Modular Equalities for Complex Reflection Arrangements

Daniela Anca Măcinic, Ştefan Papadima [1], and Clement Radu Popescu [2]

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Abstract. We compute the combinatorial Aomoto–Betti numbers $\beta_p(\mathcal{A})$ of a complex reflection arrangement. When $\mathcal{A}$ has rank at least 3, we find that $\beta_p(\mathcal{A}) \leqslant 2$, for all primes p. Moreover, $\beta_p(\mathcal{A}) = 0$ if $p > 3$, and $\beta_2(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ is the Hesse arrangement. We deduce that the multiplicity $e_d(\mathcal{A})$ of an order d eigenvalue of the monodromy action on the first rational homology of the Milnor fiber is equal to the corresponding Aomoto–Betti number, when d is prime. We give a uniform combinatorial characterization of the property $e_d(\mathcal{A}) \neq 0$, for $2 \leqslant d \leqslant 4$. We completely describe the monodromy action for full monomial arrangements of rank 3 and 4. We relate $e_d(\mathcal{A})$ and $\beta_p(\mathcal{A})$ to multinets, on an arbitrary arrangement.

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1. Introduction and statement of results

1.1. Milnor fibration and monodromy. The complement M of a degree n complex hypersurface in $\mathbb{C}^l$, $\{f = 0\}$, and the associated Milnor fibration, $f : M \to \mathbb{C}^\times$, first analysed by Milnor in his seminal book [15], attracted a lot of attention over the years. Multiplication by $\exp(\frac{2\pi\sqrt{-1}}{n})$ induces the geometric monodromy action on the associated *Milnor fiber* $F = f^{-1}(1)$, $h : F \to F$, and the *algebraic monodromy action*, $\{h_i : H_i(F, \mathbb{Q}) \to H_i(F, \mathbb{Q})\}$.

Computing h_i is a major problem in the field, when f has a non-isolated singularity at 0. Even for the defining polynomial of a (central) complex hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^l$ and $i = 1$, the answer is far from being clear. This case was tackled in the recent literature by many authors, who used a variety of tools; see for instance [20] for a brief survey. In this paper, we focus on *reflection arrangements*, associated to finite complex reflection groups.

It is well-known that every arrangement complement $M_{\mathcal{A}}$ has the homotopy type of a connected, finite CW-complex with torsion-free homology, whose first integral homology group, $H_1(M_{\mathcal{A}}, \mathbb{Z}) = \mathbb{Z}^n$, comes endowed with a natural basis, given by meridian loops around the hyperplanes.

It is also well-known that, for an arbitrary arrangement $\mathcal{A}$, h_1 induces a $\mathbb{Q}[\mathbb{Z}]$-module decomposition,

$$H_1(F_{\mathcal{A}}, \mathbb{Q}) = \bigoplus_d (\mathbb{Q}[t]/\Phi_d(t))^{e_d(\mathcal{A})}, \tag{1}$$

where Φ_d is the d-th cyclotomic polynomial, $e_d(\mathcal{A}) = 0$ if $d \nmid n$, and $e_1(\mathcal{A}) = n - 1$. See for instance [14, (1.1)].

A pleasant feature of hyperplane arrangements is the rich combinatorial structure encoded by the associated intersection lattice, $\mathcal{L}_\bullet(\mathcal{A})$, whose elements are the intersections of hyperplanes from $\mathcal{A}$, ranked by codimension and ordered by inclusion. In this context, the open monodromy action problem takes the following more precise form: are the multiplicities $e_d(\mathcal{A})$ combinatorial? If so, give a formula involving only $\mathcal{L}_\bullet(\mathcal{A})$.

1.2. Characteristic and resonance varieties. Our approach to decomposition (1) is topological, based on two types of jump loci, associated to CW-complexes having the properties recalled for $M_{\mathcal{A}}$, and the interplay between them.

The complex characteristic variety $\mathcal{V}^i_q(M)$, sitting inside the character torus $\mathbb{T}(M) := \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{C}^\times) = (\mathbb{C}^\times)^n$ is the locus of those $\rho \in \mathbb{T}(M)$ for which $\dim_{\mathbb{C}} H_i(M, \mathbb{C}_\rho) \geqslant q$, where $\mathbb{C}_\rho$ denotes the associated rank 1 local system on M.

Note that $\mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{C}) = H^1(M, \mathbb{C}) = \mathbb{C}^n$, and denote by $\exp : H^1(M, \mathbb{C}) \twoheadrightarrow \mathbb{T}(M)$ the natural exponential map. For an integer $d \geqslant 1$, let $\rho_d \in \mathbb{T}(M)$ be the exponential of the diagonal cohomology class equal to $\frac{2\pi\sqrt{-1}}{d}$, with respect to the distinguished $\mathbb{Z}$-basis. When $M = M_{\mathcal{A}}$ is an arbitrary arrangement complement and $d > 1$, it is well-known that

$$e_d(\mathcal{A}) = \dim_{\mathbb{C}} H_1(M_{\mathcal{A}}, \mathbb{C}_{\rho_d}). \tag{2}$$

See for instance [14, §2.3], and also [3, Theorem 2.5 and Corollary 6.4] for more general results of this type.
The resonance variety $\mathscr{R}^i_q(M,\Bbbk)$ over a field $\Bbbk$, sitting inside $H^1(M,\Bbbk)$, is the locus of those $\sigma \in H^1(M,\Bbbk)$ for which $\dim_{\Bbbk} H^i(H^\bullet(M,\Bbbk),\sigma*) \geqslant q$, where $\sigma*$ denotes the left multiplication by σ in the cohomology ring. When $\Bbbk$ is the prime field $\mathbb{F}_p$, denote by $\sigma_p \in H^1(M,\mathbb{F}_p)$ the diagonal cohomology class equal to 1, with respect to the distinguished $\mathbb{Z}$-basis, and define the modulo p *Aomoto-Betti number* by

$$\beta_p(M) := \dim_{\mathbb{F}_p} H^1(H^\bullet(M,\mathbb{F}_p),\sigma_p*) \tag{3}$$

When $M = M_{\mathcal{A}}$ is an arrangement complement, we will replace M by $\mathcal{A}$ in the notation.
By a celebrated theorem of Orlik and Solomon [16], the cohomology ring of $M_{\mathcal{A}}$ is combinatorial. More precisely, $\beta_p(\mathcal{A})$ may be computed from $\mathcal{L}_{\leqslant 2}(\mathcal{A})$, as well as $\mathscr{R}^1_q(\mathcal{A},\Bbbk)$, for all q and $\Bbbk$.

1.3. MODULAR BOUNDS. It follows from Theorem 11.3 in [19] that

$$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A}), \text{for all } s \geqslant 1\,, \tag{4}$$

when $\mathcal{A}$ is an arbitrary arrangement.
Actually, the above modular bound holds for all CW-complexes considered in §1.2, with the multiplicity replaced by the value from equality (2), and is in general strict, in the broader context. Our first main result says that the modular bound (4) becomes an equality, for reflection arrangements and $s = 1$.

THEOREM 1.1. *Let $\mathcal{A}$ be a complex reflection arrangement. Then $e_p(\mathcal{A}) = \beta_p(\mathcal{A})$, for all primes p. In particular, $e_p(\mathcal{A})$ is determined by $\mathcal{L}_{\leqslant 2}(\mathcal{A})$.*

1.4. AOMOTO-BETTI NUMBERS FOR REFLECTION ARRANGEMENTS. Reflection arrangements have a distinguished history, going back as far as Jordan's work from 1878 on the symmetry group of the famous Hessian configuration. A related open problem is whether the *Hessian arrangement* is the only arrangement supporting a 4-net. In Theorem 1.2(2) below, we solve this problem for reflection arrangements.
Finite complex reflection groups have been classified by Shephard and Todd [22] (see also [2], [17]). Each such group G gives rise to the complex reflection arrangement $\mathcal{A}(G)$, consisting of the fixed points of all reflections in G. Among them, we have the monomial arrangements $\mathcal{A}(m,m,l)$ in $\mathbb{C}^l$ $(l \geqslant 2)$, with defining polynomials $\Pi_{1\leqslant i<j\leqslant l}(z_i^m - z_j^m)$ $(m \geqslant 1)$, and full monomial arrangements $\mathcal{A}(m,1,l)$ in $\mathbb{C}^l$ $(l \geqslant 2)$, defined by $z_1 \ldots z_l \cdot \Pi_{1\leqslant i<j\leqslant l}(z_i^m - z_j^m)$ $(m \geqslant 2)$. We may now state our second main result.

