Normal Form for Infinite Type Hypersurfaces in $\mathbb{C}^2$ with Nonvanishing Levi Form Derivative

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Abstract. In this paper, we study real hypersurfaces M in $\mathbb{C}^2$ at points $p \in M$ of infinite type. The degeneracy of M at p is assumed to be the least possible, namely such that the Levi form vanishes to first order in the CR transversal direction. A new phenomenon, compared to known normal forms in other cases, is the presence of resonances as roots of a universal polynomial in the 7-jet of the defining function of M. The main result is a complete (formal) normal form at points p with no resonances. Remarkably, our normal form at such infinite type points resembles closely the Chern-Moser normal form at Levi-nondegenerate points. For a fixed hypersurface, its normal forms are parametrized by $S^1 \times \mathbb{R}^*$, and as a corollary we find that the automorphisms in the stability group of M at p without resonances are determined by their 1-jets at p. In the last section, as a contrast, we also give examples of hypersurfaces with arbitrarily high resonances that possess families of distinct automorphisms whose jets agree up to the resonant order.

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1. Introduction

Normal forms are known to serve as important tools to study geometric structures and their equivalence. The seminal paper [CM74] from 1974 by S.-S. Chern and J. Moser constructing a normal form for *Levi nondegenerate hypersurfaces* in complex spaces became one of the most influential in the subject. More recently, numerous authors have constructed normal forms for various classes of real hypersurfaces at points *of finite type*, see [W82], [S91], [E98a], [E98b], [EHZ05], [Kol05], [KMZ14], [KZ14a], [KZ14b]. On the other hand, much less is known for *infinite type hypersurfaces*, where we are only aware of the papers [ELZ09] and [KL14] addressing certain restricted classes of hypersurface that are not generic among infinite type ones.

The present paper is the first one dealing with a natural generic class of infinite type hypersurfaces in $\mathbb{C}^2$, which can be considered as the "most nondegenerate" among all such hypersurfaces. Namely, the class of hypersurfaces whose Levi form vanishes of order 1 at an infinite type point. The new phenomenon compared to previously known cases of normal forms for CR manifolds, is the presence of so-called *resonances*. A resonance here is an integral root of a certain invariant polynomial, called the *characteristic polynomial*, whose coefficients are polynomials in the 7-jet of the defining equation of the hypersurface. If a hypersurface has no resonances, we obtain a normal form unique up to rotations and scaling. If, on the other hand, resonances are present, the same normalization conditions are obtained for all terms except the resonant ones (of which there are always at most finitely many).

A further interesting feature of our normal form is that it closely resembles the Chern-Moser normal form, even though there are no resonances in the Chern-Moser case. For a comparison, recall the Chern-Moser normal form for a smooth Levi nondegenerate hypersurface M through 0 in $\mathbb{C}^2$: There are formal holomorphic coordinates (z, w) near $0 \in \mathbb{C}^2$ such that M is given locally by

$$\mathsf{Im}\, w = \Phi(z, \bar{z}, \mathsf{Re}\, w), \tag{1.1}$$

where the (Hermitian) formal power series expansion $\varphi(z, \bar{z}, u)$ of Φ at the origin is of the form

$$\varphi(z, \bar{z}, u) = |z|^2 + \sum_{a,b \geq 0} N_{ab}(u) z^a \bar{z}^b, \quad \left(N_{ab}(u) = \overline{N_{ba}(u)},\ u \in \mathbb{R}\right), \tag{1.2}$$

satisfying the following normalization conditions

$$N_{a0}(u) = N_{a1}(u) = 0 \quad a \geq 0, \tag{1.3}$$

and

$$N_{22}(u) = N_{32}(u) = N_{33}(u) = 0. \tag{1.4}$$

This normal form is unique modulo the action of the stability group of the sphere (($\mathsf{Im}\, w = |z|^2$ in these coordinates). Our normal form in Theorem 1.1 below is very similar.

The Chern-Moser normal form is *convergent* in the sense that if M is a real-analytic hypersurface, then the transformation to normal form is holomorphic (given by a convergent formal transformation) and the resulting equation in normal form converges to a defining equation for the transformed hypersurface. However, most known normal forms are formal (not known to be convergent, or in some cases even known to not be convergent [Kol12]), with the exception of the very recent [KZ14a], [KZ14b]. The normal form we construct in this paper is formal. We should point out, however, that there are general results concerning convergence of formal invertible mappings between real-analytic CR manifolds (see [BER00], [BMR02]) that apply in the finite type situations treated in the papers mentioned above. As a consequence, questions about biholomorphic mappings (such as, e.g., their existence) between real-analytic CR manifolds of finite type (that are also holomorphically nondegenerate; [BMR02]) can often be reduced to the analogous questions about formal mappings, and for the latter it suffices that the manifolds are in formal normal form. For the class of infinite type hypersurfaces considered in this paper, the corresponding convergence result for formal mappings between real-analytic hypersurfaces is known as well ([JL13]; cf. also the unpublished thesis [J07]).
As mentioned, there is a vast literature on normal forms for real hypersurfaces at points of finite type, but the normal form presented here is (to the best of the authors' knowledge) the first systematic result of this kind at points of infinite type. There is, however, a previous paper by the authors [ELZ09], in which new invariants are introduced for real hypersurfaces in $\mathbb{C}^2$ and a (formal) normal form is constructed for a certain class of hypersurfaces identified by conditions on these invariants. The class so identified contains some hypersurfaces of infinite type, but is in fact completely disjoint from the class considered in this paper. The main objective in [ELZ09] was to provide conditions in terms of the new invariants that would guarantee triviality (or discreteness) of the stability group of the hypersurface. The normal form in that paper was *ad hoc* and its main purpose was a means to prove the result about the stability group. There are also the results in [KL14], in which a dimension bound was proved by means of an "abstract" normal form construction (which however does not produce a normal form at all in the usual sense).

1.1. The main result. We shall now describe our main result more precisely. Let $M \subset \mathbb{C}^2$ be a smooth real hypersurface with $p \in M$. After an affine linear tranformation, we find local holomorphic coordinates (z, w), vanishing at p, such that the real tangent space to M at 0 is spanned by $\mathsf{Re}\,\partial/\partial z, \mathsf{Im}\,\partial/\partial z, \mathsf{Re}\,\partial/\partial w$, and M is given locally as a graph

$$\mathsf{Im}\, w = \varphi(z, \bar z, \mathsf{Re}\, w), \tag{1.5}$$

where $\varphi(0) = 0$ and $d\varphi(0) = 0$. We shall assume the following:

(1) M is of *infinite type* at $p = 0$, i.e., for any m, there is a holomorphic curve $\Gamma_m : \mathbb{C} \to \mathbb{C}^2$ with $\Gamma_m(0) = 0$, which is tangent to M to order m at 0.

(2) There is a smooth curve $\gamma\colon (-\epsilon,\epsilon) \to M$ with $\gamma(0) = 0$ (necessarily transverse to the complex tangent space of M at 0) such that the Levi form of M vanishes to first order at 0 along γ, i.e., $(\mathcal{L} \circ \gamma)'(0) \neq 0$, where $\mathcal{L}$ is any representative of the Levi form of M.

In view of (1) and (2), we can assume

$$\varphi(z,\bar{z},u) = z\bar{z}u + O(|(z,\bar{z},u)|^4). \tag{1.6}$$

We shall then introduce a monic polynomial $P(k, j_0^7\varphi)$ in $k \in \mathbb{C}$ and the 7-jet of φ at 0, see Definition 2.8, and the following *nonresonant condition*:

(3) $P(k, j_0^7\varphi)$ has no integral roots $k \geq 2$ (which we call *resonances*).

The polynomial $P(k) = P(k, j_0^7\varphi)$ turns out to be a CR invariant of M at $p = 0$, and is called the *characteristic polynomial.* We mention here that (3) holds for $j_0^7\varphi$ in a specific open and dense subset Ω of $J_0^7(\mathbb{C}\times\mathbb{R})$, the space of 7-jets at 0 of smooth functions in $\mathbb{C}\times\mathbb{R}$. Indeed, since P is monic in k, the set Ω is locally determined by finitely many polynomial inequalities (for a finite set of possible roots k).

Our main result is a formal normal form for the hypersurface M at $p = 0$. This normal form is unique, as is the formal transformation to normal form, modulo the action of the 2-dimensional group $S^1 \times \mathbb{R}^*$, where S^1 denotes the unit circle and $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. Since our normal form is formal, we shall formulate our result for formal hypersurfaces. For our purposes, a *formal hypersurface* through 0 in the coordinates $(z,w) \in \mathbb{C}^2$ is an object associated to a graph equation of the form (1.5), where $\varphi(z,\bar{z},u)$ is a formal power series in $z,\bar{z},u$ such that $\varphi(0) = 0$. Clearly, a smooth hypersurface M through $p = 0$ as above, defines a formal hypersurface via the Taylor series of the smooth graphing function φ in (1.5); by an abuse of notation, we shall continue to denote the formal hypersurface by M and the formal graphing power series by $\varphi(z,\bar{z},u)$. We note that two distinct smooth hypersurfaces through $p = 0$ may define the same formal hypersurface; this happens if and only if the two hypersurfaces are tangent to infinite order at 0. We also note that a smooth hypersurface M satisfies Conditions (1) and (2) above if and only if its associated formal hypersurface M satisfies the following conditions:

(1') M is of *infinite type* at $p = 0$; i.e., there is a formal holomorphic curve $\Gamma\colon \mathbb{C} \to \mathbb{C}^2$ with $\Gamma(0) = 0$, which is contained in M (formally); see [BER99].

(2') There is a formal curve $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = 0$ (necessarily transverse to the complex tangent space of M at 0) such that the Levi form $\mathcal{L}$ of M satisfies $(\mathcal{L} \circ \gamma)'(0) \neq 0$.

Also, note that Condition (3), being nonresonant, is already a condition on the Taylor series of φ.

