Overconvergent Modular Forms and Perfectoid Shimura Curves

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Received: January 18, 2016
Revised: July 28, 2016

Communicated by Otmar Venjakob

Abstract. We give a new construction of overconvergent modular forms of arbitrary weights, defining them in terms of functions on certain affinoid subsets of Scholze's infinite-level modular curve. These affinoid subsets, and a certain canonical coordinate on them, play a role in our construction which is strongly analogous with the role of the upper half-plane and its coordinate 'z' in the classical analytic theory of modular forms. As one application of these ideas, we define and study an overconvergent Eichler-Shimura map in the context of compact Shimura curves over $\mathbb{Q}$, proving stronger analogues of results of Andreatta-Iovita-Stevens.

2010 Mathematics Subject Classification: 11F33 11F33, 11F70, 11G18

Contents

1. Introduction

Let $N \geq 5$ be an integer, and let $Y_1(N)(\mathbb{C})$ be the usual analytic modular curve. A holomorphic modular form f of weight k and level N admits two rather distinct interpretations, which one might call the *algebraic* and *analytic* points of view:

Algebraic: f is a global section $\omega(f)$ of the line bundle $\omega^{\otimes k} = \omega^{\otimes k}_{E/Y_1(N)}$ on $Y_1(N)(\mathbb{C})$ (extending to $X_1(N)(\mathbb{C})$). Equivalently, f is a rule which assigns to each *test object* - an isomorphism class of triples (S, E, η) consisting of a complex analytic space S, a (generalized) elliptic curve $\pi : E \to S$ with level N structure, and η an $\mathcal{O}_S$-generator of $\omega_{E/S} := (\mathrm{Lie} E/S)^*$ - an element $\mathbf{f}(S, E, \eta) \in \mathcal{O}(S)$ such that $\mathbf{f}$ commutes with base change in S and satisfies $\mathbf{f}(S, E, a\eta) = a^{-k}\mathbf{f}(S, E, \eta)$ for any $a \in \mathcal{O}(S)^\times$. These two interpretations are easily seen to be equivalent (for simplicity we are ignoring cusps): for any test object, there is a canonical map $s : S \to Y_1(N)$ realizing E/S as the pullback of the universal curve over $Y_1(N)$, and one has $s^*\omega(f) = \mathbf{f}(S, E, \eta)\eta^{\otimes k}$.

Analytic: f is a holomorphic function on the upper half-plane $\mathfrak{h}$ of moderate growth, satisfying the transformation rule $f(\frac{az+b}{cz+d}) = (cz + d)^k f(z)$ for all $\gamma \in \Gamma_1(N)$.

How do we pass between these points of view? The key is that $\mathfrak{h}$ may be identified with the universal cover $\widetilde{Y_1(N)}$ of $Y_1(N)$, in the category of complex analytic spaces, and the pullback of the line bundle ω to $\widetilde{Y_1(N)}$ is canonically trivialized. Precisely, $\widetilde{Y_1(N)}$ consists of pairs (E,β) where $E/\mathbb{C}$ is an elliptic curve and $\beta = \{\beta_1,\beta_2\} \in H_1(E(\mathbb{C}),\mathbb{Z})$ is an oriented basis[1]. This space admits a left action of $\Gamma_1(N)$ by $\{\beta_1,\beta_2\} \mapsto \{a\beta_1+b\beta_2, c\beta_1+d\beta_2\}$. Let $p : \widetilde{Y_1(N)} \to Y_1(N)$ be the natural projection. Defining the period $z(E,\beta) \in \mathbb{C}$ by $\int_{\beta_1}\eta = z(E,\beta)\int_{\beta_2}\eta$, where $\eta \neq 0 \in H^0(E(\mathbb{C}),\Omega^1_{E/\mathbb{C}})$ is any nonzero holomorphic one-form on E, the map

$$\begin{array}{ccc}\widetilde{Y_1(N)} & \overset{\sim}{\to} & \mathfrak{h} \\ (E,\beta) & \mapsto & z(E,\beta)\end{array}$$

is a $\Gamma_1(N)$-equivariant isomorphism of Riemann surfaces. The pullback $p^*\omega$ is then trivialized by the differential η_{can} characterized by $\int_{\beta_2}\eta_{\mathrm{can}} = 1$. Defining $f(z)$ by $p^*\omega(f) = f(z)\eta_{\mathrm{can}}^{\otimes k}$, a calculation shows that $\eta_{\mathrm{can}}(\gamma z) = (cz+d)^{-1}\eta_{\mathrm{can}}(z)$, from which the transformation law of $f(z)$ follows.

In the p-adic setting a priori one only has the algebraic definition valid for integral weights. Given the importance of non-integral weights, it seems natural to hope for a direct algebraic definition of modular forms with non-integral weights. More precisely, given a continuous character $\kappa : \mathbb{Z}_p^\times \to L^\times$ for some $L/\mathbb{Q}_p$ finite, one would like to define a "p-adic modular form of weight κ and level N" as a rule which assigns to each test object an isomorphism class of triples (R,E,η) consisting of a p-adically separated and complete ring R, a (generalized) elliptic curve $E/\mathrm{Spec}R$ with level N structure, and η a generator of $\omega_{E/R}$ - an element $g(R,E,\eta) \in R\otimes_{\mathbb{Z}_p} L$ such that g commutes with base change in R and satisfies $g(R,E,r\eta) = \kappa(r)^{-1}g(R,E,\eta)$ for any $r \in R^\times$. The problem with this naive definition is that the expression $\kappa(r)$ does not make sense in general. In fact, the only characters for which this is unambiguously defined and functorial in R are the power characters $\kappa : r \to r^k, k \in \mathbb{Z}$. However, Andreatta-Iovita-Stevens and Pilloni ([AIS1, Pil]) discovered (independently) a remarkable fix to this problem, whereby for a given character κ one only allows "certain elliptic curves" and, more importantly, "certain differentials" in the definition of test objects. The admissible elliptic curves are those whose Hasse invariant has suitably small valuation. The admissible η's are defined using torsion p-adic Hodge theory and the theory of the canonical subgroup; they are not permuted by all of $R^\times$, but only by a subgroup of elements p-adically close enough to $\mathbb{Z}_p^\times$ so that $\kappa(r)$ is defined.

Our first goal in this paper is to develop a p-adic analogue of the *analytic* picture above. Of course, the most pressing question here is: what are the

[1]One also requires that β_2 generate the $\Gamma_1(N)$-structure under $H_1(E(\mathbb{C}),\mathbb{Z}/N\mathbb{Z}) \cong E(\mathbb{C})[N]$

correct analogues of $\widetilde{Y_1(N)}$, $\mathfrak{h} \subseteq \mathbb{P}^1_{\mathbb{C}}$ and z? The following theorem gives a partial answer.

Theorem 1.1. *Given N and $p \nmid N$ as above, let*

$$\mathcal{X}_\infty \sim \varprojlim_n X^{\mathrm{ad}}_{K_1(N)K(p^n)}$$

be the infinite-level perfectoid (compactified) modular curve of tame level N ([Sch4]), with its natural right action of $\mathrm{GL}_2(\mathbb{Q}_p)$. There is a natural family of $K_0(p)$-stable open affinoid perfectoid subsets $\mathcal{X}_{\infty,w} \subseteq \mathcal{X}_\infty$ parametrized by rationals $w \in \mathbb{Q}_{>0}$, with $\mathcal{X}_{\infty,w'} \subseteq \mathcal{X}_{\infty,w}$ for $w \leq w'$, and there is a canonical global section $\mathfrak{z} \in \mathcal{O}(\mathcal{X}_{\infty,w})$ (compatible under changing w) such that $\gamma^*\mathfrak{z} = \frac{a\mathfrak{z}+c}{b\mathfrak{z}+d}$ for all $\gamma \in K_0(p)$. For any κ as above and any $w \gg_\kappa 0$, the space

$$M_{\kappa,w}(N) := \left\{ f \in \mathcal{O}(\mathcal{X}_{\infty,w}) \otimes_{\mathbb{Q}_p} L \mid \kappa(b\mathfrak{z}+d) \cdot \gamma^* f = f \ \forall \gamma \in K_0(p) \right\}$$

is well-defined, and the module $M^\dagger_\kappa(N) = \lim_{w\to\infty} M_{\kappa,w}(N)$ is canonically isomorphic with the modules of overconvergent modular forms of weight κ and tame level N defined by Andreatta-Iovita-Stevens and Pilloni.

Thus, in our approach, $\mathcal{X}_{\infty,w}$ plays the role of $\widetilde{Y_1(N)}$ and the "fundamental period" $\mathfrak{z}$ defined below plays the role of z. Key to this theorem is that ω is trivialized over any $\mathcal{X}_{\infty,w}$ (by one of the "fake" Hasse invariants constructed in [Sch4]), which is similar to the fact that ω is trivialized over $\mathfrak{h}$. One reason why the latter is true is because the complex period map is not surjective onto $\mathbb{P}^1_{\mathbb{C}}$. By contrast, the p-adic period morphism $\pi_{\mathrm{HT}} : \mathcal{X}_\infty \to \mathbb{P}^1$ constructed in [Sch4] *is* surjective, but becomes non-surjective when restricted to any $\mathcal{X}_{\infty,w}$. This is the reason that our description works for overconvergent modular forms but not for classical modular forms.

The second goal of the paper is to apply our "explicit" point of view to redefine and analyze the overconvergent Eichler-Shimura map of [AIS2], which compares overconvergent modular symbols to overconvergent modular forms. Our perspective gives a short and transparent definition of these maps, and (we believe) clarifies the ideas involved. We also make use of certain new filtrations on overconvergent distribution modules to obtain a more "global" point of view on the Eichler-Shimura maps: after first defining them in the setting of Coleman families, we glue them into a morphism of coherent sheaves over the whole eigencurve.

We now describe these results in more detail.

1.1. Perfectoid modular curves and w-ordinarity. Let Y_n denote the modular curve over $\mathbb{Q}_p$ with $Y_n(S)$ parametrizing elliptic curves E/S with a point $P \in E(S)[N]$ of exact order N together with an isomorphism

$$\alpha_n : (\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E(S)[p^n].$$

Let X_n denote the usual compactification of Y_n, and let $\mathcal{Y}_n \subseteq \mathcal{X}_n$ be the associated adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. These form compatible inverse systems

$(\mathcal{X}_n)_{n\geq 1}$, $(\mathcal{Y}_n)_{n\geq 1}$ with compatible actions of $\mathrm{GL}_2(\mathbb{Q}_p)$. Fundamental to all our considerations is Scholze's construction of the infinite level modular curves $\mathcal{Y}_\infty \sim \varprojlim_n \mathcal{Y}_n$, $\mathcal{X}_\infty \sim \varprojlim_n \mathcal{X}_n$ and the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant Hodge-Tate period map $\pi_{\mathrm{HT}} : \mathcal{X}_\infty \to \mathbb{P}^1$ (a morphism of adic spaces over $\mathbb{Q}_p$).

Let us say a few more words about π_{HT}. Let $C/\mathbb{Q}_p$ be a complete algebraically closed field extension with ring of integers $\mathcal{O}_C$, and let A be an abelian variety over C of dimension g, with dual $A^\vee$. Set $\omega_A = H^0(A, \Omega^1_{A/C}) \cong \mathrm{Hom}_C(\mathrm{Lie}\, A, C)$, a C-vector space of rank g. Then we have a natural linear map $\mathrm{HT}_A : T_pA \otimes_{\mathbb{Z}_p} C \twoheadrightarrow \omega_{A^\vee}$, the *Hodge-Tate map* of A, which fits into a short exact sequence

$$0 \longrightarrow (\mathrm{Lie}\, A)(1) \overset{\mathrm{HT}^\vee_{A^\vee}(1)}{\longrightarrow} \mathrm{Hom}(T_pA^\vee, C(1)) = T_pA \otimes_{\mathbb{Z}_p} C \overset{\mathrm{HT}_A}{\longrightarrow} \omega_{A^\vee} \longrightarrow 0$$

where $-(1)$ denotes a Tate twist. If E/C is an elliptic curve and $\alpha : \mathbb{Z}_p^2 \overset{\sim}{\to} T_pE$ is a trivialization, then (E, α) defines a point in $\mathcal{Y}_\infty(C, \mathcal{O}_C)$ and π_{HT} sends (E, α) to the line $(\alpha \otimes 1)^{-1}(\mathrm{Lie}\, E) \subseteq C^2$.

Our first key definition is a new gauge for the ordinarity of an abelian variety, defined in terms of the Hodge-Tate map HT_A. Let $F_A = \mathrm{HT}_A(T_pA \otimes_{\mathbb{Z}_p} \mathcal{O}_C)$. This is an $\mathcal{O}_C$-lattice inside $\omega_{A^\vee}$.

Definition 1.2. *Let $w \in \mathbb{Q}_{>0}$.*

1. *An abelian variety A/C is* w-ordinary *if there is a basis $b_1, \ldots, b_{2g}$ of T_pA such that $\mathrm{HT}_A(b_i) \in p^w F_A$ for all $1 \leq i \leq g$.*
2. *For A/C w-ordinary and $0 < n < w+1$, the* pseudocanonical subgroup H_n of level n *is defined to be the kernel of the natural map*
$$A[p^n](C) \to F_A \otimes_{\mathcal{O}_C} \mathcal{O}_C/p^{\min(n,w)}\mathcal{O}_C$$
induced by HT_A.
3. *For A/C w-ordinary, a trivialization $\alpha : \mathbb{Z}_p^{2g} \overset{\sim}{\to} T_pA$ is* strict *if $\alpha(e_1), \ldots, \alpha(e_g) \bmod p \in A[p](C)$ form a basis for the pseudocanonical subgroup of level one.*

Let u and v be the homogeneous coordinate with respect to the standard basis $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ of $\mathbb{Q}_p^2$. We define $z = -v/u$, a coordinate function on $\mathbb{P}^1$. For $w \in \mathbb{Q}_{>0}$ we let $\mathbb{P}^1_w \subseteq \mathbb{P}^1$ be the locus

$$\mathbb{P}^1_w = \{z \mid \inf_{a \in p\mathbb{Z}_p} |z - a| \leq |p|^w\}.$$

For $\gamma \in K_0(p)$ one sees that $\gamma^* z = \frac{az+c}{bz+d}$ and that $\mathbb{P}^1_w$ is a $K_0(p)$-stable affinoid, where $K_0(p)$ is the usual Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. Define $\mathcal{X}_{\infty,w} = \pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1_w)$ and $\mathcal{Y}_{\infty,w} = \mathcal{X}_{\infty,w} \cap \mathcal{Y}_\infty$. These loci are $K_0(p)$-stable. Finally, define the *fundamental period* $\mathfrak{z} = \pi_{\mathrm{HT}}^* z \in \mathcal{O}^+_{\mathcal{X}_\infty}(\mathcal{X}_{\infty,w})$.

Theorem 1.3. *A point $(E, \alpha) \in \mathcal{Y}_\infty(C, \mathcal{O}_C)$ is contained in $\mathcal{X}_{\infty,w}$ if and only if E is w-ordinary and α is strict. Furthermore, $\mathcal{X}_{\infty,w}$ is the preimage of a* *canonical affinoid $\mathcal{X}_w \subseteq \mathcal{X}_{K_0(p)}$, and $\mathcal{X}_{\infty,w}$ is affinoid perfectoid. The $(C, \mathcal{O}_C)$-points of $\mathcal{Y}_w = \mathcal{X}_w \cap \mathcal{Y}_{K_0(p)}$ are the pairs (E, H) where E is w-ordinary and H is the pseudocanonical subgroup of level one, and $(\mathcal{X}_w)_w$ is a cofinal set of strict neighbourhoods of the ordinary multiplicative locus in $\mathcal{X}_{K_0(p)}$.*

Taking these infinite-level objects as our basic ingredients, we are able to give a short definition (Definition 2.18) of sheaves of overconvergent modular forms $\omega^{\dagger}_{\kappa,w}$ on $\mathcal{X}_w$ whose global sections yield the module $M_{\kappa,w}(N)$ of Theorem 1.1. This construction also works in families of weights. As a guide to our constructions so far, we offer the following table of analogies:

$\mathbb{C}$	$\mathbb{Q}_p$
$Y = Y_1(N)(\mathbb{C})$, a complex analytic space	$\mathcal{Y}_w \subset \mathcal{Y}_{K_0(p)}$, an adic space
$E/\mathbb{C}$ an elliptic curve	E/C a w-ordinary elliptic curve
(E,β), $\beta = \{\beta_1, \beta_2\} \in H_1(E(\mathbb{C}), \mathbb{Z})$ an oriented basis	(E,α), $\alpha : \mathbb{Z}_p^2 \xrightarrow{\sim} T_pE$ a strict trivialization
$(E,\beta) \in \tilde{Y}$, the universal cover of Y	$x = (E,\alpha) \in \mathcal{Y}_{\infty,w}(C) \subset \mathcal{X}_{\infty,w}(C)$
$\Gamma_1(N) \circlearrowright \tilde{Y} \xrightarrow{\sim} \mathfrak{h} \subset \mathbb{P}^1_{/\mathbb{C}}$	$\pi_{\mathrm{HT}} : \mathcal{X}_{\infty,w} \twoheadrightarrow \mathbb{P}^1_w \subset \mathbb{P}^1_{/\mathbb{Q}_p} \circlearrowleft K_0(p)$
$\tilde{Y} = \mathfrak{h}$ and $\overline{\mathfrak{h}}$ are contractible	$\mathcal{X}_{\infty,w}$ is affinoid perfectoid
z, the coordinate on $\mathfrak{h}$	$\mathfrak{z} \in \mathcal{O}^+(\mathcal{X}_{\infty,w})$, the fundamental period
$z = z(E,\beta)$ characterized by $\int_{\beta_1} \eta = z \int_{\beta_2} \eta$	$\mathfrak{z}(x) \in C$ characterized by $\mathrm{HT}_E(\alpha(e_1)) = \mathfrak{z}(x)\mathrm{HT}_E(\alpha(e_2))$
$\int_{\beta_2} \eta_{\mathrm{can}} = 1$	$\mathfrak{s} = \mathrm{HT}_E(\alpha(e_2))$
$\eta_{\mathrm{can}}(\gamma z) = \frac{1}{cz+d}\eta_{\mathrm{can}}(z)$	$\gamma^*\mathfrak{s} = (b\mathfrak{z}+d)\mathfrak{s}$
$(cz+d)^k \in \mathcal{O}(\mathfrak{h})$	$\kappa(b\mathfrak{z}+d) \in \mathcal{O}(\mathcal{X}_{\infty,w}) \otimes_{\mathbb{Q}_p} L$ $(w \geq w_\lambda)$
$M_k(N) = \{f \in \mathcal{O}(\mathfrak{h}) \mid f(gz) = (cz+d)^k f(z)\}$	$M_{\kappa,w}(N) = \left\{ \begin{array}{c} f \in \mathcal{O}(\mathcal{X}_{\infty,w}) \otimes_{\mathbb{Q}_p} L, \\ \kappa(b\mathfrak{z}+d)\gamma^* f = f \end{array} \right\}$
	$M^{\dagger}_{\kappa} = \lim_{w\to\infty} M_{\kappa,w}$

Table 1.1: Analogies.

As we have already mentioned, sheaves of overconvergent modular forms have been constructed previously by [AIS1] and [Pil], and their constructions (which appear slightly different on the surface) are known to give equivalent notions of overconvergent modular forms. We prove (Theorem 2.33) that our definition is also equivalent to these previous constructions. As emphasized to us by a referee, we remark that our definition of the sheaf $\omega^{\dagger}_{\kappa,w}$ is essentially as a line bundle on $\mathcal{X}_{\infty,w}$ with a descent datum to $\mathcal{X}_w$. However pro-étale descent of vector bundles is not effective in general (this follows for example from [KL2, Example 8.1.6]). In our case we show with relative ease that the resulting sheaf is indeed a line bundle.

1.2. Modular curves vs. Shimura curves. While we have written this introduction so far in the setting of modular curves, we have chosen to work with compact Shimura curves associated with an indefinite quaternion division algebra $B/\mathbb{Q}$ split at (our fixed) p in the body of the paper. There are two reasons for this. The first reason is that the local p-adic geometry of these Shimura curves is entirely analogous to the local p-adic geometry of modular curves, but the global geometry is simpler without the presence of boundary divisors, compactifications, and their attendant complications. We believe that working in a boundaryless setting helps to clarify the point of view adopted in this paper.

Having said this, many of our ideas extend to the case of modular curves. In particular, all the definitions and results in §2 have exact analogues for classical modular curves, and Theorems 1.1 and 1.3 are true as stated. The techniques we use for Shimura curves work over the open modular curve, and one may extend over the boundary using "soft" techniques (one does not need the advanced results of [Sch4, §2]).

The second reason is that our point of view on the overconvergent Eichler-Shimura map does *not* immediately generalize to modular curves. While we believe that the underlying philosophy should adapt, there are certain technical aspects of the construction (in particular Proposition 4.4) which seem difficult to adapt. As far as we can tell, this is a purely technical issue with the pro-étale site as defined in [Sch3]. In work in progress ([DT]), Diao and Tan are developing a logarithmic version of the pro-étale site, and we believe that our constructions would work well in that setting.

1.3. The overconvergent Eichler-Shimura map. From now on we work with Shimura curves attached to an indefinite quaternion division algebra $B/\mathbb{Q}_p$ split at p and use the same notation that we previously used for modular curves.

After Coleman's construction of Coleman families and the globalization of these to the Coleman-Mazur eigencurve ([Clm, CM]), different constructions of families of finite slope eigenforms and eigencurves for $\mathrm{GL}_{2/\mathbb{Q}}$ were given by Stevens ([Ste], completed by Bellaïche [Bel]) and Emerton ([Eme]). These constructions give the same eigencurve. Roughly speaking, each approach first constructs a p-adic Banach/Fréchet space (or many such spaces) of "overconvergent" objects interpolating modular forms/cohomology classes, and then creates a geometric object out of this (these) space(s). All spaces have actions of certain Hecke algebras and the fact that these give the same eigencurves amounts to saying that they contain the same finite slope systems of Hecke eigenvalues. However, not much is known about the direct relation between these spaces. One can rephrase the problem in the following way: each construction of the eigencurve remembers the spaces it came from in the form of a coherent sheaf on it, and one may ask if there are relations between these sheaves.

In [AIS2], Andreatta, Iovita and Stevens study the relationship between overconvergent modular forms and overconvergent modular symbols. While Coleman's overconvergent modular forms p-adically interpolate modular forms, Stevens's overconvergent modular symbols interpolate classical modular symbols, i.e. classes in the singular cohomology groups $H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1)$, where $\mathcal{H}^1$ is the first relative singular cohomogy of the universal elliptic curve. Classical Eichler-Shimura theory, which one may view as an elaboration of Hodge Theory for these particular varieties and coefficient systems, gives a

Hecke-equivariant isomorphism

$$H^1(Y_1(N), \mathrm{Sym}^{k-2}\,\mathcal{H}^1) \otimes_{\mathbb{Z}} \mathbb{C} \cong M_k \oplus S_k$$

where M_k is the space of weight k and level N modular forms and S_k is its subspace of cusp forms. Faltings ([Fal]) constructed a p-adic Hodge-theoretic analogue of this isomorphism, replacing singular cohomology with étale cohomology. This construction was then adapted to the overconvergent context in [AIS2].

Let us describe these ideas and our work in more detail. We refer to the main body of the paper for exact definitions. Recall ([AS, Han1]) the *overconvergent distribution modules* $\mathbf{D}_\kappa^s$ where κ is a character of $\mathbb{Z}_p^\times$ as above. It may be interpreted as a local system on $X_{K_0(p)}(\mathbb{C})$. The singular cohomology $H^1(X_{K_0(p)}(\mathbb{C}), \mathbf{D}_\kappa^s)$ is the space of overconvergent modular symbols. Following [Han2], we construct a filtration on the integral distribution module $\mathbf{D}_\kappa^{s,\circ}$ for which the corresponding topology is profinite. From this one gets a sheaf on the pro-étale site of $\mathcal{X}_{K_0(p)}$ whose cohomology is isomorphic to $H^1(X_{K_0(p)}(\mathbb{C}), \mathbf{D}_\kappa^s)$, but also carries a Galois action. To compare this to overconvergent modular forms of weight κ, one introduces a "fattened" version $\mathcal{O}\mathbf{D}_\kappa^s$ of $\mathbf{D}_\kappa^s$ which has the explicit description

$$V \mapsto (\mathbf{D}_\kappa \widehat{\otimes} \widehat{\mathcal{O}}_{\mathcal{X}_{K_0(p)}}(V_\infty))^{K_0(p)}$$

for $V \in \mathcal{X}_{K_0(p),\text{proét}}$ quasi-compact and quasi-separated (qcqs), where $V_\infty := V \times_{\mathcal{X}_{K_0(p)}} \mathcal{X}_\infty$ and $\widehat{\mathcal{O}}_{\mathcal{X}_{K_0(p)}}$ is the completed structure sheaf on $\mathcal{X}_{K_0(p),\text{proét}}$. After restricting to $\mathcal{X}_w$ it turns out to be easy to give a morphism to the completed version $\widehat{\omega}_{\kappa,w}^\dagger$: for $V \in \mathcal{X}_{w,\text{proét}}$ qcqs there is a $K_0(p)$-equivariant morphism

$$\mathbf{D}_\kappa \widehat{\otimes} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \otimes_{\mathbb{Q}_p} L$$

given on elementary tensors by

$$\mu \otimes f \mapsto \mu(\kappa(1+\mathfrak{z}x))f.$$

The formula is heavily inspired by a formula of Stevens for the comparison map between overconvergent distributions and polynomial distributions (whose cohomology computes classical modular symbols) which does not seem to be used a lot in the literature (see the paragraph before Definition 3.2). One then passes to $K_0(p)$-invariants to obtain the desired morphism of sheaves. This is our analogue of the maps denoted by $\delta_\kappa^\vee(w)$ in [AIS2]. It induces a map on cohomology groups over $\mathbb{C}_p$ which gives the desired map of spaces using that $H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \widehat{\omega}_{\kappa,w}^\dagger) \cong H^0(\mathcal{X}_{w,\mathbb{C}_p}, \omega_{\kappa,w}^\dagger \otimes_{\mathcal{O}_{\mathcal{X}_w}} \Omega^1_{\mathcal{X}_w})$. The strategy is the same as in [AIS2], except that they work with the so-called Faltings site instead of the pro-étale site. It is the presence of infinite level Shimura curves in the pro-étale sites of finite level Shimura curves that accounts for the clean explicit formulas we obtain: they provide the correct "local coordinates" for the problem at hand.

We analyze these maps by carrying out the above constructions in families of weights (as in [AIS2]). To define Galois actions one needs to work with families parametrized by certain affine formal schemes instead of the more commonly used affinoid rigid spaces. Whereas the filtrations defined in [AIS2] only work when the formal scheme is an open unit disc near the center of weight space, our filtrations are defined over arbitrary Spf R where R is finite over $\mathbb{Z}_p[[X_1, .., X_d]]$ for some d. This enables us to glue the morphisms for different families of weights into a morphism of sheaves over the whole eigencurve.

Denote by $\mathcal{C}$ the eigencurve and let $\mathcal{C}_{\mathbb{C}_p}$ be its base change to $\mathbb{C}_p$. It carries coherent sheaves $\mathbf{V}$, resp. $\mathcal{M}^\dagger$, coming from overconvergent modular symbols resp. forms. We denote by $\mathbf{V}_{\mathbb{C}_p}$, resp. $\mathcal{M}^\dagger_{\mathbb{C}_p}$, their base changes to $\mathbb{C}_p$, which may also be viewed as sheaves of $\mathcal{O}_\mathcal{C}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$-modules on $\mathcal{C}$. In the latter point of view, one may think of $\mathbb{C}_p$ as a (rather primitive) period ring. After gluing, the overconvergent Eichler-Shimura map is a morphism $\mathcal{ES} : \mathbf{V}_{\mathbb{C}_p} \to \mathcal{M}^\dagger_{\mathbb{C}_p}(-1)$ of sheaves.

Theorem 1.4 (Theorem 5.11, Theorem 5.14). *Let $\mathcal{C}^{\mathrm{sm}}$ be the smooth locus of $\mathcal{C}$.*

(1) *$\mathbf{V}$ and $\mathcal{M}^\dagger$ are locally free over $\mathcal{C}^{\mathrm{sm}}$, and the kernel $\mathcal{K}$ and image $\mathcal{I}$ of $\mathcal{ES}$ are locally projective sheaves of $\mathcal{O}_{\mathcal{C}^{\mathrm{sm}}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$-modules (or equivalently locally free sheaves on $\mathcal{C}^{\mathrm{sm}}_{\mathbb{C}_p}$). The support of $\mathcal{M}^\dagger_{\mathbb{C}_p}(-1)/\mathcal{I}$ on $\mathcal{C}_{\mathbb{C}_p}$ is Zariski closed of dimension 0.*

(2) *Let $\epsilon_{\mathcal{C}^{\mathrm{sm}}}$ be the character of $G_{\mathbb{Q}_p}$ defined by the composition*

$$G_{\mathbb{Q}_p} \xrightarrow{\epsilon} \mathbb{Z}_p^\times \xrightarrow{\chi_\mathcal{W}} \mathcal{O}_\mathcal{W}^\times \longrightarrow (\mathcal{O}^\times_{\mathcal{C}^{\mathrm{sm}}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)^\times$$

where ϵ is the p-adic cyclotomic character of $G_{\mathbb{Q}_p}$ and $\chi_\mathcal{W}$ is the universal character of $\mathbb{Z}_p^\times$. Then the semilinear action of $G_{\mathbb{Q}_p}$ on the module $\mathcal{K}(\epsilon^{-1}_{\mathcal{C}^{\mathrm{sm}}})$ is trivial.

(3) *The exact sequence*

$$0 \to \mathcal{K} \to \mathbf{V}_{\mathbb{C}_p} \to \mathcal{I} \to 0$$

is locally split. Zariski generically, the splitting may be taken to be equivariant with respect to both the Hecke- and $G_{\mathbb{Q}_p}$-actions, and such a splitting is unique.

These are stronger analogues of results in [AIS2], where the authors prove the analogous results for modular curves in some small (unspecified) open neighbourhood of the set of non-critical classical points.

We believe that our perspective on overconvergent modular forms and the overconvergent Eichler-Shimura map should generalize to higher-dimensional Shimura varieties. In particular, it should be reasonably straightforward to adapt the methods of this paper to (the compact versions of) Hilbert modular varieties.

1.4. Notation, conventions and an outline of the paper. Throughout this text, we let p denote a fixed prime.

For the purposes of this paper, a *small $\mathbb{Z}_p$-algebra* is a ring R which is reduced, p-torsion-free, and finite as a $\mathbb{Z}_p[[X_1, ..., X_d]]$-algebra for some (unspecified) $d \geq 0$. Any such R carries a canonical adic profinite topology, and is also complete for its p-adic topology. For convenience we will fix a choice of ideal $\mathfrak{a} = \mathfrak{a}_R$ defining the profinite topology (a "canonical" example of such a choice is the Jacobson radical). All constructions made using this choice will be easily verified to not depend upon it.

We will need various completed tensor product constructions, some of which are non-standard. We will need to take a form of completed tensor product between small $\mathbb{Z}_p$-algebras and various Banach spaces or other $\mathbb{Z}_p$-modules. These will always be denoted by an undecorated completed tensor product $\widehat{\otimes}$. We explain our conventions for the unadorned $\widehat{\otimes}$ in Convention 2.2 and Definitions 6.3 and 6.6 below. Any adorned $\widehat{\otimes}_A$ is a standard one, with respect to the natural topology coming from A.

We will use Huber's adic spaces as our language for non-archimedean analytic geometry in this paper. In particular, a "rigid analytic variety" will refer to the associated adic space, and all open subsets and open covers are open subsets resp. open covers of the adic space (i.e. we drop the adjective "admissible" used in rigid analytic geometry). The pro-étale site of [Sch3] is key to our constructions; we will freely use notation and terminology from that paper. For perfectoid spaces we use the language of [KL1] for simplicity (e.g. we speak of perfectoid spaces over $\mathbb{Q}_p$), but any perfectoid space appearing is a perfectoid space is the sense of [Sch1] (i.e. it lives over a perfectoid field).

Let us finish the introduction by briefly outlining the contents of the paper. In §2 we give our new definitions of sheaves of overconvergent modular forms in families and prove their basic properties, including a comparison with the definitions of [AIS1, Pil]. In §3 we recall the basic definitions from the theory of overconvergent modular symbols and define the filtrations mentioned above. We make some technical adjustments when defining slope decompositions. In particular, we do not need the concept of a weak orthonormal basis used in [AIS2]; all slope decompositions can be defined using standard orthonormal bases and formal operations. In §4 we define our overconvergent Eichler-Shimura maps, and §5 glues them over the eigencurve and proves the properties stated above.

The paper concludes with an appendix, collecting various technical results and definitions that are needed in the main text; some of these results may be of independent interest. In §6.1 we define our non-standard completed tensor products and prove some basic properties. While we only need this $\widehat{\otimes}$ for small $\mathbb{Z}_p$-algebras R, it turns out that the ring structure only serves to obfuscate the situation. Accordingly, we define $\widehat{\otimes}$ for a class of $\mathbb{Z}_p$-module that we call *profinite flat*. Throughout the text we will need to consider sheaves of rings like $\mathcal{O}_X\widehat{\otimes}R$ and $\widehat{\mathcal{O}}_X\widehat{\otimes}R$ on X where X is a rigid space and R is a small $\mathbb{Z}_p$-algebra. We prove some technical facts about these sheaves of rings and their modules in §6.2-6.3. Finally §6.4 discusses quotients of rigid spaces by finite group actions, proving a standard existence result that we were not able to locate in the literature.

