Knörrer Periodicity and Bott Periodicity

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Abstract. The goal of this article is to explain a precise sense in which Knörrer periodicity in commutative algebra and Bott periodicity in topological K-theory are compatible phenomena. Along the way, we prove an 8-periodic version of Knörrer periodicity for real isolated hypersurface singularities, and we construct a homomorphism from the Grothendieck group of the homotopy category of matrix factorizations of a complex (real) polynomial f into the topological K-theory of its Milnor fiber (positive or negative Milnor fiber).

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1 Introduction

Let k be a field. In this article, we study hypersurface rings of the form $k[x_1, \dots, x_n]/(f)$ from both an algebraic and topological point of view. An important algebraic invariant of such a ring is its homotopy category of matrix factorizations, which we denote by $[\mathrm{MF}(k[x_1, \dots, x_n], f)]$ (we recall the definition of this category in Section 2.1.1). Matrix factorizations were introduced by Eisenbud in [Eis80] as a tool for studying the homological behavior of modules over a hypersurface ring. More recently, matrix factorizations have begun appearing in a wide variety of contexts, for instance homological mirror symmetry (e.g. [KKP08], by Katzarkov-Kontsevich-Pantev) and knot theory (e.g. [KR08], by Khovanov-Rozansky). In the present work, we continue the study of an interplay between matrix factorizations and topological K-theory that was begun in the inspiring paper [BvS12] of Buchweitz-van Straten.
A fundamental result in the theory of matrix factorizations is Knörrer's periodicity theorem:

Theorem 1.1 ([Knö87] Theorem 3.1). *Suppose k is algebraically closed and $\mathrm{char}(k) \neq 2$. If $f \in (x_1, \dots, x_n) \subseteq k[[x_1, \dots, x_n]]$, there is an equivalence of categories*

$$[\mathrm{MF}(k[[x_1, \dots, x_n]], f)] \xrightarrow{\cong} [\mathrm{MF}(k[[x_1, \dots, x_n, u, v]], f + u^2 + v^2)].$$

This result plays an important role in the classification of local hypersurface rings of finite maximal Cohen-Macaulay type; we refer the reader to Chapter 9 of Leuschke-Wiegand's text [LW12] for details. Knörrer's periodicity theorem also demonstrates that one cannot recover f from its homotopy category of matrix factorizations.
The main goal of this article is explain a precise sense in which Knörrer periodicity is a manifestation of Bott periodicity in topological K-theory. In Section 2, we motivate this project with a proof of an 8-periodic version of Knörrer periodicity for isolated hypersurface singularities over the real numbers:

Theorem 1.2. *Let $f \in (x_1, \dots, x_n) \subseteq \mathbb{R}[x_1, \dots, x_n]$, and suppose $\mathbb{R}[x_1, \dots, x_n]/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{R}} \frac{\mathbb{R}[[x_1, \dots, x_n]]}{(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})} < \infty$). Then there exists an equivalence of triangulated categories*

$$[\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n]], f)] \xrightarrow{\cong} [\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n, u_1, \dots, u_8]], f - u_1^2 - \dots - u_8^2)].$$

We point out that the "period" here is exactly 8; that is, for $1 \leqslant l < 8$, it can happen that

$$[\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n]], f)] \ncong [\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n, u_1, \dots, u_l]], f - u_1^2 - \dots - u_l^2)].$$

Our proof relies heavily on machinery developed by Dyckerhoff and Toën in [Dyc11] and [Toë07]. This result draws a distinction between the maximal Cohen-Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ and those whose ground field is algebraically closed and has characteristic not equal to 2, since the latter exhibit 2-periodic Knörrer periodicity. The maximal Cohen-Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ does not seem to be well-studied, and we hope this work motivates further investigation in this direction.

The presence of 2- and 8-periodic versions of Knörrer periodicity over $\mathbb{C}$ and $\mathbb{R}$, respectively, suggests the possibility of a compatibility between Knörrer periodicity and Bott periodicity. Such a compatibility statement is formulated and proved in Section 3. We state here the version of this result over $\mathbb{C}$; a version over $\mathbb{R}$ is also proven in Section 3 (Theorems 3.32 and 3.33).

Theorem 1.3. *Suppose $f \in (x_1, \dots, x_n) \subseteq Q := \mathbb{C}[x_1, \dots, x_n]$ and either $Q/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1,\dots,x_n]]}{(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})} < \infty$) or f is quasi-homogeneous. Then there exists a commutative diagram*

$$\begin{tikzcd}
K_0[\mathrm{MF}(Q,f)] \arrow[r, "\phi_f^{\mathbb{C}}"] \arrow[dd, "K"] & KU^0(B_\epsilon, F_f) \arrow[d, "\beta"] \\
& KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2}) \arrow[d, "\mathrm{ST}_{KU}"] \\
K_0[\mathrm{MF}(Q[u,v], f + u^2 + v^2)] \arrow[r, "\phi_{f+u^2+v^2}^{\mathbb{C}}"] & KU^0(B_{\epsilon''}, F_{f+u^2+v^2})
\end{tikzcd}$$

