(1) $\mathcal{C}^{w\leq 0}$ and $\mathcal{C}^{w\geq 0}$ are extension-stable: if $A \to B \to C$ is a distinguished triangle and $A, C \in \mathcal{C}^{w\leq 0}$ (respectively $A, C \in \mathcal{C}^{w\geq 0}$) then $B \in \mathcal{C}^{w\leq 0}$ (respectively $B \in \mathcal{C}^{w\geq 0}$).*

Moreover $X \in \mathcal{C}^{w\geq 0}$ if and only if $\mathrm{Hom}(X, Y) = 0$ for all $Y \in \mathcal{C}^{w<0}$, and similarly $X \in \mathcal{C}^{w\leq 0}$ if and only if $\mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathcal{C}^{w>0}$.

(2) Bounded weight structures are non-degenerate.

(3) If $\mathcal{C}$ admits a DG-enhancement and the weight structure is bounded, then there exists a w-exact, w-conservative triangulated functor

$$t : \mathcal{C} \to K^b(\mathcal{C}^{w=0})$$

called the weight complex. *Its restriction to $\mathcal{C}^{w=0}$ is the natural inclusion.*

(4) If the weight structure is bounded and $\mathcal{C}^{w=0}$ is Karoubi-closed then so is $\mathcal{C}$.

(5) If $H \subset \mathcal{C}$ is a negative *subcategory of a triangulated category (i.e. for $X, Y \in H$ we have $\mathrm{Hom}(X, Y[n]) = 0$ for $n > 0$) generating it as a thick subcategory, then there exists a unique weight structure on $\mathcal{C}$ with $H \subset \mathcal{C}^{w=0}$. Moreover $\mathcal{C}^{w\leq 0}$ is the smallest extension-stable Karoubi-closed subcategory of $\mathcal{C}$ containing $\cup_{n\geq 0} H[n]$, and similarly for $\mathcal{C}^{w\geq 0}$. The weight structure is bounded and $\mathcal{C}^{w=0}$ is the Karoubi-closure of H in $\mathcal{C}$.*

(6) If $\mathcal{D} \subset \mathcal{C}$ is a triangulated subcategory such that $\mathcal{D}^{w\leq 0} := \mathcal{D} \cap \mathcal{C}^{w\leq 0}$ and $\mathcal{D}^{w\geq 0} := \mathcal{D} \cap \mathcal{C}^{w\geq 0}$ define a weight structure on $\mathcal{D}$ (we say the weight structure restricts *to $\mathcal{D}$) then the Verdier quotient $\mathcal{C}/\mathcal{D}$ affords a weight structure with $(\mathcal{C}/\mathcal{D})^{w\leq 0}$ the Karoubi-closure of the image of $\mathcal{C}^{w\leq 0}$ in $\mathcal{C}/\mathcal{D}$, and similarly for $(\mathcal{C}/\mathcal{D})^{w\geq 0}$, $(\mathcal{C}/\mathcal{D})^{w=0}$.*

The natural "quotient" functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is w-exact. If $X, Y \in \mathcal{C}^{w=0}$ then

$$\mathrm{Hom}(QX, QY) = \mathrm{Hom}(X, Y)/\Sigma_{Z\in\mathcal{D}^{w=0}} \mathrm{Hom}(Z, Y) \circ \mathrm{Hom}(X, Z).$$

The weight structure on $\mathcal{C}/\mathcal{D}$ is bounded if the one on $\mathcal{C}$ is.

Proof. (1) [2, Proposition 1.3.3 (1-3)]. (2) [2, Proposition 1.3.6 (3) and comment after proof]. (3) [2, Proposition 3.3.1 (I), (IV) and Section 6.3]. (4) [2, Lemma 5.2.1]. (5) [2, Theorem 4.3.2 (II) and its proof], [4, Remark 2.1.2]. (6) [2, Proposition 8.1.1]. Weight exactness holds by definition of the weight structure on $\mathcal{C}/\mathcal{D}$. $\square$

We shall call a triangulated category with a fixed weight structure a w-category.

LEMMA 18. *Let $\mathcal{C}$ be a w-category with heart H, and $H' \subset H$ an additive subcategory. Let $\mathcal{C}'$ be the thick triangulated subcategory generated by H' inside $\mathcal{C}$.*
Then the weight structure of $\mathcal{C}$ restricts to $\mathcal{C}'$. In particular, if $X \in \mathcal{C}'$ then we may choose a weight decomposition $A \to X \to X'$ (i.e. $A \in \mathcal{C}^{w\geq 0}$ and $X' \in \mathcal{C}^{w<0}$) with $A, X' \in \mathcal{C}'$.

Proof. This is just Proposition 17 (5) which says that $\mathcal{C}'$, being negatively generated by H', carries a natural unique weight structure. By the description provided we find $\mathcal{C}'^{w\leq 0} \subset \mathcal{C}^{w\leq 0}, \mathcal{C}'^{w\geq 0} \subset \mathcal{C}^{w\geq 0}$. Hence a weight decomposition in $\mathcal{C}'$ is also a weight decomposition in $\mathcal{C}$. The rest follows from the definitions. (It follows from the orthogonality characterisation that $\mathcal{C}'^{w\leq 0} = \mathcal{C}^{w\leq 0} \cap \mathcal{C}'$, but we do not need this.) $\square$

LEMMA 19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor of w-categories, and assume that the weight structure on $\mathcal{C}$ is bounded. Then F is w-exact if and only if $F(\mathcal{C}^{w=0}) \subset \mathcal{D}^{w=0}$.*

Proof. Necessity is clear, we show sufficiency. We find by induction that the thick subcategory of $\mathcal{C}$ generated by $\mathcal{C}^{w=0}$ contains $\mathcal{C}^{w\leq n} \cap \mathcal{C}^{w\geq -n}$ for all n, and hence all of $\mathcal{C}$ by boundedness. It follows that the weight structure on $\mathcal{C}$ is the one described in Proposition 17 (5), i.e. $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0}$ are obtained as extension closures of $\bigcup_{n\geq 0} \mathcal{C}^{w=n}, \bigcup_{n\leq 0} \mathcal{C}^{w=n}$. The result follows since $\mathcal{D}^{w\geq 0}, \mathcal{D}^{w\leq 0}$ are extension-stable. $\square$

LEMMA 20. *Let $\mathcal{C}$ be a w-category which is also a tensor category. Assume that $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^{w=0}$ and that tensoring is weight-bi-exact, i.e. that $\mathcal{C}^{w\leq 0} \otimes \mathcal{C}^{w\leq 0} \subset \mathcal{C}^{w\leq 0}$ and similarly for $\mathcal{C}^{w\geq 0}$.*
Then the weight complex functor is tensor whenever $\mathcal{C}$ affords a tensor DG-enhancement and Pic-injective whenever additionally the weight structure is bounded.
If moreover $\mathcal{C}$ is rigid then the dualisation $D : \mathcal{C}^{op} \to \mathcal{C}$ is w-exact (i.e. $D(\mathcal{C}^{w\geq 0}) \subset \mathcal{C}^{w\leq 0}$ and vice versa).

