Fractional Analogue of Sturm–Liouville Operator

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Abstract. In this paper we study a symmetric fractional differential operator of order 2α, $(1/2 < \alpha < 1)$. Using the extension theory a class of self–adjoint problems generated by the fractional Sturm–Liouville equation is described.

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1. Introduction

Many physical processes (diffusive processes, thermal processes and etc.) are expressed by fractional differential equations. Meanwhile, the study of boundary value problems for differential equations of fractional order is also very important to enrich and improve the fractional calculus theory. The fractional calculus has been an active field of research during several decades. In particular, the Mittag-Leffler functions are well-known in the theory of the fractional calculus, which allow us to describe phenomena in processes that progress or decay too slowly to be represented by classical functions like the exponential function and its successors. The basic properties are proved in [DN59]. Further investigations were done by Kilbas and Trujillo [KT02], Popov [P02], Jin and Rundell [JR12], and others. For more details we refer to [PS11, GKMR14] and references therein.
However, there are more open questions, for example in the spectral theory. It is well-known that the classical Sturm–Liouville equation

$$Su(x) \equiv u''(x) + q(x)u(x), \quad x \in (a, b) \tag{1.1}$$

with real $q \in C^1[a, b]$ and with boundary conditions

$$u(a) = 0, \quad u(b) = 0$$

is a self–adjoint operator in $L^2(a,b)$. Indeed, there is a class of so called 'strongly regular' boundary conditions [N67] which produce self–adjoint operators.
Nevertheless, it is unclear how to formulate a fractional analogue. Roughly speaking, fractional differential equations with the classical boundary conditions are not self–adjoint in the Hilbert space. Since self–adjointness implies a basis property of the system of root functions, mathematicians also were interested in approximation properties of fractional differential operators. For instance, the system of root functions for a fractional Sturm–Liouville type operator was investigated in [D70]. In [N77, A82] the authors studied spectral properties of the Sturm–Liouville equation with lower order fractional derivatives. More recent results can be found in [M10, RTV13, DWF14, P14, A15]. However, only non self–adjoint problems were considered in all of these papers. Fortunately, Klimek and Agrawal [KA13] found a symmetric fractional operator in the special weighted space of continuous functions. However, finding of new symmetric fractional operators is still interesting.
In this work we aim to find a symmetric fractional operator in the Hilbert space. Given a fractional differential equation of order 2α, $(1/2 < \alpha < 1)$, on an interval (a,b), the main issue is to choose 'suitable' boundary conditions to get a symmetric operator. Here, we define boundary functionals and obtain a symmetric fractional Sturm–Liouville operator in a 'suitable' Hilbert space. Using the extension theory of operators a class of self–adjoint problems is described. Finally, we derive spectral properties and allocate positive operators from the self–adjoint operators.
For applications of symmetric fractional operators to the related topics, see [KOM14, BC14, KDE15, LQ15, KM16], and for numerical realizations we refer to [AS10, ZK13, HMA14].
In subsequent works we will apply Fourier Analysis technics (see, for instance [ZK14, K15, QDH15]) in a combination with the self–adjoint fractional Sturm–Liouville operators obtained here to solve mixed problems of sub–diffusion, super–diffusion, anomalous diffusion, fractional Laplacian and others.

2. Main Results

In this section, we state main results and compile some basic definitions of fractional differential operators. For a fuller treatment the reader is referred to [SKM87, KST06] and references therein. Now we give definitions of the Riemann–Liouville fractional integrals and derivatives, and formulate the Caputo fractional derivatives. Also, we will use the sequential differentiation introduced in ([KST06], p. 394).

DEFINITION 2.1. The left and right Riemann–Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ are given by

$$I^{\alpha}_{a+}\left[f\right](t) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_a^t \left(t-s\right)^{\alpha-1} f\left(s\right) ds, \;\; t \in (a,b],$$

and

$$I_{b-}^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} (s-t)^{\alpha-1} f(s) ds, \quad t \in [a,b),$$

respectively. Here Γ denotes the Euler gamma function.