A formal (holomorphic) transformation, sending 0 to 0, is a transformation of the form $(z',w') = (F(z,w), G(z,w))$, where F and G are formal power series in z,w such that $F(0) = G(0) = 0$. The formal mapping is *invertible* if its Jacobian determinant at 0 is not zero. If the formal transformation is invertible, then we shall also refer to (z', w') as *formal (holomorphic) coordinates* at 0. We shall say that a formal transformation $(z', w') = (F(z, w), G(z, w))$ sends one formal hypersurface M into another M' if

$$\operatorname{Im} G(z, u + i\varphi) = \varphi'(F(z, u + i\varphi), \overline{F(z, u + i\varphi)}, \operatorname{Re} G(z, u + i\varphi)), \tag{1.7}$$

where $\varphi = \varphi(z, \bar{z}, u)$, and φ, φ' denote the formal graphing power series of M, M', respectively. Our main result is the following:

THEOREM 1.1. *Let M be a formal hypersurface through 0 in $\mathbb{C}^2$, satisfying Conditions (1′), (2′), and (3). Then there are formal holomorphic coordinates (z, w) at 0 such that M is given as a formal graph*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w), \tag{1.8}$$

where the formal (Hermitian) power series $\varphi(z, \bar{z}, u)$ is of the form

$$\varphi(z, \bar{z}, u) = u\left(|z|^2 + \sum_{a,b\geq 0} N_{ab}(u) z^a \bar{z}^b\right), \quad N_{ab}(u) = \overline{N_{ba}(u)}, \; u \in \mathbb{R}, \tag{1.9}$$

satisfying the following normalization conditions

$$N_{a0}(u) = N_{a1}(u) = 0 \quad a \geq 0, \tag{1.10}$$

and

$$\frac{dN_{22}}{du}(u) = \frac{dN_{32}}{du}(u) = \frac{dN_{33}}{du}(u) = 0. \tag{1.11}$$

Moreover, the only invertible formal transformations

$$z' = F(z, w), \quad w' = G(z, w) \tag{1.12}$$

that preserves the normalization (1.10) *and* (1.11) *are of the form*

$$F(z, w) = \alpha z, \; G(z, w) = sw, \quad \alpha \in S^1, \; s \in \mathbb{R}^*. \tag{1.13}$$

More generally, if M only satisfies (1′) *and* (2′)*, we can still obtain* (1.10) *as well as the non-resonant part of* (1.11)*, i.e.*

$$\frac{d^{k-1}N_{22}}{du^{k-1}}(0) = \frac{d^{k-1}N_{32}}{du^{k-1}}(0) = \frac{d^{k-1}N_{33}}{du^{k-1}}(0) = 0$$

for all non-resonant $k \geq 2$.

For a formal hypersurface M through 0 in $\mathbb{C}^2$, we shall denote by $\mathsf{Aut}_0(M)$ the stability group of M at 0, i.e., the group of invertible formal transformations that preserve M at 0. An immediate consequence of Theorem 1.1 is the following:

COROLLARY 1.2. *Let M be a formal hypersurface through 0 in $\mathbb{C}^2$, satisfying Conditions* (1′)*,* (2′)*, and* (3)*. Then, $\mathsf{Aut}_0(M)$ is a subgroup of $S^1 \times \mathbb{R}^*$.*

The realization of $\mathsf{Aut}_0(M)$ as a subgroup of $S^1 \times \mathbb{R}^*$ goes, of course, via the correspondence $(F, G) \mapsto (F_z(0), G_w(0))$ in normal coordinates. There is a vast literature of investigations concerning $\mathsf{Aut}_0(M)$ for CR manifolds, but most treat M only at finite type points. Papers that investigate $\mathsf{Aut}_0(M)$ at infinite type points include [ELZ03], [Kow02], [Kow05], [ELZ09], [JL13], [KoL14], [KoL15]. The results in Corollary 1.2 are more precise (for the class of manifolds considered here) than the results contained in these papers.

This paper is organized as follows. In Section 2 (which is broken into four subsections), the setup and normalization procedure is described and subsequently summarized in Theorem 2.10 (which readily translates into Theorem 1.1). The last subsection 2.4 there is devoted to proving the CR invariance of the characteristic polynomial and the resonances. In Section 3, an invariant description of the resonances is given. In the last section, Section 4, some examples are given.

2. Normalization

2.1. Setup. Let M be a formal hypersurface through 0 in $\mathbb{C}^2$. It is well known (see e.g. [BER99]) that there are formal holomorphic coordinates (z, w) at 0 such that M is given by a graphing equation

$$\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w), \tag{2.1}$$

where $\varphi(z, \chi, u)$ is a formal power series in (z, χ, u), which is Hermitian in (z, χ), i.e.

$$\varphi(\bar{\chi}, \bar{z}, u) = \overline{\varphi(z, \chi, u)} \tag{2.2}$$

and further satisfies the normalization

$$\varphi(z, 0, u) = \varphi(0, \chi, u) \equiv 0. \tag{2.3}$$

We shall assume in this paper that M is of infinite type at 0 (i.e., Condition (1') above), which manifests itself in the defining equation (2.1) by $\varphi(z, \bar{z}, u)$ satisfying

$$\varphi(z, \bar{z}, 0) \equiv 0. \tag{2.4}$$

Equation (2.4) implies that the formal powers series $\varphi(z, \bar{z}, u)$ has the following form

$$\varphi(z, \bar{z}, u) = \sum_{a,b \geq 0} \varphi_{ab} z^a \bar{z}^b u + \sum_{a,b \geq 0, c \geq 2} \varphi_{abc} z^a \bar{z}^b u^c, \tag{2.5}$$

and equation (2.3) implies that

$$\varphi_{0b} = \varphi_{a0} = 0, \quad \varphi_{0bc} = \varphi_{a0c} = 0, \quad \forall a, b \geq 0. \tag{2.6}$$

We shall consider the class of infinite type hypersurfaces that also satisfy Condition (2') above, which here is equivalent to $\varphi_{11} \neq 0$. It is easy to see that a linear transformation in the z-variable will make $\varphi_{11} = 1$.

2.2. Preliminary normalization. We shall normalize the defining equation of M further by making formal transformations of the form

$$z' = z + f(z,w), \quad w' = w + g(z,w), \tag{2.7}$$

where $f(z,w)$ is a power series without constant term or linear term in z, and $g(z,w)$ a power series without constant term or linear term in w. Thus, we have the expansions

$$f(z,w) = \sum_{l,k\geq 0} f_{lk} z^l w^k, \quad f_{00} = f_{10} = 0, \tag{2.8}$$

and

$$g(z,w) = \sum_{l,k\geq 0} g_{lk} z^l w^k, \quad g_{00} = g_{01} = 0. \tag{2.9}$$

We shall assume that M is initially given in the (z',w') coordinates at 0 by

$$\operatorname{Im} w' = \varphi'(z', \bar z', \operatorname{Re} w'), \tag{2.10}$$

with expansion

$$\varphi'(z',\bar z',u') = \sum_{a,b\geq 0} \varphi'_{ab}(z')^a(\bar z')^b u' + \sum_{a,b\geq 0, c\geq 2} \varphi'_{abc}(z')^a(\bar z')^b(u')^c, \tag{2.11}$$

satisfying the prenormalization described above, i.e.,

$$\varphi'_{11} = 1, \quad \varphi'_{0b} = \varphi'_{a0} = 0, \quad \varphi'_{0bc} = \varphi'_{a0c} = 0, \quad \forall a,b\geq 0. \tag{2.12}$$

We shall now describe how the transformation (2.7) affects the coefficients in the defining equation. In the new coordinates (z,w), the hypersurface M is given by the equation (2.1) and we have the basic equation

$$\varphi + \operatorname{Im} g(z, u+i\varphi) = \varphi'\big(z + f(z,u+i\varphi), \bar z + \overline{f(z,u+i\varphi)}, u + \operatorname{Re} g(z,u+i\varphi)\big), \tag{2.13}$$

where $\varphi = \varphi(z,\bar z,u)$. We shall only make transformations (2.7) that preserve the prenormalization (2.12). (Note that $\varphi'_{11} = 1$ is always preserved by the form of the transformation in (2.7).) It is well known (see [BER99]) that the prenormalization (2.3) holds in the coordinates (z,w) if and only if a defining equation $\rho(z,\bar z,w,\bar w) = 0$ of M at 0 satisfies $\rho(z,0,w,w)\equiv 0$; in our context, this means that

$$\frac{1}{2i}(g(z,w) - \bar g(0,w)) = \varphi'\left(z + f(z,w), \bar f(0,w), w + (g(z,w) - \bar g(0,w))/2\right) \tag{2.14}$$

holds identically, where the notation $\bar h(z,w) = \overline{h(\bar z,\bar w)}$ is used. We shall return to this characterization of this prenormalization in Lemma 2.3 below. For now, we just note some immediate conditions on $g(z,w)$ imposed by (2.14). Setting $w = 0$ in this identity, we see that $g(z,0)\equiv 0$, i.e.,

$$g_{l0} = 0, \quad l \geq 0. \tag{2.15}$$

Next, identifying coefficients of the monomial $z^l w$ in (2.14), using $\varphi'_{a0} = 0$ and (2.15), it is not difficult to see that we also have

$$g_{l1} = 0, \quad l \geq 0, \tag{2.16}$$

since every term in the expansion of the right hand side of (2.14) has at least two powers of w.

We also expand $\varphi(z, \bar{z}, u)$ as in (2.5); the prenormalization implies that (2.6) holds, and the form of the transformation (2.7) guarantees that we retain the identity $\varphi_{11} = 1$. We shall now normalize φ further. We use the notation

$$\Delta\varphi_{ab} := \varphi_{ab} - \varphi'_{ab}, \quad \Delta\varphi_{abc} := \varphi'_{abc} - \varphi_{abc}, \tag{2.17}$$

Our first lemma is the following, in which we assume that the prenormalization is preserved.

