Large Tilting Sheaves over Weighted Noncommutative Regular Projective Curves

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Abstract. Let $\mathbb{X}$ be a weighted noncommutative regular projective curve over a field k. The category $\operatorname{Qcoh}\mathbb{X}$ of quasicoherent sheaves is a hereditary, locally noetherian Grothendieck category. We classify all tilting sheaves which have a non-coherent torsion subsheaf. In case of nonnegative orbifold Euler characteristic we classify all large (that is, non-coherent) tilting sheaves and the corresponding resolving classes. In particular we show that in the elliptic and in the tubular cases every large tilting sheaf has a well-defined slope.

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Contents

1. Introduction

Tilting theory is a well-established technique to relate different mathematical theories. An overview of its role in various areas of mathematics can be found in [4]. One of the first results along these lines, due to Beilinson [17], establishes a connection between algebraic geometry and representation theory of finite dimensional algebras. For instance, the projective line $\mathbb{X} = \mathbb{P}_1(k)$ over a field k turns out to be closely related with the Kronecker algebra Λ, the path algebra of the quiver $\bullet \rightrightarrows \bullet$ over k. The connection is provided by the vector bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$, which is a tilting sheaf in $\operatorname{coh}\mathbb{X}$ with endomorphism ring Λ. The derived Hom-functor $\mathbf{R}\operatorname{Hom}(T, -)$ then defines an equivalence between the derived categories of $\operatorname{Qcoh}\mathbb{X}$ and $\operatorname{Mod}\Lambda$. There are many more such examples, where a noetherian tilting object T in a triangulated category $\mathcal{D}$ provides an equivalence between $\mathcal{D}$ and the derived category of $\operatorname{End}(T)$. We refer to [27, 32, 30], and to [20, 40] for the context of Calabi-Yau and cluster categories.

The weighted projective lines introduced in [27], and their generalizations in [42], called noncommutative curves of genus zero in [38], provide the basic framework for the present article. They are characterized by the existence of a tilting bundle in the category of coherent sheaves $\operatorname{coh}\mathbb{X}$. In this case the corresponding (derived-equivalent) finite-dimensional algebras are the (concealed-) canonical algebras [56, 57, 44], an important class of algebras in representation theory. A particularly interesting and beautiful case is the so-called tubular case. Here every indecomposable coherent sheaf is semistable (with respect to the slope), and the semistable coherent sheaves of slope q form a family of tubes, for every q ([45, 38]). This classification is akin to Atiyah's classification of indecomposable vector bundles over an elliptic curve [12].

The tilting objects mentioned so far are small in the sense that they are noetherian objects, and that their endomorphism rings are finite-dimensional algebras. For arbitrary rings R there is the notion of a (not necessarily noetherian or finitely generated) tilting module T, which was extended to Grothendieck categories in [23, 24].

Definition. An object T in a Grothendieck category $\vec{\mathcal{H}}$ is called *tilting* if T generates precisely the objects in $T^{\perp_1} = \{X \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(T, X) = 0\}$. The class $T^{\perp_1}$ is then called a *tilting class*.

Such "large" tilting objects in general do not produce derived equivalences in the way mentioned above. But they yield recollements of triangulated categories [15, 6, 21], still providing a strong relationship between the derived categories involved.

Large tilting modules occur frequently. For example, they arise when looking for complements to partial tilting modules, or when computing intersections of tilting classes given by classical tilting modules, and they parametrize resolving subcategories of finitely presented modules. We refer to [3] for a survey on these results.

Another reason for the interest in large tilting modules is their deep connection with localization theory. This is best illustrated by the example of a Dedekind domain R. The tilting modules over R are parametrized by the subsets $V \subseteq$ Max-Spec R, and they arise from localizations at sets of simple modules. More precisely, the universal localization $R \hookrightarrow R_V$ at the simples supported in V yields the tilting module $T_V = R_V \oplus R_V/R$, and the set $V = \emptyset$ corresponds to the regular module R, the only finitely generated tilting module [9, Cor. 6.12]. Similar results hold true in more general contexts. Over a commutative noetherian ring, the tilting modules of projective dimension one correspond to categorical localizations in the sense of Gabriel [8]. Over a hereditary ring, tilting modules parametrize universal localizations [2].
An interesting example is provided by the Kronecker algebra Λ. Here we have a complete analogy to the Dedekind case if we replace the maximal spectrum by the index set $\mathbb{X}$ of the tubular family $\mathbf{t} = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$. Indeed, the infinite dimensional tilting modules are parametrized by the subsets $V \subseteq \mathbb{X}$, and they arise from localizations at sets of simple regular modules. Again, the universal localization $\Lambda \hookrightarrow \Lambda_V$ at the simple regular modules supported in V yields the tilting module $T_V = \Lambda_V \oplus \Lambda_V/\Lambda$, and the set $V = \emptyset$ corresponds to the Lukas tilting module $\mathbf{L}$.
For arbitrary tame hereditary algebras, the classification of tilting modules is more complicated due to the possible presence of finite dimensional direct summands from non-homogeneous tubes. Infinite dimensional tilting modules are parametrized by pairs (B, V) where B is a so-called branch module, and V is a subset of $\mathbb{X}$. The tilting module corresponding to (B, V) has finite dimensional part B and an infinite dimensional part which is of the form T_V inside a suitable subcategory, see [10].

In the present paper, we tackle the problem of classifying large tilting objects in hereditary Grothendieck categories. In particular, we will consider the category Qcoh $\mathbb{X}$ of quasicoherent sheaves over a weighted noncommutative regular projective curve $\mathbb{X}$ over a field k, in the sense of [39]. We will discuss how the results described above for tame hereditary algebras extend to this more general setting.
As in module categories, a crucial role will be played by the following notion.

Definition. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category, and let $\mathcal{H}$ the class of finitely presented objects in $\vec{\mathcal{H}}$. We call a class $\mathscr{S} \subseteq \mathcal{H}$ *resolving* if it generates $\vec{\mathcal{H}}$ and has the following closure properties: $\mathscr{S}$ is closed under extensions, direct summands, and $S' \in \mathscr{S}$ whenever $0 \to S' \to S \to S'' \to 0$ is exact with $S, S'' \in \mathscr{S}$.

We will use [58] to show the following general existence result for tilting objects.

Theorem 1. [Theorem 4.4] *Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category and $\mathscr{S} \subseteq \mathcal{H}$ be resolving with* $\mathrm{pd}(S) \leq 1$ *for all $S \in \mathscr{S}$. Then there is a tilting object T in $\vec{\mathcal{H}}$ with $T^{\perp_1} = \mathscr{S}^{\perp_1}$.*

Tilting classes as above of the form $T^{\perp_1} = \mathscr{S}^{\perp_1}$ for some class $\mathscr{S}$ of finitely presented objects are said to be of *finite type*.
When $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, the category of finitely presented objects $\mathcal{H} = \operatorname{coh}\mathbb{X}$ is given by the coherent sheaves, and we have

Theorem 2. [Theorem 4.14] *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. The assignment $\mathscr{S} \mapsto \mathscr{S}^{\perp_1}$ defines a bijection between*

- *resolving classes $\mathscr{S}$ in $\mathcal{H}$, and*
- *tilting classes $T^{\perp_1}$ of finite type.*

In a module category, all tilting classes have finite type by [16]. In well behaved cases we can import this result to our situation. The complexity of the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over $\mathbb{X}$ depends on the orbifold Euler characteristic χ'_{orb}. If $\chi'_{orb}(\mathbb{X}) > 0$, then the category $\operatorname{coh}\mathbb{X}$ is of (tame) domestic type, and it is derived-equivalent to the category $\operatorname{mod} H$ for a (finite-dimensional) tame hereditary algebra H. In this case, all tilting classes have finite type, and we obtain a complete classification of all large tilting sheaves (Theorem 6.5), which - not surprisingly - is very similar to the classification in [10]. But also in the tubular case, where $\mathbb{X}$ is weighted of orbifold Euler characteristic $\chi'_{orb}(\mathbb{X}) = 0$, tilting classes turn out to always have finite type.

Before we discuss our classification results, let us give some details on the tools we will employ. Our starting point is given by the following property, which is reminiscent of the well-known splitting property (2.1) for $\operatorname{coh}\mathbb{X}$.

Theorem 3. [Theorem 3.8] *Let $T \in \operatorname{Qcoh}\mathbb{X}$ be a sheaf with $\operatorname{Ext}^1(T,T) = 0$. Then there is a split exact sequence $0 \to tT \to T \to T/tT \to 0$ where $tT \subseteq T$ denotes the (largest) torsion subsheaf of T and is a direct sum of finite length sheaves and of injective sheaves.*

This result shows that the classification of large (= non-coherent) tilting sheaves splits, roughly speaking, into two steps:

(i) The first is the classification of large tilting sheaves T which are torsion-free (that is, with $tT = 0$). This seems to be a very difficult problem in general, but it turns out that in the cases when $\mathbb{X}$ is a noncommutative curve of genus zero which is of domestic or of tubular type, we get all these tilting sheaves with the help of Theorem 1.

(ii) If, on the other hand, the torsion part tT of a large tilting sheaf T is non-zero, then it is quite straightforward to determine the shape of tT; it is a direct sum of Prüfer sheaves and a certain so-called branch sheaf B, which is coherent. We can then apply perpendicular calculus to B, in order to reduce the problem to the case that tT is a direct sum of Prüfer sheaves, or to $tT = 0$, which is the torsionfree case (i).

If Prüfer sheaves occur in the torsion part, then the corresponding torsionfree part is uniquely determined. This leads to the following, general result:

Theorem 4. [Corollary 4.12] *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve. The tilting sheaves in* $\operatorname{Qcoh}\mathbb{X}$ *which have a non-coherent torsion subsheaf are up to equivalence in bijective correspondence with pairs (B, V), where V is a non-empty subset of $\mathbb{X}$ and B is a branch sheaf.*

We will see in Section 5 that the tilting sheaf corresponding to (B, V) has coherent part B and a non-coherent part T_V formed inside a suitable perpendicular subcategory, the categorical counterpart of universal localization. In particular, the torsionfree part of T_V can be interpreted as a projective generator of the quotient category obtained from $\operatorname{Qcoh}\mathbb{X}$ by localization at the simple objects supported in V. Of course, there are also tilting sheaves given by pairs (B, V) with $V = \emptyset$. Here the non-coherent part is the Lukas tilting sheaf inside a suitable subcategory, that is, it is given by the resolving class formed by all vector bundles. Altogether, the pairs (B, V) correspond to Serre subcategories of $\operatorname{coh}\mathbb{X}$, and tilting sheaves are closely related with Gabriel localization, like in the case of tilting modules over commutative noetherian rings, cf. also [7, Sec. 5].
Let us now discuss the tubular case. Following [53], we define for every $w \in \mathbb{R}\cup\{\infty\}$ the class $\mathcal{M}(w)$ of quasicoherent sheaves of slope w. Reiten and Ringel have shown [53] that every indecomposable object has a well-defined slope. Our main result is as follows.

Theorem 5. [Theorem 8.6] *Let $\mathbb{X}$ be of tubular type. Then every large tilting sheaf in* $\operatorname{Qcoh}\mathbb{X}$ *has a well-defined slope w. If w is irrational, then there is up to equivalence precisely one tilting sheaf of slope w. If w is rational or ∞, then the large tilting sheaves of slope w are classified like in the domestic case.*

In Section 9, we will briefly discuss the elliptic case, where $\chi'_{orb}(\mathbb{X}) = 0$ and $\mathbb{X}$ is non-weighted. Some of our main results will extend to this situation. In particular, Theorem 9.1 will resemble the tubular case described above. As it turns out, this will be much easier than in the (weighted) tubular case, using an Atiyah [12] type classification, namely, that all coherent sheaves lie in homogeneous tubes.
When the orbifold Euler characteristic $\chi'_{orb}(\mathbb{X}) \geq 0$, our results also yield a classification of certain resolving classes in $\operatorname{coh}\mathbb{X}$ (see Corollaries 6.7 and 8.7 and Theorem 9.1(5)). Furthermore, Theorem 4 enables us to recover and refine some results from [14] on maximal rigid objects in tube categories (Corollary 4.19).
If $\chi'_{orb}(\mathbb{X}) < 0$, then $\operatorname{coh}\mathbb{X}$ is wild. We stress that Theorem 4 also holds in this case, but we have not attempted to classify the torsionfree large tilting sheaves in the wild case.
There is one main difference to the module case. We recall that one of the standard characterising properties of a tilting module $T \in \operatorname{Mod} R$ is the existence of an exact sequence
$$0 \to R \to T_0 \to T_1 \to 0$$
with $T_0, T_1 \in \operatorname{Add}(T)$. In contrast to $\operatorname{Mod} R$, the category $\operatorname{Qcoh}\mathbb{X}$ lacks a projective generator. When $\mathbb{X}$ has genus zero, the replacement for the ring R

in our category is a tilting bundle T_{can} whose endomorphism ring is a canonical algebra. Indeed, for every large tilting sheaf T we can always find such a tilting bundle T_{can} and a short exact sequence $0 \to T_{\text{can}} \to T_0 \to T_1 \to 0$, even with T_0, $T_1 \in \operatorname{add}(T)$. If T has a non-coherent torsion part, then we can even do this with $\operatorname{Hom}(T_1, T_0) = 0$, cf. Theorem 10.1.

Since noncommutative curves of genus zero are derived-equivalent to canonical algebras in the sense of Ringel and Crawley-Boevey [57], our results are closely related to the classification of large tilting modules over canonical algebras. The module case is treated more directly in [7], where we also address the dual concept of cotilting modules and the classification of pure-injective modules.

2. Weighted noncommutative regular projective curves

In this section we collect some preliminaries on the category of quasicoherent sheaves we are going to study, and we introduce large tilting sheaves.

The main purpose of noncommutative algebraic geometry is to study abelian categories which have the same formal properties as $\operatorname{coh}(X)$ or $\operatorname{Qcoh}(X)$ for a scheme X. These categories are regarded as the geometric objects themselves, based on the Gabriel-Rosenberg reconstruction theorem which tells us that the scheme X can be reconstructed from $\operatorname{coh}(X)$ or $\operatorname{Qcoh}(X)$. By analogy it is then convenient to use a similar terminology as for the objects of classical algebraic geometry. We refer to [64, Ch. 3].

Following this philosophy, we define the class of noncommutative curves which we will study in this paper by the axioms (NC 1) to (NC 5) below; the condition (NC 6) will follow from the others.

The axioms. A noncommutative curve $\mathbb{X}$ is given by a category $\mathcal{H}$ which is regarded as the category $\operatorname{coh}\mathbb{X}$ of *coherent sheaves* over $\mathbb{X}$. Formally it behaves like a category of coherent sheaves over a (commutative) regular projective curve over a field k (we refer to [39]):

(NC 1) $\mathcal{H}$ is small, connected, abelian and every object in $\mathcal{H}$ is noetherian;

(NC 2) $\mathcal{H}$ is a k-category with finite-dimensional Hom- and Ext-spaces;

(NC 3) There is an autoequivalence τ on $\mathcal{H}$, called Auslander-Reiten translation, such that Serre duality

$$\operatorname{Ext}^1_{\mathcal{H}}(X, Y) = \operatorname{D}\operatorname{Hom}_{\mathcal{H}}(Y, \tau X)$$

holds, where $\operatorname{D} = \operatorname{Hom}_k(-, k)$. (In particular $\mathcal{H}$ is then hereditary.)

(NC 4) $\mathcal{H}$ contains an object of infinite length.

Splitting of coherent sheaves. Assume $\mathcal{H}$ satisfies (NC 1) to (NC 4). The following rough picture of the category $\mathcal{H}$ is very useful ([47, Prop. 1.1]). Every indecomposable coherent sheaf E is either of finite length, or it is *torsionfree*, that is, it does not contain any simple sheaf; in the latter case E is also called a (vector) bundle. We thus write

$$\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0, \tag{2.1}$$

with $\mathcal{H}_+ = \operatorname{vect}\mathbb{X}$ the class of vector bundles and $\mathcal{H}_0$ the class of sheaves of finite length; we have $\operatorname{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0$. Decomposing $\mathcal{H}_0$ in its connected components we have

$$\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x,$$

where $\mathbb{X}$ is an index set (explaining the terminology $\mathcal{H} = \operatorname{coh}\mathbb{X}$) and every $\mathcal{U}_x$ is a connected uniserial length category.

Weighted noncommutative regular projective curves. Assume that $\mathcal{H}$ is a k-category satisfying properties (NC 1) to (NC 4) and the following additional condition.

(NC 5) $\mathbb{X}$ consists of infinitely many points.

Then we call $\mathbb{X}$ (or $\mathcal{H}$) a *weighted* (or *orbifold*) *noncommutative regular projective curve* over k. "Regular" can be replaced by "smooth" if k is a perfect field; we refer to [39, Sec. 7]. We refer also to [47]; we excluded certain degenerate cases described therein by our additional axiom (NC 5). It is shown in [39] that a weighted noncommutative regular projective curve $\mathbb{X}$ satisfies automatically also the following condition.

(NC 6) For all points $x \in \mathbb{X}$ there are (up to isomorphism) precisely $p(x) < \infty$ simple objects in $\mathcal{U}_x$, and for almost all x we have $p(x) = 1$.

The numbers $p(x)$ with $p(x) > 1$ are called the *weights*.

The "classical" case $\mathcal{H} = \operatorname{coh} X$ with X a regular projective curve is included in this setting. This classical case is extended into two directions: (1) curves with a noncommutative function field $k(\mathbb{X})$ are allowed; here $k(\mathbb{X})$ is a skew field which is finite dimensional over its centre, which has the form $k(X)$ for a regular projective curve X; (2) additionally (a finite number of) weights are allowed.

Weighted noncommutative regular projective curves are *noncommutative smooth proper curves* in the sense of Stafford and van den Bergh [62, Sec. 7] (where k is assumed to be algebraically closed); these categories were classified by Reiten and van den Bergh [52]. Indeed, our axioms (NC 1), (NC 2), (NC 3) are in accordance with the notion in [62]. By assuming additionally (NC 4) we avoid for instance categories which are just tubes.

Genus zero. We consider also the following condition.

(g-0) $\mathcal{H}$ admits a tilting object.

It is shown in [44] that then $\mathcal{H}$ even contains a torsionfree tilting object T_{can} whose endomorphism algebra is canonical, in the sense of [57]. We call such a tilting object *canonical*, or, by considering the full subcategory formed by the indecomposable summands of T_{can}, *canonical configuration*, cf. 5.11. We recall that $T \in \mathcal{H}$ is called *tilting*, if $\operatorname{Ext}^1(T, T) = 0$, and if for all $X \in \mathcal{H}$ we have $X = 0$ whenever $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. (This notion will be later generalized to quasicoherent sheaves.) If $\mathcal{H}$ satisfies (NC 1) to (NC 4) and (g-0), then we say that $\mathbb{X}$ is a *noncommutative curve of genus zero*; the condition (NC 5) is then automatically satisfied, we refer to [38]. The weighted projective lines, defined by Geigle-Lenzing [27], are special cases of noncommutative curves of genus zero. We remark that in the classical case $\mathcal{H} = \mathrm{coh}(X)$, where X is a (commutative) regular projective curve with structure sheaf $\mathcal{O}$, the condition (g-0) is equivalent to $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$, which means that X is of (geometric) genus zero in the classical sense; cf. Remark 2.2.

The Grothendieck group and the Euler form. The Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$ is defined as the factor of the free abelian group on the isomorphism classes on objects of $\mathcal{H}$ modulo the additivity relations on short exact sequences. We write $[X]$ for the class of a coherent sheaf X in the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$. The Grothendieck group is equipped with the Euler form, which is defined on classes of objects X, Y in $\mathcal{H}$ by

$$\langle [X], [Y] \rangle = \dim_k \mathrm{Hom}(X, Y) - \dim_k \mathrm{Ext}^1(X, Y).$$

We will usually write $\langle X, Y \rangle$, without the brackets.
In case $\mathbb{X}$ is of genus zero, $\mathcal{H}$ admits a tilting object whose endomorphism ring is a finite dimensional algebra, and thus the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of $\mathcal{H}$ is finitely generated free abelian. (From this it follows more directly that every $\mathbb{X}$ of genus zero satisfies (NC 6).)

In the following, if not otherwise specified, let $\mathcal{H} = \mathrm{coh}\,\mathbb{X}$ be a weighted noncommutative regular projective curve.

Homogeneous and exceptional tubes. For every $x \in \mathbb{X}$ the connected uniserial length categories $\mathcal{U}_x$ are called *tubes*. The number $p(x) \geq 1$ is called the *rank* of the tube $\mathcal{U}_x$. Tubes of rank 1 are called *homogeneous*, those with $p(x) > 1$ *exceptional*. We say that a point x is homogeneous (resp. exceptional) if so is the corresponding tube $\mathcal{U}_x$. If S_x is a simple sheaf in $\mathcal{U}_x$, then $\mathrm{Ext}^1(S_x, S_x) \neq 0$ in the homogeneous case, and $\mathrm{Ext}^1(S_x, S_x) = 0$ in the exceptional case. More generally, a coherent sheaf E is called *exceptional*, if E is indecomposable and E has no self-extensions. It follows then by an argument of Happel and Ringel that $\mathrm{End}(E)$ is a skew field; we refer to [50, 3.2.3]. It is well-known and easy to see that the exceptional sheaves in $\mathcal{U}_x$ are just those indecomposables of length $\leq p(x) - 1$ (which exist only for $p(x) > 1$). In particular there are only finitely many exceptional sheaves of finite length.
If $p = p(x)$, then all simple sheaves in $\mathcal{U}_x$ are given (up to isomorphism) by the Auslander-Reiten orbit $S_x = \tau^p S_x, \tau S_x, \ldots, \tau^{p-1} S_x$.
For the terminology on wings and branches in exceptional tubes we refer to Section 4.7.

Non-weighted curves. By a *(non-weighted) noncommutative regular projective curve* over the field k we mean a category $\mathcal{H} = \mathrm{coh}\,\mathbb{X}$ satisfying axioms (NC 1) to (NC 5), and additionally

(NC 6') $\mathrm{Ext}^1(S, S) \neq 0$ (equivalently: $\tau S \simeq S$) holds for all simple objects $S \in \mathcal{H}$.

This condition means that all tubes are homogeneous, that is, $p(x) = 1$ for all $x \in \mathbb{X}$; therefore these curves are also called homogeneous in [38]. For a detailed treatment of this setting we refer to [39]. We stress that thus, by abuse of language, non-weighted curves are special cases of weighted curves.

GROTHENDIECK CATEGORIES WITH FINITENESS CONDITIONS. Let us briefly recall some notions we will need in the sequel. An abelian category $\mathcal{A}$ is a *Grothendieck category*, if it is cocomplete, has a generator, and direct limits are exact. Every Grothendieck category is also complete and has an injective cogenerator. A Grothendieck category is called *locally coherent* (resp. *locally noetherian*, resp. *locally finite*) if it admits a system of generators which are coherent (resp. noetherian, resp. of finite length). In this case every object in $\mathcal{A}$ is a direct limit of coherent (resp. noetherian, resp. finite length) objects. If $\mathcal{A}$ is locally coherent then the coherent and the finitely presented objects coincide, and the full subcategory $\operatorname{fp}(\mathcal{A})$ of finitely presented objects is abelian. For more details on Grothendieck categories we refer to [26, 63, 31, 34].

THE SERRE CONSTRUCTION. $\mathcal{H} = \operatorname{coh}\mathbb{X}$ is a noncommutative noetherian projective scheme in the sense of Artin-Zhang [11] and satisfies Serre's theorem. This means that there is a positively H-graded (not necessarily commutative) noetherian ring R (with $(H, \leq)$ an ordered abelian group of rank one) such that

$$\mathcal{H} = \frac{\operatorname{mod}^H(R)}{\operatorname{mod}^H_0(R)}, \tag{2.2}$$

the quotient category of the category of finitely generated H-graded modules modulo the Serre subcategory of those modules which are finite-dimensional over k. (We refer to [38, Prop. 6.2.1], [39] and [52, Lem. IV.4.1].) With this description we can define $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ as the quotient category

$$\vec{\mathcal{H}} = \frac{\operatorname{Mod}^H(R)}{\operatorname{Mod}^H_0(R)}, \tag{2.3}$$

where $\operatorname{Mod}^H_0(R)$ denotes the localizing subcategory of $\operatorname{Mod}^H(R)$ of all H-graded torsion, that is, locally finite-dimensional, modules. The category $\vec{\mathcal{H}}$ is hereditary abelian, and a locally noetherian Grothendieck category; every object in $\vec{\mathcal{H}}$ is a direct limit of objects in $\mathcal{H}$ (therefore the symbol $\vec{\mathcal{H}}$). The full abelian subcategory $\mathcal{H}$ consists of the coherent (= finitely presented = noetherian) objects in $\vec{\mathcal{H}}$, we also write $\mathcal{H} = \operatorname{fp}(\vec{\mathcal{H}})$. Every indecomposable coherent sheaf has a local endomorphism ring, and $\mathcal{H}$ is a Krull-Schmidt category.
We remark that $\vec{\mathcal{H}}$ can, by [26, II. Thm. 1], also be recovered from its subcategory $\mathcal{H}$ of noetherian objects as the category of left-exact (covariant) k-functors from $\mathcal{H}^{\mathrm{op}}$ to $\operatorname{Mod}(k)$. We also note that our categories $\mathcal{H}$ (resp. $\vec{\mathcal{H}}$) can be described alternatively as categories $\operatorname{coh}(\mathcal{A})$ (resp. $\operatorname{Qcoh}(\mathcal{A})$) of coherent (resp. quasicoherent) modules over certain hereditary orders $\mathcal{A}$; we refer to [39, Thm. 7.11].

Prüfer sheaves. Let E be an indecomposable sheaf in a tube $\mathcal{U}_x$. By the *ray* starting in E we mean the (infinite) sequence of all the indecomposable sheaves in $\mathcal{U}_x$, which contain E as a subsheaf. The corresponding monomorphisms (inclusions) form a direct system. If the socle of E is the simple S, then the corresponding direct limit of this system is the *Prüfer sheaf* $S[\infty]$. In other words, $S[\infty]$ is the union of all indecomposable sheaves of finite length containing S (or E). Dually we define *corays* ending in E as the sequence of all indecomposable sheaves in $\mathcal{U}_x$ admitting E as a factor.
If S is a simple sheaf, then we denote by $S[n]$ the (unique) indecomposable sheaf of length n with socle S. Thus, the collection $S[n]$ ($n \geq 1$) forms the ray starting in S, and their union is $S[\infty]$. The Prüfer sheaves form an important class of indecomposable (we refer to [54]), quasicoherent, non-coherent sheaves.

Rank. Line bundles. Let $\mathcal{H}/\mathcal{H}_0$ be the quotient category of $\mathcal{H}$ modulo the Serre category of sheaves of finite length, let $\pi\colon \mathcal{H} \to \mathcal{H}/\mathcal{H}_0$ the quotient functor, which is exact. The abelian category $\mathcal{H}/\mathcal{H}_0$ is, by [47, Prop. 3.4], of the form $\mathcal{H}/\mathcal{H}_0 \simeq \operatorname{mod}(k(\mathcal{H}))$ for a unique skew field $k(\mathcal{H})$, called the *function field* of $\mathcal{H}$ (or $\mathbb{X}$). Then $\vec{\mathcal{H}}/\vec{\mathcal{H}_0} = \operatorname{Mod}(k(\mathcal{H}))$. The $k(\mathcal{H})$-dimension on $\mathcal{H}/\mathcal{H}_0$ induces the *rank* function on $\mathcal{H}$ by the formula $\operatorname{rk}(F) := \dim_{k(\mathcal{H})}(\pi F)$. It is additive on short exact sequences and thus induces a linear form $\operatorname{rk}\colon \operatorname{K}_0(\mathcal{H}) \to \mathbb{Z}$. The objects in $\mathcal{H}_0$ are just the objects of rank zero, every non-zero vector bundle has a positive rank, [47, Prop. 1.2]. The vector bundles of rank one are called *line bundles*. A line bundle L is called *special* if for each $x \in \mathbb{X}$ there is (up to isomorphism) *precisely one* simple sheaf S_x concentrated at x with

$$\operatorname{Ext}^1(S_x, L) \neq 0. \tag{2.4}$$

Special line bundles always exist, cf. [39, Prop. 1.1].
Furthermore, every non-zero morphism from a line bundle L' to a vector bundle is a monomorphism, and $\operatorname{End}(L')$ is a skew field, [47, Lem. 1.3]. Every vector bundle has a line bundle filtration, [47, Prop. 1.6].

The sheaf of rational functions. The *sheaf $\mathcal{K}$ of rational functions* is the injective envelope of any line bundle L in the category $\vec{\mathcal{H}}$; this does not depend on the chosen line bundle. Besides the Prüfer sheaves, this is another very important quasicoherent, non-coherent sheaf. It is torsionfree by [36, Lem. 14], and it is a generic sheaf in the sense of [41]; its endomorphism ring is the function field, $\operatorname{End}_{\vec{\mathcal{H}}}(\mathcal{K}) \simeq \operatorname{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L) \simeq k(\mathcal{H})$.

The derived category. Since $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$ is a hereditary category, the derived category

$$\mathcal{D} = \mathcal{D}(\vec{\mathcal{H}}) = \operatorname{Add}\Big(\bigvee_{n\in\mathbb{Z}} \vec{\mathcal{H}}[n]\Big) \tag{2.5}$$

is the repetitive category of $\vec{\mathcal{H}}$. This means: Every object in $\mathcal{D}$ can be written as $\bigoplus_{i\in I} X_i[i]$ for a subset $I \subseteq \mathbb{Z}$ and $X_i \in \vec{\mathcal{H}}$ for all i, and for all objects

X, $Y \in \vec{\mathcal{H}}$ and all integers n, m we have

$$\operatorname{Ext}^{n-m}_{\vec{\mathcal{H}}}(X,Y)=\operatorname{Hom}_{\mathcal{D}}(X[m],Y[n]).$$

The *bounded* derived category $\mathcal{D}^b=\mathcal{D}^b(\vec{\mathcal{H}})$ is the full subcategory of $\mathcal{D}$ with objects those complexes which have bounded cohomology. It has a similar repetitive structure as in (2.5), where Add is replaced by add and the subset I in $\mathbb{Z}$ as above is finite.