Acknowledgements. This collaboration grew out of discussions at the conference in Lyon in June 2013 between P.C. and C.J, and then conversations in Jussieu in October 2013 between P.C. and D.H. We thank the organizers of the Lyon conference for the wonderful gathering. All of the authors benefited from shorter or longer stays at Columbia University and the University of Oxford. D.H. would also like to thank Johan de Jong, Michael Harris and Shrenik Shah for some helpful conversations, and C.J. would like to thank Hansheng Diao for conversations relating to this work and [DT]. The authors would like to thank Judith Ludwig for pointing out a gap in our original proof of Proposition 6.24, as well as the anonymous referees for their comments and corrections.

P.C. was partially funded by EPSRC grant EP/L005190/1. D.H. received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 290766 (AAMOT). C. J. was supported by EPSRC Grant EP/J009458/1 and NSF Grants 093207800 and DMS-1128155 during the work on this paper.

2. Overconvergent modular forms

2.1. Weights and characters. In section we recall some basic notions about weights. We may define weight space as the functor from complete affinoid $(\mathbb{Z}_p, \mathbb{Z}_p)$-algebras (A, A^+) to abelian groups given by

$$(A, A^+) \mapsto \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, A^\times).$$

This functor is representable by $(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])$. The proof is well known. The key fact that makes the arguments work in this generality is that $A^\circ/A^{\circ\circ}$ is a reduced ring of characteristic p, where $A^{\circ\circ}$ is the set of topologically nilpotent elements in A. We define $\mathfrak{W} = \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ and let $\mathcal{W} = \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{rig}}$ be the associated rigid analytic weight space, with its universal character $\chi_{\mathcal{W}} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{W})^\times$.

We embed $\mathbb{Z}$ into $\mathcal{W}$ by sending k to the character $\chi_k(z) = z^{k-2}$.

Definition 2.1. (1) *A small weight is a pair $\mathcal{U} = (R_{\mathcal{U}}, \chi_{\mathcal{U}})$ where $R_{\mathcal{U}}$ is a small $\mathbb{Z}_p$-algebra and $\chi_{\mathcal{U}} : \mathbb{Z}_p^\times \to R_{\mathcal{U}}^\times$ is a continuous character such that $\chi_{\mathcal{U}}(1+p) - 1$ is topologically nilpotent in $R_{\mathcal{U}}$ with respect to the* p-adic *topology.*

(2) *An affinoid weight is a pair $\mathcal{U} = (S_{\mathcal{U}}, \chi_{\mathcal{U}})$ where $S_{\mathcal{U}}$ is a reduced Tate algebra over $\mathbb{Q}_p$ topologically of finite type and $\chi_{\mathcal{U}} : \mathbb{Z}_p^\times \to S_{\mathcal{U}}^\times$ is a continuous character.*

(3) *A weight is a pair $\mathcal{U} = (A_{\mathcal{U}}, \chi_{\mathcal{U}})$ which is either a small weight or an affinoid weight.*

We shall sometimes abbreviate $A_{\mathcal{U}}$ by A when $\mathcal{U}$ is clear from context. In either case, we can make $A_{\mathcal{U}}[\frac{1}{p}]$ into a uniform $\mathbb{Q}_p$-Banach algebra by letting $A_{\mathcal{U}}^{\circ}$ be the unit ball and equipping it with the corresponding spectral norm. We will denote this norm by $|\cdot|_{\mathcal{U}}$. Note then that there exists a smallest integer $s \geq 0$ such that $|\chi_{\mathcal{U}}(1+p)-1|_{\mathcal{U}} < p^{-\frac{1}{p^s(p-1)}}$. We denote this s by $s_{\mathcal{U}}$. When $\mathcal{U}$ is small, the existence uses that $\chi_{\mathcal{U}}(1+p)-1$ is *p-adically* topologically nilpotent. We will also make the following convention:

Convention 2.2. *Let $\mathcal{U}$ be a weight and let V be a Banach space over $\mathbb{Q}_p$. We define $V\widehat{\otimes}A_{\mathcal{U}}$ as follows:*

(1) *If $\mathcal{U}$ is small, then $V\widehat{\otimes}R_{\mathcal{U}}$ is a* mixed completed tensor product *in the sense of the appendix.*
(2) *If $\mathcal{U}$ is affinoid, then $V\widehat{\otimes}S_{\mathcal{U}} := V\widehat{\otimes}_{\mathbb{Q}_p}S_{\mathcal{U}}$.*

Remark 2.3. *Many (though not all) of the results in this paper involving a choice of some weight $\mathcal{U}$ make equally good sense whether $\mathcal{U}$ is small or affinoid. In our proofs of these results, we typically give either a proof which works uniformly in both cases, or a proof in the case where $\mathcal{U}$ is small, which is usually more technically demanding.*

When $\mathcal{U} = (R_{\mathcal{U}}, \chi_{\mathcal{U}})$ is a small weight the universal property of weight space gives us a canonical morphism

$$\mathrm{Spf}(R_{\mathcal{U}}) \to \mathfrak{W},$$

which induces a morphism

$$\mathrm{Spf}(R_{\mathcal{U}})^{\mathrm{rig}} \to \mathcal{W}.$$

When $\mathcal{U} = (S_{\mathcal{U}}, \chi_{\mathcal{U}})$ is an affinoid weight we get an induced morphism

$$\mathrm{Spa}(S_{\mathcal{U}}, S_{\mathcal{U}}^{\circ}) \to \mathcal{W}.$$

We make the following definition:

Definition 2.4. (1) *A small weight $\mathcal{U} = (R_{\mathcal{U}}, \chi_{\mathcal{U}})$ is said to be* open *if $R_{\mathcal{U}}$ is normal and the induced morphism $\mathrm{Spf}(R_{\mathcal{U}})^{\mathrm{rig}} \to \mathcal{W}$ is an open immersion.*
(2) *An affinoid weight $\mathcal{U} = (S_{\mathcal{U}}, \chi_{\mathcal{U}})$ is said to be* open *if the induced morphism $\mathrm{Spa}(S_{\mathcal{U}}, S_{\mathcal{U}}^{\circ}) \to \mathcal{W}$ is an open immersion.*
(3) *A weight $\mathcal{U} = (A_{\mathcal{U}}, \chi_{\mathcal{U}})$ is said to be* open *if it is either a small open weight or an affinoid open weight.*

Note that if $\mathcal{U}$ is a small weight such that $\mathcal{U}^{rig} \to \mathcal{W}$ is an open immersion, then the normalization of $\mathcal{U}$ is a small open weight.

Let B be any uniform $\mathbb{Q}_p$-Banach algebra. Let us say that a function $f : \mathbb{Z}_p \to B$ is s-analytic for some nonnegative integer s if, for any fixed $a \in \mathbb{Z}_p$, there is some $\varphi_{f,a} \in B\langle T\rangle$ such that $\varphi_{f,a}(x) = f(p^s x + a)$ for all $x \in \mathbb{Z}_p$. In other words, f can be expanded in a convergent power series on any ball of radius p^{-s}. This is naturally a Banach space which we denote by $\mathcal{C}^{s-\mathrm{an}}(\mathbb{Z}_p, B)$.

Theorem 2.5 (Amice). *The polynomials* $e_j^s(x) = \lfloor p^{-s}j \rfloor! \left(\begin{array}{c} x \\ j \end{array} \right)$ *form an orthonormal basis of* $\mathcal{C}^{s-\mathrm{an}}(\mathbb{Z}_p, B)$ *for any uniform* $\mathbb{Q}_p$*-Banach algebra* B*. Furthermore, we have* $e_j^s(\mathbb{Z}_p + p^s B^\circ) \subseteq B^\circ$.

Proof. This is well known, see e.g. [Col, Theorem 1.7.8] and its proof. □

We can now prove that characters extend over a bigger domain.

Proposition 2.6. *Let* $\mathcal{U} = (A_{\mathcal{U}}, \chi_{\mathcal{U}})$ *be a weight and let* B *be any uniform* $\mathbb{Q}_p$*-Banach algebra. Then for any* $s \in \mathbb{Q}_{>0}$ *such that* $s \geq s_{\mathcal{U}}$*,* $\chi_{\mathcal{U}}$ *extends canonically to a character*

$$\chi_{\mathcal{U}} : B_s^\times \to (A_{\mathcal{U}}^\circ \widehat{\otimes}_{\mathbb{Z}_p} B^\circ)^\times \subset (A_{\mathcal{U}} \widehat{\otimes} B)^\times,$$

where $B_s^\times := \mathbb{Z}_p^\times \cdot (1 + p^{s+1} B^\circ) \subseteq (B^\circ)^\times$ *and* $p^{s+1}B^\circ$ *is shorthand for* $\{b \in B^\circ \mid |b| \leq p^{-s-1}\}$ *(where* $| - |$ *is the spectral norm on* B*).*

Proof. Without loss of generality we may assume that s is an integer (e.g by replacing s with $\lfloor s \rfloor$). We may decompose any $b \in B_s^\times$ uniquely as $b = \omega(b) \langle b \rangle$, with $\omega(b) \in \mu_{p-1}$ and $\langle b \rangle \in 1 + p\mathbb{Z}_p + p^{s+1}B^\circ$. We will show that for any $s \geq s_{\mathcal{U}}$ and $b \in B_s^\times$, the individual terms of the series

$$\begin{aligned} f(b) \quad &= \quad \chi_{\mathcal{U}}(\omega(b)) \sum_{j=0}^{\infty} (\chi_{\mathcal{U}}(1+p) - 1)^j \left(\begin{array}{c} \frac{\log\langle b \rangle}{\log(1+p)} \\ j \end{array} \right) \\ \text{“} = \text{”} \quad &\quad \chi_{\mathcal{U}}(\omega(b)) \cdot \chi(1+p)^{\frac{\log\langle b \rangle}{\log 1+p}} \end{aligned}$$

lie in $A_{\mathcal{U}}^\circ \otimes_{\mathbb{Z}_p} B^\circ$ and tend to zero p-adically, so this series converges to an element of $A_{\mathcal{U}}^\circ \widehat{\otimes}_{\mathbb{Z}_p} B^\circ$ and *a fortiori* to an element of $A_{\mathcal{U}} \widehat{\otimes} B$. We claim this series defines a canonical extension of $\chi_{\mathcal{U}}$.

Using the well known formula for the p-adic valuation of factorials we see that $v_p(\lfloor p^{-s}j \rfloor!) \leq \frac{j}{p^s(p-1)}$. In particular, writing

$$\chi_{\mathcal{U}}(\omega(b)) \sum_{j=0}^{\infty} (\chi_{\mathcal{U}}(1+p) - 1)^j \left(\begin{array}{c} x \\ j \end{array} \right) = \chi_{\mathcal{U}}(\omega(b)) \sum_{j=0}^{\infty} \frac{(\chi_{\mathcal{U}}(1+p) - 1)^j}{\lfloor p^{-s}j \rfloor!} e_j^s(x),$$

our assumption on s implies that $|\chi_{\mathcal{U}}(1+p) - 1|_{\mathcal{U}} < p^{-\frac{1}{p^s(p-1)}}$, so we see that this series converges to an element of $A_{\mathcal{U}}^\circ \widehat{\otimes}_{\mathbb{Z}_p} B^\circ$ for any $x \in B$ such that $e_j^s(x) \in B^\circ$ for all j. By Theorem 2.5 it then suffices to verify that if $b \in B_s^\times$ then $x = \frac{\log\langle b \rangle}{\log 1+p} \in \mathbb{Z}_p + p^s B^\circ$. But we know that the function $\eta(b) = \frac{\log\langle b \rangle}{\log 1+p}$ defines a homomorphism from $1 + p\mathbb{Z}_p + p^{s+1}B^\circ$ to $\mathbb{Z}_p + p^s B^\circ$, so we are done. The character property follows by calculating directly from the definitions. Finally, to show that this character extends $\chi_{\mathcal{U}}$, note that for $b \in \mu_{p-1} \times (1+p)^{\mathbb{Z}_{\geq 0}}$, $f(b)$ becomes a finite sum which equals $\chi_{\mathcal{U}}(b)$ by the binomial theorem, so $f|_{\mathbb{Z}_p^\times} = \chi_{\mathcal{U}}$ by continuity. □

2.2. Shimura curves. We write C for an algebraically closed field containing $\mathbb{Q}_p$, complete with respect to a valuation $v : C \to \mathbb{R} \cup \{+\infty\}$ with $v(p) = 1$ (so v is nontrivial), and we write $\mathcal{O}_C$ for the valuation subring. Fix an embedding $\mathbb{C}_p \subseteq C$. We fix a compatible set of p^n-th roots of unity in $\mathbb{C}_p$ and use this choice throughout to ignore Tate twists (over any C). We let B denote an indefinite non-split quaternion algebra over $\mathbb{Q}$, with discriminant d which we assume is *prime to* p. We fix a maximal order $\mathcal{O}_B$ of B as well as an isomorphism $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$, and write G for the algebraic group over $\mathbb{Z}$ whose functor of points is

$$R \mapsto (\mathcal{O}_B \otimes_{\mathbb{Z}} R)^{\times},$$

where R is any ring. We fix once and for all a neat compact open subgroup $K^p \subseteq G(\widehat{\mathbb{Z}}^p)$ such that $K^p = \prod_{\ell \neq p} K_\ell$ for compact open subgroups $K_\ell \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ and (for simplicity) $\det(K^p) = (\widehat{\mathbb{Z}}^p)^{\times}$.

Recall (e.g. from [Buz1, §1]) that a *false elliptic curve* over a $\mathbb{Z}[\frac{1}{d}]$-scheme S is a pair $(A/S, i)$ where A is an abelian surface over S and $i : \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$ is an injective ring homomorphism. We refer to [Buz1] for more information and definitions regarding false elliptic curves, in particular the definition of level structures. Let $X_{\mathrm{GL}_2(\mathbb{Z}_p)}$ be the moduli space of false elliptic curves with K^p-level structure as a scheme over $\mathbb{Z}_p$. We denote by $\mathcal{X}_{\mathrm{GL}_2(\mathbb{Z}_p)}$ the Tate analytification of its generic fibre, viewed as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. For any compact open subgroup $K_p \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ we use a subscript $-_{K_p}$ to denote the same objects with a K_p-level structure added. We will mostly use the standard compact open subgroups $K_0(p^n)$ or $K(p^n)$, for $n \geq 1$. Since we will mostly work with the Shimura curves with $K_0(p)$-level structure, we make the following convention:

Convention 2.7. *We define $X := X_{K_0(p)}$, $\mathcal{X} = \mathcal{X}_{K_0(p)}$, et cetera. A Shimura curve with no level specified has $K_0(p)$-level at p.*

The following striking theorem of Scholze is key to all constructions in this paper.

Theorem 2.8 (Scholze). *There exist a perfectoid space $\mathcal{X}_\infty$ over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ such that*

$$\mathcal{X}_\infty \sim \varprojlim_n \mathcal{X}_{K(p^n)}.$$

It carries an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and there exists a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism

$$\pi_{\mathrm{HT}} : \mathcal{X}_\infty \to \mathbb{P}^1$$

of adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\mathbb{P}^1 = V_1 \cup V_2$ denote the standard affinoid cover. Then $\mathcal{V}_1 = \pi_{\mathrm{HT}}^{-1}(V_1)$ and $\mathcal{V}_2 = \pi_{\mathrm{HT}}^{-1}(V_2)$ are both affinoid perfectoid, and there exists an N and affinoid opens $S_1, S_2 \subseteq \mathcal{X}_{K(p^N)}$ such that $\mathcal{V}_i$ is the preimage of S_i. Moreover we have $\omega = \pi_{\mathrm{HT}}^{*}\mathcal{O}(1)$ on $\mathcal{X}_\infty$, where ω is obtained by pulling back the usual ω (defined below) from any finite level $\mathcal{X}_{K(p^n)}$.

A few remarks are in order. For the definition of $\sim$ we refer to [SW, Definition 2.4.1]. This theorem is essentially a special case of [Sch4, Theorem IV.1.1] except for the difference in base field and the target of π_{HT}; there one obtains a perfectoid space over some algebraically closed complete $C/\mathbb{Q}_p$ and π_{HT} takes values in a larger (partial) flag variety. The version here is easily deduced in the same way; we now sketch the argument. The tower $(\mathcal{X}_{K(p^n)})_n$ embeds into the tower of Siegel threefolds (over $\mathbb{Q}_p$), and the same argument as in the proof of [Sch4, Theorem IV.1.1] gives the existence of $\mathcal{X}_\infty$ and a map π_{HT} which takes values in the partial flag variety $\mathcal{F}l$ of GSp_4 with respect to the Siegel parabolic. Using the $M_2(\mathbb{Z}_p)$-action (see below) one sees that it takes values in $\mathbb{P}^1 \subseteq \mathcal{F}l$. Since the first version of this paper, such results have appeared in the case of general Hodge type Shimura varieties; see [CS, Theorems 2.1.2 and 2.1.3]. Finally, one easily sees that the standard affinoid opens of $\mathbb{P}^1$ come by pullback from standard affinoid opens of $\mathbb{P}^5$ via the embeddings $\mathbb{P}^1 \subseteq \mathcal{F}l \subseteq \mathbb{P}^5$, where $\mathcal{F}l \subseteq \mathbb{P}^5$ is the Plücker embedding.

Let us now discuss some standard constructions and define the sheaf ω mentioned in Theorem 2.8. For any false elliptic curve A over some $\mathbb{Z}_p$-scheme S the p-divisible group $A[p^\infty]$ carries an action $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$. Put $\mathcal{G}_A = eA[p^\infty]$, where $e \in M_2(\mathbb{Z}_p)$ is an idempotent that we will fix throughout the text (take e.g. $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$). This is a p-divisible group over S of height 2 and we have $A[p^\infty] \cong \mathcal{G}_A^{\oplus 2}$ functorially; we will fix this isomorphism. For all purposes $\mathcal{G}_A$ behaves exactly like the p-divisible group of an elliptic curve and we may use it to define ordinarity, supersingularity, level structures et cetera. We will often just write $\mathcal{G}$ instead of $\mathcal{G}_A$ if the false elliptic curve A is clear from the context. The line bundle ω is the dual of $e(\mathrm{Lie}(A^{\mathrm{univ}}/X))$, where A^{univ} is the universal false elliptic curve. We will also write $\mathcal{G}^{\mathrm{univ}} = \mathcal{G}_{A^{\mathrm{univ}}}$. The same definitions and conventions apply to the adic versions, and to other level structures.

Next, we specify the right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $(C, \mathcal{O}_C)$-points on both sides of the Hodge-Tate period map $\mathcal{X}_\infty \to \mathbb{P}^1$. First we consider $\mathbb{P}^1$: $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ acts from the left on C^2 (viewed as column vectors) and a line $L \subseteq C^2$ is sent by g to $g^\vee(L)$, where $g \mapsto g^\vee$ is the involution

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g^\vee = \det(g) g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This defines a right action. A $(C, \mathcal{O}_C)$-point of $\mathcal{X}_\infty$ consists of a false elliptic curve A/C and an isomorphism $\alpha : \mathbb{Z}_p^2 \to T_p\mathcal{G}$ (and the K^p-level structure which we ignore). Let $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and fix $n \in \mathbb{Z}$ such that $p^n g \in M_2(\mathbb{Z}_p)$ but $p^{n-1} g \notin M_2(\mathbb{Z}_p)$. For $m \in \mathbb{Z}_{\geq 0}$ sufficiently large the kernel of $p^n g^\vee$ modulo p^m stabilizes and we denote the corresponding subgroup of $\mathcal{G}[p^m]$ under α by H. We define $(A, \alpha).g$ to be $(A/H^{\oplus 2}, \beta)$, where β is defined as the composition

$$\mathbb{Z}_p^2 \xrightarrow{p^n g} \mathbb{Q}_p^2 \xrightarrow{\alpha} V_p\mathcal{G} \xrightarrow{(f^\vee)_*^{-1}} V_p(\mathcal{G}/H).$$

Here $H^{\oplus 2}$ is viewed as a subgroup scheme of $A[p^\infty]$ via the functorial isomorphism $A[p^\infty] \cong \mathcal{G}^{\oplus 2}$, $V_p(-)$ denotes the rational Tate module and $(f^\vee)_* : V_p(\mathcal{G}/H) \to V_p\mathcal{G}$ is the map induced from the dual of the natural isogeny $f : \mathcal{G} \to \mathcal{G}/H$ (note that β is isomorphism onto $T_p(\mathcal{G}/H)$). In particular, if $g \in \mathrm{GL}_2(\mathbb{Z}_p)$, then $(A,\alpha).g = (A, \alpha \cdot g)$ where $(\alpha\cdot g)(e_1) = a\alpha(e_1) + c\alpha(e_2)$, $(\alpha \cdot g)(e_2) = b\alpha(e_1) + d\alpha(e_2)$. Here and everywhere else in the text e_1 and e_2 are the standard basis vectors $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ of $\mathbb{Z}_p^2$.

2.3. w-ordinary false elliptic curves. Let H be a finite flat group scheme over $\mathcal{O}_C$ killed by p^n. We let ω_H denote the dual of $\mathrm{Lie}(H)$. It is a torsion $\mathcal{O}_C$-module and hence isomorphic to $\bigoplus_i \mathcal{O}_C/a_i\mathcal{O}_C$ for some finite set of $a_i \in \mathcal{O}_C$. The degree $\deg(H)$ of H is defined to be $\sum_i v(a_i)$. The Hodge-Tate map HT_H is the morphism of *fppf* abelian sheaves $H \to \omega_{H^\vee}$ over $\mathcal{O}_C$ defined on points by

$$f \in H = (H^\vee)^\vee \mapsto f^*(dt/t) \in \omega_{H^\vee}$$

where we view f as a morphism $f : H^\vee \to \mu_{p^n}$, $dt/t \in \omega_{\mu_{p^n}}$ is the invariant differential and $-^\vee$ denotes the Cartier dual. We will often abuse notation and use HT_H for the map on $\mathcal{O}_C$-points, and there one may identify the $\mathcal{O}_C$-points of H with the C-points of its generic fibre.

Now let G be a p-divisible group over $\mathcal{O}_C$. Taking the inverse limit over the Hodge-Tate maps for the $G[p^n]$ we obtain a morphism $\mathrm{HT}_G : T_pG \to \omega_{G^\vee}$, which we will often linearize by tensoring the source with $\mathcal{O}_C$. Taking this morphism for $G^\vee$ and dualizing it we obtain a morphism $\mathrm{Lie}(G) \to T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_C$. Putting these morphisms together we get a sequence

$$0 \to \mathrm{Lie}(G) \to T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to \omega_{G^\vee} \to 0$$

which is in fact a complex with cohomology groups killed by $p^{1/(p-1)}$ ([FGL, Théorème II.1.1]).

Let A/C be a false elliptic curve. Then A has good reduction and we will denote its unique model over $\mathcal{O}_C$ by $\mathcal{A}$. We have the Hodge-Tate sequence of $\mathcal{G}_\mathcal{A}[p^\infty]$:

$$0 \to \mathrm{Lie}(\mathcal{G}_\mathcal{A}) \to T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to \omega_{\mathcal{G}_\mathcal{A}^\vee} \to 0.$$

Here we have dropped the subscript $-_\mathcal{A}$ in the notation of the Tate module for simplicity; this should not cause any confusion. We will write HT_A for $\mathrm{HT}_{\mathcal{G}_\mathcal{A}[p^\infty]}$. The image and kernel of HT_A are free $\mathcal{O}_C$-modules of rank 1 that we will denote by F_A and F_A^1 respectively. Note that $p^{1/(p-1)}\omega_{\mathcal{G}_\mathcal{A}^\vee} \subseteq F_A \subseteq \omega_{\mathcal{G}_\mathcal{A}^\vee}$.

Recall that e_1 and e_2 are the standard basis vectors of $\mathbb{Z}_p^2$ and let w be a positive rational number.

Definition 2.9. *Let A/C be a false elliptic curve with model $\mathcal{A}/\mathcal{O}_C$. Let $w \in \mathbb{Q}_{>0}$.*

(1) *Let α be a trivialization of $T_p\mathcal{G}$. We say that α is w-ordinary if $\mathrm{HT}_A(\alpha(e_1)) \in p^w F_A$.*
(2) *A is called w-ordinary if there is a w-ordinary trivialization of $T_p\mathcal{G}$.*

Note that if A is w-ordinary, then it also w'-ordinary for all $w' < w$. Note also that A is ordinary (in the classical sense) if and only if it is ∞-ordinary (i.e. A-ordinary for all $w > 0$).

Definition 2.10. *Let A/C be a w-ordinary false elliptic curve and assume that $n \in \mathbb{Z}_{\geq 1}$ is such that $n < w + 1$. Then the kernel of the morphism $\mathcal{G}[p^n](C) \to F_A/p^{\min(n,w)}F_A$ induced by HT_A is an étale subgroup scheme H_n of $\mathcal{G}_A[p^n]$ isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ which we call the pseudocanonical subgroup of level n.*

We will use the notation H_n to denote the pseudocanonical subgroup of level n (when it exists) whenever the false elliptic curve A is clear from the context. When there are multiple false elliptic curves in action we will use the notation $H_{n,A}$. Since H_n is naturally equipped with an inclusion into $\mathcal{G}_A[p^n]$ we may take its schematic closure inside $\mathcal{G}_{\mathcal{A}}[p^n]$. This is a finite flat group scheme of rank p^n over $\mathcal{O}_C$ with generic fibre H_n and we will abuse notation and denote it by H_n as well.

When $n = 1$ we will refer to H_1 simply as the pseudocanonical subgroup and drop "of level 1". Note that if $\alpha : T_p\mathcal{G} \to \mathbb{Z}_p^2$ is a w-ordinary trivialization with $n - 1 < w \leq n$ then $(\alpha^{-1} \bmod p^n)|_{\mathbb{Z}/p^n\mathbb{Z}\oplus 0}$ trivializes the pseudocanonical subgroup. We record a simple lemma:

Lemma 2.11. *Let A/C be a false elliptic curve and let α be a w-ordinary trivialization of $T_p\mathcal{G}$. Assume that $w > n \in \mathbb{Z}_{\geq 1}$ and let $m \leq n$ be a positive integer. Then $A/H_{m,A}^{\oplus 2}$ is $(w - m)$-ordinary, and for any $m' \in \mathbb{Z}$ with $m < m' \leq n$, $H_{m'-m,A/H_m^{\oplus 2}} = H_{m',A}/H_{m,A}$.*

Proof. Let $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ denote the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$. Then $(A, \alpha).g = (A/H_{m,A}^{\oplus 2}, \beta)$ where β is defined by this equality. Let f denote the natural isogeny $\mathcal{G} \to \mathcal{G}/H_m$. From the definitions we get a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_p^2 \arrow[r, "\alpha"] \arrow[d, "g^\vee"] & T_p\mathcal{G} \arrow[r, "\mathrm{HT}_A"] \arrow[d, "f_*"] & F_A \arrow[d, "(f^\vee)^*"] \\
\mathbb{Z}_p^2 \arrow[r, "\beta"] & T_p(\mathcal{G}/H_m) \arrow[r, "\mathrm{HT}_{A/H_m^{\oplus 2}}"] & F_{A/H_m^{\oplus 2}}
\end{tikzcd}$$

and direct computation gives that $p^m \,\mathrm{HT}_{A/H_m^{\oplus 2}}(\beta(e_1)) = (f^\vee)^* \,\mathrm{HT}_A(\alpha(e_1))$. Since $\mathrm{HT}_A(\alpha(e_1)) \in p^w F_A$ we see that $\mathrm{HT}_{A/H_m^{\oplus 2}}(\beta(e_1)) \in p^{w-m} F_{A/H_m^{\oplus 2}}$ which proves the first assertion. For the second assertion, observe that by definition $H_{m'-m,A/H_m^{\oplus 2}}$ and $H_{m',A}/H_{m,A}$ are generated by $\beta(e_1) \bmod p^{m'-m}$ and $f(\alpha(e_1)) \bmod p^m$ respectively, and that these are equal. $\square$

Remark 2.12. *The commutativity of the diagram in the proof above is also what essentially proves the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of the Hodge-Tate period map π_{HT}, and the first assertion may be viewed more transparently as a direct consequence* *of this equivariance for the element g. Note also that the second assertion of the Lemma mirrors properties of the usual canonical subgroups of higher level.*

Next we recall some calculations from Oort-Tate theory which are recorded in [Far, §6.5 Lemme 9] (we thank an anonymous referee for pointing out this reference). For the last statement, see Proposition 1.2.8 of [Kas] (for further reference see Remark 1.2.7 of *loc.cit* and §3 of [Buz2]; note that these references treat elliptic curves but the results carry over *verbatim*).

Lemma 2.13. *Let H be a finite flat group scheme over $\mathcal{O}_C$ of degree p. Then H is isomorphic to $\mathrm{Spec}(\mathcal{O}_C[Y]/(Y^p - aY))$ for some $a \in \mathcal{O}_C$ and determined up to isomorphism by $v(a)$, and the following holds:*

1. *$\omega_H = (\mathcal{O}_C/a\mathcal{O}_C).dY$ and hence $\deg(H) = v(a)$.*
2. *The image of the (linearized) Hodge-Tate map $\mathrm{HT}_{H^\vee} : H^\vee(C) \otimes \mathcal{O}_C \to \omega_H$ is equal to $(c\mathcal{O}_C/a\mathcal{O}_C).dY$, where $v(c) = (1 - v(a))/(p-1)$.*

Moreover, if A/C is a false elliptic curve such that $H \subseteq \mathcal{G}_{\mathcal{A}}[p]$ and $\deg(H) > 1/(p+1)$, then H is the canonical subgroup of $\mathcal{G}_{\mathcal{A}}$.

We will use these properties freely in this section. Using this we can now show that the pseudocanonical subgroup coincides with the canonical subgroup for sufficiently large w (as a qualitative statement this is implicit in [Sch4], cf. Lemma III.3.8).

Lemma 2.14. *Let A/C be a w-ordinary false elliptic curve and assume that $p/(p^2-1) < w \leq 1$. Then H_1 is the canonical subgroup of $\mathcal{G}_{\mathcal{A}}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1(C) \arrow[r] \arrow[d, "\mathrm{HT}_{H_1}"'] & \mathcal{G}_{\mathcal{A}}[p](C) \arrow[d, "\mathrm{HT}_{\mathcal{G}_{\mathcal{A}}[p]}"] \\ 0 \arrow[r] & \omega_{H_1^\vee} \arrow[r] & \omega_{\mathcal{G}_{\mathcal{A}}[p]^\vee} \end{tikzcd}$$

with exact rows. We have $\omega_{\mathcal{G}_{\mathcal{A}}[p]^\vee} = \omega_{\mathcal{G}_{\mathcal{A}}^\vee}/p\omega_{\mathcal{G}_{\mathcal{A}}^\vee}$. Note that H_1 is an Oort-Tate group scheme and hence isomorphic to $\mathrm{Spec}(\mathcal{O}_C[Y]/(Y^p - aY))$ for some $a \in \mathcal{O}_C$ with $v(a) = \deg(H_1)$ and $H_1^\vee$ is isomorphic to $\mathrm{Spec}(\mathcal{O}_C[Y]/(Y^p - bY))$ with $ab = p$. Fix a generator $s \in H_1(C)$. By choosing generators the inclusion $\omega_{H_1^\vee} \to \omega_{\mathcal{G}_{\mathcal{A}}[p]^\vee}$ may be written as $\mathcal{O}_C/b\mathcal{O}_C \to \mathcal{O}_C/p\mathcal{O}_C$ where the map is multiplication by a, and $\mathrm{HT}_{H_1^\vee}(s)$ has valuation $v(a)/(p-1)$. Since A is w-ordinary we have $a\,\mathrm{HT}_{H_1^\vee}(s) = \mathrm{HT}_{\mathcal{A}[p]}(s) \in p^w\omega_{\mathcal{A}[p]^\vee}$ and hence $pv(a)/(p-1) \geq w$, i.e. $\deg(H_1) \geq (p-1)w/p$. By our assumption on w we deduce $\deg(H_1) > 1/(p+1)$ and hence that H_1 is the canonical subgroup. $\square$

Remark 2.15. *As emphasized by a referee, one may also bound the Hodge height of $\mathcal{G}_A$, at least under stronger assumptions on w (recall that the Hodge height is the valuation of the Hasse invariant, truncated by 1). For example,* *using* [FGL, Théorème II.1.1] *and* [AIP, Proposition 3.1.2], *one sees that if* $\frac{1}{2}+\frac{1}{p-1}<w\leq 1$ *and* $p\geq 5$, *then the Hodge height of* $\mathcal{G}_A$ *is* $\leq 1-w+\frac{1}{p-1}$.

We will now, somewhat overdue, discuss the interpretation of w-ordinarity in terms of the Hodge-Tate period map. Define a coordinate z on $\mathbb{P}^1$ by letting z correspond to the line spanned by $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ with $z=-y/x$ and for $w\in\mathbb{Q}_{>0}$ define $U_w\subseteq\mathbb{P}^1$ to be the locus $|z|\leq|p^w|$. If $w\leq n$, then this locus is $K_0(p^n)$-stable. It is a rational subset of V_1. Now define

$$\mathcal{U}_{\infty,w}=\pi_{\mathrm{HT}}^{-1}(U_w).$$

This is a $K_0(p^n)$-stable open subspace of $\mathcal{X}_\infty$. In fact, $\mathcal{U}_{\infty,w}$ is affinoid perfectoid since U_w is a rational subset of $V_1\subseteq\mathbb{P}^1$ and therefore $\mathcal{U}_{\infty,w}$ is a rational subset of $\mathcal{V}_1$ since π_{HT} is adic. The conclusion follows from [Sch1, Theorem 6.3(ii)]. Note that, directly from the definition, the $(C,\mathcal{O}_C)$-points of $\mathcal{U}_{\infty,w}$ are exactly the pairs (A,α) for which α is w-ordinary.