Lemma 2.1. *For a fixed $l \geq 2$, the following transformation rule holds, modulo a non-constant term polynomial in f_{a0}, with $a < l$, whose coefficients depend only on the coefficients of $\varphi'(z', \bar{z}', u')$:*

$$\Delta\varphi_{l1} = -f_{l0}. \tag{2.18}$$

Proof. If we identify coefficients of $z^l \bar{z} u$ in the expansion of (2.13), then we get φ_{l1} only on the left hand side in view of (2.16). Let us examine the right hand side. We note that $\varphi'(z', \bar{z}', \bar{u}')$ has at least one power of u', and $u + \mathsf{Re}\, g(z, u + i\varphi)$ has at least one power of u and any term involving g has at least two powers of u and cannot contribute to a term $z^l \bar{z} u$. A factor $\bar{z}$ can only come from $\bar{z} + \overline{f(z, u + i\varphi)}$ and $\bar{f}$ will contribute another power of u. Thus, the only terms of the form $z^l \bar{z} u$ on the right will be from $\varphi'_{a1}(z + f(z, 0))^a \bar{z} u$ for $a \leq l$. Since $\varphi'_{11} = 1$, the conclusion of Lemma 2.1 follows. □

It follows that we may perform an additional initial normalization of the defining equation of M and require, in addition to the prenormalization above, that $\varphi_{l1} = 0$ for $l \geq 2$. In what follows, we shall assume that this is part of the prenormalization, i.e., in addition to (2.12), we also assume

$$\varphi'_{l1} = 0, \quad l \geq 2, \tag{2.19}$$

and we shall consider only transformations that preserve this form. It follows from Lemma 2.1 (and the fact that f has no constant term or linear term in z) that we must impose

$$f_{l0} = 0, \quad l \geq 0. \tag{2.20}$$

It is not difficult to see that if we require (2.20), then

$$\Delta\varphi_{ab} = 0 \tag{2.21}$$

for all a, b. For convenience of notation, we shall therefore drop the $'$ on φ'_{ab} and simply write φ_{ab}.

We also have the following lemma:

Lemma 2.2. *For fixed $l \geq 3$, $k \geq 2$, the following transformation rule holds, modulo a non-constant term polynomial in $f_{a,b-1}, \bar{f}_{a,b-1}, g_{ab}, \bar{g}_{0b}, \varphi_{a1b}, \bar{\varphi}_{a1b}$, with $b < k$, whose coefficients depend on the coefficients of $\varphi'(z', \bar{z}', u')$:*

$$\Delta\varphi_{l1k} = \frac{k-1}{2}\, g_{l-1,k} - f_{l,k-1} - 2\,\varphi'_{l2}\bar{f}_{0,k-1}. \tag{2.22}$$

Proof. We identify the coefficients of $z^l \bar{z} u^k$ in the expansion of (2.13). By examining the expansion of $\mathsf{Im}\, g(z, u+i\varphi)$ and using the prenormalization conditions, we observe that on the left hand side we get

$$\begin{aligned} &\varphi_{l1k} + \frac{k}{2}\, g_{l-1,k} + \sum_{\substack{2 \leq c' \leq k-1 \\ 1 \leq a' \leq l-1}} \frac{k+1-c'}{2}\, g_{l-a',k+1-c'}\varphi_{a'1c'} + \\ &\qquad + \sum_{2 \leq c' \leq k-1} (k+1-c')\, \mathsf{Re}\, g_{0,k+1-c'}\varphi_{l1c'}, \end{aligned} \tag{2.23}$$

which is equal to

$$\varphi_{l1k} + \frac{k}{2} g_{l-1,k}$$

modulo a non-constant term polynomial in $g_{ab}, \bar{g}_{0b}, \varphi_{a'1c'}, \bar{\varphi}_{a'1c'}$, with $a, a' < l$, $b, c' < k$. On the right hand side, we examine the term (with the understanding that $\varphi'_{ab1} = \varphi'_{ab} = \varphi_{ab}$)

$$\varphi'_{abc}(z + f(z+i\varphi))^a(\bar{z} + \overline{f(z, u+i\varphi)})^b(u + \mathsf{Re}\, g(z, u+i\varphi))^c \tag{2.24}$$

and first observe that if any term from the expansion of $\varphi(z, \bar{z}, u)$ is involved, then it can only be of the form φ_{a1b}, which contributes one power of $\bar{z}$ and b powers of u. The contribution from $\bar{z} + \overline{f(z, u+i\varphi)}$ can then only be through a factor of the form $\bar{f}_{0b'}$, which will contribute at least one power of u. Since the term $u + \mathsf{Re}\, g(z, u+i\varphi)$ always contributes at least a factor of u as well, we conclude (after some thought) that the term that involves φ will be a non-constant term polynomial in $f_{a,b-1}, \bar{f}_{0,b-1}, g_{ab}, \bar{g}_{0b}, \varphi_{a1b}, \bar{\varphi}_{a1b}$, with $b < k$. If φ is not involved in the term on the right, then we can only get the single power $\bar{z}$ from $\bar{z} + \overline{f(z, u+i\varphi)}$. If $c \geq 2$ in (2.24), then we of course get the term φ'_{l1k} if $c = k$ and we pick the single term that does not involve f, $\bar{f}$, or $\mathsf{Re}\, g$. If the latter are involved, we note that $u + \mathsf{Re}\, g(z, u+i\varphi)$ contributes at least two powers of u and no term from $\mathsf{Re}\, g(z, u+i\varphi)$ can involve a g_{ab} with $b \geq k-1$. We conclude that any contribution from f or $\bar{f}$ must be through a factor of $f_{a,b-1}, \bar{f}_{a,b-1}$ with $b < k$. Thus, beside the term φ'_{l1k}, the terms arising from (2.24), with $c \geq 2$, will be a non-constant term polynomial in $f_{a,b-1}, \bar{f}_{0,b-1}, g_{ab}, \bar{g}_{0b}$, with $b < k$. Finally, if $c = 1$ in (2.24), then we get the contribution

$$f_{l,k-1} + \frac{1}{2} g_{l-1,k}$$

from the term with $a = b = 1$ (recall $\varphi_{11} = 1$ and $\varphi'_{ab} = \varphi_{ab}$ by our prenormalization) and the contribution $2\varphi_{l2}\bar{f}_{0,k-1}$ from the term with $a = l, b = 2$, but the remaining terms will be a nonconstant term polynomial in $f_{a,b-1}, \bar{f}_{a,b-1}, g_{ab}, \bar{g}_{ab}$, with $b < k$. This completes the proof. ☐

We now return to see what the characterization (2.14) of the prenormalization (2.12) yields for the coefficients g_{lk}:

Lemma 2.3. *For each $k \geq 2$ there are*

(i) *a non-constant term polynomial P_k in the variables $f_{0,b-1}$, $\bar{f}_{0,b-1}$, and $\mathsf{Re}\, g_{0b}$, with $b < k$, whose coefficients depend only on the coefficients of $\varphi'(z', \bar{z}', u')$, and*

(ii) *for each $l \geq 1$, a non-constant term polynomial Q_{lk} in $f_{a,b-1}$, $\bar{f}_{0,b-1}$, g_{ab}, and $\bar{g}_{0b}$, with $a \leq l$ and $b < k$, whose coefficients depend only on the coefficients of $\varphi'(z', \bar{z}', u')$,*

with the following property: The transformation (2.7) *preserves the prenormalization* (2.12) *if and only if $\mathsf{Im}\, g_{0k} = 0$ modulo the value of P_k for every $k \geq 2$, and $g_{1k} = \bar{f}_{0,k-1}$, $g_{lk} = 0$ both modulo the value of Q_{lk} for every $l \geq 2$ and $k \geq 2$.*

Proof. To find P_k in (i), we identify coefficients of the monomial w^k in (2.14). On the left hand side, we get $\mathsf{Im}\, g_{0k}$. On the right hand side, we note that $\varphi'(z', \bar{z}', u')$ contributes at least one power each of $z', \bar{z}', u'$. It is clear that any term in the expansion of the right hand side of (2.14) that contributes w^k will have a coefficient that is a product of $f_{0,b-1}$, $\bar{f}_{0,b-1}$, and $\mathsf{Re}\, g_{0b}$, with $b < k$, and a coefficient from the expansion of $\varphi'(z', \bar{z}', u')$. This establishes the existence of P_k in (i).

To find Q_{lk} in (ii), we identify coefficients of the monomial $z^l w^k$ in (2.14). The only contribution on the left hand side is $g_{lk}/2i$. Since we are also requiring the normalization (2.19) and have already established (2.15), (2.16), (2.20), every contribution to the coefficient of $z^l w^k$ is a product of $f_{a,b-1}$, $\bar{f}_{0,b-1}$, g_{ab}, $\bar{g}_{0b}$, with $b < k$, and a coefficient in the expansion of $\varphi'(z', \bar{z}', u')$, except when $l = 1$ in which case there is a coefficient of the form $\bar{f}_{0,k-1}$; for $l \geq 2$, the analogous term $\varphi_{l1} \bar{f}_{0,k-1}$ vanishes by (2.19).

The conclusion in (ii) now follows. □

Remark 2.4. The conditions on the coefficients g_{lk} in Lemma 2.3 above can also be derived by considering the transformation rules for Δ_{l0k} stemming from (2.13). For reasons that will become apparent in the next section, it will be convenient to do so specifically for the coefficients g_{1k}.

Lemma 2.2 suggests an additional prenormalization ($\varphi'_{l1k} = 0$, $l \geq 3$, $k \geq 2$), and this lemma together with Lemma 2.3 leads to an induction scheme that can be summarized in the following proposition:

Proposition 2.5. *In addition to the prenormalization given by* (2.12) *and* (2.19), *the following prenormalization*

$$\varphi'_{l1k} = 0, \quad l \geq 3, \ k \geq 2, \tag{2.25}$$

can be achieved. Any transformation of the form (2.7) *that preserves the prenormalization given by* (2.12), (2.19), *and* (2.25), *satisfies* (2.15), (2.16), (2.20), *and the coefficients*

$$f_{l+1,k-1}, \ \mathsf{Im}\, g_{0k}, \ g_{lk}, \quad k, l \geq 2, \tag{2.26}$$

are given by non-constant term polynomials, whose coefficients depend only on the coefficients of $\varphi'(z', \bar{z}', u')$*, in the variables*

$$f_{0,k-1},\ f_{1,k-1},\ f_{2,k-1},\ \mathsf{Re}\, g_{0k},\ g_{1k}, \quad k \geq 2, \tag{2.27}$$

and their complex conjugates.