where F_f, $F_{u^2+v^2}$, and $F_{f+u^2+v^2}$ denote the Milnor fibers of f, u^2+v^2, and $f+u^2+v^2$; $\epsilon, \epsilon', \epsilon'' > 0$; B_ϵ, $B_{\epsilon'}$, and $B_{\epsilon''}$ are closed balls of radius ϵ, ϵ', and ϵ'' in $\mathbb{C}^n$, $\mathbb{C}^2$, and $\mathbb{C}^{n+2}$, respectively; K is induced by the Knörrer functor; β is the Bott periodicity isomorphism; and ST_{KU} is given by the product in relative K-theory followed by the inverse of the map induced by pullback along the Sebastiani-Thom homotopy equivalence.

The Sebastiani-Thom homotopy equivalence to which we refer in Theorem 1.3 is discussed in Section 3.1.2.

The key construction in this section yields the horizontal maps above; specifically, given a polynomial f over the complex (real) numbers, we build a map $\Phi_f^{\mathbb{C}}$ ($\Phi_f^{\mathbb{R}}$) that assigns to a matrix factorization of a complex (real) polynomial f a class in the topological K-theory of the Milnor fiber (positive or negative Milnor fiber) of f; this map first appeared in [BvS12] in the setting of complex isolated hypersurface singularities. We prove that this construction induces a map $\phi_f^{\mathbb{C}}$ ($\phi_f^{\mathbb{R}}$) on the Grothendieck group of the homotopy category of matrix factorizations of f, and we show that it recovers the Atiyah-Bott-Shapiro construction when f is a non-degenerate quadratic over $\mathbb{R}$ or $\mathbb{C}$. The Atiyah-Bott-Shapiro construction, introduced in Part III of [ABS64], provides the classical link between $\mathbb{Z}/2\mathbb{Z}$-graded modules over Clifford algebras and vector bundles over spheres; the maps $\phi_f^{\mathbb{C}}$ and $\phi_f^{\mathbb{R}}$ we discuss in Section 3 can be thought of as providing a more general link between algebra and topology.

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2 Knörrer periodicity over $\mathbb{R}$

In this section, we recall some foundational material concerning matrix factorizations in commutative algebra, and we exhibit an 8-periodic version of Knörrer periodicity for matrix factorization categories associated to isolated hypersurface singularities over the real numbers.

2.1 Matrix factorization categories

We provide some background on matrix factorization categories. Fix a commutative algebra Q over a field k and an element f of Q. Henceforth, when we use the term "dg category", we mean "k-linear differential $\mathbb{Z}/2\mathbb{Z}$-graded category". We cite results on differential $\mathbb{Z}$-graded categories from [Toë11] several times throughout this section; we refer the reader to Section 5.1 of [Dyc11] for a discussion as to how one may reformulate the results in [Toë11] so that they apply to the $\mathbb{Z}/2\mathbb{Z}$-graded setting.

2.1.1 Definitions and some properties

Definition 2.1. The dg category $\mathrm{MF}(Q, f)$ of *matrix factorizations* of f over Q is given by the following:
Objects in $\mathrm{MF}(Q, f)$ are pairs (P, d), where P is a finitely generated projective $\mathbb{Z}/2\mathbb{Z}$-graded Q-module, and d is an odd-degree endomorphism of P such that $d^2 = f \cdot \mathrm{id}_P$. Henceforth, we will often denote an object (P, d) in $\mathrm{MF}(Q, f)$ by just P.

The morphism complex of a pair of matrix factorizations P, P', which we will denote by $\mathrm{Hom}_{\mathrm{MF}}(P, P')$, is the $\mathbb{Z}/2\mathbb{Z}$-graded module of Q-linear maps from P to P' equipped with the differential ∂ given by

$$\partial(\alpha) = d' \circ \alpha - (-1)^{|\alpha|}\alpha \circ d$$

for homogeneous maps $\alpha : P \to P'$.

We will often express an object P in $\mathrm{MF}(Q, f)$ with the notation

$$P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0,$$

where P_1, P_0 are the odd and even degree summands of P, and d_1, d_0 are the restrictions of d to P_1 and P_0, respectively.
A degree 0 morphism α in $\mathrm{MF}(Q, f)$ can be represented by a diagram of the following form:

$$\begin{array}{ccccc} P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & P_1 \\ \alpha_1 \downarrow & & \alpha_0 \downarrow & & \downarrow \alpha_1 \\ P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{d_0'} & P_1' \end{array}$$

It is straightforward to check that α is a cycle if and only if this diagram commutes. In fact, if $f \in Q$ is a non-zero-divisor, it is easy to see that the left square commutes if and only if the right square commutes.

Remark 2.2. If P_1 and P_0 are free and f is non-zero-divisor, P_1 and P_0 must have the same rank.