In this article we will want to vary w in order to capture all weights when defining overconvergent modular forms. A minor disadvantage of the loci $\mathcal{U}_{\infty,w}$ is that they are stable under different compact open subgroups as w varies. We will instead define related loci that have the advantage that they all stable under the action of $K_0(p)$ (independent of w).

Definition 2.16. *Let* $w\in\mathbb{Q}_{>0}$ *and let* A *be a* w*-ordinary false elliptic curve over* C. *Let* α *be a trivialisation of* $T_p\mathcal{G}$. *We say that* α *is* strict *if* (α mod p)$|_{\mathbb{Z}/p\mathbb{Z}\oplus 0}$ *trivializes the pseudocanonical subgroup.*

Note that if α is w-ordinary then it is strict. Given (A,α) with A w-ordinary and α strict, the orbit of (A,α) in $\mathcal{X}_\infty$ under $K_0(p)$ consists exactly of the pairs (A,β) for which β is strict. We define $\mathbb{P}^1_w\subseteq V_1$ to be the rational subsets whose $(C,\mathcal{O}_C)$-points correspond to

$$\{z\in\mathcal{O}_C\mid\exists s\in p\mathbb{Z}_p\,:\,|z-s|\leq|p^w|\}.$$

One checks directly that if $\gamma\in K_0(p)$ and $|z|,|s|<1$, then $|\gamma^\vee z-\gamma^\vee s|=|z-s|$. Thus $\mathbb{P}^1_w$ is stable under $K_0(p)$ and moreover $\mathbb{P}^1_w=U_w.K_0(p)$. We then define

$$\mathcal{X}_{\infty,w}=\pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1_w).$$

By the discussion above these are $K_0(p)$-stable and a $(C,\mathcal{O}_C)$-point (A,α) of $\mathcal{X}_\infty$ is in $\mathcal{X}_{\infty,w}$ if and only if A is w-ordinary and α is strict.

Theorem 2.17. *Let* $w\in\mathbb{Q}_{>0}$.

(1) *There is a unique affinoid open rigid subspace* $\mathcal{X}_w\subseteq\mathcal{X}$ *whose* $(C,\mathcal{O}_C)$*-points are exactly the pairs* (A,H) *with* A *a* w*-ordinary false elliptic curves (with* K^p*-level structure) and* H *its pseudocanonical subgroup.*
(2) *Let* $q:\mathcal{X}_\infty\to\mathcal{X}$ *denote the projection map. Then* $\mathcal{X}_{\infty,w}=q^{-1}(\mathcal{X}_w)$.
(3) *The sets* $(\mathcal{X}_w)_w$ *form a cofinal set of open neighbourhoods of the closure of the ordinary-multiplicative locus in* $\mathcal{X}$.

Proof. Write, for any level $K_p \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$, q_{K_p} for the projection map $\mathcal{X}_\infty \to \mathcal{X}$. We prove (1) and (2) together. Indeed, we define $\mathcal{X}_w := q(\mathcal{X}_{\infty,w})$. Since q is pro-étale, $\mathcal{X}_w$ is open and quasicompact ([Sch3], Lemma 3.10(iv)) and the characterization of the $(C, \mathcal{O}_C)$-points of $\mathcal{X}_w$ follows directly from the characterization of the $(C, \mathcal{O}_C)$-points of $\mathcal{X}_{\infty,w}$. Note that uniqueness is clear since quasicompact open subsets of quasiseparated rigid analytic varieties (viewed as adic spaces) are determined by their classical points (cf. [Sch1, Theorem 2.21]). This also finishes the proof (1), except that we need to show that $\mathcal{X}_w$ is affinoid.

By the definition of the inverse limit topology we may find m and a quasicompact open subset $W \subseteq \mathcal{X}_{K(p^m)}$ such that $q_{K(p^m)}^{-1}(W) = \mathcal{X}_{\infty,w}$. We claim that $W = q_{K(p^m)}(\mathcal{X}_{\infty,w})$. Indeed, both sets are determined by their $(C, \mathcal{O}_C)$-points and $q_{K(p^m)}$ is surjective on $(C, \mathcal{O}_C)$-points. Thus it remains to see that if r is the natural map $\mathcal{X}_{K(p^m)} \to \mathcal{X}$, then $r^{-1}(\mathcal{X}_w) = W$, which we can check on $(C, \mathcal{O}_C)$-points. This follows from the surjectivity of q_{K_p} on $(C, \mathcal{O}_C)$-points for arbitrary K_p and the fact that any trivialization that maps down to the pseudocanonical subgroup is automatically strict (in the sense that one/any lift to a trivialization of the whole Tate module is strict). This proves (2). Finally we finish the proof of (1) by showing that $\mathcal{X}_w$ is affinoid. To see this we use $\mathcal{X}_{\infty,w} = q^{-1}(\mathcal{X}_w)$. Since $\mathcal{X}_{\infty,w}$ is a rational subset of $\mathcal{V}_1$ we may find some large N and an affinoid $S \subseteq \mathcal{X}_{K(p^N)}$ such that $\mathcal{X}_{\infty,w} = q_{K(p^N)}^{-1}(S)$ (this follows from Theorem 2.8 and the fact that rational subsets come from finite level). Then $\mathcal{X}_w$ is the quotient of S by the finite group $K_0(p)/K(p^N)$ and hence affinoid by Corollary 6.26.

We now prove (3). As a qualitative result it follows by general topology arguments using the constructible topology, cf. [Sch4, Lemma III.3.8]. Here we deduce a quantitative version using the Lubin-Katz theory of the canonical subgroup. For a statement of the results we need phrased in terms of the degree function see Proposition 1.2.8 of [Kas] (as before this reference treats elliptic curves but the results carry over *verbatim*). Without loss of generality we restrict our attention to $w = 1, 2, 3, ...$ On $\mathcal{X}_1$, H_1 is the canonical subgroup and $\deg(H_1) \geq (p-1)/p$. Let $h = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Then $h(\mathcal{X}_{\infty,w}) = \mathcal{X}_{\infty,w+1}$ for all w. Moreover, if $(A, \alpha) \in \mathcal{X}_{K(p^\infty),1}(C, \mathcal{O}_C)$ and $(A, \alpha).h = (A/H^{\oplus 2}, \beta)$, then H is anticanonical and hence of degree $(1 - \deg(H_1))/p$. Hence $\deg(\mathcal{G}[p]/H) = 1 - (1 - \deg(H_1))/p > 1/2$ and $\mathcal{G}[p]/H$ is the canonical subgroup. Write δ_n for the degree of the canonical subgroup of $(h^n).A$. Then we get the recurrence relation $\delta_{n+1} = (p-1)/p + \delta_n/p$ and from $\delta_1 \geq (p-1)/p$ we deduce that

$$\delta_n \geq \delta_n' := \frac{p-1}{p} \sum_{i=0}^{n-1} \frac{1}{p^i} = 1 - \frac{1}{p^n}.$$

We have therefore proved that if $(A, H) \in \mathcal{X}_n(C, \mathcal{O}_C)$ then $deg(H) \geq \delta_n'$. As $\delta_n' \to 1$ as $n \to \infty$, the result follows since the loci $\{(A, H) \mid deg(H) \geq r\}$

for $r \in (0,1)$ form a cofinal set of open neighbourhoods of the closure of the ordinary-multiplicative locus (which is the locus $\{(A,H) \mid deg(H)=1\})$. $\square$

2.4. The fundamental period and a non-vanishing section. Recall the coordinate z on $\mathbb{P}^1$ defined earlier. The action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ is

$$z.g = g^\vee z = \frac{az+c}{bz+d}.$$

Below, whenever we have a matrix $g \in \mathrm{GL}_2$ we will use a,b,c,d as above to denote its entries. Note that z defines a function in $H^0(V_1, \mathcal{O}^+_{\mathbb{P}^1})$ and by composing with π_{HT} we obtain a function $\mathfrak{z} \in H^0(\mathcal{V}_1, \mathcal{O}^+_{\mathcal{X}_\infty})$ which we will call the fundamental period. We will use the same notation to denote its restriction to $\mathcal{X}_{\infty,w}$ for any $w>0$; by definition $\mathfrak{z} \in H^0(\mathcal{X}_{\infty,w}, p\mathbb{Z}_p + p^w\mathcal{O}^+_{\mathcal{X}_\infty})$. Note that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(p)$ then

$$\gamma^*\mathfrak{z} = \frac{a\mathfrak{z}+c}{b\mathfrak{z}+d}$$

as functions on $\mathcal{X}_{\infty,w}$. Next we wish to trivialize $\mathcal{O}(1)$ over $S=\{z \neq \infty\} \subseteq \mathbb{P}^1$ by defining a non-vanishing section. Everything we write in this section is standard but we repeat it since we will need some explicit formulas. The section we are after is algebraic so we will momentarily work in the realm of algebraic geometry. Explicitly it is given by the following formula: The line bundle $\mathcal{O}(1)$ has a geometric incarnation with total space $(\mathrm{GL}_2 \times \mathbb{A}^1)/B$ where B denotes the lower triangular Borel in GL_2 and acts on $\mathbb{A}^1$ by $g.x = d^{-1}x$ (this equivariant structure may differ from the canonical one, but it is the one corresponding to the Hodge-Tate sequence on $\mathcal{X}_\infty$). Global sections correspond to functions $f : \mathrm{GL}_2 \to \mathbb{A}^1$ satisfying $f(gh) = d(h)^{-1}f(g)$ for $g \in \mathrm{GL}_2$, $h \in B$. There is a morphism of equivariant vector bundles

$$(\mathrm{GL}_2/B) \times \mathbb{A}^2 \to (\mathrm{GL}_2 \times \mathbb{A}^1)/B,$$

given by

$$\left(g, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto (g, dx-by)\,.$$

Here the left hand side has the GL_2-action $g.(h,v) = (gh, (g^\vee)^{-1}v)$. Then the global section of $(\mathrm{GL}_2/B) \times \mathbb{A}^2$ given by the constant function $g \mapsto e_2$ maps to global section of $\mathcal{O}(1)$ given by the function

$$s : \mathrm{GL}_2 \to \mathbb{A}^1;$$

$$s(g) = -b(g).$$

Note that $s=0$ if and only $g \in B$ so s is invertible on S. Let us now return to the rigid analytic world and let $W \subseteq S$ be an open subset such $h^\vee(W) \subseteq W$ for some $h \in \mathrm{GL}_2(C)$. Then by direct calculation

$$(sh^\vee)(g) = d(g)b(h) - b(g)d(h).$$

Thus
$$\frac{s\circ h^\vee}{s}(g)=\frac{d(h)b(g)-b(h)d(g)}{b(g)}.$$
This is a function on W. Taking $g=g_z=\left(\begin{smallmatrix}0 & -1\\ 1 & z\end{smallmatrix}\right)$ we see that
$$\frac{s\circ h^\vee}{s}(z)=b(h)z+d(h).$$
We may then pull s back via π_{HT} to get (compatible) non-vanishing sections $\mathfrak{s}\in H^0(\mathcal{X}_{\infty,w},\omega)$ for all w satisfying
$$\frac{\gamma^*\mathfrak{s}}{\mathfrak{s}}=b\mathfrak{z}+d$$
for all $\gamma\in K_0(p)$. We remark that $\mathfrak{s}$ is one of the "fake" Hasse invariants constructed in [Sch4].

2.5. A perfectoid definition of overconvergent modular forms. We will give definitions of sheaves of overconvergent modular forms with prescribed small or affinoid weight. Recall the forgetful morphism $q:\mathcal{X}_{\infty,w}\to\mathcal{X}_w\subseteq\mathcal{X}$.

Definition 2.18. *Let $\mathcal{U}$ be a weight and let $w\in\mathbb{Q}_{>0}$ be such that $w\geq 1+s_{\mathcal{U}}$. We define a sheaf $\omega^\dagger_{\mathcal{U},w}$ on $\mathcal{X}_w$ by*
$$\omega^\dagger_{\mathcal{U},w}(U)=\left\{f\in\mathcal{O}_{\mathcal{X}_{\infty,w}}(q^{-1}(U))\widehat{\otimes}A_{\mathcal{U}}\mid \gamma^*f=\chi_{\mathcal{U}}(b\mathfrak{z}+d)^{-1}f\ \forall\gamma\in K_0(p)\right\}$$
where $U\subseteq\mathcal{X}_w$ is a qcqs open subset.

We remark that, since $b\mathfrak{z}+d\in\mathbb{Z}_p^\times(1+p^w\mathcal{O}^+_{\mathcal{X}_{\infty,w}})$, the assumption $w\geq 1+s_{\mathcal{U}}$ ensures that $\chi_{\mathcal{U}}(b\mathfrak{z}+d)$ is well defined (by Proposition 2.6). Define
$$\Delta_0(p)=\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{M}_2(\mathbb{Z}_p)\cap\mathrm{GL}_2(\mathbb{Q}_p)\mid c\in p\mathbb{Z}_p,\ d\in\mathbb{Z}_p^\times\}.$$
This is a submonoid of $\mathrm{GL}_2(\mathbb{Q}_p)$ containing $K_0(p)$, and it stabilizes $\mathcal{X}_{\infty,w}$ for all w. We can form a sheaf $\mathcal{F}$ on $\mathcal{X}_{\infty,w}$ by $\mathcal{F}(U_\infty)=\mathcal{O}_{\mathcal{X}_{\infty,w}}(U_\infty)\widehat{\otimes}A_{\mathcal{U}}$ for $U_\infty\subseteq\mathcal{X}_{\infty,w}$ qcqs (*not* necessarily of the form $q^{-1}(U)$), and we may equip it with a $\Delta_0(p)$-equivariant structure by the rule
$$\gamma\cdot_{\mathcal{U}}-:\ \mathcal{F}(U_\infty)\to\mathcal{F}(\gamma^{-1}U_\infty);$$
$$\gamma\cdot_{\mathcal{U}}f=\chi_{\mathcal{U}}(b\mathfrak{z}+d)\gamma^*f$$
for $\gamma\in\Delta_0(p)$. Then $\omega^\dagger_{\mathcal{U},w}=(q_*\mathcal{F})^{K_0(p)}$ and we obtain an action of the double cosets $K_0(p)\gamma K_0(p)$ for $\gamma\in\Delta_0(p)$ on $\omega^\dagger_{\mathcal{U},w}$, given by the following standard formula:
$$[K_0(p)\gamma K_0(p)]\,:\,\omega^\dagger_{\mathcal{U},w}(V)\to\omega^\dagger_{\mathcal{U},w}(U);$$
$$[K_0(p)\gamma K_0(p)].f=\sum_i\chi_{\mathcal{U}}(b_i\mathfrak{z}+d_i)\gamma_i^*f$$
where $U\subseteq\mathcal{X}_w$ is qcqs, $V\subseteq\mathcal{X}_w$ is the image of U under $[K_0(p)\gamma K_0(p)]$ viewed as a correspondence on $\mathcal{X}_w$, and $K_0(p)\gamma K_0(p)=\coprod_i\gamma_iK_0(p)$ is a coset decomposition. This gives us the Hecke action at p for $\omega^\dagger_{\mathcal{U},w}$. As in the complex case, we may also view this action as an action by correspondences. We give

a few remarks on this for Hecke operators at primes $\ell \neq p$ below; the action at p is similar. The statements analogous to Proposition 2.19 that are needed follow from Proposition 4.7.

For the Hecke actions at $\ell \neq p$, let us momentarily introduce the tame level into our notation, writing $\omega^{\dagger}_{\mathcal{U},w,K^p}$ for the sheaf on $\mathcal{X}_{w,K^p} \subset \mathcal{X}_{K_0(p)K^p}$. The construction of Hecke operators is then immediate from the following proposition.

Proposition 2.19. (1) *For any $g \in G(\mathbb{A}_f^p)$ with $g : \mathcal{X}_{w,K^p} \overset{\sim}{\to} \mathcal{X}_{w,g^{-1}K^pg}$ the associated isomorphism of Shimura curves, there is a canonical isomorphism $g^*\omega^{\dagger}_{\mathcal{U},w,g^{-1}K^pg} \cong \omega^{\dagger}_{\mathcal{U},w,K^p}$.*

(2) *For any inclusion $K_2^p \subset K_1^p$ of tame levels with $\pi : \mathcal{X}_{w,K_2^p} \to \mathcal{X}_{w,K_1^p}$ the associated finite étale projection of Shimura curves, there is a canonical isomorphism*

$$\pi_*\omega^{\dagger}_{\mathcal{U},w,K_2^p} \cong \omega^{\dagger}_{\mathcal{U},w,K_1^p} \otimes_{\mathcal{O}_{\mathcal{X}_{w,K_1^p}}} \pi_*\mathcal{O}_{\mathcal{X}_{w,K_2^p}}$$

In particular there is a canonical $\mathcal{O}_{\mathcal{X}_{w,K_1^p}}$-linear trace map

$$\pi_*\omega^{\dagger}_{\mathcal{U},w,K_2^p} \to \omega^{\dagger}_{\mathcal{U},w,K_1^p}$$

Proof. (1) follows immediately from the definition of the sheaves together with the fact that $g^*\mathfrak{z} = \mathfrak{z}$; the latter follows from [Sch4, Theorem IV.1.1(iv)].

For (2) we consider the pullback diagram

$$\begin{tikzcd}
\mathcal{X}_{\infty,w,K_2^p} \arrow[r, "q_2"] \arrow[d, "\pi_\infty"] & \mathcal{X}_{w,K_2^p} \arrow[d, "\pi"] \\
\mathcal{X}_{\infty,w,K_1^p} \arrow[r, "q_1"] & \mathcal{X}_{w,K_1^p}
\end{tikzcd}$$

of adic spaces. Given a rational subset $U \subset \mathcal{X}_{w,K_1^p}$, we then have

$$\begin{aligned}
\mathcal{O}(q_2^{-1}\pi_1 U) &= \mathcal{O}(\pi_\infty^{-1}q_1^{-1}U) \\
&\cong \mathcal{O}(q_1^{-1}U)\widehat{\otimes}_{\mathcal{O}(U)}\mathcal{O}(\pi^{-1}U) \\
&\cong \mathcal{O}(q_1^{-1}U) \otimes_{\mathcal{O}(U)} \mathcal{O}(\pi^{-1}U)
\end{aligned}$$

where in the final line we use the fact that $\mathcal{O}(\pi^{-1}U)$ is a finite projective $\mathcal{O}(U)$-Banach module. Applying $-\widehat{\otimes}A_{\mathcal{U}}$ and making use of Lemma 6.7, we get a canonical isomorphism

$$\mathcal{O}(q_2^{-1}\pi_1^{-1}U)\widehat{\otimes}A_{\mathcal{U}} \cong \left(\mathcal{O}(q_1^{-1}U)\widehat{\otimes}A_{\mathcal{U}}\right) \otimes_{\mathcal{O}(U)} \mathcal{O}(\pi^{-1}U).$$

Since $\pi_\infty^*\mathfrak{z} = \mathfrak{z}$, this isomorphism is equivariant for the $\chi_{\mathcal{U}}$-twisted action of $K_0(p)$. Passing to $K_0(p)$-invariants for the twisted action, the left-hand side then becomes $\omega^{\dagger}_{\mathcal{U},w,K_2^p}(\pi^{-1}U) = (\pi_*\omega^{\dagger}_{\mathcal{U},w,K_2^p})(U)$, while the right-hand side becomes $\omega^{\dagger}_{\mathcal{U},w,K_1^p}(U) \otimes_{\mathcal{O}(U)} (\pi_*\mathcal{O}_{\mathcal{X}_{w,K_2^p}})(U)$, so (2) follows. □

We now define our spaces of overconvergent modular forms.

Definition 2.20. *Let $\mathcal{U}$ be a weight and let $w \in \mathbb{Q}_{>0}$ be such that $w \geq 1 + s_{\mathcal{U}}$.*

(1) *We define the space of w-overconvergent modular forms $\mathcal{M}_{\mathcal{U}}^{\dagger,w}$ of weight $\mathcal{U}$ by*
$$\mathcal{M}_{\mathcal{U}}^{\dagger,w} = H^0(\mathcal{X}_w, \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}} \Omega^1_{\mathcal{X}_w}).$$

(2) *We define the space of overconvergent modular forms $\mathcal{M}_{\mathcal{U},w}^{\dagger}$ of weight $\mathcal{U}$ by*
$$\mathcal{M}_{\mathcal{U}}^{\dagger} = \varinjlim_{w} \mathcal{M}_{\mathcal{U}}^{\dagger,w}.$$

The spaces $\mathcal{M}_{\mathcal{U}}^{\dagger,w}$ are $\mathbb{Q}_p$-Banach spaces, and $\mathcal{M}_{\mathcal{U}}^{\dagger}$ is an LB-space. Using the functoriality of $\Omega^1_{\mathcal{X}}$ one may define Hecke operators on $\mathcal{M}_{\mathcal{U},w}^{\dagger}$ in the same way that we did for $\omega_{\mathcal{U},w}^{\dagger}$.

Remark 2.21. *The reason we have formulated the definition in this way is because it fits naturally with our discussion of the overconvergent Eichler-Shimura map in §4-5. It would be more fitting with our desire to be "explicit" to consider the space*
$$H^0(\mathcal{X}_w, \omega_{\mathcal{U}+2,w}^{\dagger})$$
where $\mathcal{U}+2$ is the weight $(R_{\mathcal{U}}, z \mapsto \chi_{\mathcal{U}}(z)z^2)$. Note that this shift by 2 is forced upon us by our convention to associate $k \in \mathbb{Z}$ with the character $z \mapsto z^{k-2}$. Since ω is trivialized over $\mathcal{X}_{\infty,w}$ by $\mathfrak{s}$ one sees that $\omega_{\mathcal{U}+2,w}^{\dagger} \cong \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}} \omega^2$. By the Kodaira-Spencer isomorphism $\Omega^1_{\mathcal{X}} \cong \omega^2$ we then have that $\omega_{\mathcal{U}+2,w}^{\dagger} \cong \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}} \Omega^1_{\mathcal{X}_w}$. Thus the two definitions give the same spaces. However, it is well known that the Kodaira-Spencer isomorphism fails to be equivariant for the natural *Hecke actions on both sides. Indeed it is customary in the theory of modular forms to renormalize the natural Hecke action on $\omega^{\otimes k}$ by multiplying T_ℓ by ℓ^{-1} (more precisely one multiplies the action of any double coset by $\det^{-1}$). The Kodaira-Spencer isomorphism is then Hecke-equivariant. Thus, by defining $\mathcal{M}_{\mathcal{U},w}^{\dagger}$ the way we have we can use the natural Hecke actions; there is no need to normalize.*

For a discussion of how the normalized Hecke action on $\omega^{\otimes k}$ corresponds to the natural Hecke action on $\omega^{\otimes k-2} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}}$, see [FC, p. 257-258].

2.6. Locally projective of rank one. In this subsection we prove that $\omega_{\mathcal{U},w}^{\dagger}$ is locally projective of rank one as a sheaf of $\mathcal{O}_{\mathcal{X}_w} \widehat{\otimes} A_{\mathcal{U}}$-modules.

First we prove a general lemma. Let A_∞ be a uniform $\mathbb{Q}_p$-Banach algebra equipped with an action of a profinite group G by continuous homomorphisms. Let $A = A_\infty^G$. We record the following easy facts:

Proposition 2.22. *A is a closed subalgebra of A_∞, hence carries an induced structure of a uniform $\mathbb{Q}_p$-Banach algebra, and $A^\circ = (A_\infty^\circ)^G$.*

We then have:

Lemma 2.23. *Keep the above notations and assumptions.*

(1) *Let M be any profinite flat $\mathbb{Z}_p$-module (in particular M could be a small $\mathbb{Z}_p$-algebra). We equip M with the trivial G-action. Then $(A_\infty \widehat{\otimes} M)^G = A\widehat{\otimes} M$.*
(2) *Let V be a Banach space over $\mathbb{Q}_p$ (in particular V could be a reduced affinoid $\mathbb{Q}_p$-algebra topologically of finite type). Equip V with the trivial G-action. Then $(A_\infty \widehat{\otimes}_{\mathbb{Q}_p} V)^G = A\widehat{\otimes}_{\mathbb{Q}_p} V$.*

Proof. To prove (1), we choose a pseudobasis $(e_i)_{i\in I}$ of M and follow the computation in Proposition 6.4 to see that the natural map $A^\circ \widehat{\otimes} M \to A^\circ_\infty \widehat{\otimes} M$ is the inclusion

$$\prod_{i\in I} A^\circ e_i \subseteq \prod_{i\in I} A^\circ_\infty e_i$$

with the action of G on the right hand side being coordinate-wise. Thus the natural map $A\widehat{\otimes} M \to A_\infty \widehat{\otimes} M$ is the inclusion of bounded sequences in A indexed by I into bounded sequences in A_∞ indexed by I, with the G-action on the latter being coordinate-wise. The desired statement now follows from the definition of A as A_∞^G.
The proof of (2) is similar, using an orthonormal basis instead of a pseudobasis. □

We record the following fact, which is certainly implicit in [Sch3] (our main reason for recording it is that some assumption on the sheaf $\mathcal{F}$ seems to be needed).

Lemma 2.24. *Let Y be a rigid analytic variety, let G be a profinite group and assume that $Y_\infty \in Y_{\text{proét}}$ is a Galois G-cover of Y. Let $U \in Y_{\text{proét}}$ be quasicompact and quasiseparated and set $U_\infty := U \times_Y Y_\infty$; this is a Galois G-cover of U. Let $\mathcal{F}$ be a sheaf on $Y_{\text{proét}}$ that comes via pullback from $Y_{\text{ét}}$. Then $\mathcal{F}(U) = \mathcal{F}(U_\infty)^G$.*

Proof. Recall that, by the definitions, we may find a system of open normal subgroups G_j of G such that $G \cong \varprojlim_j G/G_j$, and a compatible system of Galois G/G_j-covers Y_j of Y such that $Y_\infty = \varprojlim_j Y_j$ is a pro-étale presentation. Pick a pro-étale presentation $U = \varprojlim_i U_i$; we get $U_\infty = \varprojlim_{i,j} U_i \times_Y Y_j$. Note that U_∞ is quasicompact and quasiseparated by [Sch3, Proposition 3.12(v)]. We may then compute

$$\begin{aligned}
\mathcal{F}(U_\infty)^G &= \left(\varinjlim_{i,j} \mathcal{F}(U_i \times_Y Y_j)\right)^G \\
&= \varinjlim_{i,j} \left(\mathcal{F}(U_i \times_Y Y_j)^G\right) \\
&= \varinjlim_{i,j} \mathcal{F}(U_i) = \mathcal{F}(U).
\end{aligned}$$

Here we have used [Sch3, Lemma 3.16] for the first and fourth equality and that $U_i \times_Y Y_j \to U_i$ is a Galois G/G_j-cover for the third equality. For the second equality we use that $\mathcal{F}(U_i \times_Y Y_j) \to \mathcal{F}(U_{i'} \times_Y Y_{j'})$ is injective for large enough i, since $U_{i'} \times_Y Y_{j'} \to U_i \times_Y Y_j$ is an étale cover for large enough i (by the definition of a pro-étale presentation), and that direct limits commute with taking invariants if the maps in the direct system are eventually injective. $\square$

Remark 2.25. *An elaboration of this argument allows one to deduce the full "Cartan–Leray spectral sequence"*

$$E_2^{pq} = H^p_{cts}(G, H^q(U_\infty, \mathcal{F})) \implies H^{p+q}(U, \mathcal{F})$$

from the Čech-to-derived functor spectral sequence, under the same assumptions on U and $\mathcal{F}$ (here $H^q(U_\infty, \mathcal{F})$ is given the discrete topology). This is implicit in [Sch3] *and used repeatedly there (though it should be noted that it is the Čech-to-derived functor spectral sequence itself that is referred to as the Cartan–Leray spectral sequence in* [Sch3]*, following SGA 4).*

Lemma 2.26. *Let $K_p \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ be an open compact subgroup and let $U \subseteq \mathcal{X}_{K_p}$ be an open subset. Put $U_\infty = q_{K_p}^{-1}(U) \subseteq \mathcal{X}_\infty$. Then $\mathcal{O}^+_{\mathcal{X}_{K_p}}(U) = \mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty)^{K_p}$, and hence $\mathcal{O}_{\mathcal{X}_{K_p}}(U) = \mathcal{O}_{\mathcal{X}_\infty}(U_\infty)^{K_p}$.*

Proof. We work on the pro-étale site of $\mathcal{X}_{K_p}$. We may assume that U is quasicompact; the general case follows by gluing. $U_\infty \to U$ is a perfectoid object of $\mathcal{X}_{K_p,\text{proét}}$ which is Galois with group K_p, and Lemma 2.24 implies that $(\mathcal{O}^+_{\mathcal{X}_{K_p}}/p^m)(U) = (\mathcal{O}^+_{\mathcal{X}_{K_p}}/p^m)(U_\infty)^{K_p}$ for all m. Taking the inverse limit we get, by definition, that $\widehat{\mathcal{O}}^+_{\mathcal{X}_{K_p}}(U) = \widehat{\mathcal{O}}^+_{\mathcal{X}_{K_p}}(U_\infty)^{K_p}$. By [Sch3, Corollary 6.19] we have $\widehat{\mathcal{O}}_{\mathcal{X}_{K_p}}(U) = \mathcal{O}_{\mathcal{X}_{K_p}}(U)$. It then follows from [Sch3, Lemma 4.2(ii),(v)] that $\widehat{\mathcal{O}}^+_{\mathcal{X}_{K_p}}(U) = \mathcal{O}^+_{\mathcal{X}_{K_p}}(U)$, which finishes the proof (note that $\widehat{\mathcal{O}}^+_{\mathcal{X}_{K_p}}(U_\infty) = \mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty)$). $\square$

Now let $U \subseteq \mathcal{X}_w$ be any rational subset, U_n its preimage in $\mathcal{X}_{K(p^n)}$ and put $U_\infty = q^{-1}(U) \subseteq \mathcal{X}_{\infty,w}$. Note that $U_n \to U$ is finite étale and so $\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)$ is a finite projective $\mathcal{O}_{\mathcal{X}}(U)$-module since $\mathcal{O}_{\mathcal{X}}(U)$ is Noetherian. Suppose that $\omega|_U$ is free, and choose a nowhere vanishing section (i.e. generator) $\eta_U \in H^0(U, \omega)$. Define $t_U \in \mathcal{O}_{\mathcal{X}_\infty}(U_\infty)$ by the equality

$$\mathfrak{s} = t_U \cdot q^*\eta_U.$$

Proposition 2.27. *We have $\gamma^* t_U = (b\mathfrak{z} + d)t_U$ for any $\gamma \in K_0(p)$, and t_U is a unit. Furthermore, for any $m \in \mathbb{Z}_{\geq 1}$ we may choose some large $n = n(m)$ and elements*

$$\begin{aligned} t_U^{(n)} &\in 1 + p^m \mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty), \\ s_{U,n} &\in \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)^\times, \end{aligned}$$

such that $t_U = t_U^{(n)} s_{U,n}$.

Proof. Since neither $\mathfrak{s}$ nor $q^*\eta_U$ vanish, t_U is a unit, and

$$(b\mathfrak{z} + d)t_U q^*\eta_U = (b\mathfrak{z} + d)\mathfrak{s} = \gamma^*\mathfrak{s} = (\gamma^* t_U) q^*\eta_U$$

since $\gamma^* q^* \eta_U = q^* \eta_U$, hence $\gamma^* t_U = (b\mathfrak{z} + d) t_U$. For the second sentence, first choose $M, N \in \mathbb{Z}_{\geq 0}$ such that $|p^M| \leq |p^N t_U| \leq 1$ (possible by quasicompactness of U_∞ and invertibility of t_U). Since $\mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty) = (\varinjlim \mathcal{O}^+_{\mathcal{X}_{K(p^n)}}(U_n))^\wedge$ (p-adic completion) we may find an $n = n(m)$ and $s \in \mathcal{O}^+_{\mathcal{X}_{K(p^n)}}(U_n)$ such that $p^N t_U - s \in p^{M+m} \mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty)$. Then we set $s_{U,n} = p^{-N} s$ and $t_U^{(n)} = t_U / s_{U,n}$ and these do the job. $\square$

This has several consequences - in particular, since $s_{U,n}$ is fixed by $K(p^n)$ and $t_U^{(n)}$ is p-adically close to 1, the element $\chi_{\mathcal{U}}(t_U^{(n)}) \in (\mathcal{O}_{\mathcal{X}_\infty}(U_\infty) \widehat{\otimes} A_{\mathcal{U}})^\times$ is well-defined (for m large enough) and satisfies $\gamma^*(\chi_{\mathcal{U}}(t_U^{(n)})) = \chi_{\mathcal{U}}(b\mathfrak{z} + d) \chi_{\mathcal{U}}(t_U^{(n)})$ for all $\gamma \in K(p^n)$. Recall that

$$\omega^\dagger_{\mathcal{U},w}(U) := \left\{ f \in \mathcal{O}_{\mathcal{X}_\infty}(U_\infty) \widehat{\otimes} A_{\mathcal{U}} \mid \gamma^* f = \chi_{\mathcal{U}}(b\mathfrak{z} + d)^{-1} f \, \forall \gamma \in K_0(p) \right\}.$$

Thus, given any $f \in \omega^\dagger_{\mathcal{U},w}(U)$, the element $f \cdot \chi_{\mathcal{U}}(t_U^{(n)}) \in \mathcal{O}_{\mathcal{X}_\infty}(U_\infty) \widehat{\otimes} A_{\mathcal{U}}$ lies in $(\mathcal{O}_{\mathcal{X}_\infty}(U_\infty) \widehat{\otimes} A_{\mathcal{U}})^{K(p^n)}$, which is equal to $\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \widehat{\otimes} A_{\mathcal{U}}$ by Lemma 2.23 and Lemma 2.26.