Generalized Serre duality. It follows easily from [35, Thm. 4.4] that on $\vec{\mathcal{H}}$ we have Serre duality in the following sense. Let τ be the Auslander-Reiten translation on $\mathcal{H}$ and τ^- its (quasi-) inverse. For all $X\in\mathcal{H}$ and all $Y\in\vec{\mathcal{H}}$ we have

$$\mathrm{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(X,Y)=\operatorname{Hom}_{\vec{\mathcal{H}}}(Y,\tau X)\quad\text{and}\quad\operatorname{Ext}^1_{\vec{\mathcal{H}}}(Y,X)=\mathrm{D}\operatorname{Hom}_{\vec{\mathcal{H}}}(\tau^-X,Y),$$

with D denoting the duality $\operatorname{Hom}_k(-,k)$.

Purity. The notion of purity is of great importance in our setting. For details we refer to [51, Ch. 5].

(1) A short exact sequence $\eta\colon 0\to A\overset{\alpha}{\to}B\overset{\beta}{\to}C\to 0$ in $\vec{\mathcal{H}}$ is called *pure-exact*, if for every $F\in\mathcal{H}$ (that is, F finitely presented) the induced sequence $\operatorname{Hom}(F,\eta):0\to\operatorname{Hom}(F,A)\to\operatorname{Hom}(F,B)\to\operatorname{Hom}(F,C)\to 0$ is exact. In this case α (resp. β) is called a pure monomorphism (resp. pure epimorphism), and A a pure subobject of B.

(2) An object $E\in\vec{\mathcal{H}}$ is called *pure-injective* if for every pure-exact sequence $0\to A\to B\to C\to 0$ the induced sequence $0\to\operatorname{Hom}(C,E)\to\operatorname{Hom}(B,E)\to\operatorname{Hom}(A,E)\to 0$ is exact.

(3) An object $E\in\vec{\mathcal{H}}$ is called Σ-*pure-injective* if the coproduct $E^{(I)}$ is pure-injective for every set I.

Lemma 2.1. *Every coherent sheaf $F\in\mathcal{H}$ is pure-injective.*

Proof. If μ is a pure-exact sequence in $\vec{\mathcal{H}}$, then $\operatorname{Hom}_{\vec{\mathcal{H}}}(\tau^-F,\mu)$ is exact. Since $\operatorname{Ext}^2_{\vec{\mathcal{H}}}(-,-)$ vanishes, this amounts to exactness of $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(\tau^-F,\mu)$, and hence of $\mathrm{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(\tau^-F,\mu)$, which in turn is equivalent to exactness of $\operatorname{Hom}_{\vec{\mathcal{H}}}(\mu,F)$ by Serre duality. This gives the claim. □

Almost split sequences. Since the objects of $\mathcal{H}$ are pure-injective, it follows directly from [35, Prop. 3.2] that the category $\mathcal{H}$ has almost split sequences which also satisfy the almost split properties in the larger category $\vec{\mathcal{H}}$; more precisely: for every indecomposable $Z\in\mathcal{H}$ there is a non-split short exact sequence

$$0\to X\xrightarrow{\alpha}Y\xrightarrow{\beta}Z\to 0$$

in $\mathcal{H}$ with $X=\tau Z$ indecomposable such that for every object $Z'\in\vec{\mathcal{H}}$ any morphism $Z'\to Z$ that is not a retraction factors through β (and equivalently, for every object $X'\in\vec{\mathcal{H}}$ any morphism $X\to X'$ that is not a section factors through α).

Hereditary orders. For the details on notions and results in this and the following subsections we refer to [39]. Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over k. Let $\bar{p}$ be the least common multiple of the weights $p(x)$. The centre of the function field $k(\mathcal{H})$ is of the form $k(X)$, the function field of a unique regular projective curve X over k. We call X the *centre curve* of $\mathcal{H}$. The dimension $[k(\mathcal{H}) : k(X)]$ is finite, a square number, which we denote by $s(\mathcal{H})^2$. We call $s(\mathcal{H})$ the *skewness* of $\mathcal{H}$ (or $\mathbb{X}$). The (closed) points of X are in one-to-one correspondence to the (closed) points of $\mathbb{X}$. Let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf of X. For every $x \in X$ we have the local rings $(\mathcal{O}_x, \mathfrak{m}_x)$, and the residue class field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. For all $x \in \mathbb{X}$ there are the ramification indices $e_\tau(x) \geq 1$. There exist only finitely many points $x \in \mathbb{X}$ with $p(x)e_\tau(x) > 1$. By a result of Reiten and van den Bergh [52], [39, Thm. 7.11] the category $\mathcal{H}$ can be realized as $\mathcal{H} = \operatorname{coh}(\mathcal{A})$, the category of coherent $\mathcal{A}$-modules, where $\mathcal{A}$ is a torsionfree coherent sheaf of hereditary $\mathcal{O}$-orders in a full matrix algebra over $k(\mathcal{H})$. Moreover, $\vec{\mathcal{H}} = \operatorname{Qcoh}(\mathcal{A})$.

If $\mathbb{X}$ is weighted then there is an underlying non-weighted curve $\mathbb{X}_{nw}$, which follows from (NC 6) by perpendicular calculus [28], cf. [39, Prop. 1.1]. We have $\bar{p} = 1$ (that is, $\mathbb{X} = \mathbb{X}_{nw}$) if and only if $\mathcal{A}$ is a maximal order.

Structure sheaf. We now define the *structure sheaf* L of $\mathcal{H} = \operatorname{coh}(\mathcal{A})$ to be a line bundle with the following properties: in the non-weighted case ($\bar{p} = 1$) we set $L_{\mathcal{A}} = \mathcal{A}_{\mathcal{A}}$, and in the weighted case ($\bar{p} > 1$) we let L be a special line bundle corresponding to the structure sheaf of the underlying non-weighted curve via perpendicular calculus, cf. [39, Prop. 1.1]. In the following we will always consider the pair $(\mathcal{H}, L)$, that is, $\mathcal{H}$ equipped with structure sheaf L. We recall that $k(\mathcal{H}) = \operatorname{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L)$.

Orbifold Euler characteristic and representation type. One defines the *average Euler form* $\langle\langle E, F\rangle\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, F\rangle$, and then the normalized *orbifold Euler characteristic* of $\mathcal{H}$ by $\chi'_{orb}(\mathbb{X}) = \frac{1}{s(\mathcal{H})^2 \bar{p}^2} \langle\langle L, L\rangle\rangle$. If k is perfect, one has a nice formula to compute the Euler characteristic:

$$\chi'_{orb}(\mathbb{X}) = \chi'(X) - \frac{1}{2} \sum_x \Big(1 - \frac{1}{p(x)e_\tau(x)}\Big)[k(x) : k]. \tag{2.6}$$

Here, $\chi'(X) = \dim_k \operatorname{Hom}_X(\mathcal{O}, \mathcal{O}) - \dim_k \operatorname{Ext}^1_X(\mathcal{O}, \mathcal{O})$ is the normalized Euler characteristic of the centre curve X (or of $\operatorname{coh}(X)$; cf. also [39, Rem. 13.11 (1)]). If k is not perfect, there is still a similar formula, we refer to [39, Cor. 13.13].

The orbifold Euler characteristic determines the representation type of the category $\mathcal{H} = \operatorname{coh} \mathbb{X}$ (see also Theorem 2.3 below):

- $\mathbb{X}$ is domestic: $\chi'_{orb}(\mathbb{X}) > 0$
- $\mathbb{X}$ is elliptic: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ non-weighted ($\bar{p} = 1$)
- $\mathbb{X}$ is tubular: $\chi'_{orb}(\mathbb{X}) = 0$, and $\mathbb{X}$ properly weighted ($\bar{p} > 1$)
- $\mathbb{X}$ is wild: $\chi'_{orb}(\mathbb{X}) < 0$.

In this paper we will prove some general results for all representation types, and we will obtain finer and complete classification results in the cases of non-negative orbifold Euler characteristic.

Remark 2.2. (1) If $\mathbb{X}$ is non-weighted with structure sheaf L, then we call the number $g(\mathbb{X}) = [\operatorname{Ext}^1(L,L) : \operatorname{End}(L)]$ the *genus* of $\mathbb{X}$. The condition $g(\mathbb{X}) = 1$ is equivalent to the elliptic case. In case $g(\mathbb{X}) \geq 1$ there does not exist any exceptional object in $\mathcal{H}$; this follows readily from the Riemann-Roch formula [39, Prop. 9.1]. Now it follows with [38, 0.5.4] that the condition $g(\mathbb{X}) = 0$ is equivalent to condition (g-0); actually, in this case there is a tilting bundle of the form $T = L \oplus \overline{L}$ with $\overline{L}$ indecomposable of rank one or two, and $\operatorname{End}(T)$ is a tame hereditary k-algebra.

(2) If $\mathbb{X}$ is weighted then $\mathcal{H} = \operatorname{coh}\mathbb{X}$ contains a tilting bundle (that is, $\mathcal{H}$ satisfies (g-0)) if and only if $g(\mathbb{X}_{nw}) = 0$. In other words, $\mathcal{H}$ satisfies (g-0) if the genus, in the non-orbifold sense, is zero. This follows from (1) with [42, Thm. 4.3].

Degree and slope. We define the *degree* function $\deg\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$, by

$$\deg(F) = \frac{1}{\kappa\varepsilon}\langle\!\langle L, F\rangle\!\rangle - \frac{1}{\kappa\varepsilon}\langle\!\langle L, L\rangle\!\rangle\operatorname{rk}(F), \tag{2.7}$$

with $\kappa = \dim_k \operatorname{End}(L)$ and ε the positive integer such that the resulting linear form $\mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ becomes surjective. We have $\deg(L) = 0$, and deg is positive and τ-invariant on sheaves of finite length. The *slope* of a non-zero coherent sheaf F is defined as $\mu(F) = \deg(F)/\operatorname{rk}(F) \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Moreover, F is called *stable* (*semistable*, resp.) if for every non-zero proper subsheaf F' of F we have $\mu(F') < \mu(F)$ (resp. $\mu(F') \leq \mu(F)$).
More details on these numerical invariants will be given in 5.10.

Stability. The stability notions are very useful for the classification of vector bundles (we refer to [27, Prop. 5.5], [47], [38, Prop. 8.1.6], [39]):

Theorem 2.3. *Let $\mathcal{H} = \operatorname{coh}\mathbb{X}$ be a weighted noncommutative regular projective curve over k.*

(1) *If $\chi'_{orb}(\mathbb{X}) > 0$ (domestic type), then every indecomposable vector bundle is stable and exceptional. Moreover,* $\operatorname{coh}\mathbb{X}$ *admits a tilting bundle.*

(2) *If $\chi'_{orb}(\mathbb{X}) = 0$ (elliptic or tubular type), then every indecomposable coherent sheaf is semistable. If $\mathbb{X}$ is tubular (that is, $\bar{p} > 1$), then* $\operatorname{coh}\mathbb{X}$ *admits a tilting bundle. If $\mathbb{X}$ is elliptic (that is, $\bar{p} = 1$) then every indecomposable coherent sheaf E is non-exceptional and satisfies $\tau E \simeq E$.*

(3) *If $\chi'_{orb}(\mathbb{X}) < 0$, then every Auslander-Reiten component in $\mathcal{H}_+ = \operatorname{vect}\mathbb{X}$ is of type $\mathbb{Z}A_\infty$, and $\mathcal{H}$ is of wild representation type.* ($\operatorname{coh}\mathbb{X}$ *may or may not satisfy (g-0).)* □

Orthogonal and generated classes. Let $\mathcal{X}$ be a class of objects in $\vec{\mathcal{H}}$. We will use the following notation:

$$\begin{aligned} \mathcal{X}^{\perp_0} &= \{F \in \vec{\mathcal{H}} \mid \operatorname{Hom}(\mathcal{X}, F) = 0\}, \quad \mathcal{X}^{\perp_1} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(\mathcal{X}, F) = 0\}, \\ {}^{\perp_0}\mathcal{X} &= \{F \in \vec{\mathcal{H}} \mid \operatorname{Hom}(F, \mathcal{X}) = 0\}, \quad {}^{\perp_1}\mathcal{X} = \{F \in \vec{\mathcal{H}} \mid \operatorname{Ext}^1(F, \mathcal{X}) = 0\}, \\ \mathcal{X}^{\perp} &= \mathcal{X}^{\perp_0} \cap \mathcal{X}^{\perp_1}, \quad {}^{\perp}\mathcal{X} = {}^{\perp_0}\mathcal{X} \cap {}^{\perp_1}\mathcal{X}. \end{aligned}$$

Following [28] we call ${}^{\perp}\mathcal{X}$ (resp. $\mathcal{X}^{\perp}$) the *left-perpendicular* (resp. *right-perpendicular*) category of $\mathcal{X}$. By $\operatorname{Add}(\mathcal{X})$ (resp. $\operatorname{add}(\mathcal{X})$) we denote the class of all direct summands of direct sums of the form $\bigoplus_{i \in I} X_i$, where I is any set (resp. finite set) and $X_i \in \mathcal{X}$ for all i. By $\operatorname{Gen}(\mathcal{X})$ we denote the class of all objects Y *generated by* $\mathcal{X}$, that is, such that there is an epimorphism $X \to Y$ with $X \in \operatorname{Add}(\mathcal{X})$ (and similarly $\operatorname{gen}(\mathcal{X})$ with $\operatorname{add}(\mathcal{X})$).

Let $(I, \leq)$ be an ordered set and $\mathcal{X}_i$ classes of objects for all $i \in I$, in any additive category. We write $\bigvee_{i \in I} \mathcal{X}_i$ for $\operatorname{add}(\bigcup_{i \in I} \mathcal{X}_i)$ if additionally $\operatorname{Hom}(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $i < j$ is satisfied. In particular, notation like $\mathcal{X}_1 \vee \mathcal{X}_2$ and $\mathcal{X}_1 \vee \mathcal{X}_2 \vee \mathcal{X}_3$ makes sense (where $1 < 2 < 3$).

The following induction technique will be very important.

Reduction of weights. Let S be an exceptional simple sheaf. In other words, S lies on the mouth of a tube, with index x, of rank $p(x) > 1$. Then the right perpendicular category $S^{\perp}$ is equivalent to $\operatorname{Qcoh} \mathbb{X}'$, where $\mathbb{X}'$ is a curve such that the rank $p'(x)$ of the tube of index x is $p'(x) = p(x) - 1$ and all other weights and all the numbers $e_{\tau}(y)$ are preserved. We refer to [28] for details. From the formula (2.6) (and [39, Cor. 13.13], which holds over any field) of the orbifold Euler characteristic we see $\chi'(\mathbb{X}') > \chi'(\mathbb{X})$, and we conclude readily that $\mathbb{X}'$ is of domestic type if $\mathbb{X}$ is tubular or domestic. By similar reasons, $\mathbb{X}'$ is of genus zero if so is $\mathbb{X}$.

Tubular shifts. If $x \in \mathbb{X}$ is a point of weight $p(x) \geq 1$, then there is an autoequivalence σ_x of $\mathcal{H}$ (which extends to an autoequivalence of $\vec{\mathcal{H}}$), called the *tubular shift* associated with x. We refer to [44, (S10)] and [38, Sec. 0.4] for more details. These are generalizations of the tubular mutations [49], and they are also related to the Seidel-Thomas twists [59]; in case $p(x) = 1$ the tubular shift σ_x actually agrees with the Seidel-Thomas twist T_E with $E = S_x$ the simple sheaf at x, since this is spherical in the sense that $\operatorname{Ext}^1(E, E) \simeq \operatorname{End}(E)$ is a finite dimensional skew field (in [59] only the case $\operatorname{End}(S_x) = k$ is considered). We just recall that for every vector bundle E there is a universal exact sequence

$$0 \to E \to \sigma_x(E) \to E_x \to 0, \tag{2.8}$$

where $E_x = \bigoplus_{j=0}^{p(x)-1} \operatorname{Ext}^1(\tau^j S_x, E) \otimes \tau^j S_x \in \mathcal{U}_x$ with the tensor product taken over the skew field $\operatorname{End}(S_x)$. We also write

$$\sigma_x(E) = E(x) \quad \text{and} \quad (\sigma_x)^n(E) = E(nx),$$

and we will use the more handy notation

$$E_x = \bigoplus_{j=0}^{p(x)-1} (\tau^j S_x)^{e(j,x,E)}$$

with the exponents given by the multiplicities

$$e(j,x,E) = [\mathrm{Ext}^1(\tau^j S_x, E) : \mathrm{End}(S_x)],$$

the $\mathrm{End}(S_x)$-dimension of $\mathrm{Ext}^1(\tau^j S_x, E)$. In the particular case when $E = L$ is the structure sheaf (which is a special line bundle), and S_x is such that $\mathrm{Hom}(L, S_x) \neq 0$, we have $e(j,x,L) = e(x)$ for $j = p(x) - 1$ and $= 0$ otherwise.

Tilting sheaves. Let $\vec{\mathcal{H}}$ be a Grothendieck category, for instance $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.

Definition 2.4. An object $T \in \vec{\mathcal{H}}$ is called a *tilting object* or *tilting sheaf* if $\mathrm{Gen}(T) = T^{\perp_1}$. Then $\mathrm{Gen}(T)$ is called the associated *tilting class*.

This definition is inspired by [23, Def. 2.3], but we dispense with the self-smallness assumption made there. In a module category, we thus recover the definition of a *tilting module* (of projective dimension one) from [25].

We recall that the *projective dimension* $\mathrm{pd}(X)$ of an object X in $\vec{\mathcal{H}}$ is defined to be the smallest integer $n \geq -1$ such that $\mathrm{Ext}^{n+1}(X, -) = 0$ holds, and ∞, if no such n exists. Here, Ext-groups are defined via injective resolutions.

Lemma 2.5 ([23, Prop. 2.2]). *An object $T \in \vec{\mathcal{H}}$ is tilting if and only if the following conditions are satisfied:*

(TS0) *T has projective dimension* $\mathrm{pd}(T) \leq 1$.

(TS1) $\mathrm{Ext}^1(T, T^{(I)}) = 0$ *for every cardinal I.*

(TS2) $T^{\perp} = 0$, *that is: if $X \in \vec{\mathcal{H}}$ satisfies* $\mathrm{Hom}(T,X) = 0 = \mathrm{Ext}^1(T,X)$, *then* $X = 0$.

We will mostly consider hereditary categories $\vec{\mathcal{H}}$ where (TS0) is automatically satisfied. In case $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ with $\mathbb{X}$ of genus zero, we will also consider the following condition, where $T_{\mathrm{can}} \in \mathcal{H}$ is a tilting bundle such that $\mathrm{End}(T_{\mathrm{can}}) = \Lambda$ is a canonical algebra, that is, T_{can} is a fixed canonical configuration.

(TS3) There are an autoequivalence σ on $\mathcal{H}$ and an exact sequence

$$0 \to \sigma(T_{\mathrm{can}}) \to T_0 \to T_1 \to 0$$

such that $\mathrm{Add}(T_0 \oplus T_1) = \mathrm{Add}(T)$; if this can be realized with the additional property $\mathrm{Hom}(T_1, T_0) = 0$, then we say that T satisfies condition (TS3+).

Since $\sigma(T_{\mathrm{can}})$ is a tilting bundle, (TS3) implies (TS2). As it will turn out, in case of genus zero, all tilting sheaves we construct will satisfy (TS3), and some will even satisfy (TS3+), see Example 4.22, Corollary 8.8, and Section 10.

Let $\vec{\mathcal{H}}$ additionally be locally coherent with $\mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$.

Lemma 2.6. *Let $T \in \vec{\mathcal{H}}$ be tilting.*

(1) $\mathrm{Gen}(T) = \mathrm{Pres}(T)$*, the class of objects in* $\vec{\mathcal{H}}$ *which are cokernels of morphisms of the form* $T^{(J)} \to T^{(I)}$.

(2) $T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) = \mathrm{Add}(T)$.

(3) *If* $X \in \mathcal{H}$ *is coherent having a local endomorphism ring and* $X \in \mathrm{Add}(T)$*, then* X *is a direct summand of* T.

Proof. (1) The same proof as in [25, Lemma 1.2] works here.
(2) Is an easy consequence of (1).
(3) Since X is coherent, we get $X \in \mathrm{add}(T)$. Since X has local endomorphism ring, the claim follows. □

Definition 2.7. Two tilting objects $T, T' \in \vec{\mathcal{H}}$ are *equivalent*, if they generate the same tilting class. This is equivalent to $\mathrm{Add}(T) = \mathrm{Add}(T')$. A tilting sheaf $T \in \vec{\mathcal{H}}$ is called *large* if it is not equivalent to a coherent tilting sheaf.

For the rest of this section we assume that $\mathbb{X}$ *is of genus zero and* $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ *with a fixed special line bundle* L.

Tilting bundles and concealed-canonical algebras. We fix a tilting bundle $T_{\mathrm{cc}} \in \mathcal{H}$. Its endomorphism ring Σ is a concealed-canonical k-algebra. Every concealed-canonical algebra arises in this way, we refer to [44]. Especially for $T_{\mathrm{cc}} = T_{\mathrm{can}}$, a canonical configuration, we get a canonical algebra. We remark that T_{cc} is in particular a noetherian tilting object in $\vec{\mathcal{H}}$. It is well-known that T_{cc} is a (compact) generator of $\mathcal{D}$ inducing an equivalence

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{D}}(T_{\mathrm{cc}}, -)\colon \mathcal{D}(\mathrm{Qcoh}\,\mathbb{X}) \longrightarrow \mathcal{D}(\mathrm{Mod}\,\Sigma)$$

of triangulated categories (cf. [18, Prop. 1.5] and [33, Thm. 8.5]). Via this equivalence the module category $\mathrm{Mod}\,\Sigma$ can be identified (like in [43, Thm. 3.2] and [41]) with the full subcategory $\mathrm{Add}(\mathcal{T}_{\mathrm{cc}} \vee \mathcal{F}_{\mathrm{cc}}[1])$ of $\mathcal{D}$, where $(\mathcal{T}_{\mathrm{cc}}, \mathcal{F}_{\mathrm{cc}})$ is the torsion pair in $\vec{\mathcal{H}}$ given by $\mathcal{T}_{\mathrm{cc}} = \mathrm{Gen}(T_{\mathrm{cc}}) = {T_{\mathrm{cc}}}^{\perp_1}$ and $\mathcal{F}_{\mathrm{cc}} = {T_{\mathrm{cc}}}^{\perp_0}$. This torsion pair induces a split torsion pair $(\mathcal{Q}, \mathcal{C}) = (\mathcal{F}_{\mathrm{cc}}[1], \mathcal{T}_{\mathrm{cc}})$ in $\mathrm{Mod}\,\Sigma$. Moreover, $\mathrm{mod}\,\Sigma = (\mathcal{T}_{\mathrm{cc}} \cap \mathcal{H}) \vee (\mathcal{F}_{\mathrm{cc}} \cap \mathcal{H})[1]$.

Correspondences between tilting objects. Following [16], we call a tilting sheaf $T \in \vec{\mathcal{H}}$ of *finite type* if the tilting class $T^{\perp_1}$ is determined by a class of finitely presented objects $\mathscr{S} \subseteq \mathcal{H}$ such that $T^{\perp_1} = \mathscr{S}^{\perp_1}$. If T is of finite type, then $\mathscr{S} := {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is the largest such class. We are now going to see that all tilting sheaves lying in $\mathcal{T}_{\mathrm{cc}}$ are of finite type.
We call an object T in the triangulated category $\mathcal{D}^b = \mathcal{D}^b(\mathrm{Qcoh}\,\mathbb{X})$ a *tilting complex* if the following two conditions hold.

(TC1) $\mathrm{Hom}_{\mathcal{D}}(T, T^{(I)}[n]) = 0$ for all cardinals I and all $n \in \mathbb{Z}$, $n \neq 0$.

(TC2) If $X \in \mathcal{D}^b$ satisfies $\mathrm{Hom}_{\mathcal{D}}(T, X[n]) = 0$ for all $n \in \mathbb{Z}$, then $X = 0$.

Proposition 2.8. *The following statements are equivalent for* $T \in \mathcal{T}_{\mathrm{cc}}$ *(viewed as a complex concentrated in degree zero).*

(1) T *is a tilting sheaf in* $\vec{\mathcal{H}}$.

(2) *T is a tilting complex in $\mathcal{D}^b$.*
(3) *T is a tilting module in $\operatorname{Mod}\Sigma$ (of projective dimension at most one).*

Moreover, every tilting sheaf $T \in \vec{\mathcal{H}}$ lying in $\mathcal{T}_{\mathrm{cc}}$ is of finite type.

Proof. Clearly (2) implies (1) and (3). We show that (1) implies (2). Since $\vec{\mathcal{H}}$ is hereditary, $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, T^{(I)}) = 0$ is equivalent to $\operatorname{Hom}_{\mathcal{D}}(T, T^{(I)}[n]) = 0$ for all $n \neq 0$. Let $X = \bigoplus_{i=-s}^{s} X_i \in \mathcal{D}^b$ be such that $X_i \in \vec{\mathcal{H}}[i]$, and assume

$$\operatorname{Hom}_{\mathcal{D}}(T, X_i[n]) = 0 \text{ for all } n \in \mathbb{Z} \text{ and all } i. \tag{2.9}$$

Since $X_i[-i] \in \vec{\mathcal{H}}$, this implies for $n = -i$ and $n = -i + 1$ the condition

$$\operatorname{Hom}_{\vec{\mathcal{H}}}(T, X_i[-i]) = 0 = \operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X_i[-i]).$$

By (1) we conclude $X_i[-i] = 0$, and thus $X_i = 0$. Finally, we conclude $X = 0$. The proof that (3) implies (2) is similar. We just have to observe that condition (2.9) yields $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X_i[-i]) = 0$, that is, $X_i[-i] \in \operatorname{Gen}(T) \subseteq \mathcal{T}_{\mathrm{cc}}$, and thus X_i is, up to shift in the derived category, a Σ-module.
Assume that T satisfies condition (1). In order to show that T is of finite type, we set $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ and verify $\mathscr{S}^{\perp_1} = T^{\perp_1}$. The inclusion $\mathscr{S}^{\perp_1} \supseteq T^{\perp_1}$ is trivial. Further, since $T \in \mathcal{T}_{\mathrm{cc}}$, we have $T_{\mathrm{cc}} \in \mathscr{S}$, and thus $\mathscr{S}^{\perp_1} \subseteq \mathcal{T}_{\mathrm{cc}}$ consists of Σ-modules. We view T as a tilting Σ-module and exploit the corresponding result in $\operatorname{Mod}\Sigma$ from [16]. It states that the tilting class ${T_\Sigma}^{\perp_1} = \{X \in \operatorname{Mod}\Sigma \mid \operatorname{Ext}^1_\Sigma(T, X) = 0\}$ is determined by a class $\widetilde{\mathscr{S}} = {}^{\perp_1}({T_\Sigma}^{\perp_1}) \cap \operatorname{mod}\Sigma$ of finitely presented modules of projective dimension at most one, that is, ${T_\Sigma}^{\perp_1} = \widetilde{\mathscr{S}}^{\perp_1}$. Notice that $\widetilde{\mathscr{S}} \subseteq \mathcal{T}_{\mathrm{cc}}$. Otherwise there would be an indecomposable $F \in \mathcal{F}_{\mathrm{cc}}$ with $F[1] \in \widetilde{\mathscr{S}}$. Then $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, \tau F) = \operatorname{D}\operatorname{Hom}_{\vec{\mathcal{H}}}(F, T) = \operatorname{D}\operatorname{Ext}^1_\Sigma(F[1], T) = 0$, that is, $\tau F \in \operatorname{Gen}(T) \subseteq \mathcal{T}_{\mathrm{cc}}$, and $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T_{\mathrm{cc}}, \tau F) = 0$. But also $\operatorname{Hom}_{\vec{\mathcal{H}}}(T_{\mathrm{cc}}, \tau F) = \operatorname{D}\operatorname{Ext}^1_{\vec{\mathcal{H}}}(F, T_{\mathrm{cc}}) = \operatorname{D}\operatorname{Hom}_{\mathcal{D}}(F[1], T_{\mathrm{cc}}[2]) = \operatorname{D}\operatorname{Ext}^2_\Sigma(F[1], T_{\mathrm{cc}}) = 0$ since $\operatorname{pdim}_\Sigma F[1] \leq 1$, and so $F[1] = 0$, a contradiction.
Now any object X in $\mathcal{T}_{\mathrm{cc}}$ can be viewed both in $\operatorname{Mod}\Sigma$ and $\vec{\mathcal{H}}$, and the functors $\operatorname{Ext}^1_\Sigma(X, -)$ and $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(X, -)$ coincide on $\mathcal{T}_{\mathrm{cc}}$. In particular, $\widetilde{\mathscr{S}} \subseteq \mathscr{S}$, and if X is a sheaf in $\mathscr{S}^{\perp_1}$, then X is a Σ-module with $\operatorname{Ext}^1_\Sigma(S, X) = 0$ for all $S \in \widetilde{\mathscr{S}}$, hence $\operatorname{Ext}^1_{\vec{\mathcal{H}}}(T, X) = \operatorname{Ext}^1_\Sigma(T, X) = 0$, that is, $X \in T^{\perp_1}$. This finishes the proof. $\square$

We will construct and classify a certain class of large tilting sheaves independently of the representation type, even independently of the genus, namely the tilting sheaves with a large torsion part. A complete classification of all large tilting sheaves will be obtained in the domestic and the tubular (that is: in the non-wild) genus zero cases.

The domestic case is akin to the tame hereditary case:

Tame hereditary algebras. There is a tilting bundle T_{cc} such that $H = \mathrm{End}(T_{\mathrm{cc}})$ is a tame hereditary algebra if and only if $\mathbb{X}$ is of domestic type. In this case it follows from Proposition 2.8 that the large tilting H-modules (of projective dimension at most one), as classified in [10], correspond (up to equivalence) to the large tilting sheaves in $\operatorname{Qcoh}\mathbb{X}$. Indeed, recall that T_{cc} induces a torsion pair $(\mathcal{T}_{\mathrm{cc}}, \mathcal{F}_{\mathrm{cc}})$ in $\operatorname{Qcoh}\mathbb{X}$ and a split torsion pair $(\mathcal{Q}, \mathcal{C})$ in $\operatorname{Mod} H$. By [10, Thm. 2.7] every large tilting H-module lies in the class $\mathcal{C} \subseteq \operatorname{Mod} H$, and it will be shown in Proposition 6.3 below that every large tilting sheaf lies in $\mathcal{T}_{\mathrm{cc}}$.

3. Torsion, torsionfree, and divisible sheaves

In this section let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$*, where* $\mathbb{X}$ *is a weighted noncommutative regular projective curve over a field* k*.* Our main aim is to prove that every tilting sheaf splits into a direct sum of indecomposable sheaves of finite length, Prüfer sheaves, and a torsionfree sheaf.