THEOREM 1.2. *For a complex reflection arrangement $\mathcal{A}$ of rank at least 3, the following hold.*

(1) *If $p > 3$, then $\beta_p(\mathcal{A}) = 0$.*
(2) *$\beta_2(\mathcal{A}) \neq 0 \Leftrightarrow \beta_2(\mathcal{A}) = 2 \Leftrightarrow \mathcal{A}$ supports a 4-net $\Leftrightarrow \mathcal{A}$ is the Hessian arrangement.*

(3) *The only cases when* $\beta_3(\mathcal{A}) \neq 0$ *are:* $\mathcal{A}(m,1,3)$ *with* $m \equiv 1 \pmod 3$, *where* $\beta_3 = 1$; $\mathcal{A}(m,m,3)$ *with* $m \geqslant 2$, *where* $\beta_3 = 1$ *if* $m \not\equiv 0 \pmod 3$ *and otherwise* $\beta_3 = 2$; $\mathcal{A}(m,m,4)$, *where* $\beta_3 = 1$.
(4) *In particular,* $\beta_p(\mathcal{A}) \leqslant 2$, *for all primes* p.

We imposed the rank condition since the Aomoto-Betti numbers for arrangements of rank at most 2 are known (see for instance [14]). Moreover, when $\mathcal{A} = \mathcal{A}(m,m,2)$ with $m \equiv 0 \pmod p$ it is easy to see that $\beta_p(\mathcal{A}) = m - 2$. Thus, the conclusion of Theorem 1.2(4) no longer holds, for $m > 4$.
The resonance varieties $\mathscr{R}^1_q(\mathcal{A},\mathbb{C})$ are quite well-understood, due to work by Falk, Libgober, Marco-Buzunáriz and Yuzvinsky; see [10], [13]. There are results in positive characteristic which show a different qualitative behaviour of resonance in this case; see for instance Falk [9]. The complete picture over $\mathbb{F}_p$ largely remains a mystery. Our Theorems 1.2 and 1.1, together with recent vanishing results due to Dimca and Sticlaru ([6], [7]), verify the strong modular conjecture from [20], for the important class of complex reflection arrangements.

1.5. A COMBINATORIAL NON-TRIVIALITY TEST. Dimca, Ibadula and Măcinic asked in [4] the following natural question: if $d > 1$ and $e_d(\mathcal{A}) \neq 0$, does this imply that $\rho_d \in \exp\, \mathscr{R}^1_1(\mathcal{A},\mathbb{C})$? A positive answer (for all d) would imply that the non-triviality of h_1 is combinatorial, since the converse implication is known, for all values of d.

THEOREM 1.3. *If* $\mathcal{A}$ *is a complex reflection arrangement, then the above question has a positive answer, for* $2 \leqslant d \leqslant 4$.

We derive Theorem 1.3 and Theorem 1.1 from Theorem 1.2 with the aid of a general result (proved in Theorem 4.3) that relates combinatorial structures on arrangements satisfying the key multinet axiom introduced by Falk-Yuzvinsky in [10] to the algebraic monodromy action and the Aomoto-Betti numbers of an arrangement. The tools from our paper also enable us to give a complete, combinatorial description in Proposition 4.4 for the monodromy action on $H_1(F_{\mathcal{A}},\mathbb{Q})$, in the case of full monomial arrangements of rank 3 and 4. Related results may be found in [12] and [20]. By Theorems 1.1, 1.2 and [6, 7], this complete, combinatorial description holds for arbitrary complex reflection arrangements.

2. NON-EXCEPTIONAL REFLECTION ARRANGEMENTS

We first compute the Aomoto-Betti numbers of monomial and full monomial arrangements.

2.1. THE CLASSIFICATION. (cf. [22], [2], [17])
A finite reflection group G decomposes as a product of irreducible factors of the same kind. At the level of arrangements, $\mathcal{A}(G \times G')$ is the product $\mathcal{A}(G) \times \mathcal{A}(G')$, and the corresponding complements satisfy $M(G \times G') = M(G) \times M(G')$. (Whenever convenient, we will abbreviate notation and replace $\mathcal{A}(G)$ by G, when speaking about associated objects.)

The irreducible reflection arrangements of rank at least 3 comprise the monomial and full monomial arrangements, $\mathcal{A}(m,m,l)$ $(m \geqslant 2, l \geqslant 3$ or $m = 1, l \geqslant 4)$ and $\mathcal{A}(m,1,l)$ $(l \geqslant 3)$, plus the exceptional arrangements $\mathcal{A}(G_{23}) - \mathcal{A}(G_{37})$. The Hessian arrangement is $\mathcal{A}(G_{25})$. See for instance [17] for the notations.

2.2. VANISHING CRITERIA. Given an arbitrary arrangement $\mathcal{A}$ and an r–flat $X \in \mathcal{L}_r(\mathcal{A})$, set $\mathcal{A}_X := \{H \in \mathcal{A} | H \supseteq X\}$, and define the multiplicity of X to be equal to $|\mathcal{A}_X|$.
Using the distinguished $\mathbb{Z}$-basis, we identify an element $\eta \in H^1(M_{\mathcal{A}}, \mathbb{F}_p)$ with the family $\{\eta_H \in \mathbb{F}_p\}_{H \in \mathcal{A}}$. We denote by $Z_p(\mathcal{A}) \subseteq H^1(M_{\mathcal{A}}, \mathbb{F}_p)$ the kernel of $\sigma_p *$.
Definition (3) implies that $\beta_p(\mathcal{A}) = \dim_{\mathbb{F}_p} Z_p(\mathcal{A}) - 1$. Our computations are based on the following well-known result (see e.g. [20]).

LEMMA 2.1. *An element η belongs to $Z_p(\mathcal{A})$ if and only if, for any $X \in \mathcal{L}_2(\mathcal{A})$,*

$$\begin{cases} \sum_{H \in \mathcal{A}_X} \eta_H = 0 & \text{if} \quad |\mathcal{A}_X| \equiv 0 \pmod p \\ \eta_H = \eta_{H'}, \forall\, H, H' \in \mathcal{A}_X & \text{if} \quad |\mathcal{A}_X| \not\equiv 0 \pmod p \end{cases}$$

Clearly, $\beta_p(\mathcal{A}) = 0$ if and only if $\eta \in Z_p(\mathcal{A})$ implies that η is constant. A first useful vanishing criterion is due to Yuzvinsky.

LEMMA 2.2. *([25]) If $|\mathcal{A}| \not\equiv 0 \pmod p$, then $\beta_p(\mathcal{A}) = 0$.*

A second convenient situation is the following.