Furthermore, for any $\gamma \in G_n := K_0(p)/K(p^n)$, we have

$$\begin{aligned} \gamma^*(f \cdot \chi_{\mathcal{U}}(t_U^{(n)})) &= \chi_{\mathcal{U}}(b\mathfrak{z} + d)^{-1} \cdot f \cdot \chi_{\mathcal{U}}(\gamma^* t_U^{(n)}) \\ &= f \cdot \chi_{\mathcal{U}}(t_U^{(n)}) \cdot \chi_{\mathcal{U}} \left(\frac{\gamma^* t_U^{(n)}}{(b\mathfrak{z} + d) t_U^{(n)}} \right) \\ &= f \cdot \chi_{\mathcal{U}}(t_U^{(n)}) \cdot \chi_{\mathcal{U}} \left(\frac{s_{U,n}}{\gamma^* s_{U,n}} \right) \\ &= f \cdot \chi_{\mathcal{U}}(t_U^{(n)}) \cdot \chi_{\mathcal{U}}(j_{U,n}(\gamma))^{-1} \end{aligned}$$

where $j_{U,n}$ is the cocycle

$$\begin{aligned} j_{U,n}(\gamma) : G_n &\to \mathbb{Z}_p^\times \cdot \left(1 + p^{\min(w,m)} \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)^\circ \right) \\ \gamma &\mapsto \frac{\gamma^* s_{U,n}}{s_{U,n}}. \end{aligned}$$

In summary, we find that the map $f \mapsto f \cdot \chi_{\mathcal{U}}(t_U^{(n)})$ defines an $\mathcal{O}_{\mathcal{X}}(U)$-*module isomorphism* of $\omega^\dagger_{\mathcal{U},w}(U)$ onto the subspace of functions $f_0 \in \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \widehat{\otimes} A_{\mathcal{U}}$ fixed by the twisted action

$$\begin{aligned} \gamma : \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \widehat{\otimes} A_{\mathcal{U}} &\to \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \widehat{\otimes} A_{\mathcal{U}} \\ f_0 &\mapsto \chi_{\mathcal{U}}(j_{U,n}(\gamma)) \cdot \gamma^* f_0 \end{aligned}$$

of G_n. Since G_n is a finite group, the usual idempotent

$$e_n = \frac{1}{|G_n|} \sum_{\gamma \in G_n} \chi_{\mathcal{U}}(j_{U,n}(\gamma))^{-1} \gamma$$

in the group algebra $(\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)\widehat{\otimes}A_{\mathcal{U}})[G_n]$ defines an $\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}}$-module splitting of the inclusion $\omega^{\dagger}_{\mathcal{U},w}(U)\subseteq \mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)\widehat{\otimes}A_{\mathcal{U}}$. This realizes $\omega^{\dagger}_{\mathcal{U},w}(U)$ as a direct summand of a finite projective $\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}}$-module, and therefore $\omega^{\dagger}_{\mathcal{U},w}(U)$ is finite projective over $\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}}$ as desired. We may now prove the main result of this section:

Theorem 2.28. *We have $\omega^{\dagger}_{\mathcal{U},w}=\mathrm{Loc}(\omega^{\dagger}_{\mathcal{U},w}(\mathcal{X}_w))$ and $\omega^{\dagger}_{\mathcal{U},w}(\mathcal{X}_w)$ is a finite projective $\mathcal{O}_{\mathcal{X}}(\mathcal{X}_w)\widehat{\otimes}A_{\mathcal{U}}$-module of rank* 1. *Moreover, $\omega^{\dagger}_{\mathcal{U},w}$ is étale locally free.*

Proof. To prove that $\omega^{\dagger}_{\mathcal{U},w}=\mathrm{Loc}(\omega^{\dagger}_{\mathcal{U},w}(\mathcal{X}_w))$ and that it is locally projective of finite rank it suffices, by Theorem 6.20, to prove that $\omega^{\dagger}_{\mathcal{U},w}$ is a coherent $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}A_{\mathcal{U}}$-module (it is then locally projective by the above). To do this, we work locally using the U above. We wish to show that for $V\subseteq U$ (without loss of generality assume V rational) the natural map

$$(\mathcal{O}_{\mathcal{X}}(V)\widehat{\otimes}A_{\mathcal{U}})\otimes_{(\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}})}\omega^{\dagger}_{\mathcal{U},w}(U)\to\omega^{\dagger}_{\mathcal{U},w}(V)$$

is an isomorphism. To see this, note that by applying Lemma 6.7 twice the natural map

$$(\mathcal{O}_{\mathcal{X}}(V)\widehat{\otimes}A_{\mathcal{U}})\otimes_{(\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}})}(\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)\widehat{\otimes}A_{\mathcal{U}})\to\mathcal{O}_{\mathcal{X}_{K(p^n)}}(V_n)\widehat{\otimes}A_{\mathcal{U}}$$

is an isomorphism (note that $\mathcal{O}_{\mathcal{X}}(V)\otimes_{\mathcal{O}_{\mathcal{X}}(U)}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)\cong\mathcal{O}_{\mathcal{X}_{K(p^n)}}(V_n)$). This isomorphism matches up the idempotents on both sides, giving us the desired isomorphism. To prove that the rank is one note that $\mathcal{O}_{\mathcal{X}}(U)\to\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)$ is Galois with group G_n and by applying Lemma 6.7 twice we may deduce that $\mathcal{O}_{\mathcal{X}}(U)\widehat{\otimes}A_{\mathcal{U}}\to\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)\widehat{\otimes}A_{\mathcal{U}}$ is also Galois with group G_n. Moreover, the twisted action whose invariants give $\omega^{\dagger}_{\mathcal{U},w}(U)$ is a Galois descent datum, so we see that $\omega^{\dagger}_{\mathcal{U},w}(U)$ is the descent of a rank 1 free module, hence rank 1 projective as desired. Finally, note that this also proves that $\omega_{\mathcal{U},w}$ becomes trivial over U_n, which gives the final statement of the theorem. $\square$

The techniques of this proof also yield the following result.

Lemma 2.29. *Let $\mathcal{U}=(A_{\mathcal{U}},\chi_{\mathcal{U}})$ be a weight, and let $i:A_{\mathcal{U}}\to A_{\mathcal{Z}}$ be a surjection such that $\mathcal{Z}=(A_{\mathcal{Z}},\chi_{\mathcal{Z}}=i\circ\chi_{\mathcal{U}})$ is also a weight and such that* $\ker(i)$ *is generated by a regular element $x\in A_{\mathcal{U}}$. Then we have a natural exact sequence of sheaves*

$$0\to\omega^{\dagger}_{\mathcal{U},w}\xrightarrow{\cdot x}\omega^{\dagger}_{\mathcal{U},w}\to\omega^{\dagger}_{\mathcal{Z},w}\to 0$$

on $\mathcal{X}_w$, and an exact sequence of global sections

$$0\to H^0(\mathcal{X}_w,\omega^{\dagger}_{\mathcal{U},w})\xrightarrow{\cdot x}H^0(\mathcal{X}_w,\omega^{\dagger}_{\mathcal{U},w})\to H^0(\mathcal{X}_w,\omega^{\dagger}_{\mathcal{Z},w})\to 0.$$

Proof. There is certainly a (not necessarily exact) sequence of sheaves

$$0\to\omega^{\dagger}_{\mathcal{U},w}\xrightarrow{\cdot x}\omega^{\dagger}_{\mathcal{U},w}\to\omega^{\dagger}_{\mathcal{Z},w}\to 0,$$

and we check exactness of this sequence on a basis of suitably small open subsets $U \subset \mathcal{X}_w$ as in the proof of Theorem 2.28. By assumption we have a short exact sequence

$$0 \to A_{\mathcal{U}} \stackrel{\cdot x}{\to} A_{\mathcal{U}} \to A_{\mathcal{Z}} \to 0$$

of $A_{\mathcal{U}}$-modules. Tensoring this sequence over $A_{\mathcal{U}}$ with $A_{\mathcal{U}}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)$, we obtain by the flatness of $A_{\mathcal{U}}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n)$ over $A_{\mathcal{U}}$ (proved in the same way as Lemma 6.13(2)) a short exact sequence

$$0 \to A_{\mathcal{U}}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \stackrel{\cdot x}{\to} A_{\mathcal{U}}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \to A_{\mathcal{Z}}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_{K(p^n)}}(U_n) \to 0.$$

Applying the idempotent e_n as above gives a short exact sequence

$$0 \to \omega^{\dagger}_{\mathcal{U},w}(U) \stackrel{\cdot x}{\to} \omega^{\dagger}_{\mathcal{U},w}(U) \to \omega^{\dagger}_{\mathcal{Z},w}(U) \to 0$$

as desired.

Taking cohomology, we note that $H^1(\mathcal{X}_w, \omega^{\dagger}_{\mathcal{U},w}) = 0$ by Theorem 2.28 and Proposition 6.16, and the lemma follows. ☐

2.7. Comparison with other definitions of overconvergent modular forms. In this section we take $\mathcal{V} = (S_{\mathcal{V}}, \chi_{\mathcal{V}})$ to be an open affinoid weight, and we will compare our definition of overconvergent modular forms of weight $\mathcal{V}$ with those in the literature (all known to be equivalent). More specifically we will compare it to that of [Pil], trivially modified to our compact Shimura curves.

We now recall the definition of the "Pilloni torsor". This is the object denoted by $F_n^{\times}$ in [Pil] but we will use the notation $\mathcal{T}(n,v)$. For any $n \geq 1$ and any $v < \frac{p-1}{p^n}$, $\mathcal{T}(n,v)$ is a rigid space equipped with a smooth surjective morphism $\mathrm{pr} : \mathcal{T}(n,v) \to \mathcal{X}(v)$, where $\mathcal{X}(v) \subseteq \mathcal{X}$ is the locus where the Hodge height is $\leq v$. For any point $x \in \mathcal{X}(v)(C, \mathcal{O}_C)$, the $(C, \mathcal{O}_C)$-points in the fiber $\mathrm{pr}^{-1}(x)$ consists of the differentials $\eta \in \omega_{\mathcal{G}_A}$ that reduce to an element in the image of the Hodge-Tate map $(H_n^{\vee})^{\mathrm{gen}} \to \omega_{H_n}$, where H_n is the canonical subgroup of $\mathcal{G}$ of level n and $(H_n^{\vee})^{\mathrm{gen}} \subseteq H_n^{\vee}$ denotes the subset of generators. The set $\mathrm{pr}^{-1}(x)$ is a torsor for the group $\mathbb{Z}_p^{\times}(1 + p^{n - \frac{p^n-1}{p-1}\mathrm{Hdg}(x)}\mathcal{O}_C)$, and we think of $\mathcal{T}(n,v)$ as an open subspace of $\mathcal{T}$, where $\mathcal{T}$ is the total space of $e(0^*\Omega^1_{\mathcal{A}^{\mathrm{univ}}/\mathcal{X}(v)})$ (here $0 : \mathcal{X}(v) \to \mathcal{A}^{\mathrm{univ}}$ is the zero section; this is a line bundle on $\mathcal{X}(v)$). We remark that the canonical action of $\mathbb{Z}_p^{\times}$ on $\mathcal{T}$ preserves $\mathcal{T}(n,v)$ (as should be clear from the description of the fibres above). This induces an action on functions.

If we fix $w = n - \frac{p^n-1}{p-1}v$, then $\mathcal{X}(v) \subseteq \mathcal{X}_w$; this follows for example from [AIP, Proposition 3.2.1]. We will compare our construction of w-overconvergent modular forms over $\mathcal{X}(v)$ to the notion defined in [Pil]. For the purpose of the comparison, a *Pilloni form* of weight $\mathcal{V}$ over a quasi-compact open $U \subseteq \mathcal{X}(v)$ is defined to be an element $\mathbf{f} \in \mathcal{O}_{\mathcal{T}(n,v)}(\mathrm{pr}^{-1}(U))\widehat{\otimes}_{\mathbb{Q}_p} S_{\mathcal{V}}$ (i.e a function on $\mathrm{pr}^{-1}(U) \times \mathcal{V} \subseteq \mathcal{T}(n,v) \times \mathcal{W}$. Here we abuse notation and write $\mathcal{V}$ also for the image of the natural map into $\mathcal{W}$ induced by $\mathcal{V}$) such that $z.\mathbf{f} = \chi_{\mathcal{V}}^{-1}(z)\mathbf{f}$ for all $z \in \mathbb{Z}_p^{\times}$.

Now fix $U \subseteq \mathcal{X}(v)$ as used in the previous section, such that we have a nowhere vanishing $\eta_U \in \omega(U)$. We freely use the notation of the previous section with one exception: we use r instead of the already used letter n. Thus we have functions $t_U^{(r)} \in 1 + p^m \mathcal{O}^+_{\mathcal{X}_\infty}(U_\infty)$ and $s_{U,r} \in \mathcal{O}_{\mathcal{X}_{K(p^r)}}(U_r)^\times$ such that

$$\mathfrak{s} = t_U^{(r)} s_{U,r} \eta_U .$$

Let $\mathcal{T}_{U,r}(n,v) = \mathcal{T}(n,v) \times_U U_r$. It is an open subset of $\mathcal{T}_{U,r} := \mathcal{T} \times_U U_r$. It inherits commuting actions of $G_r := K_0(p)/K(p^r)$ and $\mathbb{Z}_p^\times$.

Lemma 2.30. *Assume that $m \geq n$ and $r \geq n$. Then $s_{U,r}\eta_U$ trivializes $\mathcal{T}_{U,r}(n,v)$.*

Proof. We think of $s_{U,r}\eta_U$ as a section $U_r \to \mathcal{T}_{U,r}$ and we wish to show that the image lands inside $\mathcal{T}_{U,r}(n,v)$, for which it is enough to argue on geometric points. Take a $(C, \mathcal{O}_C)$-point $\bar{x}$ and lift it to a $(C, \mathcal{O}_C)$-point x of $\mathcal{X}_\infty$. Then we see that

$$(s_{U,r}\eta_U)(\bar{x}) = ((t_U^{(r)})^{-1}\mathfrak{s})(x).$$

If $\bar{x} = (A, \bar{\alpha}) \in \mathcal{Y}^{gd}_{K(p^r)}$ we put $x = (A, \alpha)$. Then $\mathfrak{s}(A,\alpha) = \mathrm{HT}_A(\alpha(e_2))$ by definition and this maps to a generator of $H_n^\vee$ via the canonical map $\mathcal{G}[p^n] \cong \mathcal{G}[p^n]^\vee \to H_n^\vee$, so $\mathfrak{s}(A,\alpha)$ lies in the fibre of $\mathcal{T}_{U,r}(n,v)$ over $(A,\bar{\alpha})$. This proves that $\mathfrak{s}(x)$ lies in the fibre of $\mathcal{T}_{U,r}(n,v)$ over $\bar{x}$ for all x. The result follows since $t_U^{(r)}$ is small. □

The assumption $m \geq n$ and $r \geq n$ will be in force throughout the rest of this section so that the Lemma applies. Recall what we proved in the process of proving Theorem 2.28: w-overconvergent modular forms f of weight $\mathcal{V}$ over U identifies, via the map $f \mapsto f_0 := \chi_{\mathcal{V}}(t_U^{(r)})f$, with functions $f_0 \in \mathcal{O}_{\mathcal{X}(p^r)}(U_r)\widehat{\otimes}_{\mathbb{Q}_p} S_{\mathcal{V}}$ such that

$$\gamma^* f_0 = \chi_{\mathcal{V}}(j_{U,r}(\gamma))^{-1} f_0$$

for all $\gamma \in G_r$.

Proposition 2.31.
1. *The space of Pilloni forms of weight $\mathcal{V}$ over U is isomorphic, via pullback, to the space of functions $\mathbf{g}$ on $\mathcal{T}_{U,r}(n,v) \times \mathcal{V}$ such that $z.\mathbf{g} = \chi_{\mathcal{V}}(z)^{-1}\mathbf{g}$ for all $z \in \mathbb{Z}_p^\times$ and $\gamma^*\mathbf{g} = \mathbf{g}$ for all $\gamma \in G_r$.*
2. *The space of w-overconvergent modular forms of weight $\mathcal{V}$ over U is isomorphic to the space of functions g_0 on $\mathcal{T}_{U,r}(n,v) \times \mathcal{V}$ such that $z.g_0 = g_0$ for all $z \in \mathbb{Z}_p^\times$ and $\gamma^* g_0 = \chi_{\mathcal{V}}(j_{U,r}(\gamma))^{-1} g_0$ for all $\gamma \in G_r$. The isomorphism is the map $f \mapsto f_0$ above composed with pullback from $U_r \times \mathcal{V}$ to $T_{U,r}(n,v) \times \mathcal{V}$.*

Proof. (1) follows from the fact that $\mathrm{pr}^{-1}(U) \times \mathcal{V}$ is the quotient of $T_{U,r}(n,v) \times \mathcal{V}$ by G_r. The proof of (2) is a similar descent. □

To identify the two spaces, it remains to go from one kind of function on $\mathcal{T}_{U,r}(n,w) \times \mathcal{V}$ to the other. The key lies in trivializing the cocycle $\chi_{\mathcal{V}}(j_{U,r}(\gamma))$. Note that $s_{U,r}\eta_U$, as a nowhere vanishing section of $\mathcal{T}_{U,r}$ also canonically defines a function on $\mathcal{T}_{U,r}$ *minus the zero section* which we will denote by $(s_{U,r}\eta_U)^\vee$. The same applies to η_U itself and we have $(s_{U,r}\eta_U)^\vee = s_{U,r}^{-1}\eta_U^\vee$. By restriction we obtain functions on $\mathcal{T}_{U,r}(n,v)$.

Lemma 2.32. $(s_{U,r}\eta_U)^\vee \in \mathbb{Z}_p^\times.(1+p^w\mathcal{O}^+_{\mathcal{T}_{U,r}}(\mathcal{T}_{U,r}(n,v)))$.

Proof. This follows directly from Lemma 2.30. □

This implies that we may apply $\chi_{\mathcal{V}}$ to $(s_{U,r}\eta_U)^\vee$, and we may therefore define a $\mathcal{O}_{\mathcal{X}(v)}(U)\widehat{\otimes}_{\mathbb{Q}_p}S_{\mathcal{V}}$-module isomorphism

$$\Phi : \mathcal{O}_{\mathcal{T}_{U,r}}(\mathcal{T}_{U,r}(n,v))\widehat{\otimes}_{\mathbb{Q}_p}S_{\mathcal{V}} \to \mathcal{O}_{\mathcal{T}_{U,r}}(\mathcal{T}_{U,r}(n,v))\widehat{\otimes}_{\mathbb{Q}_p}S_{\mathcal{V}}$$

which is simply multiplication by $\chi_{\mathcal{V}}((s_{U,r}\eta_U)^\vee)$:

$$\Phi(h) = \chi_{\mathcal{V}}((s_{U,r}\eta_U)^\vee)h.$$

Theorem 2.33. *The image of the space of functions in part (1) of Proposition 2.31 under Φ is the space of functions in part (2) of Proposition 2.31. Moreover, the induced isomorphism between the space of w-overconvergent modular forms of weight $\mathcal{V}$ over U and the space of Pilloni forms of weight $\mathcal{V}$ over U is independent of all choices and hence the isomorphisms for varying U glue together to an isomorphism of sheaves over $\mathcal{X}_w$.*

Proof. For the first part, the key thing to notice is that

$$j_{U,r}(\gamma) = \frac{(s_{U,r}\eta_U)^\vee}{\gamma^*(s_{U,r}\eta_U)^\vee}$$

and hence that

$$\chi_{\mathcal{V}}(j_{U,r}(\gamma))^{-1} = \frac{\chi_{\mathcal{V}}(\gamma^*(s_{U,r}\eta_U)^\vee)}{\chi_{\mathcal{V}}((s_{U,r}\eta_U)^\vee)}.$$

The rest is then a straightforward computation. For the second part we remark that the composite isomorphism from w-overconvergent modular forms to Pilloni forms is given by

$$f \mapsto \chi_{\mathcal{V}}(t_U^{(r)})\chi_{\mathcal{V}}((s_{U,r}\eta_U)^\vee)^{-1}f$$

and then descending the right hand side to $\mathrm{pr}^{-1}(U)\times\mathcal{V}$. Morally, the right hand side is equal to $\chi_{\mathcal{V}}(\mathfrak{s}^\vee)^{-1}f$ (recall that $t_U^{(r)}s_{U,r}\eta_U = \mathfrak{s}$) and hence independent of the choices made. To turn this into a rigorous argument is straightforward but tedious and notationally cumbersome. We leave the details to the interested reader. □

3. Overconvergent modular symbols

We will recall some material on overconvergent modular symbols in the form we need. Most of these constructions are probably well known with the exception of certain filtrations defined in [Han2].

3.1. Basic definitions and the filtrations. Let $\mathbf{A}^s$ be the affinoid ring over $\mathbb{Q}_p$ defined by

$$\mathbf{A}^s = \{f : \mathbb{Z}_p \to \mathbb{Q}_p \mid f \,\text{analytic on each}\, p^s\mathbb{Z}_p - \text{coset}\}.$$

We let $\mathbf{A}^{s,\circ}$ denote the subring of powerbounded elements of $\mathbf{A}^s$. Given a weight $\mathcal{U}$, consider the module

$$\mathbf{A}^{s,\circ}_{\mathcal{U}} = \mathbf{A}^{s,\circ}\widehat{\otimes}_{\mathbb{Z}_p} A^\circ_{\mathcal{U}}$$

where the completion is the *p-adic completion*. Recall the Amice basis $(e^s_j)_{j\geq 0}$ of $\mathbf{A}^{s,\circ}$ from Theorem 2.5. Using it we may write $\mathbf{A}^{s,\circ}_{\mathcal{U}}$ as

$$\mathbf{A}^{s,\circ}_{\mathcal{U}} = \widehat{\bigoplus}_{j\geq 0} A^\circ_{\mathcal{U}} e^s_j$$

and hence view elements of $\mathbf{A}^{s,\circ}_{\mathcal{U}}$ as functions $\mathbb{Z}_p \to A^\circ_{\mathcal{U}}$. For any $s \geq s_{\mathcal{U}}$ and any $d \in \mathbb{Z}_p^\times, c \in p\mathbb{Z}_p$, $x \mapsto \chi_{\mathcal{U}}(cx+d)$ defines an element of $\mathbf{A}^{s,\circ}_{\mathcal{U}}$ (by calculations very similar to those in the proof of Proposition 2.6), and we then consider $\mathbf{A}^{s,\circ}_{\mathcal{U}}$ endowed with the right $\Delta_0(p)$-action

$$(f \cdot_{\mathcal{U}} \gamma)(x) = \chi_{\mathcal{U}}(cx+d) f\left(\frac{ax+b}{cx+d}\right);$$

one checks without too much trouble that $f \cdot_{\mathcal{U}} \gamma \in \mathbf{A}^{s,\circ}_{\mathcal{U}}$. We set $\mathbf{D}^{s,\circ}_{\mathcal{U}} = \mathrm{Hom}_{A^\circ_{\mathcal{U}}}(\mathbf{A}^{s,\circ}_{\mathcal{U}}, A^\circ_{\mathcal{U}})$ and $\mathbf{D}^s_{\mathcal{U}} = \mathbf{D}^{s,\circ}_{\mathcal{U}}[\frac{1}{p}]$, with the dual left action. It is the continuous $A_{\mathcal{U}}[\frac{1}{p}]$-dual of the $A_{\mathcal{U}}[\frac{1}{p}]$-Banach module $\mathbf{A}^s_{\mathcal{U}} := \mathbf{A}^{s,\circ}_{\mathcal{U}}[\frac{1}{p}]$. Note that

$$\mathbf{D}^{s,\circ}_{\mathcal{U}} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{A}^{s,\circ}, A^\circ_{\mathcal{U}}).$$

The Amice basis gives an orthonormal $A_{\mathcal{U}}[\frac{1}{p}]$-basis of $\mathbf{A}^s_{\mathcal{U}}$ and induces an isomorphism

$$\mathbf{D}^s_{\mathcal{U}} \xrightarrow{\sim} \prod_{j\geq 0} A^\circ_{\mathcal{U}}$$

given by

$$\mu \mapsto (\mu(e^s_j))_{j\geq 0}.$$

Proposition 3.1. *Let $\mathcal{U} = (R_{\mathcal{U}}, \chi_{\mathcal{U}})$ be a small weight and let $s \geq 1 + s_{\mathcal{U}}$.*

(1) $\mathbf{D}^{s,\circ}_{\mathcal{U}}$ *admits a decreasing $\Delta_0(p)$-stable filtration by sub-$R_{\mathcal{U}}$-modules*

$$\mathbf{D}^{s,\circ}_{\mathcal{U}} = \mathrm{Fil}^0\mathbf{D}^{s,\circ}_{\mathcal{U}} \supset \mathrm{Fil}^1\mathbf{D}^{s,\circ}_{\mathcal{U}} \supset \cdots \supset \mathrm{Fil}^i\mathbf{D}^{s,\circ}_{\mathcal{U}} \supset \cdots$$

such that each quotient $\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}$ is a finite abelian group of exponent p^k, the group $K(p^{s+k})$ acts trivially on $\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}$, and $\mathbf{D}^{s,\circ}_{\mathcal{U}} \cong \varprojlim_k \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}$.

(2) $\mathbf{D}^{s,\circ}_{\mathcal{U}}$*, with the topology induced by the submodules $(\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}})_{k\geq 0}$, is a profinite flat $\mathbb{Z}_p$-module.*

Proof. We will only recall how the filtrations are constructed and refer to [Han2, §2.2] for the details (note that (2) is a consequence of (1)). The module $\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}$ is defined to be the kernel of the natural map

$$\mathbf{D}^{s,\circ}_{\mathcal{U}} \to \mathbf{D}^{s-1,\circ}_{\mathcal{U}}/\mathfrak{a}^k_{\mathcal{U}}\mathbf{D}^{s-1,\circ}_{\mathcal{U}},$$

where we recall that $\mathfrak{a}_{\mathcal{U}}$ is our fixed choice of ideal of definition for the profinite topology on $R_{\mathcal{U}}$. □

Let $k \geq 2$ be an integer and let A be a ring. We let $\mathscr{L}_k(A)$ denote the module of polynomials $A[X]^{\deg \leq k-2}$ with left $M_2(A)$-action

$$(\delta \cdot_k p)(X) = (d+bX)^{k-2} p\left(\frac{c+aX}{d+bX}\right),$$

and set in particular $\mathscr{L}_k = \mathscr{L}_k(\mathbb{Q}_p)$ and $\mathscr{L}_k^\circ = \mathscr{L}_k(\mathbb{Z}_p)$. By direct calculation, the map

$$\begin{array}{rcl} \rho_k : \mathbf{D}_k^{s,\circ} & \to & \mathscr{L}_k^\circ \\ \mu & \mapsto & \int (1+Xx)^{k-2}\mu(x) = \\ & & = \sum_{j=0}^{k-2} \left(\begin{array}{c} k-2 \\ j \end{array} \right) \mu(x^j) X^j \end{array}$$

is $\Delta_0(p)$-equivariant.

Definition 3.2. *The integration map in weight k, denoted i_k, is the $\Delta_0(p)$-equivariant map $i_k : \mathbf{D}_{\mathcal{U}}^{s,\circ} \to \mathscr{L}_k^\circ$ defined by $i_k = \rho_k \circ \sigma_k$.*

3.2. Slope decompositions. We will recall material from [AS] and [Han1] in order to define slope decompositions on the spaces of overconvergent modular symbols that we are interested in. We will mildly abuse notation by writing $X(\mathbb{C})$ for the complex Shimura curve of level $K := K^p K_0(p)$ (viewed as a Riemann surface). Any $K_0(p)$-module M defines a local system on $X(\mathbb{C})$ which we will also denote by M. If M in addition is a $\Delta_0(p)$-module then we get induced Hecke actions as well. The spaces of overconvergent modular symbols that we are interested in are

$$H^1(X(\mathbb{C}), \mathbf{D}_{\mathcal{U}}^s)$$

for open weights $\mathcal{U}$. We will give these spaces slope decompositions using the methods of [Han1]. Almost everything goes through *verbatim* and we will content ourselves with a brief discussion. Throughout this section, h will denote a non-negative rational number. For definitions and generalities on slope decompositions that we will use we refer to [JN, §2.2] (we remark that the notion introduced in [AS, §4] needs slight tweaking for the purpose of constructing the whole eigencurve, as opposed to just local pieces).

Let us start by recalling the general setup from [Han1, §2.1]. We have the functorial *adelic* (co)chain complexes $C_\bullet^{\mathrm{ad}}(K,-)$ and $C_{\mathrm{ad}}^\bullet(K,-)$ whose (co)homology functorially computes homology and cohomology of local systems attached to $\Delta_0(p)$-modules, respectively. Fix once and for all a choice of triangulation of $X(\mathbb{C})$. This choice gives us functorial complexes $C_\bullet(K,-)$ and $C^\bullet(K,-)$, called *Borel-Serre complexes*, which are chain homotopic to $C_\bullet^{\mathrm{ad}}(K,-)$ and $C_{\mathrm{ad}}^\bullet(K,-)$ respectively. We fix such a chain homotopy once and for all. The key features of the Borel-Serre complexes are that there are non-negative integers $r(i)$ such that $r(i) = 0$ for $i < 0$ and for i sufficiently large, and such that for any $R[\Delta_0(p)]$-module M (where R is any commutative ring),

$$C_i(K, M) \cong M^{r(i)}$$

functorially as R-modules (and similarly for $C^\bullet(K, -)$, with the same integers $r(i)$). Therefore the total complex $\bigoplus_i C_i(K, M)$ inherits properties of M, such as being Banach if R is a $\mathbb{Q}_p$-Banach algebra (and also orthonormalizability). In this case, we have a canonical and functorial topological duality isomorphism

$$C^\bullet(K, \mathrm{Hom}_{R,cts}(M, P)) \cong \mathrm{Hom}_{R,cts}(C_\bullet(K, M), P)$$

where M and P are Banach R-modules.

We now follow [Han1, §3.1] using the orthonormalizable $A_{\mathcal{U}}[\frac{1}{p}]$-module $\mathbf{A}^s_{\mathcal{U}}$, where $\mathcal{U} = (A_{\mathcal{U}}, \chi_{\mathcal{U}})$ is an open weight. Our U_p-operator is given by the double coset $K_0(p) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_0(p)$ and the formalism gives us lifts to our Borel-Serre complexes that we denote by $\tilde{U}$. We note that $\tilde{U} \in \mathrm{End}_{A_{\mathcal{U}}[\frac{1}{p}]}(C_\bullet(K, \mathbf{A}^s_{\mathcal{U}}))$ is compact and we denote its Fredholm determinant by $F_{\mathcal{U}}(X) \in A_{\mathcal{U}}[\frac{1}{p}][[X]]$. The proof of [Han1, Proposition 3.1.1] goes through for small weights to show that this definition is independent of s. Thus the existence of a slope $\leq h$-decomposition of $C_\bullet(K, \mathbf{A}^s_{\mathcal{U}})$ is equivalent to the existence of a slope $\leq h$-factorization of $F_{\mathcal{U}}(X)$. If $\mathcal{V}$ is another open weight with $\mathcal{V}^{\mathrm{rig}} \subseteq \mathcal{U}^{\mathrm{rig}}$, then the relation $\mathbf{A}^s_{\mathcal{V}} \cong \mathbf{A}^s_{\mathcal{U}} \widehat{\otimes}_{A_{\mathcal{U}}[\frac{1}{p}]} A_{\mathcal{V}}[\frac{1}{p}]$ implies that the $F_{\mathcal{U}}$ glue to a power series $F(X) \in \mathcal{O}_{\mathscr{W}}(\mathscr{W})[[X]]$. Also [Han1, Proposition 3.1.2] goes through in this more general setting: the slope $\leq h$-subcomplex $C_\bullet(K, \mathbf{A}^s_{\mathcal{U}})_{\leq h}$ of $C_\bullet(K, \mathbf{A}^s_{\mathcal{U}})$ is independent of s (if it exists) and if $\mathcal{V}^{\mathrm{rig}} \subseteq \mathcal{U}^{\mathrm{rig}}$ then we have a canonical isomorphism

$$C_\bullet(K, \mathbf{A}^s_{\mathcal{U}})_{\leq h} \otimes_{A_{\mathcal{U}}[\frac{1}{p}]} A_{\mathcal{V}}[\frac{1}{p}] \cong C_\bullet(K, \mathbf{A}^s_{\mathcal{V}})_{\leq h}.$$

We say that $(\mathcal{U}, h)$ is a *slope datum* if $C_\bullet(K, \mathbf{A}^s_{\mathcal{U}})$ has a slope $\leq h$-decomposition or equivalently if $F_{\mathcal{U}}$ has a slope $\leq h$-factorization. We have the following version of [Han1, Proposition 3.1.3]:

Proposition 3.3. *Assume that $(\mathcal{U}, h)$ is a slope datum and that $\mathcal{V} = (S_{\mathcal{V}}, \chi_{\mathcal{V}})$ is an open affinoid weight with $\mathcal{V} \subseteq \mathcal{U}^{\mathrm{rig}}$. Then there is a canonical isomorphism*

$$H_*(X(\mathbb{C}), \mathbf{A}^s_{\mathcal{U}})_{\leq h} \otimes_{A_{\mathcal{U}}[\frac{1}{p}]} S_{\mathcal{V}} \cong H_*(X(\mathbb{C}), \mathbf{A}^s_{\mathcal{V}})_{\leq h}$$

for any $s \geq s_{\mathcal{U}}$.