Proof. If $\mathcal{D}$ is a negative DG tensor category, then $H^0(\mathcal{D})$ is tensor in a natural way and the weight complex functor t manifestly respects the tensor structure. If $\mathcal{C}$ is a tensor DG category with the property that $H^n(\mathrm{Hom}(X, Y)) = 0$ for all $X, Y \in \mathcal{D}$ and $n > 0$ then the good truncation $\tau_{\leq 0}\mathcal{D}$ is tensor in a natural way, and the quasi-equivalence $\tau_{\leq 0}\mathcal{D} \to \mathcal{D}$ is a tensor equivalence.

Hence the weight complex functor is tensor as soon as there is any tensor DG enhancement of $\mathcal{C}^{w=0}$. Moreover by Proposition 17 (3) if the weight structure is bounded then t is w-conservative. Since it induces an isomorphism on hearts it is a fortiori Pic-injective. This proves the first part.
For the second part, let $X \in \mathcal{C}$. The category $\mathcal{C}$ being rigid means that there exists an object DX such that $\otimes DX$ is both right and left adjoint to $\otimes X$.
If $X \in \mathcal{C}^{w\geq 0}$ and $Y \in \mathcal{C}^{w>0}$ then $\mathrm{Hom}(Y, DX) = \mathrm{Hom}(Y \otimes X, \mathbb{1}) = 0$ because $Y \otimes X \in \mathcal{C}^{w>0}$ whereas $\mathbb{1} \in \mathcal{C}^{w=0}$. It follows that $DX \in \mathcal{C}^{w\leq 0}$ by Proposition 17 (1). The case of $X \in \mathcal{C}^{w\leq 0}$ is similar. □

We point out that for any field k, the category $\mathbf{DM}^{gm}(k, \mathbb{F})$ carries a canonical weight structure [3]. (Note that the perfectness assumption in that article can be dispensed with by passing to the equivalent category $\mathbf{DM}^{gm}(k^p, \mathbb{F})$.) It is bounded and $\mathbf{DM}^{gm}(k, \mathbb{F})$ is also a rigid tensor category with the tensor structure satisfying the assumptions of Lemma 20. The base change functors $f^* : \mathbf{DM}^{gm}(l, \mathbb{F}) \to \mathbf{DM}^{gm}(l', \mathbb{F})$ for $f : Spec(l') \to Spec(l)$ are w-exact by Lemma 19. The heart of the weight structure is $Chow(k^p, \mathbb{F})$ which contains $Chow(k, \mathbb{F})$ as a full subcategory by Lemma 1.
In the remainder of this section we will be dealing with the following situation. The coefficient ring $\mathbb{F}$ is a finite field of characteristic p (necessarily $p \neq e$, where e is the exponential characteristic of the ground field k). For every extension l/k we are given a set $S_l \subset SmProj(l)$ such that for all closed points $x \in X \in S_l$ we have $p|deg(x)$. Recall the categories $\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$ of Section 2. We will assume that they are stable by base change, i.e. that for $X \in S_l$ and l'/l another extension we have $MX_{l'} \in \langle S_{l'} \rangle^{\otimes,T}_{Chow(l',\mathbb{F})}$.
We write $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ for the thick triangulated subcategory of $\mathbf{DM}(l, \mathbb{F})$ generated by $\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \subset Chow(l, \mathbb{F}) \subset \mathbf{DM}(l, \mathbb{F})^{w=0}$. It is tensor. The categories $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ are also stable by base change in the sense that if $f : Spec(l') \to Spec(l)$ is a field extension then $f^*(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})) \subset \mathbf{D}\langle S \rangle\mathbf{TM}(l', \mathbb{F})$. By Proposition 17 (5) the weight structure on $\mathbf{DM}^{gm}(l, \mathbb{F})$ restricts to $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$, and the heart is $\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$.
We write $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$ for the Karoubi-closed tensor subcategory of $Chow(l, \mathbb{F})$ generated by Tate twists of motives of varieties in S_l (i.e. this is $\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$ "without the Tate motives"). By Proposition 7 this subcategory consists of Tate-free objects. Let $\langle S_l \rangle^{tri} \subset \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ be the thick triangulated subcategory generated by $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$. As before, the weight structure restricts to $\langle S_l \rangle^{tri}$. We write $\varphi^l_0 : \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$ for the Verdier quotient.

Proposition 21. *The category $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$ carries natural weight and tensor structures, and φ^l_0 is a w-exact tensor functor. The composite*

$$Tate(\mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$$

is a full embedding with essential image $\left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}$.

Proof. The existence of the weight structure and weight exactness is Proposition 17 (6). This also says that $\left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S\rangle^{tri}\right)^{w=0}$ is generated as a Karoubi-closed subcategory by $\varphi_0^l\left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})^{w=0}\right)$. If $M \in \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})^{w=0} = \langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$ then we may write $M \approx M' \oplus T$ with T a Tate and $M' \in \langle S_l\rangle^{\otimes}_{Chow(l,\mathbb{F})}$, by Proposition 7. Thus $\varphi_0^l(M) \approx \varphi_0^l(T)$ and so $\varphi_0^l : Tate(\mathbb{F}) \to \left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S\rangle^{tri}\right)^{w=0}$ is essentially surjective up to (relative) Karoubi-closing. We shall show it is fully faithful whence its essential image is (absolutely) Karoubi-closed and so $\varphi_0^l : Tate(\mathbb{F}) \to \left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S\rangle^{tri}\right)^{w=0}$ will be an equivalence. But by the description in Proposition 17 (6) it suffices to prove that any morphism between Tate objects factoring through $\langle S_l\rangle^{\otimes}_{Chow(l,\mathbb{F})}$ is zero. This follows from Lemma 4.
For the existence of the tensor structure we need $\langle S\rangle^{tri} \otimes \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F}) \subset \langle S\rangle^{tri}$; then φ_0^l is automatically tensor. Considering generators, it suffices to show that $\langle S_l\rangle^{\otimes}_{Chow(l,\mathbb{F})} \otimes \langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \subset \langle S_l\rangle^{\otimes}_{Chow(l,\mathbb{F})}$. This follows from Proposition 7. □

Let l/k be any extension. We write $\Phi^l : \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) \to K^b(Tate(\mathbb{F}))$ for the composite

$$\begin{aligned}\Phi^l : \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F}) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S_l\rangle^{tri} \\ \xrightarrow{t} K^b\left(\left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S\rangle^{tri}\right)^{w=0}\right) \approx K^b(Tate(\mathbb{F}))\end{aligned}$$

of base change, the Verdier quotient functor φ_0^l, and the weight complex t. It is a w-exact triangulated tensor functor. We can now state the main theorem of this section.

Theorem 22. *Let k be a ground field of exponential characteristic e, $\mathbb{F}$ a finite field of characteristic $p \neq e$. Suppose given for each field extension l/k a set $S_l \subset SmProj(l)$ and a function $ex = ex_l : S_l \to \mathbb{N}$. Assume that the following hold (for all fields l/k):*

(1) For $x \in X \in S_l$ closed, $p|deg(x)$.

(2) If l'/l is a field extension and $X \in S_l$ has no rational point over l', then $X_{l'}$ is isomorphic to an object of $S_{l'}$ and $ex(X_{l'}) \leq ex(X)$.

(3) If l'/l is a field extension and $X \in S_l$ has a rational point over l', then $MX_{l'}$ is a summand of a motive of the form

$$T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$, $X_i^{(j)} \in S_{l'}$ and $ex(X_i^{(j)}) < ex(X)$ for all i, j.