LEMMA 2.3. *Assume that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.*

(1) *$H_1(M_{\mathcal{A}}, \mathbb{C}_{\rho_d}) = 0$, for all $d > 1$.*
(2) *$\beta_p(\mathcal{A}) = 0$, for all primes p.*

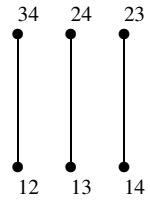
Proof. Assuming the contrary, we infer from [18, Proposition 13.1] that $\rho_d \in \mathscr{V}_1^1(M_{\mathcal{A}_1} \times M_{\mathcal{A}_2}) = \{1\} \times \mathscr{V}_1^1(M_{\mathcal{A}_2}) \cup \mathscr{V}_1^1(M_{\mathcal{A}_1}) \times \{1\}$, respectively $\sigma_p \in \mathscr{R}_1^1(M_{\mathcal{A}_1} \times M_{\mathcal{A}_2}, \mathbb{F}_p) = \{0\} \times \mathscr{R}_1^1(M_{\mathcal{A}_2}, \mathbb{F}_p) \cup \mathscr{R}_1^1(M_{\mathcal{A}_1}, \mathbb{F}_p) \times \{0\}$, in contradiction with the fact that all coordinates of ρ_d (respectively σ_p) are different from 1 (respectively 0). □

By Lemma 2.3(2), we only need to compute $\beta_p(G)$ for an irreducible complex reflection group G.
To state the third vanishing criterion, we need to introduce certain simple graphs, associated to an arrangement $\mathcal{A}$ and an integer $k \geqslant 2$, with vertex set $\mathcal{A}$. The edges of $\Gamma_k(\mathcal{A})$ are defined by the condition $|\mathcal{A}_{H \cap H'}| \not\equiv 0 \pmod k$, for $k > 2$, and by $|\mathcal{A}_{H \cap H'}|$ is either odd or equal to 2, for $k = 2$. The defining property for $\Gamma_{(k)}(\mathcal{A})$ is $|\mathcal{A}_{H \cap H'}| = k$. Note that $\Gamma_{(2)}(\mathcal{A})$ is a subgraph of $\Gamma_p(\mathcal{A})$, for all primes p. The equivalence relation on $\mathcal{A}$ associated to the edge paths of $\Gamma_k(\mathcal{A})$, respectively $\Gamma_{(k)}(\mathcal{A})$, will be denoted by $\sim_k$, respectively $\sim_{(k)}$.

EXAMPLE 2.4. Let $\mathcal{A} = \mathcal{A}(1,1,l)$ be the braid arrangement in $\mathbb{C}^l$, with hyperplanes labeled by the two-element subsets of $\{1, \ldots, l\}$. For $ij \neq st$, the multiplicity of the 2–flat determined by the hyperplanes $z_i = z_j$ and $z_s = z_t$ is 2 if $|\{i,j,s,t\}| = 4$ and 3 if $|\{i,j,s,t\}| = 3$. Let p be a prime. It follows from the above definition that $\Gamma_p(\mathcal{A})$ is a complete graph (with full edge set) if $p \neq 3$. For $p = 3$, ij and st are connected by an edge of $\Gamma_3(\mathcal{A})$ if and only if $|\{i,j,s,t\}| = 4$. We infer that the graph $\Gamma_3(\mathcal{A})$ is discrete

(with no edges) when $l \leqslant 3$, has 3 connected components when $l = 4$ (see the picture of the corresponding graph below), and is connected when $l \geqslant 5$.

$\Gamma_3(\mathcal{A}(1,1,4))$:



We obtain the following immediate consequences of Lemma 2.1.

LEMMA 2.5. *Each of the properties below implies that* $\beta_p(\mathcal{A}) = 0$.

(1) *The graph* $\Gamma_p(\mathcal{A})$ *is connected.*
(2) *The graph* $\Gamma_{(2)}(\mathcal{A})$ *is connected.*
(3) *For all* $X \in \mathcal{L}_2(\mathcal{A})$, $p \nmid |\mathcal{A}_X|$.

2.3. INTERSECTION LATTICES. The Aomoto-Betti numbers for $\mathcal{A}(1,1,l)$ were computed in [14]. They verify all statements from Theorems 1.2 and 1.1. Hence, we may suppose from now on that $m \geqslant 2$.
We need to describe $\mathcal{L}_{\leqslant 2}(\mathcal{A})$. It will be convenient to label the various hyperplanes as follows. Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{m})$, $(H_i)\ z_i = 0$ for all $1 \leqslant i \leqslant l$, and $(H_{ij^\alpha})\ z_i - \omega^\alpha z_j = 0$, for $1 \leqslant i < j \leqslant l$ and $\alpha \in \mathbb{Z}/m\mathbb{Z}$. We go on by listing the 2-flats (identified with the corresponding subarrangements $\mathcal{A}_X$).

CASE I: $\mathcal{A}(m,1,l), l \geqslant 4$:
$\mathrm{I_a}$: $\{H_i, H_j, H_{ij^\alpha}(\alpha \in \mathbb{Z}/m\mathbb{Z})\}$, with multiplicity $m+2$;
$\mathrm{I_b}$: $\{H_{ij^\alpha}, H_{jk^\beta}, H_{ik^{\alpha+\beta}}\}$, with multiplicity 3;
$\mathrm{I_c}$: $\{H_{ij^\alpha}, H_{kh^\beta}\}$, with multiplicity 2;
$\mathrm{I_d}$: $\{H_i, H_{jk^\alpha}\}$, with multiplicity 2.
CASE II: $\mathcal{A}(m,m,l), l \geqslant 4$: types $\mathrm{I_b}$ and $\mathrm{I_c}$, plus $\{H_{ij^\alpha}(\alpha \in \mathbb{Z}/m\mathbb{Z})\}$, with multiplicity m.
CASE III: $\mathcal{A}(m,1,3)$: types $\mathrm{I_a}$, $\mathrm{I_b}$ and $\mathrm{I_d}$.
CASE IV: $\mathcal{A}(m,m,3)$: types $\mathrm{I_b}$ and II.

2.4. β_p-VANISHING. We will use Lemma 2.5 to treat the cases when $\beta_p(\mathcal{A}) = 0$ in the non-exceptional families. To simplify things, we suppress H from notation and identify the hyperplanes with their labels, i and ij^α.
We claim that, for $\mathcal{A} = \mathcal{A}(m,1,l)$ with $l \geqslant 4, \Gamma_{(2)}(\mathcal{A})$ is connected. Indeed, given $i < j$ we may find $h < k$ with i, j, h, k distinct. Hence, $i \sim_{(2)} hk^0 \sim_{(2)} j$ and $ij^\alpha \sim_{(2)} k$, which proves connectivity. Similar arguments lead to the following conclusions. If $\mathcal{A} = \mathcal{A}(m,1,l)$ with $l = 3$ and $p \neq 3$, then $\Gamma_p(\mathcal{A})$ is connected; for $p = 3$ and $m \not\equiv 1 \pmod 3, \Gamma_3(\mathcal{A})$ is connected. The remaining full monomial Aomoto-Betti numbers, $\beta_3(m,1,3)$ with $m \equiv 1 \pmod 3$, will be computed later on.
If $\mathcal{A} = \mathcal{A}(m,m,l)$ with $l \geqslant 5$, then $\Gamma_{(2)}(\mathcal{A})$ is connected. For $l = 3,4$ and $p \neq 3, \Gamma_p(\mathcal{A})$ is connected. So, for monomial arrangements, only β_3 in ranks 3 and 4 remains to be calculated.