Define $Z^0\mathrm{MF}(Q, f)$ to be the category with the same objects as $\mathrm{MF}(Q, f)$ and with morphisms given by the degree 0 cycles in $\mathrm{MF}(Q, f)$. When Q is regular with finite Krull dimension and f is a regular element of Q (i.e. f is a non-unit, non-zero-divisor), $Z^0\mathrm{MF}(Q, f)$ is an exact category with the evident family of exact sequences ([Orl03] Section 3.1).
The *homotopy category*, $[\mathrm{MF}(Q, f)]$, of the dg category $\mathrm{MF}(Q, f)$ is defined to be the quotient of $Z^0\mathrm{MF}(Q, f)$ by morphisms that are boundaries in $\mathrm{MF}(Q, f)$. That is, objects in $[\mathrm{MF}(Q, f)]$ are the same as those of $\mathrm{MF}(Q, f)$, and the morphisms in $[\mathrm{MF}(Q, f)]$ between objects P, P' are classes in $H^0\mathrm{Hom}_{\mathrm{MF}}(P, P')$.

DEFINITION 2.3. We call a matrix factorization *trivial* if it is a direct sum of matrix factorizations that are isomorphic in $Z^0\mathrm{MF}(Q, f)$ to either

$$E \underset{\mathrm{id}_E}{\overset{f\cdot\mathrm{id}_E}{\rightleftarrows}} E$$

or

$$E \underset{f\cdot\mathrm{id}_E}{\overset{\mathrm{id}_E}{\rightleftarrows}} E$$

for some finitely generated projective Q-module E.

The following result gives an alternative characterization for when a morphism in $Z^0\mathrm{MF}(Q,f)$ is a boundary in $\mathrm{MF}(Q,f)$; the straightforward proof is omitted.

PROPOSITION 2.4. *A morphism $\alpha : P \to P'$ in $Z^0\mathrm{MF}(Q,f)$ is a boundary in $\mathrm{MF}(Q,f)$ if and only if it factors through a trivial matrix factorization in $Z^0\mathrm{MF}(Q,f)$.*

We conclude this section with a technical result that will be used in the proof of Proposition 3.19:

PROPOSITION 2.5. *Let $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$ be a matrix factorization of f over Q. Assume f is a non-zero-divisor. Then the following are equivalent:*

(1) $\mathrm{coker}(d_1)$ *is isomorphic to L/fL for some projective Q-module L.*

(2) There exists a trivial matrix factorization E and a matrix factorization E' that is isomorphic in $Z^0\mathrm{MF}(Q,f)$ to one of the form

$$F \underset{f}{\overset{\mathrm{id}_F}{\rightleftarrows}} F$$

such that $P \oplus E'$ is isomorphic to E in $Z^0\mathrm{MF}(Q,f)$.

We will use the following general fact about idempotent complete categories. We suspect that this result is well-known to experts; we omit the purely formal proof.

LEMMA 2.6. *Let $\mathcal{C}$ be an idempotent complete additive category, and let $\mathcal{E}$ be a collection of objects in $\mathcal{C}$ that is*

- *closed under isomorphisms,*
- *closed under finite coproducts, and*
- *closed under taking summands; that is, whenever X is an object in $\mathcal{C}$ such that id_X factors through an object in $\mathcal{E}$, X is an object in $\mathcal{E}$.*

Denote by $\mathcal{L}$ the quotient of $\mathcal{C}$ by those morphisms that factor through an object in $\mathcal{E}$. If X and Y are objects in $\mathcal{C}$, their images in $\mathcal{L}$ are isomorphic if and only if there exist objects E_X, E_Y in $\mathcal{E}$ such that

$$X \oplus E_X \cong Y \oplus E_Y.$$

We now prove Proposition 2.5:

Proof. (2) $\Rightarrow$ (1): Since the cokernel of d_1 is isomorphic to the cokernel of

$$d_1 \oplus \mathrm{id}_F : P_1 \oplus F \to P_0 \oplus F,$$

we may assume P is trivial. In this case, the result is obvious.

(1) ⇒ (2): Choose a projective Q-module L such that there exists an isomorphism

$$\operatorname{coker}(d_1) \xrightarrow{\cong} L/fL.$$

We have Q-projective resolutions

$$0 \to P_1 \xrightarrow{d_1} P_0 \to \operatorname{coker}(d_1) \to 0$$

$$0 \to L \xrightarrow{f} L \to L/fL \to 0$$

Thus, there exist maps

$$\beta_i : P_i \to L,\ \gamma_i : L \to P_i$$

for $i = 0, 1$ making the following diagrams commute:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\beta_1"'] & P_0 \arrow[r] \arrow[d, "\beta_0"'] & \operatorname{coker}(d_1) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & L \arrow[r] & L/fL \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\gamma_1"'] & L \arrow[r] \arrow[d, "\gamma_0"'] & L/fL \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & \operatorname{coker}(d_1) \arrow[r] & 0
\end{tikzcd}$$