Proof. When $\mathcal{U}$ is affinoid this is exactly [Han1, Proposition 3.1.3], so assume without loss of generality that $\mathcal{U} = (A_{\mathcal{U}}, \chi_{\mathcal{U}})$ is small. The same proof will go through if we can verify that $A_{\mathcal{U}}[\frac{1}{p}] \to S_{\mathcal{V}}$ is flat. This is standard. One proof goes as follows: Consider the adic space $\mathrm{Spa}(A_{\mathcal{U}}, A_{\mathcal{U}})$ and pick a rational subset V in its generic fibre which contains $\mathcal{V}$. Then $\mathcal{O}(V) \to S_{\mathcal{V}}$ is flat since it corresponds to an open immersion of affinoids in rigid geometry, and

$A_{\mathcal{U}} \to \mathcal{O}(V)$ is flat since it is a rational localization in the theory of adic spaces when the rings of definition are Noetherian. $\square$

We may then give the analogue of the above proposition for $H^*(X(\mathbb{C}), \mathbf{D}_{\mathcal{U}}^s)$ (cf. [Han1, Proposition 3.1.5]).

Proposition 3.4. *Assume that $(\mathcal{U}, h)$ is a slope datum. Then $C^\bullet(K, \mathbf{D}_{\mathcal{U}}^s)$, and hence $H^*(X(\mathbb{C}), \mathbf{D}_{\mathcal{U}}^s)$, admit slope $\leq h$-decompositions. If furthermore $\mathcal{V} \subseteq \mathcal{U}^{\mathrm{rig}}$ is an open affinoid weight, then there are canonical isomorphisms*

$$C^\bullet(K, \mathbf{D}_{\mathcal{U}}^s)_{\leq h} \otimes_{A_{\mathcal{U}}[\frac{1}{p}]} S_{\mathcal{V}} \cong C^\bullet(K, \mathbf{D}_{\mathcal{V}}^s)_{\leq h}$$

and

$$H^*(X(\mathbb{C}), \mathbf{D}_{\mathcal{U}}^s)_{\leq h} \otimes_{A_{\mathcal{U}}[\frac{1}{p}]} S_{\mathcal{V}} \cong H^*(X(\mathbb{C}), \mathbf{D}_{\mathcal{V}}^s)_{\leq h}.$$

Proof. Using the duality $C^\bullet(K, \mathbf{D}_{\mathcal{U}}^s) \cong \mathrm{Hom}_{A_{\mathcal{U}}[\frac{1}{p}], cts}(C_\bullet(K, \mathbf{A}_{\mathcal{U}}^s), A_{\mathcal{U}}[\frac{1}{p}])$ and the flatness of $A_{\mathcal{U}}[\frac{1}{p}] \to S_{\mathcal{V}}$ the proof of [Han1, Proposition 3.1.5] goes through *verbatim.* $\square$

4. Sheaves on the pro-étale site

In the next two sections we will often consider Shimura curves over $\mathbb{C}_p$ as well as $\mathbb{Q}_p$. Any rigid analytic variety we have defined may be base changed from $\mathbb{Q}_p$ to $\mathbb{C}_p$, and we will denote this base change by a subscript $-_{\mathbb{C}_p}$, e.g. $\mathcal{X}_{\mathbb{C}_p}$. The space $\mathcal{X}_{\mathbb{C}_p}$ may be considered as an object of $\mathcal{X}_{\text{proét}}$ using the pro-étale presentation $\mathcal{X}_{\mathbb{C}_p} = \varprojlim_K \mathcal{X}_K$, where $K/\mathbb{Q}_p$ is finite and $\mathcal{X}_K$ denotes the base change of $\mathcal{X}$ to K. Note that the slice $\mathcal{X}_{\text{proét}}/\mathcal{X}_{\mathbb{C}_p}$ is equivalent to $\mathcal{X}_{\mathbb{C}_p,\text{proét}}$ as sites (use [Sch3, Proposition 3.15] and [Sch1, Proposition 7.4]). We define $\mathcal{X}_\infty := \varprojlim_K \mathcal{X}_{\infty,K}$ (with K as above) where the inverse limit is taken in the category of perfectoid spaces. Equivalently, we may define it as (the perfectoid space corresponding to) the perfectoid object $\varprojlim_K \mathcal{X}_{\infty,K}$ in $\mathcal{X}_{\text{proét}}$; it lives in $\mathcal{X}_{\text{proét}}/\mathcal{X}_{\mathbb{C}_p} \cong \mathcal{X}_{\mathbb{C}_p,\text{proét}}$. Similar remarks apply to $\mathcal{X}_w$, $\mathcal{X}_{\infty,w}$ et cetera.

4.1. A handy lemma. Let X be a rigid analytic variety over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, G a profinite group, and X_∞ a perfectoid space with $X_\infty \to X$ a pro-étale G-covering of X, i.e. $X_\infty \sim \varprojlim_j X_j$ where $(X_j)_{j\in J}$ is an inverse system of rigid analytic varieties finite étale over X equipped with a right action of G, such that each $X_j \to X$ is a finite étale G/G_j-covering for $G_j \subseteq G$ a cofinal sequence of normal open subgroups. We can and often will view X_∞ as an object in the pro-étale site of X. Let M be a profinite flat $\mathbb{Z}_p$-module equipped with a continuous left G-action. Define a sheaf on the pro-étale site of X by

$$(M \widehat{\otimes} \mathcal{O}_X)(V) = \left(M \widehat{\otimes} \widehat{\mathcal{O}}_X(V \times_X X_\infty) \right)^G,$$

where V is a qcqs object in $X_{\text{proét}}$. Here the right action of G on the tower $(X_i)_i$ induces a left action on $\widehat{\mathcal{O}}_X(V_\infty)$, and the action indicated is the diagonal left action $g \cdot (m \otimes f) = gm \otimes g^* f$.

Lemma 4.1. *Given any presentation $M = \varprojlim_i M_i$, $M_i = M/I_i$ as in Definition 6.3 with each I_i preserved by G, the sheaf $M\widehat{\otimes}\mathcal{O}_X$ coincides with the sheaf*

$$\widehat{\mathbb{Q}}_p \otimes_{\widehat{\mathbb{Z}}_p} \varprojlim_i \left(\nu^* \widetilde{M}_i \otimes_{\mathbb{Z}_p} \mathcal{O}_X^+\right),$$

where $\widetilde{M}_i$ is the locally constant sheaf on $X_{\text{ét}}$ associated with M_i, and we recall that $\nu : X_{\text{proét}} \to X_{\text{ét}}$ is the canonical morphism of sites.

Proof. We may check this on affinoid perfectoid $V \in X_{\text{proét}}$ since these form a basis of $X_{\text{proét}}$ (and are qcqs). Set $V_\infty = V \times_X X_\infty$, this is also affinoid perfectoid (see e.g. the proof of [Sch3, Lemma 4.6]). Each M_i is finite abelian group of p-power exponent. The key observation is then that there is an almost equality

$$M_i \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X^+(V_\infty) =^a (\nu^* \widetilde{M}_i \otimes_{\mathbb{Z}_p} \mathcal{O}_X^+)(V_\infty)$$

using [Sch3, Lemma 4.10]. Taking G-invariants we see that, since $V_\infty \to V$ is a G-cover,

$$\left(M_i \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_X^+(V_\infty)\right)^G =^a (\nu^* \widetilde{M}_i \otimes_{\mathbb{Z}_p} \mathcal{O}_X^+)(V)$$

by Lemma 2.24. Now take inverse limits and invert p, and use that these operations commute with taking G-invariants. □

4.2. Sheaves of overconvergent distributions. We introduce certain pro-étale sheaves that compute overconvergent modular symbols and carry a Galois action. When $\mathbb{L}$ is a locally constant constructible sheaf on the étale site of a rigid analytic variety, we may pull it back to a sheaf $\nu^*\mathbb{L}$ on the pro-étale site. We will often abuse notation and denote $\nu^*\mathbb{L}$ by $\mathbb{L}$ as well; by [Sch3, Corollary 3.17] there is little harm in this. For the rest of this section we will let $\mathcal{U} = (R_\mathcal{U}, \chi_\mathcal{U})$ be a *small* weight. Note that the $K_0(p)$-modules $\mathbf{D}_\mathcal{U}^{s,\circ}/\mathrm{Fil}^k\mathbf{D}_\mathcal{U}^{s,\circ}$ define locally constant constructible sheaves on the étale site $\mathcal{X}_{\text{ét}}$.

Definition 4.2. *Let $\mathcal{U}$ be a small weight and let $s \geq 1 + s_\mathcal{U}$. Set*

$$\begin{aligned}\mathbf{V}_\mathcal{U}^{s,\circ} &= \varprojlim_k H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}_\mathcal{U}^{s,\circ}/\mathrm{Fil}^k\mathbf{D}_\mathcal{U}^{s,\circ}) \\ &= \varprojlim_k H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}_\mathcal{U}^{s,\circ}/\mathrm{Fil}^k\mathbf{D}_\mathcal{U}^{s,\circ})\end{aligned}$$

and

$$\mathbf{V}_{\mathcal{U},\mathcal{O}_{\mathbb{C}_p}}^{s,\circ} = \varprojlim_k \left(H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}_\mathcal{U}^{s,\circ}/\mathrm{Fil}^k\mathbf{D}_\mathcal{U}^{s,\circ}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}\right).$$

We also define $\mathbf{V}_\mathcal{U}^s = \mathbf{V}_\mathcal{U}^{s,\circ}[\frac{1}{p}]$ and $\mathbf{V}_{\mathcal{U},\mathbb{C}_p}^s = \mathbf{V}_{\mathcal{U},\mathcal{O}_{\mathbb{C}_p}}^{s,\circ}[\frac{1}{p}]$.

These objects carry natural global Galois actions. We have the following comparison:

Proposition 4.3. *There is a canonical Hecke-equivariant isomorphism $\mathbf{V}_\mathcal{U}^s \cong H^1(X(\mathbb{C}), \mathbf{D}_\mathcal{U}^s)$.*

Proof. Artin's comparison theorem between étale and singular cohomology gives us canonical Hecke-equivariant isomorphisms

$$H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \cong H^1(X(\mathbb{C}), \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}})$$

for all k. Now take inverse limits and invert p. For this to do the job we need to be able to commute the inverse limit and the H^1 on the right hand side. But we may do this since the relevant higher inverse limits vanish, by finiteness of $\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}$ and by applying [Sch3, Lemma 3.18]. □

We now introduce a sheaf on $\mathcal{X}_{\text{proét}}$ that computes overconvergent modular symbols.

Proposition 4.4. *Let $\mathcal{U}$ be a small weight. There is a canonical Hecke- and Galois-equivariant isomorphism*

$$\mathbf{V}^s_{\mathcal{U},\mathbb{C}_p} \cong H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}\mathbf{D}^s_{\mathcal{U}})$$

where $\mathcal{O}\mathbf{D}^s_{\mathcal{U}}$ is the sheaf on $\mathcal{X}_{\text{proét}}$ defined by

$$\mathcal{O}\mathbf{D}^s_{\mathcal{U}} := \left(\varprojlim_k \left((\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^+_{\mathcal{X}_{\mathbb{C}_p}}\right)\right)\left[\frac{1}{p}\right].$$

Proof. By [Sch3, Theorem 5.1]), we have an almost isomorphism

$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} \overset{a}{\cong} H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, (\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^+_{\mathcal{X}_{\mathbb{C}_p}})$$

of $\mathcal{O}_{\mathbb{C}_p}$-modules. Passing to the inverse limit over k, note that on one hand, the higher inverse limits of any $H^i_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}$ vanish since $H^i_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}})$ is finite (by the finiteness properties of $\mathrm{Fil}^\bullet\mathbf{D}^{s,\circ}_{\mathcal{U}}$), thus $H^i_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}$ satisfy the Mittag-Leffler condition. On the other hand the higher inverse limit

$$R^i \varprojlim_k ((\mathbf{D}^{s,\circ}_{\mathcal{U}}/\mathrm{Fil}^k\mathbf{D}^{s,\circ}_{\mathcal{U}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^+_X)$$

vanishes for $i > 0$. This follows from [Sch3, Lemma 3.18] upon noting that (in the notation of Lemma 4.1, with $X = \mathcal{X}_{\mathbb{C}_p}$, $X_\infty = \mathcal{X}_{\infty,\mathbb{C}_p}$ and $G = K_0(p)$) objects of the form V_∞ give a basis of $\mathcal{X}_{\mathbb{C}_p,\text{proét}}$ satisfying the hypotheses of that lemma. Thus we may commute the inverse limit with taking cohomology on the right hand side. Inverting p we get the result. □

Let $V = \varprojlim_i V_i \to \mathcal{X}$ be a pro-étale presentation of an affinoid perfectoid object of $\mathcal{X}_{\text{proét}}$, and let $V_\infty = V \times_{\mathcal{X}} \mathcal{X}_\infty$. This is still affinoid perfectoid and $V_\infty \to V$ is pro-étale. The following is immediate from Lemma 4.1:

Lemma 4.5. *Let $\mathcal{U}$ be a small weight. The sheaf $\mathcal{O}\mathbf{D}^s_{\mathcal{U}}$ admits the following explicit description on qcqs $V \in \mathcal{X}_{\text{proét}}$:*

$$\mathcal{O}\mathbf{D}^s_{\mathcal{U}}(V) = \left(\mathbf{D}^{s,\circ}_{\mathcal{U}} \widehat{\otimes} \widehat{\mathcal{O}}_{\mathcal{X}}(V_\infty)\right)^{K_0(p)}.$$

We remark that the sheaf $V_\infty \mapsto \mathbf{D}_{\mathcal{U}}^{s,\circ}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}}(V_\infty)$ on $\mathcal{X}_{\text{proét}}/\mathcal{X}_\infty$ is $\Delta_0(p)$-equivariant and the induced Hecke action on $\mathcal{O}\mathbf{D}_{\mathcal{U}}^s$ agrees with the natural action by correspondences.

4.3. Completed sheaves of overconvergent modular forms. Let $q : \mathcal{X}_\infty \to \mathcal{X}$ be the natural projection as before. Recall that for any weight $\mathcal{U}$ and any $w \geq 1 + s_{\mathcal{U}}$, we have the sheaf $\omega_{\mathcal{U},w}^{\dagger}$ of overconvergent modular forms of weight $\mathcal{U}$ on the analytic site of $\mathcal{X}_w$, whose sections over a rational subset $U \subseteq \mathcal{X}_w$ are given by

$$\omega_{\mathcal{U},w}^{\dagger}(U) = \left\{ f \in \mathcal{O}_{\mathcal{X}_\infty}(U_\infty)\widehat{\otimes}A_{\mathcal{U}} \mid \gamma^* f = \chi_{\mathcal{U}}(b\mathfrak{z}+d)^{-1} f \ \forall \gamma \in K_0(p) \right\}.$$

We may similarly define a sheaf $\omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}$ on the analytic site of $\mathcal{X}_{w,\mathbb{C}_p}$ by

$$\omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}(U) = \left\{ f \in \mathcal{O}_{\mathcal{X}_{\infty,\mathbb{C}_p}}(U_\infty)\widehat{\otimes}A_{\mathcal{U}} \mid \gamma^* f = \chi_{\mathcal{U}}(b\mathfrak{z}+d)^{-1} f \ \forall \gamma \in K_0(p) \right\}$$

with $U \subseteq \mathcal{X}_{w,\mathbb{C}_p}$ quasi-compact. We now define completed versions of these sheaves.

Definition 4.6. *We define a sheaf* $\widehat{\omega}_{\mathcal{U},w}^{\dagger}$ *on* $\mathcal{X}_{w,\text{proét}}$ *by*

$$\widehat{\omega}_{\mathcal{U},w}^{\dagger}(V) = \left\{ f \in \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)\widehat{\otimes}A_{\mathcal{U}} \mid \gamma^* f = \chi_{\mathcal{U}}(b\mathfrak{z}+d)^{-1} f \ \forall \gamma \in K_0(p) \right\},$$

where $V \in \mathcal{X}_{w,\text{proét}}$ *is qcqs and* $V_\infty = V \times_{\mathcal{X}_w} \mathcal{X}_{\infty,w}$. *We let* $\widehat{\omega}_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}$ *denote the restriction of* $\widehat{\omega}_{\mathcal{U},w}^{\dagger}$ *to the slice* $\mathcal{X}_{w,\text{proét}}/\mathcal{X}_{w,\mathbb{C}_p} \cong \mathcal{X}_{w,\mathbb{C}_p,\text{proét}}$.

Let us write η for the canonical morphism of sites from the étale site to the analytic site, and put $\lambda = \eta \circ \nu$.

Proposition 4.7. *There are canonical isomorphisms*

$$\widehat{\omega}_{\mathcal{U},w}^{\dagger} \cong \lambda^* \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}})$$

and

$$\widehat{\omega}_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} \cong \lambda^* \omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_{w,\mathbb{C}_p}}\widehat{\otimes}A_{\mathcal{U}}).$$

This has the following consequences:

(1) *We have a canonical isomorphism*

$$\widehat{\omega}_{\mathcal{U},w}^{\dagger} \cong \nu^* \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}})$$

and similarly for the $\mathbb{C}_p$*-sheaves, where we abuse the notation and write* $\omega_{\mathcal{U},w}^{\dagger}$ *also for the étale sheaf* $\eta^*\omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_{w,an}}\widehat{\otimes}A_{\mathcal{U}}} (\mathcal{O}_{\mathcal{X}_{w,\text{ét}}}\widehat{\otimes}A_{\mathcal{U}})$ *(where we have written out "an" and "et" for clarity).*

(2) $\omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}$ *is the base change of* $\omega_{\mathcal{U},w}^{\dagger}$ *from* $\mathcal{X}_w$ *to* $\mathcal{X}_{w,\mathbb{C}_p}$ *(see Lemma 6.21) and hence* $\omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} = \mathrm{Loc}(\omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}(\mathcal{X}_{w,\mathbb{C}_p}))$.

(3) *We have a canonical isomorphism* $\nu_*\widehat{\omega}_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} \cong \omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}$ *and*

$$R^1\nu_*\widehat{\omega}_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} = \omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}} \Omega^1_{\mathcal{X}_{w,\mathbb{C}_p}}(-1).$$

Furthermore, $R^i\nu_*\widehat{\omega}_{\mathcal{U},w,\mathbb{C}_p}^{\dagger}$ *vanishes for* $i \geq 2$.

Proof. The isomorphisms are proven by repeating much of the proof of Theorem 2.28 (including the preliminary lemmas). We focus on the $\mathbb{Q}_p$-sheaves; the case of the $\mathbb{C}_p$-sheaves is identical. Let $V \in \mathcal{X}_{w,\text{proét}}$ be an affinoid perfectoid with image U in $\mathcal{X}_w$ and assume without loss of generality that U is a rational subset of $\mathcal{X}_w$ and that $\omega|_U$ is trivial. We use the notation of the proof of Theorem 2.28 freely. Arguing in the same way we obtain isomorphisms

$$\widehat{\omega}^{\dagger}_{\mathcal{U},w}(V) \cong (\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_n)\widehat{\otimes}A_{\mathcal{U}})^{G_n}$$

and

$$(\nu^*\omega^{\dagger}_{\mathcal{U},w} \otimes_{\mathcal{O}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_w}\widehat{\otimes}A_{\mathcal{U}}))(V) \cong$$
$$\cong (\widehat{\mathcal{O}}_{\mathcal{X}_w}(U_n)\widehat{\otimes}A_{\mathcal{U}})^{G_n} \otimes_{\widehat{\mathcal{O}}_{\mathcal{X}_w}(U)\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_w}(V)\widehat{\otimes}A_{\mathcal{U}})$$

where $V_n = U_n \times_U V$ and G_n acts by the twisted action $f_0 \mapsto \chi_{\mathcal{U}}(j_{U,n}(\gamma))\gamma^* f_0$. Applying Lemma 6.7 twice one obtains a G_n-equivariant isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_n)\widehat{\otimes}A_{\mathcal{U}} \cong (\widehat{\mathcal{O}}_{\mathcal{X}_w}(U_n)\widehat{\otimes}A_{\mathcal{U}}) \otimes_{\widehat{\mathcal{O}}_{\mathcal{X}_w}(U)\widehat{\otimes}A_{\mathcal{U}}} (\widehat{\mathcal{O}}_{\mathcal{X}_w}(V)\widehat{\otimes}A_{\mathcal{U}}).$$

Taking invariants we obtain we desired isomorphism.

Assertion (1) then follows by transitivity of pullbacks. To prove (2), first evaluate the isomorphism for $\widehat{\omega}^{\dagger}_{\mathcal{U},w}$ on $U \subseteq \mathcal{X}_{w,\mathbb{C}_p}$ to see that $\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p}$ restricted to the analytic site of $\mathcal{X}_{w,\mathbb{C}_p}$ is the base change of $\omega^{\dagger}_{\mathcal{U},w}$, and then restrict the second isomorphism to the analytic site of $\mathcal{X}_{w,\mathbb{C}_p}$ to get the first part (2). The second then follows from Lemma 6.21.

To prove (3) we apply the projection formula to the isomorphism in (1) to get

$$R^i\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p} \cong \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p} \otimes_{(\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}\widehat{\otimes}A_{\mathcal{U}})} R^i\nu_*(\widehat{\mathcal{O}}_{\mathcal{X}_{w,\mathbb{C}_p}}\widehat{\otimes}A_{\mathcal{U}}),$$

and the result then follows from Corollary 6.9. $\square$

Corollary 4.8. *The Leray spectral sequence*

$$H^i_{\text{ét}}(\mathcal{X}_{w,\mathbb{C}_p}, R^j\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p}) \Rightarrow H^{i+j}_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p})$$

induces a canonical Hecke- and Galois-equivariant isomorphism

$$H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p}) \cong H^0(\mathcal{X}_{w,\mathbb{C}_p}, \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}} \Omega^1_{\mathcal{X}_{w,\mathbb{C}_p}})(-1)$$

Proof. By Proposition 4.7 $R^j\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p}$ is a locally projective $\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}\widehat{\otimes}A_{\mathcal{U}}$-module for all j and hence has no higher cohomology by (the étale version of) Proposition 6.15. Therefore the Leray spectral sequence degenerates at the E_2-page, giving us isomorphisms

$$H^0_{\text{ét}}(\mathcal{X}_{w,\mathbb{C}_p}, R^j\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p}) \cong H^j_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p})$$

for all j. Now apply Proposition 4.7 again.

For Hecke-equivariance we focus on the Hecke operators away from p; the Hecke equivariance at p follows by the same arguments but is notationally slightly different. We will only check compatibility for the trace maps (compatibility for pullbacks follow from functoriality), and we put ourselves in the situation of Proposition 2.19. By the first isomorphisms in Proposition 4.7, the trace map t of Proposition 2.19(2) induces a trace map

$$\widehat{t} : f_{\text{proét},*}\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p} \to \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p}.$$

We need to check that the diagram

$$\begin{tikzcd}
H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p,K_2^p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p}) \arrow[r] \arrow[d, "\widehat{t}"'] & H^0(\mathcal{X}_{w,\mathbb{C}_p,K_2^p}, \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p} \otimes \Omega^1)(-1) \arrow[d] \\
H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p,K_1^p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p}) \arrow[r] & H^0(\mathcal{X}_{w,\mathbb{C}_p,K_1^p}, \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p} \otimes \Omega^1)(-1)
\end{tikzcd}$$

commutes, where the left hand vertical arrow is induced by $\hat{t}$ as in Lemma 4.9(2) and the right hand vertical arrow is induced by t in the obvious way. We do this in two steps. For the first step, we note that Lemma 4.9(3) gives a commutative diagram

$$\begin{tikzcd}
H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p,K_2^p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p}) \arrow[r] \arrow[d, "\hat{t}"'] & H^0(\mathcal{X}_{w,\mathbb{C}_p,K_2^p}, R^1\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p}) \arrow[d, "R^1\nu_*\hat{t}"] \\
H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p,K_1^p}, \widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p}) \arrow[r] & H^0(\mathcal{X}_{w,\mathbb{C}_p,K_1^p}, R^1\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p}).
\end{tikzcd}$$

For the second step, we claim that the diagram

$$\begin{tikzcd}
R^1\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p} \arrow[r] \arrow[d, "R^1\nu_*\hat{t}"'] & \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_2^p} \otimes \Omega^1(-1) \arrow[d, "t"] \\
R^1\nu_*\widehat{\omega}^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p} \arrow[r] & \omega^{\dagger}_{\mathcal{U},w,\mathbb{C}_p,K_1^p} \otimes \Omega^1(-1)
\end{tikzcd}$$

commutes, where the horizontal arrows are the isomorphisms of Proposition 4.7(3); this follows upon combining the first isomorphism of Proposition 4.7, the pullback-pushforward adjunction of ν, and Lemma 4.10. □

Lemma 4.9. *Let $f : X \to Y$ be a finite étale morphism of rigid analytic varieties.*

(1) *The functor $f_{\text{proét},*}$ is exact.*
(2) *Suppose we are given abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ on $X_{\text{proét}}$ and $Y_{\text{proét}}$, respectively, together with a "trace" map $t : f_{\text{proét},*}\mathcal{F} \to \mathcal{G}$ of abelian sheaves on $Y_{\text{proét}}$. Then t induces a canonical map $t : H^n_{\text{proét}}(X, \mathcal{F}) \to H^n_{\text{proét}}(Y, \mathcal{G})$ together with derived trace maps $R^i\nu_*t : f_{\text{ét},*}R^i\nu_{X,*}\mathcal{F} \to R^i\nu_{Y,*}\mathcal{G}$ of abelian sheaves on $Y_{\text{ét}}$.*
(3) *The maps in (2) are compatible with the Grothendieck spectral sequences for $R\Gamma_{\text{proét}} = R\Gamma_{\text{ét}} \circ R\nu_*$, i.e. the maps t and $R^i\nu_*t$ fit into compatible*

morphisms

$$\begin{tikzcd} H^i_{\text{ét}}(X, R^j\nu_{X,*}\mathcal{F}) \arrow[r, Rightarrow] \arrow[d, "R^j\nu_* t"'] & H^{i+j}_{\text{proét}}(X, \mathcal{F}) \arrow[d, "t"] \\ H^i_{\text{ét}}(Y, R^j\nu_{Y,*}\mathcal{G}) \arrow[r, Rightarrow] & H^{i+j}_{\text{proét}}(Y, \mathcal{G}) \end{tikzcd}$$

of pro-étale cohomology groups and of the spectral sequences computing them.

Proof. To prove right-exactness of $f_{\text{proét},*}$ we need to check that it preserves surjections. For this we may work étale locally on Y, so we may pick an étale cover which splits f and the assertion is then trivial.

For (2) and (3), we argue as follows. By (1), the Leray spectral sequence gives an isomorphism $H^n_{\text{proét}}(X, \mathcal{F}) \cong H^n_{\text{proét}}(Y, f_{\text{proét},*}\mathcal{F})$, so composing this with the evident map $H^n_{\text{proét}}(Y, f_{\text{proét},*}\mathcal{F}) \to H^n_{\text{proét}}(Y, \mathcal{G})$ gives the claimed map on pro-étale cohomology. On the other hand, a direct calculation gives a natural isomorphism $f_{\text{ét},*}\nu_{X,*} \cong \nu_{Y,*}f_{\text{proét},*}$ as functors $\text{Sh}(X_{\text{proét}}) \to \text{Sh}(Y_{\text{ét}})$. Indeed, both functors send a sheaf $\mathcal{F}$ to the sheaf associated with the presheaf $U \mapsto \mathcal{F}(U \times_Y X)$ (where $U \times_Y X$ is regarded as an element of $X_{\text{proét}}$). Since all these functors preserve injectives, we may pass to total derived functors, getting $Rf_{\text{ét},*}R\nu_{X,*} \cong R\nu_{Y,*}Rf_{\text{proét},*}$ as functors $D^+(\text{Sh}(X_{\text{proét}})) \to D^+(\text{Sh}(Y_{\text{ét}}))$. Since $f_{\text{proét},*}$ and $f_{\text{ét},*}$ are both exact functors, this becomes $f_{\text{ét},*}R\nu_{X,*} \cong R\nu_{Y,*}f_{\text{proét},*}$. Applying this to $\mathcal{F}$ and composing with the evident map $R\nu_{Y,*}f_{\text{proét},*}\mathcal{F} \to R\nu_{Y,*}\mathcal{G}$ induced by t gives a map $R\nu_*t : f_{\text{ét},*}R\nu_{X,*}\mathcal{F} \to R\nu_{Y,*}\mathcal{G}$, and we obtain the maps $R^i\nu_*t$ upon passing to cohomology sheaves. Finally, (3) follows by applying $R\Gamma_{Y,\text{ét}}$ to the map $R\nu_*t$, in combination with the isomorphism $R\Gamma_{X,\text{ét}}R\nu_{X,*} \cong R\Gamma_{Y,\text{ét}}f_{\text{ét},*}R\nu_{X,*}$. □

Lemma 4.10. *Maintain the setup of the previous lemma, and suppose X and Y are smooth and are defined over a complete algebraically closed extension $C/\mathbb{Q}_p$. Take $\mathcal{F} = \widehat{\mathcal{O}}_X$ and $\mathcal{G} = \widehat{\mathcal{O}}_Y$, and let t be the natural $\widehat{\mathcal{O}}_Y$-linear trace map $\widehat{\text{tr}} : f_{\text{proét},*}\widehat{\mathcal{O}}_X \to \widehat{\mathcal{O}}_Y$. Then $R^i\nu_*\widehat{\text{tr}} : f_{\text{ét},*}R^i\nu_{X,*}\widehat{\mathcal{O}}_X \to R^i\nu_{Y,*}\widehat{\mathcal{O}}_Y$ coincides with the composite map*

$$f_{\text{ét},*}R^i\nu_{X,*}\widehat{\mathcal{O}}_X \cong f_{\text{ét},*}\Omega^i_X(-i) \cong \Omega^i_Y(-i) \otimes_{\mathcal{O}_Y} f_{\text{ét},*}\mathcal{O}_X \xrightarrow{1\otimes\text{tr}} \Omega^i_Y(-i) \cong R^i\nu_{Y,*}\widehat{\mathcal{O}}_Y.$$

Proof. This follows from a careful reading of the proof of Lemma 3.24 of [Sch2]. Let us give a very brief outline of the argument. One considers a short exact sequence S_X

$$0 \to \hat{\mathbb{Z}}_p(1) \to \varprojlim_{\times p} \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

as in Lemma 3.24 of [Sch2]. We have also an analogous short exact sequence S_Y on Y and a norm map $f_{\text{proét},*}S_X \to S_Y$ on $Y_{\text{proét}}$. To both S_X and S_Y we can associate commutative diagrams as in [Sch2, Lemma 3.24] and one shows that they fit together into a commutative cube. Chasing through this cube gives us a lemma for $i = 1$. The general case follows from this as in [Sch2, Proposition 3.23] by taking exterior products. $\square$

4.4. The overconvergent Eichler-Shimura map. Working with the proétale site of $\mathcal{X}_w$ has the advantage of allowing us to define the overconvergent Eichler-Shimura map at the level of sheaves explicitly, using the cover $\mathcal{X}_{\infty,w}$ and the fundamental period $\mathfrak{z}$.

Fix a *small* weight $\mathcal{U}$. The following proposition is the key step in the construction of the overconvergent Eichler-Shimura map.

Proposition 4.11. *Let $w \geq 1 + s_{\mathcal{U}}$ and $s \geq 1 + s_{\mathcal{U}}$. Let $V_\infty \in \mathcal{X}_{w,\text{proét}}/\mathcal{X}_{\infty,w}$ be qcqs. We have a map*

$$\beta_{\mathcal{U}} : \mathbf{D}_{\mathcal{U}}^{s,\circ}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to R_{\mathcal{U}}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$$

defined on pure tensors by $\beta_{\mathcal{U}} : \mu\otimes f \mapsto \mu(\chi_{\mathcal{U}}(1+\mathfrak{z}x))f$. Here we use Proposition 2.6, with $B = \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$, to be able to apply $\chi_{\mathcal{U}}$ to $1+\mathfrak{z}x$ for any $x \in \mathbb{Z}_p$. This defines a morphism of sheaves on $\mathcal{X}_{w,\text{proét}}/\mathcal{X}_{\infty,w}$. $\beta_{\mathcal{U}}$ satisfies the equivariance relation

$$\beta_{\mathcal{U}}(\gamma^* h) = \chi_{\mathcal{U}}(b\mathfrak{z} + d)\gamma^*\beta_{\mathcal{U}}(h)$$

for any $\gamma \in \Delta_0(p)$, $V_\infty \in \mathcal{X}_{w,\text{proét}}$ qcqs and $h \in \mathbf{D}_{\mathcal{U}}^{s,\circ}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(\gamma^{-1}V_\infty)$. In particular, pushing forward to $\mathcal{X}_{w,\text{proét}}$ and passing to $K_0(p)$-invariants, $\beta_{\mathcal{U}}$ induces an $R_{\mathcal{U}}$-linear and Hecke equivariant map

$$\delta_{\mathcal{U}} : \mathcal{O}\mathbf{D}_{\mathcal{U}}^s \to \widehat{\omega}_{\mathcal{U},w}^{\dagger}$$

of abelian sheaves on $\mathcal{X}_{w,\text{proét}}$.