2.5. THE REMAINING NON-EXCEPTIONAL CASES. A mod 3 cocycle $\eta \in Z_3(\mathcal{A})$ is a family of elements of $\mathbb{F}_3$, η_k and η_{ij^α}, satisfying the equations from Lemma 2.1, for any $X \in \mathcal{L}_2(\mathcal{A})$.
CASE $\mathcal{A} = \mathcal{A}(m,1,3)$ with $m \equiv 1 (\mathrm{mod}\ 3)$.
The equations coming from 2-flats of type $\mathrm{I_d}$ say that $\eta_{jk^\alpha} = \eta_i$, where i is the third element of $\{1,2,3\}$. The equations of type $\mathrm{I_b}$ become equivalent to $\eta_1 + \eta_2 + \eta_3 = 0$, while type $\mathrm{I_a}$ equations say that $\eta_i + \eta_j + m\eta_k = 0$, for all $i \neq j \neq k$. We infer that $\beta_3(m,1,3) = 1$, as asserted in Theorem 1.2.
CASE $\mathcal{A} = \mathcal{A}(m,m,4)$.
The equations of type $\mathrm{I_c}$ say that $\eta_{ij^\alpha} = \eta_{ij}$, and $\eta_{34} = \eta_{12}, \eta_{24} = \eta_{13}, \eta_{23} = \eta_{14}$. Type $\mathrm{I_b}$ equations reduce then to $\eta_{12} + \eta_{13} + \eta_{14} = 0$, while type II conditions follow from $\eta_{ij^\alpha} = \eta_{ij}$. Again, $\beta_3(m,m,4) = 1$, as claimed.
CASE $\mathcal{A} = \mathcal{A}(m,m,3)$ with $m \not\equiv 0 (\mathrm{mod}\ 3)$.
The equations of type II say that $\eta_{ij^\alpha} = \eta_{ij}$, and the type $\mathrm{I_b}$ conditions then reduce to $\eta_{12} + \eta_{23} + \eta_{13} = 0$. This shows that $\beta_3(m,m,3) = 1$, as asserted.
CASE $\mathcal{A} = \mathcal{A}(m,m,3)$ with $m = 3n$.
Set $\eta_{12^\alpha} = a_\alpha, \eta_{23^\alpha} = b_\alpha, \eta_{13^\alpha} = c_\alpha$. With this notation, the equations of type $\mathrm{I_b}$ are equivalent to the system

$$a_\alpha + b_\beta + c_{\alpha+\beta} = 0, \forall\, \alpha,\beta \in \mathbb{Z}/m\mathbb{Z}, \tag{5}$$

while the conditions of type II read

$$\sum a_\alpha = \sum b_\beta = \sum c_\gamma = 0. \tag{6}$$

We first solve the system (5), as follows. It implies that $a_\alpha + b_\beta = a_{\alpha'} + b_{\beta'}$, if $\alpha+\beta = \alpha'+\beta'$, in particular $a_\alpha - a_0 = b_\alpha - b_0 := d_\alpha$, for all α, and $d_\alpha + d_\beta = d_{\alpha'} + d_{\beta'}$, if $\alpha+\beta = \alpha'+\beta'$. We infer that $d_\alpha = \alpha d_1$, for all α. Hence $a_\alpha = a_0 + \alpha d_1, b_\alpha = b_0 + \alpha d_1$ and $c_\alpha = -a_0 - b_0 - \alpha d_1$, which solves the system (5). In particular, its solution space is 3-dimensional.
Finally, it is an easy matter to check that (5) $\Rightarrow$ (6), since $m = 3n$. Therefore, $\beta_3(3n,3n,3) = 2$, as asserted.
This proves Theorem 1.2 for non-exceptional reflection arrangements.

3. EXCEPTIONAL REFLECTION ARRANGEMENTS

We finish the proof of Theorem 1.2, by computing the Aomoto-Betti numbers of the exceptional complex reflection arrangements of rank at least 3, $G_{23} - G_{37}$.

3.1. THE GROUPS G_{31}, G_{32}, G_{33}. CASE $\mathcal{A} = \mathcal{A}(G_{31})$. The hyperplanes of $\mathcal{A}$ live in $\mathbb{C}^4$. Their defining equations are as follows (see [11]). Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{4})$. The hyperplanes of $\mathcal{A}$ are:

- (H_i) $z_i = 0$ $(1 \leqslant i \leqslant 4)$;
- (H_{ij^β}) $z_i - \omega^\beta z_j = 0$ $(1 \leqslant i < j \leqslant 4,\ \beta \in \mathbb{Z}/4\mathbb{Z})$;
- $(H_{\overline{\alpha}})$ $z_1 + \sum_{2\leqslant i\leqslant 4} \omega^{\alpha_i} z_i = 0$ $(\overline{\alpha} = (\alpha_2,\alpha_3,\alpha_4) \in (\mathbb{Z}/4\mathbb{Z})^3, \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \ (\mathrm{mod}\ 2))$.

By Lemma 2.5(2), it is enough to show that $\Gamma_{(2)}(G_{31})$ is connected. This can be seen as follows. Clearly, the 2-flat $H_k \cap H_{ij^\beta}$ has multiplicity 2, when $i \neq j \neq k$. This implies that $i \sim_{(2)} j \sim_{(2)} kh^\beta$, for all $1 \leqslant i < j \leqslant 4$, $1 \leqslant k < h \leqslant 4$ and $\beta \in \mathbb{Z}/4\mathbb{Z}$. Given any $H_{\overline{\alpha}}$, it is not hard to see that the multiplicity of $H_{\overline{\alpha}} \cap H_{12^{\alpha_2}}$ is 2. This proves connectivity, as claimed.

CASE $\mathcal{A} = \mathcal{A}(G_{32})$. Set $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$. The arrangement $\mathcal{A}$ consists of the following hyperplanes in $\mathbb{C}^4$ (see [24]):

- (H_i) $z_i = 0$ $(1 \leqslant i \leqslant 4)$;
- $(H_{1^{\alpha\beta}})$ $z_2 + \omega^\alpha z_3 + \omega^\beta z_4 = 0$;
- $(H_{2^{\alpha\beta}})$ $z_1 + \omega^\alpha z_3 - \omega^\beta z_4 = 0$;
- $(H_{3^{\alpha\beta}})$ $z_1 - \omega^\alpha z_2 + \omega^\beta z_4 = 0$;
- $(H_{4^{\alpha\beta}})$ $z_1 + \omega^\alpha z_2 - \omega^\beta z_3 = 0$ $(\alpha, \beta \in \mathbb{Z}/3\mathbb{Z})$.

Clearly, the 2-flats $H_i \cap H_j$ $(i \neq j)$ and $H_i \cap H_{i^{\alpha\beta}}$ have multiplicity 2. This shows that $\Gamma_{(2)}(G_{32})$ is connected and we are done.

CASE $\mathcal{A} = \mathcal{A}(G_{33})$. Here, $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$ and the hyperplanes (in $\mathbb{C}^6$) are as follows (see [2, 21]):

- (H_{ij^β}) $z_i - \omega^\beta z_j = 0$ $(1 \leqslant i < j \leqslant 4,\ \beta \in \mathbb{Z}/3\mathbb{Z})$;
- $(H_{\overline{\alpha}})$ $\sum_{1 \leqslant i \leqslant 4} \omega^{\alpha_i} z_i + z_5 + z_6 = 0$ $(\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{Z}/3\mathbb{Z})^4,\ \sum \alpha_i = 0)$.

Plainly, $ij^\beta \sim_{(2)} kh^0 \sim_{(2)} ij^{\beta'}$, for all β, β' (where $i \neq j \neq k \neq h$). Like in the Case $G = G_{31}$, it can be checked that $H_{\overline{\alpha}} \cap H_{ij^\beta}$ has multiplicity 2, if $\beta = \alpha_j - \alpha_i$. We infer that $\Gamma_{(2)}(G_{33})$ is connected, and we are done.

3.2. MORE VANISHING CRITERIA. We will no longer need defining equations to settle the remaining cases. We will use instead a couple of new vanishing arguments. For the beginning, let us recall from [17, pp. 224-225] the following very useful properties of reflection groups and arrangements, derived from a key result of Steinberg [23]. For any complex reflection group G and any $X \in \mathcal{L}_r(\mathcal{A}(G))$, the fixer subgroup $G_X := \{g \in G \mid gx = x,\ \forall x \in X\}$ is a reflection group, and $\mathcal{A}(G)_X = \mathcal{A}(G_X)$ is again a reflection arrangement, of rank r. By the construction of $\mathcal{A}(G)$, the group G acts on the arrangement $\mathcal{A}(G)$, hence on the intersection lattice $\mathcal{L}_\bullet(G)$. Let us denote by $\mathcal{O}_X$ the G-orbit of $X \in \mathcal{L}(G)$. Let $\mathrm{Type}(X)$ be the isomorphism type of the reflection group G_X. It follows from [17, Lemma 6.88] that the type is constant on each orbit $\mathcal{O}_X$. Moreover, Table C from [17] gives $|\mathcal{A}(G)|$ and the orbit partition of $\mathcal{L}_\bullet(G)$ for all exceptional groups, in terms of types of orbits.