Hence, we have maps

$$h_P : P_0 \to P_1,\ h_L : L \to L$$

such that

$$\gamma_1 \circ \beta_1 - \mathrm{id}_{P_1} = h_P \circ d_1,\ \gamma_0 \circ \beta_0 - \mathrm{id}_{P_0} = d_1 \circ h_P.$$

$$\beta_1 \circ \gamma_1 - \mathrm{id}_L = fh_L,\ \beta_0 \circ \gamma_0 - \mathrm{id}_L = fh_L.$$

We have commutative diagrams

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, "h_P \circ d_1"] & P_0 \arrow[d, "h_P"] \\
P_1 \arrow[r, "\mathrm{id}_{P_1}"] \arrow[d, "\mathrm{id}_{P_1}"] & P_1 \arrow[d, "d_1"] \\
P_1 \arrow[r, "d_1"] & P_0
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "f \cdot \mathrm{id}_L"] \arrow[d, "f \cdot h_L"] & L \arrow[d, "h_L"] \\
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "\mathrm{id}_L"] & L \arrow[d, "f \cdot \mathrm{id}_L"] \\
L \arrow[r, "f \cdot \mathrm{id}_L"] & L
\end{tikzcd}$$

Denote by $\mathcal{E}$ the collection of matrix factorizations of f over Q isomorphic in $Z^0\mathrm{MF}(Q,f)$ to a matrix factorization of the form

$$E \underset{f}{\overset{\mathrm{id}_E}{\rightleftarrows}} E.$$

Notice that $Z^0\mathrm{MF}(Q,f)$ is an idempotent complete additive category, and $\mathcal{E}$ is closed under direct sums and direct summands in $Z^0\mathrm{MF}(Q,f)$. Letting $\mathcal{L}$ denote the quotient of $Z^0\mathrm{MF}(Q,f)$ by those morphisms that factor through an object in $\mathcal{E}$, we have that

$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \cong (L \underset{\mathrm{id}_L}{\overset{f}{\rightleftarrows}} L)$$

in $\mathcal{L}$. The result now follows from Lemma 2.6.

□

2.1.2 Triangulated structure

Suppose Q is regular with finite Krull dimension and f is a regular element of Q. A feature of the homotopy category $[\mathrm{MF}(Q,f)]$ is that it may be equipped with a triangulated structure in the following way ([Orl03] Section 3.1):
The shift functor maps the object

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$$

to the object

$$P[1] = (P_0 \underset{-d_1}{\overset{-d_0}{\rightleftarrows}} P_1).$$

Given a morphism $\alpha : (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \to (P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0')$ in $Z^0\mathrm{MF}(Q,f)$, the *mapping cone* of α is defined as follows:

$$\mathrm{cone}(\alpha) = (P_0' \oplus P_1 \underset{\begin{pmatrix} d_1' & \alpha_0 \\ 0 & -d_0 \end{pmatrix}}{\overset{\begin{pmatrix} d_0' & \alpha_1 \\ 0 & -d_1 \end{pmatrix}}{\rightleftarrows}} P_1' \oplus P_0)$$

There are canonical morphisms $P' \to \mathrm{cone}(\alpha)$ and $\mathrm{cone}(\alpha) \to P[1]$ in $Z^0\mathrm{MF}(Q,f)$. Taking the distinguished triangles in $[\mathrm{MF}(Q,f)]$ to be the triangles isomorphic in $[\mathrm{MF}(Q,f)]$ to those of the form

$$P \xrightarrow{\alpha} P' \to \mathrm{cone}(\alpha) \to P[1],$$

$[\mathrm{MF}(Q,f)]$ may be equipped with the structure of a triangulated category.

The Grothendieck group, $K_0[\mathrm{MF}(Q,f)]$, of the triangulated category $[\mathrm{MF}(Q,f)]$ is defined to be the free abelian group generated by isomorphism classes of $[\mathrm{MF}(Q,f)]$ modulo elements of the form $[P_1]-[P_2]+[P_3]$, where P_1, P_2, and P_3 fit into a distinguished triangle in the following way:

$$P_1 \to P_2 \to P_3 \to P_1[1].$$

Remark 2.7. The category $\mathrm{MF}(Q,f)$ is not always triangulated in the dg sense (see Section 4.4 of [Toë11] for the definition of a triangulated dg category). When $\mathrm{MF}(Q,f)$ is triangulated in the dg sense, the induced triangulated structure on $[\mathrm{MF}(Q,f)]$ agrees with the triangulated structure just described.