Definition 2.2. The left Riemann–Liouville fractional derivative D_{a+}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ is defined by

$$D_{a+}^{\alpha}[f](t) = \frac{d}{dt} I_{a+}^{1-\alpha}[f](t), \quad \forall t \in (a,b].$$

Similarly, the right Riemann–Liouville fractional derivative D_{b-}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ is given by

$$D_{b-}^{\alpha}[f](t) = -\frac{d}{dt} I_{b-}^{1-\alpha}[f](t), \quad \forall t \in [a,b).$$

Definition 2.3. The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ are defined by

$$\mathcal{D}_{a+}^{\alpha}[f](t) = D_{a+}^{\alpha}[f(t) - f(a)], \quad t \in (a,b],$$

and

$$\mathcal{D}_{b-}^{\alpha}[f](t) = D_{b-}^{\alpha}[f(t) - f(b)], \quad t \in [a,b),$$

respectively.

Consider the expression

$$\mathcal{L}u := \mathcal{D}_{a+}^{\alpha}\left(D_{b-}^{\alpha}(u)\right) \tag{2.1}$$

in $L^2(a,b)$. Here we assume that $\frac{1}{2} < \alpha < 1$.
Now, we define and characterize a space generated by the Caputo–Riemann–Liouville equation (2.1).

Theorem 2.1. *The space*

$$H^{2\alpha}(a,b) = \left\{u \in L^{2\alpha}(a,b): \ \mathcal{D}_{a+}^{\alpha} D_{b-}^{\alpha} u \in L^2(a,b)\right\}$$

closed with respect to the norm

$$\|u\|_{H^{2\alpha}(a,b)} := \|u\|_{L^2(a,b)} + \|\mathcal{D}_{a+}^{\alpha} D_{b-}^{\alpha} u\|_{L^2(a,b)}$$

is a Banach space. Here, $L^{2\alpha}(a,b)$ is a Hölder space of the order 2α. Furthermore, $H^{2\alpha}(a,b)$ is the Hilbert space with the inner product

$$(u,v)_{H^{2\alpha}(a,b)} := (u,v) + (\mathcal{D}_{a+}^{\alpha} D_{b-}^{\alpha} u, \mathcal{D}_{a+}^{\alpha} D_{b-}^{\alpha} v),$$

where $(\cdot,\cdot)$ is the inner product of the space $L^2(a,b)$.

Define $\mathcal{L}_m$ as an operator acting from $L^2(a,b)$ to $L^2(a,b)$ by the formula (2.1) with the domain

$$D(\mathcal{L}_m) = \left\{u \in H^{2\alpha}(a,b) : \xi_1^-(u) = \xi_2^-(u) = \xi_1^+(u) = \xi_2^+(u) = 0\right\},$$

where functionals $\xi_1^-(u)$, $\xi_2^-(u)$, $\xi_1^+(u)$, $\xi_2^+(u)$ are defined as follows:

$$\begin{aligned} \xi_1^-(u) &:= I_{b-}^{1-\alpha}[u](a), \;\; \xi_2^-(u) := I_{b-}^{1-\alpha}[u](b), \\ \xi_1^+(u) &:= D_{b-}^{\alpha}[u](a), \;\; \xi_2^+(u) := D_{b-}^{\alpha}[u](b). \end{aligned} \tag{2.2}$$

Also, we introduce the operator by the action (2.1)

$$\mathcal{L}_M : L^2(a,b) \longrightarrow L^2(a,b)$$

with the domain $D(\mathcal{L}_M) := \big\{u \in H^{2\alpha}(a,b)\big\}$.
Note that to investigate the time–fractional diffusion equation the authors of [GLY15] used the fractional Sobolev space explored by Adams [A99]. They showed that the space is equivalent to the Hilbert space induced by a second order differential operator.
Using the following class of matrices, we describe self-adjoint problems in Theorem 2.2. Also, an analogue of the strongly regular boundary conditions is obtained.

DEFINITION 2.4. We say that

$$\theta := \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix}$$

is a SA–matrix if it can be represented in one of the following forms:

$$\begin{pmatrix} 1 & 0 & r & c \\ 0 & 1 & -c & d \end{pmatrix}, \; \begin{pmatrix} d & 1 & 0 & r \\ c & 0 & 1 & d \end{pmatrix},$$

$$\begin{pmatrix} 1 & d & r & 0 \\ 0 & c & -d & 1 \end{pmatrix}, \; \begin{pmatrix} r & c & 1 & 0 \\ -c & d & 0 & 1 \end{pmatrix},$$

for $r, c, d \in \mathbb{R}$. The matrices

$$\begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \end{pmatrix} \text{and} \begin{pmatrix} \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\ \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \end{pmatrix}$$

are not distinguished.