Here we are interpreting $\chi_{\mathcal{U}}(1 + \mathfrak{z}x)$ as an element of $\mathbf{A}_{\mathcal{U}}^{s,\circ}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$, and extending μ by linearity and continuity to an $R_{\mathcal{U}}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$-linear map $\mathbf{A}_{\mathcal{U}}^{s,\circ}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to R_{\mathcal{U}}\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$.

Proof. First we check that $\beta_{\mathcal{U}}$ is well defined, i.e. that the description on pure tensors extends by continuity. To do this, note that the description on tensors define compatible maps

$$(\mathbf{D}_{\mathcal{U}}^{s-1,\circ}/\mathfrak{a}_{\mathcal{U}}^k\mathbf{D}_{\mathcal{U}}^{s-1,\circ}) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to (R_{\mathcal{U}}/\mathfrak{a}_{\mathcal{U}}^k) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$$

for all k. By the definition of the filtrations we then get compatible maps

$$(\mathbf{D}_{\mathcal{U}}^{s,\circ}/\mathrm{Fil}^k\mathbf{D}_{\mathcal{U}}^{s,\circ}) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to (R_{\mathcal{U}}/\mathfrak{a}_{\mathcal{U}}^k) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty).$$

Taking limits and inverting p we get the desired map. It is then enough to check equivariance on pure tensors $h = \mu \otimes f$. We compute

$$\begin{aligned}
\beta_{\mathcal{U}}(\gamma h) &= (\gamma \cdot_{\mathcal{U}} \mu)(\chi_{\mathcal{U}}(1+\mathfrak{z}x))\gamma^* f \\
&= \mu\left(\chi_{\mathcal{U}}(cx+d)\chi_{\mathcal{U}}(1+\mathfrak{z}\frac{ax+b}{cx+d})\right)\gamma^* f \\
&= \mu\left(\chi_{\mathcal{U}}(cx+d+\mathfrak{z}(ax+b))\right)\gamma^* f \\
&= \mu\left(\chi_{\mathcal{U}}(b\mathfrak{z}+d)\chi_{\mathcal{U}}(1+x\frac{a\mathfrak{z}+c}{b\mathfrak{z}+d})\right)\gamma^* f \\
&= \chi_{\mathcal{U}}(b\mathfrak{z}+d)\mu\left(\chi_{\mathcal{U}}(1+x\frac{a\mathfrak{z}+c}{b\mathfrak{z}+d})\right)\gamma^* f \\
&= \chi_{\mathcal{U}}(b\mathfrak{z}+d)\gamma^*\left(\mu\left(\chi_{\mathcal{U}}(1+\mathfrak{z}x)\right)f\right) \\
&= \chi_{\mathcal{U}}(b\mathfrak{z}+d)\gamma^*\beta_{\mathcal{U}}(h)
\end{aligned}$$

and arrive at the desired conclusion. We leave the Hecke equivariance away from p to the reader. $\square$

The map $\delta_{\mathcal{U}}$ is our version of the map $\delta_k^\vee(w)$ from [AIS2, §4.2]. We may then define the overconvergent Eichler-Shimura map at the level of spaces in the same way as in [AIS2]:

Definition 4.12. *Let $\mathcal{U}$ be a small weight. The weight $\mathcal{U}$ overconvergent Eichler-Shimura map $ES_{\mathcal{U}}$ is given as the composite*

$$\begin{aligned}
&\mathbf{V}^s_{\mathcal{U},\mathbb{C}_p} \cong \\
&\cong H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}\mathbf{D}^s_{\mathcal{U}}) \stackrel{\text{res}}{\to} H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \mathcal{O}\mathbf{D}^s_{\mathcal{U}}) \stackrel{\delta_{\mathcal{U}}}{\to} H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \hat{\omega}^\dagger_{\mathcal{U},w,\mathbb{C}_p}) \cong \\
&\qquad \cong H^0(\mathcal{X}_{w,\mathbb{C}_p}, \omega^\dagger_{\mathcal{U},w,\mathbb{C}_p} \otimes \Omega^1_{\mathcal{X}_{w,\mathbb{C}_p}})(-1) =: \mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p}(-1)
\end{aligned}$$

where the first map is induced by restricting cohomology classes along the inclusion $\mathcal{X}_w \subseteq \mathcal{X}$. $ES_{\mathcal{U}}$ is Hecke- and $G_{\mathbb{Q}_p}$-equivariant by Propositions 4.3, 4.4, 4.11 and Corollary 4.8.

The above definition gives also a map ES_κ for any individual point $\kappa \in \mathcal{W}(\overline{\mathbb{Q}}_p)$, since any such weight is both small and affinoid. Note also that the construction of $ES_{\mathcal{U}}$ is functorial in $\mathcal{U}$: if we have a morphism $\mathcal{U} \to \mathcal{U}'$ compatible with the characters, then we have a commutative diagram

$$\begin{tikzcd}
\mathbf{V}^s_{\mathcal{U}',\mathbb{C}_p} \arrow[r, "ES_{\mathcal{U}'}"] \arrow[d] & \mathcal{M}^{\dagger,w}_{\mathcal{U}',\mathbb{C}_p}(-1) \arrow[d] \\
\mathbf{V}^s_{\mathcal{U},\mathbb{C}_p} \arrow[r, "ES_{\mathcal{U}}"] & \mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p}(-1)
\end{tikzcd}$$

where the vertical maps are the natural ones. In particular, when $\mathcal{U}$ is a small open weight and $\kappa \in \mathcal{U}^{\mathrm{rig}}$, we have a commutative diagram

$$\begin{tikzcd} \mathbf{V}^s_{\mathcal{U},\mathbb{C}_p} \arrow[r, "ES_{\mathcal{U}}"] \arrow[d] & \mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p}(-1) \arrow[d] \\ \mathbf{V}^s_{\kappa,\mathbb{C}_p} \arrow[r, "ES_{\kappa}"] & \mathcal{M}^{\dagger,w}_{\kappa,\mathbb{C}_p}(-1). \end{tikzcd}$$

4.5. Factorization for weights $k \geq 2$. To gain some control of the overconvergent Eichler-Shimura map we will prove that it factors through the p-adic Eichler-Shimura map defined by Faltings ([Fal]) for integral weights $k \in \mathbb{Z}_{\geq 2}$. The key step is to prove this factorization on the level of sheaves, which is the goal of this section. In our setup, this factorization turns out to be rather transparent.

Let $V \in \mathcal{X}_{\text{proét}}$ be qcqs and put $V_\infty = V \times_{\mathcal{X}} \mathcal{X}_\infty$. Let $k \geq 2$. By Lemma 4.5 we know that

$$\mathcal{O}\mathbf{D}^s_k(V) = (\mathbf{D}^s_k \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}}(V_\infty))^{K_0(p)}.$$

Let $\mathcal{G}^{\mathrm{univ}}$ be the universal p-divisible group over $\mathcal{X}$. We define $\mathcal{T}$ to be the relative Tate module of $\mathcal{G}^{\mathrm{univ}}$ viewed as a sheaf on $\mathcal{X}_{\text{proét}}$. Let ω be as before. We then define variants of those sheaves on $\mathcal{X}_{\text{proét}}$ by

$$\widehat{\mathcal{T}} = \mathcal{T} \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}};$$

$$\widehat{\omega} = \lambda^* \omega \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}.$$

Let us write $\widehat{\mathcal{V}}_k = \mathrm{Sym}^{k-2}(\mathcal{T}) \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}} = \mathrm{Sym}^k(\widehat{\mathcal{T}})$ and identify $\mathscr{L}^\circ_k$ with $\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$ via the isomorphism sending X^i to $e_1^i e_2^{k-2-i}$ $(0 \leq i \leq k-2)$. This identifies the $M_2(\mathbb{Z}_p)$-action on $\mathscr{L}^\circ_k$ with the standard left action on $\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$.

Lemma 4.13. *For any $V \in \mathcal{X}_{\text{proét}}$ qcqs, we have $\widehat{\mathcal{V}}_k(V) \cong (\mathscr{L}_k \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}}(V_\infty))^{K_0(p)}$.*

Proof. Follows from Lemma 4.1 upon noting that $\mathcal{T}(V_\infty) \cong \mathscr{L}^\circ_1$ functorially in V and $\Delta_0(p)$-equivariantly via the universal trivialization over $\mathcal{X}_\infty$. □

The above lemma and Lemma 4.5 gives us a map

$$\mathcal{O}\mathbf{D}^s_k(V) \to \widehat{\mathcal{V}}_k(V)$$

induced by the integration map

$$i_k : \mathbf{D}^s_k \to \mathscr{L}_k$$

defined in Definition 3.2.

Let $\mathfrak{s} \in H^0(\mathcal{X}_{\infty,w}, \omega)$ be the non-vanishing section defined in §2.4. Recall that

$$\widehat{\omega}^\dagger_{k,w}(V) = \{f \in \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) | \gamma^* f = (b\mathfrak{z} + d)^{-k} f\}$$

for $V \in \mathcal{X}_{w,\text{proét}}$ qcqs.

Lemma 4.14. *We have*

$$\widehat{\omega}^{\dagger}_{k,w} \cong (\widehat{\omega}^{\otimes k-2})|_{\mathcal{X}_w}$$

via the map

$$f \mapsto f \cdot \mathfrak{s}^{\otimes k-2}$$

Proof. For V qcqs the map induces an isomorphism between $\widehat{\omega}^{\dagger}_{k,w}(V)$ and the set $\{\eta \in \widehat{\omega}^{\otimes k-2}(V_\infty) \mid \gamma^*\eta = \eta\}$ using the transformation rule for $\mathfrak{s}$. By an argument similar to that in the (first half of the) proof of Lemma 2.26 the latter is functorially isomorphic to $\widehat{\omega}^{\otimes k-2}(V)$. $\square$

Recall the linearized Hodge-Tate map

$$\widehat{\mathcal{T}} \to \widehat{\omega}$$

as a map of sheaves of $\mathcal{X}_{\text{proét}}$. Taking $(k-2)$-th symmetric powers we get a map

$$\widehat{\mathcal{V}}_k \to \widehat{\omega}^{\otimes k-2}.$$

Restricting to $\mathcal{X}_w$ and using Lemma 4.14 we get a map

$$v_k : \widehat{\mathcal{V}}_k \to (\widehat{\omega}^{\otimes k-2})|_{\mathcal{X}_{\infty,w}} \cong \widehat{\omega}^{\dagger}_{k,w}$$

which we want to describe explicitly:

Lemma 4.15. *Let $k \geq 2$ and let $V_\infty \in \mathcal{X}_{w,\text{proét}}/\mathcal{X}_\infty$ be qcqs. Define a map*

$$\mathscr{L}_k \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)$$

by $X^i \mapsto \mathfrak{z}^i$ for $0 \leq i \leq k-2$. This is an $\widehat{\mathcal{O}}_{\mathcal{X}}$-linear and $\Delta_0(p)$-equivariant morphism of sheaves on $\mathcal{X}_{w,\text{proét}}/\mathcal{X}_\infty$. Pushing forward to $\mathcal{X}_{w,\text{proét}}$ and taking $K_0(p)$-invariants we get a map $\widehat{\mathcal{V}}_k \to \widehat{\omega}^{\dagger}_{k,w}$. Evaluating this on $V \in \mathcal{X}_{w,\text{proét}}$ qcqs and putting $V_\infty = V \times_{\mathcal{X}} \mathcal{X}_{\infty,w}$, we obtain a map

$$(\mathscr{L}_k \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty))^{K_0(p)} \to \widehat{\omega}^{\dagger}_{k,w}(V).$$

which is equal to v_k (using Lemma 4.13).

Proof. We describe the map $\widehat{\mathcal{V}}_k \to \widehat{\omega}^{\otimes k-2}$ and then use the isomorphism of the previous lemma. Over V_∞, the sheaf $\widehat{\omega}^{\otimes k-2}$ is trivialized by $\mathfrak{s}^{\otimes k-2}$, so v_k is morphism

$$\mathscr{L}_k \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty) \to \widehat{\mathcal{O}}_{\mathcal{X}_w}(V_\infty)\mathfrak{s}^{\otimes k-2}.$$

From the definitions we see that it sends X^i to $\mathrm{HT}(\alpha(e_1))^i\,\mathrm{HT}(\alpha(e_2))^{k-2-i}$ where α denotes the trivialization of $\mathcal{T}$ over V_∞ and HT denotes the Hodge-Tate map. By definition, $\mathfrak{s} = \mathrm{HT}(\alpha(e_2))$ and over $\mathcal{X}_w$ we have the relation $\mathrm{HT}(\alpha(e_1)) = \mathfrak{z}\,\mathrm{HT}(\alpha(e_2))$. Thus X^i is sent to $\mathfrak{z}^i\mathfrak{s}^{\otimes k-2}$. Finally observe that the isomorphism of Lemma 4.14 sends $\mathfrak{s}^{\otimes k-2}$ to 1. $\square$

We can now prove the factorization theorem at the level of sheaves for our overconvergent Eichler-Shimura map.

Proposition 4.16. *For $k \geq 2$ we have a commutative diagram:*

$$\begin{tikzcd} \mathcal{O}\mathbf{D}_k^s \arrow[r, "\mu \mapsto \mu((1+\mathfrak{z}x)^{k-2})"] \arrow[d, "\mu \mapsto \mu((1+Xx)^{k-2})"'] & \widehat{\omega}_{k,w}^{\dagger} \\ \widehat{\mathcal{V}}_k \arrow[ur, "X^i \mapsto \mathfrak{z}^i"'] & \end{tikzcd}$$

Proof. It follows from our prior discussion and Lemma 4.15. $\square$

Remark 4.17. *This diagram, which is a diagram of sheaves over $\mathcal{X}_w$, may be viewed as the restriction of the following diagram of sheaves over $\mathcal{X}$:*

$$\begin{tikzcd} \mathcal{O}\mathbf{D}_k^s \arrow[r, "\mu \mapsto \mu((1+\mathfrak{z}x)^{k-2})\mathfrak{s}^{\otimes k-2}"] \arrow[d, "\mu \mapsto \mu((1+Xx)^{k-2})"'] & \widehat{\omega}^{\otimes k-2} \\ \widehat{\mathcal{V}}_k \arrow[ur, "X^i \mapsto \mathfrak{z}^i\mathfrak{s}^{\otimes k-2}"'] & \end{tikzcd}$$

using the isomorphism of Lemma 4.14. Note that although $\mathfrak{z}$ and $\mathfrak{s}$ do not extend to $\mathcal{X}_\infty$, the products $\mathfrak{z}^i\mathfrak{s}^{\otimes k-2}$ do extend to $\mathcal{X}_\infty$ for all $0 \leq i \leq k-2$, so the formulas in the diagram make sense. This will cause our overconvergent Eichler-Shimura maps to have very large kernels at classical weights $k \geq 2$ (see §5.2).

5. The overconvergent Eichler-Shimura map over the eigencurve

In this section we prove our main results concerning the overconvergent Eichler-Shimura map. These results are analogous to the main results of [AIS2]. However, the payoff for working with arbitrary small open weights is that we may glue our overconvergent Eichler-Shimura maps for different small open weights into a morphism of coherent sheaves over the eigencurve. This allows us to work over rather arbitrary regions of weight space and with general slope cutoffs, as opposed to the rather special open discs inside the analytic part of weight space used in [AIS2].

5.1. Sheaves on the eigencurve. Let $\mathcal{U}$ be a small weight and let $s \geq 1+s_{\mathcal{U}}$. Recall we have defined the modules

$$\mathbf{V}_{\mathcal{U}}^s = H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathbf{D}_{\mathcal{U}}^s);$$

$$\mathbf{V}_{\mathcal{U},\mathbb{C}_p}^s \cong H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}\mathbf{D}_{\mathcal{U}}^s);$$

$$\mathcal{M}_{\mathcal{U}}^{\dagger,w} = H^0(\mathcal{X}_w, \omega_{\mathcal{U},w}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_w}} \Omega^1_{\mathcal{X}_w});$$

$$\mathcal{M}_{\mathcal{U},\mathbb{C}_p}^{\dagger,w} = H^0(\mathcal{X}_{w,\mathbb{C}_p}, \omega_{\mathcal{U},w,\mathbb{C}_p}^{\dagger} \otimes_{\mathcal{O}_{\mathcal{X}_{w,\mathbb{C}_p}}} \Omega^1_{\mathcal{X}_{w,\mathbb{C}_p}}).$$

In this subsection we spread the finite-slope pieces of these modules into sheaves on the whole eigencurve $\mathcal{C} = \mathcal{C}_{K^p}$, and we glue our Eichler-Shimura maps into a morphism of sheaves.

Recall the rigid analytic weight space $\mathcal{W} = \mathrm{Spf}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^{\mathrm{rig}}$, with its universal character $\chi_{\mathcal{W}} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{W})^\times$. If $\mathcal{U}$ is an open weight, we write $\mathcal{O}^{bd}(\mathcal{U}^{\mathrm{rig}})$ for the ring of bounded functions on $\mathcal{U}^{\mathrm{rig}}$. This is a Banach $\mathbb{Q}_p$-algebra equal to $A_{\mathcal{U}}[\frac{1}{p}]$ as a dense subring, and with a natural map $\mathcal{O}(\mathcal{W}) \to \mathcal{O}(\mathcal{U}^{\mathrm{rig}})$. Recall that we have defined the Fredholm determinant

$$F_{\mathcal{U}}(T) = \det(1 - \widetilde{U}T | C_\bullet(K, \mathbf{A}^s_{\mathcal{U}}))$$

in §3.2. It is an element of $\mathcal{O}(\mathcal{U}^{\mathrm{rig}})^{bd}\{\{T\}\}$, the ring of entire power series with coefficients in $\mathcal{O}^{bd}(\mathcal{U}^{rig})\{\{T\}\}$, and we showed that it glues to a Fredholm series $F(T) \in \mathcal{O}(\mathcal{W})\{\{T\}\}$ (see e.g. [Han1, Definition 4.1.1] for the definition of a Fredholm series). Now let $\mathcal{V} = (S_{\mathcal{V}}, \chi_{\mathcal{V}})$ be an open *affinoid* weight. Arguing as in [Pil, §5.2] we see that $\mathcal{M}^{\dagger,w}_{\mathcal{V}}$ satisfies Buzzard's property (Pr) over $S_{\mathcal{V}}$ (i.e. it is a direct summand of an orthonormalizable $S_{\mathcal{V}}$-module, see the paragraph before Lemma 2.11 of [Buz3]). Since U_p is compact we get a Fredholm determinant

$$G_{\mathcal{V}}(T) = \det(1 - U_p T | \mathcal{M}^{\dagger,w}_{\mathcal{V}})$$

in $\mathcal{O}(\mathcal{V})\{\{T\}\}$ and these glue to a Fredholm series $G(T) \in \mathcal{O}(\mathcal{W})\{\{T\}\}$. Let us set $H(T) = F(T)G(T)$, this is also a Fredholm series. Recall ([Han1, §4.1]) that a *Fredholm hypersurface* is a closed subvariety of $\mathcal{W} \times \mathbb{A}^1$ cut out by a Fredholm series. Given a Fredholm series f we write $\mathscr{Z}(f)$ for the corresponding Fredholm hypersurface. Recall that, if $h \in \mathbb{Q}_{\geq 0}$ and $W \subseteq \mathcal{W}$ is open, then f has a slope $\leq h$-decomposition in $\mathcal{O}^{bd}(W)\{\{T\}\}$ if and only if the natural map $\mathscr{Z}_{W,h}(f) := \mathscr{Z}(f) \cap (W \times \mathbb{B}[0,h]) \to \mathbb{B}[0,h]$ is finite flat, where $\mathbb{B}[0,h] \subseteq \mathbb{A}^1$ is the closed disc around 0 of radius p^h. We say that (W,h) is *slope-adapted* for f if these equivalent conditions hold. We have the following key result:

Lemma 5.1. *There exists a collection of pairs $(\mathcal{V}_i, h_i)$, with $\mathcal{V} \subseteq \mathcal{W}$ open and affinoid and $h_i \in \mathbb{Q}_{\geq 0}$, such that $(\mathcal{V}_i, h_i)$ is slope-adapted for H for all i and the $(\mathscr{Z}_{\mathcal{V}_i,h_i}(H))_i$ form an open (admissible) cover of $\mathscr{Z}(H)$. Moreover, for each i we may find a small open weight $\mathcal{U}_i$ such that $\mathcal{V}_i \subseteq \mathcal{U}_i^{\mathrm{rig}}$ and $(\mathcal{U}_i^{\mathrm{rig}}, h_i)$ is slope-adapted for H.*

Proof. See [Han2, Lemmas 2.3.1-2.3.4] (the first part is a theorem of Coleman-Mazur and Buzzard and is the key step in the "eigenvariety machine", cf. [Buz3, Theorem 4.6]). □

Fix collections $\mathcal{V}_i, \mathcal{U}_i$ and h_i satisfying the conclusions of Lemma. For now, let $h \in \mathbb{Q}_{\geq 0}$ and let $\mathcal{U}$ be any small open weight such that $(\mathcal{U}^{rig}, h)$ is slope-adapted for H, and let $\mathcal{V} \subseteq \mathcal{U}^{\mathrm{rig}}$ be an open affinoid weight. Recall our overconvergent Eichler-Shimura map

$$ES_{\mathcal{U}} : \mathbf{V}^s_{\mathcal{U},\mathbb{C}_p} \to \mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p}(-1).$$

Since $(\mathcal{U}^{\mathrm{rig}}, h)$ is slope-adapted for H (and hence for F and G) and $ES_{\mathcal{U}}$ is Hecke-equivariant we may take slope $\leq h$-parts on both sides to obtain an $R_{\mathcal{U}}[\frac{1}{p}]$-linear map

$$ES_{\mathcal{U},\leq h} : \mathbf{V}^s_{\mathcal{U},\mathbb{C}_p,\leq h} \to \mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p,\leq h}(-1)$$

of finite projective $R_{\mathcal{U}}[\frac{1}{p}]$-modules. We may compose with this map the natural map $\mathcal{M}^{\dagger,w}_{\mathcal{U},\mathbb{C}_p,\leq h}(-1) \to \mathcal{M}^{\dagger,w}_{\mathcal{V},\mathbb{C}_p,\leq h}(-1)$ and tensor the source with $S_{\mathcal{V}}$ to obtain an $S_{\mathcal{V}}$-linear map

$$ES_{\mathcal{V},\leq h} : \mathbf{V}^s_{\mathcal{V},\mathbb{C}_p,\leq h} \cong \mathbf{V}^s_{\mathcal{U},\mathbb{C}_p,\leq h} \otimes_{R_{\mathcal{U}}[\frac{1}{p}]} S_{\mathcal{V}} \to \mathcal{M}^{\dagger,w}_{\mathcal{V},\mathbb{C}_p,\leq h}(-1)$$

where the first isomorphism is that of Proposition 3.4.

Let us now recall the eigencurves for $B^\times$ of tame level K^p constructed out of overconvergent modular forms resp. overconvergent modular symbols, using the perspective of [Han1, §4]. Let Σ_0 denote the finite set of primes ℓ for which $K_\ell \ncong \mathrm{GL}_2(\mathbb{Z}_\ell)$ and let $\Sigma = \Sigma_0 \cup \{p\}$. We let $\mathbb{T}$ denote the abstract Hecke algebra generated by commuting formal variables T_ℓ, S_ℓ for $\ell \notin \Sigma$ and U_p. Then $\mathbb{T}$ acts on all spaces of modular forms and modular symbols used in this article. We may construct two eigenvariety data ([Han1, Definition 4.2.1]) $\mathfrak{D}_{\mathcal{M}^\dagger} = (\mathcal{W}, \mathscr{Z}(H), \mathcal{M}^\dagger, \mathbb{T}, \psi_{\mathcal{M}^\dagger})$ and $\mathfrak{D}_{\mathbf{V}} = (\mathcal{W}, \mathscr{Z}(H), \mathbf{V}, \mathbb{T}, \psi_{\mathbf{V}})$. Here $\mathcal{M}^\dagger$ and $\mathbf{V}$ are the coherent sheaves on $\mathscr{Z}(H)$ obtained by gluing the $\varinjlim_w \mathcal{M}^{\dagger,w}_{\mathcal{V}_i,\leq h_i}$ resp. the $\varprojlim_s \mathbf{V}^s_{\mathcal{V}_i,\leq h_i}$ over the $\mathscr{Z}_{\mathcal{V}_i,h_i}(H)$ (see [Han1, Proposition 4.3.1] and the discussion following it for the construction of $\mathbf{V}$; the construction of $\mathcal{M}^\dagger$ is similar but easier). Given these, [Han1, Theorem 4.2.2] allows us to construct the eigenvarieties $\mathcal{C}_{\mathcal{M}^\dagger}$ and $\mathcal{C}_{\mathbf{V}}$ together with coherent sheaves $\mathcal{M}^\dagger$ on $\mathcal{C}_{\mathcal{M}^\dagger}$ resp. $\mathbf{V}$ on $\mathcal{C}_{\mathbf{V}}$ (this is a mild abuse of notation; we will have no further reason to consider the sheaves with the same name over $\mathscr{Z}(H)$). Since the following proposition is well known we only give a brief sketch of the proof:

Proposition 5.2. *There is a canonical isomorphism $\mathcal{C}_{\mathcal{M}^\dagger} \cong \mathcal{C}_{\mathbf{V}}$ compatible with the projection maps down to $\mathcal{W}$ and the actions of $\mathbb{T}$.*

Proof. This follows by applying [Han1, Theorem 5.1.2] twice (once in each direction), upon noting that both eigencurves are reduced and equi-dimensional (of dimension 1; they are "unmixed" in the terminology of [Han1]). For these applications one should modify the eigenvariety data, replacing $\mathscr{Z}(H)$ in $\mathfrak{D}_{\mathcal{M}^\dagger}$ with the support of $\mathcal{M}^\dagger$ and similarly for $\mathfrak{D}_{\mathbf{V}}$. The relevant very Zariski dense subsets are then constructed using the control/classicality theorems of Stevens and Coleman together with the usual Eichler-Shimura isomorphism (see [Han1, Theorem 3.2.5] for a general version of the control theorem for overconvergent modular symbols, and see e.g. [Joh, Theorem 4.16] for a proof of Coleman's control theorem in the context of the compact Shimura curves used here). $\square$

In light of this we will identify the two eigencurves and simply denote it by $\mathcal{C}$. It carries two coherent sheaves $\mathcal{M}^\dagger$ and $\mathbf{V}$ which are determined by canonical isomorphisms

$$\mathcal{M}^\dagger(\mathcal{C}_{\mathcal{V}_i,h_i}) \cong \varinjlim_w \mathcal{M}^{\dagger,w}_{\mathcal{V}_i,\le h_i} \cong \mathcal{M}^{\dagger,w}_{\mathcal{V}_i,\le h_i};$$

$$\mathbf{V}(\mathcal{C}_{\mathcal{V}_i,h_i}) \cong \varprojlim_s \mathbf{V}^s_{\mathcal{V}_i,\le h_i} \cong \mathbf{V}^s_{\mathcal{V}_i,\le h_i}$$

for all i and sufficiently large w resp. s, where $\mathcal{C}_{\mathcal{V}_i,h_i}$ is the preimage of $\mathscr{X}_{\mathcal{V}_i,h_i}(H)$ in $\mathcal{C}$. We form the sheaves $\mathcal{M}^\dagger_{\mathbb{C}_p} := \mathcal{M}^\dagger \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ and $\mathbf{V}_{\mathbb{C}_p} := \mathbf{V}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. They are determined by canonical isomorphisms

$$\mathcal{M}^\dagger_{\mathbb{C}_p}(\mathcal{C}_{\mathcal{V}_i,h_i}) \cong \varinjlim_w \mathcal{M}^{\dagger,w}_{\mathcal{V}_i,\mathbb{C}_p,\le h_i} \cong \mathcal{M}^{\dagger,w}_{\mathcal{V}_i,\mathbb{C}_p,\le h_i};$$

$$\mathbf{V}_{\mathbb{C}_p}(\mathcal{C}_{\mathcal{V}_i,h_i}) \cong \varprojlim_s \mathbf{V}^s_{\mathcal{V}_i,\mathbb{C}_p,\le h_i} \cong \mathbf{V}^s_{\mathcal{V}_i,\mathbb{C}_p,\le h_i}$$

for any sufficiently large w resp. s. We may also naturally view these sheaves as coherent sheaves on the base change $\mathcal{C}_{\mathbb{C}_p}$ of $\mathcal{C}$ from $\mathbb{Q}_p$ to $\mathbb{C}_p$. They are then determined by the obvious modifications of the above canonical isomorphisms.

With these preparations we may now state the main theorem of this section which glues the maps $ES_{\mathcal{V}_i,\le h_i}$:

Theorem 5.3. *There exists a canonical Hecke and Galois-equivariant morphism*

$$\mathcal{ES} : \mathbf{V}_{\mathbb{C}_p} \to \mathcal{M}^\dagger_{\mathbb{C}_p}(-1)$$

of coherent sheaves of $\mathcal{O}_\mathcal{C}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$-modules on $\mathcal{C}$ (or coherent sheaves of $\mathcal{O}_{\mathcal{C}_{\mathbb{C}_p}}$-modules on $\mathcal{C}_{\mathbb{C}_p}$), which glues $(ES_{\mathcal{V}_i,\le h_i})_i$.

Proof. The $ES_{\mathcal{V}_i,\le h_i}$ induce maps over the cover $(\mathcal{C}_{\mathcal{V}_i,h_i})_i$ using the canonical isomorphisms. Checking that they glue is tedious but straightforward, using the functoriality of the maps $ES_\mathcal{U}$ in the small open weight $\mathcal{U}$ as well as the naturality of the construction of $ES_{\mathcal{V},\le h}$ for $\mathcal{V} \subseteq \mathcal{U}^{rig}$, $\mathcal{V}$ open affinoid and $\mathcal{U}$ small open such that $(\mathcal{U}^{rig}, h)$ is slope-adapted for H. We leave the remaining details to the reader. □

The morphism $\mathcal{ES}$ is our main object of study in this section. Our main technique to study it is via specialization to non-critical classical points of $\mathcal{C}$ (recall that a classical point x is non-critical if $h < k-1$, where h is the slope and k is the weight of x). Let $\kappa \in \mathcal{W}(L)$ with $L/\mathbb{Q}_p$ finite and let $(\mathcal{V}, h)$ be slope-adapted with $\mathcal{V}$ connected and affinoid and $\kappa \in \mathcal{V}(L)$. By [Han1, Theorem 3.3.1] there is a second quadrant spectral sequence (with s big enough)

$$E_2^{pq} = \mathrm{Tor}_{-p}^{\mathcal{O}_\mathcal{W}(\mathcal{V})}(H^q(K, \mathbf{D}^s_\mathcal{V})_{\le h}, L) \implies H^{p+q}(K, \mathbf{D}^s_\kappa)_{\le h}$$

which degenerates at the E_2-page since $\mathcal{O}_\mathcal{W}(\mathcal{V})$ is a Dedekind domain. This gives us a short exact sequence

$$0 \to \mathbf{V}(\mathcal{C}_{\mathcal{V},h}) \otimes_{\mathcal{O}_\mathcal{W}(\mathcal{V})} L \to H^1(K, \mathbf{D}^s_\kappa)_{\le h} \to \mathrm{Tor}_1^{\mathcal{O}_\mathcal{W}(\mathcal{V})}(H^2(K, \mathbf{D}^s_\mathcal{V})_{\le h}, L) \to 0.$$

Now assume that $x \in \mathcal{C}(L')$ is a classical non-critical point of weight $k = \kappa$ (with L'/L a finite field extension). After localizing the above short exact sequence at x it becomes an isomorphism

$$(\mathbf{V}(\mathcal{C}_{\mathcal{V},h}) \otimes_{\mathcal{O}_{\mathcal{W}(\mathcal{V})}} L)_x \cong (H^1(K, \mathbf{D}_k^s)_{\leq h})_x.$$

This follows from the fact that x does not occur in $H^2(K, \mathbf{D}_{\mathcal{V}}^s)_{\leq h}$, which by [Han1, Proposition 4.5.2] follows from the fact that it does not occur in $H^2(K, \mathbf{D}_k^s)_{\leq h}$, which in turn follows from the control theorem. Using the control theorem again, the right hand side of the isomorphism is canonically isomorphic to $H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{V}_k)_x$. From this we deduce that the fiber of $\mathbf{V}$ at x is the largest semisimple quotient of the generalized eigenspace in $H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{V}_k)$ associated with x.

By a similar but simpler analysis (using Lemma 2.29) one sees that the fibre of $\mathcal{M}^\dagger$ at an arbitrary point $x \in \mathcal{C}(\overline{\mathbb{Q}}_p)$ is equal to the largest semisimple quotient of the generalized eigenspace in $H^0(\mathcal{X}_w, \omega^\dagger_{\kappa,w} \otimes_{\mathcal{O}_{\mathcal{X}_w}} \Omega^1_{\mathcal{X}_w})$ associated with x where w is sufficiently large. One may of course apply the control theorem if x is non-critical and classical to gain a further refinement. Similarly, the analogous statements apply to the same objects based changed from $\mathbb{Q}_p$ to $\mathbb{C}_p$.

We will need a few extras fact about $\mathcal{C}$ before we proceed to analyze $\mathcal{ES}$.