Remark 2.8. When Q is a regular local ring and f is a regular element of Q, one has an equivalence of triangulated categories

$$[\mathrm{MF}(Q,f)] \xrightarrow{\cong} \underline{\mathrm{MCM}}(Q/(f)),$$

where $\underline{\mathrm{MCM}}(Q/(f))$ denotes the *stable category* of maximal Cohen-Macaulay (MCM) modules over the ring $Q/(f)$. The stable category of MCM modules is obtained by taking the quotient of the category of MCM modules over $Q/(f)$ by those morphisms that factor through a projective $Q/(f)$-module. The above equivalence is given, on objects, by

$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \mapsto \operatorname{coker}(d_1).$$

Matrix factorizations were first defined by Eisenbud in [Eis80]; this interplay between matrix factorizations and MCM modules over hypersurface rings provided the original motivation for the study of matrix factorization categories.

2.1.3 Stabilization

Assume now that Q is a regular local ring of Krull dimension n, and suppose f is a regular element of Q. Denote by $\mathrm{D^b}(Q/(f))$ the bounded derived category of $Q/(f)$, and set $\mathrm{Perf}(Q/(f))$ to be the full subcategory of $\mathrm{D^b}(Q/(f))$ given by perfect complexes. $\mathrm{Perf}(Q/(f))$ is a thick subcategory of $\mathrm{D^b}(Q/(f))$; define $\underline{\mathrm{D^b}}(Q/(f))$ to be the Verdier quotient of $\mathrm{D^b}(Q/(f))$ by $\mathrm{Perf}(Q/(f))$. In [Buc86], Buchweitz calls this quotient the *stabilized derived category* of $Q/(f)$.
By [Buc86], the functor

$$\underline{\mathrm{MCM}}(Q/(f)) \to \underline{\mathrm{D^b}}(Q/(f))$$

that sends an MCM module M to the complex with M concentrated in degree 0 is a triangulated equivalence. Hence, composing with the equivalence in Remark 2.8, one has an equivalence

$$[\mathrm{MF}(Q,f)] \to \underline{\mathrm{D^b}}(Q/(f))$$

Following [Dyc11], given an object C in $\underline{\mathrm{D}^{\mathrm{b}}}(Q/(f))$, we denote by C^{stab} the isomorphism class in $[\mathrm{MF}(Q,f)]$ corresponding to C under the above equivalence ("stab" stands for "stabilization"). In particular, thinking of the residue field k of $Q/(f)$ as a complex concentrated in degree 0, we may associate to k an isomorphism class k^{stab} in $[\mathrm{MF}(Q,f)]$. We now construct an object E_f in $\mathrm{MF}(Q,f)$ that represents k^{stab}; this construction appears in [Dyc11]. Choose a regular system of parameters $x_1,\dots,x_n$ for Q, and consider the Koszul complex

$$(\bigoplus_{i=0}^{n} \bigwedge\nolimits^i Q^n, s_0)$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded complex of free Q-modules with even (odd) degree piece given by the direct sum of the even (odd) exterior powers of Q^n. Here, s_0 denotes the $\mathbb{Z}/2\mathbb{Z}$-folding of the Koszul differential associated to $x_1,\dots,x_n$. Choose an expression of $f \in Q$ of the form

$$f = g_1x_1 + \cdots + g_nx_n.$$

Fix a basis $e_1,\dots,e_n$ of Q^n, and set s_1 to be the odd-degree endomorphism of $\bigoplus_{i=0}^n \bigwedge^i Q^n$ given by exterior multiplication on the left by $g_1e_1 + \cdots + g_ne_n$. Define

$$E_f := (\bigoplus_{i=0}^{n} \bigwedge\nolimits^i Q^n, s_0 + s_1).$$

It is easy to check that E_f is a matrix factorization of f. By Corollary 2.7 in [Dyc11], E_f represents k^{stab} in $[\mathrm{MF}(Q,f)]$. In particular, E_f does not depend on the choice of regular system of parameters $x_1,\dots,x_n$ or coefficients $g_1,\dots,g_n$ up to homotopy equivalence. Henceforth, we shall denote the dg algebra $\mathrm{End}_{\mathrm{MF}}(E_f)$ by $A_{(Q,f)}$.

2.2 The tensor product of matrix factorizations

Let k be a field. We begin this section with a technical definition:

DEFINITION 2.9. Suppose Q is a commutative algebra over k and $f \in Q$. If the pair (Q,f) satisfies

- Q is essentially of finite type over k
- Q is equidimensional of dimension n
- The module $\Omega^1_{Q/k}$ of Kähler differentials is locally free of rank n
- The zero locus of $df \in \Omega^1_{Q/k}$ is a 0-dimensional scheme supported on a unique closed point $\mathfrak{m}$ of $\mathrm{Spec}(Q)$ with residue field k and $f \in \mathfrak{m}$

we shall call the pair $Q/(f)$ an *isolated hypersurface singularity*, or IHS.

Remark 2.10. Our IHS condition above is precisely condition (B) in Section 3.2 of [Dyc11]. As noted in loc. cit., if $Q/(f)$ and $Q'/(f')$ are IHS, $Q \otimes_k Q'/(f \otimes 1 + 1 \otimes f')$ is as well.