Proof. The proof is a straightforward induction on $k \geq 2$ using Lemmas 2.3 and 2.2, together with the normalizations (2.15), (2.16), (2.20). Details are left to the reader. □

2.3. Complete normalization. Our aim now is to find a final (complete) normalization of the defining equation of M at 0 that uniquely determines the variables in (2.27), and has the property that this normalization is preserved only when these variables vanish. Proposition 2.5 will then imply that the only transformation of the form (2.7) that preserves this complete normalization is the identity mapping.

We shall assume that the prenormalizations in the previous subsection are preserved (although as alluded to in that section, we will also study the transformation rules for $\Delta\varphi_{10k}$). Recall that our prenormalization implies that $\varphi'_{ab} = \varphi_{ab}$ and we will drop $'$ on φ'_{ab} so simplify the notation. The main technical result in this paper is contained in the following lemma.

Lemma 2.6. *The transformation rules are given as follows, modulo non-constant term polynomials, whose coefficients are given by the coefficients in the expansion of* $\varphi'(z', \bar{z}', u')$*, in the variables consisting of* $f_{a,b-1}, g_{ab}$ *and their complex conjugates, with* $b < k$*:*

$$\begin{aligned}
\Delta\varphi_{10k} &= \frac{1}{2i} g_{1k} - \overline{f}_{0,k-1}, \qquad \Delta\varphi_{11k} = (k-1)\mathsf{Re}\, g_{0k} - 2\mathsf{Re}\, f_{1,k-1} \\
\Delta\varphi_{21k} &= \left(\frac{k}{2} - \frac{1}{2}\right) g_{1k} + (i(k-1) - 2\varphi_{22})\bar{f}_{0,k-1} - f_{2,k-1} \\
\Delta\varphi_{22k} &= -6\mathsf{Re}\,(\varphi_{23}\bar{f}_{0,k-1}) + (k-1)\varphi_{22}\mathsf{Re}\, g_{0k} + 2(k-1)\mathsf{Im}\, f_{1,k-1} \\
&\quad - 4\varphi_{22}\mathsf{Re}\, f_{1,k-1}, \\
\Delta\varphi_{32k} &= \left(\frac{k-1}{2}\varphi_{22} + \frac{i}{2}\binom{k}{2} - \frac{ik}{2}\right) g_{1k} \\
&\quad + \left(\binom{k-1}{2} + 3i(k-1)\varphi_{22} - 3\varphi_{33}\right)\bar{f}_{0,k-1} - 4\varphi_{42} f_{0,k-1} \\
&\quad + (k-1)\varphi_{32}\mathsf{Re}\, g_{0k} - \varphi_{32}(5\mathsf{Re}\, f_{1,k-1} + i\mathsf{Im}\, f_{1,k-1}) \\
&\quad - (i(k-1) + \varphi_{22}) f_{2,k-1} \\
\Delta\varphi_{33k} &= \mathsf{Re}\,((k-1)\varphi_{23} g_{1k}) + 8\mathsf{Re}\,(((k-1)i\varphi_{23} - \varphi_{34})\bar{f}_{0,k-1}) \\
&\quad + \left((k-1)\varphi_{33} - \binom{k}{3} + \binom{k}{2}\right)\mathsf{Re}\, g_{0k} + ((k-1) - 6\varphi_{33})\,\mathsf{Re}\, f_{1,k-1} \\
&\quad + 6(k-1)\varphi_{22}\mathsf{Im}\, f_{1,k-1} - 4\mathsf{Re}\,(\varphi_{23} f_{2,k-1}),
\end{aligned}$$

where $k \geq 2$ *and we use the convention* $\binom{a}{b} = 0$ *whenever* $a < b$*.*

Proof. We begin by inspecting the terms arising from the expansion of the left-hand side of (2.13):

$$\sum \mathsf{Im}\, g_{lk} z^l (u + i \sum \varphi_{abc})^k. \tag{2.28}$$

The only term with no φ_{abc} that is relevant to the transformation rules in Lemma 2.6 is $\mathsf{Im}\, g_{1k} z u^k$ which contributes to $\Delta\varphi_{10k}$ with coefficient $\frac{1}{2i}$. Furthermore, factors φ_{abc} with $c \geq 2$ cannot contribute since they make the total degree in u greater than k. It remains to consider terms with one or several factors φ_{ab}. In view of the preservation of (2.12) and (2.19), these can only be

$$\varphi_{11} = 1, \quad \varphi_{22}, \quad \varphi_{23}, \quad \varphi_{32}, \quad \varphi_{33}. \tag{2.29}$$

Recall that, by Lemma 2.3, all coefficients of g except $\mathsf{Re}\, g_{0k}$ and g_{1k} are determined by $f_{a,b-1}, g_{ab}$, and their complex conjugates, with $b < k$.

We next inspect terms with g_{1k} that appear as

$$\frac{1}{2i} g_{1k} z (u + i \sum \varphi_{ab} z^a \bar{z}^b u)^k. \tag{2.30}$$

The term with single factor φ_{11} contributes as $\frac{1}{2i} ik g_{1k} z^2 \bar{z} u^k$ to $\Delta\varphi_{21k}$. The term with single factor φ_{22} contributes as $\frac{1}{2i} ik \varphi_{22} g_{1k} z^3 \bar{z}^2 u^k$ to $\Delta\varphi_{32k}$. The term with single factor φ_{23} contributes as $\frac{1}{2i} ik \varphi_{23} g_{1k} z^3 \bar{z}^3 u^k$ to $\Delta\varphi_{33k}$. The term with single factor φ_{32} contributes as its conjugate $\frac{1}{2i} ik \varphi_{32} \bar{g}_{1k} z^3 \bar{z}^3 u^k$ to $\Delta\varphi_{33k}$. The factor φ_{33} has no contribution to the identities in the lemma. Further, the term with the square of φ_{11} contributes as $\frac{1}{2i} i^2 \binom{k}{2} g_{1k} z^3 \bar{z}^2 u^k$ to $\Delta\varphi_{32k}$. Other products $\varphi_{ab}\varphi_{cd}$ have no contribution. Also terms with more than 2 factors φ_{ab} have no contribution.

Next consider terms with $\mathsf{Re}\, g_{0k}$ that appear as

$$\mathsf{Im}\, (u + i \sum \varphi_{ab} z^a \bar{z}^b u)^k \mathsf{Re}\, g_{0k}. \tag{2.31}$$

The term with single factor φ_{11} contributes as $kz\bar{z}u^k \mathsf{Re}\, g_{0k}$ to $\Delta\varphi_{11k}$. The term with single factor φ_{22} contributes as $k\varphi_{22} z^2 \bar{z}^2 u^k \mathsf{Re}\, g_{0k}$ to $\Delta\varphi_{22k}$. The terms with single factors φ_{32} and φ_{23} contribute as $\mathsf{Im}\, (ik\varphi_{32} z^3 \bar{z}^2 + ik\varphi_{23} z^2 \bar{z}^3) u^k \mathsf{Re}\, g_{0k}$ to $\Delta\varphi_{32k}$. The term with single factor φ_{33} contributes as $k\varphi_{33} z^3 \bar{z}^3 u^k \mathsf{Re}\, g_{0k}$ to $\Delta\varphi_{33k}$. Next, there is no contribution from terms with products $\varphi_{ab}\varphi_{cd}$ because of the reality of φ. Finally, the term with the cube of φ_{11} contributes as $\mathsf{Im}\, (i^3 \binom{k}{3} z^3 \bar{z}^3 u^k) \mathsf{Re}\, g_{0k}$ to $\Delta\varphi_{33k}$. Other terms have no contribution.

We now inspect the terms on the right-hand side of (2.13) that contribute with minus. Those containing $f_{l,k-1}$ and g_{lk} arise from the expansion of

$$\begin{aligned} &-\sum \varphi_{ab} (z + \sum f_{l,k-1} z^l (u + i\varphi)^{k-1})^a \\ &\qquad \times (\bar{z} + \sum \bar{f}_{l,k-1} \bar{z}^l (u - i\varphi)^{k-1})^b (u + \mathsf{Re} \sum g_{lk} z^l (u + i\varphi)^k), \end{aligned} \tag{2.32}$$

where the φ_{ab} that occur are technically φ'_{ab} but we recall that $\varphi'_{ab} = \varphi_{ab}$ as a consequence of (2.20).

We first collect the terms with g_{1k} that appear as

$$-\varphi_{ab} z^a \bar{z}^b \frac{1}{2} z(u+i\varphi)^k g_{1k}. \tag{2.33}$$

For $(a,b)=(1,1)$ we obtain $-\frac{1}{2}g_{1k}$ contributing to $\Delta\varphi_{21k}$ and $-\frac{1}{2}ik\varphi_{11}g_{1k} = -\frac{ik}{2}g_{1k}$ contributing to $\Delta\varphi_{32k}$. For $(a,b)=(2,2)$ we obtain $-\varphi_{22}\frac{1}{2}g_{1k}$ contributing to $\Delta\varphi_{32k}$. For $(a,b)=(2,3)$ we obtain $-\varphi_{23}\frac{1}{2}g_{1k}$ contributing to $\Delta\varphi_{33k}$. For $(a,b)=(3,2)$ we obtain its conjugate $-\varphi_{32}\frac{1}{2}\bar{g}_{1k}$ contributing to the same term. Other terms have no contribution.
We next consider the terms with g_{0k} that appear as

$$-\varphi_{ab} z^a \bar{z}^b \mathsf{Re}\,(u+i\varphi)^k \mathsf{Re}\, g_{0k}. \tag{2.34}$$