Definition 3.1. Let $V \subseteq \mathbb{X}$ be a subset. A quasicoherent sheaf F is called V*-torsionfree* if $\operatorname{Hom}(S_x, F) = 0$ for all $x \in V$ and all simple sheaves $S_x \in \mathcal{U}_x$. In case $V = \mathbb{X}$ the sheaf F is *torsionfree*. We set

$$\mathscr{S}_V = \coprod_{x \in V} \mathcal{U}_x$$

and denote by

$$\mathcal{F}_V = \mathscr{S}_V{}^{\perp_0}$$

the class of V-torsionfree sheaves.

Similarly, a quasicoherent sheaf D is called V*-divisible* if $\operatorname{Ext}^1(S_x, D) = 0$ for all $x \in V$ and for all simple sheaves $S_x \in \mathcal{U}_x$. In case $V = \mathbb{X}$ we call D just *divisible*. We denote by

$$\mathcal{D}_V = \mathscr{S}_V{}^{\perp_1}$$

the class of V-divisible sheaves. It is closed under direct summands, set-indexed direct sums, extensions and epimorphic images. Furthermore, we call D *precisely* V*-divisible* if D is V-divisible, and if $\operatorname{Ext}^1(S, D) \neq 0$ for every simple sheaf $S \in \mathscr{S}_{\mathbb{X}\setminus V}$.

Remark 3.2. The class $\mathscr{S}_V$ is a Serre subcategory in $\mathcal{H} = \operatorname{fp}(\vec{\mathcal{H}})$, its direct limit closure $\mathcal{T}_V = \vec{\mathscr{S}}_V$ is a localizing subcategory in $\vec{\mathcal{H}}$ of finite type, and $(\mathcal{T}_V, \mathcal{F}_V)$ is a hereditary torsion pair in $\vec{\mathcal{H}}$. In particular, the canonical quotient functor $\pi\colon \vec{\mathcal{H}} \to \vec{\mathcal{H}}/\mathcal{T}_V$ has a right-adjoint $s\colon \vec{\mathcal{H}}/\mathcal{T}_V \to \vec{\mathcal{H}}$ which commutes with direct limits. The class of V-torsionfree and V-divisible sheaves

$$\mathscr{S}_V{}^{\perp} = \mathcal{T}_V{}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V \tag{3.1}$$

is a full exact subcategory of $\vec{\mathcal{H}}$, that is, the inclusion functor $j\colon \mathscr{S}_V{}^{\perp} \to \vec{\mathcal{H}}$ is exact and induces an isomorphism $\operatorname{Ext}^1_{\mathscr{S}_V{}^{\perp}}(A, B) \simeq \operatorname{Ext}^1_{\vec{\mathcal{H}}}(A, B)$ for all $A, B \in \mathscr{S}_V{}^{\perp}$. In particular, $\operatorname{Ext}^1_{\mathscr{S}_V{}^{\perp}}$ is right exact, so that the category $\vec{\mathcal{H}}/\mathcal{T}_V \simeq \mathscr{S}_V{}^{\perp}$

is hereditary. For details we refer to [28, Prop. 1.1, Prop. 2.2, Cor. 2.4], [31, Thm. 2.8], [34, Lem. 2.2, Thm. 2.6, Thm. 2.8, Cor. 2.11].

We note that in case $V=\mathbb{X}$ the subclass $\mathscr{S}_{\mathbb{X}}=\mathcal{H}_0$ of $\mathcal{H}$ is the class of finite length sheaves, $\mathcal{T}=\mathcal{T}_{\mathbb{X}}$ in $\vec{\mathcal{H}}$ forms the class of torsion sheaves, $\mathcal{F}=\mathcal{F}_{\mathbb{X}}$ the class of torsionfree sheaves, and $\mathcal{F}\cap\mathcal{H}=\operatorname{vect}\mathbb{X}$ the class of vector bundles.

Lemma 3.3. *Let $X\in\vec{\mathcal{H}}$. Let tX be the largest subobject of X which lies in $\mathcal{T}$, the* torsion subsheaf *of X. Then the quotient X/tX is torsionfree, and the canonical sequence*

$$\eta\colon 0\to tX\to X\to X/tX\to 0$$

is pure-exact.

Proof. Clearly, X/tX is torsionfree. Let $F\in\mathcal{H}$. We know that $F=F_+\oplus F_0$, where F_+ is a vector bundle and F_0 is of finite length. It follows that $\operatorname{Ext}^1(F,tX)=\operatorname{Ext}^1(F_+,tX)\oplus\operatorname{Ext}^1(F_0,tX)$. The left summand is zero by Serre duality, since every vector bundle is torsionfree. Moreover, $\operatorname{Hom}(F_0,X/tX)=0$, so $\operatorname{Hom}(F,X)\to\operatorname{Hom}(F,X/tX)$ is surjective. □

Lemma 3.4. *A quasicoherent sheaf is injective if and only if it is divisible.*

Proof. Trivially every injective sheaf is divisible. Conversely, every divisible sheaf Q is L'-injective for every line bundle L': this means that if $L''\subseteq L'$ is a sub line bundle of L', then every morphism $f\in\operatorname{Hom}(L'',Q)$ can be extended to L'. Indeed, there is commutative diagram with exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L'' \arrow[r] \arrow[d, "f"'] & L' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Q \arrow[r] & X \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

with E of finite length. Since Q is divisible, the lower sequence splits, and it follows that f lifts to L'. This shows that Q is L'-injective. Since the line bundles form a system of generators of $\vec{\mathcal{H}}$, we obtain by the version [63, V. Prop. 2.9] of Baer's criterion that Q is injective in $\vec{\mathcal{H}}$. □

Remark 3.5. By the closure properties mentioned above, the class $\mathcal{D}$ of divisible sheaves is a torsion class. Given an object $X\in\vec{\mathcal{H}}$, we denote by dX the largest divisible subsheaf of X. Since dX is injective,

$$X\simeq dX\oplus X/dX.$$

The sheaves with $dX=0$, called *reduced*, form the torsion-free class corresponding to the torsion class $\mathcal{D}$.

Proposition 3.6.

(1) *The indecomposable injective sheaves are (up to isomorphism) the sheaf $\mathcal{K}$ of rational functions and the Prüfer sheaves $S[\infty]$ ($S\in\mathcal{H}$ simple).*

(2) *Every torsion sheaf F is of the form*

$$F = \bigoplus_{x \in \mathbb{X}} F_x \quad \text{with} \quad F_x \in \vec{\mathcal{U}}_x \text{ unique,} \tag{3.2}$$

and there are pure-exact sequences

$$0 \to E_x \to F_x \to P_x \to 0 \tag{3.3}$$

in $\vec{\mathcal{U}}_x$ with E_x a direct sum of indecomposable finite length sheaves and P_x a direct sum of Prüfer sheaves (for all $x \in \mathbb{X}$).

(3) *Every sheaf of finite length is Σ-pure-injective.*

Proof. (1) It is well-known that in a locally noetherian category every injective object is a direct sum of indecomposable injective objects. Every indecomposable injective object has a local endomorphism ring and is the injective envelope of each of its non-zero subobjects. For details we refer to [26].
Let E be an indecomposable injective sheaf. We consider its torsion part tE. If $tE \neq 0$, then E has a simple subsheaf S. It follows that E is injective envelope of S, and thus it contains the direct family $S[n]$ ($n \geq 1$) and its union $S[\infty]$. We claim that $E = S[\infty]$. Indeed, it is easy to see that $S[\infty]$ is uniserial, with each proper subobject of the form $S[n]$ for some $n \geq 1$. If there were a simple object U with $0 \neq \operatorname{Ext}^1(U, S[\infty]) = \operatorname{D}\operatorname{Hom}(S[\infty], \tau U)$, then there would be a surjective map $S[\infty] \to \tau U$, whose kernel would have to be a (maximal) subobject of $S[\infty]$, hence of the form $S[n]$, which is impossible since $S[\infty]$ has infinite length. It follows that $S[\infty]$ is divisible, thus injective, and we conclude $E = S[\infty]$.
If, on the other hand, $tE = 0$, then E is torsionfree and contains a line bundle L' as a subobject. Then E is the injective envelope of L'. In the quotient category $\mathcal{H}/\mathcal{H}_0$ the structure sheaf L and L' become isomorphic ([47]), and thus (by definition of the morphism spaces in the quotient category) there is a third line bundle L'' which maps non-trivially to both, L' and L. It follows that L' has the same injective envelope as L, namely $\mathcal{K}$.
(2) The torsion class $\mathcal{T}$ is a hereditary (cf. [52, Prop. A.2]) locally finite Grothendieck category with injective cogenerator given by the direct sum of all the Prüfer sheaves. We have the coproduct of (locally finite) categories

$$\mathcal{T} = \coprod_{x \in \mathbb{X}} \vec{\mathcal{U}}_x, \tag{3.4}$$

from which we derive (3.2).

In order to proof the existence of a sequence (3.3), we show that $\vec{\mathcal{U}}_x$ *coincides with the category of torsion modules over a certain bounded hereditary noetherian prime ring*, and then we apply the similar result [61, Thm. 1] for modules.

To this end we briefly recall some notions, cf. [64, Ch. 4]: let M_R be a topological module over the topological ring R; then M is called pseudo-compact if it is Hausdorff, complete, and its topology is generated by submodules of finite colength; the ring R is called pseudo-compact if R_R is. Moreover, M_R is called discrete if its topology is discrete; this is the case if and only if the right annihilator ideals $\mathrm{Ann}(x)$ are open for every $x \in M$.

Let now $\mathcal{U} = \mathcal{U}_x$ be a tube of rank $p \geq 1$, with simple objects $S, \tau S, \ldots, \tau^{p-1}S$, and E the injective cogenerator of $\vec{\mathcal{U}}$ given by $\bigoplus_{j=0}^{p-1} \tau^j S[\infty]$. Its (opposite) endomorphism algebra $R = \mathrm{End}(E)^{\mathrm{op}}$ is a pseudo-compact ring: a basis of a suitable (Gabriel) topology is given by the right ideals $I(U)$ of endomorphisms of E annihilating U (for $U \in \mathcal{U}$). By [26, IV.4. Cor. 1] the category $\vec{\mathcal{U}}$ is dual to $\mathrm{PC}(R)$, the category of pseudo-compact R-modules, the duality is given by the functor $X \mapsto \mathrm{Hom}(X, E)$; note that in [26] left modules are considered, whereas we consider right modules, like in [64]. Since $\mathrm{soc}(E) = \bigoplus_{i=0}^{p-1} \tau^i S$, we get $R/\mathrm{rad}(R) \simeq \mathrm{End}(\mathrm{soc}(E)) \simeq D^p$ as k-algebras, with $D = \mathrm{End}(\tau^i S)$, by [26, IV.4. Prop. 12]. In particular, the simple R-modules are finite dimensional. It follows that R is cofinite in the sense of [64]. From [64, Prop. 4.10] we get that $R^{\mathrm{op}} = \mathrm{End}(E)$ is also pseudo-compact, and $\mathrm{PC}(R)^{\mathrm{op}} \simeq \mathrm{Dis}(R^{\mathrm{op}})$. Thus, $\vec{\mathcal{U}}$ is equivalent to $\mathrm{Dis}(R^{\mathrm{op}})$.

We now show that “discrete module” coincides with “torsion module”. Using the special shape of $\vec{\mathcal{U}}$, it follows from [1] (cf. also [39, Prop. 13.4]) that $R^{\mathrm{op}} \simeq H_p(V, \mathfrak{m})$, given by matrices $(a_{ij}) \in \mathrm{M}_p(V)$ with $a_{ij} \in \mathfrak{m}$ for $j > i$; here $V = \mathrm{End}(\tau^i S[\infty])$ is a (noncommutative) complete local principal ideal domain with maximal ideal $\mathfrak{m}$, so that every non-zero one-sided ideal is a power of $\mathfrak{m}$. In particular, R^{op} is a complete semiperfect, bounded hereditary noetherian prime ring. By [65, Prop. 3.22] the topology on R^{op} is the J-adic one, with J the Jacobson radical, which is generated by a normal and regular element. Since moreover, by the special shape of R^{op}, each non-zero ideal contains a power of J, we readily see that $M \in \mathrm{Mod}(R^{\mathrm{op}})$ is discrete if and only if each element in M is annihilated by a power of J, or equivalently, each element in M is annihilated by a non-zero ideal. This means that M is torsion in the sense of [55, p. 373]. In particular, then each element in M is annihilated by a regular element. The converse is also true: by [63, Sec. IV.6.3.] each regular element generates an essential right ideal, which, by boundedness, contains a non-zero ideal.

We summarize: *The category $\vec{\mathcal{U}}$ coincides with the category of those R^{op}-modules M which are torsion in the sense that each element of M is annihilated by a regular element.* Now, in the terminology of [61], the sequence (3.3) expresses that E_x is a *basic submodule* of the torsion module F_x, and the existence of such a pure submodule is given by [61, Thm. 1].

(3) Each indecomposable R-module F of finite length has finite endolength, since it is finite dimensional over k, by the argument from the preceding part. From [66, Beisp. 2.6 (1)] we obtain that F is a Σ-pure-injective R-module. Since an object M in a locally noetherian category is pure-injective if and only if the summation map $M^{(I)} \to M$ factors through the canonical embedding

$M^{(I)} \to M^I$ for every I (we refer to [51, Thm. 5.4]), we conclude that F is Σ-pure-injective also in $\vec{\mathcal{H}}$. □

If F is a torsion sheaf like in (3.2), we call the set of those $x \in \mathbb{X}$ with $F_x \neq 0$ the *support* of F. If the support of F is of the form $\{x\}$, we say F is *concentrated* at x.

Corollary 3.7. *Let $F \in \vec{\mathcal{H}}$ be a torsion sheaf.*

(1) *There is a pure-exact sequence*

$$0 \to E \xrightarrow{\subseteq} F \to F/E \to 0 \tag{3.5}$$

such that E is a direct sum of finite length sheaves and F/E is injective.

(2) *If F has no non-zero direct summand of finite length, then F is a direct sum of Prüfer sheaves.*

(3) *If F is a reduced torsion sheaf and $E_1, \dots, E_n$ are the only indecomposable direct summands of F of finite length, then F is pure-injective and isomorphic to $\bigoplus_{j=1}^n {E_j}^{(I_j)}$ for suitable sets I_j.*

(4) *If F is indecomposable, then F is either of finite length or a Prüfer sheaf.*

Proof. (1) The direct sum of all pure-exact sequences (3.3) ($x \in \mathbb{X}$) is pure-exact.
(2) This follows from (1) by purity. (Locally, in x, we can also refer to [60, Thm. 10].)
(3) We consider the pure-exact sequence (3.5). By assumption, E must be of the form $\bigoplus_{j=1}^n {E_j}^{(I_j)}$ (indeed, since E is pure in F, its direct summands of finite length, being pure-injective, are also direct summands of F). Now E is, by part (3) of Proposition 3.6, pure-injective, and thus $F \simeq E \oplus F/E$. Since F is reduced, we conclude $F \simeq E$.
(4) This follows readily from (2). □

The following basic splitting property will be crucial for our treatment of large tilting sheaves.

Theorem 3.8. *Let $T \in \vec{\mathcal{H}}$ be a sheaf such that $\operatorname{Ext}^1(T,T) = 0$ holds.*

(1) *The torsion part tT is a direct sum of Prüfer sheaves and exceptional sheaves of finite length. Accordingly, it is pure-injective.*

(2) *The canonical exact sequence $0 \to tT \to T \to T/tT \to 0$ splits.*

Proof. By Lemma 3.3 it suffices to prove part (1). By Lemma 2.1 the assertion is true in case tT is coherent. If tT does not admit any non-zero summand of finite length, then we conclude from Corollary 3.7 (2) that tT is a direct sum of Prüfer sheaves, and then tT is in particular pure-injective. Let now E be an indecomposable summand of tT of finite length. The composition of embeddings $E \to tT \to T$ gives a surjection $\operatorname{Ext}^1(T,T) \to \operatorname{Ext}^1(E,T)$, showing that $\mathrm{Ext}^1(E,T)=0$. Forming the push-out, the projection $tT\to E$ yields the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & tT \arrow[r] \arrow[d, two heads] & T \arrow[r] \arrow[d] & T/tT \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & T' \arrow[r] & T/tT \arrow[r] & 0.
\end{tikzcd}$$

Using Serre duality $\mathrm{Ext}^1(T/tT,E) = \mathrm{D}\operatorname{Hom}(\tau^- E, T/tT) = 0$, the lower sequence splits, showing that there is an epimorphism $T\to E$. This gives a surjective map $\mathrm{Ext}^1(E,T)\to\mathrm{Ext}^1(E,E)$, showing that $\mathrm{Ext}^1(E,E)=0$. Therefore E must belong to an exceptional tube of some rank $p>1$, and has length $<p$. Thus there are only finitely many such E. From Corollary 3.7 and Remark 3.5 we conclude that tT is a direct sum of copies of these finitely many exceptionals of finite length and of Prüfer sheaves. This proves the theorem. $\square$

Given a tilting sheaf $T\in\vec{\mathcal{H}}$, we will often write

$$T = T_+ \oplus T_0$$

with $T_0 = tT$ the *torsion* and $T_+\simeq T/tT$ the *torsionfree part* of T. We will say that T has a *large torsion part* if tT is large in the sense that there is no coherent sheaf E such that $\mathrm{Add}(tT)=\mathrm{Add}(E)$.

4. Tilting sheaves induced by resolving classes

In this section we introduce the notion of a resolving class, and we employ it to construct the torsionfree Lukas tilting sheaf $\mathbf{L}$ and the tilting sheaves $T_{(B,V)}$. We further classify all tilting sheaves with large torsion part, and we establish a bijection between resolving classes and tilting classes of finite type.

4.1. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category with $\mathcal{H}=\mathrm{fp}(\vec{\mathcal{H}})$. Let T be a tilting object of finite type in $\vec{\mathcal{H}}$, that is,

$$\mathcal{B} := \mathrm{Gen}(T) = T^{\perp_1} = \mathscr{S}^{\perp_1}$$

for some $\mathscr{S}\subseteq\mathcal{H}$, which we choose to be the largest class with this property

$$\mathscr{S} = {}^{\perp_1}\mathcal{B}\cap\mathcal{H}.$$

Applying $\mathrm{Ext}^1(S,-)$ to the sequence

$$0\to X\to E(X)\to E(X)/X\to 0 \tag{4.1}$$

where $X\in\vec{\mathcal{H}}$ is arbitrary and $E(X)$ is its injective envelope, we see that

(o) $\mathscr{S}$ consists of objects S with $\mathrm{pd}_{\vec{\mathcal{H}}}(S)\le 1$.

We list further properties of $\mathscr{S}$ that can be verified by the reader:

(i) $\mathscr{S}$ is closed under extensions;
(ii) $\mathscr{S}$ is closed under direct summands;
(iii) $S'\in\mathscr{S}$ whenever $0\to S'\to S\to S''\to 0$ is exact with $S, S''\in\mathscr{S}$.

Definition 4.2. Let $\vec{\mathcal{H}}$ be a locally coherent Grothendieck category. We call a class $\mathscr{S} \subseteq \mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$ *resolving* if it satisfies (i), (ii), (iii), and generates $\vec{\mathcal{H}}$.

Remark 4.3. A generating system $\mathscr{S} \subseteq \mathcal{H}$ is resolving whenever it is closed under extensions and subobjects. In case $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$ the converse also holds true; we refer to Corollary 4.17 below.

Theorem 4.4. *Let $\vec{\mathcal{H}}$ be locally coherent and $\mathscr{S}$ a resolving class such that $\mathrm{pd}_{\vec{\mathcal{H}}}(S) \leq 1$ for all $S \in \mathscr{S}$. Then there is a tilting object T in $\vec{\mathcal{H}}$ with $T^{\perp_1} = \mathscr{S}^{\perp_1}$.*

Proof. The class $\mathcal{B} = \mathscr{S}^{\perp_1}$ is pretorsion, that is, it is closed under direct sums (recall that $\mathscr{S} \subseteq \mathcal{H}$ consists of finitely presented objects) and epimorphic images (here we need the assumption on the projective dimension). Further, it is special preenveloping as $({}^{\perp_1}\mathcal{B}, \mathcal{B})$ is a complete cotorsion pair, see [58, Sec. 1.3 and Cor. 2.15]. By assumption, $\mathscr{S}$ contains a system of generators $(G_i, i \in I)$ for $\vec{\mathcal{H}}$. Set $G = \bigoplus_{i \in I} G_i$, and take a special $\mathcal{B}$-preenvelope of G, i.e. a short exact sequence

$$0 \to G \to T_0 \to T_1 \to 0 \tag{4.2}$$

where $T_0 \in \mathcal{B}$ and $T_1 \in {}^{\perp_1}\mathcal{B}$. Since $\mathcal{B}$ is pretorsion, also $T_1 \in \mathcal{B}$, and $T = T_0 \oplus T_1$ satisfies $\operatorname{Gen}(T) \subseteq \mathcal{B}$. We claim that T is the desired tilting object. Indeed, for every $X \in \vec{\mathcal{H}}$ there is a natural isomorphism

$$\operatorname{Ext}^1\Big(\bigoplus_{i \in I} G_i, X\Big) \simeq \prod_{i \in I} \operatorname{Ext}^1(G_i, X). \tag{4.3}$$

(This we get from the natural isomorphism $\operatorname{Hom}(\bigoplus_{i \in I} G_i, X) \simeq \prod_{i \in I} \operatorname{Hom}(G_i, X)$ by applying $\operatorname{Hom}(G_i, -)$ and $\operatorname{Hom}(\bigoplus_{i \in I} G_i, -)$ to the exact sequence (4.1).) Since $G_i \in \mathscr{S}$ for all $i \in I$, we deduce

$$\operatorname{Ext}^1(G, X) = 0 \quad \text{for all } X \in \mathcal{B}. \tag{4.4}$$

Hence $G \in {}^{\perp_1}\mathcal{B}$, and (4.2) shows that T_0 and T belong to ${}^{\perp_1}\mathcal{B}$ as well. So

$$\operatorname{Gen}(T) \subseteq \mathcal{B} \subseteq T^{\perp_1}.$$

Let now $X \in T^{\perp_1}$. Since G is a generator, there is an epimorphism $G^{(J)} \to X$ and a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & G^{(J)} \arrow[r] \arrow[d] & (T_0)^{(J)} \arrow[r] \arrow[d] & (T_1)^{(J)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & X' \arrow[r] & (T_1)^{(J)} \arrow[r] & 0.
\end{tikzcd}$$

Since $X \in {T_1}^{\perp_1}$ and thus by (4.3) also $X \in ({T_1}^{(J)})^{\perp_1}$, the lower sequence splits. Therefore we get an epimorphism ${T_0}^{(J)} \to X$, showing that $X \in \operatorname{Gen}(T)$. We conclude that T is a tilting object with $\operatorname{Gen}(T) = \mathcal{B}$. $\square$

Let now $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$*, where* $\mathbb{X}$ *is a weighted noncommutative regular projective curve over a field* k*.* We exhibit two applications of the theorem. The first one is quite easy.

Proposition 4.5. *Let* $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$*, where* $\mathbb{X}$ *is a weighted noncommutative regular projective curve. There is a torsionfree large tilting sheaf* $\mathbf{L}$*, called* Lukas tilting sheaf*, such that* $\mathbf{L}^{\perp_1} = (\operatorname{vect}\mathbb{X})^{\perp_1}$*.*

Proof. The class $\mathscr{S} = \operatorname{vect}\mathbb{X}$ is resolving. By Theorem 4.4 there is a tilting sheaf $\mathbf{L}$ with $(\operatorname{vect}\mathbb{X})^{\perp_1} = \mathbf{L}^{\perp_1}$. We show that $\mathbf{L}$ is torsionfree. Assume that $\mathbf{L}$ has a non-zero torsion part T_0. By Theorem 3.8 this is a direct summand of $\mathbf{L}$. Then

$$(\operatorname{vect}\mathbb{X})^{\perp_1} = \mathbf{L}^{\perp_1} \subseteq {T_0}^{\perp_1} \cap (\operatorname{vect}\mathbb{X})^{\perp_1} \subsetneq (\operatorname{vect}\mathbb{X})^{\perp_1},$$

where the last inclusion is proper because there exists a simple sheaf S with $\operatorname{Hom}(S, T_0) \neq 0$ and thus $\tau S \in (\operatorname{vect}\mathbb{X})^{\perp_1} \setminus {T_0}^{\perp_1}$. Thus we get a contradiction. We conclude that $T_0 = 0$. Clearly, $\mathbf{L}$ is then also large. □

We record the following observation for later reference.

Lemma 4.6. $\mathbf{L}^{\perp_1}$ *contains the class* $\mathcal{D}_V$ *of* V*-divisible sheaves for any* $\emptyset \neq V \subseteq \mathbb{X}$*.*

Proof. With the notation of Definition 3.1, we have ${\mathscr{S}_V}^{\perp_1} = {}^{\perp_0}\mathscr{S}_V$ and $(\operatorname{vect}\mathbb{X})^{\perp_1} = {}^{\perp_0}\operatorname{vect}\mathbb{X}$ by Serre duality. Let F be a sheaf such that there is a non-zero morphism to a vector bundle, and consequently also to a line bundle. Since every non-zero subsheaf of a line bundle is a line bundle again, there is even an epimorphism from F to a line bundle. This line bundle maps onto a simple sheaf concentrated at $x \in V$. We conclude that F is not V-divisible. □

The second application is the classification of all tilting sheaves having a large torsion part. We first introduce some terminology.

4.7. Branch sheaves. Let $\mathcal{U} = \mathcal{U}_x$ be a tube of rank $p > 1$. We recall that an indecomposable sheaf $E \in \mathcal{U}$ is exceptional (that is, $\operatorname{Ext}^1(E, E) = 0$) if and only if its length is $\leq p-1$; in particular, there are only finitely many such E. If E is exceptional in $\mathcal{U}$, then we call the collection $\mathcal{W}$ of all the subquotients of E the *wing rooted in* E. The set of all simple sheaves in $\mathcal{W}$ is called the *basis* of $\mathcal{W}$. It is of the form $S, \tau^- S, \ldots, \tau^{-(r-1)}S$ for an exceptional simple sheaf S and an integer r with $1 \leq r \leq p-1$ which equals the length of the *root* E; we call such a set of simples a *segment* in $\mathcal{U}$, and we say that two wings (or segments) in $\mathcal{U}$ are *non-adjacent* if the segments of their bases (or the segments) are disjoint and their union consists of $< p$ simples and is not a segment [46, Ch. 3].

We remark that the full subcategory $\operatorname{add}\mathcal{W}$ of $\mathcal{H}$ is equivalent to the category of finite-dimensional representations of the linearly oriented Dynkin quiver $\vec{\mathbb{A}}_r$, cf. [46, Ch. 3]. By [56, p. 205] any tilting object B in the category $\operatorname{add}\mathcal{W}$ has precisely r non-isomorphic indecomposable summands $B_1, \ldots, B_r$ forming a so-called *connected branch* B in $\mathcal{W}$: one of the B_i is isomorphic to the root

E, and for every j the wing rooted in B_j contains precisely ℓ_j indecomposable summands of B, where ℓ_j is the length of B_j. In particular, for every j we have a (full) *subbranch* of B rooted in B_j; if B_j is different from the root of $\mathcal{W}$, we call this subbranch *proper*.

Following [46, Ch. 3], we call a sheaf B of finite length a *branch sheaf* if it is a multiplicity free direct sum of connected branches in pairwise non-adjacent wings; it then follows that $\operatorname{Ext}^1(B,B)=0$.

Every branch sheaf B decomposes into $B=\bigoplus_{x\in\mathbb{X}} B_x$; of course $B_x\neq 0$ only if x is one of the finitely many exceptional points $x_1,\dots,x_t$, and there are only finitely many isomorphism classes of branch sheaves.

Given a non-empty subset $V\subseteq\mathbb{X}$, we can also write

$$B=B_{\mathfrak{i}}\oplus B_{\mathfrak{e}}$$

where $B_{\mathfrak{e}}$ is supported in $\mathbb{X}\setminus V$ and $B_{\mathfrak{i}}$ in V. In such case we will say that $B_{\mathfrak{e}}$ is *exterior* and $B_{\mathfrak{i}}$ is *interior* with respect to V.

We now turn to the main result of this section. It states that any choice of a non-empty subset $V\subseteq\mathbb{X}$ and a branch sheaf B determines a unique tilting sheaf T with large torsion part, and every such tilting sheaf arises in this way. More precisely, the set V is the support of the non-coherent (Prüfer) summands in the torsion part tT of T, while B collects the coherent summands of tT. Furthermore, the summand $B_{\mathfrak{i}}$ of B which is interior with respect to V determines the rays contributing a Prüfer summand to T.

Theorem 4.8. *Let $\vec{\mathcal{H}}=\operatorname{Qcoh}\mathbb{X}$, where $\mathbb{X}$ is a weighted noncommutative regular projective curve.*

(1) *Let $\emptyset\neq V\subseteq\mathbb{X}$ and $B\in\mathcal{H}_0$ be a branch sheaf. There is, up to equivalence, a unique large tilting sheaf $T=T_+\oplus T_0$ whose whose torsionfree part T_+ is V-divisible, and whose torsion part is given by*

$$T_0=B\oplus\bigoplus_{x\in V}\bigoplus_{j\in\mathcal{R}_x}\tau^j S_x[\infty], \tag{4.5}$$

where the non-empty sets $\mathcal{R}_x\subseteq\{0,\dots,p(x)-1\}$ are uniquely determined by B, see (4.8).

(2) *Every tilting sheaf with large torsion part is, up to equivalence, as in (1).*

Notation. Let $\emptyset\neq V\subseteq\mathbb{X}$ and $B=B_{\mathfrak{i}}\oplus B_{\mathfrak{e}}$ be a branch sheaf with interior and exterior part with respect to V given by $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively. The large tilting sheaf from Theorem 4.8 will be denoted by

$$T_{(B,V)}=T_{(B_{\mathfrak{i}},V)}\oplus B_{\mathfrak{e}}. \tag{4.6}$$

For the proof we need several preparations. We start by describing the torsion part of a tilting sheaf.