Suppose Q and Q' are commutative algebras over k, $f \in Q$, and $f' \in Q'$. Given objects P and P' in $\mathrm{MF}(Q, f)$, $\mathrm{MF}(Q', f')$, one can form their *tensor product* over k:

$$((P_1 \otimes_k P'_0) \oplus (P_0 \otimes_k P'_1) \underset{\begin{pmatrix} d_0 \otimes \mathrm{id}_{P'_0} & -\mathrm{id}_{P_1} \otimes d'_1 \\ \mathrm{id}_{P_0} \otimes d'_0 & d_1 \otimes \mathrm{id}_{P'_1} \end{pmatrix}}{\overset{\begin{pmatrix} d_1 \otimes \mathrm{id}_{P'_0} & \mathrm{id}_{P_0} \otimes d'_1 \\ -\mathrm{id}_{P_1} \otimes d'_0 & d_0 \otimes \mathrm{id}_{P'_1} \end{pmatrix}}{\rightleftarrows}} (P_0 \otimes_k P'_0) \oplus (P_1 \otimes_k P'_1)).$$

We will denote the tensor product by $P \otimes_{\mathrm{MF}} P'$. This construction first appeared in [Yos98]; it can be thought of as a $\mathbb{Z}/2\mathbb{Z}$-graded analogue of the tensor product of complexes. It is straightforward to check that $P \otimes_{\mathrm{MF}} P'$ is an object in $\mathrm{MF}(Q \otimes_k Q', f \otimes 1 + 1 \otimes f')$. In fact, setting $f \oplus f' := f \otimes 1 + 1 \otimes f' \in Q \otimes_k Q'$, and noting that there is a canonical map of complexes

$$\mathrm{Hom}_{\mathrm{MF}}(P, L) \otimes_k \mathrm{Hom}_{\mathrm{MF}}(P', L') \to \mathrm{Hom}_{\mathrm{MF}}(P \otimes_{\mathrm{MF}} P', L \otimes_{\mathrm{MF}} L'),$$

we have the following:

Proposition 2.11. *There is a dg functor*

$$\mathrm{ST}_{\mathrm{MF}} : \mathrm{MF}(Q, f) \otimes_k \mathrm{MF}(Q', f') \to \mathrm{MF}(Q \otimes_k Q', f \oplus f')$$

that sends an object (P, P') to $P \otimes_{\mathrm{MF}} P'$.

Remark 2.12. It is straightforward to verify that $\mathrm{ST}_{\mathrm{MF}}$ induces a pairing

$$K_0[\mathrm{MF}(Q, f)] \otimes K_0[\mathrm{MF}(Q', f')] \to K_0[\mathrm{MF}(Q \otimes_k Q', f \oplus f')].$$

Remark 2.13. The "ST" in the name $\mathrm{ST}_{\mathrm{MF}}$ stands for "Sebastiani-Thom", since this tensor product operation is related to the Sebastiani-Thom homotopy equivalence discussed in Section 3.1.2. A precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence is illustrated by the proof of Proposition 3.29 below; see Remark 3.31 for further details.

Now, suppose $Q/(f)$ and $Q'/(f')$ are IHS. Set $Q'' := Q \otimes_k Q'$. We will denote by $\widehat{Q}$ the $\mathfrak{m}$-adic completion of $Q_{\mathfrak{m}}$, where $\mathfrak{m}$ is as in the definition of IHS. Define $\widehat{Q'}$ and $\widehat{Q''}$ similarly, and let

$$\phi : \widehat{Q} \otimes_k \widehat{Q'} \to \widehat{Q''}$$

denote the canonical ring homomorphism. ϕ induces a dg functor

$$\mathrm{MF}(\phi) : \mathrm{MF}(\widehat{Q} \otimes_k \widehat{Q'}, f \oplus f') \to \mathrm{MF}(\widehat{Q''}, f \oplus f').$$

Set $\widehat{\mathrm{ST}}_{\mathrm{MF}}$ to be the composition of $\mathrm{MF}(\phi)$ with the tensor product functor

$$\mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f') \to \mathrm{MF}(\widehat{Q} \otimes_k \widehat{Q'}, f \oplus f').$$

Proposition 2.14. *If $Q/(f)$ and $Q'/(f')$ are* IHS*,*

$$\widehat{\mathrm{ST}}_{\mathrm{MF}} : \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f') \to \mathrm{MF}(\widehat{Q''}, f \oplus f')$$

is a Morita equivalence of dg categories.

Remark 2.15. We emphasize that Proposition 2.14 is really a straightforward application of several results in [Dyc11]; we include a proof for completeness. We refer the reader to Section 4.4 of [Toë11] for the definition of a Morita equivalence of dg categories.