Then the family $\{\Phi^l\}_l$, as l runs through finitely generated extensions of k, is w-conservative (so in particular conservative) and Pic-injective.

We note that (2) and (3) imply that $\langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$ are stable by base change, i.e. we are in the situation we have been discussing. Also (1) implies that none of the $X \in S_l$ have rational points over l. The somewhat obscure functions ex_l are necessary to make an induction step in the proof work. We will always use $ex = \dim$ in applications.

Before proving the result we explain how to compute Φ^l in the case that k is perfect (but l need not be).

Proposition 23. *Assume in addition that k is perfect. Let l/k be a field extension.*

There exists an essentially unique additive functor $\Phi^l_0 : \langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \to Tate(l,\mathbb{F})$ such that $\Phi^l_0|_{Tate(l,\mathbb{F})} = \mathrm{id}$ and $\Phi^l_0(M) = 0$ if M is Tate-free. It is tensor and the following diagram commutes (up to natural isomorphism; the lower horizontal arrow is base change of Chow motives):

$$\begin{array}{ccc}
\mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) & \xrightarrow{\Phi^l} & K^b(Tate(\mathbb{F})) \\
{\scriptstyle t}\downarrow & & \uparrow{\scriptstyle \Phi^l_0} \\
K^b\left(\langle S_k\rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) & \longrightarrow & K^b\left(\langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})}\right)
\end{array}$$

Proof. Certainly Φ^l_0 is essentially unique, using e.g. Proposition 7. The functor $t\circ\varphi^l_0$ satisfies the required properties, so Φ^l_0 exists. It is tensor by construction. To establish the commutativity claim, consider the diagram

$$\begin{array}{ccc}
\mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) & \xrightarrow{t} & K^b\left(\langle S_k\rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) \\
\downarrow & & \downarrow \\
\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F}) & \xrightarrow{t} & K^b\left(\langle S_l\rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) \\
{\scriptstyle \varphi^l_0}\downarrow & & \downarrow{\scriptstyle \Phi^l_0} \\
\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S_l\rangle^{tri} & \xrightarrow{t} & K^b(Tate(\mathbb{F})).
\end{array}$$

It suffices to prove that the two squares commute (up to natural isomorphism). This is most readily seen using DG-enhancements: let $\mathcal{D}(r)$ be a functorial negative DG-enhancement of $\langle S_r\rangle^{\otimes,T}_{Chow(r,\mathbb{F})} \subset \mathbf{D}\langle S\rangle\mathbf{TM}(r,\mathbb{F})$, for fields r/k. (In other words $\mathcal{D}(r)$ is a DG-category with the same objects as $\langle S_r\rangle^{\otimes,T}_{Chow(r,\mathbb{F})}$ and mapping complexes concentrated in non-positive degrees, such that these mapping complexes compute morphisms in $\mathbf{D}\langle S\rangle\mathbf{TM}(r,\mathbb{F})$. Finally we ask that when varying r, the assignment $r \mapsto \mathcal{D}(r)$ is a (pseudo-)functor and $H^*(\mathcal{D}(r)) \to \mathbf{D}\langle S\rangle\mathbf{TM}(r,\mathbb{F})$ is a (pseudo-)natural transformation.) Then it suffices to establish strict commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{D}(k) & \longrightarrow & \mathcal{D}(k)_0 \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}(l) & \longrightarrow & \mathcal{D}(l)_0 \approx \langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \\
\big\downarrow & & \Phi^l_0 \big\downarrow \\
\mathcal{D}(l)/\langle S_l \rangle^{tri} & \longrightarrow & \left(\mathcal{D}(l)/\langle S_l \rangle tri\right)_0 \approx Tate(\mathbb{F}),
\end{array}$$

where $\mathcal{D}_0$ for a negative DG-category means zero-truncation. (Indeed the previous diagram is obtained by passing to $Ho(\text{Pre-Tr}(\bullet))$.) The upper square commutes by definition and the lower square commutes if and only if it commutes on degree zero morphisms, which is true essentially by definition of Φ^l_0. □

We establish Theorem 22 through a series of lemmas.

Lemma 24. *Let $\mathcal{C}$ be a w-category, $X \in \mathcal{C}^{w\le 0}$. Suppose given weight decompositions $A \to X \to X'$ and $B[1] \to X' \to X''$ (i.e. $A, B \in \mathcal{C}^{w\ge 0}$, $X' \in \mathcal{C}^{w<0}$ and $X'' \in \mathcal{C}^{w<-1}$).*
Then $A, B \in \mathcal{C}^{w=0}$ and for $T \in \mathcal{C}^{w=0}$ there is a natural exact sequence

$$\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to 0.$$

Proof. We have $A, B \in \mathcal{C}^{w=0}$ by (the dual of) [2, Proposition 1.3.3 (6)]. There is an exact sequence

$$\mathrm{Hom}(T, X'[-1]) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, X') = 0$$

where the last term is zero because $T \in \mathcal{C}^{w\ge 0}, X' \in \mathcal{C}^{w<0}$. In particular $\mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X)$ is surjective. Applying the same reasoning to $\mathrm{Hom}(T, X'[-1])$ we find that $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, X'[-1])$ is surjective and hence

$$\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to 0$$

is exact. This concludes the proof. □

Corollary 25. *Let $X \in \mathbf{D}\langle S \rangle \mathbf{TM}(k, \mathbb{F})^{w\le 0}$ have a weight decomposition $T \to X \to X'$ with $T \in Tate(\mathbb{F})$ (and $X' \in \mathbf{D}\langle S \rangle \mathbf{TM}(k, \mathbb{F})^{w<0}$). Suppose that $\varphi^k(X) \in \left(\mathbf{D}\langle S \rangle \mathbf{TM}(k, \mathbb{F})/\langle S_k \rangle^{tri}\right)^{w<0}$.*
Then for $T' \in Tate(\mathbb{F})$ we have $\mathrm{Hom}(T', X) = 0$.

Proof. Let $B[1] \to X' \to X''$ be a further weight decomposition. Naturality in the above lemma yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}(T', B) & \xrightarrow{\gamma} & \mathrm{Hom}(T', T) & \to & \mathrm{Hom}(T', X) & \to & 0 \\
\alpha\Big\downarrow & & \beta\Big\downarrow & & \Big\downarrow & & \\
\mathrm{Hom}(\varphi^k(T'), \varphi^k(B)) & \xrightarrow{\delta} & \mathrm{Hom}(\varphi^k(T'), \varphi^k(T)) & \to & \mathrm{Hom}(\varphi^k(T'), \varphi^k(X)) & \to & 0.
\end{array}$$

Since φ^k is weight exact we have $\mathrm{Hom}(\varphi^k(T'), \varphi^k(X)) = 0$ and so δ is surjective. The construction of φ^k (in particular Proposition 21) implies that α is surjective and β is an isomorphism. It follows that γ is surjective, whence $\mathrm{Hom}(T', X) = 0$. This concludes the proof. $\square$

The main work in proving our theorem is the following lemma. We let $\varphi^l : \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F}) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l, \mathbb{F})/\langle S_l\rangle^{tri}$ be the composite of φ_0^l and base change.