Lemma 4.9. *Let T be a tilting sheaf and x an exceptional point of weight $p=p(x)>1$ such that $(tT)_x\neq 0$. There are two possible cases:*

(1) *"Exterior branch": $(tT)_x$ contains no Prüfer sheaf, but at most $p-1$ indecomposable summands of finite length, which are arranged in connected branches in pairwise non-adjacent wings.*

(2) *"Interior branch": $(tT)_x$ contains precisely s Prüfer sheaves, where $1 \leq s \leq p$, and precisely $p-s$ indecomposable summands of finite length. The latter lie in wings of the following form: if $S[\infty]$, $\tau^{-r}S[\infty]$ are summands of T with $2 \leq r \leq p$, but the Prüfer sheaves $\tau^{-}S[\infty], \ldots, \tau^{-(r-1)}S[\infty]$ in between are not, then there is a (unique) connected branch in the wing $\mathcal{W}$ rooted in $S[r-1]$ that occurs as a summand of T.*

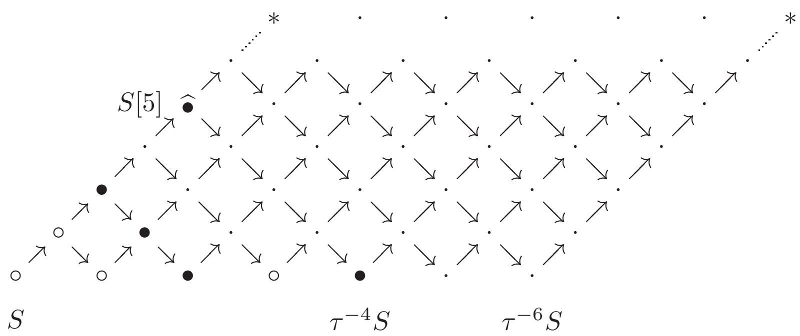

Figure 4.1. Lemma 4.9 (2) with $r=6$, $\bullet$ = a branch B, $\widehat{\bullet}$ = its root, $*$ = Prüfer summand of T; $\circ$ = undercut $B^{>}$ as in (4.9)

Proof. Given a simple object $S \in \mathcal{U}_x$, the corresponding Prüfer sheaf $S[\infty]$ is $S[p]$-filtered, and thus by [58, Prop. 2.12] we have

(4.7) $S[\infty]$ is a summand of T $\Leftrightarrow$ ${}^{\perp_1}(T^{\perp_1})$ contains the ray $\{S[n] \mid n \geq 1\}$.

If no such ray exists, then $(tT)_x$ has at least one indecomposable summand of finite length, and it is well-known that all such summands are arranged in branches in pairwise non-adjacent wings, compare [46, Ch. 3].

Assume now that, say, $S[\infty]$ and $\tau^{-r}S[\infty]$ are summands of T, but no Prüfer sheaf "in between" is a summand, where $2 \leq r \leq p$ (when $r=p$, there is precisely one Prüfer summand). We show that $S[r-1]$ is a summand of T. By (4.7) this is equivalent to show $\operatorname{Ext}^1(T, S[r-1]) = 0$. If this is not the case, then $\operatorname{Hom}(\tau^{-}S[r-1], T) \neq 0$, and thus there exists an indecomposable summand E of T lying on a ray starting in $\tau^{-}S[r-1], \ldots, \tau^{-(r-2)}S[2]$ or $\tau^{-(r-1)}S$. But for such an E we have $0 \neq \operatorname{D}\operatorname{Hom}(\tau^{-}E, \tau^{-r}S[\infty]) = \operatorname{Ext}^1(\tau^{-r}S[\infty], E)$, contradicting the fact that T has no self-extension. Thus $S[r-1]$ is a direct summand of T. The latter argument also shows that every indecomposable summand of T of finite length and lying on a ray starting in $S, \tau S, \dots, \tau^{-(r-1)}S$ actually lies in the wing $\mathcal{W}$ rooted in $S[r-1]$.

We claim that the direct sum B of all indecomposable summands of T lying in $\mathcal{W}$ forms a tilting object in $\operatorname{add}\mathcal{W}$. We have $\operatorname{Ext}^1(B,B)=0$. Assume that B is not a tilting object in $\mathcal{W}$. Then there is an indecomposable $E \in \mathcal{W}$, not a direct summand of B, such that $\operatorname{Ext}^1(E\oplus B, E\oplus B)=0$. Let E' be the indecomposable quotient of $S[r-1]$ such that E embeds into E'. We have a short exact sequence $0 \to F \to S[r-1] \to E' \to 0$ with indecomposable $F \in \mathcal{W}$. Let T_+ be the torsionfree part of T. Then exactness of $0 = \operatorname{Hom}(F, T_+) \to \operatorname{Ext}^1(E', T_+) \to \operatorname{Ext}^1(S[r-1], T_+) = 0$ shows $\operatorname{Ext}^1(E', T_+)=0$, and then also $\operatorname{Ext}^1(E, T_+) = 0$. Moreover $\operatorname{Ext}^1(T_+, E) = \operatorname{D}\operatorname{Hom}(\tau^- E, T_+) = 0$, and since $E \in \mathcal{W}$, there are no extensions between E and Prüfer summands of T. We conclude that $E \in T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) = \operatorname{Add}(T)$, a contradiction. Thus B is tilting, and it forms a connected branch.

Doing this with every "gap" between Prüfer sheaves in $(tT)_x$, one sees that $(tT)_x$ contains precisely $p-s$ indecomposable summands of finite length. $\square$

Lemma 4.10. *In the preceding lemma, the torsionfree part T_+ of T belongs to $\mathcal{W}^{\perp_1}$ for every wing $\mathcal{W}$ occurring in (1) or (2), and it is even x-divisible in case (2).*

Proof. The first part of the statement is shown as in the preceding proof. In case (2) it then remains to check that T_+ has no extensions with the simple objects in $\mathcal{U}_x$ which do not belong to the wings defined by the Prüfer summands of T. Let $\mathcal{W}$ be such wing and E such simple object, that is, $E \notin \mathcal{W}$, but $\tau E \in \mathcal{W}$. Assume $0 \neq \operatorname{Ext}^1(E, T_+) \simeq \operatorname{D}\operatorname{Hom}(T_+, \tau E)$. Since $\operatorname{Hom}(T_+, \tau\mathcal{W}) = 0$, repeated application of the almost split property yields an indecomposable object U on the ray starting in S such that $\operatorname{Hom}(T_+, \tau U) \neq 0$. By Serre duality $\operatorname{Ext}^1(U, T_+) \neq 0$, and since U embeds in $S[\infty]$, also $\operatorname{Ext}^1(S[\infty], T_+) \neq 0$, a contradiction. $\square$

As mentioned above, the interior branch sheaves and the Prüfer sheaves occurring in the torsion part of a tilting sheaf are interrelated. In the situation of Lemma 4.9 (2), we denote by $\mathcal{R}_x$ the set of cardinality s of all $j \in \{0, \dots, p(x)-1\}$ such that the Prüfer sheaf $\tau^j S[\infty]$ is a direct summand of T. Each such set defines a unique collection

$$\mathcal{W} = \{\tau^j S[\infty] \mid j \in \mathcal{R}_x\}^{\perp_1} \cap \mathcal{U}_x$$

of pairwise non-adjacent wings in the exceptional tube $\mathcal{U}_x$, whereas the branch B, viewed as collection of indecomposable sheaves, is given as

$$B = \operatorname{Add}(T) \cap \mathcal{U}_x.$$

In particular, this shows that a tilting sheaf T' with a different branch $B' \neq B$ in $\mathcal{U}_x$ will have ${T'}^{\perp_1} \neq T^{\perp_1}$, that is, T and T' cannot be equivalent.

Conversely, every non-zero branch sheaf in $\mathcal{U}_x$ – which we will often identify with the set of its indecomposable summands – defines a unique collection $\mathcal{W}$

of pairwise non-adjacent wings in $\mathcal{U}_x$, and this defines uniquely the set $\mathcal{R}_x$; namely, if $S, \tau^- S, \ldots, \tau^{-(r-1)} S$ is a basis of one of the wings in $\mathcal{W}$, we have

$$\mathcal{R}_x = \{j = 0, \ldots, p(x) - 1 \mid \tau^{j+1} S \notin \mathcal{W}\}. \tag{4.8}$$

We now consider a pair (B, V) given by a branch sheaf $B \in \mathcal{H}$ and a subset $V \subseteq \mathbb{X}$, and we associate a resolving class to it. For the moment $V = \emptyset$ is permitted. In case $V \neq \emptyset$, the corresponding tilting sheaf T given by Theorem 4.4 will have the properties required by Theorem 4.8.

The resolving class $\mathscr{S}$ associated to (B, V) will consist of all vector bundles, of the rays given by the sets $\mathcal{R}_x$ in (4.8), and of some objects determined by B. Up to τ-shift, these objects will lie in the wings defined by B, namely, in the part which lies "under" B, in a sense that we are going to explain below.

Let us fix some notation. Recall that $B = \bigoplus_{x \in \mathbb{X}} B_x$ where each B_x is a direct sum of connected branches in pairwise non-adjacent wings in $\mathcal{U}_x$. For every x denote by $\mathcal{W}_x$ the collection of all such wings, and for every $x \in V$ let $\mathcal{R}_x$ be the associated non-empty subset of $\{0, \ldots, p(x) - 1\}$ defined by (4.8).

In order to determine the part of $\mathcal{W}_x$ lying "under" B_x, we will have to distinguish two cases. In fact, when B_x is exterior with respect to V, it turns out that we have to consider $\tau \mathcal{W}_x$ rather than $\mathcal{W}_x$.

Given a connected branch C with associated wing $\mathcal{W}_C$, let us call the set

$$C^{>} := \begin{cases} C^{\perp_0} \cap \mathcal{W}_C & \text{if } C \text{ is interior,} \\ C^{\perp_0} \cap \tau \mathcal{W}_C & \text{if } C \text{ is exterior,} \end{cases} \tag{4.9}$$

the *undercut* of C. The undercut $B^{>}$ of the branch sheaf B is the union of the undercuts of all its connected branch components. The undercut is illustrated in Figure 4.1 above. Another example is shown in Figure 10.1.

LEMMA 4.11. *Let $V \subseteq \mathbb{X}$ and $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ be a branch sheaf.*

(1) *With the notation above, the class*

$$\mathscr{S} = \operatorname{add}\Big(\operatorname{vect} \mathbb{X} \cup \tau^-(B^{>}) \cup \bigcup_{x \in V} \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\} \Big) \tag{4.10}$$

is resolving.

(2) *If T is a tilting sheaf with $T^{\perp_1} = \mathscr{S}^{\perp_1}$, then $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$, the torsionfree part T_+ is V-divisible, and the torsion part is given by*

$$T_0 = B \oplus \bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty].$$

Proof. (1) The class $\mathscr{S}$ is clearly closed under subobjects. A simple case by case analysis shows that $\mathscr{S}$ is also closed under extensions. For instance, if $0 \to A \to E \to C \to 0$ is a short exact sequence with A a vector bundle and $C \in \mathscr{S}$ indecomposable of finite length, then $E = E_+ \oplus E_0$, with E_+ a vector bundle and E_0 of finite length; it follows that E_0 is isomorphic to a subobject of C, and thus $E_0 \in \mathscr{S}$, and then $E \in \mathscr{S}$. Compare also [10, p. 36 from line

-19]. Since $\mathscr{S}$ contains the system of generators $\operatorname{vect}\mathbb{X}$, we conclude that it is resolving.
(2) By Serre duality, an indecomposable coherent sheaf $E \in \mathcal{H}$ belongs to ${}^{\perp_1}(T^{\perp_1})$ if and only if $\tau E \in (T^{\perp_1})^{\perp_0} = (\mathscr{S}^{\perp_1})^{\perp_0} = ({}^{\perp_0}\tau\mathscr{S})^{\perp_0}$. We claim that this is further equivalent to $\tau E \in \tau\mathscr{S}$, that is, $E \in \mathscr{S}$. Indeed, the claim is shown by arguing inside the abelian category $\mathcal{H}$ as in [53, Lem. 1.3], keeping in mind that $\tau\mathscr{S}$ is closed under subobjects and extensions by part (1).
We thus have $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. It follows from (4.7) that T has precisely the Prüfer summands $\tau^j S_x[\infty]$ with $x \in V$ and $j \in \mathcal{R}_x$. In particular, T_+ is V-divisible by Lemma 4.10. Furthermore,

$$\mathscr{S}^{\perp_1} \cap \mathscr{S} = T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = \operatorname{Add}(T) \cap \mathcal{H}, \tag{4.11}$$

and we now show that this class further coincides with $\operatorname{add}(B)$.
Let $\mathcal{W}$ be the union of non-adjacent wings associated to B, and let B_1 and B_2 be two indecomposable summands of B. Then $0 = \operatorname{Ext}^1(B_1, B_2) = \operatorname{D}\operatorname{Hom}(B_2, \tau B_1)$. Thus $\tau B_1 \in B^{\perp_0}$. If B_1 is either exterior, or interior with $\tau B_1 \in \mathcal{W}$, then $\tau B_1 \in B^{>}$, that is, $B_1 \in \tau^-(B^{>}) \subseteq \mathscr{S}$. If, on the other hand, B_1 is interior with $\tau B_1 \notin \mathcal{W}$, then $B_1 \in \mathscr{S}$ by definition of $\mathcal{R}_x$. Moreover, we have $\operatorname{Ext}^1(\tau^-(B^{>}), B_1) = \operatorname{D}\operatorname{Hom}(B_1, B^{>}) = 0$, and then $\operatorname{Ext}^1(\tau^j S_x[n], B_1) = \operatorname{D}\operatorname{Hom}(B_1, \tau^{j+1} S_x[n]) = 0$, for any $x \in V$ and $j \in \mathcal{R}_x$, shows that $B_1 \in \mathscr{S}^{\perp_1}$.
Conversely, let $E \in \mathscr{S} \cap \mathscr{S}^{\perp_1}$ be indecomposable. By (4.11) we have that E is a summand of T, in particular E is exceptional and belongs to an exceptional tube. If E is supported in V, then it is a summand of $B_{\mathfrak{i}}$ by Lemma 4.9 and the fact that the connected parts of B form tilting objects in the corresponding wings. If E is not supported in V, then it belongs to $\tau^-(C^{>})$ for a connected branch component C of $B_{\mathfrak{e}}$. Since $\tau^-(C^{>}) = {}^{\perp_1}C \cap \mathcal{W}_C$ where $\mathcal{W}_C$ is the wing associated to C, we infer again that E is a summand of $B_{\mathfrak{e}}$.
We conclude that T_0 is given by $B \oplus \bigoplus_{x\in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty]$, as desired. □

We can now complete our classification of tilting sheaves with large torsion part.

Proof of Theorem 4.8. (1) By the preceding lemma there exists a (large) tilting sheaf with the claimed properties.
(2) Let now $T = T_+ \oplus T_0$ be any tilting sheaf with a non-coherent torsion part T_0. From Lemma 4.9 we infer that T_0 is of the form $B \oplus \bigoplus_{x\in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty]$. It is sufficient to show that the class $\mathscr{S}$ from (4.10) satisfies $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$, since this will imply $T^{\perp_1} = \mathscr{S}^{\perp_1}$, as desired.
By Lemma 4.10 the torsionfree part T_+ of T is V-divisible. From Lemma 4.6 we infer $T_+ \in (\operatorname{vect}\mathbb{X})^{\perp_1}$. Since also $T_0 \in (\operatorname{vect}\mathbb{X})^{\perp_1}$ by Serre duality, we conclude $\operatorname{Ext}^1(X, T) = 0$ for any vector bundle X, hence $\operatorname{vect}\mathbb{X} \subseteq {}^{\perp_1}(T^{\perp_1})$.
Next, we show $\tau^-(B^{>}) \subseteq {}^{\perp_1}(T^{\perp_1})$. If $E \in \tau^-(B_{\mathfrak{i}}{}^{>})$, then $\operatorname{Ext}^1(E, B) = \operatorname{D}\operatorname{Hom}(B, \tau E) = 0$ by definition of the undercut. Since T_+ and the Prüfer sheaves are V-divisible, we get $\operatorname{Ext}^1(E, T) = 0$ and $E \in {}^{\perp_1}(T^{\perp_1})$. If $E \in \tau^-(B_{\mathfrak{e}}{}^{>})$, then it belongs to $\tau^-(C^{>}) = {}^{\perp_1}C \cap \mathcal{W}_C$ for a connected

branch component C of $B_{\mathfrak{c}}$ with associated wing $\mathcal{W}_C$. It follows $\mathrm{Ext}^1(E,B) = \mathrm{D}\,\mathrm{Hom}(B,\tau E) = 0$, and $\mathrm{Ext}^1(E,T_+) = 0$ by Lemma 4.10, so again $E \in {}^{\perp_1}(T^{\perp_1})$.
Finally, if E belongs to a ray $\{\tau^j S_x[n] \mid n \geq 1\}$ with $x \in V$ and $j \in \mathcal{R}_x$, then $E \in {}^{\perp_1}(T^{\perp_1})$ by (4.7).
Altogether we have shown $\mathscr{S} \subseteq {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. In order to prove the reverse inclusion, let $E \in \mathcal{H}$ be indecomposable with $E \in {}^{\perp_1}(T^{\perp_1})$. By definition of $\mathscr{S}$, we can assume that E is of finite length, and further, if concentrated at a point $x \in V$, that it has the form $\tau^j S_x[n]$ with $j \notin \mathcal{R}_x$. This means $\tau^j S_x \in \tau^- \mathcal{W}$ by (4.8), so there is a connected branch component C of $B_{\mathfrak{i}}$ with associated wing $\mathcal{W}_C$ such that $\tau^j S_x \in \tau^- \mathcal{W}_C$. Since C is a summand of T, we have $E \in {}^{\perp_1}C \cap \tau^- \mathcal{W}_C = \tau^-(C^{>}) \subseteq \mathscr{S}$.
It remains to check the case when E is concentrated at a point $x \notin V$. Notice that $\mathrm{Hom}(T,\tau E) \simeq \mathrm{D}\,\mathrm{Ext}^1(E,T) = 0$ implies $\mathrm{Ext}^1(T,\tau E) \neq 0$ by condition (TS2). But the latter amounts to $\mathrm{Ext}^1(B_{\mathfrak{c}},\tau E) \neq 0$, or equivalently, $\mathrm{Hom}(E,B_{\mathfrak{c}}) \neq 0$. Let $0 \neq f : E \to B_{\mathfrak{c}}$. If E is simple, f is a monomorphism, and $E \in \mathscr{S}$ because $B_{\mathfrak{c}} \in \tau^-(B_{\mathfrak{c}}{}^{>}) \subseteq \mathscr{S}$ and $\mathscr{S}$ is closed under subjects. If E has length $\ell > 1$, we consider the short exact sequence $0 \to \mathrm{Ker}\, f \to E \to \mathrm{Im}\, f \to 0$ where $\mathrm{Im}\, f$ belongs to $\mathscr{S} \subseteq {}^{\perp_1}(T^{\perp_1})$ and $\mathrm{Ker}\, f \in {}^{\perp_1}(T^{\perp_1})$. Proceeding by induction on ℓ and using that $\mathscr{S}$ is closed under extensions, we conclude that $E \in \mathscr{S}$, which completes the proof. □

Corollary 4.12. *Let $\vec{\mathcal{H}} = \mathrm{Qcoh}\,\mathbb{X}$ with $\mathbb{X}$ a weighted noncommutative regular projective curve. There is a bijection between the equivalence classes of tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part, and the set of pairs (B,V) given by a branch sheaf $B \in \mathcal{H}$ and a subset $\emptyset \neq V \subseteq \mathbb{X}$.* □

Remark 4.13. It is well known that the hereditary torsion pairs in $\mathrm{Qcoh}\,\mathbb{X}$ are in bijection with the Serre subcategories of $\mathrm{coh}\,\mathbb{X}$. As explained in [7, Sec. 5.2], this bijection restricts to a bijective correspondence between the hereditary torsion pairs $(\mathcal{T},\mathcal{F})$ with non-trivial $\mathcal{F}$ (or equivalently, such that $\mathcal{F}$ generates $\mathrm{Qcoh}\,\mathbb{X}$) and the Serre subcategories consisting of finite length objects. Moreover, one easily verifies that the Serre subcategories of $\mathrm{add}\,\mathcal{H}_0$ are precisely the small additive closures of unions of tubes and pairwise non-adjacent wings. In other words, there is a surjective map from the set of all pairs (B,V) given by a branch sheaf B and a subset $V \subseteq \mathbb{X}$, and the Serre subcategories of $\mathrm{add}\,\mathcal{H}_0$. This map is not injective in general, because different branch sheaves can give rise to the same wings. In the non-weighted case, however, the parametrization of tilting sheaves reduces to the subsets $V \subseteq \mathbb{X}$, and we obtain a bijection between tilting sheaves and faithful hereditary torsion pairs in $\mathrm{Qcoh}\,\mathbb{X}$, in perfect analogy with the classification of tilting modules over commutative noetherian rings from [8]. For more details we refer to [7, Sec. 5.2].

A correspondence. Next, we establish an analogue of [5, Thm. 2.2] stating that the resolving subclasses of $\mathcal{H}$ correspond bijectively to tilting classes of finite type. As we will see below, in the domestic and in the tubular cases every tilting class is of finite type.

Theorem 4.14. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$. The assignments $\Phi\colon \mathscr{S} \mapsto \mathscr{S}^{\perp_1}$ and $\Psi\colon \mathcal{B} \mapsto {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$ define mutually inverse bijections between*

- *resolving classes $\mathscr{S}$ in $\mathcal{H}$, and*
- *tilting classes $\mathcal{B} = T^{\perp_1}$ with $T \in \vec{\mathcal{H}}$ tilting of finite type.*

For the proof of the Theorem, we need the following observations.

Remark 4.15. In the situation of Lemma 4.9 (2), the right perpendicular category $\mathcal{W}^{\perp}$ of a wing $\mathcal{W}$ rooted in $S[r-1]$ coincides with the right perpendicular category to its basis $S, \tau^{-}S, \dots, \tau^{-(r-2)}S$. If B forms a (connected) branch in $\mathcal{W}$, then also $B^{\perp} = \mathcal{W}^{\perp}$, and when forming this perpendicular category, the r rays starting in the simple objects $S, \tau^{-}S, \dots, \tau^{-(r-2)}S, \tau^{-(r-1)}S$ and the corresponding Prüfer sheaves are turned into a single ray $\tau^{-(r-1)}S[rn]$, $n \geq 1$, and a single Prüfer sheaf $S[\infty]$.

Lemma 4.16 (Perpendicular Lemma). *Let $B \in \mathcal{H}$ be a branch sheaf. Let $T \in \vec{\mathcal{H}}$ be a sheaf such that $T \in B^{\perp}$.*

(1) *We have $B^{\perp} \simeq \operatorname{Qcoh}\mathbb{X}'$, where $\mathbb{X}'$ is a noncommutative regular projective curve with reduced weights $1 \leq p_i' \leq p_i$.*

(2) *$T \oplus B$ is a (large) tilting sheaf in $\vec{\mathcal{H}}$ if and only if T is a (large) tilting sheaf in $\vec{\mathcal{H}}' = \operatorname{Qcoh}\mathbb{X}'$.*

Proof. (1) This follows from the preceding remark.
(2) It is clear that $T \oplus B$ satisfies (TS1) if and only if so does T. We assume that $T \oplus B$ satisfies (TS2). Let $X \in \vec{\mathcal{H}}'$ such that $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. Since $\vec{\mathcal{H}}' = B^{\perp}$ we get $\operatorname{Hom}(T \oplus B, X) = 0 = \operatorname{Ext}^1(T \oplus B, X)$, and hence $X = 0$ follows, and T satisfies (TS2). Conversely, let T satisfy (TS2). Let $X \in \vec{\mathcal{H}}$ with $\operatorname{Hom}(T \oplus B, X) = 0 = \operatorname{Ext}^1(T \oplus B, X)$. Then in particular $X \in B^{\perp} = \vec{\mathcal{H}}'$, and also $\operatorname{Hom}(T, X) = 0 = \operatorname{Ext}^1(T, X)$. Then $X = 0$, so that $T \oplus B$ satisfies (TS2). □

Proof of Theorem 4.14. $\Phi(\mathscr{S}) = \mathscr{S}^{\perp_1}$ defines a map between the named sets by Theorem 4.4. By the discussion in 4.1 we see that $\mathscr{S} := \Psi(\mathcal{B}) = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$ satisfies conditions (i), (ii) and (iii) for resolving. Notice that $\mathscr{S}$ is even closed under subobjects since $\operatorname{Qcoh}\mathbb{X}$ is hereditary. We show that $\mathscr{S}$ also generates $\vec{\mathcal{H}}$.

First we show that $\mathscr{S}$ contains a non-zero vector bundle. Let $\mathscr{S}' \subseteq \mathcal{H}$ with $\mathcal{B} = \mathscr{S}'^{\perp_1}$. Then

$$\mathscr{S}' \subseteq {}^{\perp_1}(\mathscr{S}'^{\perp_1}) \cap \mathcal{H} = \mathscr{S}. \tag{4.12}$$

We assume that $\mathscr{S}$ does not contain any non-zero vector bundle, which we will lead to a contradiction. Then $\mathscr{S}' \subseteq \mathcal{H}_0$. Let T be tilting with $\mathcal{B} = T^{\perp_1}$. Since a coherent X lies in ${}^{\perp_1}\mathcal{B}$ if and only if $\operatorname{Ext}^1(X, T) = 0$, we get $\operatorname{Hom}(T, E) \neq 0$ for every non-zero vector bundle E. If T is additionally torsionfree, then we infer $\mathrm{Ext}^1(T,F)=0$ for all finite length sheaves F. It follows from (TS2) that T is a generator for $\vec{\mathcal{H}}$, and then also projective. From Serre duality we conclude that there is no non-zero morphism from a vector bundle to T, which is impossible. If on the other hand, T has a large torsion part, then by Lemma 4.10 the torsionfree part T_+ is x-divisible for (at least) one point x. But T, and then also T_+, maps epimorphic to some line bundle L', and L' maps non-trivially to a simple sheaf S_x concentrated at x, thus $\mathrm{Hom}(T_+,S_x)\neq 0$, contradicting the x-divisibility. The final case to consider is that the torsion part T_0 is a branch sheaf B. By Lemma 4.16 then T_+ is torsionfree tilting in $B^\perp=\mathrm{Qcoh}\,\mathbb{X}'\subseteq\vec{\mathcal{H}}$. Since $\mathrm{vect}\,\mathbb{X}'=\mathrm{vect}\,\mathbb{X}\cap B^\perp$ (the inclusion of the right perpendicular category is rank-preserving, by [28, Prop. 9.6]), we infer that T_+ maps non-trivially to any non-zero vector bundle over $\mathbb{X}'$, and we get a contradiction by the torsionfree case treated before. Thus in any case, $\mathscr{S}$ contains a non-zero vector bundle.

Since $\mathscr{S}$ is closed under subobjects, it contains also a line bundle L'. By [52, Lem. IV.4.1], [39, Rem. 3.8] there is a suitable product σ of tubular shifts such that (L',σ) forms an ample pair, and there is a monomorphism $\sigma^{-1}L'\to L'$. We conclude that $\mathscr{S}$ contains the system of generators $\{\sigma^{-n}L'\mid n\geq 0\}$ for $\vec{\mathcal{H}}$.

We have thus shown that Φ and Ψ define maps between the named sets. Now, from (4.12) we infer $\Psi\Phi(\mathscr{S})\supseteq\mathscr{S}$. The converse inclusion follows from [53, Lem. 1.3] as in the proof of Lemma 4.11 (2). Thus $\Psi\Phi(\mathscr{S})=\mathscr{S}$. Moreover, $\Phi\Psi(\mathcal{B})=({}^{\perp_1}\mathcal{B}\cap\mathcal{H})^{\perp_1}\supseteq({}^{\perp_1}\mathcal{B})^{\perp_1}\supseteq\mathcal{B}$. Since $\mathcal{B}$ is of finite type, there is $\mathscr{S}'\subseteq\mathcal{H}$ such that $\mathcal{B}=\mathscr{S}'^{\perp_1}$, and from (4.12) we conclude $\mathscr{S}'\subseteq\Psi(\mathcal{B})$, hence $\Phi\Psi(\mathcal{B})=\mathcal{B}$. This completes the proof of the theorem. □

Corollary 4.17. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}}=\mathrm{Qcoh}\,\mathbb{X}$. A generating system $\mathscr{S}\subseteq\mathcal{H}$ is resolving if and only if it is closed under extensions and subobjects.* □

We further have the following immediate consequence of Theorem 4.4.

Corollary 4.18. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and $\vec{\mathcal{H}}=\mathrm{Qcoh}\,\mathbb{X}$. If $\mathscr{S}'\subseteq\mathcal{H}$ is a set containing at least one non-zero vector bundle, then there is a tilting sheaf $T\in\vec{\mathcal{H}}$ with $T^{\perp_1}=\mathscr{S}'^{\perp_1}$.*

Proof. Let $\mathcal{B}=\mathscr{S}'^{\perp_1}$. Then $\mathscr{S}:={}^{\perp_1}\mathcal{B}\cap\mathcal{H}$ satisfies $\mathscr{S}^{\perp_1}=\mathcal{B}$, it is closed under extensions and subobjects, and we see as in the proof of Theorem 4.14 that it contains a generating system. Thus $\mathscr{S}$ is resolving, and the claim follows from Theorem 4.4. □

Maximal rigid objects in a (large) tube. Let $\vec{\mathcal{U}}$ be the direct limit closure of a tube $\mathcal{U}$ in $\vec{\mathcal{H}}$. Recall from Section 3 that $\vec{\mathcal{U}}$ is an exact subcategory of $\vec{\mathcal{H}}$, and it is itself a hereditary locally finite Grothendieck category, cf. also [19]. Following [14], we call an object U in $\vec{\mathcal{U}}$ *rigid* if $\mathrm{Ext}^1(U,U)=0$, and *maximal rigid* if it is rigid and every indecomposable $Y\in\vec{\mathcal{U}}$ satisfying $\mathrm{Ext}^1(U\oplus Y,U\oplus Y)=0$ is a direct summand of U. This definition relies on the fact that every rigid object U has an indecomposable decomposition. Indeed, up to multiplicities, U is a finite direct sum of indecomposables, which are either Prüfer sheaves or exceptional coherent sheaves, cf. Theorem 3.8. U is said to be of *Prüfer type* if it has a Prüfer summand. Finally, two maximal rigid objects are said to be *equivalent* if they have the same indecomposable direct summands.

As a consequence of the discussion above, we can recover and refine results from [14, Sec. 5].