Proof. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the maximal ideals of Q and Q' arising in Definition 2.9. Suppose $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$ have Krull dimensions n and m, respectively. $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$ are regular local rings; choose regular systems of parameters $x_1, \dots, x_n$ and $y_1, \dots, y_m$ in $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$, and choose expressions

$$f = g_1 x_1 + \cdots + g_n x_n$$

$$f' = h_1 y_1 + \cdots + h_m y_m$$

of f and f'. Use these expressions to construct the dga's $A_{(Q_{\mathfrak{m}}, f)}$ and $A_{(Q'_{\mathfrak{m}'}, f')}$, as in Section 2.1.3.
Note that $x_1, \dots, x_n$ and $y_1, \dots, y_m$ form regular systems of parameters in $\widehat{Q}$ and $\widehat{Q'}$ as well, so we may use these expressions to construct $A_{(\widehat{Q}, f)}$ and $A_{(\widehat{Q'}, f')}$. Also, $x_1 \otimes 1, \dots, x_n \otimes 1, 1 \otimes y_1, \dots, 1 \otimes y_m$ is a regular system of parameters in $Q''_{\mathfrak{m}''}$, where $\mathfrak{m}'' := \mathfrak{m} \otimes 1 + 1 \otimes \mathfrak{m}'$, so we may use the expression

$$f \oplus f' = (g_1 x_1 \otimes 1) + \cdots + (g_n x_n \otimes 1) + (1 \otimes h_1 y_1) + \cdots + (1 \otimes h_m y_m)$$

to construct $A_{(Q''_{\mathfrak{m}''}, f \oplus f')}$ and $A_{(\widehat{Q''}, f \oplus f')}$.
By Section 6.1 of [Dyc11], we have a quasi-isomorphism

$$F : A_{(Q_{\mathfrak{m}}, f)} \otimes_k A_{(Q'_{\mathfrak{m}'}, f')} \xrightarrow{\cong} A_{(Q''_{\mathfrak{m}''}, f \oplus f')}.$$

We also have a canonical map

$$G : A_{(\widehat{Q}, f)} \otimes_k A_{(\widehat{Q'}, f')} \to A_{(\widehat{Q''}, f \oplus f')}.$$

By the proof of Theorem 5.7 in [Dyc11], the inclusions

$$A_{(Q_{\mathfrak{m}}, f)} \hookrightarrow A_{(\widehat{Q}, f)}$$

$$A_{(Q'_{\mathfrak{m}'}, f')} \hookrightarrow A_{(\widehat{Q'}, f')}$$

$$A_{(Q''_{\mathfrak{m}''}, f \oplus f')} \hookrightarrow A_{(\widehat{Q''}, f \oplus f')}$$

are all quasi-isomorphisms. Since a tensor product of Morita equivalences is again a Morita equivalence ([Toë11] Section 4.4), it follows that the map

$$A_{(Q_{\mathfrak{m}}, f)} \otimes_k A_{(Q'_{\mathfrak{m}'}, f')} \to A_{(\widehat{Q}, f)} \otimes_k A_{(\widehat{Q'}, f')}$$

is a Morita equivalence.

We have the following commutative square:

$$\begin{tikzcd}
A_{(Q_{\mathfrak{m}},f)} \otimes_k A_{(Q'_{\mathfrak{m}'},f')} \arrow[r] \arrow[d] & A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')} \arrow[d, "G"] \\
A_{(Q''_{\mathfrak{m}''},f\oplus f')} \arrow[r] & A_{(\widehat{Q''},f\oplus f')}
\end{tikzcd}$$

It follows that G is a Morita equivalence.

One may think of a dga as a dg category with a single object. Adopting this point of view, we have inclusion functors

$$i : A_{(\widehat{Q},f)} \hookrightarrow \mathrm{MF}(\widehat{Q}, f)$$

$$j : A_{(\widehat{Q'},f')} \hookrightarrow \mathrm{MF}(\widehat{Q'}, f')$$

$$l : A_{(\widehat{Q''},f\oplus f')} \hookrightarrow \mathrm{MF}(\widehat{Q''}, f \oplus f')$$

Combining Theorem 5.2 and Lemma 5.6 in [Dyc11], we conclude that i, j, and l are Morita equivalences. In particular, we have that

$$i \otimes j : A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')} \to \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f')$$

is a Morita equivalence.

Finally, consider the following commutative diagram:

$$\begin{tikzcd}
A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')} \arrow[r, "i\otimes j"] \arrow[d, "G"] & \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f') \arrow[d, "\widehat{\mathrm{ST}}_{\mathrm{MF}}"] \\
A_{(\widehat{Q''},f\oplus f')} \arrow[r, "l"] & \mathrm{MF}(\widehat{Q''}, f \oplus f')
\end{tikzcd}$$

Since the left-most vertical map and both horizontal maps are Morita equivalences, $\widehat{\mathrm{ST}}_{\mathrm{MF}}$ is as well.

□

Remark 2.16. Using Theorem 4.11 of [Dyc11] along with a similar argument to the one above, one may show that, under the assumptions of Proposition 2.14, the functor

$$\mathrm{MF}(Q, f) \otimes_k \mathrm{MF}(Q', f') \to \mathrm{MF}(Q'', f \oplus f')$$

given by tensor product of matrix factorizations is also a Morita equivalence.