For $(a,b)=(1,1)$ we obtain $-\mathsf{Re}\, g_{0k}$ contributing to $\Delta\varphi_{11k}$ and $\binom{k}{2}\varphi_{11}^2 g_{0k} = \binom{k}{2} g_{0k}$ contributing to $\Delta\varphi_{33k}$. For $(a,b)=(2,2)$ we obtain $-\varphi_{22}\mathsf{Re}\, g_{0k}$ contributing to $\Delta\varphi_{22k}$. For $(a,b)=(3,2)$ we obtain $-\varphi_{32}\mathsf{Re}\, g_{0k}$ contributing to $\Delta\varphi_{32k}$. For $(a,b)=(3,3)$ we obtain $-\varphi_{33}\mathsf{Re}\, g_{0k}$ contributing to $\Delta\varphi_{33k}$.
As our final consideration we deal with the terms involving $f_{l,k-1}$. We begin with terms involving $f_{0,k-1}$ that arise as

$$-\varphi_{ab}(a z^{a-1}\bar{z}^b f_{0,k-1}(u+i\varphi)^{k-1} + b z^a \bar{z}^{b-1}\bar{f}_{0,k-1}(u-i\varphi)^{k-1})u. \tag{2.35}$$

For $(a,b)=(1,1)$ we obtain $-\bar{f}_{0,k-1}$ contributing to $\Delta\varphi_{10k}$, $-\varphi_{11}\bar{f}_{0,k-1}(-i)(k-1)\varphi_{11} = i(k-1)\bar{f}_{0,k-1}$ contributing to $\Delta\varphi_{21k}$, $-\varphi_{11}\bar{f}_{0,k-1}(-i)^2\binom{k-1}{2}\varphi_{11}^2 = \binom{k-1}{2}\bar{f}_{0,k-1}$ and $-\varphi_{11}\bar{f}_{0,k-1}(-i)(k-1)\varphi_{22} = i(k-1)\varphi_{22}\bar{f}_{0,k-1}$ both contributing to $\Delta\varphi_{32k}$, and $-\varphi_{11}\bar{f}_{0,k-1}(-i)(k-1)\varphi_{23} = i(k-1)\varphi_{23}\bar{f}_{0,k-1}$ and its conjugate $-\varphi_{11}f_{0,k-1}i(k-1)\varphi_{32} = -i(k-1)\varphi_{32}f_{0,k-1}$ both contributing to $\Delta\varphi_{33k}$.
Next, for $(a,b)=(2,2)$ we obtain $-2\varphi_{22}\bar{f}_{0,k-1}$ contributing to $\Delta\varphi_{21k}$, $-2\varphi_{22}(k-1)(-i)\varphi_{11}\bar{f}_{0,k-1} = 2i(k-1)\varphi_{22}\bar{f}_{0,k-1}$ contributing to $\Delta\varphi_{32k}$.
For $(a,b)=(2,3)$ and $(a,b)=(3,2)$ we obtain $-3\varphi_{23}\bar{f}_{0,k-1}$ and its conjugate $-3\varphi_{32}f_{0,k-1}$ contributing to $\Delta\varphi_{22k}$, $-3\varphi_{23}\bar{f}_{0,k-1}(k-1)(-i\varphi_{11}) = 3(k-1)i\varphi_{23}\bar{f}_{0,k-1}$ and its conjugate $-3\varphi_{32}f_{0,k-1}(k-1)(i\varphi_{11}) = -3(k-1)i\varphi_{32}f_{0,k-1}$ contributing to $\Delta\varphi_{33k}$.
For $(a,b)=(3,3)$ we obtain $-3\varphi_{33}\bar{f}_{0,k-1}$ contributing to $\Delta\varphi_{32k}$.
For $(a,b)=(4,2)$ we obtain $-4\varphi_{42}f_{0,k-1}$ also contributing to $\Delta\varphi_{32k}$.
For $(a,b)=(3,4)$ and $(a,b)=(4,3)$ we obtain $-4\varphi_{34}\bar{f}_{0,k-1}$ and its conjugate $-4\varphi_{43}f_{0,k-1}$ both contributing to $\Delta\varphi_{33k}$.
Other terms have no contribution.
We next treat terms involving $f_{1,k-1}$ that arise as

$$-\varphi_{ab}(a f_{1,k-1}(u+i\varphi)^{k-1} + b\bar{f}_{1,k-1}(u-i\varphi)^{k-1})z^a\bar{z}^b u. \tag{2.36}$$

For $(a,b)=(1,1)$ we obtain $-f_{1,k-1}-\bar{f}_{1,k-1} = -2\mathsf{Re}\, f_{1,k-1}$ contributing to $\Delta\varphi_{11k}$,

$$-\varphi_{11}(f_{1,k-1}(k-1)i\varphi_{11} + \bar{f}_{1,k-1}(k-1)(-i\varphi_{11})) = 2(k-1)\mathsf{Im}\, f_{1,k-1}$$

contributing to $\Delta\varphi_{22k}$, and

$$-\varphi_{11}(f_{1,k-1}(k-1)i\varphi_{22}+\bar f_{1,k-1}(k-1)(-i\varphi_{22}))=-2(k-1)\mathsf{Re}\,(i\varphi_{22}f_{1,k-1})$$

and

$$-\varphi_{11}(f_{1,k-1}\binom{k-1}{2}(i\varphi_{11})^2+\bar f_{1,k-1}\binom{k-1}{2}(-i\varphi_{11})^2)=2\binom{k-1}{2}\mathsf{Re}\,f_{1,k-1}$$

both contributing to $\Delta\varphi_{33k}$.
For $(a,b)=(2,2)$ we obtain $-\varphi_{22}(2f_{1,k-1}+2\bar f_{1,k-1})=-4\varphi_{22}\mathsf{Re}\,f_{1,k-1}$ contributing to $\Delta\varphi_{22k}$ and

$$-\varphi_{22}(2f_{1,k-1}(k-1)i\varphi_{11}+2\bar f_{1,k-1}(k-1)(-i\varphi_{11}))=4(k-1)\varphi_{22}\mathsf{Im}\,f_{1,k-1}$$

contributing to $\Delta\varphi_{33k}$.
For $(a,b)=(3,2)$ we obtain $-\varphi_{32}(3f_{1,k-1}+2\bar f_{1,k-1})=-\varphi_{32}(5\mathsf{Re}\,f_{1,k-1}+i\mathsf{Im}\,f_{1,k-1})$ contributing to $\Delta\varphi_{32k}$.
Finally for $(a,b)=(3,3)$ we obtain $-\varphi_{32}(3f_{1,k-1}+3\bar f_{1,k-1})=-6\varphi_{33}\mathsf{Re}\,f_{1,k-1}$ contributing to $\Delta\varphi_{33k}$.
It remains to deal with terms involving $f_{2,k-1}$ that arise as

$$-\varphi_{ab}(af_{2,k-1}z(u+i\varphi)^{k-1}+b\bar f_{2,k-1}\bar z(u-i\varphi)^{k-1})z^a\bar z^b u. \tag{2.37}$$

For $(a,b)=(1,1)$ we obtain $-f_{2,k-1}$ contributing to $\Delta\varphi_{21k}$, and

$$-f_{2,k-1}(k-1)(i\varphi_{11})=-i(k-1)f_{2,k-1}$$

contributing to $\Delta\varphi_{32k}$.
For $(a,b)=(2,2)$ we obtain $-\varphi_{22}f_{2,k-1}$ contributing to $\Delta\varphi_{32k}$.
For $(a,b)=(2,3)$ we obtain $-\varphi_{23}2f_{2,k-1}$ and for $(a,b)=(3,2)$ its conjugate $-\varphi_{32}2\bar f_{2,k-1}$ both contributing to $\Delta\varphi_{33k}$. Other terms have no contribution. □

Extracting real and imaginary parts we obtain from Lemma 2.6 the following identity, modulo a vector of non-constant term polynomials, whose coefficients are given by the coefficients in the expansion of $\varphi'(z',\bar z',u')$, in the variables consisting of $f_{a,b-1},g_{ab}$ and their complex conjugates, with $b<k$:

$$\begin{pmatrix}\mathsf{Re}\,\Delta\varphi_{10k}\\ \mathsf{Im}\,\Delta\varphi_{10k}\\ \Delta\varphi_{11k}\\ \mathsf{Re}\,\Delta\varphi_{21k}\\ \mathsf{Im}\,\Delta\varphi_{21k}\\ \Delta\varphi_{22k}\\ \mathsf{Re}\,\Delta\varphi_{32k}\\ \mathsf{Im}\,\Delta\varphi_{32k}\\ \Delta\varphi_{33k}\end{pmatrix}=A\begin{pmatrix}\mathsf{Re}\,g_{1k}\\ \mathsf{Im}\,g_{1k}\\ \mathsf{Re}\,f_{0,k-1}\\ \mathsf{Im}\,f_{0,k-1}\\ \mathsf{Re}\,g_{0k}\\ \mathsf{Re}\,f_{1,k-1}\\ \mathsf{Im}\,f_{1,k-1}\\ \mathsf{Re}\,f_{2,k-1}\\ \mathsf{Im}\,f_{2,k-1}\end{pmatrix}, \tag{2.38}$$

where the matrix A is explicitly given, but a bit too large to write down here. We have, however, the following lemma:

LEMMA 2.7. *The determinant of A is of the form*

$$\det A = \frac{1}{4}(k-1)\det B, \tag{2.39}$$

where $\det B$ *is a polynomial in k of degree 7, whose leading coefficient is* $2^4/3$. *Moreover, the coefficients of the polynomial* $\det B$ *depend only on* φ_{ab} *with* $a, b \le 4$ *and* $a + b \le 7$.

DEFINITION 2.8. For a formal hypersurface $M \subset \mathbb{C}^2$, given by (2.10) at $0 \in M$ in any coordinate system satisfying the prenormalization described in Proposition 2.5, we define its *characteristic polynomial* $P(k, j_0^7\varphi)$ to be $(3/2^4)\det B$ (so that P is monic in k). We call an integer $k \ge 2$ a *resonance* for M (at 0) if $P(k, j_0^7\varphi) = 0$. Then M is said to be *nonresonant* if there are no resonances.