THEOREM 2.2. *Let θ is a SA–matrix. Then $\mathcal{L}_\theta$ introduced by*

$$\mathcal{D}_{a+}^{\alpha} D_{b-}^{\alpha} u(x) = f(x), \;\; a < x < b,$$

for $u \in H^{2\alpha}(a,b)$ with conditions

$$\begin{aligned} \theta_{11}\xi_1^-(u) + \theta_{12}\xi_2^-(u) + \theta_{13}\xi_1^+(u) + \theta_{14}\xi_2^+(u) = 0, \\ \theta_{21}\xi_1^-(u) + \theta_{22}\xi_2^-(u) + \theta_{23}\xi_1^+(u) + \theta_{24}\xi_2^+(u) = 0, \end{aligned}$$

is a self–adjoint extension of $\mathcal{L}_m$ in $H^{2\alpha}(a,b)$.

3. Properties of Fractional Operators

Now we formulate some well–known properties of fractional operators [SKM87, KST06].

Property 3.1. *(cf.* [KST06]*, p. 73, p. 76, p. 96) Let $f \in L^1(a,b)$ and $0 < \alpha, \beta < 1$. Then, equations*

$$I^{\alpha}_{a+} I^{\beta}_{a+} f(x) = I^{\alpha+\beta}_{a+} f(x),$$
$$I^{\alpha}_{b-} I^{\beta}_{b-} f(x) = I^{\alpha+\beta}_{b-} f(x),$$

and

$$I^{\alpha}_{b-} D^{\alpha}_{b-} f(x) = f(x) - I^{1-\alpha}_{b-} f(a) \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}$$

are satisfied a.e. in $[a,b]$. If a function f is absolutely continuous, then

$$I^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} f(x) = f(x) - f(a)$$

holds for almost all $x \in [a,b]$.

Property 3.2. *(cf.* [SKM87]*, p. 87) Let $\alpha, \beta > 0$, and C is a constant. Then for all $\varepsilon \in [0,1]$ the function*

$$f(x) = C\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}(b-x-\varepsilon)^{\beta-1}_{*} = \begin{cases} 0, \; b-x \le \varepsilon; \\ \\ C\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}(b-x-\varepsilon)^{\beta-1}, \; b-x > \varepsilon; \end{cases}$$

satisfies

$$I^{\alpha}_{b-} f(x) = \begin{cases} 0, \; b-x \le \varepsilon; \\ \\ C(b-x-\varepsilon)^{\alpha+\beta-1} \; b-x > \varepsilon; \end{cases}$$

almost everywhere on $[a,b]$.

Property 3.3. *Let $0 < \alpha < 1$ and $f \in L^2(a,b)$. Then for arbitrary $\varepsilon \in (a,b)$*

$$f(x) = C(b-x-\varepsilon)^{\alpha-1}_{*} = \begin{cases} 0, \; b-x \le \varepsilon; \\ \\ C(b-x-\varepsilon)^{\alpha-1}, \; b-x > \varepsilon; \end{cases}$$

satisfies

$$D^{\alpha}_{b-} f(x) = 0$$

almost everywhere on $[a,b]$.

Property 3.4. *Let $0 < \alpha < 1$ and $f \in L^2(a,b)$. Then for all $\varepsilon \in (a,b)$ the function*

$$f(x) = C\theta(b-x-\varepsilon) = \begin{cases} 0, \; b-x \le \varepsilon, \\ \\ C, \; b-x > \varepsilon, \end{cases} \quad C = const, \;\; a < x < b,$$

satisfies the equality

$$\mathcal{D}^{\alpha}_{a+} f(x) = 0, \;\; a < x < b,$$

where $\theta(x)$ is the Heaviside function.

Property 3.5. *Fix $\varepsilon \in (a,b)$. Let $0<\alpha<1$ and $f \in L^2(a,b)$. Then for any C_1, C_2 the function*

$$f(x) = C_1(b-x-\varepsilon)_*^{\alpha-1} + C_2(b-x-\varepsilon)_*^{\alpha},$$

satisfies

$$D_{b-}^{\alpha} f(x) = C_2 \theta(b-x-\varepsilon)$$

for almost all $x \in [a,b]$.