This leads to the quick computation of the sets

$$\mathscr{P}(G) := \{p \text{ prime} \mid \exists\, X \in \mathcal{L}_2(G) \text{ such that } |\mathcal{A}(G_X)| \equiv 0 \pmod{p}\}$$

In particular, $\mathscr{P}(G) \subseteq \{2, 3, 5\}$, for every exceptional arrangement of rank at least 3. We infer from Lemma 2.5(3) that $\beta_p(G) = 0$, if $p > 5$. Hence, we may suppose from now on that $p \leqslant 5$.

For an arbitrary arrangement $\mathcal{A}$, $H \in \mathcal{A}$ and a prime p, we define

$$m_p(H) = 1 + \sum (|\mathcal{A}_X| - 1), \tag{7}$$

where the sum is taken over those $X \in \mathcal{L}_2(\mathcal{A})$ such that $X \subseteq H$ and $|\mathcal{A}_X| \not\equiv 0 \pmod{p}$.
The numbers from (7) may be extracted from Table C in [17], for exceptional reflection arrangements of rank at least 3. This is based on the following fact, valid for an arbitrary reflection group G. For $X, Y \in \mathcal{L}(G)$, let $u(X, Y)$ be the number of $Z \in \mathcal{L}(G)$ such that $Z \in \mathcal{O}_Y$ and $Z \subseteq X$. Clearly, this number depends only on $\mathcal{O}_X$ and $\mathcal{O}_Y$. The values $u(H, X)$ may be found in Table C, for all orbit types corresponding to $H \in \mathcal{L}_1(G)$ and $X \in \mathcal{L}_2(G)$.

LEMMA 3.1. *For an arrangement $\mathcal{A}$ and a prime p, the following hold.*

(1) *If $m_p(H) > \frac{|\mathcal{A}|}{2}$ for all $H \in \mathcal{A}$, then $\beta_p(\mathcal{A}) = 0$.*
(2) *If $m_p(H) > \frac{|\mathcal{A}|}{3}$ for all $H \in \mathcal{A}$ and $\mathcal{A}$ has no rank 2 flats of multiplicity $p \cdot r$ with $r > 1$, then $\beta_p(\mathcal{A}) = 0$.*

Proof. If $\beta_p(\mathcal{A}) \neq 0$, there is a non-constant function $\eta \in \mathbb{F}_p^{\mathcal{A}}$ satisfying all equations from Lemma 2.1. Fix $H \in \mathcal{A}$ and set $\eta_H = \alpha$. We claim that $|\{\eta = \alpha\}| \geqslant m_p(H)$. Indeed, if $X \in \mathcal{L}_2(\mathcal{A})$ is contained in H and $|\mathcal{A}_X| \not\equiv 0 \pmod{p}$, then η must have the constant value α on $\mathcal{A}_X$. An easy count of all these hyperplanes gives the claimed inequality.
In Part (1), this implies that η must be constant, a contradiction. In Part (2), we infer that η has only two distinct values. By adding constant functions and multiplying by non-zero elements in $\mathbb{F}_p$, we may assume that these values are $\eta_H = 0$ and $\eta_{H'} = 1$. By Lemma 2.1, the flat $X = H \cap H'$ has multiplicity $p \cdot r$, imposing the condition $\sum_{K \in \mathcal{A}_X} \eta_K = 0$. Since necessarily $r = 1$, we arrive again at a contradiction. □

3.3. INDUCTION ON RANK. We start with a couple of general considerations. A subarrangement $\mathcal{B} \subseteq \mathcal{A}$ is called *line-closed in* $\mathcal{A}$ (see the first definition from [8, Definition 1.1]) if $\mathcal{B}_X = \mathcal{A}_X$, for all $X \in \mathcal{L}_2(\mathcal{B})$. This property implies that the restriction map, $\mathbb{F}_p^{\mathcal{A}} \longrightarrow \mathbb{F}_p^{\mathcal{B}}$, sends $Z_p(\mathcal{A})$ into $Z_p(\mathcal{B})$. Clearly, $\mathcal{A}_Y$ is line-closed in $\mathcal{A}$, for any $Y \in \mathcal{L}(\mathcal{A})$.

LEMMA 3.2. *Let G be a complex reflection group. Assume that $2 \leqslant r < \operatorname{rank}(\mathcal{A}(G))$ and $\beta_p(\mathcal{A}(G')) = 0$, for all irreducible groups $G' \in \operatorname{Type}(\mathcal{L}_r(\mathcal{A}(G)))$. Then $\beta_p(\mathcal{A}(G)) = 0$.*

Proof. Assuming the contrary, there exist $\eta \in Z_p(G)$ and $H_1, H_2 \in \mathcal{A}(G)$ such that $\eta_{H_1} \neq \eta_{H_2}$. From our assumption on r, we may find $H_3, \ldots, H_r \in \mathcal{A}(G)$ such that $X = H_1 \cap H_2 \cdots \cap H_r \in \mathcal{L}_r(G)$. Set $\mathcal{B} = \mathcal{A}(G)_X = \mathcal{A}(G_X)$. We deduce that $\beta_p(G_X) \neq 0$. If G_X is reducible, this contradicts Lemma 2.3. Otherwise, our second assumption is violated. □

3.4. THE RANK 3 CASE. The rank 3 exceptional groups are $H_3, G_{24}, G_{25}, G_{26}$ and G_{27}. Table C from [17] provides the following information on each group:

- $\mathscr{P}(G) = \{2, 3, 5\};\ \{2, 3\};\ \{2\};\ \{2, 5\};\ \{2, 3, 5\}$;
- $|\mathcal{A}(G)| = 15; 21; 12; 21; 45$.

By Lemma 2.2 and Lemma 2.5(3), the only primes p which might give $\beta_p(G) \neq 0$ are as follows:

$$3,5\ (H_3);\ 3\ (G_{24});\ 2\ (G_{25});\ -\ (G_{26});\ 3,5\ (G_{27}).$$

Using Lemma 3.1(1), we obtain $\beta_3(H_3) = \beta_3(G_{24}) = \beta_3(G_{27}) = \beta_5(G_{27}) = 0$, and Lemma 3.1(2) gives $\beta_5(H_3) = 0$. Finally, $\beta_2(G_{25}) = 2$, cf. [20]. Thus, Theorem 1.2 is proved in this case. Indeed, the Hessian arrangement $\mathcal{A}(G_{25})$ supports a 4-net (see e.g. [26]). This implies that $\beta_2 \neq 0$, by [20]; the other implications from Theorem 1.2(2) are obvious.

Applying Lemma 3.2 for $r = 3$ and $p = 5$, we also infer that Theorem 1.2(1) holds for all complex reflection arrangements of rank at least 3. Thus, we only need to show that $\beta_2(G) = \beta_3(G) = 0$, when G is exceptional of rank at least 4, in order to complete the proof of Theorem 1.2.

3.5. THE REMAINING CASES. The only rank 5 exceptional arrangement is G_{33}, for which we know from §3.1 that all β_p vanish. By the computations from Section 2, the same thing happens for non-exceptional irreducible arrangements of rank 5. Lemma 3.2, applied for $r = 5$, guarantees then that we may reduce our proof to the rank 4 case. Here, the list is $G = F_4, G_{29}, H_4, G_{31}, G_{32}$, and the last two groups were treated in §3.1.