Corollary 4.19. *Let $\vec{\mathcal{U}}$ be the direct limit closure of a tube $\mathcal{U} = \mathcal{U}_x$ in $\vec{\mathcal{H}}$. The following statements are equivalent for an object $U \in \vec{\mathcal{U}}$.*

1. *U is maximal rigid in $\vec{\mathcal{U}}$.*
2. *U is tilting in $\vec{\mathcal{U}}$.*
3. *U is of Prüfer type and it coincides, up to multiplicities, with the summand $(tT)_x$ supported at x in the torsion part of some large tilting sheaf $T \in \vec{\mathcal{H}}$.*

Moreover, the map $U \mapsto (\mathcal{T}_U, \mathcal{F}_U)$ where $\mathcal{F}_U := {}^{\perp_1}U \cap \mathcal{U}$ and $\mathcal{T}_U := {}^{\perp_0}\mathcal{F}_U \cap \mathcal{U}$ defines a bijective correspondence between equivalence classes of such objects U and torsion pairs in $\mathcal{U}$ whose torsionfree class generates $\mathcal{U}$. If B is the coherent part of U, which is a branch sheaf, and the set $\mathcal{R}_x$ is defined as in (4.8), then the torsion pair corresponding to U is explicitly given as

$$\mathcal{F}_U = \operatorname{add}\big(\tau^-(B^>) \cup \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\big) \quad \textit{and} \quad \mathcal{T}_U = \operatorname{gen}(\tau^- B),$$

and we have

$$\mathcal{F}_U \cap {\mathcal{F}_U}^{\perp_1} = \operatorname{add}(B). \tag{4.13}$$

Proof. The implication (3)⇒(1) follows immediately from Lemma 4.9 (2).

For the implication (3)⇒(2) let $T = T_{(B,\{x\})} = T_+ \oplus U$ be a large tilting sheaf in $\vec{\mathcal{H}}$. In order to prove that U is tilting in $\vec{\mathcal{U}}$, it suffices to verify condition (TS2), that is, to show that any $X \in U^\perp \cap \vec{\mathcal{U}}$ must be zero. Let E be a direct summand of X of finite length. Then also $E \in U^\perp$. Using Serre duality we obtain moreover $E \in T^\perp$, since T_+ is torsionfree and x-divisible. Thus $E = 0$ since T is tilting. So X does not have any non-zero summand of finite length, hence it is a direct sum of Prüfer sheaves in $\vec{\mathcal{U}}$ by Corollary 3.7. Since U has a Prüfer summand (which maps onto all Prüfer sheaves in $\vec{\mathcal{U}}$), the condition $\operatorname{Hom}(U, X) = 0$ implies $X = 0$, as desired.

We now show that each of (1) or (2) implies (3). Let U be maximal rigid or tilting in $\vec{\mathcal{U}}$, and assume without loss of generality that there are no multiplicities. Then $U = B \oplus U'$ where $U' \neq 0$ is a direct sum of Prüfer sheaves and B is of finite length. If $B \neq 0$, then U' defines a collection $\mathcal{W} = U'^{\perp_1} \cap \mathcal{U}$ of pairwise non-adjacent wings in the exceptional tube $\mathcal{U}$, and we infer as in the proof of Lemma 4.9 (2) that B is a direct sum of connected branches in $\mathcal{W}$. In other words, B is a branch sheaf, and U satisfies (3), being for instance the torsion part of the tilting sheaf $T = T_{(B,\{x\})} = T_+ \oplus U$.

Moreover, by Lemma 4.11, there is a resolving subcategory $\mathscr{S}$ of $\mathcal{H}$ corresponding to $(B,\{x\})$. It has the form $\mathscr{S} = {}^{\perp_1}(T^{\perp_1})\cap\mathcal{H} = {}^{\perp_1}T\cap\mathcal{H}$, and it gives rise to a resolving subcategory $\mathscr{S}\cap\mathcal{U}$ in $\mathcal{U}$, which coincides with $\mathcal{F}_U = {}^{\perp_1}U\cap\mathcal{U}$ because T_+ is x-divisible. The explicit shape of $\mathcal{F}_U$ is an immediate consequence of (4.10). Moreover, we have $(\mathscr{S}\cap\mathcal{U})^{\perp_1}\cap\mathcal{U} = \mathscr{S}^{\perp_1}\cap\mathcal{U}$ (since $\operatorname{Ext}^1(\mathcal{H}_+,\mathcal{U}) = 0$ by Serre duality), and since $\mathscr{S}^{\perp_1} = \operatorname{Gen}(T)$, we get ${\mathcal{F}_U}^{\perp_1}\cap\mathcal{U} = \operatorname{Gen}(T)\cap\mathcal{U} = \operatorname{gen}(B)$. Thus $\mathcal{T}_U = {}^{\perp_0}\mathcal{F}_U\cap\mathcal{U} = \tau^-{\mathcal{F}_U}^{\perp_1}\cap\mathcal{U} = \operatorname{gen}(\tau^-B)$. By (4.11) we finally obtain $\mathcal{F}_U\cap{\mathcal{F}_U}^{\perp_1} = \mathscr{S}\cap\mathscr{S}^{\perp_1}\cap\mathcal{U} = \operatorname{Add}(T)\cap\mathcal{U} = \operatorname{add}(B)$, which proves (4.13).

It follows readily that $U\mapsto(\mathcal{T}_U,\mathcal{F}_U)$ defines a map between the named sets, and this map is injective since $\mathcal{F}_U$, by (4.13), determines the branch part of U, and therefore U itself. This map is also surjective: if $(\mathcal{T},\mathcal{F})$ is a torsion pair in $\mathcal{U}$ with $\mathcal{F}$ generating, then $\mathcal{F}$ is clearly resolving in $\mathcal{U}$, and we can apply Theorem 4.4 for the hereditary, locally finite Grothendieck category $\vec{\mathcal{U}}$ to obtain a tilting object U in $\vec{\mathcal{U}}$ with $U^{\perp_1} = \mathcal{F}^{\perp_1}$. As in the proof of Lemma 4.11 (2) we get $\mathcal{F} = {}^{\perp_1}U\cap\mathcal{U} = \mathcal{F}_{\mathcal{U}}$, from which the claim follows. □

GENUS ZERO. *For the rest of this section let* $\mathbb{X}$ *be of genus zero and* $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$*.* We refine the results above with the following notion.

DEFINITION 4.20. Let $\mathscr{S}$ be a class of objects in $\mathcal{H}$. We call $\mathscr{S}$ *strongly resolving* if it is closed under extensions and subobjects, and if it contains a tilting bundle T_{cc}.

REMARK 4.21. Let $\mathscr{S}\subseteq\mathcal{H}$ be a strongly resolving class containing a tilting bundle T_{cc}. Then $\mathscr{S}$ is resolving (this is verified by using that $T_{\mathrm{cc}}(-nx)\subseteq T_{\mathrm{cc}}$ by (2.8) for all $n\geq 0$ and all points $x\in\mathbb{X}$, and that the system $(T_{\mathrm{cc}}(-nx), n\geq 0)$ is generating by [38, Prop. 6.2.1]).

So we can apply Theorem 4.4 to obtain a tilting sheaf T generating the class $\mathcal{B} = \mathscr{S}^{\perp_1}$. More explicitly, any special $\mathcal{B}$-preenvelope

$$0\to T_{\mathrm{cc}}\to T_0\to T_1\to 0 \tag{4.14}$$

of T_{cc} leads to a tilting sheaf of finite type

$$T = T_0\oplus T_1$$

with $T^{\perp_1} = \mathcal{B}$ and $T\in\operatorname{Gen}(T_{\mathrm{cc}})$.

Indeed, the exact sequence $\operatorname{Ext}^1(T_1,X)\to\operatorname{Ext}^1(T_0,X)\to\operatorname{Ext}^1(T_{\mathrm{cc}},X)\to 0$ shows that $X\in T^{\perp_1}$ implies $X\in{T_{\mathrm{cc}}}^{\perp_1} = \operatorname{Gen}(T_{\mathrm{cc}})$, and the claim follows replacing G by T_{cc} in the proof of Theorem 4.4.

Notice that the sheaves T_0 and T_1 are $\mathscr{S}$-filtered in the sense of [58, Def. 2.9], and the class ${}^{\perp_1}(T^{\perp_1})$ consists precisely of the direct summands of the $\mathscr{S}$-filtered objects, see [58, Thm. 2.13 and Cor. 2.15].

EXAMPLE 4.22. (1) The system $\mathscr{S} = \operatorname{vect}\mathbb{X}$ of all vector bundles is strongly resolving, and the Lukas tilting sheaf $\mathbf{L}$ from Proposition 4.5 with $\mathbf{L}^{\perp_1} = \mathscr{S}^{\perp_1}$ is large, torsionfree and satisfies condition (TS3).

(2) Let $T = T_{(B,V)}$ where $\emptyset \neq V \subseteq \mathbb{X}$ and B is a branch sheaf. The class $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is given by (4.10), and it is strongly resolving as $\operatorname{vect}\mathbb{X} \subseteq \mathscr{S}$; we even have $T_{\text{can}} \in \mathscr{S}$. By the preceding discussion $T^{\perp_1} = \mathscr{S}^{\perp_1}$ and $T \in \operatorname{Gen}(T_{\text{can}})$. Sequence (4.14) shows that T satisfies (TS3). In fact, we will see in Theorem 10.1 that T even satisfies condition (TS3+).

5. Tilting sheaves under perpendicular calculus

Throughout this section, $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ with $\mathbb{X}$ a weighted noncommutative regular projective curve over a field k. We use perpendicular calculus (in particular Lemma 4.16) to reduce some considerations to tilting sheaves $T_V = T_{(0,V)}$ with trivial branch sheaf $B = 0$. This will allow us to obtain an explicit description of the torsionfree part T_+ of any tilting sheaf $T_{(B,V)}$ and an alternate method to determine the Prüfer summands in the torsion part.

Remark 5.1. The Perpendicular Lemma 4.16 has several applications.

(1) Let $B \in \mathcal{H}$ be a branch sheaf. Let $T \in \vec{\mathcal{H}}$ be a sheaf such that tT and B have disjoint supports and $\operatorname{Ext}^1(B,T) = 0$ holds. Then $T \in B^{\perp}$. (This follows by applying $\operatorname{Hom}(B,-)$ to the canonical exact sequence $0 \to tT \to T \to T/tT \to 0$.) Thus we can use Lemma 4.16 to reduce our considerations to tilting sheaves with trivial exterior branch part $B_{\mathfrak{e}}$.

(2) Let $\mathbb{X}$ be a noncommutative regular projective curve of weight type $(p_1,\dots,p_t)$ (with $p_i \geq 2$), and assume that $\mathbb{X}'$ arises from $\mathbb{X}$ by reduction of some weights, so that $\mathbb{X}'$ is of weight type $(p_1',\dots,p_t')$, with $1 \leq p_i' \leq p_i$. Then the classification of (large) tilting sheaves in $\operatorname{Qcoh}\mathbb{X}$ is at least as complicated as the classification in $\operatorname{Qcoh}\mathbb{X}'$. Indeed, if T' is a (large) tilting sheaf in $\operatorname{Qcoh}\mathbb{X}'$, then we can find a branch sheaf $B \in \operatorname{coh}\mathbb{X}$ such that $T = T' \oplus B$ is (large) tilting in $\operatorname{Qcoh}\mathbb{X}$: namely, we have $\operatorname{Qcoh}\mathbb{X}' \simeq \mathcal{E}^{\perp} \subseteq \operatorname{Qcoh}\mathbb{X}$ for a finite set $\mathcal{E}$ of exceptional simple sheaves; we can then take any branch sheaf B whose components lie in the wings whose bases belong to $\mathcal{E}$; then $B^{\perp} = \mathcal{E}^{\perp}$ and $T' \in B^{\perp}$. Clearly, if T_1' and T_2' are not equivalent, then $T_1' \oplus B$ and $T_2' \oplus B$ are also not equivalent.

(3) In particular: if $\mathbb{X}$ is a weighted projective line of wild type (in the sense of [27]), then $\operatorname{Qcoh}\mathbb{X}$ contains all large tilting sheaves coming from a suitable weighted projective line $\mathbb{X}'$ of tubular type.

Let us now assume that $V \neq \emptyset$ and $B_{\mathfrak{e}} = 0$. Then all the branches of $B = B_{\mathfrak{i}}$ are interrelated with Prüfer summands of $T_{(B,V)}$ as described in Lemma 4.9 (2). Let $\vec{\mathcal{H}}' = (\tau^- B)^{\perp} = \operatorname{Qcoh}\mathbb{X}'$ and $i\colon \vec{\mathcal{H}}' \to \vec{\mathcal{H}}$ the inclusion. If we define, in analogy of Definition 3.1, the class $\mathscr{S}_V'$ and its direct limit closure $\mathcal{T}_V' = \vec{\mathscr{S}_V'}$ in $\vec{\mathcal{H}}'$, then it is easy to see that we have

$$\vec{\mathcal{H}}'/\mathcal{T}_V' \simeq {\mathscr{S}_V'}^{\perp} = (\tau^- B)^{\perp} \cap (i\mathscr{S}_V')^{\perp} = {\mathscr{S}_V}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V.$$

Lemma 5.2. *Let $T = T_{(B,V)}$ be the tilting sheaf in $\vec{\mathcal{H}}$ given by (4.6) with torsionfree part T_+. We assume $B_{\mathfrak{e}} = 0$. Then*

$$T_V := T_{(0,V)} = T_+ \oplus \bigoplus_{x\in V} \bigoplus_{j\in\mathcal{R}_x} \tau^j S_x[\infty] \tag{5.1}$$

is a large tilting sheaf in $\vec{\mathcal{H}}'$.

Proof. It is sufficient to show that $T_{(0,V)}$ lies in the right-perpendicular category $(\tau^- B)^\perp$. By the definition of $\mathcal{R}_x$, and since the $\tau^j S_x[\infty]$ are injective, this is true for the direct sum of the Prüfer summands. Since T_+ is V-divisible, this also holds for T_+. □

We conclude

Corollary 5.3. *$T_{(B,V)} = T_{(B_{\mathfrak{i}},V)} \oplus B_{\mathfrak{e}}$ and $T_{(0,V)}$ have the same torsionfree part.* □

We will now deal with $T_V = T_{(0,V)}$. Its torsion part consists of Prüfer sheaves only. We consider T_V as object in $\vec{\mathcal{H}}' = \operatorname{Qcoh}\mathbb{X}' = (\tau^- B)^\perp$, and we exhibit the following explicit construction.

Let Λ' be a finite direct sum of indecomposable vector bundles F_j in $\vec{\mathcal{H}}' = \operatorname{Qcoh}\mathbb{X}'$ such that Λ' maps onto each simple sheaf in $\vec{\mathcal{H}}'$. For instance,

- by [39, Prop. 1.1], we can always find special line bundles F_j with this property (by applying suitable tubular shifts to the structure sheaf L); or
- in case $\mathbb{X}$ is of genus zero, we can take alternatively $\Lambda' = T'_{\mathrm{can}}$, a canonical configuration in $\vec{\mathcal{H}}'$. (See Remark 5.12.)

We denote by $e(j,x) = e(j,x,\Lambda')$ the $\operatorname{End}(S_x)$-dimension of $\operatorname{Ext}^1(\tau^j S_x, \Lambda')$, by $p'(x)$ the weight of x in $\mathbb{X}'$, and consider the universal sequence in $\mathcal{H}'$

$$0 \to \Lambda' \to \Lambda'(x) \to \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x)^{e(j,x)} \to 0 \tag{5.2}$$

where the $\tau^j S_x$ are the simple sheaves in $\mathcal{H}'$ concentrated at x. Since the inclusion $S_x \to S_x[\infty]$ yields a surjection $\operatorname{Ext}^1(S_x[\infty], \Lambda') \to \operatorname{Ext}^1(S_x, \Lambda')$, this induces a short exact sequence in $\vec{\mathcal{H}}' \subseteq \vec{\mathcal{H}}$

$$\eta_x \colon 0 \to \Lambda' \to \Lambda'_x \to \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \to 0. \tag{5.3}$$

Note that $\tau^j S_x[\infty]$ are also Prüfer sheaves in $\vec{\mathcal{H}}$. For $x \in V$ these short exact sequences are spliced together via

$$\operatorname{Ext}^1\Big(\bigoplus_{y\in V} \tau^j S_y[\infty], \Lambda'\Big) \simeq \prod_{y\in V} \operatorname{Ext}^1(\tau^j S_y[\infty], \Lambda'), \tag{5.4}$$

which defines

$$\eta_V\colon 0 \to \Lambda' \to \Lambda'_V \to \bigoplus_{x\in V} \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \to 0. \tag{5.5}$$

Lemma 5.4. *Λ'_V is torsionfree and precisely V-divisible.*

Proof. That Λ'_V is torsionfree and V-divisible can be shown as in the proof of [55, Prop. 5.2]. Let $y \in \mathbb{X}\setminus V$ and $S \in \mathcal{U}_y$ be simple. By applying $\operatorname{Hom}(S,-)$ to sequence (5.5) we get $\operatorname{Ext}^1(S,\Lambda'_V) \simeq \operatorname{Ext}^1(S,\Lambda') \neq 0$. Thus Λ'_V is precisely V-divisible. □

We now adopt the notation from Section 3 and interpret the sequence η_V in (5.5) in terms of localization theory.

Lemma 5.5. *Assume $V \neq \emptyset$ and $B_{\mathfrak{e}} = 0$. Let $\pi = \pi_V\colon \vec{\mathcal{H}} \to \vec{\mathcal{H}}/\mathcal{T}_V$ be the canonical quotient functor.*

(1) *In ${\mathscr{S}_V}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V$ we have $\pi\Lambda' \simeq \pi(\Lambda'_V)$.*

(2) *$\pi\Lambda'$ is a finitely presented projective generator in ${\mathscr{S}_V}^{\perp} \simeq \vec{\mathcal{H}}/\mathcal{T}_V$.*

(3) *The functor $X \mapsto \operatorname{Hom}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda', X)$ yields an equivalence*

$$\vec{\mathcal{H}}/\mathcal{T}_V \simeq \operatorname{Mod}(\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda')).$$

In particular, ${\mathscr{S}_V}^{\perp}$ is locally noetherian.

Proof. (1) This is clear by the exact sequence (5.5).
(2) Let $x \in V$. Then Λ' and $\Lambda'(nx)$ become isomorphic in $\vec{\mathcal{H}}/\mathcal{T}_V$ for all $n \in \mathbb{Z}$, which follows from (5.2). We note that every short exact sequence in $\vec{\mathcal{H}}/\mathcal{T}_V$ is isomorphic to the image of a short exact sequence in $\vec{\mathcal{H}}$ under the quotient functor π. If $A \in \mathcal{H}$, then, by [38, 0.4.6], [39], for sufficiently large $n > 0$ we have $\operatorname{Ext}^1(\Lambda'(-nx), A) = 0$, which shows that $\pi\Lambda' \simeq \pi(\Lambda'(-nx))$ is projective with respect to images of coherent objects. Since the class $\operatorname{Ker}\operatorname{Ext}^1(\pi\Lambda', -)$ is closed under direct limits, it follows that $\pi\Lambda'$ is projective. Since also, again by [38, 0.4.6], for sufficiently large $n > 0$ we have $\operatorname{Hom}(\Lambda'(-nx), A) \neq 0$, we get $\operatorname{Hom}(\pi\Lambda', \pi A) \neq 0$ for every $A \in \mathcal{H}$, and it follows easily that $\pi\Lambda'$ is a generator in the quotient category. It is finitely presented because $\operatorname{Hom}(\Lambda', -)$ and hence $\operatorname{Hom}(\pi\Lambda', -)$ preserve direct limits (we refer to Remark 3.2 and [34, Lem. 2.5]).
(3) This is a well-known result by Gabriel-Mitchell, we refer to [13, II.1]. For the last statement, recall that Λ' is noetherian, and so is $\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda')$. □

As an additional information on Λ'_V we exhibit its minimal injective resolution. We recall that the sheaf $\mathcal{K}$ of rational functions is the injective envelope of the structure sheaf L.

Proposition 5.6. *Let $\emptyset \neq V \subseteq \mathbb{X}$. There is a short exact sequence*

$$0 \to \Lambda'_V \to \Lambda'_{\mathbb{X}} \to \bigoplus_{y\in\mathbb{X}\setminus V} \bigoplus_{j=0}^{p'(y)-1} (\tau^j S_y[\infty])^{e(j,y)} \to 0. \tag{5.6}$$

This is the minimal injective resolution of Λ'_V. Moreover, $\Lambda'_{\mathbb{X}} \simeq \mathcal{K}^n$ with $n = \operatorname{rk}(\Lambda')$.

Proof. Via the identity (5.4) we have $\eta_V = (\eta_y)_{y\in V}$ and $\eta_{\mathbb{X}} = (\eta_x)_{x\in\mathbb{X}}$. Thus inclusion $\iota\colon \bigoplus_{y\in V}\bigoplus_j (\tau^j S_y[\infty])^{e(j,y)} \to \bigoplus_{x\in\mathbb{X}}\bigoplus_j (\tau^j S_x[\infty])^{e(j,x)}$ induces a map on the Ext^1-spaces, which on the products induces projection onto the components of V, and thus maps $\eta_{\mathbb{X}}$ to η_V. Thus there is a pull-back diagram

$$\begin{tikzcd}
\eta_V\colon 0 \arrow[r] & \Lambda' \arrow[r] \arrow[d, equal] & \Lambda'_V \arrow[r] \arrow[d] & \bigoplus_{y\in V}\bigoplus_{j=0}^{p'(y)-1} (\tau^j S_y[\infty])^{e(j,y)} \arrow[r] \arrow[d, "\iota"] & 0\,, \\
\eta_{\mathbb{X}}\colon 0 \arrow[r] & \Lambda' \arrow[r] & \Lambda'_{\mathbb{X}} \arrow[r] & \bigoplus_{x\in\mathbb{X}}\bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \arrow[r] & 0
\end{tikzcd}$$

that is, $\eta_V = \eta_{\mathbb{X}} \cdot \iota$. Now we get sequence (5.6) with the snake lemma. The sequence (5.5) is, for $V = \mathbb{X}$, the minimal injective resolution of Λ'; this follows from the construction of $\Lambda'_{\mathbb{X}}$ like in [53, Thm. 4.1]. Therefore $\Lambda'_{\mathbb{X}} \simeq \mathcal{K}^n$ with $n = \operatorname{rk}(\Lambda')$. From the monomorphisms $\Lambda' \to \Lambda'_V \to \Lambda'_{\mathbb{X}}$ it is then clear that the sequence (5.6) is the minimal injective resolution of Λ'_V. □

Since the sequence (5.6) lies in ${\mathscr{S}_V}^{\perp} = \operatorname{Mod}(\operatorname{End}_{\vec{\mathcal{H}}/\mathcal{T}_V}(\pi\Lambda'))$, it is also the minimal injective resolution of the projective generator $\pi\Lambda'_V$.

The main result about the torsionfree part interprets T_+ as a projective generator in the localization of $\vec{\mathcal{H}}$ (or $\vec{\mathcal{H}}'$) at V.

Proposition 5.7. $\operatorname{Add}(T_+) = \operatorname{Add}(\Lambda'_V)$.

Proof. Invoking the uniqueness statement of Theorem 4.8 it is sufficient to show that $Q = Q_+ \oplus Q_0$ with $Q_+ = \Lambda'_V$ and $Q_0 = T_0 = \bigoplus_{x\in V}\bigoplus_{j=0}^{p'(x)-1} \tau^j S_x[\infty]$ is a tilting object in $\vec{\mathcal{H}}'$. From Lemma 5.5 we deduce $\operatorname{Ext}^1(Q_+, {Q_+}^{(I)}) = 0$, and using the sequence (5.6) we see that $\operatorname{Ext}^1(Q, Q^{(I)}) = 0$ for each set I. Let $X \in \vec{\mathcal{H}}'$. We conclude that $X \in \operatorname{Gen}(Q)$ implies $X \in Q^{\perp_1}$. We have to show that the converse also holds. So, let now $X \in Q^{\perp_1}$. In particular, $X \in {Q_0}^{\perp_1}$. The embeddings $S_y \to S_y[\infty] \to Q_0$ give rise to epimorphisms $\operatorname{Ext}^1(Q_0, X) \to \operatorname{Ext}^1(S_y, X)$ for all $y \in V$, and hence X is V-divisible. Consider the short exact sequences $0 \to K \to {Q_+}^{(I)} \to B \to 0$ and $0 \to B \to X \to C \to 0$, where $I = \operatorname{Hom}(Q_+, X)$, so that B is the trace of Q_+ in X. It is sufficient to show that $C = 0$. Since X is V-divisible, the same holds for C. Moreover $\operatorname{Hom}(Q_+, C) = 0$. We show, that C is V-torsionfree. Assume, this is not the case. Then there is $y \in V$ such that $\mathrm{Hom}(S_y, C) \neq 0$. Since C (and thus also tC and $(tC)_y$) is y-divisible, we get $S_y[\infty] \subseteq (tC)_y \subseteq C$. Since $S_y[\infty]$ is injective, there is a surjection $\mathrm{Hom}(Q_+, S_y[\infty]) \to \mathrm{Hom}(\Lambda', S_y[\infty]) \neq 0$, and $\mathrm{Hom}(Q_+, C) \neq 0$ follows, a contradiction. Thus, $C \in \mathscr{S}_V{}^{\perp}$, and since $\mathrm{Hom}(Q_+, C) = 0$, we get $C = 0$ by Lemma 5.5. This finishes the proof. □

The following is a reformulation of Theorem 4.8.

Theorem 5.8. *Let $\mathbb{X}$ be a weighted noncommutative regular projective curve. The tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part are, up to equivalence, the sheaves of the form*

$$T_{(B,V)} = T_V \oplus B$$

with a subset $\emptyset \neq V \subseteq \mathbb{X}$, a branch sheaf $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ with interior and exterior part $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively, and a tilting sheaf T_V in the category $\mathrm{Qcoh}\,\mathbb{X}' = (B_{\mathfrak{e}} \oplus \tau^- B_{\mathfrak{i}})^{\perp} \subseteq \vec{\mathcal{H}}$, given as the direct sum of the middle term and the end term of the sequence (5.5). □

Corollary 5.9. *Let $\mathbb{X}$ be a (non-weighted) noncommutative regular projective curve. The tilting sheaves in $\vec{\mathcal{H}}$ having a large torsion part are, up to equivalence, the sheaves T_V with $\emptyset \neq V \subseteq \mathbb{X}$.* □

Genus zero. Before we specialize the above construction to the genus zero case in Remark 5.12 below, we need to explain some notations and concepts, which will also be used in later sections.

5.10. Numerical invariants. Each noncommutative curve of genus zero $\mathbb{X}$ has a so-called underlying tame bimodule, which is either of dimension type $(2,2)$ or $(1,4)$. In the first case we have $\varepsilon = 1$, in the second $\varepsilon = 2$. We recall that the structure sheaf L has the property that for every point $x \in \mathbb{X}$ there is precisley one simple $S_x \in \mathcal{U}_x$ with $\mathrm{Hom}(L, S_x) \neq 0$, and $\mathrm{End}(L)$ is a skew field. One then defines $\kappa = [\mathrm{End}(L) : k]$ and for every point x

$$f(x) = \frac{1}{\varepsilon}[\mathrm{Hom}(L, S_x) : \mathrm{End}(L)], \;\; e(x) = [\mathrm{Hom}(L, S_x) : \mathrm{End}(S_x)].$$

For an exceptional point x_i one writes $f_i = f(x_i)$ and $e_i = e(x_i)$. We have

$$\deg(S_x) = \frac{\bar{p}}{p(x)} f(x).$$

If k is algebraically closed, then all the numbers ε, κ, $e(x)$, $f(x)$ are equal to 1. We refer to [44], [42] and [38] for details.

5.11. Canonical configuration. Let $\mathbb{X}$ again be of genus zero and of arbitrary weight type. Let L be the structure sheaf, which is of degree 0 and hence of slope 0. Let $S_1, \dots, S_t$ be the simple exceptional sheaves such that $\mathrm{Hom}(L, S_i) \neq 0$. The exceptional vector bundles $L_i(j)$ are defined [44, Sec. 5] as the middle terms of the add(L)-couniversal sequences

$$0 \to L^{\varepsilon f_i} \to L_i(j) \to \tau^- S_i[j] \to 0, \tag{5.7}$$

for $i = 1, \dots, t$ and $j = 1, \dots, p_i - 1$. Similarly, $\overline{L}$ is defined as the middle term of the add(L)-couniversal sequence

$$0 \to L^{\varepsilon} \to \overline{L} \to S \to 0, \tag{5.8}$$

where S is a simple sheaf concentrated at a point x_0 with $p(x_0) = 1$ and $f(x_0) = 1$. The vector bundle $\overline{L}$ is exceptional, has rank $\varepsilon \in \{1, 2\}$ and slope $\bar{p}/\varepsilon$. From (5.8) we deduce that $\overline{L}$, like L, satisfies

$$\operatorname{Hom}(\overline{L}, \tau^j S_i) \neq 0 \quad \text{if and only if} \quad j \equiv 0 \operatorname{mod} p_i. \tag{5.9}$$

The collection of all vector bundles L, $\overline{L}$ and the $L_i(j)$ yields the canonical configuration (5.10) (associated with L), which we denote by T_{can}. Its endomorphism ring is a canonical algebra, cf. [44, Prop. 5.5]. Considered as full subcategory of $\mathcal{H}$ it has the following form:

(5.10)

$$\begin{tikzcd}[column sep=small]
 & L_1(1) \arrow[r] & L_1(2) \arrow[r] & \cdots \arrow[r] & L_1(p_1-2) \arrow[r] & L_1(p_1-1) \arrow[ddr] & \\
 & L_2(1) \arrow[r] & L_2(2) \arrow[r] & \cdots \arrow[r] & L_2(p_2-2) \arrow[r] & L_2(p_2-1) \arrow[dr] & \\
L \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & \vdots & & & \vdots & \overline{L} \\
 & L_t(1) \arrow[r] & L_t(2) \arrow[r] & \cdots \arrow[r] & L_t(p_t-2) \arrow[r] & L_t(p_t-1) \arrow[ur] &
\end{tikzcd}$$

By [44, 5.4 and 5.5] there are short exact sequences

$$0 \to L^{\varepsilon f_i} \to L_i(1) \to \tau^{-} S_i \to 0 \tag{5.11}$$

$$0 \to L_i(j-1) \to L_i(j) \to \tau^{-j} S_i \to 0 \tag{5.12}$$

$$0 \to L^{\varepsilon} \to \overline{L} \to S \to 0 \tag{5.13}$$

$$0 \to L_i(j) \to \overline{L}^{f_i} \to \tau^{-j} S_i[p_i - j] \to 0. \tag{5.14}$$

Remark 5.12. Let $\mathbb{X}$ be of genus zero and consider the tilting sheaf $T_{(B,V)}$ in $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X}$. Let $\Lambda = T_{\text{can}}$ be the canonical configuration (5.10). *We can choose Λ' from above as the canonical configuration T'_{can} in the category $\vec{\mathcal{H}}' = \operatorname{Qcoh} \mathbb{X}'$.*

Indeed, if a branch sheaf $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ is given, we can assume, by applying suitable tubular shifts (associated to the exceptional points) to Λ, that we have $\operatorname{Hom}(L, B_{\mathfrak{i}}) = 0 = \operatorname{Hom}(\overline{L}, B_{\mathfrak{i}})$ and $\operatorname{Hom}(L, \tau B_{\mathfrak{e}}) = 0 = \operatorname{Hom}(\overline{L}, \tau B_{\mathfrak{e}})$. Then those direct summands of Λ lying in $(B_{\mathfrak{e}} \oplus \tau^{-} B_{\mathfrak{i}})^{\perp} \simeq \operatorname{Qcoh} \mathbb{X}'$ form a canonical configuration $\Lambda' = T'_{\text{can}}$ in $\operatorname{Qcoh} \mathbb{X}'$, containing L and $\overline{L}$; it arises from Λ by removing some "non-adjacent segments" $L_i(j), L_i(j+1), \dots, L_i(j+r-2)$ from the inner parts of the arms. (Compare also [46, Thm. 3.1].)