Property 3.6. *(cf.* [KST06]*, p. 76) Let $u,v \in L^2(a,b)$ and $0<\alpha<1$. Then we have the fractional integration by parts formula*

$$\left(I_{b-}^{\beta} u(t), v(t)\right) = \left(u(t), I_{a+}^{\beta} v(t)\right).$$

4. Proofs

We begin by proving some necessary properties of the operators $\mathcal{L}_m$ and $\mathcal{L}_M$.

Lemma 4.1. *Fix $\varepsilon \in [a,b]$. A linear combination of $(b-x-\varepsilon)_*^{\alpha}$ and $(b-x-\varepsilon)_*^{\alpha-1}$ is from the kernel of $\mathcal{L}_M$ ($\mathbf{Ker}\mathcal{L}_M$).*

The proof of Lemma 4.1 follows from Properties 3.2, 3.3, 3.4 and 3.5.

Lemma 4.2. *The equation $\mathcal{L}_m u = g$ has a solution $u \in D(\mathcal{L}_m)$ if and only if there exists $f \in L^2(a,b)$ such that $(f,v)=0$ for any $v \in \mathbf{Ker}\mathcal{L}_M$:*

$$\mathcal{R}(\mathcal{L}_m) \oplus \mathbf{Ker}\mathcal{L}_M = L^2(a,b).$$

Proof. Let $f \in \mathcal{R}(\mathcal{L}_m)$. Then there exists $w \in L^2(a,b)$ such that for any $v \in \mathbf{Ker}\mathcal{L}_M$ we have

$$(f,v) = (\mathcal{L}_m w, v) = (w, \mathcal{L}_M v) = 0.$$

Fix $f \in L^2(a,b)$ with $(f,v)=0$ for all v from $\mathbf{Ker}\mathcal{L}_M$. By definition of $\mathcal{L}_M$ there is $g \in \mathrm{Dom}(\mathcal{L}_M)$ such that $\mathcal{L}_M g = f$. Easy to see that for an arbitrary function $v \in \mathbf{Ker}\mathcal{L}_M$ we obtain

$$0 = (f,v) = (\mathcal{L}_M g, v) = \sum_{i=1}^{2} [\xi_i^-(v)\xi_i^+(g) - \xi_i^-(g)\xi_i^+(v)]. \tag{4.1}$$

Indeed, informal calculations of $(\mathcal{D}_{a+}^{\alpha}\left[D_{b-}^{\alpha} u\right], v)$ proves the last equality. By changing integration order in

$$(\mathcal{D}_{a+}^{\alpha}\left[D_{b-}^{\alpha} u\right], v) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \int_a^x (x-t)^{-\alpha} \frac{d}{dt} D_{b-}^{\alpha} u(t) dt v(x) dx, \tag{4.2}$$

we get

$$\begin{aligned} &\int_a^b \int_a^x (x-t)^{-\alpha} \frac{d}{dt} D_{b-}^{\alpha} u(t) dt v(x) dx \\ &= \int_a^b \frac{d}{dt} D_{b-}^{\alpha} u(t) \int_t^b (x-t)^{-\alpha} v(x) dx dt. \end{aligned} \tag{4.3}$$

Integrating by parts in the right–side of the equation (4.3), we obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_a^b \frac{d}{dt}D_{b-}^{\alpha}u(t)\int_t^b (x-t)^{-\alpha}v(x)dxdt =$$
$$= D_{b-}^{\alpha}u(t)I_{b-}^{1-\alpha}v(t)\Big|_a^b + \left(D_{b-}^{\alpha}u, D_{b-}^{\alpha}v\right).$$

Let us calculate

$$\left(D_{b-}^{\alpha}u, D_{b-}^{\alpha}v\right) = -\int_a^b \frac{d}{dt}I_{b-}^{1-\alpha}u(t)D_{b-}^{\alpha}v(t)dt$$
$$= -I_{b-}^{1-\alpha}u(t)D_{b-}^{\alpha}v(t)\Big|_a^b + \int_a^b I_{b-}^{1-\alpha}u(t)\frac{d}{dx}D_{b-}^{\alpha}v(t)dt.$$

Now, by applying the fractional integration by parts formula (Property 3.6)

$$\left(I_{b-}^{\beta}u, v\right) = \left(u, I_{a+}^{\beta}v\right)$$

to $\left(I_{b-}^{1-\alpha}u, \frac{d}{dx}D_{b-}^{\alpha}v\right)$, we have

$$\left(I_{b-}^{1-\alpha}u, \frac{d}{dx}D_{b-}^{\alpha}v\right) = \left(u, \mathcal{D}_{a+}^{\alpha}\left[D_{b-}^{\alpha}\right]v\right).$$