Lemma 26. *Let $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w\le 0}$ and suppose that for all l/k finitely generated, $\varphi^l(X) \in \left(\mathbf{D}\langle S\rangle\mathbf{TM}(l, \mathbb{F})/\langle S_l\rangle^{tri}\right)^{w<0}$. Then $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w<0}$.*

Proof. We begin by pointing out that Lemma 16 also applies if k is not perfect. Indeed if k^p/k is the perfect closure then X_{k^p} is homeomorphic to X, so has the same set of points, and the residue field extensions of $X_{k^p} \to X$ are purely inseparable, so induce equivalences on $\mathbf{DM}(?, \mathbb{F})$. Thus the Lemma holds over k if and only if it holds over k^p.
Let $\mathcal{R}$ be the set of finite multi-subsets of $\mathbb{N}$ (i.e. the set of finite non-increasing sequences in $\mathbb{N}$). It is well-ordered lexicographically and so can be used for induction. We extend ex to a function $ex_l : \mathbf{D}\langle S\rangle\mathbf{TM}(l, \mathbb{F}) \to \mathcal{R}$. First, for $X_1, \ldots, X_n \in S_l$ put $ex(X_1, \ldots, X_n) = \{\{ex(X_1), \ldots, ex(X_n)\}\}$. Next, if $Y \in \mathbf{D}\langle S\rangle\mathbf{TM}(l, \mathbb{F})$ then there exist $X_1, \ldots, X_n \in S_l$ such that $Y \in \langle Tate(\mathbb{F}), X_1, \ldots, X_n\rangle^{tri}$, i.e. Y is in the thick tensor triangulated subcategory generated by the MX_i and the Tate motives. We let $ex(Y)$ be the minimum of $ex(X_1, \ldots, X_n)$ such that this holds. We shall abuse notation and write $ex(Y) = ex(X_1, \ldots, X_n)$ to additionally mean that $Y \in \langle Tate(\mathbb{F}), X_1, \ldots, X_n\rangle^{tri}$.
Let us observe that if $ex(Y) = ex(X_1, \ldots, X_n)$ and l'/l is an extension in which one of the X_i acquires a rational point, then $ex(Y_{l'}) < ex(Y_l)$, using assumptions (2) and (3).
We shall prove the result by induction on $ex(X)$. Note that is suffices to prove that there is a weight decomposition $A \xrightarrow{\alpha} X \to X'$ (i.e. $A \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w=0}$ and $X' \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w<0}$) with $\alpha = 0$ (because then $X' \approx X \oplus A[1]$ and so $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w<0}$, the latter being Karoubi-closed by definition).
If $ex(X) = \emptyset$ then X must must be Tate. By Lemma 18 we may choose a weight decomposition $T \xrightarrow{\alpha} X \to X'$ with $T \in Tate(\mathbb{F})$. By the corollary above

(applied to $T' = T$) we find that $\alpha = 0$. This finishes the base case of our induction.
Suppose now $ex(X) = ex(X_1, \ldots, X_n) > \emptyset$. If l/k is any extension such that one of the $X_1, \ldots, X_n$ acquires a rational point over l, then we may assume the lemma proved over l by induction, so $X_l \in \mathbf{D}\langle S\rangle\mathbf{TM}(l, \mathbb{F})^{w<0}$. Let $A \xrightarrow{\alpha} X \to X'$ be a weight decomposition; as before way may choose $A \in \langle\{X_1, \ldots, X_n\}\rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$. Write $A \approx T \oplus A'$ as in Proposition 7. I claim that $\alpha|_{A'} = 0$. It is enough to show that if Y is a product of the X_i then $\mathrm{Hom}(MY\{n\}, X) = 0$ for all n. By Lemma 16, it is enough to show that for all $n \in \mathbb{Z}$ and $p \in Y$ we have that $\mathrm{Hom}_{\mathbf{DM}(p,\mathbb{F})}(\mathbb{1}\{n\}, X_p) = 0$. But every variety has a rational point after base change to any one of its points, so $X_p \in \mathbf{D}\langle S\rangle\mathbf{TM}(p, \mathbb{F})^{w<0}$ by induction. This proves the claim.

We thus have a weight decomposition $T \oplus A' \xrightarrow{(\alpha,0)^T} X \to X'$. Let Y be a cone on $\alpha : T \to X$. We find that $X' \approx Y \oplus A'[1]$ and hence $Y \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w<0}$. Thus $T \xrightarrow{\alpha} X \to Y$ is a weight decomposition. Using the corollary again we get $\mathrm{Hom}(T, X) = 0$ and so $\alpha = 0$. This finishes the induction step. □

The rest of Theorem 22 is relatively easy to establish now. We begin with the following.

Lemma 27. *Let $\mathcal{C}, \mathcal{D}$ be w-categories with bi-w-exact tensor structures. Suppose that $\mathcal{C}$ is rigid and its weight structure is bounded.*
Let $\phi : \mathcal{C} \to \mathcal{D}$ be a w-exact tensor functor such that whenever $X \in \mathcal{C}^{w\leq 0}$ and $\phi(X) \in \mathcal{D}^{w<0}$ then $X \in \mathcal{C}^{w<0}$.
Then ϕ is w-conservative.

Proof. Let $X \in \mathcal{C}$. If $\phi(X) \in \mathcal{D}^{w\leq 0}$ then also $X \in \mathcal{C}^{w\leq 0}$. Indeed since the weight structure on $\mathcal{C}$ is bounded we have $X \in \mathcal{C}^{w\leq N}$ for some N. If $N > 0$ then the assumptions imply that $X \in \mathcal{C}^{w\leq N-1}$, and so on.
Suppose now instead that $\phi(X) \in \mathcal{D}^{w\geq 0}$. We need to show that $X \in \mathcal{C}^{w\geq 0}$. But $X \in \mathcal{C}^{w\geq 0}$ if and only if $DX \in \mathcal{C}^{w\leq 0}$ by Lemma 20 (use that $X \approx D(DX)$), and ϕ commutes with taking duals (since $\mathcal{C}$ is rigid). Thus $\phi(DX) = D\phi(X) \in \mathcal{D}^{w\leq 0}$, so $DX \in \mathcal{C}^{w\leq 0}$ and we are done. □

It follows from Lemmas 26 and 27 that $\{\varphi^l\}_l$ is a w-conservative family. But all our weight structures are bounded so the weight complex functors are w-conservative, and thus $\{\Phi^l\}_l$ is also a w-conservative family.
Finally for Pic-injectivity, let $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})$ be invertible with $\Phi^l(X) \approx \mathbb{1}$ for all l. Since $\mathbb{1} \in K^b(Tate(\mathbb{F}))^{w=0}$, w-conservativity implies that $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F})^{w=0} = \langle S_k\rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$. Write $X \approx T \oplus X'$, with T Tate and X' Tate-free. Then $\mathbb{1} \approx \Phi^k(X) = T$ and so $T \approx \mathbb{1}$. It follows that $\Phi^l(X) = \mathbb{1} \oplus \Phi^l(X') \in Tate(\mathbb{F})$. For this to be invertible we need $\Phi^l(X') = 0$. Since this is true for all l, conservativity implies that $X' = 0$. This finishes the proof of Theorem 22.

5 Application 1: Invertibililty of Affine Quadrics

We now begin to reap in the benefits of the work of the previous sections. First we construct the conservative and Pic-injective collection of functors we shall use in the remainder of this work. After that we study invertibility of affine quadrics.