6. The domestic case

In this section let $\mathbb{X}$ be a noncommutative curve of genus zero. Assume that $\mathbb{X}$ is of domestic type, that is, the normalized orbifold Euler characteristic $\chi'_{orb}(\mathbb{X})$

is positive. This means, for the degree of the line bundle $\tau L = L \otimes_{\mathcal{A}} \boldsymbol{\omega}_{\mathcal{A}} = L(\omega)$ (with $\boldsymbol{\omega}_{\mathcal{A}}$ the *dualizing sheaf* in $\mathcal{H} = \operatorname{coh}(\mathcal{A})$) we have

$$\delta(\omega) := \deg(\tau L) = -\frac{2\bar{p}s^2}{\kappa\varepsilon} \cdot \chi'_{orb}(\mathbb{X}) < 0.$$

Here, $\bar{p}$ is the least common multiple of the weights $p_1, \dots, p_t$, moreover $\kappa = \dim_k \operatorname{End}(L)$ and $s = s(\mathcal{H}) = [k(\mathcal{H}) : k(X)]^{1/2}$ the skewness. For every indecomposable vector bundle E one has the following slope formula

$$\mu(\tau E) = \mu(E) + \delta(\omega).$$

We recall the main features of the domestic case:

(D1) All indecomposable vector bundles are stable and exceptional.
(D2) If E and F are indecomposable vector bundles, then $\operatorname{Hom}(E, F) = 0$ if $\mu(E) > \mu(F)$.
(D3) If E is an indecomposable vector bundle then $\mu(\tau E) < \mu(E)$.
(D4) The collection $\mathcal{F}$ of indecomposable vector bundles F such that $0 \leq \mu(F) < -\delta(\omega)$ forms a slice in the sense of [56, 4.2], and $T_{\text{her}} := \bigoplus_{F \in \mathcal{F}} F$ is a tilting bundle having a tame hereditary algebra as endomorphism ring. We refer to [47, Prop. 6.5] (the result there is in a more general context).
(D5) There are only finitely many Auslander-Reiten orbits of vector bundles. (From (D3) it follows that $\mathcal{F}$ contains precisely one indecomposable from each Auslander-Reiten orbit, the finiteness follows from (D4).)

Lemma 6.1. *Let $\mathbb{X}$ be domestic. Let T be a torsionfree tilting sheaf. Then there is $m \in \mathbb{Z}$ such that $\operatorname{Hom}(T, E) = 0$ for every indecomposable vector bundle E with $\mu(E) < m$.*

Proof. The simple idea is the following: if T would map non-trivially to vector bundles of arbitrarily small slopes, then, using line bundle filtrations, T would be a generator for the class of all vector bundles. But by the tilting property, torsionfreeness and Serre duality we then get $\operatorname{Hom}(F, T) = 0$ for all coherent sheaves F, which is impossible. Filling this idea with details for a formal proof is quite straightforward in case of a weighted projective line, but slightly technical in the general case; we postpone these details to the appendix, cf. Lemma A.8. □

Lemma 6.2. *Assume that $\mathbb{X}$ is domestic, and that $T \in \vec{\mathcal{H}}$ is a large tilting object which is torsionfree. Then there is no non-zero morphism from T to a vector bundle.*

Proof. By the previous lemma, let m be an integer such that $\operatorname{Hom}(T, F) = 0$ for all vector bundles F with $\mu(F) < m$. Let $\mathcal{F}$ be a set of representatives of indecomposable vector bundles F with $m + \delta(\vec{\omega}) \leq \mu(F) < m$. By property (D4) the bundle $T_{\text{her}} = \bigoplus_{F \in \mathcal{F}} F$ is tilting and its endomorphism ring is a tame hereditary algebra H such that $\operatorname{Ext}^1(T_{\text{her}}, T) = 0$. Thus, by Proposition 2.8, T can be identified with an H-module.

We assume that there is a vector bundle E with $\operatorname{Hom}(T, E) \neq 0$. Our aim is to get a contradiction. By the previous lemma we can assume T does not map non-trivially to any predecessor of E (since they have smaller slopes by stability). Then every non-zero morphism $T \to E$ must be a split epimorphism, by the almost split property. Thus, T is a tilting H-module having a finite dimensional indecomposable preprojective module P (corresponding to E) as a direct summand, and then T is equivalent to a finite dimensional tilting module T' by [10, Thm. 2.7]. In other words, $\operatorname{Add}(T) = \operatorname{Add}(T')$ in $\operatorname{Mod} H$, and then also in $\vec{\mathcal{H}}$, where T' is a coherent tilting sheaf. Since T is large, this gives the desired contradiction and proves the lemma. □

PROPOSITION 6.3. *Let $\mathbb{X}$ be a domestic curve and $T \in \vec{\mathcal{H}}$ a large tilting sheaf. Then $T \in \operatorname{Gen}(T_{\mathrm{cc}})$ for every tilting bundle T_{cc}. In particular, T is of finite type.*

Proof. For $T = T_{(B,V)}$ this was already shown in Remark 4.21. Therefore we can assume that T is torsionfree. By the preceding lemma we have $\operatorname{Ext}^1(T_{\mathrm{cc}}, T) = \operatorname{D}\operatorname{Hom}(T, \tau T_{\mathrm{cc}}) = 0$, that is, $T \in \operatorname{Gen}(T_{\mathrm{cc}})$. The last statement then follows from Proposition 2.8. □

PROPOSITION 6.4. *Assume that $\mathbb{X}$ is domestic, and that $T \in \vec{\mathcal{H}}$ is a large tilting sheaf which is torsionfree. Then T is equivalent to the Lukas tilting sheaf $\mathbf{L}$.*

Proof. Since T is torsionfree, $T^{\perp_1}$ contains the class of torsion sheaves $\vec{\mathscr{S}}_{\mathbb{X}}$ by Serre duality. Then ${}^{\perp_1}(T^{\perp_1}) \cap \operatorname{coh}\mathbb{X} \subseteq \operatorname{vect}\mathbb{X}$, and by Lemma 6.2 we even have equality. Now Proposition 6.3 yields $\operatorname{Gen}(T) = \operatorname{Gen}(\mathbf{L})$, compare also Theorem 4.14. □

The main result of this section summarizes the discussions above:

THEOREM 6.5. *Let $\mathbb{X}$ be a domestic curve.*

(1) *The large tilting sheaves in $\vec{\mathcal{H}}$ are, up to equivalence, the sheaves of the form*

$$T_{(B,V)} = T_{(B_{\mathfrak{i}},V)} \oplus B_{\mathfrak{e}}$$

with a subset $V \subseteq \mathbb{X}$, a branch sheaf $B = B_{\mathfrak{i}} \oplus B_{\mathfrak{e}}$ with interior and exterior part $B_{\mathfrak{i}}$ and $B_{\mathfrak{e}}$, respectively, and a tilting sheaf $T_{(B_{\mathfrak{i}},V)}$ in the category ${B_{\mathfrak{e}}}^{\perp} = \operatorname{Qcoh}\mathbb{X}'$; here $T_{(B_{\mathfrak{i}},V)}$ with $V \neq \emptyset$ is given by Theorems 4.8 and 5.8, and $T_{(B_{\mathfrak{i}},\emptyset)} = T_{(0,\emptyset)} = \mathbf{L}'$ is the Lukas tilting sheaf in ${B_{\mathfrak{e}}}^{\perp}$.

(2) *There is a bijection between the set of equivalence classes of large tilting sheaves in $\vec{\mathcal{H}}$ and the set of pairs (B, V) given by a branch sheaf $B \in \mathcal{H}$ and a subset $V \subseteq \mathbb{X}$. Moreover, every large tilting object is uniquely determined (up to equivalence) by its torsion part.* □

As a special case we get:

Corollary 6.6. *Let $\mathbb{X}$ be a non-weighted noncommutative curve of genus zero. The large tilting sheaves in $\vec{\mathcal{H}}$ are, up to equivalence, the sheaves of the form T_V with $\emptyset \neq V \subseteq \mathbb{X}$ defined in (5.1), and the Lukas tilting sheaf $\mathbf{L}$.* □

For completeness, we record the corresponding classification of resolving classes (compare Theorem 4.14 and Lemma 4.11).

Corollary 6.7. *Let $\mathbb{X}$ be a domestic curve. The complete list of the resolving classes $\mathscr{S} \subseteq \mathcal{H}$ containing $\operatorname{vect}\mathbb{X}$ is given by*

$$\operatorname{add}\bigl(\operatorname{vect}\mathbb{X} \cup \tau^{-}(B^{>}) \cup \bigcup_{x \in V} \{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\bigr)$$

with $V \subseteq \mathbb{X}$ and B a branch sheaf. □

7. Semistability in Euler characteristic zero

Throughout this section let $\mathbb{X}$ be a weighted noncommutative projective curve of orbifold Euler characteristic zero, and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.

The main feature of the case $\chi'_{orb}(\mathbb{X}) = 0$ is that *every indecomposable coherent sheaf is semistable*, cf. Theorem 2.3. We collect here some basic properties which essentially follow from semistability and thus hold both in the tubular and in the elliptic case. Later, in the next two sections, we will have to distinguish the two cases. For general information on the tubular case we refer to [45], [41], [53, Ch. 13], [38, Ch. 8] and [39, Sec. 13], on the elliptic case to [39, Sec. 9].

Let us recall some notation. We write $\bar{p}$ for the least common multiple of the weights $p_1, \dots, p_t$, that is, $\bar{p} = 1$ if $\mathbb{X}$ is elliptic, and $\bar{p} > 1$ if $\mathbb{X}$ is tubular. Further, the slope of a non-zero object $E \in \mathcal{H}$ is defined by $\mu(E) = \frac{\deg(E)}{\operatorname{rk}(E)} \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, with $\deg(E) = \frac{1}{\kappa\varepsilon}\langle\langle L, E\rangle\rangle$, cf. (2.7).

By semistability we have the following result, similar to Atiyah's classification [12].

Theorem 7.1 ([38, Prop. 8.1.6], [39, Thm. 9.7]). *For every $\alpha \in \widehat{\mathbb{Q}}$ the full subcategory $\mathbf{t}_\alpha$ of $\mathcal{H}$ formed by the semistable sheaves of slope α is a non-trivial abelian uniserial category whose connected components form stable tubes; the tubular family $\mathbf{t}_\alpha$ is parametrized again by a weighted noncommutative regular projective curve $\mathbb{X}_\alpha$ over k which satisfies $\chi'_{orb}(\mathbb{X}_\alpha) = 0$ and is derived-equivalent to $\mathbb{X}$.* □

We can thus write

$$\mathcal{H} = \bigvee_{\alpha \in \widehat{\mathbb{Q}}} \mathbf{t}_\alpha.$$

In particular, $\mathbf{t}_\infty$ consists of the finite length sheaves.

We will need the following important application of the Riemann-Roch formula from [39, Thm. 13.8].

Lemma 7.2. *If X, $Y \in \mathcal{H}$ are indecomposable with $\mu(X) < \mu(Y)$, then there exists j with $0 \leq j \leq \bar{p} - 1$ such that $\operatorname{Hom}(X, \tau^j Y) \neq 0$.* □

Quasicoherent sheaves having a real slope. For $w \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ we define

$$\mathbf{p}_w = \bigcup_{\alpha < w} \mathbf{t}_\alpha \qquad \mathbf{q}_w = \bigcup_{w < \beta} \mathbf{t}_\beta,$$

where α, $\beta \in \widehat{\mathbb{Q}}$. Accordingly, $\mathcal{H} = \mathbf{p}_w \vee \mathbf{t}_w \vee \mathbf{q}_w$ if w is rational, and $\mathcal{H} = \mathbf{p}_w \vee \mathbf{q}_w$ if w is irrational. Moreover, let

$$\mathcal{C}_w = \mathbf{q}_w{}^{\perp_0} = {}^{\perp_1}\mathbf{q}_w \qquad \mathcal{B}_w = {}^{\perp_0}\mathbf{p}_w = \mathbf{p}_w{}^{\perp_1}$$

and

$$\mathcal{M}(w) = \mathcal{B}_w \cap \mathcal{C}_w.$$

The sheaves in $\mathcal{M}(w)$ are said to have *slope* w. Clearly, for coherent sheaves this definition of slope is equivalent to the former one, and for irrational w there are only non-coherent sheaves in $\mathcal{M}(w)$.
For $v \leq w \leq \infty$ we have $\mathcal{C}_v \subseteq \mathcal{C}_w$ and $\mathcal{B}_v \supseteq \mathcal{B}_w$. Moreover,

$$\bigcap_{w \in \widehat{\mathbb{R}}} \mathcal{C}_w = 0 \quad \text{and} \quad \bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{C}_w = \mathcal{C}_\infty = \vec{\mathcal{H}},$$

and

$$\bigcap_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{B}_\infty = {}^{\perp_0} \operatorname{vect} \mathbb{X} \quad \text{and} \quad \mathcal{H} \cap \bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{H}.$$

We note that for example $\bigoplus_{\alpha \in \widehat{\mathbb{Q}}} S_\alpha$ with $S_\alpha \in \mathbf{t}_\alpha$ quasisimple is not in $\bigcup_{w \in \widehat{\mathbb{R}}} \mathcal{B}_w$. Let $X \in \vec{\mathcal{H}}$ be a non-zero object. Let $v = \sup\{r \in \widehat{\mathbb{R}} \mid X \in \mathcal{B}_r\} \in \widehat{\mathbb{R}} \cup \{-\infty\}$ and $w = \inf\{r \in \widehat{\mathbb{R}} \mid X \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. Since $X \neq 0$ we have $v \leq w$.
In the special case, when $w = \infty$, a sheaf $X \in \vec{\mathcal{H}}$ has slope ∞ if and only if $X \in {}^{\perp_0} \operatorname{vect} \mathbb{X} = (\operatorname{vect} \mathbb{X})^{\perp_1}$. (This, as a definition, makes also sense for other representation types; *in the domestic case*, we have seen that *every large tilting sheaf has slope* ∞.)

Interval categories. The following technique is very useful in the tubular or elliptic setting. Let $\alpha \in \widehat{\mathbb{Q}}$. Denote by $\mathcal{H}\langle\alpha\rangle$ the full subcategory of $\mathcal{D}^b(\mathcal{H})$ defined by

$$\bigvee_{\beta > \alpha} \mathbf{t}_\beta[-1] \vee \bigvee_{\gamma \leq \alpha} \mathbf{t}_\gamma.$$

The abelian category $\mathcal{H}\langle\alpha\rangle$ is a HRS-tilt of $\mathcal{H}$ in $\mathcal{D}^b(\mathcal{H})$ with respect to the split torsion pair $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ in $\mathcal{H}$ given by $\mathcal{T}_\alpha = \bigvee_{\beta > \alpha} \mathbf{t}_\beta$ and $\mathcal{F}_\alpha = \bigvee_{\gamma \leq \alpha} \mathbf{t}_\gamma$, see [29, I. Thm. 3.3] and [48, Prop. 2.2]. By [38, Prop. 8.1.6], [39, Thm. 9.7] we have $\mathcal{H}\langle\alpha\rangle = \operatorname{coh} \mathbb{X}_\alpha$ for some curve $\mathbb{X}_\alpha$ with $\chi'_{orb}(\mathbb{X}_\alpha) = 0$ and being derived-equivalent to $\mathbb{X}$. (If k is algebraically closed, then $\mathbb{X}_\alpha$ is isomorphic to $\mathbb{X}$; but this is not true in general.) The rank function on $\mathcal{H}\langle\alpha\rangle$ defines a linear form $\operatorname{rk}_\alpha \colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$. A sequence $\eta \colon 0 \to E' \xrightarrow{u} E \xrightarrow{v} E'' \to 0$ with objects E', E, E'' in $\mathcal{H} \cap \mathcal{H}\langle\alpha\rangle$ is exact in $\mathcal{H}$ if and only if it is exact in $\mathcal{H}\langle\alpha\rangle$; indeed, both conditions are equivalent to $E' \xrightarrow{u} E \xrightarrow{v} E'' \xrightarrow{\eta} E'[1]$ being a triangle in $\mathcal{D}^b(\mathcal{H})$.

Lemma 7.3 (Reiten-Ringel). *For every $w \in \widehat{\mathbb{R}}$ the pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair, which is split in case $w \in \widehat{\mathbb{Q}}$.*

Proof. As in [53, Lem. 1.4] one shows that $\mathrm{Gen}(\mathbf{q}_w)$ is extension-closed; the same proof works in the locally noetherian category $\vec{\mathcal{H}}$, replacing "finite length" by "finitely presented". Then $\mathrm{Gen}(\mathbf{q}_w) = {}^{\perp_0}({\mathbf{q}_w}^{\perp_0}) = {}^{\perp_0}\mathcal{C}_w$ follows like in [53, Lem. 1.3], and thus $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair. For the splitting property in case $w = \alpha \in \widehat{\mathbb{Q}}$ we have to show that every short exact sequence $\eta\colon 0 \to X \to Y \to Z \to 0$ with $X \in \mathrm{Gen}(\mathbf{q}_\alpha)$ and $Z \in \mathcal{C}_\alpha$ splits. We may assume that X is a subobject of Y and $Z = Y/X$. If Z is finitely presented, this follows from Serre duality. For general $Z \in \mathcal{C}_\alpha$, we consider the set of subobjects U of Y such that $U \cap X = 0$ and $Y/(X+U) \in \mathcal{C}_\alpha$. Like in [53, Prop. 1.5(b)] one has a maximal such U, and as in [53, Prop. 1.5(a)] one shows $Y = X \oplus U$, so that η splits. (If one assumes that the inclusion $X + U \subsetneq Y$ is proper, then $\vec{\mathcal{H}}$ being locally noetherian allows to find Y' with $X + U \subsetneq Y' \subseteq Y$ with $Y'/(X+U)$ finitely presented. Then we proceed like in [53]. We remark that an analogue of condition (F) therein can be proved along the same lines by exploiting the fact that an indecomposable $E \in \mathcal{H}$ belongs to $\mathbf{q}_\alpha$ if and only if $\delta(E) > 0$, where $\delta = -\mathrm{rk}_\alpha$.) $\square$

Let $\alpha \in \widehat{\mathbb{Q}}$. By $\vec{\mathcal{H}}\langle\alpha\rangle$ we denote the direct limit closure of $\mathcal{H}\langle\alpha\rangle$ in $\mathcal{D}^b(\vec{\mathcal{H}})$. We have $\vec{\mathcal{H}}\langle\alpha\rangle = \mathrm{Qcoh}\,\mathbb{X}_\alpha$. If $X \in \vec{\mathcal{H}}$ has a rational slope α, then clearly $X \in \vec{\mathcal{H}} \cap \vec{\mathcal{H}}\langle\alpha\rangle$ where the intersection is formed in $\mathcal{D}^b(\vec{\mathcal{H}}) = \mathcal{D}^b(\vec{\mathcal{H}}\langle\alpha\rangle)$; in $\vec{\mathcal{H}}\langle\alpha\rangle$ then T has slope ∞. Clearly, $\mathcal{C}_\alpha = \vec{\mathcal{H}}\langle\alpha\rangle \cap \vec{\mathcal{H}}$.

Lemma 7.4. *Let $\alpha \in \widehat{\mathbb{Q}}$. For an object T in $\vec{\mathcal{H}}$ lying in $\mathcal{C}_\alpha$, the following conditions are equivalent:*

(i) *T is a tilting sheaf in $\vec{\mathcal{H}}$;*
(ii) *T is a tilting complex in $\mathcal{D}^b(\vec{\mathcal{H}})$;*
(iii) *T is a tilting sheaf in $\vec{\mathcal{H}}\langle\alpha\rangle$.*

Proof. This is shown like in Proposition 2.8. $\square$

Remark 7.5. There is an interesting class of locally coherent categories which are derived-equivalent to $\vec{\mathcal{H}}$: If w is irrational, then we define $\mathcal{H}\langle w\rangle = \bigvee_{\beta>w} \mathbf{t}_\beta[-1] \vee \bigvee_{\gamma<w} \mathbf{t}_\gamma$ and $\vec{\mathcal{H}}\langle w\rangle$ similarly as above. It is easy to see that $\mathcal{H}\langle w\rangle$ is hereditary and does not contain any simple object. Accordingly, $\vec{\mathcal{H}}\langle w\rangle$ is a Grothendieck category (we refer to [7, Sec. 2.4+2.5]) which is locally coherent but *not* locally noetherian. Moreover, $\vec{\mathcal{H}}\langle w\rangle$ is derived-equivalent to $\vec{\mathcal{H}}$, and in the tubular case it contains a finitely presented tilting object T_{can} whose endomorphism ring is a tubular canonical algebra. It is not difficult to show that there are only countably many irrational w' such that the category $\vec{\mathcal{H}}\langle w'\rangle$ (resp. $\mathcal{H}\langle w'\rangle$) is equivalent to $\vec{\mathcal{H}}\langle w\rangle$ (resp. $\mathcal{H}\langle w\rangle$). It would be of interest to get a better understanding of the "geometric meaning" of these categories.

Indecomposable quasicoherent sheaves. The following statement reflects the importance of the concept of slope in the tubular/elliptic case, also for quasicoherent sheaves.

Theorem 7.6 (Reiten-Ringel). (1) *$\operatorname{Hom}(\mathcal{M}(w'),\mathcal{M}(w))=0$ for $w<w'$.*
(2) *Every indecomposable sheaf has a well-defined slope $w\in\widehat{\mathbb{R}}$.*

Proof. (1) This follows like in [53, Thm. 13.1].
(2) We transfer the original proof for modules over a tubular algebra in [53, Thm. 13.1] to $\operatorname{Qcoh}\mathbb{X}$; we need a slight modification. Let $X\in\vec{\mathcal{H}}$ be indecomposable. Then $0\neq X\in\bigcup_{w\in\widehat{\mathbb{R}}}\mathcal{C}_w\setminus\bigcap_{w\in\widehat{\mathbb{R}}}\mathcal{C}_w$. Let $w\in\widehat{\mathbb{R}}$ be the infimum of all $\alpha\in\widehat{\mathbb{Q}}$ such that $X\in\mathcal{C}_\alpha$. Since $\mathbf{q}_w=\bigcup_{w<\alpha}\mathbf{q}_\alpha$, we have $\operatorname{Hom}(\mathbf{q}_w,X)=0$, that is, $X\in\mathcal{C}_w$.
We now show that $X\in\mathcal{B}_w={}^{\perp_0}\mathbf{p}_w$. We observe that

$$\mathcal{B}_w=\bigcap_{\alpha<w}{}^{\perp_0}\mathbf{t}_\alpha$$

and $\operatorname{Gen}(\mathbf{q}_\alpha)\subseteq{}^{\perp_0}\mathbf{t}_\alpha$. Hence, if $X\notin\mathcal{B}_w$, then there is a rational $\beta<w$ with $X\notin\operatorname{Gen}(\mathbf{q}_\beta)$. But $(\operatorname{Gen}(\mathbf{q}_\beta),\mathcal{C}_\beta)$ is a split torsion pair, and since X is indecomposable, we get $X\in\mathcal{C}_\beta$. Since $\beta<w$ this gives a contradiction to the choice of w. □

Remark 7.7. If T is a *noetherian* tilting object in $\vec{\mathcal{H}}$ (that is, $T\in\mathcal{H}$ (which exists if and only if $\bar{p}>1$)), then T does *not* have any slope. In fact, if $T=T_1\oplus\ldots\oplus T_n$ with pairwise nonisomorphic indecomposable T_i, then n coincides with the rank of the Grothendieck group $\mathrm{K}_0(\mathcal{H})$. If T would have a slope α, then each summand T_i would be exceptional of slope α, hence lying in one of the finitely many exceptional tubes of slope α. If such a tube has rank $p>1$, then there are at most $p-1$ indecomposable summands of T lying in this tube. If $p_1,\ldots,p_t$ are the weights of $\mathbb{X}$, then $n=|\mathrm{K}_0(\mathcal{H})|=\sum_{i=1}^t(p_i-1)+2>\sum_{i=1}^t(p_i-1)\geq n$, giving a contradiction.

Proposition 7.8. *Let $w\in\widehat{\mathbb{R}}$. There is a large tilting sheaf $\mathbf{L}_w$ of slope w.*

Proof. Applying Theorem 4.4 to the strongly resolving subcategory $\operatorname{add}(\mathbf{p}_w)$, we get a tilting sheaf T with $\operatorname{Gen}(T)=\mathscr{S}^{\perp_1}={\mathbf{p}_w}^{\perp_1}=\mathcal{B}_w$. Moreover, by the tilting property clearly $T\in{}^{\perp_1}\mathcal{B}_w$, which is a subclass of $\mathcal{C}_w$. By the preceding remark, T is large. □

Let $T\in\vec{\mathcal{H}}$ be a tilting object of rational slope α. Then in $\vec{\mathcal{H}}\langle\alpha\rangle$ the splitting property of Theorem 3.8 holds, that is, the canonical exact sequence $0\to t_\alpha(T)\to T\to T/t_\alpha(T)\to 0$ in $\vec{\mathcal{H}}\langle\alpha\rangle$ splits, where $t_\alpha(T)$ denotes the torsion subsheaf of T in $\vec{\mathcal{H}}\langle\alpha\rangle$.

Definition 7.9. Let T be a tilting sheaf of slope w. We call T a *torsionfree* tilting sheaf, if either w is irrational, or if $w=\alpha\in\widehat{\mathbb{Q}}$ and $t_\alpha(T)=0$.

Lemma 7.10. *For every $w\in\widehat{\mathbb{R}}$ the tilting sheaf $\mathbf{L}_w$ is torsionfree.*

Proof. For irrational w there is nothing to show. Switching to the category $\vec{\mathcal{H}}\langle\alpha\rangle$ if $w = \alpha$ is rational, we can assume without loss of generality that $w = \infty$. Then the claim follows from Proposition 4.5. $\square$

We will now treat the elliptic case and the tubular case separately, starting with the tubular case.

8. The tubular case

Throughout this section let $\mathbb{X}$ be a tubular noncommutative curve of genus zero and $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$.

Lemma 8.1. *Let $\alpha \in \widehat{\mathbb{Q}}$. Let $T \in \vec{\mathcal{H}}$ be a large tilting sheaf with $T \in \mathcal{C}_\alpha$ and $t_\alpha(T) \neq 0$. Then T has slope α.*

Proof. Switching to the category $\vec{\mathcal{H}}\langle\alpha\rangle = \operatorname{Qcoh}\mathbb{X}_\alpha$, we can assume without loss of generality that $\alpha = \infty$. (This will just simplify the notation.) If tT contains a non-coherent summand, then with Theorem 5.8 we get that T has slope ∞, since $T \in \mathcal{B}_\infty$ follows from 4.6. If, on the other hand, tT only consists of coherent summands (necessarily only finitely many indecomposables) then T/tT is a torsionfree tilting sheaf in

$$\operatorname{Qcoh}\mathbb{X}' = tT^{\perp} \subseteq \vec{\mathcal{H}},$$

where $\mathbb{X}'$ is a curve with reduced weights, thus of domestic type. By [28, Prop. 9.6] the induced inclusion $\operatorname{coh}\mathbb{X}' \subseteq \mathcal{H}$ is rank-preserving. The torsionfree sheaf T/tT is equivalent to the Lukas tilting sheaf $\mathbf{L}' \in \operatorname{Qcoh}\mathbb{X}'$ by Proposition 6.4. We show that $\mathbf{L}'$, as object in $\vec{\mathcal{H}}$, has slope ∞. We assume this is not the case. Then there is $\beta < \infty$ with $\operatorname{Hom}(\mathbf{L}', \mathbf{t}_\beta) \neq 0$. Since in $\vec{\mathcal{H}}$ every vector bundle has a line bundle filtration, it follows that there is a line bundle L with $\operatorname{Hom}(\mathbf{L}', L) \neq 0$. Since non-zero subobjects of line bundles are line bundles, we can assume without loss of generality that there is an epimorphism $f\colon \mathbf{L}' \to L$. Let E be an indecomposable vector bundle over $\mathbb{X}'$, considered as object in $\vec{\mathcal{H}}$. Let $x_0 \in \mathbb{X}$ be a homogeneous point. The support of tT is disjoint from x_0, and thus the associated tubular shift automorphism σ_0 fixes tT. Then $E(nx_0) \in \operatorname{vect}\mathbb{X}'$ for all $n > 0$: indeed, by definition of the tubular shift there is an exact sequence $0 \to E \to E(nx_0) \to C \to 0$ in $\vec{\mathcal{H}}$ with C lying in the tube $\mathcal{U}_{x_0}$; then $E, C \in tT^{\perp}$ implies $E(nx_0) \in tT^{\perp}$, having the same rank as E. By [44, (S15)], for $n \gg 0$ we have $\operatorname{Hom}(L, E(nx_0)) \neq 0$. We get $\operatorname{Hom}(\mathbf{L}', E(nx_0)) \neq 0$, which also holds in the full subcategory $\operatorname{Qcoh}\mathbb{X}'$, and gives a contradiction since in $\operatorname{Qcoh}\mathbb{X}'$ one has $\mathbf{L}' \in {}^{\perp_0}\operatorname{vect}\mathbb{X}'$. Thus $\mathbf{L}'$ has slope ∞, and so has T, which is equivalent to $\mathbf{L}' \oplus tT$. $\square$

In the tubular case, the tilting bundle T_{cc} can be chosen such that its indecomposable summands have arbitrarily small slopes. This will imply that tilting sheaves have finite type. The following statement is crucial.