Proof of Lemma 2.7. By performing elementary row operations on A (left to the diligent reader), we can bring A to the form:
and expanding the determinant, we see that we can write $\det A = \frac{1}{4}(k-1)\det B$, where

$$B = \begin{pmatrix} -6\mathsf{Re}\,\varphi_{32} & 6\mathsf{Im}\,\varphi_{32} & -2\varphi_{22} & 2(k-1) \\ \begin{matrix} 2k^2-4k+3+2\varphi_{22}^2 \\ -3\varphi_{33} - 4\mathsf{Re}\,\varphi_{42} \end{matrix} & 4(k-1)\varphi_{22} + 4\mathsf{Im}\,\varphi_{42} & -3\mathsf{Re}\,\varphi_{32} & \mathsf{Im}\,\varphi_{32} \\ 4(k-1)\varphi_{22} - 4\mathsf{Im}\,\varphi_{42} & \begin{matrix} -2k^2+4k-3-2\varphi_{22}^2 \\ +3\varphi_{33} - 4\mathsf{Re}\,\varphi_{42} \end{matrix} & -3\mathsf{Im}\,\varphi_{32} & -\mathsf{Re}\,\varphi_{32} \\ \begin{matrix} -8\mathsf{Re}\,\varphi_{43} + 8\varphi_{22}\mathsf{Re}\,\varphi_{32} \\ +2(k-1)\mathsf{Im}\,\varphi_{32} \end{matrix} & \begin{matrix} 8\mathsf{Im}\,\varphi_{43} - 8\varphi_{22}\mathsf{Im}\,\varphi_{32} \\ +2(k-1)\mathsf{Re}\,\varphi_{32} \end{matrix} & \begin{matrix} \frac{2k^2-4k+6}{3} \\ -4\varphi_{33} \end{matrix} & \begin{matrix} 6(k-1) \\ \times\varphi_{22} \end{matrix} \end{pmatrix}. \tag{2.40}$$

The statement of the lemma now readily follows. □

Remark 2.9. We note here that it is not necessary to require the full prenormalization given in Proposition 2.5 in order to guarantee that the expression (2.40) for the matrix B above gives rise to the characteristic polynomial. Indeed, it is enough to require that φ' just satisfies (2.19), since in this case, (2.20) implies that (2.21) holds; i.e., we must have $f(z, 0) = z$ for any transformation respecting the prenormalization (2.19), and hence $\varphi'_{ab} = \varphi_{ab}$.

If M is in nonresonant, as described in Definition 2.8, then it follows from (2.38) that we can inductively require the following additional normalization for $k \ge 2$:

$$\varphi_{10k} = \varphi_{11k} = \varphi_{21k} = \varphi_{22k} = \varphi_{32k} = \varphi_{33k} = 0, \tag{2.41}$$

which will completely determine the variables (2.27), i.e.,

$$f_{0,k-1},\ f_{1,k-1},\ f_{2,k-1},\ \mathsf{Re}\,g_{0k},\ g_{1k},$$

in Proposition 2.5. It follows from (2.38), and a straighforward induction on $k \ge 2$ using also Proposition 2.5, that the only transformation preserving the complete normalization described above is the identity mapping. More generally, if M has resonances k, we can still obtain the equations (2.41) for all non-resonant k. We summarize this result in the following theorem.

Theorem 2.10. *Let M be a formal hypersurface through* 0 *in* $\mathbb{C}^2$*, satisfying the assumptions described in Subsection* 2.1*. Assume furthermore that M is in general position at* 0*. Then there are formal holomorphic coordinates* (z, w) *at* 0 *such that M is given as a formal graph*

$$\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, g), \tag{2.42}$$

where the formal (Hermitian) power series $\varphi(z, \bar{z}, u)$ *is of the form*

$$\varphi(z, \bar{z}, u) = \sum_{a,b\geq 0} \varphi_{ab} z^a \bar{z}^b u + \sum_{\substack{a,b\geq 0\\ k\geq 2}} \varphi_{abc} z^a \bar{z}^b u^k \tag{2.43}$$

satisfying the following normalization conditions

$$\varphi_{11} = 1,\ \varphi_{a0} = \varphi_{l1} = \varphi_{a0k} = \varphi_{l+1,1k} = 0, \quad a \geq 0,\, k, l \geq 2 \tag{2.44}$$

and

$$\varphi_{10k} = \varphi_{11k} = \varphi_{21k} = \varphi_{22k} = \varphi_{32k} = \varphi_{33k} = 0, \quad k \geq 2. \tag{2.45}$$

Moreover, the only formal transformation of the form

$$z' = z + f(z, w), \quad w' = w + g(z, w), \tag{2.46}$$

where f and g are formal holomorphic power series with $f(0, 0) = g(0, 0) = f_z(0, 0) = g_w(0, 0) = 0$*, that preserves the normalization* (2.44) *and* (2.45) *is the identity, i.e.,* $f \equiv g \equiv 0$*.*
Furthermore, without assuming M to be in general position, we still obtain its formal normalization given by all equations (2.44) *and those in* (2.45) *for all non-resonant k.*

Remark 2.11. We note that there is some redundancy in the conditions (2.44) and (2.45). The reason we present the conditions in this way here is so that the reader can keep track of which conditions come from the prenormalizations in Subsection 2.2 (those in (2.44)) and which come from the final normalization in Subsection 2.3 (those in (2.45)). In Theorem 1.1, we have eliminated this duplication of conditions, and present the results in a form that closely mimics the Chern-Moser normal form.

To round out the discussion, we note that a general invertible transformation

$$(z', w') = (F(z'', w''), G(z'', w''))$$

preserving the normalization in Theorem 2.10 can be factored as $(z, w) = (\alpha z'', s w'')$ composed with a transformation of the form (2.46); in order to preserve the real tangent space to M at 0, we need to require $s \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, and in order to preserve $\varphi_{11} = 1$, we must require $|\alpha| = 1$. Since the linear transformation $(z, w) = (\alpha z'', s w'')$ preserves the normalization, we conclude that the group $G := S^1 \times \mathbb{R}^*$ acts on the space of normal forms and the isotropy group of M at 0 is a subgroup of G. Moreover, the uniqueness part of Theorem 2.10 implies the following: *Any formal holomorphic transformation that preserves the normal form in Theorem* 2.10 *is of the form* $(z, w) \mapsto (\alpha z, s w)$

with $(\alpha, s) \in S^1 \times \mathbb{R}^*$. Theorem 1.1 now follows easily by writing the defining equation of M in the form

$$\operatorname{\mathsf{Im}} w = \operatorname{\mathsf{Re}} w \left(|z|^2 + \sum_{a,b \geq 0} N_{ab}(\operatorname{\mathsf{Re}} w) z^a \bar{z}^b \right),$$

and translating the conditions in Theorem 2.10 into conditions on $N_{ab}(u)$.

2.4. CR invariance of the characteristic polynomial and resonances. In this subsection, we address the issue of CR invariance of the characteristic polynomial $P(k, j_0^7\varphi)$ introduced in Definition 2.8. Of course, given a preliminary normalization as in Proposition 2.5 (or just the weaker normalization given by (2.19), as noted in Remark 2.9), then $P(k, j_0^7\varphi)$ is uniquely determined, but *a priori* a different preliminary normalization may result in a different characteristic polynomial. It follows from the normalization procedure above that any other preliminary normalization can be obtained from a given one by applying the linear transformations $(z, w) \mapsto (\alpha z, sw)$, where $\alpha \in S^1$ and $s > 0$. Clearly, the coefficients φ_{ab} are unaffected by the transformation $(z, w) \mapsto (z, sw)$, so it remains to investigate how transformations $(z, w) \mapsto (\alpha z, w)$ with $\alpha = e^{it}$ transform $P(k, j_0^7\varphi)$. We claim that $P(k, j_0^7\varphi)$ is invariant under such a transformation, which proves the invariance of the characteristic polynomial and the resonances.

Proposition 2.12. *Let $M \subset \mathbb{C}^2$ be a formal hypersurface, given by* (2.10) *at $0 \in M$ in any coordinate system satisfying the prenormalization described in Proposition* 2.5*, and let $P(k) = P(k, j_0^7\varphi)$ denote its characteristic polynomial defined in Definition* 2.8*. Then, the polynomial $P(k)$ is independent of the preliminary normalization chosen.*

Proof. By definition, $P(k)$ is the monic polynomial $(3/2^4)\det B$, where B is given by (2.40). By the remarks preceding the proposition, it suffices to check that the action $(z, w) \mapsto (e^{it}z, w)$, for $t \in \mathbb{R}$, on the preliminary normalization leaves $P(k)$ unchanged. We observe that this action does not change the coefficients $\varphi_{22}, \varphi_{33}$, and changes the coefficients $\varphi_{32}, \varphi_{43}, \varphi_{42}$ by

$$\varphi_{32} \mapsto e^{it}\varphi_{32}, \quad \varphi_{43} \mapsto e^{it}\varphi_{43}, \quad \varphi_{42} \mapsto e^{2it}\varphi_{42}. \tag{2.47}$$

It will be convenient here to work instead directly with the complex system resulting from Lemma 2.6 rather than the real system in (2.38). We recall that by reality of φ, we have $\varphi_{kl} = \overline{\varphi_{lk}}$. Thus, we shall consider the following system, given by Lemma 2.6:

$$\begin{pmatrix} \Delta\varphi_{10k} \\ \Delta\varphi_{01k} \\ \Delta\varphi_{11k} \\ \Delta\varphi_{21k} \\ \Delta\varphi_{12k} \\ \Delta\varphi_{22k} \\ \Delta\varphi_{32k} \\ \Delta\varphi_{23k} \\ \Delta\varphi_{33k} \end{pmatrix} = \Xi \begin{pmatrix} g_{1k} \\ \bar{g}_{1k} \\ f_{0,k-1} \\ \bar{f}_{0,k-1} \\ \mathsf{Re}\, g_{0k} \\ f_{1,k-1} \\ \bar{f}_{1,k-1} \\ f_{2,k-1} \\ \bar{f}_{2,k-1} \end{pmatrix}, \tag{2.48}$$

where the 9×9 matrix Ξ can be explicitly computed from the right hand side of the equations in Lemma 2.6.