We will be dealing with quadratic forms. If l is a field and ϕ is a non-degenerate quadratic form over l, we write $Y_\phi = Proj(\phi = 0)$ for the projective quadric. This does not really make sense if $\dim \phi = 1$ in which case we put $Y_\phi = \emptyset$ by convention. Given $a \in l^\times$ we put $Y_\phi^a = Proj(\phi = aZ^2)$ and $X_\phi^a = Spec(\phi = a)$. All of these varieties are smooth.

Fix a perfect field k of exponential characteristic $e \neq 2$ and coefficient ring A containing $1/e$. We denote by $\mathbf{QM}(k, A)$ the Karoubi-closed tensor subcategory of $Chow(k, A)$ generated by the (motives of) smooth projective quadrics over k, and the Tate motives.

By [16, Property (14.5.6)] the category $Chow(k, A)$ embeds into $\mathbf{DM}^{gm}(k, A)$. We write $\mathbf{DQM}^{gm}(k, A)$ for the thick triangulated subcategory of $\mathbf{DM}^{gm}(k, A)$ generated by $\mathbf{QM}(k, A)$. This is a tensor category.

We write $\mathbf{QM}(k) = \mathbf{QM}(k, \mathbb{Z}[1/e])$ and $\mathbf{DQM}^{gm}(k) = \mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$. As promised, these categories contain the motives of all (smooth) affine quadrics.

Lemma 28. *If ϕ is a non-degenerate quadratic form over the perfect field k of characteristic not two, and $a \in k^\times$, then the affine quadric X_ϕ^a satisfies $M(X_\phi^a) \in \mathbf{DQM}^{gm}(k, A)$.*

Proof. We have $X_\phi^a = Y_\phi^a \setminus Y_\phi$ and $M(Y_\phi^a), M(Y_\phi), \mathbb{1}\{1\} \in \mathbf{DQM}^{gm}(k, A)$, so the result follows from the Gysin triangle. □

We recall the following result.

Lemma 29 ((Rost)). *Let ϕ be an isotropic non-degenerate quadratic form. Then there exists a non-degenerate form ψ such that*

$$M(Y_\phi) \approx \mathbb{1} \oplus M(Y_\psi)\{1\} \oplus \mathbb{1}\{\dim Y_\phi\}.$$

Moreover for $a \in k^\times$ the natural "inclusion" $M(Y_\phi) \to M(Y_\phi^a)$ is given by

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & s\{1\} & i\{1\} \end{pmatrix} : \mathbb{1} \oplus \mathbb{1}\{\dim Y_\phi\} \oplus M(Y_\psi)\{1\} \to \mathbb{1} \oplus \mathbb{1}\{\dim Y_\phi + 1\} \oplus M(Y_\psi^a)\{1\},$$

where $i : M(Y_\psi) \to M(Y_\psi^a)$ is the natural "inclusion" and $s : \mathbb{1}\{\dim Y_\psi^a\} \to M(Y_\psi^a)$ is the fundamental class (dual of the structure map).

Proof. This is a result about Chow motives.

It is basically [19, Proposition 2]. Rost starts with $\phi = \mathbb{H} \perp \psi$, but this is equivalent to ϕ having a rational point.

For the explicit form of the "inclusion", note first that all matrix entries shown as zero have to be so for dimensional reasons. The entries "id" and "$i\{1\}$" follow from naturality of Rost's construction. For the final entry, we can argue as follows. Note that $\mathbb{Z} = CH^0(Y^a_\psi) = \mathrm{Hom}(\mathbb{1}\{\dim Y^a_\psi + 1\}, MY^a_\psi\{1\}) \approx \mathrm{Hom}(\mathbb{1}\{\dim Y^a_\psi + 1\}, MY_\phi) = CH^1(Y_\phi)$. The induced map we are interested in corresponds under this identification to the cycle class of the closed subvariety $Y_\phi \subset Y^a_\phi$. So up to verifying a sign (which is irrelevant for all our applications), it is enough to show that this class is a generator, which one sees for example by considering the embedding into ambient projective space. □

Lemma 30. *For a field extension l/k let S_l be the set of anisotropic projective smooth quadrics over l, and let $ex_l : S_l \to \mathbb{N}$ be the dimension function $ex(X) = \dim X$. Then Theorem 22 applies, with $\mathbb{F} = \mathbb{F}_2$.*

We note that $\mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F}_2) = \mathbf{DQM}^{gm}(k, \mathbb{F}_2)$, in the notation of the Theorem.

Proof. Points on an anisotropic quadric have degree divisible by two by Springer's theorem [14, Chapter 7, Theorem 2.3], hence condition (1) holds. Condition (2) is satisfied essentially by definition. Finally condition (3) follows from Lemma 29. □

It follows from Lemma 29 that motives of quadrics are geometrically Tate. Let $f : Spec(k^s) \to Spec(k)$ be a separable closure. It follows that the weight complex functor $t : \mathbf{DQM}^{gm}(k^s) \to K^b(Chow(k^s, \mathbb{Z}[1/e]))$ takes values in $K^b(Tate(\mathbb{Z}[1/e]))$. We write Ψ for the composite

$$\Psi : \mathbf{DQM}^{gm}(k) \xrightarrow{f^*} \mathbf{DQM}^{gm}(k^s) \xrightarrow{t} K^b(Tate(\mathbb{Z}[1/e])).$$

Let $g : Spec(l) \to Spec(k)$ be any field extension and $\alpha : \mathbb{Z}[1/e] \to \mathbb{F}_2$ be the natural surjection. Via Lemma 30 and Theorem 22 we obtain functors $\Phi^l : \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to K^b(Tate(\mathbb{F}_2))$. We abuse notation and denote the composite with change of coefficients $\mathbf{DQM}^{gm}(k) \xrightarrow{L\alpha_\#} \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to K^b(Tate(\mathbb{F}_2))$ also by Φ^l.

Theorem 31. *The functors Ψ, Φ^l are tensor triangulated. Together (as l ranges over all finitely generated extensions of k) they are conservative and Pic-injective.*

Proof. The functors are composites of tensor triangulated functors, so are tensor triangulated.
By Theorem 14 the collection $f^*, \{L\alpha_{p\#}\}_p$ (where p ranges over all primes) is conservative and Pic-injective. Since all weight complex functors are conservative and Pic-injective by Lemma 20, the collection $tf^*, \{tL\alpha_{p\#}\}_p$ is conservative and Pic-injective. We have $tf^* = \Psi$. By Theorem 22 we may replace $tL\alpha_{2\#}$ in our collection by $\{\Phi^l\}_l$.