As a result, we get

$$\begin{aligned} \left(\mathcal{D}_{a+}^{\alpha}\left[D_{b-}^{\alpha}u\right], v\right) &- \left(u, \mathcal{D}_{a+}^{\alpha}\left[D_{b-}^{\alpha}\right]v\right) \\ &= D_{b-}^{\alpha}u(t)I_{b-}^{1-\alpha}v(t)\Big|_a^b - I_{b-}^{1-\alpha}u(t)D_{b-}^{\alpha}v(t)\Big|_a^b. \end{aligned} \tag{4.4}$$

Further, using the notations of (2.2) we obtain (4.1) from the formula (4.4). Lemma 4.1 implies that the kernel of $\mathcal{L}_M$ consists of the infinity amount of linear independent functions. Due to the arbitrariness of v from (4.1) we have

$$\xi_i^-(g) = \xi_i^+(g) = 0, \quad i = 1, 2.$$

Hence $f \in \mathcal{R}(\mathcal{L}_m)$. The proof is complete. □

COROLLARY 4.1. *$D(\mathcal{L}_m)$ is dense in $L^2(a,b)$.*

Proof. Let $g \in L^2(a,b)$ be orthogonal to the lineal $\mathrm{Dom}(\mathcal{L}_m)$. Find a function v which is an arbitrary solution of the equation $\mathcal{L}_M v = g$. Then for any $u \in \mathrm{Dom}(\mathcal{L}_m)$ we have

$$0 = (u, g) = (u, \mathcal{L}_M v) = (\mathcal{L}_m u, v).$$

By Lemma 4.2 we get $v \in \mathbf{Ker}\mathcal{L}_M$. Therefore, $g = \mathcal{L}_M v = 0$. The lemma is proved. □

4.1. PROOF OF THEOREM 2.1. By Corollary 4.1 the operator $\mathcal{L}_m$ is closable in $L^2(a,b)$. Then $(\mathcal{L}_m)^* = \mathcal{L}_M$ in $L^2(a,b)$. Hence, it follows that $\mathcal{L}_M$ is a closed operator. Thereby, $H^{2\alpha}(a,b)$ is the Banach space (see [DS63]). The second part can be proved by checking the axioms of the Hilbert space.

4.2. Proof of Theorem 2.2. Since for any $u, v \in D(\mathcal{L}_m)$, we have

$$(\mathcal{L}_m u, v) = (u, \mathcal{L}_m v),$$

then by definition [N67] $\mathcal{L}_m$ is a Hermitian operator. By virtue of Corollary 4.1 the operator $\mathcal{L}_m$ is a symmetric operator. Thereby, the operator $\mathcal{L}_\theta$ is self–adjoint if

$$D(\mathcal{L}_\theta) = D(\mathcal{L}_\theta^*). \tag{4.5}$$

Finally, the proof of Theorem 2.2 follows from (4.4) by the direct calculations.

5. On some Spectral Properties

Theorem 5.1. *For the self–adjoint operator $\mathcal{L}_\theta$ with θ of the form*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \theta_{21} & 0 & \theta_{23} & 0 \end{pmatrix},$$

the following facts are true:
[(i)] $\mathcal{L}_\theta^{-1}$ *is a compact operator in $L^2(a,b)$.*
[(ii)] *The spectrum is discrete and real valued, and the system of eigenfunctions is a complete orthogonal basis in $L^2(a,b)$.*

Proof. (i) If $\theta_{21} \neq 0$ and $\theta_{21} \neq \theta_{23}$ then the inverse operator represents as

$$\mathcal{L}_\theta^{-1} f(x) = \frac{\theta_{21}}{\theta_{21} - \theta_{23}} \frac{(b-x)^\alpha}{(b-a)\Gamma(\alpha+1)} I_{a+}^{\alpha+1} f(b) + I_{b-}^\alpha I_{a+}^\alpha f(x),$$

and, if $\theta_{21} = 0$ then the representation has the form

$$\mathcal{L}_\theta^{-1} f(x) = I_{b-}^\alpha I_{a+}^\alpha f(x), \;\; a < x < b.$$