CASE $G = F_4$. The irreducible rank 3 types are listed in Table C from [17]: B_3 and C_3; in both cases, the arrangement $\mathcal{A}(G')$ is $\mathcal{A}(2,1,3)$, for which all β_p vanish, cf. Section 2. We may conclude by resorting to Lemma 3.2 for $r = 3$.

CASE $G = G_{29}$. The list of irreducible rank 3 types is: $G' = A_3, B_3, G(4,4,3)$. Taking $r = 3$ and $p = 2$ in Lemma 3.2, we deduce from Section 2 that $\beta_2(G_{29}) = 0$. Since $|\mathcal{A}(G_{29})| = 40, \beta_3(G_{29}) = 0$, by Lemma 2.2.

CASE $G = H_4$. The irreducible types of $\mathcal{L}_3(G)$ are: $G' = A_3, H_3$. Again by Lemma 3.2 and previous computations, $\beta_2(H_4) = 0$. Finally, $\beta_3(H_4) = 0$, as follows from Lemma 3.1(1).

The proof of Theorem 1.2 is complete.

4. MULTINETS AND JUMP LOCI

In this section we prove Theorems 1.1 and 1.3. Along the way, we establish a useful general result that relates combinatorial structures on arrangements satisfying the main multinet axiom to the algebraic monodromy of its Milnor fibration and its Aomoto-Betti numbers.

4.1. MULTINETS AND WEIGHTED PARTITIONS. The work of Falk and Yuzvinsky from [10] gives, among other things, a description of the resonance variety of an arrangement $\mathcal{A}$, $\mathscr{R}^1_1(\mathcal{A},\mathbb{C})$, in terms of *multinets* on the associated matroid. A $k-$multinet on $\mathcal{L}_{\leqslant 2}(\mathcal{A})$ is a partition Π with $k \geqslant 3$ non-empty blocks, $\mathcal{A} = \bigsqcup_{\alpha\in[k]} \mathcal{A}_\alpha$, together with a function, $\mathfrak{m} : \mathcal{A} \to \mathbb{Z}_{>0}$, satisfying certain axioms. The most important is the following:

For any $H \in \mathcal{A}_\alpha$ and $H' \in \mathcal{A}_\beta$ with $\alpha \neq \beta$, and every $\gamma \in [k]$,

$$n_X := \sum_{K\in\mathcal{A}_\gamma\cap\mathcal{A}_X} \mathfrak{m}_K \tag{8}$$

is independent of γ, where $X = H \cap H' \in \mathcal{L}_2(\mathcal{A})$.
Let Π be a partition of $\mathcal{A}$ with $k \geqslant 3$ non-empty blocks, as above. Let $\mathfrak{m} : \mathcal{A} \to \mathbb{Z}$ be an assignment of arbitrary integer weights to the hyperplanes of $\mathcal{A}$.

DEFINITION 4.1. The pair $\mathscr{N} = (\Pi, \mathfrak{m})$ is a *weighted k-partition* if axiom (8) is satisfied. We will say that $\mathscr{N}$ is *h-reduced* ($h \geqslant 1$) if $\mathfrak{m}_K \equiv 1 \pmod h$, for all $K \in \mathcal{A}$.

The underlying partition of a k-multinet, together with its positive weight function, is a weighted k-partition. Moreover, the usual notion of reduced multinet corresponds to $h = 1$.
We will need a result from [20] related to axiom (8). To recollect it, we start with a few notations. Set $H_{\mathcal{A}} := H_1(M_{\mathcal{A}}, \mathbb{Z})$ and denote by $\{a_H\}_{H \in \mathcal{A}}$ the distinguished $\mathbb{Z}$-basis. Let S be $\mathbb{CP}^1 \backslash \{k \text{ points}\}$ and set $H_S := H_1(S, \mathbb{Z}) = \mathbb{Z} - \operatorname{span}\langle c_\alpha | \alpha \in [k] \rangle / \sum_{\alpha \in [k]} c_\alpha$, where c_α is the class of a small loop in S around the point α.
Let $\cup_{\mathcal{A}} : \bigwedge^2 H^1(M_{\mathcal{A}}, \mathbb{Z}) \to H^2(M_{\mathcal{A}}, \mathbb{Z})$ be the cup product. Recalling from [16] that $H^\bullet(M_{\mathcal{A}}, \mathbb{Z})$ has no torsion, we denote by $\nabla_{\mathcal{A}} : H_2(M_{\mathcal{A}}, \mathbb{Z}) \to \bigwedge^2 H_{\mathcal{A}}$ the $\mathbb{Z}$-dual comultiplication map.
The next result improves Theorem 2.4 from [10], in several ways. The hypothesis of Proposition 4.2 is reduced to the key axiom (*iii*) from [10, Definition 2.1]. The weight function $\mathfrak{m}$ may take arbitrary integer values, while in [10] positivity plays a crucial role. The conclusion in [10] is that $\operatorname{im}(\phi \otimes \Bbbk)^* \subseteq H^1(M_{\mathcal{A}}, \Bbbk)$ is an isotropic subspace, for a characteristic 0 field $\Bbbk$ (in positive characteristic, an additional condition on $\mathfrak{m}$ is needed in [10]), while Proposition 4.2 gives the conclusion over $\mathbb{Z}$. For the reader's convenience, we include the proof.

PROPOSITION 4.2 ([20]). *Let $\mathscr{N} = (\Pi, \mathfrak{m})$ be a weighted k-partition. Then $\bigwedge^2 \phi \circ \nabla_{\mathcal{A}} = 0$, where $\phi : H_{\mathcal{A}} \to H_S$ sends a_H to $\mathfrak{m}_H c_\alpha$, for $H \in \mathcal{A}_\alpha$. Therefore, $\cup_{\mathcal{A}} \circ \bigwedge^2 \phi^* = 0$, by taking $\mathbb{Z}$-duals.*

Proof. Let $\mathfrak{h}(\mathcal{A})$ be the $\mathbb{Z}$-form of the holonomy Lie algebra of $\mathcal{A}$, appearing in Proposition 5.2 from [20]. By definition, this is the graded $\mathbb{Z}$-Lie algebra quotient of the free $\mathbb{Z}$-Lie algebra generated by $H_{\mathcal{A}}$, $\mathbb{L}^\bullet(H_{\mathcal{A}})$, graded by bracket length, by the graded Lie ideal generated by $\operatorname{im}(\nabla_{\mathcal{A}}) \subseteq \bigwedge^2 H_{\mathcal{A}}$, where $\bigwedge^2 H_{\mathcal{A}}$ is identified with $\mathbb{L}^2(H_{\mathcal{A}})$ via the Lie bracket. Let $\mathbb{L}^\bullet(\phi) : \mathbb{L}^\bullet(H_{\mathcal{A}}) \to \mathbb{L}^\bullet(H_S)$ be the graded $\mathbb{Z}$-Lie algebra map extending ϕ. Our claim says that $\mathbb{L}^\bullet(\phi)$ factors through $\mathfrak{h}(\mathcal{A})$.
To check this, we recall from [20, (30)] that the defining Lie relations of $\mathfrak{h}(\mathcal{A})$ are:

$$\sum_{K \in \mathcal{A}_X} [a_K, a_L], \quad \text{for} \quad X \in \mathcal{L}_2(\mathcal{A}) \quad \text{and} \quad L \in \mathcal{A}_X .$$

Thus, we have to show that $[\sum_{K \in \mathcal{A}_X} \phi(a_K), \phi(a_L)] = 0 \in \mathbb{L}^2(H_S)$.
There are two cases to consider. When X is mono-coloured, i.e., $\mathcal{A}_X \subseteq \mathcal{A}_\alpha$ for some $\alpha \in [k]$, by construction $\phi(a_K) \in \mathbb{Z} \cdot c_\alpha$, for all $K \in \mathcal{A}_X$, and we are done. Otherwise, $X = H \cap H'$, with $H \in \mathcal{A}_\alpha$, $H' \in \mathcal{A}_\beta$ and $\alpha \neq \beta$. Again by construction, $\sum_{K \in \mathcal{A}_X} \phi(a_K) = \sum_{\gamma \in [k]} (\sum_{K \in \mathcal{A}_\gamma \cap \mathcal{A}_X} \mathfrak{m}_K) c_\gamma$, which equals $n_X(\sum_{\gamma \in [k]} c_\gamma)$, by axiom (8). This implies that $\sum_{K \in \mathcal{A}_X} \phi(a_K) = 0 \in \mathbb{L}^1(H_S)$, by the definition of H_S, which completes the proof. □

4.2. RELATING WEIGHTED PARTITIONS TO JUMP LOCI. We are now ready to state our result, keeping the previous notation.

THEOREM 4.3. *Assume that $\mathcal{N} = (\Pi, \mathfrak{m})$ is a k-reduced weighted k-partition. Then the following hold, for all divisors p, d of k with p prime and $d > 1$.*

(1) $\rho_d(\mathcal{A}) \in \exp\, \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, *in particular* $e_d(\mathcal{A}) > 0$.
(2) $\beta_p(\mathcal{A}) \neq 0$.