It remains to deal with $L\alpha_{p\#}$ at odd p. Let $M \in \mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$. By repeated application of Lemma 29 we can find an extension L/k (which we may assume Galois) of degree a power of 2, such that M_L is in the triangulated subcategory generated by the Tate motives. In particular $t(L\alpha_{p\#}M_L) \approx L\alpha_{p\#}\Psi(M)$ (as complexes of Tate motives). Since $[L : k]$ is a power of two, base change along L/k is conservative in odd characteristic by Proposition 9. Thus if $\Psi(M) \simeq 0$ then also $L\alpha_{p\#}M \simeq 0$ and our collection is conservative.
We need to work a bit harder for Pic-injectivity. Let $M \in \mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$ be invertible with $\Phi^l(M) \simeq \mathbb{1}[0]$ for all l/k and $\Psi(M) \simeq \mathbb{1}[0]$. Then we know that $L\alpha_{2\#}(M) \simeq \mathbb{1}$ by Theorem 22. We also have $t(M_L) = \Psi(M) \simeq \mathbb{1}$, so $M_L \simeq \mathbb{1}$ by Lemma 20. Consider the mod 2 Bockstein sequence

$$\begin{gathered}\mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M[-1]) = 0 \to \mathrm{Hom}(\mathbb{1}, M) \xrightarrow{2} \mathrm{Hom}(\mathbb{1}, M) \to \\ \to \mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M) \to \mathrm{Hom}(\mathbb{1}, M[1]) \xrightarrow{2} \mathrm{Hom}(\mathbb{1}, M[1]) \to \\ \to \mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M[1]) = 0.\end{gathered}$$

The extremal terms are zero because $L\alpha_{2\#}M \simeq \mathbb{1}$, and for the same reason we have that $\mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M) \approx \mathbb{F}_2$. Thus $\mathrm{Hom}(\mathbb{1}, M)$ has no 2-torsion, whereas $\mathrm{Hom}(\mathbb{1}, M[1])$ has no 2-cotorsion. The composite $M \to M_L \to M$ of base change and transfer is multiplication by $[L : k] = 2^N$. We conclude that $\mathrm{Hom}(\mathbb{1}, M)$ injects into $\mathrm{Hom}_L(\mathbb{1}_L, M_L) \approx \mathbb{Z}[1/e]$ and that the kernel of $\mathrm{Hom}(\mathbb{1}, M[1]) \to \mathrm{Hom}_L(\mathbb{1}_L, M_L[1]) = 0$ (i.e. the whole group) is contained in the 2^N-torsion. But multiplication by 2 is surjective on $\mathrm{Hom}(\mathbb{1}, M[1])$, whence so is multiplication by 2^N, and we conclude that $\mathrm{Hom}(\mathbb{1}, M[1]) = 0$. Consequently we have $\mathrm{Hom}(\mathbb{1}, M) \approx \mathbb{Z}[1/e]$ (since it is an ideal of $\mathbb{Z}[1/e]$ with a non-vanishing quotient, i.e. $\mathbb{F}_2$).
We shall now apply Proposition 15. As we have seen $M_L \simeq \mathbb{1}$, so we obtain a $G = Gal(L/k)$-action on $\mathrm{Hom}(\mathbb{1}, M_L) \approx \mathbb{Z}[1/e]$, i.e. a group homomorphism $\kappa_M : G \to \mathbb{Z}[1/e]^\times$. Since e is prime we have $\mathbb{Z}[1/e]^\times = \{\pm 1\} \times \{e^k | k \in \mathbb{Z}\}$ and since G is finite the image of κ_M must be contained in $\{\pm 1\}$. Note that if $\kappa_M = 1$ then $M \simeq \mathbb{1}$ and we are done. Indeed it suffices by Theorem 14 to show that $L\alpha_{p\#}M \simeq \mathbb{1}$ for odd p. Since $(L\alpha_{p\#}M)_L \simeq \mathbb{1}$, by Proposition 15 this happens if and only if an appropriate Galois action is trivial, but this action is just the reduction $G \xrightarrow{\kappa_M} \mathbb{Z}[1/e]^\times \to (\mathbb{Z}/p)^\times$. So assume now that κ_M is non-trivial.
Let $\beta : \mathbb{Z}[1/e] \to \mathbb{Z}[1/(2e)]$ be the natural map. Note that $\kappa_M : G \to \{\pm 1\}$ has a kernel index 2, i.e. corresponds to a quadratic subextension $k \subset k_2 \subset L$. I claim that $L\beta_\# M \simeq L\beta_\# \tilde{M}Spec(k_2)$. Indeed this follows from Proposition 15 applied to $A = \mathbb{Z}[1/(2e)]$, where f^* becomes conservative, and the observation that $\kappa_{\tilde{M}Spec(k_2)} = \kappa_M$.
In particular we must have $\mathrm{Hom}(\mathbb{1}, L\beta_\# \tilde{M}Spec(k_2)) \approx \mathrm{Hom}(\mathbb{1}, M) \otimes_{\mathbb{Z}[1/e]} \mathbb{Z}[1/(2e)] = \mathbb{Z}[1/(2e)]$, by Proposition 12 and our previous computation. But one may compute easily that $\mathrm{Hom}(\mathbb{1}, L\beta_\# \tilde{M}Spec(k_2)) = 0$. This contradiction concludes the proof. $\square$

Note that if A is a PID, then $Pic(K^b(Tate(A))) = \mathbb{Z} \oplus \mathbb{Z}$. Consequently we have the following corollary.

COROLLARY 32. *The abelian group $Pic(\mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e]))$ is torsion-free (where k is a perfect field of exponential characteristic $e \neq 2$).*

As the proof of the theorem shows, this is completely false with $\mathbb{Z}[1/(2e)]$ coefficients (or in the étale topology), where we have $Pic = \mathbb{Z} \oplus \mathbb{Z} \oplus F$ where F is an $\mathbb{F}_2$-vector space.
We can now prove that affine quadrics are invertible.

THEOREM 33. *Let k be a perfect field of characteristic not two, ϕ a non-degenerate quadratic form over k and $a \in k^\times$. Then $\tilde{M}(X_\phi^a)$ is invertible in $\mathbf{DM}^{gm}(k, \mathbb{Z}[1/e])$.*

Proof. We have $\tilde{M}X_\phi^a \in \mathbf{DQM}^{gm}(k)$ by Lemma 28 and so we can use Theorem 31. Since the category $\mathbf{DQM}^{gm}(k)$ is generated by rigid objects (Chow motives) it is rigid and so conservative tensor functors detect invertibility, by standard arguments. We thus need to show that $\Psi(\tilde{M}X_\phi^a)$ is invertible and that for each l/k, $\Phi^l(\tilde{M}X_\phi^a)$ is invertible.
Let $d+2 = \dim \phi$. Let us put $V_\phi^a = D(MX_\phi^a)\{d+1\}$ and $\tilde{V}_\phi^a = D(\tilde{M}X_\phi^a)\{d+1\}$. Then $\tilde{M}X_\phi^a$ is invertible if and only if $\tilde{V}_\phi^a$ is. From the closed inclusion $i : Y_\phi \to Y_\phi^a$ with complement X_ϕ^a we get the dual Gysin triangle

$$MY_\phi \xrightarrow{i} MY_\phi^a \to V_\phi^a.$$

It follows that $t(V_\phi^a) = [MY_\phi \xrightarrow{i} \dot{M}Y_\phi^a]$. Here the dot is used to indicate the term of degree zero in the chain complex. Dualising the defining triangle of $\tilde{M}X_\phi^a$ we obtain

$$\mathbb{1}\{d+1\} \xrightarrow{s} V_\phi^a \to \tilde{V}_\phi^a,$$

where s is the fundamental class (dual of the structure map). Hence we finally obtain

$$t(\tilde{V}_\phi^a) = [MY_\phi \oplus \mathbb{1}\{d+1\} \xrightarrow{(i,s)} \dot{M}Y_\phi^a] =: C(\phi).$$