Lemma 5.4. *Let $\kappa \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ be a weight and let $s \geq s_\kappa$. Then $H^0(K, \mathbf{D}_\kappa^s) = 0$.*

Proof. Write G^1 for the closed subgroup of G of elements of reduced norm one. Under our fixed isomorphism $G(\mathbb{Z}_p) \cong \mathrm{GL}_2(\mathbb{Z}_p)$ we have $G^1(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Z}_p)$. To prove the lemma, it suffices to show that $H^0(\Gamma, \mathbf{D}_\kappa^s) = 0$ for any congruence subgroup $\Gamma \subset G^1(\mathbb{Z})$ contained in $K_0(p)$. By the p-adic continuity of the Γ-action on $\mathbf{D}_\kappa^s$, we have

$$H^0(\Gamma, \mathbf{D}_\kappa^s) = H^0(\Gamma_p, \mathbf{D}_\kappa^s),$$

where Γ_p denotes the p-adic closure of Γ in $\mathrm{GL}_2(\mathbb{Z}_p)$. By [Rap, Lemma 2.7] and the Zariski density of Γ in G^1 (see e.g. [PR, Theorem 4.10]), Γ_p contains an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$, and in particular contains a nontrivial element $u = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for some $a \in \mathbb{Z}_p, a \neq 0$. Note that u acts on $\mu \in \mathbf{D}_\kappa^s$ by $(u \cdot \mu)(f(x)) = \mu(f(x+a))$; in particular this action does not depend on κ. Since any element of $H^0(\Gamma_p, \mathbf{D}_\kappa^s) \subseteq \mathbf{D}_\kappa^s = \mathbf{D}^s \otimes_{\mathbb{Q}_p} L$ (L the residue field of κ) is fixed by u, it now suffices to show that no nonzero element of $\mathbf{D}^s$ is fixed by u. To see this, note that the Amice transform

$$\mu \mapsto A_\mu(T) = \int (1+T)^x \mu(x) \in \mathbb{Q}_p[[T]]$$

defines a $\mathbb{Q}_p$-linear injection of $\mathbf{D}^s$ (regarded as a ring under convolution) into a subring of $\mathbb{Q}_p[[T]]$ (cf. [Col, §1.8] for more detailed statements). An easy calculation shows that $A_{u \cdot \mu}(T) = (1+T)^a A_\mu(T)$, so if $u \cdot \mu = \mu$ then $((1+T)^a - 1) \cdot A_\mu(T) = 0$, and since $\mathbb{Q}_p[[T]]$ is a domain this implies $A_\mu(T) = 0$ and then $\mu = 0$ as desired. $\square$

Lemma 5.5. *The sheaves $\mathcal{M}^\dagger$ and $\mathbf{V}$ are torsion free.*

Proof. We start with $\mathcal{M}^\dagger$. This is a local statement so we may work over some $\mathcal{C}_{\mathcal{V},h}$ coming from a slope-adapted $(\mathcal{V},h)$ with $\mathcal{V}$ affinoid. Then $\mathcal{O}_{\mathcal{C}}(\mathcal{C}_{\mathcal{V},h})$ is finite over $\mathcal{O}_{\mathcal{W}}(\mathcal{V})$, so it suffices to show torsion freeness over $\mathcal{O}_{\mathcal{W}}(\mathcal{V})$, which is clear by definition.

We now prove torsion-freeness for $\mathbf{V}$. Working locally as above it is enough to show that $H^1(K,\mathbf{D}^s_{\mathcal{V}})_{\leq h}$ is a torsion-free $\mathcal{O}_{\mathcal{W}}(\mathcal{V})$-module, where $\mathcal{V}\subseteq\mathcal{W}$ is affinoid and $(\mathcal{V},h)$ is slope-adapted. Let $\kappa\in\mathcal{V}(\overline{\mathbb{Q}}_p)$ be a weight, cutting out a maximal ideal $\mathfrak{m}_\kappa$ and let $L:=\mathcal{O}_{\mathcal{W}}(\mathcal{V})/\mathfrak{m}_\kappa$. It is enough to show to that the $\mathfrak{m}_\kappa$-torsion vanishes for all κ. This torsion is equal to $\mathrm{Tor}_1^{\mathcal{O}_{\mathcal{W}}(\mathcal{V})}(H^1(K,\mathbf{D}^s_{\mathcal{V}})_{\leq h},L)$. Using the Tor-spectral sequence ([Han1, Theorem 3.3.1]) one sees that this Tor-group is a subquotient of $H^0(K,\mathbf{D}^s_\kappa)_{\leq h}$, which vanishes by the previous Lemma. □

Next, assume that our tame level is of the form $K_1(N)$ for some N with $(N,dp)=1$. Let $\mathcal{C}^{\mathrm{nc}}\subset\mathcal{C}$ denote the set of non-critical classical crystalline points for which the roots of the pth Hecke polynomial are distinct.[2] For each positive divisor $M|N$, let $\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ denote the set of points x in $\mathcal{C}^{\mathrm{nc}}$ for which the tame Artin conductor of the Galois representation ρ_x associated to x is exactly Md. Note that the tame Artin conductor is sometimes just called the tame conductor; see the discussion in [Bel] before Lemma IV.4.3. Let $\mathcal{C}_{M-\mathrm{new}}$ denote the Zariski-closure of $\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ in $\mathcal{C}$. The following lemma was stated without proof for modular curves in [Han2].

Lemma 5.6. *Assume that the tame level is of the form $K_1(N)$. For any $M|N$, $\mathcal{C}_{M-\mathrm{new}}$ is a union of irreducible components of $\mathcal{C}$, and $\mathcal{C}=\cup_{M|N}\mathcal{C}_{M-\mathrm{new}}$.*

Proof. Adapting the proof of [Bel, Lemma IV.4.7], one shows that $\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ is an *accumulation subset* of $\mathcal{C}$: each point $x\in\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ has a neighborhood basis of affinoids $U\subset\mathcal{C}$ for which $U\cap\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ is Zariski-dense in U. This property implies that each irreducible component of the Zariski-closure $\mathcal{C}_{M-\mathrm{new}}$ has positive dimension. Since $\mathcal{C}$ is equidimensional of dimension one, we deduce from [Con, Corollary 2.2.7] that $\mathcal{C}_{M-\mathrm{new}}$ is a union of irreducible components of $\mathcal{C}$. Since $\mathcal{C}^{\mathrm{nc}}=\cup_{M|N}\mathcal{C}^{\mathrm{nc}}_{M-\mathrm{new}}$ is a Zariski-dense accumulation subset of $\mathcal{C}$, the remainder of the lemma is clear. □

5.2. Faltings's Eichler-Shimura map. In this section we adapt Faltings's construction ([Fal]) of the p-adic Eichler-Shimura morphism to our setting, and use it to give a precise description of ES_k at weights $k\geq 2$. Fix $k\in\mathbb{Z}_{\geq 2}$. Recall the morphism

$$\widehat{\mathcal{V}}_k\to\widehat{\omega}^{\otimes k-2}$$

defined in §4.5. It induces a map

$$H^1_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathbb{C}_p},\widehat{\mathcal{V}}_k)\to H^1_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathbb{C}_p},\widehat{\omega}^{\otimes k-2}).$$

[2] This last condition is conjecturally automatic.

Note that we have an exact sequence
(1)
$$0 \to H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2}) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2}) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1) \to 0$$

coming from the spectral sequence $H^i_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, R^j\nu_*\widehat{\omega}^{\otimes k-2}) \Rightarrow H^{i+j}_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2})$, which degenerates at the E_2-page by an argument similar to (but simpler than) the proof of Proposition 4.7.

Proposition 5.7. *The composite map*
$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2}) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1)$$
is surjective, and the kernel is isomorphic to $H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(k)$ *as a Hecke- and Galois module.*

Proof. Recall the Hodge-Tate sequence
$$0 \to \widehat{\omega}^{\otimes -1}(1) \to \widehat{\mathcal{T}} \to \widehat{\omega} \to 0$$
for $\mathcal{G}^{\text{univ}}$. Since the morphism $\widehat{\mathcal{V}}_k \to \widehat{\omega}^{\otimes k-2}$ comes from the map $\widehat{\mathcal{T}} \to \widehat{\omega}$ by taking $(k-2)$-th symmetric powers, we see that its kernel Ker_k sits in a short exact sequence
$$0 \to \widehat{\omega}^{\otimes 2-k}(k-2) \to \text{Ker}_k \to Q \to 0$$
where Q is simply defined to be the quotient. We get exact sequences
$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \text{Ker}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2});$$
$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}(k)) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \text{Ker}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, Q).$$
We have an isomorphism $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_j) \cong H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \text{Sym}^j \mathcal{T}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ for all $j \geq 0$. From the Hodge-Tate sequence one deduces that Q carries a filtration with non-zero graded pieces $\widehat{\omega}^{\otimes i}$ for $i = 4-k, 6-k, ..., k-4$ (up to twists). Using this filtration and the sequences (1) with $k-2$ replaced by $i = 4-k, 6-k, ..., k-4$ together with the usual Eichler-Shimura isomorphism for quaternionic Shimura curves one sees that $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k)$ and $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, Q)$ do not have any Hecke eigenvalues in common. Thus, looking at the second exact sequence above, we see that any generalized Hecke eigenvector in $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \text{Ker}_k)$ that is not killed by the map $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \text{Ker}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k)$ must come from $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}(k-2))$.

Hence we have an exact sequence
$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}(k-2)) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2}).$$
By the exact sequence (1) and looking at Hecke eigenvalues again we get an exact sequence
$$H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}(k-2)) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1).$$
We now apply a similar argument to $H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}(k-2))$. Replacing $k-2$ with $2-k$ in (1) we have the exact sequence
$$0 \to H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes 2-k}) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes 2-k}) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes 2-k} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1) \to 0.$$

Arguing with Hecke eigenvalues as above and using Serre duality we get a sequence

(2)
$$0 \to H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes 2-k}(k-2)) \to H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1) \to 0$$

which is exact in the middle. By Serre duality

$$H^1_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes 2-k}(k-2)) \cong H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(k-2).$$

Counting dimensions, we see that (2) is a short exact sequence as desired. $\square$

We have a diagram

$$\begin{tikzcd}
H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}\mathbf{D}^s_k) \arrow[r, "ES_k"] \arrow[d] & H^1_{\text{proét}}(\mathcal{X}_{w,\mathbb{C}_p}, \widehat{\omega}^\dagger_{k,w}) \arrow[r, "\sim"] & H^0_{\text{ét}}(\mathcal{X}_{w,\mathbb{C}_p}, \omega^\dagger_{k,w} \otimes \Omega^1_{\mathcal{X}_{w,\mathbb{C}_p}})(-1) \\
H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \arrow[r] & H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \widehat{\omega}^{\otimes k-2}) \arrow[r] \arrow[u] & H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1) \arrow[u]
\end{tikzcd}$$

which commutes by Remark 4.17 and the functoriality of the remaining maps. Note that the right vertical map is injective by analytic continuation. Its image is, by definition, the space of classical modular forms. We then have:

Proposition 5.8. *Let $k \geq 2$. The image of ES_k is contained in the space of classical modular forms. Moreover ES_k is surjective on slope $\leq h$-parts when $h < k-1$.*

Proof. The first statement is clear from the diagram. To see the second, note that both the left and right vertical maps in the diagram above are isomorphisms on slope $\leq$ h-parts when $h < k-1$ by the control theorems of Stevens and Coleman. To conclude, use that the map $H^1(\mathcal{X}_{\mathbb{C}_p}, \widehat{\mathcal{V}}_k) \to H^0_{\text{ét}}(\mathcal{X}_{\mathbb{C}_p}, \omega^{\otimes k-2} \otimes \Omega^1_{\mathcal{X}_{\mathbb{C}_p}})(-1)$ is surjective by Proposition 5.7. $\square$

Note that the kernel of ES_k is big: it is infinite-dimensional with finite codimension.

5.3. Results. In this section we deduce some properties of our overconvergent Eichler-Shimura map $\mathcal{ES}$ and the sheaves $\mathbf{V}$ and $\mathcal{M}^\dagger$. The results are very similar to those of [AIS2, §6], but we are able to prove them in a more global form. Let us denote by $\mathcal{C}^{\text{sm}}$ the smooth locus of $\mathcal{C}$. It contains the étale locus $\mathcal{C}^{\text{ét}}$ of the weight map, and both these loci are Zariski open. Furthermore $\mathcal{C}^{\text{ét}}$ contains the set $\mathcal{C}^{\text{nc}}$ of non-critical classical crystalline points for which the roots of the p-Hecke polynomial are distinct. Let us explicitly record a lemma in rigid geometry that makes arguments involving Zariski density simpler. We will (sometimes implicitly) apply it in the remainder of this section.

Lemma 5.9. *Let X be a smooth rigid space over a non-archimedean field K, and assume that $(V_n)_{n=1}^\infty$ is an increasing cover of X by affinoids. Assume that $S \subseteq X$ is a very Zariski dense set of points. Then we may find an increasing* *cover* (U_n) *of* X *by affinoids such that* $S \cap U_n$ *is very Zariski dense in* U_n *for all* n.

Proof. We define U_n to be the Zariski closure of $S \cap V_n$ in V_n. By definition U_n is a union of components of V_n, hence affinoid, and it remains to show that $X = \bigcup_{n=1}^{\infty} U_n$. For this it suffices to show that, for fixed n and a fixed component C of V_n, there exists $m \geq n$ such that $C \subseteq U_m$. Fix n and C. C is a subset of a (global) component D of X, so by Zariski density we pick a point $s \in S \cap D$. If $m \geq n$, let C_m denote the component of V_m such that $C \subseteq C_m$. Define, for $m \geq n$,

$$T_m = \{x \in V_m \mid \exists k \geq m \,:\, i_{km}(x) \in C_k\}$$

where $i_{km} : V_m \to V_k$ is the inclusion map. Then T_m is a union of components and we claim that $s \in T_m$ for some m. To see this, note that $T_m = V_m \cap T_{m+1}$ for all $m \geq n$, so $T := \bigcup_{m \geq n} T_m$ is an open subset of X with open complement. Hence if $s \notin T$, s could not lie on the same component as C. Therefore there is some m for which $s \in T_m$, and hence some k for which $s \in C_k$, which implies that $C \subseteq U_k$ by the very Zariski density of S. $\square$

Remark 5.10. *Note that* $\mathcal{C}^{\mathrm{sm}}$ *may be covered by an increasing union of affinoids. To see this note first that this is true for* $\mathcal{C}$ *since it is finite over* $\mathcal{W} \times \mathbb{A}^1$*, which is quasi-Stein. If we pick an increasing affinoid cover* (V_n) *of* $\mathcal{C}$*, then* $\mathcal{C}^{\mathrm{sm}} = \bigcup_n V_n^{\mathrm{sm}}$ *and* V_n^{sm} *is Zariski dense in* V_n*, with complement cut out by the functions* $f_{n,1}, ..., f_{n,k_n}$ *say. We may choose* $\epsilon_n \in p^{\mathbb{Q}}$ *tending to* 0 *such that*

$$\{|f_{n,1}|, ..., |f_{n,k_n}| < \epsilon_n\} \cap V_m \subseteq \{|f_{m,1}|, ..., |f_{m,k_m}| < 2^{n-m}\epsilon_m\}$$

for all $m \leq n$*, where we use a general fact that the sets* $(\{|g_i| \leq \epsilon\})_{\epsilon \in p^{\mathbb{Q}}}$ *define a cofinal system of neighbourhoods of* $\{|g_i| = 0\}$*. Then we claim that the sets*

$$U_n = \{|f_{n,1}|, ..., |f_{n,k_n}| \geq \epsilon_n\}$$

are affinoid and cover $\mathcal{C}^{\mathrm{sm}}$*. Indeed, they cover* $\mathcal{C}^{\mathrm{sm}}$ *by definition, and they are affinoid since they are rational subsets of the* V_n*.*

With this we state a simple consequence of Proposition 5.8:

Theorem 5.11. *View* $\mathcal{ES}$ *as a morphism of coherent sheaves on* $\mathcal{C}_{\mathbb{C}_p}$*. Then* $\mathcal{ES}$ *is surjective outside a Zariski-closed subset of* $\mathcal{C}_{\mathbb{C}_p}$ *of dimension* 0*, and the support of* $\mathrm{coker}(\mathcal{ES})$ *is disjoint from the set of non-critical points.*

Proof. Since $\mathrm{coker}(\mathcal{ES})$ is coherent its support is Zariski-closed. Hence the first statement follows from the second: if the support is not of dimension 0, then it must contain some component of $\mathcal{C}$ and hence non-critical points. To prove the second statement it suffices to prove that $\mathcal{ES}$ is surjective on fibres at non-critical points. Let x be such a point, with weight k. By the computation of the fibres in §5.1 and the computation of $\mathcal{ES}$ we see that it amounts to the Eichler-Shimura map ES_k restricted to maximal semisimple quotients of the generalized eigenspaces for x. Since x is non-critical this map is surjective by Proposition 5.8. $\square$

Let us raise the following question:

Question: Is the support of coker $\mathcal{ES}$ precisely the set of x with weight $k \in \mathbb{Z}_{\geq 2}$ such that ES_k fails to be surjective on generalized eigenspaces for x?

Next let us state some results on the structure of $\mathbf{V}$ and $\mathcal{M}^\dagger$ over $\mathcal{C}^{\mathrm{sm}}$. For any $n \in \mathbb{Z}_{\geq 1}$ we let $\tau(n)$ denote the number of positive divisors of n.

Theorem 5.12. *Assume, for simplicity, that that the tame level K^p is of the form $\prod_{\ell \nmid \mathrm{disc}(B)} K_1(N) \times \prod_{\ell \mid \mathrm{disc}(B)} \mathcal{O}_{B_\ell}^\times$ for some $N \geq 3$. Let $M \in \mathbb{Z}_{\geq 1}$ be a divisor of N.*

(1) *$\mathbf{V}$ and $\mathcal{M}^\dagger$ are locally free on $\mathcal{C}^{\mathrm{sm}}$. The rank of $\mathbf{V}$ over $\mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ is $2\tau(N/M)$ and the rank of $\mathcal{M}^\dagger$ over $\mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ is $\tau(N/M)$.*
(2) *Let ℓ be a prime not dividing $Np \cdot \mathrm{disc}(B)$. Then $\mathbf{V}$ is unramified at ℓ and the trace of Frob_ℓ on $\mathbf{V}$ over $\mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ is $\tau(N/M) \cdot T_\ell$, where we view T_ℓ as a global function on $\mathcal{C}$.*
(3) *Let $x \in \mathcal{C}^{\mathrm{sm}}_{N-\mathrm{new}}(\overline{\mathbb{Q}}_p)$. Then the fibre of $\mathbf{V}$ at x is the Galois representation attached to x via the theory of pseudorepresentations.*

Proof. First we prove (1). By Lemma 5.5 $\mathbf{V}$ and $\mathcal{M}^\dagger$ are torsion free coherent sheaves over $\mathcal{C}^{\mathrm{sm}}$, which is smooth of pure dimension 1. Therefore $\mathbf{V}$ and $\mathcal{M}^\dagger$ are locally free. To compute the ranks over $\mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ it suffices to compute the dimension of the fibres at points in $\mathcal{C}^{\mathrm{nc}} \cap \mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$. This is a computation using Atkin-Lehner theory, the control theorems of Stevens resp. Coleman and the classical Eichler-Shimura isomorphism (see e.g [Bel, Lemma IV.6.3]).

To prove (2), note that the statements are true after specializing to points in $\mathcal{C}^{\mathrm{nc}}$ (using Stevens's control theorem and the well known structure of the Galois representations $H^1_{\text{ét}}(\mathcal{X}, \mathcal{V}_k)$). By Zariski density of $\mathcal{C}^{\mathrm{nc}} \cap \mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ in $\mathcal{C}^{\mathrm{sm}}_{M-\mathrm{new}}$ we may then conclude. Note that, to check that it is unramified, one may reduce to the affinoid case by Lemma 5.9 and Remark 5.10, where it becomes a purely ring-theoretic statement. Localizing (ring-theoretically), one may then reduce to the free case, where one can check on matrix coefficients using Zariski density.

Finally, (3) follows from (2). □

Remark 5.13. *The assumption on K^p in the above theorem is only to simplify the exposition. The theorem and the definition of the $\mathcal{C}_{M-\mathrm{new}}$ may be adapted to arbitrary tame levels $K^p = \prod_{\ell \neq p} K_\ell$ with essentially the same proof, except for the explicit formulas for the ranks.*

We remark that the statement that $\mathbf{V}$ is unramified at $\ell \nmid Np \cdot \mathrm{disc}(B)$ also follows from the same fact for the $\mathbf{V}^s_{\mathcal{U}}$ (for $\mathcal{U}$ small open), which in turn follows from the construction upon noting that all members of the tower $(\mathcal{X}_{K_p})_{K_p}$ are analytifications of schemes over $\mathbb{Q}$ with good reduction at ℓ.

Finally, we prove that $\mathcal{ES}$ generically gives us Hodge-Tate filtrations/decompositions on $\mathbf{V}$.

Theorem 5.14. *We work over $\mathcal{C}^{\mathrm{sm}}$.*

(1) *The kernel $\mathcal{K}$ and image $\mathcal{I}$ of $\mathcal{ES}$ are locally projective sheaves of $\mathcal{O}_{\mathcal{C}^{\mathrm{sm}}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$-modules, and may also be viewed as locally free sheaves on $\mathcal{C}^{sm}_{\mathbb{C}_p}$.*

(2) *Let $\epsilon_{\mathcal{C}^{\mathrm{sm}}}$ be the character of $G_{\mathbb{Q}_p}$ defined by the composition*

$$G_{\mathbb{Q}_p} \xrightarrow{\epsilon} \mathbb{Z}_p^\times \xrightarrow{\chi_{\mathcal{W}}} \mathcal{O}_{\mathcal{W}}^\times \longrightarrow (\mathcal{O}^\times_{\mathcal{C}^{\mathrm{sm}}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)^\times$$

where ϵ is the p-adic cyclotomic character of $G_{\mathbb{Q}_p}$ and $\chi_{\mathcal{W}}$ is the universal character of $\mathbb{Z}_p^\times$. Then the semilinear action of $G_{\mathbb{Q}_p}$ on the module $\mathcal{K}(\epsilon^{-1}_{\mathcal{C}^{\mathrm{sm}}})$ is trivial.

(3) *The exact sequence*

$$0 \to \mathcal{K} \to \mathbf{V}_{\mathbb{C}_p} \to \mathcal{I} \to 0$$

is locally split. Zariski generically, the splitting may be taken to be equivariant with respect to both the Hecke- and $G_{\mathbb{Q}_p}$-actions, and such a splitting is unique.

Proof. Part (1) is clear since $\mathbf{V}$ and $\mathcal{M}^\dagger$ are locally free on $\mathcal{C}^{\mathrm{sm}}$ and $\mathcal{O}_{\mathcal{C}^{\mathrm{sm}}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ is locally a Dedekind domain. The second statement follows similarly.

For part (2) we use Lemma 5.9 and Remark 5.10 to reduce to the affinoid case with a very Zariski dense set of points in $\mathcal{C}^{\mathrm{nc}}$. Taking a further cover we may also assume that $\mathcal{K}$ is free (perhaps after making a finite extension, but this will not effect the rest of the argument). The statement now follows from the family version of Sen theory ([Se1], [Se2]) , using the fact that the Sen operator ϕ of $\mathcal{K}(\epsilon^{-1}_{\mathcal{C}^{\mathrm{sm}}})$ vanishes on points of $\mathcal{C}^{\mathrm{nc}}$ and hence on $\mathcal{C}^{\mathrm{sm}}$ by Zariski density of $\mathcal{C}^{\mathrm{nc}}$ and analyticity of ϕ. The argument proceeds exactly as the proof of [AIS2, Theorem 6.1(c)], to which we refer for more details.

Finally we prove part (3), arguing more or less exactly as in the proof of [AIS2, Theorem 6.1(d)], to which we refer for more details. Once again we may work over an affinoid $U \subseteq \mathcal{C}^{\mathrm{sm}}$ with a Zariski dense set of points $U \cap \mathcal{C}^{\mathrm{nc}}$. We may without loss of generality assume that $\mathcal{K}$, $\mathbf{V}_{\mathbb{C}_p}$ and $\mathcal{I}$ are free (once again, one might need to make a finite extension, but this does not effect the rest of the argument and we will ignore it). The short exact sequence is an extension which defines an element in $H^1(G_{\mathbb{Q}_p}, \mathcal{H})$ where

$$\mathcal{H} := \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}(U)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p}(\mathcal{I}, \mathcal{K}).$$

Let ϕ be the Sen operator of $\mathcal{H}$. By work of Sen it is well-known that $\det(\phi)$ kills $H^1(G_{\mathbb{Q}_p}, \mathcal{H})$. Specializing at points in $U \cap \mathcal{C}^{\mathrm{nc}}$ one sees that $\det(\phi)$ does not vanish identically on any component of U. Thus, localizing with respect to $\det(\phi)$, we find a Zariski open subset of U over which the extension is split as semilinear $G_{\mathbb{Q}_p}$-representations. Moreover, if we further remove the finite set of points whose weight is $-1 \in \mathbb{Z}$, then $\mathcal{I}$ and $\mathcal{K}$ have distinct Hodge-Tate weights fibre-wise (by using part (2) and that $\mathcal{I}$ has constant Hodge-Tate weight -1). Thus there can be no non-zero Galois equivariant homomorphisms between $\mathcal{K}$ and $\mathcal{I}$. Since the Hecke action commutes with the $G_{\mathbb{Q}_p}$-action, this implies that the $G_{\mathbb{Q}_p}$-splitting must be Hecke stable as well. □

6. Appendix

In this appendix we define a "mixed" completed tensor product that we will use in the main text, and prove some basic properties as well as a few technical results that we need.

6.1. Mixed completed tensor products. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be its ring of integers, with uniformizer ϖ. Our "mixed completed tensor products" will be denoted by an unadorned $\widehat{\otimes}$. The base field is assumed to be implicit in the notation. In the main text we will always take $K = \mathbb{Q}_p$ so this should not cause any problem. We starting by making the following definition:

Definition 6.1. *Let M be a topological $\mathcal{O}$-module.*

1. *M is called* linear-topological *if there exists a basis of neighbourhoods of* 0 *consisting of $\mathcal{O}$-submodules.*
2. *We will say that M is a* profinite flat *$\mathcal{O}$-module if M is flat over $\mathcal{O}$ (i.e torsion-free), linear-topological and compact.*

Let us remark that if M is a profinite flat $\mathcal{O}$-module then the topology on M is profinite, which justifies this terminology. We also remark that such M are exactly the $\mathcal{O}$-modules which are projective and pseudocompact in the language of [SGA3, Exposé VII_B, §0]. Let us recall the following structure theorem for profinite flat $\mathcal{O}$-modules, specialized from [SGA3, Exposé VII_B, 0.3.8]:

Proposition 6.2. *M is a profinite flat $\mathcal{O}$-module if and only if it is isomorphic to $\prod_{i\in I}\mathcal{O}$ equipped with the product topology, for some set I.*

We note that a set of elements $(e_i)_{i\in I}$ in M such that $M \cong \prod_{i\in I}\mathcal{O}e_i$ is called a *pseudobasis*. In what follows 'ϖ-adically complete" always means complete and separated.

Definition 6.3. *Let M be a profinite flat $\mathcal{O}$-module and let X be any $\mathcal{O}$-module. We define*

$$X\widehat{\otimes}M := \varprojlim_i (X \otimes_{\mathcal{O}} M/I_i)$$

where (I_i) runs through any cofinal set of neighbourhoods of 0 *consisting of $\mathbb{Z}_p$-submodules.*

By abstract nonsense this is independent of the choice of system of neighbourhoods of 0. Note that if A is an $\mathcal{O}$-algebra and R is a small $\mathcal{O}$-algebra, then $A\widehat{\otimes}R$ comes with a natural structure of a ring, as is seen by choosing the neighbourhoods to be ideals. To compute $X\widehat{\otimes}M$ in a useful form we fix a pseudobasis $(e_i)_{i\in I}$ and define, for $J \subseteq I$ finite and $n \geq 1$ an integer, $M_{J,n}$ to be the submodule corresponding to

$$M_{J,n} := \prod_{i\in J}\varpi^n\mathcal{O}e_i \times \prod_{i\notin J}\mathcal{O}e_i.$$

This forms a system of neighbourhoods of 0. Then we have:

Proposition 6.4. *The functor $X \mapsto X\widehat{\otimes}M$ is isomorphic to the functor $X \mapsto \prod_{i\in I}\widehat{X}$, where $\widehat{X}$ is the ϖ-adic completion of X.*

Proof. We compute

$$\begin{aligned} X\widehat{\otimes}M &= \varprojlim_{J,n} X\otimes_{\mathcal{O}} M/M_{J,n} \\ &= \varprojlim_{J}(\varprojlim_{n} X\otimes_{\mathcal{O}} M/M_{J,n}) \\ &\cong \varprojlim_{J}(\varprojlim_{n} X\otimes_{\mathcal{O}} \prod_{i\in J}\mathcal{O}/\varpi^n) \\ &= \varprojlim_{J}\left(\prod_{i\in J}(\varprojlim_{n} X\otimes_{\mathcal{O}} \mathcal{O}/\varpi^n)\right) \\ &= \varprojlim_{J}\left(\prod_{i\in J}\widehat{X}\right) \\ &= \prod_{i\in I}\widehat{X}. \end{aligned}$$

□

Corollary 6.5. *We have*

(1) *$X \mapsto X\widehat{\otimes}M$ is exact on ϖ-adically complete X.*
(2) *If $0 \to X \to Y \to Q \to 0$ is an exact sequence of $\mathcal{O}$-modules where X is ϖ-adically complete and Q is killed by ϖ^N for some integer $N \geq 0$, then Y is ϖ-adically complete and the natural map $(X\widehat{\otimes}M)[1/\varpi] \to (Y\widehat{\otimes}M)[1/\varpi]$ is an isomorphism.*

Proof. (1) follows from exactness of products in the category of $\mathcal{O}$-modules. For (2) note that Q is ϖ-adically complete and $Q\widehat{\otimes}M$ is killed by ϖ^N, so the second statement will follow from the first and (1). To see that Y is ϖ-adically complete first note that $\varpi^N Y \subseteq X$ (i.e. Q killed by ϖ^N) so

$$\varpi^{N+k}Y \subseteq \varpi^k X \subseteq \varpi^k Y$$

for all $k \geq 0$ and hence we can compute $\widehat{Y}$ as $\varprojlim_k Y/\varpi^k X$. Thus we get exact sequences

$$0 \to X/\varpi^k X \to Y/\varpi^k X \to Q \to 0$$

for all k and hence an exact sequence

$$0 \to X = \widehat{X} \to \widehat{Y} \to Q$$

by left exactness of inverse limits. This sits in a natural diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] & \widehat{Y} \arrow[r] & Q & \end{tikzcd}$$

with exact rows and the snake lemma now implies that the map $Y \to \widehat{Y}$ is an isomorphism as desired. $\square$

In particular we can now define $-\widehat{\otimes}M$ on K-Banach spaces (this is why we are calling it a "mixed" completed tensor product). It is canonically independent of the choice of unit ball, but we also make a more general definition.

Definition 6.6. *Let V be a K-vector space and let $V^\circ \subseteq V$ be an $\mathcal{O}_K$-submodule such that $V^\circ[1/\varpi] = V$. Then we define*

$$V\widehat{\otimes}M := (V^\circ\widehat{\otimes}M)[1/\varpi]$$

where the choice of V° is implicit (the choice will in general affect the result).

If V is naturally a Banach space we will always take V° to be an open and bounded $\mathcal{O}$-submodule, and Corollary 6.5(2) implies that $V\widehat{\otimes}M$ is independent of the choice.
We finish with a lemma needed in the main text.

Lemma 6.7. *Let S be a K-Banach algebra and let V be a finite projective S-module and U a Banach S-module. Let M be a profinite flat $\mathcal{O}_K$-module. Then we have a natural isomorphism $V \otimes_S (U\widehat{\otimes}M) \cong (V \otimes_S U)\widehat{\otimes}M$. If V and U in addition are Banach S-algebras and M is a small $\mathcal{O}_K$-algebra, then this is an isomorphism of rings.*

Proof. Pick an open bounded $\mathcal{O}_K$-subalgebra S_0 of S and pick open bounded S_0-submodules $V_0 \subseteq V$ and $U_0 \subseteq U$. Moreover, pick $\mathcal{O}_K$-submodules (I_i) of M that form a basis of neighbourhoods of 0. Then we have natural maps

$$V_0 \otimes_{S_0} (\varprojlim_i (U_0 \otimes_{\mathcal{O}_K} M/I_i)) \to V_0 \otimes_{S_0} U_0 \otimes_{\mathcal{O}_K} M/I_i$$

for all i which give us a natural map

$$V_0 \otimes_{S_0} (U_0\widehat{\otimes}M) \to (V_0 \otimes_{S_0} U_0)\widehat{\otimes}M.$$

Now invert ϖ to get the desired map. If V and U are Banach S-algebras we may choose everything so that V_0 and U_0 are subrings, and if M is a small $\mathcal{O}_K$-algebra we may choose the I_i to be ideals, hence we see that this is a ring homomorphism. To see that it is an isomorphism note that if V is in addition free we may choose V_0 to be a finite free S_0-module. The assertion is then clear. In general, write V as a direct summand of a finite free S-module. $\square$

6.2. Mixed completed tensor products on rigid-analytic varieties.

Let X be a quasicompact and separated smooth rigid analytic variety over an algebraically closed and complete extension C of K, of pure dimension n. Here K is a finite extension of $\mathbb{Q}_p$ as in the previous subsection and we use the same notation $\mathcal{O} = \mathcal{O}_K, \varpi$ et cetera. In this subsection we aim to prove:

Proposition 6.8. *We have a canonical isomorphism*

$$R^i\nu_*(\widehat{\mathcal{O}}_X\widehat{\otimes}M) \cong (R^i\nu_*\widehat{\mathcal{O}}_X)\widehat{\otimes}M$$

of sheaves on $X_{\text{ét}}$.