Proof. Part (2). Since the weighted partition is k-reduced and $p|k$, $\phi \otimes \Bbbk$ is surjective, by construction, for $\Bbbk = \mathbb{F}_p$ and $\mathbb{C}$. It follows from Proposition 4.2 that $\mathrm{im}(\phi \otimes \Bbbk)^* \subseteq H^1(M_{\mathcal{A}}, \Bbbk)$ is a $(k-1)$-dimensional subspace, isotropic with respect to the cup product. The linear map sending each c_α to $1 \in \mathbb{F}_p$ defines an element of $H^1(S, \mathbb{F}_p)$, denoted $\sigma_p(S)$. Clearly, $\phi^*(\sigma_p(S)) = \sigma_p(\mathcal{A})$, since $\mathcal{N}$ is in particular p-reduced. By definition (3), $\beta_p(\mathcal{A}) \neq 0$ as claimed, since $k \geqslant 3$.
Part (1). Note that $\phi : H_{\mathcal{A}} \to H_S$ induces homomorphisms $\phi^* : \mathbb{T}(S) \to \mathbb{T}(M_{\mathcal{A}})$ and $\phi^* : H^1(S, \mathbb{C}) \hookrightarrow H^1(M_{\mathcal{A}}, \mathbb{C})$, compatible with the surjective exponential maps of S and $M_{\mathcal{A}}$. The map sending each c_α to $\exp(\frac{2\pi\sqrt{-1}}{k})$ defines an element of the character torus, $\rho_k(S) \in \mathbb{T}(S)$. Plainly, $\phi^*(\rho_k(S)) = \rho_k(\mathcal{A})$, since $\mathcal{N}$ is k-reduced. We also have $(\rho_k)^{k/d} = \rho_d$, for both S and $\mathcal{A}$, since we assumed that d divides k. Hence, $\phi^*(\rho_d(S)) = \rho_d(\mathcal{A}) \in \exp\, \phi^*(H^1(S, \mathbb{C})) \subseteq \exp\, \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, where the last inclusion follows from the argument in Part (2). Indeed, we know that $\phi^*(H^1(S, \mathbb{C}))$ is an isotropic subspace in $H^1(M_{\mathcal{A}}, \mathbb{C})$, of dimension at least 2, and we may simply use the definition of $\mathscr{R}^1_1$. The conclusion $e_d(\mathcal{A}) > 0$ is a direct consequence of equality (2), since it is well-known that $\exp\, \mathscr{R}^1_1(\mathcal{A}, \mathbb{C}) \subseteq \mathscr{V}^1_1(M_{\mathcal{A}})$; see e.g. [5, Theorem D] for a more general result. □

4.3. REDUCED WEIGHTED PARTITIONS ON COMPLEX REFLECTION ARRANGEMENTS. We begin the proof of Theorem 1.3. Let $\mathcal{A}$ be a complex reflection arrangement and assume $e_d(\mathcal{A}) > 0$, with $2 \leqslant d \leqslant 4$. We will show that $\rho_d(\mathcal{A}) \in \exp\, \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$ with the aid of Theorem 4.3(1), which requires the existence of a certain weighted partition on $\mathcal{A}$.
An easy preliminary remark is that the question from [4] always has a positive answer, for any arrangement $\mathcal{A}$ of rank at most 2. To see this, note first that the assumption $e_d(\mathcal{A}) \neq 0$ is equivalent to $\rho_d \in \mathscr{V}^1_1(M_{\mathcal{A}})$, by equality (2). When $\mathrm{rank}(\mathcal{A}) \leqslant 2$, it is known that $\mathscr{V}^1_1(M_{\mathcal{A}}) = \exp\, \mathscr{R}^1_1(\mathcal{A}, \mathbb{C})$, so the conclusion follows trivially. Consequently, we may also suppose that the rank is at least 3. On the other hand, $e_d(\mathcal{A}) > 0$ and $d = p^s$ together imply, via the modular bound (4), that $\beta_p(\mathcal{A}) > 0$. Therefore, $\mathcal{A}$ must be either the Hessian arrangement, or one of the arrangements from Theorem 1.2(3). To apply Theorem 4.3(1), we need to describe suitable weighted partitions on these arrangements, in each case.
The Hessian arrangement supports a reduced 4-multinet (actually, a 4-net). The monomial arrangement $\mathcal{A}(m, m, 3)$ has a reduced 3-multinet (in fact, a 3-net), as noted in [10].
A (non-reduced) 3-multinet on the full monomial arrangement $\mathcal{A}(m, 1, 3)$ was constructed in [10]. It is immediate to check that the weighted partition associated to this multinet is 3-reduced, when $m \equiv 1 \pmod 3$.

The last case is $\mathcal{A} = \mathcal{A}(m,m,4)$, with hyperplanes (H_{ij^μ}) $z_i - \omega^\mu z_j = 0$, where $1 \leqslant i < j \leqslant 4$, $\mu \in \mathbb{Z}/m\mathbb{Z}$ and $\omega = \exp(\frac{2\pi\sqrt{-1}}{m})$. We define a partition Π with three blocks, $\{H_{ij^\mu}, H_{kh^\nu} \mid \mu,\nu \in \mathbb{Z}/m\mathbb{Z}\}$, and set $\mathfrak{m} \equiv 1$ on $\mathcal{A}$. It is straightforward to verify axiom (8) by using the description of 2-flats given in §2.3. (Actually, this is a 3-net on $\mathcal{L}_{\leqslant 2}(\mathcal{A})$.)

4.4. PROOF OF THEOREM 1.3 COMPLETED. In case $\mathcal{A} = \mathcal{A}(G_{25})$, d must be 2 or 4. We may take $k = 4$ in Theorem 4.3 to obtain the desired conclusion. The remaining cases, described in Theorem 1.2(3), lead to $d = 3$. Taking $k = 3$, we conclude as before. □