The functor Ψ is computed by first applying geometric base change, so ϕ becomes completely split. In particular it has to be isotropic. An induction on dimension using Lemma 34 below shows that we may reduce to $\dim \phi = 1$ or 2, i.e. $\{x^2 = 1\}$ or $\{xy = 1\}$ (recall that completely split quadrics are characterised by their dimension, so we can choose any non-degenerate model quadric of the correct dimension). But $\tilde{M}(\{x^2 = 1\}) = \mathbb{1}$ and $\tilde{M}(\{xy = 1\}) = \tilde{M}(\mathbb{G}_m)$ are both invertible.
Dealing with Φ^l is a bit harder.
The expression $C(\phi) \in K^b(\mathbf{QM}(k))$ makes sense even if k is not perfect. Using Proposition 23 it suffices to prove: if l/k is any field extension, then $\Phi_0^l C(\phi_l)$ is invertible. We drop the subscript zero from now on. We may as well prove: if k

is any field and ϕ is any non-degenerate quadratic form over k, then $\Phi^k(C(\phi))$ is invertible. By Lemma 34 below, if $\phi \approx \psi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$. We may thus assume that either ϕ is anisotropic, or $\phi = \mathbb{H}$, or ϕ is of dimension one.

If $\phi = \mathbb{H}$ then $Y_\phi \approx Spec(k \times k)$, $Y^a_\phi \approx \mathbb{P}^1$ and the result follows easily. If ϕ is of dimension one then $MY_\phi = 0$ and either $MY^a_\phi = \mathbb{1} \oplus \mathbb{1}$ or $MY^a_\phi = M(k')$, where k'/k is a quadratic extension. Again the result follows easily.

So we may assume that ϕ is anisotropic. There are three cases. If $\phi \perp \langle -a \rangle$ is also anisotropic, then none of MY_ϕ, MY^a_ϕ afford Tate summands, by Proposition 7. Thus $\Phi^k(C(\phi)) = \mathbb{1}\{d+1\}[1]$ is invertible.

If $\phi \perp \langle -a \rangle$ is isotropic, then $\phi \perp \langle -a \rangle = \psi \perp \mathbb{H}$. Suppose that ψ has dimension greater than one. Then by (the contrapositive of) Lemma 35 below, ψ is anisotropic. It follows that $MY^a_\phi \approx \mathbb{1} \oplus \mathbb{1}\{d+1\} \oplus MY^a_\psi$ and $\Phi^k(C(\phi)) = [\mathbb{1}\{d+1\} \to \dot{\mathbb{1}} \oplus \mathbb{1}\{d+1\}]$. The component $\mathbb{1}\{d+1\} \to \mathbb{1}\{d+1\}]$ comes from the fundamental class of M^a_ψ and so is an isomorphism. Thus $\Phi^k(C(\phi)) \simeq \mathbb{1}$ is invertible.

Finally it might be that ψ has dimension one. Then $Y^a_\phi \approx \mathbb{P}^1$ whereas MY_ϕ affords no Tate summands, and the result follows as in the case of dimension greater than one. This concludes the proof. $\square$

Lemma 34. *Notation as in the theorem. If $\phi = \psi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$.*

Proof. Using the explicit form for the inclusion $MY_\phi \to MY^a_\phi$ from Lemma 29 we find that

$$C(\phi) = [(\mathbb{1} \oplus \mathbb{1}\{d\} \oplus MY_\psi\{1\}) \oplus \mathbb{1}\{d+1\} \xrightarrow{\alpha} \dot{\mathbb{1}} \oplus \mathbb{1}\{d+1\} \oplus MY^a_\psi\{1\}],$$

where α is given by the matrix

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & s\{1\} & i\{1\} & 0 \end{pmatrix}.$$

Here f comes from the fundamental class and so is an isomorphism. It follows that $C(\phi) \approx C(\psi)\{1\} \oplus cone(\mathrm{id}_{\mathbb{1}})[-1] \oplus cone(\mathrm{id}_{\mathbb{1}\{d+1\}})[-1] \simeq C(\psi)\{1\}$. This is the desired result. $\square$

Lemma 35. *If $\phi \perp \langle a \rangle \approx \psi \perp \mathbb{H} \perp \mathbb{H}$, then ϕ is isotropic.*

Proof. Let $X = Y_{\langle a \rangle \perp \phi}$. Then $Y_\phi = X \cap \{X_0 = 0\}$. Since $\langle a \rangle \perp \phi \approx \psi \perp \mathbb{H} \perp \mathbb{H}$, we find that $Y_{\mathbb{H}\perp\mathbb{H}} \subset X$. Then $Y_\phi \cap Y_{\mathbb{H}\perp\mathbb{H}} = Y_{\mathbb{H}\perp\mathbb{H}} \cap \{X_0 = 0\}$ (intersecting inside X). Now we know that after a *linear* change of coordinates $(X_0 : \cdots : X_r) \mapsto (T_0 : \cdots : T_r)$ the subvariety $Y_{\mathbb{H}\perp\mathbb{H}}$ of X is given by the equations $T_0T_1 + T_2T_3 = 0$, $T_i = 0$ for $i > 3$. Thus $Y_\phi \cap Y_{\mathbb{H}\perp\mathbb{H}}$ is obtained by adding a further *linear* constraint in the T_0, T_1, T_2, T_3. It is easy to see that there must be a rational, non-zero solution, so Y_ϕ has a rational point. This was to be shown. $\square$

6 Application 2: Po Hu's Conjectures for Motives

In this final section we prove a version for motives of Po Hu's conjectures [12, Conjecture 1.4]. We retain notation from the previous section.
For $\underline{a} = (a_1, \ldots, a_n) \in (k^\times)^n$, $b \in k^\times$ let us put

$$U^b_{\underline{a}} = X^b_{\langle\langle a_1, \ldots, a_n \rangle\rangle},$$

where $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is the n-fold Pfister quadric associated with the symbol $\underline{a}$. We use notation such as $\underline{a}, a' = (a_1, \ldots, a_n, a') \in (k^\times)^{n+1}$ for concatenation of tuples.

Theorem 36. *Let k be a perfect field of characteristic not two, and $\underline{a} \in (k^\times)^n, b \in k^\times$.*
In $\mathbf{DM}^{gm}(k)$ there is an isomorphism

$$\tilde{M}(U^1_{\underline{a},b}) \otimes \tilde{M}(U^b_{\underline{a}})[1] \approx \tilde{M}(U^1_{\underline{a}})\{2^n\}. \tag{6.1}$$

To prove this, we have to recall some facts about *Rost motives.* If $\underline{a} \in (k^\times)^n$, then there is the associated Rost motive $R_{\underline{a}} \in \mathbf{QM}(k)$. Recall that one has $H^1_{et}(k, \mathbb{F}_2) = k^\times/2$, and hence cup product yields a natural map $\partial = \partial^k : (k^\times)^n \to H^n_{et}(k, \mathbb{Z}/2)$. The Rost motives have the remarkable property that $R_{\underline{a}}$ is irreducible if and only if $\partial(\underline{a}) \neq 0$. In fact there are canonical maps

$$\mathbb{1}\{2^{n-1} - 1\} \to R_{\underline{a}} \to \mathbb{1} \tag{6.2}$$

(which we call structure maps) and if $\partial(\underline{a}) = 0$ then this is a splitting distinguished triangle. The same statements hold true with $\mathbb{F}_2$ coefficients. These results follow from the work of a number of people, see [18] for an overview.
The relationship between Rost motives and $U^b_{\underline{a}}$ is encapsulated in the following proposition.