Here $U \mapsto (\widehat{\mathcal{O}}_X \widehat{\otimes} M)(U) = \widehat{\mathcal{O}}_X(U) \widehat{\otimes} M$ for $U \in X_{\text{proét}}$ qcqs defines a sheaf on $X_{\text{proét}}$ and the completed tensor product is computed using $\widehat{\mathcal{O}}_X^+(U) \subseteq \widehat{\mathcal{O}}_X(U)$. The former is ϖ-adically complete by [Sch3, Lemma 4.2(iii)], so we have exactness of $-\widehat{\otimes} M$ and hence a sheaf. We use $\mathrm{Im}((R^i\nu_*\widehat{\mathcal{O}}_X^+)(U) \to (R^i\nu_*\widehat{\mathcal{O}}_X)(U))$ to compute the completed tensor product on the right hand side and claim that this is a sheaf, but these statements are not obvious and we will prove them below.

We note the following important corollary to the above proposition.

Corollary 6.9. *We have*

$$R^i\nu_*(\widehat{\mathcal{O}}_X \widehat{\otimes} M) \cong \Omega_X^i(-i) \otimes_{\mathcal{O}_X} (\mathcal{O}_X \widehat{\otimes} M)$$

as sheaves on $X_{\text{ét}}$.

Proof. This follows from Proposition 6.8, Lemma 6.7 and the canonical isomorphism $R^i\nu_*(\widehat{\mathcal{O}}_X) \cong \Omega_X^i(-i)$ (Proposition 3.23 of [Sch2]) together with the finite projectivity of the latter sheaf. □

We will need several lemmas before we can prove Proposition 6.8.

Lemma 6.10. *The presheaf* $U \mapsto H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$ *on* $X_{\text{ét}}$ *is a sheaf, where* $\mathcal{O}_X^+(U)$ *has its* ϖ*-adic topology and the trivial* $\mathbb{Z}_p^n$*-action.*

Proof. This follows from the computation in [Sch3, Lemma 5.5]; $H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$ is isomorphic to $\bigwedge^i \mathcal{O}_X^+(U)^n$ in a way compatible with completed tensor products. □

Put $T^n = \mathrm{Spa}(C\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle, \mathcal{O}_C\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle)$. We let $\widetilde{T}^n$ be the inverse limit over $n \geq 0$ of $\mathrm{Spa}(C\langle X_1^{\pm 1/p^n}, ..., X_n^{\pm 1/p^n}\rangle, \mathcal{O}_C\langle X_1^{\pm 1/p^n}, ..., X_n^{\pm 1/p^n}\rangle)$. This is an affinoid perfectoid (see [Sch3, Example 4.4]).

Lemma 6.11. *Let* $U \in X_{\text{ét}}$ *and and assume that there is a map* $U \to T^n$ *which is a composition of rational embeddings and finite étale maps. Pull back the affinoid perfectoid* $\mathbb{Z}_p^n$*-cover* $\widetilde{T}^n$ *to obtain an affinoid perfectoid* $\mathbb{Z}_p^n$*-cover* $\widetilde{U}$ *of* U*. Then:*

(1) *There is a canonical injection*

$$H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U)) \to H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+)$$

with cokernel killed by p.

(2) $H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X) \cong H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X(U))$.

(3) $H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+ \widehat{\otimes} M)^a \cong (H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+)\widehat{\otimes} M)^a$ *(where* $-^a$ *denotes the associated almost* $\mathcal{O}_C$*-module).*

(4) $H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X \widehat{\otimes} M) \cong H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X)\widehat{\otimes} M$.

Proof. We start with (1). The Cartan–Leray spectral sequence (cf. Remark 2.25) gives us

$$H^p_{cts}(\mathbb{Z}_p^n, H^q(\widetilde{U}_{\text{proét}}, \mathcal{O}_X^+/\varpi^m)) \implies H^{p+q}(U_{\text{proét}}, \mathcal{O}_X^+/\varpi^m)$$

for all m. Since the $H^q(\widetilde{U}_{\text{proét}}, \mathcal{O}_X^+/\varpi^m)$ are almost zero for all $q \geq 1$ ([Sch3, Lemma 4.10], see the proof of (i) and (v)) we conclude that

$$H^p(U_{\text{proét}}, \mathcal{O}_X^+/\varpi^m)^a = H^p(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(\widetilde{U}))^a.$$

Next, note that the inverse system

$$(H^p(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(\widetilde{U}))_m$$

has surjective transition maps by the proof of [Sch3, Lemma 5.5]. Since the inverse system $(\mathcal{O}_X^+/\varpi^m)_m$ has almost vanishing higher inverse limits on the pro-étale site (by an easy application of the almost version of [Sch3, Lemma 3.18]) we conclude that there is an almost isomorphism

$$\varprojlim H^p(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(\widetilde{U})^a \cong H^p(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+)^a.$$

By [Sch3, Lemma 5.5] we have

$$\bigwedge^p (\mathcal{O}_X^+/\varpi^m)(U)^n \cong H^p(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(U)) \to H^p(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(\widetilde{U}))$$

with cokernel killed by p, compatibly in m; in fact the proof shows that the injection is split. It remains to verify that $\varprojlim H^q(\mathbb{Z}_p^n, (\mathcal{O}_X^+/\varpi^m)(U)) \cong H^q_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$. This is standard; one may e.g. argue as in the proof of [NSW, Theorem 2.5.7] (note that the relevant $\varprojlim^1$ vanishes by the isomorphism above). This finishes the proof of (1).

For (2), use (1) and invert ϖ - this commutes with taking cohomology by quasicompactness.

For (3) we view $-\widehat{\otimes} M$ as the functor of taking self products over some index set I. We remark that $H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+)^a$ is ϖ-adically complete by part (1), [Sch3, Lemma 5.5] and the almost version of Corollary 6.5(3). Thus we are left with verifying that cohomology for $\widehat{\mathcal{O}}_X^+$ almost commutes with arbitrary products. Writing $\prod_{i\in I}$ as $\varprojlim_J \prod_{i \in J}$ for all $J \subseteq I$ finite we are left with showing that the relevant higher inverse limits vanish, which is easy using the Mittag-Leffler condition and [Sch3, Lemma 3.18].

Finally we invert ϖ in (3) to deduce (4). □

LEMMA 6.12. *The sheaf $R^i\nu_*\widehat{\mathcal{O}}_X$ is equal to the sheaf $U \mapsto H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X(U))$, and $(R^i\nu_*\widehat{\mathcal{O}}_X)\widehat{\otimes} M$ may be computed using $H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U)) \subseteq R^i\nu_*\widehat{\mathcal{O}}_X(U)$.*

Proof. We may work étale locally on X, so assume without loss of generality that there is a map $X \to T^n$ which is a composition of rational embeddings and finite étale maps. By abstract nonsense $R^i\nu_*\widehat{\mathcal{O}}_X$ is the sheafification of the presheaf $U \mapsto H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X)$. Let us write $\mathcal{F}$ for this presheaf. By part (1) of the previous Lemma there is an exact sequence

$$0 \to H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U)) \to H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+) \to Q^i(U) \to 0$$

where $Q^i(U)$ is simply defined as the quotient. This is an exact sequence of presheaves on X^w_{et}. Moreover $Q^i(U)$ is killed by p for all U. Write $\mathcal{G}$ for the presheaf $U \mapsto H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$, $\mathcal{F}^+$ for $U \mapsto H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+)$ and $\mathcal{Q}$ for $U \mapsto Q^i(U)$. Then the exact sequence above sheafifies to

$$0 \to \mathcal{G} \to (\mathcal{F}^+)^{sh} = R^i\nu_*\widehat{\mathcal{O}}_X^+ \to \mathcal{Q}^{sh} \to 0$$

($-^{sh}$ for sheafification) since $\mathcal{G}$ is already a sheaf by Lemma 6.10, and $\mathcal{Q}^{sh}$ is killed by p since $\mathcal{Q}$ is. Inverting ϖ we get that $H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))[1/\varpi] \cong R^i\nu_*\widehat{\mathcal{O}}_X(U)$. For the other assertion of the lemma, note that $\mathrm{Im}(R^i\nu_*\widehat{\mathcal{O}}_X^+(U) \to R^i\nu_*\widehat{\mathcal{O}}_X(U))$ coincides with $H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$ under the identification $H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))[1/\varpi] \cong R^i\nu_*\widehat{\mathcal{O}}_X(U)$, and is therefore a sheaf (as was asserted in Proposition 6.8). $\square$

Proof of Proposition 6.8: Lemma 6.12 gives us that

$$((R^i\nu_*\widehat{\mathcal{O}}_X)\widehat{\otimes}M)(U) = (H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))\widehat{\otimes}M)[1/\varpi].$$

On the other hand $R^i\nu_*(\widehat{\mathcal{O}}_X\widehat{\otimes}M)$ is the sheaf associated with the presheaf

$$U \mapsto H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X\widehat{\otimes}M)$$

by abstract nonsense. Working locally again, assume that there is a map $U \to T^n$ as above. We may then compute

$$\begin{aligned} H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X\widehat{\otimes}M) &= H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X)\widehat{\otimes}M \\ &= (\mathrm{Im}(H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X^+) \to H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X))\widehat{\otimes}M)[1/\varpi] \\ &= (H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))\widehat{\otimes}M)[1/\varpi] \end{aligned}$$

using Lemma 6.11 (4) and the proof of Lemma 6.12. Therefore $U \mapsto H^i(U_{\text{proét}}, \widehat{\mathcal{O}}_X\widehat{\otimes}M)$ is actually defines sheaf when restricting to the basis of such U (since $U \mapsto H^i_{cts}(\mathbb{Z}_p^n, \mathcal{O}_X^+(U))$ is a sheaf), so it must equal $R^i\nu_*(\widehat{\mathcal{O}}_X\widehat{\otimes}M)$ on these U. This gives the desired isomorphism. $\square$

6.3. Analogues of the theorems of Tate and Kiehl. Let X be a quasicompact and quasiseparated rigid analytic variety over some complete nonarchimedean field K of *characteristic zero*. For simplicity let us work with the site given by the basis of the topology consisting of quasicompact open subsets together with finite covers (we will specialize this further eventually). Given a small $\mathbb{Z}_p$-algebra R, we may form the sheaf $\mathcal{R}(U) = \mathcal{O}_X(U)\widehat{\otimes}R$ on X (U is qc open). In this section we are going to prove analogues of the theorems of Tate and Kiehl for $\mathcal{O}_X$ for $\mathcal{R}$ under the assumption that X is affinoid and that K is discretely valued. The assumption that K is discretely valued makes it easy to prove various flatness assertions. We suspect that this assumption can be dropped with some more work but we will not need the extra generality. We will however prove a few statements that we need without this assumption on K. For simplicity we work with the topology on X given by rational subsets and finite covers. We start with some basic properties.

Lemma 6.13. *Let X be affinoid and assume that K is discretely valued.*

(1) *For every rational $U \subseteq X$, $\mathcal{R}(U)$ is Noetherian.*
(2) *If $V \subseteq U \subseteq X$ are rational, then the map $\mathcal{R}(U) \to \mathcal{R}(V)$ is flat.*
(3) *If U is rational and $(U_i)_i$ is a finite rational cover of U, then the natural map $\mathcal{R}(U) \to \prod_i \mathcal{R}(U_i)$ is faithfully flat.*

Proof. To prove (1), choose surjections $\mathcal{O}_K\langle X_1, \dots, X_m\rangle \twoheadrightarrow \mathcal{O}_X(U)$, $\mathbb{Z}_p[[T_1, \dots, T_n]] \twoheadrightarrow R$ and use the image of $\mathbb{Z}_p\langle X_1, \dots, X_m\rangle$ to compute $\mathcal{O}_X(U)\widehat{\otimes}R$ (this is valid by the open mapping theorem). Then we get a surjection

$$\mathcal{O}_K\langle X_1, \dots, X_m\rangle\widehat{\otimes}\mathbb{Z}_p[[T_1, \dots, T_n]] \twoheadrightarrow \mathcal{O}_X(U)\widehat{\otimes}R$$

and by direct computation the left hand side is

$$(\mathcal{O}_K\langle X_1, \dots, X_m\rangle[[T_1, \dots, T_n]])[1/p]$$

, which is Noetherian (since K is discretely valued), hence the right hand side is Noetherian.

Next we prove (2). Pick functions $f_1, \dots, f_n, g$ such that $V = \{|f_1|, \dots, |f_n| \leq |g| \neq 0\}$. We factor $\mathcal{R}(U) \to \mathcal{R}(V)$ as

$$\mathcal{O}_X(U)\widehat{\otimes}R \to \mathcal{O}_X(U)[1/g] \otimes_{\mathcal{O}_X(U)} (\mathcal{O}_X(U)\widehat{\otimes}R) \to \mathcal{O}_X(V)\widehat{\otimes}R.$$

The first morphism is flat since $\mathcal{O}_X(U) \to \mathcal{O}_X(U)[1/g]$ is, so it suffices to show that the second morphism is flat. It is obtained from

$$\mathcal{O}_X^\circ(U)[f_1/g, \dots, f_n/g] \otimes_{\mathcal{O}_X^\circ(U)} (\mathcal{O}_X^\circ(U)\widehat{\otimes}R) \to \mathcal{O}_X^\circ(V)\widehat{\otimes}R$$

by inverting p, so it suffices to show that this map is flat. But the right hand side is the completion of the left hand side with respect to the ideal generated by the images of the generators of the maximal ideal of R. Since the ring on the left hand side is Noetherian (it is finitely generated over $\mathcal{O}_X^\circ(U)\widehat{\otimes}R$, which is Noetherian by the same proof as in (1)), we conclude that the map is flat.

For (3) we use a more geometric argument. By part (2) the map

$$\mathcal{O}_X(U)\widehat{\otimes}R \to \prod_i \mathcal{O}_X(U_i)\widehat{\otimes}R$$

is flat. To see that it is faithfully flat, it suffices to show that it induces a surjection on spectra of *maximal* ideals, by [Bou, §§I.3.5, Proposition 9]. To simplify notation, put $S = \mathcal{O}_X(U)$ and $S_i = \mathcal{O}_X(U_i)$. Since we are only interested in maximal ideals, we will momentarily work with classical rigid spaces rather than adic spaces in this proof (only). R, S° and the S_i° are formally of finite type over $\mathcal{O}_K$, and the unadorned completed tensor products $S^\circ\widehat{\otimes}R$ etc. are completed tensor products over $\mathcal{O}_K$ in the category of topological $\mathcal{O}_K$-algebras formally of finite type. Recall Berthelot's generic fibre functor $(-)^{rig}$ (see e.g. [dJ, §7]). By [dJ, Proposition 7.2.4(g)] the generic fibre functor commutes with fibre products, so the morphism

$$\bigsqcup_i U_i \times_{\mathrm{Sp}\,K} \mathrm{Spf}(R)^{rig} = \bigsqcup_i \mathrm{Spf}(S_i^\circ \widehat{\otimes} R)^{rig} \to$$
$$\to \mathrm{Spf}(S^\circ \widehat{\otimes} R)^{rig} = U \times_{\mathrm{Sp}\,K} \mathrm{Spf}(R)^{rig}$$

is surjective (it is a finite open cover). By [dJ, Lemma 7.1.9] we deduce that we have a surjection

$$\bigsqcup_i \mathrm{MaxSpec}(S_i \widehat{\otimes} R) = \mathrm{MaxSpec}\left(\prod_i S_i \widehat{\otimes} R\right) \to \mathrm{MaxSpec}(S \widehat{\otimes} R),$$

which is what we wanted to prove. □

Having established this Lemma we now view $\mathcal{R}$ as a sheaf of Banach algebras on X by picking the unit ball of $\mathcal{R}(U)$ to be $\mathcal{R}^\circ(U) = \mathcal{O}_X^\circ(U) \widehat{\otimes} R$. Then all restriction maps are norm-decreasing, hence continuous, and if we view R as a profinite flat $\mathbb{Z}_p$-module and choose a pseudobasis with index set I, the Banach space structure on $\mathcal{R}(U)$ is the same as the natural Banach space structure on the set of bounded sequences in $\mathcal{O}_X(U)$ indexed by I (of course the multiplication is very different from the pointwise multiplication on the latter). We will follow [AW, §5] (which in turn follows [FvdP, §4.5]) closely in the proof of our analogue of Kiehl's theorem.

Definition 6.14. *Let $U \subseteq X$ be rational and let M be a finitely generated $\mathcal{R}(U)$-module. For every $V \subseteq U$ rational, we define*

$$\mathrm{Loc}(M)(V) := \mathcal{R}(V) \otimes_{\mathcal{R}(U)} M.$$

This is a presheaf of $\mathcal{R}$-modules of U.

Proposition 6.15 (Tate's theorem). *For every rational $U \subseteq X$, $\check{H}^i(\mathfrak{U}, \mathcal{R}) = 0$ for all $i \geq 1$ and all finite rational covers $\mathfrak{U}$ of U, i.e. the augmented Čech complex for $\mathfrak{U}$ is exact. As a consequence, $H^i(U, \mathcal{R}) = 0$ for all $i \geq 1$.*

Proof. The second statement is a direct consequence of the first by a well known theorem of Cartan. For the first, pick a pseudobasis $(e_i)_{i\in I}$ of R, considered as a profinite flat $\mathbb{Z}_p$-module. For a cover $\mathfrak{U}$ as above, write $C^\bullet_{aug}(\mathfrak{U}, -)$ for the augmented Čech complex functor. Then, using the pseudobasis, we have

$$C^\bullet_{aug}(\mathfrak{U}, \mathcal{R}) = C^\bullet_{aug}(\mathfrak{U}, \mathcal{R}^\circ)[1/p] \cong \left(\prod_{i\in I} C^\bullet_{aug}(\mathfrak{U}, \mathcal{O}_X^\circ)\right)[1/p].$$

Note that the cohomology groups $\check{H}^i(\mathfrak{U}, \mathcal{O}_X^\circ)$ are bounded p-torsion for $i \geq 1$ by Tate's theorem and the open mapping theorem, hence by exactness of products and the above displayed equation so are the cohomology groups $\check{H}^i(\mathfrak{U}, \mathcal{R}^\circ)$ for $i \geq 1$. Thus $\check{H}^i(\mathfrak{U}, \mathcal{R})$ vanish as desired. □

Proposition 6.16. *Assume that K is discretely valued. Then functor* Loc *defines a full exact embedding of abelian categories from the category of finitely generated $\mathcal{R}(U)$-modules into the category of $\mathcal{R}$-modules on U. Moreover, $\check{H}^i(\mathfrak{U}, \mathrm{Loc}(M)) = 0$ and $H^i(U, \mathrm{Loc}(M)) = 0$ for all $i \geq 1$ and all finite rational covers $\mathfrak{U}$ of U for all finitely generated $\mathcal{R}(U)$-modules M. If K is not discretely valued,* $\mathrm{Loc}(M)$ *is still a sheaf with vanishing higher cohomology groups if M is a finitely generated projective module.*

Proof. We start with the sheaf/vanishing assertions. Again, the statement about derived functor cohomology follows from that of Čech cohomology by Cartan's theorem, and the assertion about Čech cohomology implies that $\mathrm{Loc}(M)$ is a sheaf. Pick a cover $\mathfrak{U}$. Then by Proposition 6.15 the complex $C^\bullet_{aug}(\mathfrak{U}, \mathcal{R})$ is exact. If M is projective, then it is flat and hence $C^\bullet_{aug}(\mathfrak{U}, \mathcal{R}) \otimes_{\mathcal{R}(U)} M = C^\bullet_{aug}(\mathfrak{U}, \mathrm{Loc}(M))$ is exact. If K is discretely valued, then by the flatness properties of $\mathcal{R}$ all terms are flat $\mathcal{R}(U)$-modules and we may deduce that $C^\bullet_{aug}(\mathfrak{U}, \mathcal{R}) \otimes_{\mathcal{R}(U)} M = C^\bullet_{aug}(\mathfrak{U}, \mathrm{Loc}(M))$ is exact for arbitrary finitely generated M. In either case, it follows that $\mathrm{Loc}(M)$ is a sheaf with vanishing higher Čech cohomology.

Now let K be discretely valued. To see that the functor is fully faithful, note that $\mathrm{Loc}(M)$ is generated by global sections as an $\mathcal{R}$-module. For exactness, one checks that if $f : M \to N$ is a morphism of finitely generated $\mathcal{R}(U)$-modules, then $\ker(\mathrm{Loc}(f)) = \mathrm{Loc}(\ker(f))$ and similarly for images and cokernels (in particular, images and cokernels in this sub-abelian category turn out to be equal to the presheaf images and cokernels). □

Remark 6.17. *Propositions 6.15 and 6.16 also hold for the étale site of X, with the same proofs with obvious modifications.*

We may now introduce coherent $\mathcal{R}$-modules.

Definition 6.18. *Assume that K is discretely valued. For a cover $\mathfrak{U}$ of U as above a sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on U is called* $\mathfrak{U}$-coherent *if for every $U_i \in \mathfrak{U}$ there exists an $\mathcal{R}(U_i)$-module M_i such that $\mathcal{F}|_{U_i} \cong \mathrm{Loc}(M_i)$.*

$\mathcal{F}$ is said to be coherent *if it is $\mathfrak{U}$-coherent for some $\mathfrak{U}$.*

The equivalence of this definition and the usual definition is standard, cf. e.g. [AW, §5.2].

Before proving Kiehl's theorem we need a simple lemma. Pick $f \in \mathcal{O}_X(X)$ and consider the standard cover of X consisting of $X(f) = \{|f| \leq 1\}$ and $X(1/f) = \{|f| \geq 1\}$. We denote the intersection by $X(f, 1/f)$. We let s denote the restriction map $\mathcal{O}_X(X(1/f)) \to \mathcal{O}_X(X(f, 1/f))$.

Lemma 6.19. *With notation as above:*

1. *The image of s is dense.*
2. *The image of the restriction map $\mathcal{R}(X(1/f)) \to \mathcal{R}(X(f, 1/f))$ is dense for the natural Banach algebra structure on the target.*

Proof. Statement (1) is well known, but since there seems to be a little confusion in the literature let us give the one-sentence proof: s is the completion of the identity map on $\mathcal{O}_X(X)[1/f]$ with respect to the group topology generated by $(\varpi^n \mathcal{O}_X^\circ(X)[1/f])_{n\geq 0}$ on the source and the group topology generated by $(\varpi^n \mathcal{O}_X^\circ(X)[f,1/f])_{n\geq 0}$ on the target.

To prove (2), pick a pseudobasis $(e_i)_{i\in I}$ for R. Then the map is the natural map from bounded I-sequences in $\mathcal{O}_X(X(1/f))$ to bounded I-sequences in $\mathcal{O}_X(X(f,1/f))$, which has dense image since s does. $\square$

We may now follow the proof of Kiehl's theorem from [AW, §5.3-5.5]. One firstly establishes an auxiliary lemma ([AW, Lemma 5.3]), which only uses the Banach algebra structure and vanishing of cohomology (in our setting this is Proposition 6.15). Using this auxiliary lemma one proves [AW, Theorem 5.4] on surjections of certain maps (upon noting that, in their notation, it is erroneously asserted in §5.3 that s_1 rather than s_2 has dense image, see Lemma 6.19). Then Ardakov and Wadsley establish a weak form of Kiehl's theorem for the covering $\{X(f), X(1/f)\}$ ([AW, Corollary 5.4]), using [AW, Theorem 5.4] and properties of Loc functor, which is Proposition 6.16 in our setting. From this, [AW, Theorem 5.5] follows as well. Following their arguments, we get

Theorem 6.20 (Kiehl's Theorem). *Assume that K is discretely valued and let $\mathcal{F}$ be a coherent $\mathcal{R}$-module on X. Then there exists a finitely generated $\mathcal{R}(X)$-module M such that $\mathcal{F} \cong \mathrm{Loc}(M)$. Moreover, if $\mathcal{F}$ is locally projective (i.e. there is a cover $\mathfrak{U}$ such that $\mathcal{F}(U)$ is projective for all $U \in \mathfrak{U}$), then M is projective.*

Proof. For the last part, use that $\mathcal{R}(X) \to \prod_{U\in\mathfrak{U}} \mathcal{R}(U)$ is faithfully flat and that projectivity for finitely presented modules may be checked after a faithfully flat base extension. $\square$

We record a simple base change lemma.

Lemma 6.21. *Assume that K is discretely valued and that C is the completion of an algebraic closure $\overline{K}$ of K. Let X/K be an affinoid rigid space and let M be a finitely generated projective $\mathcal{R}(X)$ module. Let X_C be the base change of X to C and let $f : X_C \to X$ be the natural map, and let $\mathcal{R}_C$ be the sheaf $U \mapsto \mathcal{O}_{X_C} \widehat{\otimes} R$ on X_C. Then we have natural isomorphisms $M \otimes_{\mathcal{R}(X)} \mathcal{R}_C(X) \cong M \otimes_K C$ and*

$$f^{-1}(\mathrm{Loc}(M)) \otimes_{f^{-1}\mathcal{R}} \mathcal{R}_C \cong \mathrm{Loc}(M \otimes_K C).$$

Proof. The proof of the first assertion is similar to the proof of Lemma 6.7. Using this, the second assertion follows straight from the definitions upon noting that f is open, which makes it easy to compute f^{-1}. $\square$

Remark 6.22. *When working with affinoid weights in the main part of the paper we will often need analogues of the results of this section (as well as Lemma 6.7) when one replaces $-\widehat{\otimes} R$ by $-\widehat{\otimes}_{\mathbb{Q}_p} S$ in the definition of $\mathcal{R}$, where S is a reduced $\mathbb{Q}_p$-Banach algebra of topologically of finite type. In this case these* *results are classical results in rigid geometry, or straightforward consequences of such results.*

6.4. Quotients of rigid spaces by finite groups. In this section we show that if X is a rigid space and G is a finite group acting on X from the right such that X has a cover by G-stable affinoid open subsets, then the quotient X/G exists in the category of rigid spaces. If $X = \mathrm{Spa}(A, A^\circ)$ is affinoid, then the quotient $X/G := \mathrm{Spa}(A^G, (A^G)^\circ)$ exists in the category of *affinoid* rigid spaces by [BGR, §6.3.3 Proposition 2]. The purpose of this section is to show that this construction glues. In fact, we show that the quotient is a quotient in Huber's ambient category $\mathscr{V}$ ([Hub, §2]), hence in the category of adic spaces. This is presumably well known, but we have not been able to find a reference.

We start with some general discussion on quotients. Let $\underline{X} = (X, \mathcal{O}_X, (v_x)_{x\in X}) \in \mathscr{V}$ and let G be a finite group acting on $\underline{X}$ from the right. Consider the quotient topological space $Y = X/G$ and let $\pi : X \to Y$ be the quotient map. We may consider the sheaf of rings $\mathcal{O}_Y := (\pi_*\mathcal{O}_X)^G$ on Y. For $y \in Y$ we pick $x \in X$ such that $\pi(x) = y$ and let v_y denote the valuation on $\mathcal{O}_{Y,y}$ given by composing v_x with the natural map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. This is independent of the choice of x (and representatives for the valuations v_x). Then we have:

Lemma 6.23. *With notation as above, the sheaf $\mathcal{O}_Y := (\pi_*\mathcal{O}_X)^G$ is a sheaf of complete topological rings on Y and $(Y, \mathcal{O}_Y, (v_y)_{y\in Y})$ is in $\mathscr{V}$ and is the categorical quotient of X by G in $\mathscr{V}$.*

Proof. We prove that $\mathcal{O}_Y$ is a sheaf of complete topological rings, the rest is then immediate from the definitions. First, note that $\mathcal{O}_Y$ is a sheaf since invariants are left exact. Second, note that taking invariants preserve completeness (for example, argue with nets). Finally, taking invariants preserves topological embeddings, so $\mathcal{O}_Y$ is a sheaf of complete topological rings. □

Now let K be a complete non-archimedean field with ring of integers $\mathcal{O}_K$, and fix an element $\varpi \in \mathcal{O}_K$ with $0 < |\varpi| < 1$. Let B be a K-algebra of topologically finite type, and assume that B carries a left action over K by a finite group G. Then $A := B^G$ is a K-algebra of topologically finite type and the inclusion $A \to B$ is a finite homomorphism by [BGR, §6.3.3 Proposition 2]. We wish to show that $Y := \mathrm{Spa}(A, A^\circ)$ is the quotient of $X := \mathrm{Spa}(B, B^\circ)$ in $\mathscr{V}$ by verifying the conditions of Lemma 6.23.

Proposition 6.24. *$Y = X/G$ as topological spaces.*

Proof. On topological spaces π clearly factors through X/G. π is closed since it is finite. We know from commutative algebra (the going-up theorem) that $\mathrm{MaxSpec}(B) \twoheadrightarrow \mathrm{MaxSpec}(A)$ as sets, hence π is surjective on classical points and is therefore surjective since it is closed and classical points are dense. Since π is closed and surjective it then suffices to show that the fibers of π are G-orbits, i.e. that $Y = X/G$ as sets.

To show this, we first note that π fits into a G-equivariant commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d] & Y \arrow[d] \\ \mathrm{Spv}(B) \arrow[r, "\pi'"] & \mathrm{Spv}(A) \end{tikzcd}$$

with injective vertical arrows; here, following Huber, $\mathrm{Spv}(R)$ denotes the set of all equivalence classes of valuations on a given ring R. It now suffices to show that the fibers of π' are G-orbits. To see this, choose any valuation $y \in \mathrm{Spv}(A)$, with support $\mathfrak{p} = \mathfrak{p}_y = \mathrm{Ker}\, y \in \mathrm{Spec}(A)$. By [Bou, §§V.2.2 Theorem 2(i)], G acts transitively on the set of supports $\mathfrak{q}_x \in \mathrm{Spec}(B)$ of valuations $x \in \pi'^{-1}(y)$. Therefore, choosing any $\mathfrak{q} = \mathfrak{q}_x$ of this form, we need to show that the stabilizer $G_\mathfrak{q}$ of $\mathfrak{q} \in \mathrm{Spec}(B)$ acts transitively on the set of valuations $x \in \mathrm{Spv}(\mathrm{Frac}(B/\mathfrak{q}))$ extending the given valuation $y \in \mathrm{Spv}(\mathrm{Frac}(A/\mathfrak{p}))$. By [Bou, §§V.2.2 Theorem 2(ii)], $\mathrm{Frac}(B/\mathfrak{q})$ is a finite normal extension of $\mathrm{Frac}(A/\mathfrak{p})$ and $G_\mathfrak{q}$ surjects onto $\mathrm{Aut}(\mathrm{Frac}(B/\mathfrak{q})/\mathrm{Frac}(A/\mathfrak{p}))$, so this follows from [ZS, §VI.7 Corollary 3].

□

Corollary 6.25. *$Y = (X/G, (\pi_*\mathcal{O}_X)^G, (v_{xG}))$ and is therefore the quotient of $\mathrm{Spa}(B, B^\circ)$ by G in $\mathscr{V}$.*

Proof. We know that $Y = X/G$ on topological spaces and it suffices to verify that $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$ since then the equality of the valuations follows directly. There is a natural map $\mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ and hence it suffices to prove that this is an isomorphism on rational subsets $\{|f_1|, \dots, |f_n| \le |h| \neq 0\}$ on Y, i.e. that

$$B\langle f_1/h, \dots, f_n/h\rangle^G = B^G\langle f_1/h, \dots, f_n/h\rangle.$$

To prove this, note that we have an exact sequence

$$0 \to B^G \to B \overset{a\mapsto(ga-a)_{g\in G}}{\longrightarrow} \prod_{g\in G} B_{(g)}$$

of finite B^G-modules. Note also that $B^G\langle \frac{f_1}{h}, \dots, \frac{f_n}{h}\rangle$ is flat over B^G and that

$$B\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle = B\widehat{\otimes}_{B^G} B^G\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle = B \otimes_{B^G} B^G\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle,$$

since B is module-finite over B^G. Then applying $- \otimes_{B^G} B^G\langle \frac{f_1}{h}, \dots, \frac{f_n}{h}\rangle$ to the exact sequence above gives

$$0 \to B^G\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle \to B\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle \overset{a\mapsto(ga-a)_{g\in G}}{\longrightarrow} \prod_{g\in G} B\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle_{(g)}.$$

Since the kernel of the third arrow here is $B\langle \frac{f_1}{h}, \dots, \frac{f_n}{h}\rangle^G$, we get

$$B^G\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle = B\langle \tfrac{f_1}{h}, \dots, \tfrac{f_n}{h}\rangle^G$$

as desired. □

Corollary 6.26. *Let X be a rigid space over K with a right action of a finite group G and assume that there is a cover of X by G-stable affinoids. Then the quotient X/G of X by G exists in $\mathscr{V}$ and is a rigid space. Moreover, the natural map $X \to X/G$ is finite and if X is affinoid then X/G is affinoid.*

Proof. Consider $(X/G, (\pi_*\mathcal{O}_X)^G, (v_{xG}))$ where $\pi : X \to X/G$ is the quotient map on topological spaces. If this is a rigid space then the remaining statements hold by Lemma 6.23 and Corollary 6.25. To see that it is a rigid space let $(U_i)_{i\in I}$ be a cover of G-stable affinoids. By G-stability $\pi^{-1}(\pi(U_i)) = U_i$ for all i and hence $\pi(U_i) = U_i/G$ is open for all i. By Corollary 6.25 $(U_i/G, (\pi_*\mathcal{O}_{U_i})^G, (v_{xG})) = (X/G, (\pi_*\mathcal{O}_X)^G, (v_{xG}))|_{U_i}$ is an affinoid rigid space for all i, and we are done. $\square$

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