Indeed, it implies compactness of $\mathcal{L}_\theta^{-1}$ in $L^2(a,b)$.
(ii) From compactness of the operator $\mathcal{L}_\theta^{-1}$ follows discreteness of the spectrum and that the system of eigenfunctions is a complete orthogonal basis in $L^2(a,b)$. Self–adjointness of $\mathcal{L}_\theta$ implies [N67] that all eigenvalues are real. □

Theorem 5.2. *Let θ has one of the following forms*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

$$\begin{pmatrix} \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Then for all $\rho \in \mathbb{R}$ the operator $\mathcal{L}_\theta$ is positive in $L^2(a,b)$.

Proof. It is enough to show that

$$(\mathcal{D}_{a+}^\alpha \left[D_{b-}^\alpha u\right], u) \geq 0.$$

Let us calculate

$$(\mathcal{D}_{a+}^\alpha \left[D_{b-}^\alpha u\right], u) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \int_a^x (x-t)^{-\alpha} \frac{d}{dt} D_{b-}^\alpha u(t) dt \, u(x) dx.$$

By changing integration order, we obtain

$$\int_a^b \int_a^x (x-t)^{-\alpha} \frac{d}{dt} D_{b-}^{\alpha} u(t) dt\, u(x) dx =$$
$$= \int_a^b \frac{d}{dt} D_{b-}^{\alpha} u(t) \int_t^b (x-t)^{-\alpha} u(x) dx dt.$$

Integrating by parts in the right–side of the last integral, we get

$$\frac{1}{\Gamma(1-\alpha)} \int_a^b \frac{d}{dt} D_{b-}^{\alpha} u(t) \int_t^b (x-t)^{-\alpha} u(x) dx dt =$$
$$= D_{b-}^{\alpha} u(t) I_{b-}^{1-\alpha} u(t) \Big|_a^b + \left(D_{b-}^{\alpha} u, D_{b-}^{\alpha} u \right).$$

Finally, from conditions (5.1) and (5.2) we have

$$D_{b-}^{\alpha} u(t) I_{b-}^{1-\alpha} u(t) \Big|_a^b = 0.$$

Thereby, the theorem is proved. □

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References

[A82] T.S. Aleroev, The Sturm-Liouville problem for a second-order differential equation with fractional derivatives in the lower terms. *Differentsialye Uravneniya.* 18, No 2 (1982), 341–342.

[A99] R.A. Adams, Sobolev Spaces. Academic Press, New York (1999).

[A15] A. Ansari, On finite fractional Sturmiouville transforms. *Integral Transforms and Special Functions.* 26, No 1 (2015), 51–64.

[AS10] S. Abbasbandy and A. Shirzadi, Homotopy analysis method for multiple solutions of the fractional Sturm–Liouville problems. *Numer. Algorithms.* 54, No 4 (2010), 521–532.

[BC14] T. Blaszczyk, M. Ciesielski, Numerical solution of fractional Sturm–Liouville equation in integral form. *Fractional Calculus and Applied Analysis.* 17, No 2 (2014), 307–320.

[D70] M.M. Dzhrbashyan, A boundary value problem for a Sturm-Liouville type differential operator of fractional order. *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* 5, No 2 (1970), 71–96.

[DN59] M.M. Dzhrbashyan and A.B. Nersesjan, *Izv. Akad. Nauk. Armjan SSR Ser. Fiz.-Mat. Nauk.* 12, No. 5 (1959), 17–42.

[DS63] N. Dunford and J.T. Schwartz, *Linear Operators. Part 2: Spectral Theory.* New York: Interscience, (1963).

[DWF14] J.-S. Duan, Z. Wang, and S.-Z. Fu, The zeros of the solutions of the fractional oscillation equation. *Fract. Calc. Appl. Anal.* 17, No 1 (2014), 10–22.

[GKMR14] R. Gorenflo, A.A. Kilbas, F. Mainardi, and S.V. Rogosin, Mittag-Leffler functions, related topics and applications. Springer Monographs in Mathematics. Springer, Heidelberg, (2014).

[GLY15] R. Gorenflo, Yu. Luchko, and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces. *Fract. Calc. Appl. Anal.* 18, No. 3 (2015), 799–820.