Proposition 37. *For $\underline{a} \in (k^\times)^n, b \in k^\times$ there is a distinguished triangle*

$$\tilde{M}(U^b_{\underline{a}}) \to R_{\underline{a},b} \to R_{\underline{a}}\{2^{n-1}\} \oplus \mathbb{1}.$$

Here $R_{\underline{a},b} \to \mathbb{1}$ is the structure map, and the composite

$$\mathbb{1}\{2^n - 1\} \to R_{\underline{a},b} \to R_{\underline{a}}\{2^{n-1}\}$$

is the $\{2^{n-1}\}$ twist of the structure map $\mathbb{1}\{2^{n-1} - 1\} \to R_{\underline{a}}$.

Proof. This is essentially [13, proof of Proposition 5.5].
We know that $U := U^b_{\underline{a}}$ is the complement of $X := Y_{\langle\langle \underline{a} \rangle\rangle}$ in $Y := Y^b_{\langle\langle \underline{a} \rangle\rangle}$. By the work of Rost [19, Theorem 17 and Proposition 19], if we put $R_n := R_{\underline{a},b}$ and $R_{n-1} = R_{\underline{a}}$, then

$$M(Y) = R_n \oplus \bigoplus_{k=1}^{2^{n-1}-1} R_{n-1}\{k\} := R_n \oplus R',$$

$$M(X) = \bigoplus_{k=0}^{2^{n-1}-1} R_{n-1}\{k\} := R_{n-1} \oplus R'$$

and the natural map $M(X) \to M(Y)$ is the identity on R'.
The localisation triangle $M^c(X) = M(X) \to M^c(Y) = M(Y) \to M^c(U)$ fits into the following commutative diagram of (distinguished) triangles:

$$\begin{tikzcd}
R' \arrow[r, equal] \arrow[d] & R' \arrow[d] & \\
M(X) \arrow[r] \arrow[d] & M(Y) \arrow[r] \arrow[d] & M^c(U) \\
R_{n-1} & R_n &
\end{tikzcd}$$

An application of the octahedral axiom yields a distinguished triangle $R_{n-1} \to R_n \to M^c(U)$. Noting that $DM^c(U) = M(U)\{-(2^n-1)\}, DR_n = R_n\{-(2^n-1)\}$ and $DR_{n-1} = R_{n-1}\{-(2^{n-1}-1\}$, by dualising and twisting the triangle, we find a distinguished triangle $M(U) \to R_n \to R_{n-1}\{2^{n-1}\}$. Adding in the copy of $\mathbb{1}$ implied in $\tilde{M}(U)$, we get the claimed triangle with the correct map $R_n \to \mathbb{1}$.
To see the second claim about the differential, the important point is that in the triangle $R_{n-1} \to R_n \to M^c(U)$ the map $R_{n-1} \to R_n$ is induced from the inclusion $M(X) \to M(Y)$ by passing to the appropriate summands. It follows that $R_{n-1} \to R_n \to \mathbb{1}$ is the structure map of $R_{n-1} \to \mathbb{1}$. The desired result now follows by dualising. □

Proof of Theorem 36. By Lemma 28, we have $\tilde{M}(U_{\underline{a}}^b) \in \mathbf{DQM}^{gm}(k)$, etc. We also know by Theorem 33 that both sides of equation (6.1) are invertible. Hence if $F : \mathbf{DQM}^{gm}(k) \to \mathcal{C}$ is a Pic-injective functor, it suffices to prove that $F(LHS) \approx F(RHS)$.
Of course we use the Pic-injective collection from Theorem 31.
¿From Proposition 37 we know that

$$t(\tilde{M}(U_{\underline{a}}^b) = [\dot{R}_{\underline{a},b} \to R_{\underline{a}}\{2^n\} \oplus \mathbb{1}],$$

and we also know certain things about the differential. To compute Ψ, we have to consider geometric base change, where the triangle (6.2) is splitting distinguished. One obtains

$$\Psi(\tilde{M}(U_{\underline{a}}^b)) = [\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}\{2^{n-1}\} \oplus \mathbb{1}\{2^n-1\} \oplus \mathbb{1}]$$

and from the information about the differential given in proposition 37 we deduce that $\Psi(\tilde{M}(U_{\underline{a}}^b) \simeq \mathbb{1}\{2^{n-1}\}[-1]$. Thus $\Psi(LHS) \approx \Psi(RHS)$ reads

$$\mathbb{1}\{2^n\}[-1] \otimes \mathbb{1}\{2^{n-1}\}[-1][1] \approx \mathbb{1}\{2^{n-1}\}[-1]\{2^n\},$$

which is certainly true.
Now let l/k be an arbitrary field extension. We need to prove $\Phi^l(LHS) \approx \Phi^l(RHS)$. This involves $R_{\underline{a}}, R_{\underline{a},b}, R_{\underline{a},1}$ and $R_{\underline{a},b,1}$. Depending on l these may

Table 1: Terms needed to compute Φ^l.

	$\partial^l(\underline{a}, b) \neq 0$	$\partial^l(\underline{a}, b) = 0$ but $\partial^l(\underline{a}) \neq 0$
$\Phi^l(U^1_{\underline{a},b})$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^{n+1}-1\} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^{n+1}-1\} \to \dots$
		$\dots \to \mathbb{1}\{2^n\} \oplus \mathbb{1}\{2^{n+1}-1\} \oplus \mathbb{1}]$
$\Phi^l(U^b_{\underline{a}})$	$[\dot{0} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$
$\Phi^l(U^1_{\underline{a}})$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$

Table 2: Terms needed to compute Φ^l, simplified form.

	$\partial^l(\underline{a}, b) \neq 0$	$\partial^l(\underline{a}, b) = 0$ but $\partial^l(\underline{a}) \neq 0$
$\Phi^l(U^1_{\underline{a},b})$	$\mathbb{1}\{2^{n+1}-1\}$	$\mathbb{1}\{2^n\}[-1]$
$\Phi^l(U^b_{\underline{a}})$	$\mathbb{1}[-1]$	$\mathbb{1}\{2^n-1\}$
$\Phi^l(U^1_{\underline{a}})$	$\mathbb{1}\{2^n-1\}$	$\mathbb{1}\{2^n-1\}$

or may not split into Tate motives, so may or may not survive Φ. We see that $R_{\underline{a},1}$ and $R_{\underline{a},b,1}$ always split (because $\partial^l(1) = 0$), and that $R_{\underline{a},b}$ splits whenever $R_{\underline{a}}$ splits (because $\partial(\underline{a}, b) = \partial(\underline{a}) \cup \partial(b)$).

If $R_{\underline{a}}$ splits then everything is split and Φ^l is just mod two reduction of Ψ, so we know the equation is satisfied. Thus there are just two cases and three things in each to compute, which we gather in Table 1.

The differentials can again be figured out using Proposition 37. Using these one can simplify the expressions. We have gathered the results in Table 2.

To complete the proof, we check that $\Phi^l(LHS) \approx \Phi^l(RHS)$ in both cases. This is easy. $\square$

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