4.5. PROOF OF THEOREM 1.1. We may suppose that $\operatorname{rank}(\mathcal{A}) \geqslant 3$, since otherwise the conclusion is known (see [14, p. 773]). By the modular bound (4) and Theorem 1.2, $e_p(\mathcal{A}) = \beta_p(\mathcal{A})$, when $\mathcal{A}$ is not $\mathcal{A}(G_{25})$ or one of the arrangements listed in Theorem 1.2(3). Moreover, we have to verify the conclusion only for $p = 2$ (in case $\mathcal{A}(G_{25})$) or for $p = 3$ (in the remaining cases).
The equality $e_2(G_{25}) = \beta_2(G_{25}) = 2$ is well-known (see e.g. [20]). When $\mathcal{A}$ is not $\mathcal{A}(m,m,3)$ with $m \equiv 0 \pmod 3$, we know that $\beta_3(\mathcal{A}) = 1$. In these cases, we may use the 3-reduced weighted 3-partitions from §4.3, for $d = k = 3$, exactly as in §4.4, to obtain that $e_3(\mathcal{A}) > 0$. Now we are done, since the modular bound (4) implies that $e_3(\mathcal{A}) \leqslant 1$.
The last case, $\mathcal{A} = \mathcal{A}(m,m,3)$ with $m = 3n$ and $p = 3$, when $\beta_3(\mathcal{A}) = 2$, requires a more careful treatment.
The (relabeled) hyperplanes of $\mathcal{A}$ are $\{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}} \mid \alpha,\beta,\gamma \in \mathbb{Z}/m\mathbb{Z}\}$. We recall from §2.3 the two types of 2-flats: $\{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ and $\{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}}\}$ with $\alpha+\beta+\gamma = 0$.
We have to show that $e_3(\mathcal{A}) \geqslant 2$, in order to finish the proof. To this end, we need two reduced weighted 3-partitions on $\mathcal{A}$. The first one, $\mathcal{N}$, is constructed in [10]. The blocks of the partition Π are given by $\{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}_{1\leqslant i<j\leqslant 3}$. The blocks of the second one, $\mathcal{N}'$, are defined by $\{H_{ij^\alpha} \mid 1 \leqslant i < j \leqslant 3, \alpha \equiv \tau \pmod 3\}_{\tau\in\mathbb{F}_3}$, with α replaced by $-\alpha$ when $\{i,j\} = \{1,3\}$.
Let us check axiom (8) for $\mathcal{N}'$. The 2-flats X appearing in axiom (8) clearly coincide with those $\mathcal{A}_X$ that contain two hyperplanes with different colours with respect to Π'. For $X = \{H_{ij^\alpha} \mid \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ we find that $n_X = n$. For $X = \{H_{12^\alpha},\ H_{23^\beta},\ H_{13^{-\gamma}}\}$ with $\alpha+\beta+\gamma = 0$, the condition on colours translates to $\alpha \not\equiv \beta \not\equiv \gamma \pmod 3$, and implies that $n_X = 1$. Hence Π' defines a reduced weighted 3-partition $\mathcal{N}'$.
Consider the two (surjective) homomorphisms from Proposition 4.2, $\phi, \phi' : H_{\mathcal{A}} \to H_S$. Clearly, $\phi^*\mathbb{T}(S) = \exp\ \phi^*H^1(S,\mathbb{C})$ is a positive dimensional subtorus of $\mathbb{T}(M_{\mathcal{A}})$, and similarly for ϕ'. Moreover, $\phi^*\mathbb{T}(S) \subseteq \mathcal{V}_1^1(M_{\mathcal{A}})$, since $\phi^*H^1(S,\mathbb{C}) \subseteq H^1(M_{\mathcal{A}},\mathbb{C})$ is isotropic of dimension 2, hence contained in $\mathscr{R}_1^1(\mathcal{A},\mathbb{C})$, and likewise for ϕ'. Therefore, we may find two irreducible components of $\mathcal{V}_1^1(M_{\mathcal{A}})$, W and W', such that $\phi^*\mathbb{T}(S) \subseteq W$ and $\phi'^*\mathbb{T}(S) \subseteq W'$. On the other hand, $\rho_3(\mathcal{A}) \in W \cap W'$, by the argument from the proof of Theorem 4.3(1).

In this situation, it follows from a result of Artal Bartolo, Cogolludo and Matei [1, Proposition 6.9] that $\rho_3(\mathcal{A}) \in \mathscr{V}_2^1(M_\mathcal{A})$, if $W \neq W'$. Hence, $e_3(\mathcal{A}) \geqslant 2$, by equality (2), and we are done.
Suppose then that $W = W'$. We know from [10] that actually $W = \phi^*\mathbb{T}(S)$, since ϕ comes from a 3-net. Taking tangent spaces at the origin $1 \in \mathbb{T}(M_\mathcal{A})$, we infer that $\phi'^* H^1(S,\mathbb{C}) \subseteq \phi^* H^1(S,\mathbb{C})$.
We identify $H^1(M_\mathcal{A},\mathbb{C})$ with $\mathbb{C}^\mathcal{A}$, using the distinguished $\mathbb{Z}$-basis. In this way, the subspace $\phi^* H^1(S,\mathbb{C})$ (respectively $\phi'^* H^1(S,\mathbb{C})$) is identified with the subset of those elements $\eta \in \mathbb{C}^\mathcal{A}$ (respectively $\eta' \in \mathbb{C}^\mathcal{A}$) taking the constant values a, b, c (respectively a', b', c') on the blocks of Π (respectively Π'), where $a+b+c = a'+b'+c' = 0$. Now, it is an easy matter to check that $\phi^* H^1(S,\mathbb{C}) \cap \phi'^* H^1(S,\mathbb{C}) = 0$. This contradiction finishes the proof of Theorem 1.1. □

4.6. FULL MONODROMY ACTION. It follows from decomposition (1) that the characteristic polynomial $\Delta_\mathcal{A}(t) = (t-1)^{|\mathcal{A}|-1}\Pi_{1<d|n}\Phi_d(t)^{e_d(\mathcal{A})}$ encodes the full monodromy action on $H_1(F_\mathcal{A},\mathbb{Q})$.
The approach via modular bounds works only for prime power monodromy multiplicities, $e_{p^s}(\mathcal{A})$. One way to avoid this inconvenience is to impose restrictions on multiplicities of 2-flats, like in [20] for instance, to arrive at full monodromy computations. Unfortunately, as we saw in §2.3, arbitrarily high flat multiplicities may appear for non-exceptional complex reflection arrangements.
Even in this kind of situation, there is hope related to the following well-known vanishing criterion (see e.g. [14]): if $d \nmid |\mathcal{A}_X|$ for any $X \in \mathcal{L}_2(\mathcal{A})$, then $e_d(\mathcal{A}) = 0$. It turns out that this works for full monomial arrangements of small rank. The result below verifies in particular the strong form of the conjecture from [20].

PROPOSITION 4.4. *For $\mathcal{A} = \mathcal{A}(m,1,l)$, with $l = 3$ or 4, $\Delta_\mathcal{A}(t) = (t-1)^{|\mathcal{A}|-1}(t^2+t+1)$, if $l = 3$ and $m \equiv 1 (mod\ 3)$, and $\Delta_\mathcal{A}(t) = (t-1)^{|\mathcal{A}|-1}$, otherwise.*

Proof. We have to compute $e_d(\mathcal{A})$ for all divisors $d > 1$ of $|\mathcal{A}|$. If d is prime, this was done in Theorem 1.1 and Theorem 1.2. It follows from §2.3 that the 2-flats of $\mathcal{A}$ have multiplicities $2, 3$ or $m+2$, $|\mathcal{A}| = 3(m+1)$ for $l = 3$ and $|\mathcal{A}| = 2(3m+2)$ for $l = 4$. If d is not prime and $e_d(\mathcal{A}) \neq 0$, the vanishing criterion forces $m \equiv -2 \pmod d$. Writing that $|\mathcal{A}| \equiv 0 \pmod d$, we obtain for $l = 3$ that $3 \equiv 0 \pmod d$, a contradiction, and $8 \equiv 0 \pmod d$, for $l = 4$. In the second case, the modular bound implies that $\beta_2(\mathcal{A}) > 0$, contradicting Theorem 1.2(2). □

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Daniela Anca Măcinic
Simion Stoilow Institute of Mathematics
P.O. Box 1-764
RO-014700 Bucharest
Romania
Anca.Macinic@imar.ro

Ştefan Papadima
Simion Stoilow Institute of Mathematics
P.O. Box 1-764
RO-014700 Bucharest
Romania
Stefan.Papadima@imar.ro

Clement Radu Popescu
Simion Stoilow Institute of Mathematics
unit no. 4
P.O. Box 1-764
RO-014700 Bucharest
Romania
Radu.Popescu@imar.ro