[HMA14] M.A. Hajji, Q.M. Al-Mdallal, and F.M. Allan, An efficient algorithm for solving higher-order fractional Sturm–Liouville eigenvalue problems. *J. Comput. Phys.* 272 (2014), 550–558.

[JR12] B. Jin and W. Rundell, An inverse Sturm-Liouville problem with a fractional derivative. *J. Comput. Phys.* 231, No. 14 (2012), 4954–4966.

[K15] M. Klimek, 2D space-time fractional diffusion on bounded domainpplication of the fractional Sturm-Liouville theory. *Methods and Models in Automation and Robotics (MMAR).* (2015) 309–314.

[KA13] M. Klimek and O.P. Agrawal, Fractional Sturm–Liouville problem. *Computers and Mathematics with Applications.* 66, No 5 (2013), 795–812.

[KDE15] H. Khosravian-Arab, M. Dehghan, and M.R. Eslahchi, Fractional Sturm–Liouville boundary value problems in unbounded domains: Theory and applications. *Journal of Computational Physics.* 299 (2015), 526–560.

[KL13] M. Klimek and M. Lupa, Reflection symmetric formulation of generalized fractional variational calculus. *Fractional Calculus and Applied Analysis.* 16, No 1 (2013), 243–261

[KM16] M. Klimek, A.B. Malinowska, and T. Odzijewicz, Applications of the fractional Sturm–Liouville problem to the space-time fractional diffusion in a finite domain. *Fract. Calc. Appl. Anal.* 19, No. 2 (2016), 516–550.

[KOM14] M. Klimek, T. Odzijewicz, and A.B. Malinowska, Variational methods for the fractional Sturm–Liouville problem. *Journal of Mathematical Analysis and Applications.* 416, No 1 (2014), 402–426.

[KST06] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland, Mathematics studies, (2006).

[KT02] A.A. Kilbas and J.J. Trujillo, Differential equations of fractional order: methods, results and problems. *II. Appl. Anal.* 81, No 2 (2002), 435–493.

[LQ15] J. Li, J. Qi, Eigenvalue problems for fractional differential equations with right and left fractional derivatives. *Applied Mathematics and Computation.* 256, (2015), 1–10.

[M10] Q.M. Al-Mdallal, On the numerical solution of fractional Sturmiouville problems. *International Journal of Computer Mathematics.* 87, No 12 (2010), 2837–2845.

[N67] M.A. Naimark, *Linear Differential Operators*, Frederic Ungar, New York (1967).

[N77] A.M. Nakhushev, Sturm-Liouville problem for an ordinary differential equation of second order with fractional derivatives in the lower-order terms. *Doklady Akademiia Nauk SSSR.* 234, (1977), 308–311.

[P02] A.Yu. Popov, On zeros of a certain family of Mittag-Leffler functions. *J. Math. Sci. (N. Y.).* 144, No 4 (2007), 4228–4231; translated from Sovrem. Mat. Prilozh. 35 (2005), 28–30 (Russian).

[P14] L. Plociniczak, Eigenvalue asymptotics for a fractional boundary-value problem. *Applied Mathematics and Computation.* 241 (2014), 125–128.

[PS11] A.Yu. Popov and A.M. Sedletskii, Distribution of roots of Mittag-Leffler functions. *J. Math. Sci. (N. Y.).* 190, No 2 (2013), 209–409; translation from Sovrem. Mat. Fundam. Napravl. 40 (2011), 3–171 (Russian).

[QDH15] L. Qiu, W. Deng, and J.S. Hesthaven, Nodal discontinuous Galerkin methods for fractional diffusion equations on 2D domain with triangular meshes. *Journal of Computational Physics* 298 (2015), 678–694.

[RTV13] M. Rivero, J.J. Trujillo, and M.P. Velasco, A fractional approach to the Sturm–Liouville problem. *Cent. Eur. J. Phys.* 11, No 10 (2013), 1246–1254.

[SKM87] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional order integrals and derivatives and some applications.* Minsk: Nauka i tekhnika. (1987) (Russian).

[ZK13] M. Zayernouri and G. Karniadakis, Fractional Sturm–Liouville eigenproblems: theory and numerical approximation. *J. Comput. Phys.* 252 (2013), 495–517.

[ZK14] M. Zayernouri and G.E. Karniadakis, Discontinuous spectral element methods for time-and space-fractional advection equations. *SIAM Journal on Scientific Computing.* 36, No 4 (2014), B684–B707.

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