If we now denote by $\Xi(e^{it})$ the matrix obtained by making the substitutions (2.47) (and their complex conjugates) in Ξ, then in view of Lemma 2.7 and the definition of the characteristic polynomial $P(k)$ it suffices to show that the rank of $\Xi(e^{it})$ for fixed k is constant in t. This follows immediately from the observation, whose simple verification is left to the reader, that

$$\Xi(e^{it}) = D_1(e^{it})\, \Xi\, D_2(e^{-it}) \tag{2.49}$$

where $D_1(\lambda)$, $D_2(\lambda)$ are the 9×9 diagonal matrices with

$$\begin{aligned} D_1(\lambda) &:= D(\lambda, \lambda^{-1}, 1, \lambda, \lambda^{-1}, 1, \lambda, \lambda^{-1}, 1), \\ D_2(\lambda) &:= D(\lambda^{-1}, \lambda, \lambda, \lambda^{-1}, 1, 1, 1, \lambda^{-1}, \lambda) \end{aligned} \tag{2.50}$$

and where $D(\lambda_1, \dots, \lambda_j)$ denotes the diagonal $j \times j$ matrix whose diagonal entries are $\lambda_1, \dots, \lambda_j$. We note that $\det D_1(e^{it}) = \det D_2(e^{it}) = 1$, which means that in fact $\det \Xi(e^{it})$ is independent of t. We also note that

$$\Delta\varphi_k \mapsto D_1(e^{it})^{-1} \Delta\varphi_k, \quad H_k \mapsto D_2(e^{it}) H_k$$

are the natural actions of the circle S^1 on the coefficient matrices $\Delta\varphi_k$ and H_k on the left and right in (2.48), respectively, under rotations $(z, w) \mapsto (e^{it} z, w)$. This completes the proof of Proposition 2.12. □

3. Invariant description of resonances

3.1. Formal jet spaces along formal submanifolds. Let S be a formal submanifold through 0 in $\mathbb{R}^m$ of codimension d, i.e. defined by a $\mathbb{R}^d$-valued formal power series map ρ_S with rank d at 0. A *formal function k-jet along S* is an equivalence class of formal functions in $\mathbb{R}^m$, where two functions are *k-equivalent* when they coincide together with their partial derivatives up to order k along S, i.e. modulo the ideal generated by the components of ρ_S. Similarly *k-jets of formal transformations along S* are defined as equivalence classes of invertible formal transformations of $\mathbb{R}^m$ preserving S.

We denote by $J^k_S(\mathbb{R}^m)$ and $J^k_S(\mathbb{R}^m, \mathbb{R}^m)$ the space of all formal function k-jets along S and that of formal transformation jets respectively. The space $J^k_S(\mathbb{R}^m)$ has a *canonical structure of an $\mathbb{R}$-algebra* induced by the algebra structure on formal functions with respect to addition and multiplication. Similarly the space $J^k_S(\mathbb{R}^m, \mathbb{R}^m)$ has a *canonical group structure* with respect to composition. More invariantly, these jet spaces can be defined for formal functions and maps of any formal manifold (instead of $\mathbb{R}^m$) with obvious transformation rule with respect to formal coordinate changes, in particular, also for any smooth manifold at any fixed point.

3.2. Bundle structure on formal jet spaces. We have obvious truncation maps

$$\pi\colon J^k_S(\mathbb{R}^m) \to J^{k-1}_S(\mathbb{R}^m), \quad \pi\colon J^k_S(\mathbb{R}^m, \mathbb{R}^m) \to J^{k-1}_S(\mathbb{R}^m, \mathbb{R}^m), \tag{3.1}$$

and write

$$K^k_S(\mathbb{R}^m) := \pi^{-1}(0) \subset J^k_S(\mathbb{R}^m), \quad K^k_S(\mathbb{R}^m, \mathbb{R}^m) := \pi^{-1}(\mathsf{id}) \subset J^k_S(\mathbb{R}^m, \mathbb{R}^m),$$

for the preimages of the zero and the identity jet respectively. Note that $K^k_S(\mathbb{R}^m)$ is a subalgebra of $J^k_S(\mathbb{R}^m)$, whereas the group operation of $J^k_S(\mathbb{R}^m, \mathbb{R}^m)$ induces a canonical *vector space structure* on $K^k_S(\mathbb{R}^m, \mathbb{R}^m)$.
Furthermore, both maps (3.1) define canonical affine bundle structures on their corresponding jet spaces with $K^k_S(\mathbb{R}^m)$ and $K^k_S(\mathbb{R}^m, \mathbb{R}^m)$ respectively acting transitively and freely on fibers by means of affine transformations.

3.3. Tangential and normal formal jet spaces. We call a k-jet $\Lambda \subset K^k_S(\mathbb{R}^m, \mathbb{R}^m)$ *tangential* if can be represented by a map $H = \mathsf{id}+h$ with $h(\mathbb{R}^m) \subset S$. Equivalently, tangential k-jets can be described as those represented by maps H satisfying the identity

$$D^k H(\underbrace{T\mathbb{R}^m \times \ldots \times T\mathbb{R}^m}_{k \text{ times}}) \subset TS$$

along S, where $D^k H\colon T\mathbb{R}^m \times \ldots \times T\mathbb{R}^m \to T\mathbb{R}^m$ is the total k-th derivative (regarded as a map with formal power series coefficients). Tangential k-jets form a vector subspace $T^k_S(\mathbb{R}^m, \mathbb{R}^m)$ of $K^k_S(\mathbb{R}^m, \mathbb{R}^m)$. We further call its corresponding quotient space

$$N^k_S(\mathbb{R}^m, \mathbb{R}^m) := K^k_S(\mathbb{R}^m, \mathbb{R}^m)/T^k_S(\mathbb{R}^m, \mathbb{R}^m)$$

the *normal k-jet space* and write

$$\nu\colon K^k_S(\mathbb{R}^m, \mathbb{R}^m)/T^k_S(\mathbb{R}^m, \mathbb{R}^m) \to N^k_S(\mathbb{R}^m, \mathbb{R}^m)$$

for the canonical projection. In our notation $H = (f, g)$ for S given by $w = 0$, tangential and normal k-jets correspond to the components f_k and g_k respectively.

3.4. Formal holomorphic jet spaces. Denote by

$$HJ^k_S(\mathbb{C}^2, \mathbb{C}^2) \subset J^k_S(\mathbb{C}^2, \mathbb{C}^2)$$

the submanifold of *holomorphic jets*, i.e. those representable by holomorphic power series. Then clearly

$$HK^k_S(\mathbb{C}^2, \mathbb{C}^2) := HJ^k_S(\mathbb{C}^2, \mathbb{C}^2) \cap K^k_S(\mathbb{C}^2, \mathbb{C}^2) \tag{3.2}$$

becomes a $\mathbb{R}$-vector subspace of $K_S^k(\mathbb{C}^2, \mathbb{C}^2)$, which is canonically a $\mathbb{C}$-vector space.

3.5. Formal jet spaces of real infinite type hypersurfaces. To any formal infinite type real hypersurface M through 0 in $\mathbb{C}^2$, given by a formal equation $\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w)$, we associate the formal k-jet $j_S^k\varphi$ along the formal complex submanifold S in M through 0. Similarly, to every formal holomorphic (or even real-analytic) transformation H of $\mathbb{C}^n$, we associate its formal k-jet $j_S^k H$ along S.

3.6. A universal family of generalized Chern-Moser operators. The transformation of M defined by φ into M' defined by φ' via a map H preserving S, induces a canonical transformation map

$$A^k \colon J_S^k(\mathbb{C}^2, \mathbb{C}^2) \times J_S^k(\mathbb{R}^3) \to J_S^k(\mathbb{R}^3), \quad (j_S^k H, j_S^k \varphi) \xmapsto{A^k} j_S^k \varphi'. \tag{3.3}$$

More specifically, we shall consider k-jets $j_S^k H$ whose $(k-1)$-jet truncations are tangential, i.e. such that

$$j_S^k H \in \pi^{-1}(T_S^{k-1}(\mathbb{C}^2, \mathbb{C}^2)).$$

In our notation from previous sections, with S given by $w = 0$, this corresponds to the k-jets represented by

$$H = \mathsf{id} + \left(\sum_a f_{a,k-1} z^a w^{k-1} + \sum_b f_{b,k} z^b w^k, \, \sum_c g_{c,k} z^c w^k \right).$$

The subspace

$$\pi^{-1}(T_S^{k-1}(\mathbb{C}^2, \mathbb{C}^2)) \subset J_S^k(\mathbb{C}^2, \mathbb{C}^2)$$

inherits the canonical affine bundle structure over the space of tangential $T_S^{k-1}(\mathbb{C}^2, \mathbb{C}^2)$, on whose fibers $K_S^k(\mathbb{C}^2, \mathbb{C}^2)$ acts freely and transitively by means of affine transformations. Then the vector subspace of tangential k-jets $T_S^k(\mathbb{C}^2, \mathbb{C}^2) \subset K_S^k(\mathbb{C}^2, \mathbb{C}^2)$ also acts freely (but of course not transitively) on the same fibers. Consider the quotient bundle

$$Q_S^k(\mathbb{C}^2, \mathbb{C}^2) := \pi^{-1}(T_S^{k-1}(\mathbb{C}^2, \mathbb{C}^2))/T_S^k(\mathbb{C}^2, \mathbb{C}^2)$$

modulo this action. In our notation, jet classes in $Q_S^k(\mathbb{C}^2, \mathbb{C}^2)$ are represented by maps

$$H = \mathsf{id} + \left(\sum_a f_{a,k-1} z^a w^{k-1}, \, \sum_c g_{c,k} z^c w^k \right).$$

Furthermore, it follows from the chain rule applied to (formal) coordinate changes preserving S that the group structure on $J_S^k(\mathbb{C}^2, \mathbb{C}^2)$ induces a canonical $\mathbb{C}$-vector space structure on $Q_S^k(\mathbb{C}^2, \mathbb{C}^2)$.