Lemma 8.2. *For any large tilting sheaf $T \in \vec{\mathcal{H}}$ there is $\alpha \in \widehat{\mathbb{Q}}$ with $T \in \mathcal{B}_\alpha$.*

Proof. If T has a non-trivial torsion part, then T has slope ∞ by Lemma 8.1. Thus we will assume in the following that T is torsionfree.
Let $\mathcal{B} = \mathrm{Gen}(T)$ and $\mathscr{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$. Suppose there is *no* α with $T \in \mathcal{B}_\alpha$. We will lead this to a contradiction.

(1) *There are infinitely many and arbitrarily small* α *with* $\mathrm{Hom}(T, \mathbf{t}_\alpha) \neq 0$. Indeed, otherwise there would be some α with $\mathbf{p}_\alpha \subseteq \mathscr{S}$, and then

$$T \in \mathcal{B} \subseteq ({}^{\perp_1}\mathcal{B})^{\perp_1} \subseteq \mathscr{S}^{\perp_1} \subseteq {\mathbf{p}_\alpha}^{\perp_1} = \mathcal{B}_\alpha,$$

which is not the case by our assumption.

(2) *There is no* α *such that* $\mathscr{S} \cap \mathbf{p}_\alpha = \emptyset$. Indeed, if there were such α, then $\mathrm{Hom}(T, X) \neq 0$ for all $X \in \mathbf{p}_\alpha$. So, for any line bundle L in $\mathbf{p}_\alpha$, the trace L' of T in L would be a non-zero line bundle again. Applying $\mathrm{Ext}^1(T, -)$ to the short exact sequence $0 \to L' \to L \to C \to 0$ would give $\mathrm{Ext}^1(T, L) = 0$, as T is torsionfree and C has finite length. Then, given a point $x \in \mathbb{X}$ and an integer $n \geq 1$, we would infer $\mathrm{Ext}^1(T, L(nx)) = 0$ from the exact sequence $0 \to L \to L(nx) \to F \to 0$ with F of finite length. Now, since any line bundle L in $\mathcal{H}$ satisfies $L(-nx) \in \mathbf{p}_\alpha$ for $n \gg 0$, we would conclude that $\mathrm{Hom}(L, T) = \mathrm{D}\,\mathrm{Ext}^1(T, \tau L) = 0$ holds for all line bundles, and using line bundle filtrations, even for all vector bundles. But this is clearly impossible, since $T \neq 0$, as torsionfree object, is a direct limit of vector bundles. This proves (2).

Thus $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$ for infinitely many and arbitrarily small α.

(3) *Every indecomposable* $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ *is exceptional.* Indeed, let $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ be indecomposable, and let $\beta < \alpha$ with $\mathrm{Hom}(T, \mathbf{t}_\beta) \neq 0$. Considering images, there is an indecomposable $B \in \mathcal{H}$ with $B \in \mathrm{Gen}(T)$ and slope $\mu(B) < \alpha$. By Lemma 7.2 we have $\mathrm{Hom}(B, \tau^j X) \neq 0$ for some integer j. If we assume that X is not exceptional, we can even show $\mathrm{Hom}(B, \tau X) \neq 0$. Indeed, this is clear if X lies in a homogeneous tube, which means $\tau X = X$, while for X lying in an exceptional tube of rank $p > 1$ we know from Lemma 7.2 that B maps non-trivially into a quasisimple object of the tube, and by the almost split property it follows inductively that B maps non-trivially into each object from the tube which has quasilength $\geq p$, so in particular into τX. Now we get $\mathrm{Ext}^1(X, T) = \mathrm{D}\,\mathrm{Hom}(T, \tau X) \neq 0$, which is a contradiction to $X \in \mathscr{S} \subseteq {}^{\perp_1}\mathcal{B}$. This shows (3).

We now fix an indecomposable, exceptional $X \in \mathscr{S} \cap \mathbf{t}_\alpha$ lying in a tube of rank $p > 1$.

(4) *There is an indecomposable* Y *in the same tube and of the same quasi-length as* X *such that* $\mathrm{Hom}(T, Y) \neq 0$. In order to show this, we start with an arbitrary indecomposable Z of quasi-length p in the same tube. Since $\tau^- Z$ is not exceptional, and thus not in $\mathscr{S}$, we have $\mathrm{Hom}(T, Z) \neq 0$. Then, considering the almost split sequences, we get inductively that T maps non-trivially to an object of quasi-length ℓ for any $\ell < p$: given $0 \neq f \in \mathrm{Hom}(T, Z)$ where Z is indecomposable of quasilength ℓ, with $2 \leq \ell \leq p$, there is an irreducible monomorphism ι ending in Z and an irreducible epimorphism π starting in Z,

and either $\pi f \neq 0$, or f factors through ι; in both cases T maps non-trivially to an object in the tube of quasilength $\ell - 1$. This shows (4).

(5) *There is an indecomposable coherent direct summand B of T of slope $\mu(B) \leq \alpha$.* Indeed, since for the fixed X as above $\operatorname{Hom}(T, \tau X) = 0$, we can assume that the object Y from (4) satisfies $\operatorname{Hom}(T, Y) \neq 0$ and $\operatorname{Hom}(T, \tau Y) = 0$. We conclude $\operatorname{Ext}^1(Y, T) = \operatorname{D}\operatorname{Hom}(T, \tau Y) = 0$, thus $Y \in \mathscr{S}$. Let B be an indecomposable summand of the trace of T in Y. Since $B \subseteq Y$, we conclude $\operatorname{Ext}^1(B, T) = 0$, hence $B \in \mathscr{S}$. Thus $B \in \mathcal{B} \cap {}^{\perp_1}\mathcal{B}$, and by Lemma 2.6, B is a direct summand of T, of slope $\mu(B) \leq \alpha$.

Repeating these arguments for slope smaller than $\mu(B)$ we get inductively an infinite sequence of indecomposable coherent sheaves $B_1, B_2, B_3, \ldots$ lying in $\operatorname{add}(T)$, and with slopes $\mu(B_1) > \mu(B_2) > \mu(B_3) > \ldots$. We conclude $\operatorname{Ext}^1(B_i, B_j) = 0$ for all i, j and $\operatorname{Hom}(B_i, B_j) = 0$ for all $i < j$. So, for all n, the sequence $(B_n, \ldots, B_2, B_1)$ is exceptional in $\mathcal{H}$. This gives our desired contradiction, since the length of exceptional sequences in $\mathcal{H}$ is bounded by the (finite!) rank of the Grothendieck group $\operatorname{K}_0(\mathcal{H})$. $\square$

Proposition 8.3. *Let $\mathbb{X}$ be tubular. Every tilting sheaf $T \in \vec{\mathcal{H}}$ is of finite type.*

Proof. By Lemma 8.2 there is $\alpha \in \widehat{\mathbb{Q}}$ with $T \in \mathcal{B}_\alpha = \mathbf{p}_\alpha{}^{\perp_1}$. Then $\operatorname{Ext}^1(\mathbf{p}_\alpha, T) = 0$, and choosing a tilting bundle $T_{\mathrm{cc}} \in \mathbf{p}_\alpha$, we get $\operatorname{Ext}^1(T_{\mathrm{cc}}, T) = 0$. Now we can apply Proposition 2.8. $\square$

The proof above also shows that $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$ is a strongly resolving subcategory of $\mathcal{H}$ such that $\operatorname{Gen}(T) = \mathscr{S}^{\perp_1}$. Now let us start conversely with a strongly resolving subcategory.

Lemma 8.4. *Let $\alpha \in \widehat{\mathbb{Q}}$ and $\mathscr{S} \subseteq \mathcal{C}_\alpha \cap \mathcal{H}$ be strongly resolving.*

(1) *There is a tilting sheaf $T \in \vec{\mathcal{H}}$ with $T \in \mathcal{C}_\alpha$ and $T^{\perp_1} = \mathscr{S}^{\perp_1}$. Moreover:*
(2) *If $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$, then $t_\alpha(T) \neq 0$.*
(3) *If $\mathscr{S} \cap \mathbf{t}_\alpha = \emptyset$, then $t_\alpha(T) = 0$.*

Proof. (1) By Theorem 4.4 there is a tilting sheaf $T \in \vec{\mathcal{H}}$ with $\mathscr{S}^{\perp_1} = T^{\perp_1}$. Moreover, there is an exact sequence (4.14) with $T = T_0 \oplus T_1$, and by Remark 4.21 the summands T_0 and T_1 are $\mathscr{S}$-filtered. Since $\mathcal{C}_\alpha = {}^{\perp_1}\mathbf{q}_\alpha$ is closed under filtered direct limits (which follows from [58, Prop. 2.12]), we get T_0, $T_1 \in \mathcal{C}_\alpha$, thus T is in $\mathcal{C}_\alpha$.

(2) Assume that $t_\alpha(T) = 0$. Let $S \in \mathscr{S} \cap \mathbf{t}_\alpha$ be indecomposable. Then $\operatorname{Ext}^1(T, S) = \operatorname{D}\operatorname{Hom}(\tau^- S, T) = 0$, that is, $S \in T^{\perp_1}$. For every $X \in T^{\perp_1} = \mathscr{S}^{\perp_1}$ we have $\operatorname{Ext}^1(S, X) = 0$, and thus $S \in {}^{\perp_1}(T^{\perp_1})$. Since, by Lemma 2.6, $T^{\perp_1} \cap {}^{\perp_1}(T^{\perp_1}) = \operatorname{Add}(T)$ we get $S \in \operatorname{Add}(T)$, and then S is a summand of $t_\alpha(T)$, contradiction. Thus $t_\alpha(T) \neq 0$.

(3) Assume that $t_\alpha(T) \neq 0$. By Lemma 8.1 then T has slope α, so $\operatorname{Gen}(T) \subseteq \mathcal{B}_\alpha$, and we even have equality since $\mathscr{S} \subseteq \mathbf{p}_\alpha$. So T is torsionfree by Lemma 7.10, contradiction. $\square$

The main result of this section is the following.

Theorem 8.5. *Let $\mathbb{X}$ be tubular. Every large tilting sheaf T has a slope $w \in \widehat{\mathbb{R}}$.*

Proof. Let $\mathcal{B} = \operatorname{Gen}(T) = T^{\perp_1}$ and $\mathscr{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{H}$. Define $w = \inf\{r \in \widehat{\mathbb{R}} \mid T \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. This is well-defined. We show that T has slope w. By properties of the infimum we have $T \in \mathcal{C}_w$, but $T \notin \mathcal{C}_v$ for all $v < w$. We have to show that $T \in \mathcal{B}_w$. By the preceding lemma T is of finite type, in other words, $T^{\perp_1} = \mathscr{S}^{\perp_1}$. For every rational number $\alpha < w$ let

$$\mathscr{S}_\alpha = \mathscr{S} \cap \mathcal{C}_\alpha.$$

Since $\mathscr{S}$ is strongly resolving by Lemma 8.2, the same holds for $\mathscr{S}_\alpha$. Since $T \notin \mathcal{C}_\alpha$, the set of all rational numbers $\alpha < w$ with $\mathscr{S} \cap \mathbf{t}_\alpha \neq \emptyset$ is not bounded by a smaller number than w; this follows from Lemma 8.4 and since T is determined by $\mathscr{S}$. Thus there is a sequence of rational numbers

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < w$$

with $\lim_{i\to\infty} \alpha_i = w$ and

$$\mathscr{S} \cap \mathbf{t}_{\alpha_i} \neq \emptyset. \tag{8.1}$$

By Lemma 8.4 there is a tilting object T_i with ${T_i}^{\perp_1} = {\mathscr{S}_{\alpha_i}}^{\perp_1}$ and $T_i \in \mathcal{C}_{\alpha_i}$ and with $t_{\alpha_i}(T_i) \neq 0$. Now, by Lemma 8.1 the tilting object T_i has slope α_i. Then we get $\operatorname{Gen}(T_i) \subseteq \operatorname{Gen}(\mathbf{L}_{\alpha_i}) = \mathcal{B}_{\alpha_i}$ (the largest tilting class of slope α_i). Since $\mathscr{S}_{\alpha_i} \subseteq \mathscr{S}$, we get

$$\mathcal{B}_{\alpha_i} \supseteq {\mathscr{S}_{\alpha_i}}^{\perp_1} \supseteq \mathscr{S}^{\perp_1} \ni T$$

for all i, and thus $T \in \bigcap_{i\geq 1} \mathcal{B}_{\alpha_i} = \mathcal{B}_w$. □

Theorem 8.6. *Let $\mathbb{X}$ be a noncommutative curve of genus zero of tubular type.*

(1) *The sheaves $\mathbf{L}_w$ with $w \in \widehat{\mathbb{R}}$ are, up to equivalence, the unique torsion-free large tilting sheaves (in the sense of Definition 7.9).*

(2) *The equivalence classes of large non-torsionfree tilting sheaves are in bijective correspondence with triples (α, B, V), where $\alpha \in \widehat{\mathbb{Q}}$, $V \subseteq \mathbb{X}_\alpha$ and $B \in \operatorname{add} \mathbf{t}_\alpha$ is a branch sheaf, and $(B, V) \neq (0, \emptyset)$.*

Proof. (1) Let T be a torsionfree tilting sheaf of slope w. Then $T^{\perp_1} \subseteq \mathcal{B}_w = {\mathbf{L}_w}^{\perp_1}$. Hence we have ${}^{\perp_1}({\mathbf{L}_w}^{\perp_1}) \cap \mathcal{H} = \operatorname{add}(\mathbf{p}_w) = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$; the last equality follows, since T generates every sheaf of finite length. Now $T^{\perp_1} = {\mathbf{L}_w}^{\perp_1}$ follows from Proposition 8.3.

(2) Every large non-torsionfree tilting sheaf T has a slope $\alpha \in \widehat{\mathbb{Q}}$. By Lemma 7.4, T is a large tilting sheaf in $\vec{\mathcal{H}}\langle\alpha\rangle$, having a non-zero torsion part $t_\alpha(T)$. We now apply Theorems 5.8 and 6.5 to the category $\vec{\mathcal{H}}\langle\alpha\rangle$. The non-torsionfree tilting sheaves of slope α are given by

- $T_{(B,V)}$ with $\emptyset \neq V \subseteq \mathbb{X}$ (here $t_\alpha(T)$ is non-coherent);
- $\mathbf{L}' \oplus B$, with $0 \neq B \in \operatorname{add} \mathbf{t}_\alpha$ a branch sheaf and $\mathbf{L}' \in B^\perp = \operatorname{Qcoh} \mathbb{X}'_\alpha$ the Lukas tilting sheaf over the domestic curve $\mathbb{X}'_\alpha$ (here $t_\alpha(T) = B$ is coherent).

This finishes the proof. □

We say that a resolving class $\mathscr{S} \subseteq \mathcal{H}$ *has slope* w if $\mathbf{p}_w \subseteq \mathscr{S}$ and $\mathscr{S}$ does not contain any indecomposable of slope $\beta > w$.

Corollary 8.7. *For a tubular curve $\mathbb{X}$, the complete list of the resolving classes $\mathscr{S}$ in $\mathcal{H} = \operatorname{coh}\mathbb{X}$ having a slope is given by*

- $\operatorname{add}\mathbf{p}_w$ *with* $w \in \widehat{\mathbb{R}}$*; and*
- $\operatorname{add}\big(\mathbf{p}_\alpha \cup \tau^-(B^>) \cup \bigcup_{x\in V}\{\tau^j S_x[n] \mid j \in \mathcal{R}_x,\, n \in \mathbb{N}\}\big)$ *with* $\alpha \in \widehat{\mathbb{Q}}$*,* $V \subseteq \mathbb{X}_\alpha$*,* $B \in \operatorname{add}\mathbf{t}_\alpha$ *a branch sheaf, and* $(B, V) \neq (0, \emptyset)$*.*

Proof. By Theorem 8.6, the list contains precisely the resolving classes corresponding to the large tilting sheaves under the bijection of Theorem 4.14, and they all have a slope. Conversely, let $\mathscr{S}$ be resolving having a slope w and T a tilting sheaf such that $T^{\perp_1} = \mathscr{S}^{\perp_1}$. If w is irrational, then $\mathscr{S} = \operatorname{add}\mathbf{p}_w$. If, on the other hand, $w = \alpha \in \widehat{\mathbb{Q}}$, then $\operatorname{Add}(T) \cap \mathcal{H} = \mathscr{S} \cap \mathscr{S}^{\perp_1} \subseteq \operatorname{add}(\mathbf{p}_\alpha \cup \mathbf{t}_\alpha) \cap \mathbf{p}_\alpha{}^{\perp_1} \subseteq \operatorname{add}\mathbf{t}_\alpha$, that is, all coherent summands of T belong to the same tubular family, and therefore T cannot be coherent. $\square$

Corollary 8.8 (Property (TS3)). *Let T_{can} be the canonical tilting bundle. Let $T \in \vec{\mathcal{H}}$ be a large tilting sheaf. Then for any homogeneous point x_0 and $n \gg 0$ there is a short exact sequence*

$$0 \to T_{\mathrm{can}}(-nx_0) \to T_0 \to T_1 \to 0$$

with $\operatorname{add}(T_0 \oplus T_1) = \operatorname{add}(T)$.

Proof. If T has slope w, choose $n \gg 0$ such that all indecomposable summands of $T_{\mathrm{can}}(-nx_0)$ have slope smaller than w. $\square$

9. The elliptic case

The tubular case, where all indecomposable coherent sheaves lie in tubes, is very similar (but weighted) to Atiyah's classification of indecomposable vector bundles over elliptic curves [12]. There are even more affinities between elliptic and tubular curves, see [22]. It is thus natural to expect a similar classification of large tilting sheaves as in the tubular case. But there are some technical differences: Since these curves are non-weighted, that is, do not have exceptional tubes, there is *no* coherent tilting sheaf. Reduction arguments using perpendicular calculus as in the proof of Lemma 8.1 are not possible. Moreover, the Grothendieck groups of elliptic curves are *not* finitely generated abelian, hence the (last part of the) proof of the crucial Lemma 8.2 does not work in the elliptic case. Additionally, we do not know whether in the elliptic case all tilting sheaves are of finite type. On the other hand, because all tubes are homogeneous, some arguments are even easier. For instance, in the elliptic case Lemma 7.2 has the stronger form

$$X, Y \in \mathcal{H} \text{ indecomposable, } \mu(X) < \mu(Y) \quad \Rightarrow \quad \operatorname{Hom}(X, Y) \neq 0. \tag{9.1}$$

Examples are the "classical" (commutative) elliptic curves over an algebraically closed field, and the real elliptic curves: the Klein bottle, the annulus, the Möbius band and the elliptic Witt curves, [39, Ex. 12.2].

For every rational α and each $\vec{\mathcal{H}}\langle\alpha\rangle$, Corollary 5.9 yields tilting sheaves $T_{\alpha,V}$ of slope α with non-zero torsion part supported in $\emptyset \neq V \subseteq \mathbb{X}_\alpha$.

Theorem 9.1. *Let $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$ be the category of quasicoherent sheaves over a noncommutative elliptic curve.*

(1) *Every tilting sheaf in $\vec{\mathcal{H}}$ has a slope $w \in \widehat{\mathbb{R}}$.*

(2) *For every $w \in \widehat{\mathbb{R}}$ there is a tilting sheaf $\mathbf{L}_w$ with ${\mathbf{L}_w}^{\perp_1} = \mathcal{B}_w$ which is torsionfree (in the sense of Definition 7.9).*

(3) *For every $\alpha \in \widehat{\mathbb{Q}}$ and every non-empty $V \subseteq \mathbb{X}_\alpha$ there is, up to equivalence, precisely one tilting sheaf T of slope α with $t_\alpha(T)$ supported in V, namely $T = T_{\alpha,V}$.*

(4) *Every tilting sheaf of finite type is equivalent to one listed in (2) or (3).*

(5) *The resolving subclasses of $\mathcal{H}$ are given precisely by $\operatorname{add}\mathbf{p}_w$ with $w \in \widehat{\mathbb{R}}$, and $\operatorname{add}\big(\mathbf{p}_\alpha \cup \bigcup_{x\in V}\mathbf{t}_{\alpha,x}\big)$ with $\alpha \in \widehat{\mathbb{Q}}$ and $\emptyset \neq V \subseteq \mathbb{X}_\alpha$.*

Proof. (1), (2), (3) We show that every tilting sheaf $T \in \vec{\mathcal{H}}$ has a slope $w \in \widehat{\mathbb{R}}$. To this end, let $w = \inf\{r \in \widehat{\mathbb{R}} \mid T \in \mathcal{C}_r\} \in \widehat{\mathbb{R}}$. We assume first that w is rational; then without loss of generality $w = \infty$. If $tT \neq 0$, then T is by Corollary 5.9 of the form T_V with $\emptyset \neq V \subseteq \mathbb{X}$ (in particular, we also have uniqueness in this case). Let now T be torsionfree. Then $\operatorname{vect}\mathbb{X} \subseteq {}^{\perp_1}T$. Indeed, otherwise one finds a line bundle L', say of slope $\alpha < \infty$, such that T maps onto L'. By (9.1), L' maps non-trivially to *each* vector bundle of slope $> \alpha$. Since, by torsionfreeness, all simple sheaves lie in $\operatorname{Gen}(T)$, it follows that all line bundles of slope $> \alpha$ lie in $\operatorname{Gen}(T)$. Let E be an indecomposable vector bundle of slope $> \alpha$. Then L' is a subsheaf of E, and we find a line bundle L'' with $L' \subseteq L'' \subseteq E$ such that $E' = E/L''$ is torsionfree, thus a line bundle. Since $\operatorname{rk}(E') = \operatorname{rk}(E) - 1$ and $\mu(E') \geq \mu(E) > \alpha$ we see by induction that every indecomposable vector bundle of slope $> \alpha$ lies in $\operatorname{Gen}(T) = T^{\perp_1}$. By Serre duality $\operatorname{Hom}(\mathbf{q}_\beta, T) = 0$ for all rational β with $\alpha < \beta < \infty$. But then $T \in \mathcal{C}_\alpha$, which gives a contradiction to the choice of w $(= \infty)$. It follows that T has slope ∞, moreover $\mathscr{S} := {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H} = \operatorname{vect}\mathbb{X}$.
Let now w be irrational. We have to show $T \in {\mathbf{p}_w}^{\perp_1}$. Otherwise, there is a rational $\alpha < w$ such that $\operatorname{Hom}(T, \mathbf{t}_\alpha) \neq 0$. Considering images, we can assume with loss of generality that there is an epimorphism in this set. Then it is easy to see that there is $x \in \mathbb{X}_\alpha$ such that T generates a tube $\mathbf{t}_{\alpha,x}$. Then it follows like in [53, Rem. 13.3], that T generates all coherent objects E of all rational slopes β with $\alpha < \beta \leq \infty$. But this means, by Serre duality, that for all those E we have $\operatorname{Hom}(E, T) = 0$, and thus $T \in \mathcal{C}_\alpha$. This is a contradiction to the choice of w. We conclude $T \in {\mathbf{p}_w}^{\perp_1} = \mathcal{B}_w$, and T has slope w. (We remark that this argument for irrational w also applies to the torsionfree case when w is rational.)
Finally, for every $w \in \widehat{\mathbb{R}}$ there is a torsionfree tilting sheaf $\mathbf{L}_w$. Indeed, $\mathscr{S} = \operatorname{add}\mathbf{p}_w$ generates $\vec{\mathcal{H}}$ and is thus resolving. The claim now follows from Theorem 4.4.

(4) Let T be tilting of finite type, $T^{\perp_1} = \mathscr{S}^{\perp_1}$ for some $\mathscr{S} \subseteq \mathcal{H}$ which we choose as $\mathscr{S} = {}^{\perp_1}(T^{\perp_1}) \cap \mathcal{H}$. By (1), T has a slope w. If T has a non-trivial torsion part, then T is equivalent to a tilting sheaf in (3) by Corollary 5.9. So we assume that T is torsionfree. Since a coherent object X is in $\mathscr{S}$ if and only if $\operatorname{Ext}^1(X,T)=0$, we have $\mathbf{p}_w \subseteq \mathscr{S}$: indeed, $\operatorname{Ext}^1(\mathbf{t}_\alpha, T) = \operatorname{D}\operatorname{Hom}(T, \mathbf{t}_\alpha) = 0$ for all rational $\alpha < w$ by slope reasons. Furthermore, if $X \in \mathbf{q}_w$, then $\operatorname{Ext}^1(T, \tau X) = 0$ as $T \in \mathcal{C}_w = {}^{\perp_1}\mathbf{q}_w$, so $\operatorname{Ext}^1(X,T) \simeq \operatorname{D}\operatorname{Hom}(T,\tau X) \neq 0$, and $X \notin \mathscr{S}$. Finally, in case $w \in \widehat{\mathbb{Q}}$, it follows as in Lemma 8.4 that $\mathscr{S} \subseteq \mathcal{C}_w \cap \mathcal{H}$ satisfies $\mathscr{S} \cap \mathbf{t}_w = \emptyset$. We thus conclude $\mathscr{S} = \operatorname{add}\mathbf{p}_w$, and T is equivalent to the tilting sheaf $\mathbf{L}_w$ from (2).

(5) Using (4), the claim follows from Theorem 4.14 and Lemma 4.11. □

10. Combinatorial descriptions and an example

Let $\mathbb{X}$ be a noncommutative curve of genus zero, of arbitrary weight type. In this section we further investigate the large tilting sheaves $T_{(B,V)}$ with $V \neq \emptyset$. We already know that they are of finite type and satisfy condition (TS3). We give an explicit construction for the sequence in (TS3), and we verify the stronger property (TS3+).

We denote by Λ a canonical tilting bundle T_{can}, as in Remark 5.12. By copresenting each indecomposable summand of T_{can} by summands of $T_{(B,V)}$ we will prove the following.

Theorem 10.1. *Let $\mathbb{X}$ be of genus zero and $T = T_{(B,V)}$ with $V \neq \emptyset$ as in (4.6). The canonical configuration $T_{\text{can}} = \Lambda$ has an $\operatorname{add}(T)$-copresentation as follows:*

$$0 \to T_{\text{can}} \to T_0' \oplus B_0 \to T_1' \oplus B_1 \to 0 \tag{10.1}$$

with $T_0' \in \operatorname{add}(\Lambda_V')$ torsionfree, $T_1' \in \operatorname{add}(\bigoplus_{x \in V} \bigoplus_{j \in \mathcal{R}_x} \tau^j S_x[\infty])$ and B_0, $B_1 \in \operatorname{add}(B)$ such that $\operatorname{Hom}(B_1, B_0) = 0$; moreover, in T_1' all Prüfer summands $\tau^j S_x[\infty]$ of T occur and $\operatorname{add}(B_0 \oplus B_1) = \operatorname{add}(B)$.

As a first preparation we have the following simple fact.

Lemma 10.2. *Let B be a connected branch and B' a proper subbranch of B, rooted in $Z \in B$. Then one of the following two cases holds.*

1. *There is an epimorphism $X \to Z$ with $X \in B \setminus B'$, and then there is no non-zero morphism from B' to $B \setminus B'$.*
2. *There is a monomorphism $Z \to Y$ with $Y \in B \setminus B'$, and then there is no non-zero morphism from $B \setminus B'$ to B'.*

Proof. Since B' is proper, it is clear that there is either an epimorphism $X \to Z$ or a monomorphism $Z \to Y$ with X or Y in $B \setminus B'$, respectively. Let $\mathcal{W}'$ be the wing rooted in Z. Since B' forms a tilting object in $\mathcal{W}'$, it is clear, that $\mathcal{W}'$ is disjoint with $B \setminus B'$. Let $U \in B'$ and $V \in B \setminus B'$. Assume the first case, and $\operatorname{Hom}(U,V) \neq 0$. Then V lies on a ray starting in the basis of $\mathcal{W}'$, but not in $\mathcal{W}'$. We then get $\operatorname{Hom}(X, \tau V) \neq 0$. By Serre duality we get $\operatorname{Ext}^1(V,X) \neq 0$, which gives a contradiction because of $\operatorname{Ext}^1(B,B)=0$. The second case follows similarly. □

10.3. Let $T = T_{(B,V)}$ be a given large tilting sheaf with $V \neq \emptyset$. For the moment we assume, for notational simplicity, that B is an *inner* branch sheaf. Let us explain the strategy we are going to pursue for the proof of the theorem.

Step 1: *Initial step.* We start with the canonical configuration $\Lambda = T_{\text{can}}$ in $\vec{\mathcal{H}} = \operatorname{Qcoh}\mathbb{X}$, which consists of arms between L and $\overline{L}$, compare (5.10). By applying suitable tubular shifts to Λ, we can assume without loss of generality that $\operatorname{Hom}(L, B) = 0 = \operatorname{Hom}(\overline{L}, B)$. We then form $\vec{\mathcal{H}}' = \operatorname{Qcoh}\mathbb{X}' = (\tau^- B)^\perp$. Then the subconfiguration Λ' of indecomposable summands of Λ lying in $(\tau^- B)^\perp$ forms a canonical configuration $\Lambda' = T'_{\text{can}}$ in $\vec{\mathcal{H}}'$, containing L and $\overline{L}$, compare Remark 5.12. For Λ' we have the copresentation

$$0 \to \Lambda' \to \Lambda'_V \to \bigoplus_{x\in V} \bigoplus_{j=0}^{p'(x)-1} (\tau^j S_x[\infty])^{e(j,x)} \to 0. \tag{10.2}$$

from (5.5), which is already of the desired form with respect to the theorem we want to prove; by construction, it gives an $\operatorname{add}(T)$-copresentation of each indecomposable summand of Λ'. It remains to compute suitable copresentations for each indecomposable summand of Λ not in Λ', and then to take the direct sum of all of these sequences with (10.2). This will be done inductively working in each connected branch component, starting with the root of that component. Let us consider one such component lying in a wing $\mathcal{W}$ rooted in, say, $S[r-1]$ with $2 \leq r \leq p$, concentrated at a point $x \in V$. We will call $S[\infty]$ the Prüfer sheaf *above* $\mathcal{W}$.