Now, it follows from our calculations in previous sections that the maps A^k in (3.3) induce canonical maps

$$C^k \colon Q_S^k(\mathbb{C}^2, \mathbb{C}^2) \times J_S^1(\mathbb{R}^3) \to K_S^k(\mathbb{R}^3), \tag{3.4}$$

which are linear in the first argument $Q^k_S(\mathbb{C}^2,\mathbb{C}^2)$. More precisely, the maps C^k are given by

$$C^k(\Lambda,\lambda) = A^k(j^j_S H, j^k_S\varphi) - j^k_S\varphi, \quad \Lambda = [j^k_S H],\ \lambda = j^1_S(\varphi), \tag{3.5}$$

where $[j^k_S H]$ denotes the equivalence class of a k-jet $j^k_S H \in \pi^{-1}(T^{k-1}_S(\mathbb{C}^2,\mathbb{C}^2))$ in $Q^k_S(\mathbb{C}^2,\mathbb{C}^2)$. The maps (3.4) can be regarded as *analogues of the components of the Chern-Moser operator* for Levi-nondegenerate hypersurfaces. Note that while the Chern-Moser operator only depends on the Levi form, (3.4) is the universal family of linear operators parametrized by 1-jets of defining functions φ of the infinite type (formal) real hypersurface $M \subset \mathbb{C}^2$.

3.7. Jet interpretation of the normalization conditions. Our initial prenormalization conditions (2.12) (with $c \le 1$) and (2.19) are readily seen to be equivalent to

$$j^1_S\varphi = j^1_S(z\bar{z}u) \mod (z\bar{z})^2, \tag{3.6}$$

with (2.15), (2.16), (2.20) implying

$$j^0_S H = \mathsf{id}, \quad j^1_S H \in T^1_S(\mathbb{C}^2,\mathbb{C}^2)$$

for any transformation H preserving (3.6). To obtain an invariant description for spaces of partial derivatives in both z and $\bar{z}$ we need to consider *bi-jet spaces*

$$J^{a,b}_0(S\times\overline{S})$$

of (a,b)-equivalence classes of (formal) power series in $(z,\bar{z})$, where z denotes coordinates on a (formal) complex manifold S, and two series are (a,b)-equivalent when their partial derivatives coincide at 0 up to order a in z and b in $\bar{z}$. We shall also use the partial jet spaces $J^{a,\cdot}(S\times\overline{S})$, where only the differentiation order in z is bounded by a. Using this terminology in our normal form, we consider in our normalization process components of φ representing its iterated jets

$$j^{1,\cdot}_0 j^k_S\varphi \in J^{1,\cdot}_0(S\times\overline{S}), \quad j^{3,3}_0 j^k_S\varphi \in J^{3,3}_0(S\times\overline{S}).$$

Invariantly, we are projecting our generalized Chern-Moser operators (3.4) onto the fiber product

$$J^{1,\cdot}_0(S\times\overline{S}) \times_{J^{1,1}_0(S\times\overline{S})} J^{3,3}_0(S\times\overline{S}). \tag{3.7}$$

3.8. Invariant description of the resonances. Denote by Π^k the canonical projection from $J^k_S(\mathbb{R}^3)$ onto the fiber product (3.7). Then it follows from our normal form construction that *k is a resonance if and only if the composition $\Pi^k \circ C^k$ has a nontrivial kernel in the first component $Q^k_S(\mathbb{C}^2,\mathbb{C}^2)$.*
In view of Proposition 2.5 and Lemma 2.6, an equivalent characterization can be obtained as follows: *k is a resonance if and only if the projection of C^k to $J^{3,3}_0(S\times\overline{S})$ is not surjective.*

4. Examples

We conclude this paper by giving a few examples.

Example 4.1. Consider a hypersurface $M \subset \mathbb{C}^2$ of the following form

$$\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w), \quad \varphi(z, \bar{z}, u) = u|z|^2 + u^2\psi(z, \bar{z}, u), \tag{4.1}$$

where $\psi(z, \bar{z}, u)$ is such that $\varphi(z, \bar{z}, u)$ satisfies (2.12) and (2.19), for example,

$$\psi(z, \bar{z}, u) = \theta(|z|^2, u)$$

where $\theta(x, u)$ satisfies $\theta_x(0, u) = 0$. In view of (2.21) all terms involving φ_{ab} in $\det B$ in (2.40) are the same as in the normal form, and hence are 0, and we compute

$$\det B = \frac{2}{3}k(2k + 3)(k - 1)(2k^2 - 3k + 2)^2.$$

Since the roots of $2k^2 - 3k + 2$ are not real, we conclude that M is nonresonant at 0. Therefore, we can put M into normal form as described in Theorem 1.1, i.e., eliminate terms of the form $|z|^4u^k$, $z^3\bar{z}^2u^k$, and $|z|^6u^k$ in $\psi(z, \bar{z}, u)$). The stability group of M is a subgroup of S^1, unless ψ after normalization vanishes completely.

Example 4.2. If M is given by an equation of the form

$$\mathsf{Im}\, w = \mathsf{Re}\, w \left(|z|^2 + \frac{C}{4}|z|^4 + \frac{D}{36}|z|^6 + O(|z|^8) \right) + O((\mathsf{Re}\, w)^2),$$

then the characteristic polynomial (modulo a multiplicative constant making it monic in k) is given by

$$\begin{aligned} &(k-1)\left|24(k-1)^2 + 6iC(k-1) + 3C^2 - D + 12\right|^2 \times \\ &\qquad\qquad \times \left(48(k-1)^2 + 27C^2 - 8D + 96\right). \end{aligned} \tag{4.2}$$

The first two factors do not have any integral roots $k \geq 2$ provided that $C \neq 0$. In this case, there is for any integer $k \geq 2$ an unique D such that the characteristic polynomial has exactly that resonance k. If on the other hand, $C = 0$, the characteristic polynomial (modulo a multiplicative constant) is given by

$$(k-1)\left(D - 24k^2 + 48k - 36\right)^2 \left(D - 6\left(k^2 - 2k + 3\right)\right).$$

The reader can easily check that the last two factors have no real roots if $D < -12$, one root each at $k = 1$ if $D = -12$, and two (distinct) roots each, symmetric about about $k = 1$ when $D > -12$. In particular, for $C = 0$, the hypersurface M has either zero or two resonances.

We note that if M satisfies Conditions (1') and (2') in the introduction, but has a resonance at $k = k_0 \geq 2$, then we cannot in general achieve the normalization (2.45) at $k = k_0$. We can make a choice of the derivatives of f and g in (2.27) at $k = k_0$, and then proceed with the normalization for $k > k_0$ (until the next resonance, if it exists). However, the choice of (2.27) at $k = k_0$ will in general affect the corresponding coefficients (2.45). Therefore, the existence of a resonance $k = k_0$ does not necessarily imply that the derivatives (2.27) at $k = k_0$ of an automorphism of M is not determined by previous ones. There are, however, known examples of $M \subset \mathbb{C}^2$ satisfying Conditions (1') and (2'), whose stability groups at 0 are not determined by 1-jets (see [Kow02], [Z02], [KL14]). Such hypersurfaces cannot be in general position (i.e., must have resonances) at 0 by Corollary 1.2, and the failure of 1-jet determination is caused by the resonances. We mention two such examples here (of the form in Example 4.2), where a resonance $k = k_0$ actually corresponds to indeterminacy of the derivatives (2.27) for $k = k_0$ in automorphisms of M; more examples can be found in the list in [KL14].

Example 4.3. For a positive integer m, consider the following $M_m \subset \mathbb{C}^2$,

$$\mathsf{Im}\, w = i\mathsf{Re}\, w \frac{1 - q_m(2m|z|^2)}{1 + q_m(2m|z|^2)} = \mathsf{Re}\, w \left(|z|^2 + \left(\frac{2m^2}{3} + \frac{1}{3} \right) |z|^6 + \dots \right), \tag{4.3}$$

where

$$q_m(x) = e^{(i/m) \arcsin x}. \tag{4.4}$$

It is readily checked that M_m satisfies (1') and (2') at 0, and comparing with the formula for the characteristic polynomial in (4.2) (with $C = 0$), we see that it is given by

$$(k-1)\left((k-1)^2 - 4m^2\right)\left((k-1)^2 - m^2\right)^2.$$

Its resonances are therefore given by $k = m + 1$ and $k = 2m + 1$. We note (cf. [Z02], [KL14]) that the following local 1-parameter family of biholomorphisms belong to its stability group at 0:

$$H_t(z, w) = \left(\frac{z}{(1 - tw^{2m})^{1/2}}, \frac{w}{(1 - tw^{2m})^{1/2m}} \right), \quad t \in \mathbb{R}. \tag{4.5}$$

We note that the jets $j_0^{2m} H_t$ agree for all $t \in \mathbb{R}$, but the derivatives (2.27) for $k = 2m + 1$ depend on the parameter t.

Example 4.4. The following example (corresponding to $C \neq 0$ in Example 4.2) illustrates that even a single resonance can be responsible for the presence of automorphisms not determined by their 1-jets. We let $q_T(|z|^2)$ be the unique solution to

$$u q_T'(u) = \frac{\tan(q_T(u))}{1 + T\tan(q_T(u))}, \quad q_T(0) = 0, \ q_T'(0) = 1,$$

where $T \in \mathbb{R} \setminus \{0\}$. Then, for any $m \in \mathbb{N}$, the hypersurface $M_{m,T}$ defined by

$$\begin{aligned} \mathsf{Im}\, w = & \qquad \mathsf{Re}\, w \tan\left(\tfrac{q_T(m|z|^2)}{m} \right) \\ = & \quad \mathsf{Re}\, w \left(|z|^2 - mT|z|^4 + \left(\tfrac{2 + m^2(9T^2+1)}{6} \right) |z|^6 + \dots \right) \end{aligned}$$

also satisfies (1') and (2') at 0. It has an infinitesimal CR automorphism given by

$$X = \frac{1}{m} \left(\frac{1}{2} + iT \right) z w^m \frac{\partial}{\partial z} + w^{m+1} \frac{\partial}{\partial w}, \tag{4.6}$$

as can be seen from a computation carried out in [KL14, Lemma 10 and 4]. The resonances of its characteristic polynomial, by the observation that the first two factors of the characteristic polynomial in (4.2) do not have any if $T \neq 0$, are the integral roots $k \geq 2$ of the polynomial

$$k^2 - 2k + 1 - m^2;$$

that is, only $k = m + 1$ occurs. The infinitesimal CR automorphism (4.6) illustrates the failure of the conclusion in Corollary 1.2 in this case.

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