Step 2: *Induction start with root.* Note that $S[r] \in (\tau^- B)^\perp \simeq \operatorname{Qcoh}\mathbb{X}'$ becomes simple. The basis of $\mathcal{W}$ is given by the simple sheaves $S, \tau^- S, \ldots, \tau^{-(r-2)} S$. This segment of simples corresponds to a segment of direct summands of Λ' lying in the inner of one arm. We denote this segment by $L(1), \ldots, L(r-1)$, so that there are epimorphisms

$$L(i) \twoheadrightarrow \tau^{-i+1} S \qquad i = 1, \ldots, r-1. \tag{10.3}$$

(We will do this for every branch component, and then we will need, of course, a shift of indices. In the notation of (5.10) the segment $L(1), \ldots, L(r-1)$ is $L_i(j), \ldots, L_i(j+r-2)$ for some arm-index i and some j.) With this "calibration" the sequence (5.7) becomes

$$0 \to L(0) \to L(r-1) \to S[r-1] \to 0 \tag{10.4}$$

where $L(0)$ is a predecessor of $L(1)$, either still in the inner of the same arm, or $L(0) = L^{\varepsilon f(x)}$; in any case $L(0) \in \operatorname{add}(\Lambda')$. This means that for $L(0)$ we already have a copresentation. Using Lemma 10.5 below, we will get a copresentation for $L(r-1)$, which will be compatible with the statement of our theorem.
We will then proceed in a similar way with the other members of the connected branch B, going down the branch inductively, as described in the next step.

Step 3: *Induction step.* We introduce further notation. We define

$$W_{ij} = S[i]/S[i-j] \in \mathcal{W}$$

for $i = 1, \dots, r-1$; $j = 1, \dots, i$, where $S[0] = 0$. We call W_{ij} *wing objects*, and the pair of indices (i, j) *wing pairs*. The length of W_{ij} is j; we say that i is the *level* and $i - j$ the *colevel* of W_{ij}. So W_{ij} is uniquely determined by its level and colevel, which fix the ray and coray W_{ij} belongs to. Applying the construction of an $\operatorname{add}(L)$-couniversal extension to the short exact sequences $0 \to W_{jj} \to W_{ii} \to W_{i,i-j} \to 0$, and recalling that we have $\operatorname{Hom}(L, \mathcal{W}) = 0$, we deduce from [44, Prop. 5.1] that there are short exact sequences

$$0 \to L(j) \to L(i) \to W_{i,i-j} \to 0 \tag{10.5}$$

for $1 \le j < i$. We assume now that $W_{i,i-j}$ be part of B. The (direct) neighbours of smaller length in the same component of the branch might be

$$\begin{tikzcd}
 & W_{i,i-j} \arrow[ddr] & \\
W_{i-\ell,i-j-\ell} \arrow[ur] & & \\
 & & W_{i,i-j-s}
\end{tikzcd}$$

where $W_{i-\ell,i-j-\ell} \to W_{i,i-j}$ denotes a composition of ℓ irreducible monomorphisms and $W_{i,i-j} \to W_{i,i-j-s}$ a composition of s irreducible epimorphisms. In this situation we compute an $\operatorname{add}(T)$-copresentation of $L(i-\ell)$ and $L(j+s)$, respectively, if $\operatorname{add}(T)$-copresentations of $L(j)$ or $L(i)$, respectively, are already known. In other words: having already exploited $W_{i,i-j}$ for computing a suitable copresentation of an indecomposable summand of Λ, we will then use its lower neighbours for computing copresentations for further summands. The two different kinds of neighbours are reflected by the following two lemmas. Roughly speaking, the first lemma (treating the epimorphism case) adds the branch summand $W_{i,i-j-s}$ to the end term, the second (treating the monomorphism case) the branch summand $W_{i-\ell,i-j-\ell}$ to the middle term in the copresentation of Λ.

Lemma 10.4. *Let $(i, i-j)$ and $(i, i-j-s)$ be wing pairs and assume that $W_{i,i-j}$ and $W_{i,i-j-s}$ are summands of B. Assume there is an exact sequence*

$$0 \to L(i) \to \mathcal{G} \oplus B_0 \to P \oplus B_1 \to 0$$

such that

(i) *B_0, $B_1 \in \operatorname{add}(B)$ are disjoint with the subbranch rooted in $W_{i,i-j-s}$;*
(ii) *$\operatorname{Hom}(B_1, B_0) = 0$;*
(iii) *$\mathcal{G}$ is torsionfree and x-divisible;*
(iv) *P is a direct sum of copies of the Prüfer sheaf $S[\infty]$ above the wing $\mathcal{W}$.*

Then there is an exact sequence

$$0 \to L(j+s) \to \mathcal{G} \oplus B_0 \to P \oplus W_{i,i-j-s} \oplus B_1 \to 0$$

with $\operatorname{Hom}(W_{i,i-j-s}, B_0) = 0$.

Proof. The sequence

$$0 \to L(j+s) \to L(i) \to W_{i,i-j-s} \to 0$$

together with the given sequence yields the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L(j+s) \arrow[r] \arrow[d, equal] & L(i) \arrow[r] \arrow[d] & W_{i,i-j-s} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L(j+s) \arrow[r] & \mathcal{G} \oplus B_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & P \oplus B_1 \arrow[r, equal] \arrow[d] & P \oplus B_1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The right column splits, since $W_{i,i-j-s}$ and B_1 as summands of the branch B are Ext-orthogonal, and since $\operatorname{Ext}^1(P, W_{i,i-j-s}) = \operatorname{D}\operatorname{Hom}(\tau^- W_{i,i-j-s}, P) = 0$. Because of (i) we get $\operatorname{Hom}(W_{i,i-j-s}, B_0) = 0$ from Lemma 10.2. □

Lemma 10.5. *Let $(i, i-j)$ and $(i-\ell, i-j-\ell)$ be wing pairs such that $W_{i,i-j}$ and $W_{i-\ell,i-j-\ell}$ are summands of B (the case $\ell = 0$ is permitted). Assume there is an exact sequence*

$$0 \to L(j) \to \mathcal{G} \oplus B_0 \to P \oplus B_1 \to 0$$

such that $B_0, B_1 \in \operatorname{add}(B)$ are disjoint from the subbranch rooted in $W_{i-\ell,i-j-\ell}$, $\operatorname{Hom}(B_1, B_0) = 0$, $\mathcal{G}$ is torsionfree and x-divisible, and P is a direct sum of copies of the Prüfer sheaf above the wing $\mathcal{W}$. Then there is an exact sequence

$$0 \to L(i-\ell) \to \mathcal{G} \oplus B_0 \oplus W_{i-\ell,i-j-\ell} \to P \oplus B_1 \to 0,$$

and $\operatorname{Hom}(B_1, W_{i-\ell,i-j-\ell}) = 0$.

Proof. There is the push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L(j) \arrow[r] \arrow[d] & \mathcal{G} \oplus B_0 \arrow[r] \arrow[d] & P \oplus B_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L(i-\ell) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & P \oplus B_1 \arrow[r] & 0 \\
 & W_{i-\ell,i-j-\ell} \arrow[r, equal] \arrow[d] & W_{i-\ell,i-j-\ell} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Now, since $\mathcal{G}$ is x-divisible and $W_{i-\ell,i-j-\ell}$ and B_0 as summands of the branch B are Ext-orthogonal, the middle column splits. Moreover, $\operatorname{Hom}(B_1, W_{i-\ell,i-j-\ell}) = 0$ follows again from Lemma 10.2. □

10.6. Let now B be an *exterior* branch part, the inner branch parts already treated. We proceed similarly to the inner case. We briefly explain the differences. By applying suitable tubular shifts to Λ, we can assume without loss of generality that $\mathrm{Hom}(L,\tau B)=0=\mathrm{Hom}(\overline{L},\tau B)$. We then form $\vec{\mathcal{H}}'=\mathrm{Qcoh}\,\mathbb{X}'=B^{\perp}$. Then the subconfiguration Λ' of indecomposable summands of Λ lying in $B^{\perp}$ forms a canonical configuration $\Lambda'=T'_{\mathrm{can}}$ in $\vec{\mathcal{H}}'$, containing L and $\overline{L}$. Note that $\tau S[r]\in B^{\perp}\simeq \mathrm{Qcoh}\,\mathbb{X}'$ becomes simple. The basis of a wing $\mathcal{W}$ corresponding to a connected component of B is given by the simple sheaves (concentrated at x) $S,\tau^{-}S,\dots,\tau^{-(r-2)}S$. This segment of simples corresponds to a segment of direct summands of Λ' lying in the inner of one arm. We denote this segment by $L(1),\dots,L(r-1)$, so that there are epimorphisms

$$L(i)\twoheadrightarrow \tau^{-i+2}S \qquad i=1,\dots,r-1. \tag{10.6}$$

This yields a short exact sequence

$$0\to L(1)\to L(r)\to S[r-1]\to 0 \tag{10.7}$$

where $L(r)$ is either in the inner of the same arm, or $L(r)=\overline{L}^{f(x)}$. (We refer to the diagram in [44, p. 536].) Thus the desired copresentation of $L(r)$ is already given. Then, for $L(1)$ and for the induction step we have modified versions of Lemma 10.4 and 10.5, just taking into account the different notation (10.6).

Proof of Theorem 10.1. Let Λ be a given canonical configuration, considered as full subcategory of $\mathcal{H}$. As usual we write $B=B_{\mathfrak{i}}\oplus B_{\mathfrak{e}}$ with respect to V. We can assume that the canonical configuration Λ' in $(\tau^{-}B_{\mathfrak{i}}\oplus B_{\mathfrak{e}})^{\perp}\simeq\mathrm{Qcoh}\,\mathbb{X}'$ is a subconfiguration of Λ, containing L and $\overline{L}$. Recall that B decomposes into $B=\bigoplus_{i=1}^{t}B_{x_i}$ over the exceptional points $x_1,\dots,x_t$, and each B_{x_i} (in case it is nonzero) is a direct sum of finitely many connected branches in non-adjacent wings. Then

$$\Lambda=\Lambda'\oplus\bigoplus_{i=1}^{t}\bigoplus_{\ell}L_i(\ell)$$

for suitable ℓ, forming finitely many non-adjacent segments in $\{1,\dots,p_i-1\}$, corresponding to the connected branches as described above.
Step 1 yields the $\mathrm{add}(T)$-copresentation

$$0\to\Lambda'\to T'_0\to T'_1\to 0 \tag{10.8}$$

of Λ', given by (10.2). We then have to compute suitable copresentations for the $L_i(\ell)$. By forming the direct sum we will get the desired copresentation for Λ. This can be done separately by performing Step 2 and Step 3 for every connected branch (using Lemma 10.4 and 10.5 and keeping in mind the modifications in 10.6).

We still have to show that in this way we obtain $\mathrm{add}(T)$-copresentations of *all* indecomposable summands of Λ. It is enough to do this for every single wing $\mathcal{W}$ involved, say $\mathcal{W}$ is rooted in $S[r-1]$, and the corresponding summands of Λ are given by $L(1),\dots,L(r-1)$. (So this notation applies to the inner case, the exterior is treated similarly.) The kernel of the epimorphism $L(r-1) \to S[r-1] = W_{r-1,r-1}$ is (a power of) an indecomposable summand of Λ', and from Lemma 10.5 (case $\ell = 0$) we get an add(T)-copresentation of $L(r-1)$. Let W_{ij} be a summand of B, different from the root $S[r-1]$. Then W_{ij} has a unique upper neighbour Z in B. There are two cases:

(a) There is an epimorphism $Z \to W_{ij}$. Then Lemma 10.4 gives a copresentation of $L(i-j)$ where $i-j$ is the colevel of W_{ij}.

(b) There is a monomorphism $W_{ij} \to Z$. Then Lemma 10.5 gives a copresentation of $L(i)$ where i is the level of W_{ij}.

So either the level or the colevel determines the index of the summand of Λ we can treat with the help of W_{ij}. In both cases the obtained index lies between 1 and $r-2$. Assume now that there are two different summands W_{ij} and $W_{k\ell}$ of B, which are also different from the root of $\mathcal{W}$, and which yield the *same* index under the procedure above. We consider the upper neighbours of U of W_{ij} and V of $W_{k\ell}$. If there are epimorphisms $U \to W_{ij}$ and $V \to W_{k\ell}$, then we conclude that the colevels of W_{ij} and $W_{k\ell}$ coincide; similarly if there are monomorphisms $W_{ij} \to U$ and $W_{k\ell} \to V$, then the levels of both coincide. In the mixed case, when there is a monomorphism $W_{ij} \to U$ and an epimorphism $V \to W_{k\ell}$, then the level of W_{ij} is the colevel of $W_{k\ell}$. In all these cases it is easy to see that there are non-zero extensions between one of these objects and the other or the upper neighbour of the other, which gives a contradiction. Indeed, if W_{ij} and $W_{k\ell}$ have the same colevel, they belong to the same ray and $i \neq k$, say $i < k$. Then $\operatorname{Ext}^1(W_{kl}, U) = \mathrm{D}\operatorname{Hom}(U, \tau W_{kl}) \neq 0$. The level case is similar. In the mixed case, let c be the level of $W_{ij} = W_{cj}$ and the colevel of $W_{k\ell} = W_{k,k-c}$. Then W_{ij} lies on the coray ending in $W_{c,1}$ and $\tau W_{k\ell} = W_{k-1,\ell}$ lies on the ray starting in $W_{c,1}$, so $\operatorname{Ext}^1(W_{kl}, W_{ij}) = \mathrm{D}\operatorname{Hom}(W_{ij}, \tau W_{k\ell}) \neq 0$.

It follows that the $r-1$ summands of the branch B yield copresentations for $r-1$ distinct indecomposable summands of Λ, which are then necessarily given by $L(1), \dots, L(r-1)$. $\square$

We now illustrate the procedure, which can be done for each involved exceptional tube separately. In the following example we have two wings in the same tube to consider. (Note that compared with Lemmas 10.4 and 10.5 by a matter of notation there are unavoidable shifts of indices.)

EXAMPLE 10.7. In the following we will use the numerical invariants from 5.10 and the short exact sequences from 5.11, which are the building blocks of the canonical configuration (5.10). Let Λ be a canonical algebra of weight type given by the sequence (11), the only exceptional point given by x, let $V = \{x\}$ and $e = e(x)$, $f = f(x)$, $d = ef$ and $\varepsilon \in \{1, 2\}$ be the numerical type of $\mathbb{X}$. Then Λ is realized as canonical configuration

$$L \to L(1) \to L(2) \to L(3) \to L(4) \to \ldots \to L(9) \to L(10) \to \overline{L}$$

in $\mathcal{H}$. Let

$$B = S[4] \oplus \tau^{-2} S[2] \oplus \tau^{-2} S \oplus S \oplus S'[3] \oplus S'[2] \oplus \tau^{-} S'$$

be a branch, where we assume that S is simple with $\mathrm{Hom}(L, \tau^2 S) \neq 0$ and $S' = \tau^{-6} S$. Then $\mathrm{Hom}(L(i+2), \tau^{-i} S) \neq 0$ for $i = -1, 0, \dots, 8$. There are two connected components of B, lying in the wings rooted in $S[4]$ and $S'[3]$, respectively. The situation is illustrated in Figure 10.1, where the indecomposable summands of the branch B are denoted by $\bullet$, the roots of the two wings by $\widehat{\bullet}$. The two vertical lines indicate the identification by the τ-period. We also exhibit the undercuts by $\circ$, and the four Prüfer sheaves belonging to $T_{(B,V)}$ by the symbol $*$ over the corresponding ray. We have

$$\Lambda' = L \oplus L(1) \oplus L(6) \oplus L(7) \oplus \overline{L} \in (\tau^- B)^\perp.$$

There are the universal exact sequences in $(\tau^- B)^\perp = \mathrm{Qcoh}\,\mathbb{X}'$ (where the only weight of $\mathbb{X}'$ is given by $p' = 5$)

$$0 \to L \to \mathcal{G} \to \tau S[\infty]^e \to 0 \tag{10.9}$$

$$0 \to \overline{L} \to \mathcal{G}^\varepsilon \to \tau S[\infty]^{\varepsilon e} \to 0 \tag{10.10}$$

$$0 \to L(i+2) \to \mathcal{G}_i^{\varepsilon f} \to \tau^{-(i+1)} S[\infty]^{\varepsilon d} \to 0 \quad \text{for } i = -1,\ 4,\ 5. \tag{10.11}$$

with torsionfree, indecomposable $\mathcal{G}$, $\mathcal{G}_i$; note that $\mathcal{G}, \mathcal{G}_i \in (\tau^- B)^\perp$, and thus these objects are x-divisible. Their direct sum gives the short exact sequence

$$0 \to \Lambda' \to \Lambda'_V \to \tau S[\infty]^{(1+\varepsilon)e} \oplus S[\infty]^{\varepsilon d} \oplus \tau^{-5} S[\infty]^{\varepsilon d} \oplus S'[\infty]^{\varepsilon d} \to 0$$

where $\Lambda'_V = \mathcal{G}^{1+\varepsilon} \oplus \mathcal{G}_{-1}^{\varepsilon f} \oplus \mathcal{G}_4^{\varepsilon f} \oplus \mathcal{G}_5^{\varepsilon f}$. This was Step 1.

We now treat the first branch. This corresponds to the segment $L(2)$, $L(3)$, $L(4)$, $L(5)$ of Λ. Step 2: Applying Lemma 10.5 (to the sequence $0 \to L(1) \to L(5) \to S[4] \to 0$) gives the exact sequence

$$0 \to L(5) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \to S[\infty]^{\varepsilon d} \to 0. \tag{10.12}$$

Step 3: Applying Lemma 10.4 again yields

$$0 \to L(3) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \to \tau^{-2} S[2] \oplus S[\infty]^{\varepsilon d} \to 0. \tag{10.13}$$

Now applying Lemma 10.5 two times yields

$$0 \to L(4) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S[4] \oplus \tau^{-2} S \to \tau^{-2} S[2] \oplus S[\infty]^{\varepsilon d} \to 0 \tag{10.14}$$

and

$$0 \to L(2) \to \mathcal{G}_{-1}^{\varepsilon f} \oplus S \to S[\infty]^{\varepsilon d} \to 0. \tag{10.15}$$

The second branch corresponds to the segment $L(8)$, $L(9)$, $L(10)$ of Λ. Step 2, and then Step 3, which is applying Lemma 10.5 two times and then Lemma 10.4, yields the exact sequences

$$0 \to L(10) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \to S'[\infty]^{\varepsilon d} \to 0, \tag{10.16}$$

then

$$0 \to L(9) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \oplus S'[2] \to S'[\infty]^{\varepsilon d} \to 0 \tag{10.17}$$

and finally

$$0 \to L(8) \to \mathcal{G}_5^{\varepsilon f} \oplus S'[3] \oplus S'[2] \to \tau^- S' \oplus S'[\infty]^{\varepsilon d} \to 0 \tag{10.18}$$

Forming the direct sum of all 12 short exact sequences (10.9)–(10.18) we get the add(T)-copresentation of Λ as in Theorem 10.1.

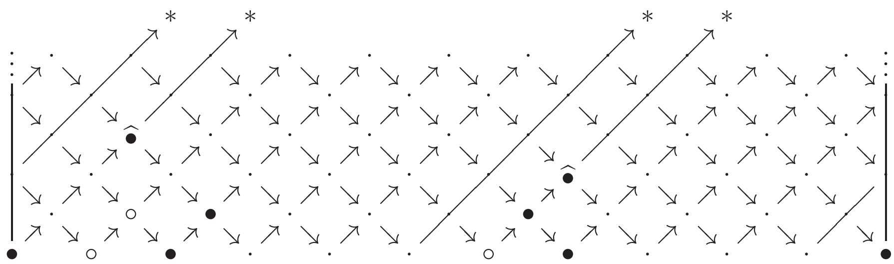

Figure 10.1. The branches of Example 10.7

Appendix A. Slope arguments in the domestic case

In this appendix we complement the arguments given in the proof of Lemma 6.1 by more details. Most of them are well-established for weighted projective lines, see for example [43, Thm. 2.7]. Here we see that in general we have to be careful with the special line bundles.

Let $\mathbb{X}$ be an *arbitrary* noncommutative curve of genus zero. Recall that a line bundle L' is called *special* if for every exceptional point x_i there is *precisely one* simple sheaf S_i concentrated at x_i with $\operatorname{Hom}(L', S_i) \neq 0$. Every autoequivalence σ of $\mathcal{H}$ induces an autoequivalence of $\mathcal{H}_0$ and thus of $\mathcal{H}/\mathcal{H}_0$, and is therefore rank-preserving. Hence, if L' is a special line bundle, then so is $\sigma L'$. For Geigle-Lenzing weighted projective lines [27] it is well-known (see [43, 2.1]) that for each degree $\vec{x}$ we have

$$\operatorname{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{\omega}+\vec{c})) \neq 0 \text{ if } \operatorname{Hom}(\mathcal{O}, \mathcal{O}(\vec{x})) = 0.$$

Since $\operatorname{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{\omega}+\vec{c})) = \operatorname{D}\operatorname{Ext}^1(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{x}))$, when we write $L = \mathcal{O}$ and $\overline{L}$ replaces $\mathcal{O}(\vec{c})$, the following statement is the generalization of this to noncommutative curves of genus zero (of arbitrary weight type).

Lemma A.1. *Let $\mathbb{X}$ be a noncommutative curve of genus zero and X be an indecomposable vector bundle. Then* $\operatorname{Hom}(L, X) \neq 0$ *or* $\operatorname{Ext}^1(\overline{L}, X) \neq 0$ *holds.*

In the domestic case this is a special case of [47, Prop. 4.1].

Proof. Assume that $\operatorname{Hom}(L, X) = 0 = \operatorname{Ext}^1(\overline{L}, X)$. We now apply $\operatorname{Hom}(-, X)$ to several of the exact sequences above. Sequence (5.13) gives

$$0 \to \operatorname{Hom}(S, X) \to \operatorname{Hom}(\overline{L}, X) \to \operatorname{Hom}(L^{\varepsilon}, X) \to$$
$$\to \operatorname{Ext}^1(S, X) \to \operatorname{Ext}^1(\overline{L}, X) \to \operatorname{Ext}^1(L^{\varepsilon}, X) \to 0$$

and from the assumptions we conclude that all terms are zero. Applying then $\operatorname{Hom}(-, X)$ to (5.14) shows $\operatorname{Ext}^1(L_i(j), X) = 0$. Similarly, (5.11) and (5.12) inductively yield $\mathrm{Hom}(L_i(j), X) = 0$. Altogether this gives that $\mathrm{Hom}(T_{\mathrm{can}}, X) = 0 = \mathrm{Ext}^1(T_{\mathrm{can}}, X)$, and since T_{can} is a tilting object we get $X = 0$, a contradiction. □

Lemma A.2. *Let $\mathbb{X}$ be domestic. Let L be a special line bundle. Let F be an indecomposable vector bundle of slope $\mu(F) - \mu(L) > \bar{p}/\varepsilon + \delta(\omega)$. Then* $\mathrm{Hom}(L, F) \neq 0$.

Proof. For every special line bundle L we can form a canonical configuration like (5.10), see [44]. Then L does not necessarily have degree zero, but still $\mu(\overline{L}) - \mu(L) = \bar{p}/\varepsilon$.
We assume $\mathrm{Hom}(L, F) = 0$. Then by Lemma A.1 $\mathrm{Ext}^1(\overline{L}, F) \neq 0$, and by Serre duality, $\mathrm{Hom}(F, \tau\overline{L}) \neq 0$. But by assumption

$$\mu(F) > \bar{p}/\varepsilon + \delta(\omega) + \mu(L) = \mu(\overline{L}) + \delta(\omega) = \mu(\tau\overline{L}),$$

which contradicts the stability of indecomposable vector bundles in the domestic case (Theorem 2.3). □

Remark A.3. In the domestic case every indecomposable vector bundle is exceptional. In particular this is true for every line bundle. But there are domestic cases where not every line bundle is special. Take for example the domestic symbol $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. It tells us that there is precisely one exceptional point x, and this point satisfies $p(x) = 2$, $f(x) = 1$ and $e(x) = 2$. (For the general definition of a symbol of a genus zero curve we refer to [37].) Let now L be a special line bundle which maps onto the simple S_x. The kernel then is a line bundle L'. One verifies that $[L']$ is a 1-root in $\mathrm{K}_0(\mathbb{X})$ and that $\mathrm{Hom}(L', S_x) \neq 0$ and $\mathrm{Hom}(L', \tau S_x) \neq 0$. Hence L' is not special.

Lemma A.4. *Let $\mathbb{X}$ be domestic. Let T be a torsionfree tilting sheaf. Assume that for every $n \in \mathbb{Z}$ there is a* special *line bundle L_n with $\mu(L_n) < n$ such that $\mathrm{Hom}(T, L_n) \neq 0$. If L' is an arbitrary line bundle, then also* $\mathrm{Hom}(T, L') \neq 0$.

Proof. Let L' be a line bundle. Choose $n \in \mathbb{Z}$ such that $n < \mu(L') - \bar{p}/\varepsilon - \delta(\omega)$. Then $\mu(L') - \mu(L_n) > \bar{p}/\varepsilon + \delta(\omega)$, and by Lemma A.2, since L_n is special, we have $\mathrm{Hom}(L_n, L') \neq 0$. Hence there is a monomorphism $L_n \to L'$. Since $\mathrm{Hom}(T, L_n) \neq 0$ we get $\mathrm{Hom}(T, L') \neq 0$ as well. □

In order to remove the word "special" from the preceding lemma, we use the Riemann-Roch formula (see [39])

$$\text{(A.1)} \qquad \frac{1}{\bar{p}\kappa}\langle\langle X, Y\rangle\rangle = -\frac{\varepsilon}{2}\delta(\omega)\cdot \mathrm{rk}(X)\cdot\mathrm{rk}(Y) + \frac{\varepsilon}{\bar{p}}\begin{vmatrix} \mathrm{rk}(X) & \mathrm{rk}(Y) \\ \deg(X) & \deg(Y)\end{vmatrix}$$

where

$$\langle\langle X, Y\rangle\rangle = \sum_{j=0}^{\bar{p}-1}\langle X, \tau^{-j}Y\rangle$$

is the average Euler form. In particular, if L' and L are line bundles with $\mu(L) = \deg(L) \geq \deg(L') = \mu(L')$ then (by $\delta(\omega) < 0$) we have $\langle\langle L', L\rangle\rangle >$

0. Since, by stability, $\mathrm{Ext}^1(L',\tau^{-j}L) = \mathrm{D}\,\mathrm{Hom}(\tau^{-j-1}L,L') = 0$, and thus $\langle L',\tau^{-j}L\rangle = \dim_k \mathrm{Hom}(L',\tau^{-j}L)$, we obtain $\mathrm{Hom}(L',\tau^{-j}L) \neq 0$ for some $j \in \{0,\dots,\bar{p}-1\}$. It follows that there is a monomorphism $L' \to \tau^{-j}L$, where $\mu(\tau^{-j}L) = \mu(L) - j\cdot\delta(\omega) \leq \mu(L) - (\bar{p}-1)\cdot\delta(\omega)$.
If L is a special line bundle, then also $\tau^n L$ is special for every integer n, and has slope $\mu(\tau^n L) = \mu(L) + n\cdot\delta(\omega)$. So, if L' is a given line bundle, then there is a *special* line bundle L of slope $\mu(L)$ in the interval $[\mu(L'),\mu(L')-\delta(\omega)[$. With the preceding paragraph we obtain $j \in \{0,\dots,\bar{p}-1\}$ and a monomorphism $L' \to \tau^{-j}(L)$, for which $\mu(\tau^{-j}L) \leq \mu(L)-(\bar{p}-1)\cdot\delta(\omega) < \mu(L')-\bar{p}\cdot\delta(\omega)$ holds. To summarize:

Lemma A.5. *Let $\mathbb{X}$ be domestic. For every line bundle L' there is a* special *line bundle L with a monomorphism $L' \to L$, so that the distance of slopes*

$$0 \leq \mu(L) - \mu(L') < -\bar{p}\cdot\delta(\omega)$$

is bounded by a constant. □

We then have: if L' is such that $\mathrm{Hom}(T,L') \neq 0$, then, since there is a monomorphism $L' \to L$, also $\mathrm{Hom}(T,L) \neq 0$. As a consequence we get: if we find line bundles L' of arbitrarily small slope with $\mathrm{Hom}(T,L') \neq 0$, then we also find *special* line bundles L of arbitrarily small slope with $\mathrm{Hom}(T,L) \neq 0$. Therefore we now have the stronger version of Lemma A.4.

Lemma A.6. *Let $\mathbb{X}$ be domestic. Let T be a torsionfree tilting sheaf. Assume that for every $n \in \mathbb{Z}$ there is a line bundle L_n with $\mu(L_n) < n$ such that* $\mathrm{Hom}(T,L_n) \neq 0$. *If L' is an arbitrary line bundle, then also* $\mathrm{Hom}(T,L') \neq 0$. □

Lemma A.7. *Let $\mathbb{X}$ be domestic. Let T be a torsionfree tilting sheaf. Then there is $n_0 \in \mathbb{Z}$ such that* $\mathrm{Hom}(T,L') = 0$ *for every line bundle L' with $\mu(L') < n_0$.*

Proof. Otherwise we would have $\mathrm{Hom}(T,L') \neq 0$ for all line bundles L' by the preceding lemma. As in the proof of Lemma 8.2, we see that for a line bundle L' the condition $\mathrm{Hom}(T,L') \neq 0$ amounts to $L' \in \mathrm{Gen}(T)$, and since every vector bundle has a line bundle filtration, we infer that all vector bundles lie in $\mathrm{Gen}(T) = T^{\perp_1}$. By Serre duality we get that *no* vector bundle (even no coherent sheaf) maps to the torsionfree sheaf T, which is a contradiction, since $\vec{\mathcal{H}}$ is locally noetherian. □

We conclude with the desired result.

Lemma A.8 (Lemma 6.1). *Let $\mathbb{X}$ be domestic. Let T be a torsionfree tilting sheaf. Then there is $m \in \mathbb{Z}$ such that* $\mathrm{Hom}(T,E) = 0$ *for every indecomposable vector bundle E with $\mu(E) < m$.*

Proof. Let $\mathcal{F}$ be the set of indecomposable vector bundles F with $0 \leq \mu(F) < -\delta(\omega)$. This is a finite set by (D5), and every indecomposable vector bundle is of the form $\tau^n F$ for some $F \in \mathcal{F}$ and some $n \in \mathbb{Z}$. For every $F \in \mathcal{F}$ we fix a line bundle filtration, which altogether form a finite collection $\mathcal{L}$ of line bundles. We denote by $\alpha = \alpha(\mathcal{F})$ the maximum of slopes of the objects in $\mathcal{L}$. Then $\alpha(\tau^m\mathcal{F}) = \alpha + m\delta(\omega)$. With the bound n_0 from Lemma A.7, for all $m \in \mathbb{Z}$ such that $\alpha + m\delta(\omega) < n_0$, we get $\mathrm{Hom}(T, \tau^m\mathcal{L}) = 0$, and thus $\mathrm{Hom}(T, \tau^m\mathcal{F}) = 0$. It follows that $\mathrm{Hom}(T, E) = 0$ for every indecomposable vector bundle E with $\mu(E) < m\delta(\omega)$. □

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