2.3 Matrix factorizations of quadratics

Fix a field k such that $\mathrm{char}(k) \neq 2$ and a finite-dimensional vector space V over k. Let $q : V \to k$ be a quadratic form, and let $\mathrm{Cliff}_k(q)$ denote the Clifford algebra associated to q. $\mathrm{Cliff}_k(q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded k-algebra; let $\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q))$ denote the category of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left modules over $\mathrm{Cliff}_k(q)$. Henceforth, when we refer to a module over a Clifford algebra, we will always mean it to be a left module.

Assume q is non-degenerate, and choose a basis $\{e_1, \dots, e_n\}$ of V with respect to which q is *diagonal*; that is,

$$q = a_1x_1^2 + \cdots + a_nx_n^2 \in S^2(V^*)$$

where the x_i comprise the dual basis corresponding to the e_i, and the a_i are nonzero elements of k. Denote by Q the localization of $S(V^*)$ at the ideal $(x_1, \dots, x_n)$.

The following theorem, due to Buchweitz-Eisenbud-Herzog, yields a relationship between Clifford modules and matrix factorizations of non-degenerate quadratic forms:

Theorem 2.17 ([BEH87]). *There is an equivalence of k-linear categories*

$$\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q)) \xrightarrow{\cong} [\mathrm{MF}(\widehat{Q}, q)].$$

Denote by Θ the explicit construction of this equivalence described in the proof of Theorem 14.7 in [Yos90].

Remark 2.18. The inclusion

$$k[x_1, \dots, x_n] \hookrightarrow \widehat{Q}$$

induces an equivalence

$$[\mathrm{MF}(k[x_1, \dots, x_n], q_n)] \xrightarrow{\cong} [\mathrm{MF}(\widehat{Q}, q_n)].$$

To see this, we first recall that every matrix factorization of q_n over $\widehat{Q}$ is isomorphic in $[\mathrm{MF}(\widehat{Q}, q_n)]$ to one with (linear) polynomial entries ([Yos90] Proposition 14.3); hence, the functor is essentially surjective.

Also, one has a commutative diagram

$$\begin{tikzcd} {[\mathrm{MF}(Q, q_n)]} \arrow[r, "\cong"] \arrow[d] & \underline{\mathrm{MCM}}(Q/(q_n)) \arrow[d] \\ {[\mathrm{MF}(\widehat{Q}, q_n)]} \arrow[r, "\cong"] & \underline{\mathrm{MCM}}(\widehat{Q}/(q_n)) \end{tikzcd}$$

The morphism sets in $\underline{\mathrm{MCM}}(Q/(q_n))$ are Artinian modules, and hence complete. Thus, the functor on the right is fully faithful, and so the functor on the left is as well.

It now follows from Theorem 4.11 in [Dyc11] that the functor

$$[\mathrm{MF}(k[x_1,\dots,x_n],q_n)] \to [\mathrm{MF}(\widehat{Q},q_n)]$$

is fully faithful.

Suppose $q' : W \to k$ is another non-degenerate quadratic form; choose a basis of W with respect to which q' is diagonal, and let $y_1,\dots,y_m$ denote the corresponding basis of W^*. As above, we may think of q' as an element of $S^2(W^*)$. Set Q' to be the localization of $S(W^*)$ at the ideal $(y_1,\dots,y_m)$.

It is well-known that the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product of $\mathrm{Cliff}_k(q)$ and $\mathrm{Cliff}_k(q')$ over k is canonically isomorphic to $\mathrm{Cliff}_k(q\oplus q')$. Further, by Remark 1.3 in [Yos98], the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product of Clifford modules is compatible, via this canonical isomorphism and the equivalence in Theorem 2.17, with the tensor product $\mathrm{ST}_{\mathrm{MF}}$ in Proposition 2.11. That is, one has a commutative diagram of k-linear categories

$$\begin{tikzcd}
\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q)) \times \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q')) \arrow[r] \arrow[d, "\Theta\times\Theta"] & \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q\oplus q')) \arrow[d, "\Theta"] \\
{[\mathrm{MF}(Q,q)] \times [\mathrm{MF}(Q',q')]} \arrow[r, "\mathrm{ST}_{\mathrm{MF}}"] & {[\mathrm{MF}(Q\otimes_k Q', q\oplus q')]}
\end{tikzcd}$$

Let C be a rank 1 free $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}_k(q)$-module. If $\dim(V) = 1$ and $q = x^2$, it is easy to check that the isomorphism class of $\Theta(C)$ is k^{stab}, where k^{stab} is as defined in Section 2.1.3. Further, $E_q \otimes_{\mathrm{MF}} E_{q'} \cong E_{q\oplus q'}$, where E_q, $E_{q'}$, and $E_{q\oplus q'}$ are as in Section 2.1.3 ([Dyc11] Section 6.1). Thus, we have:

Proposition 2.19. *If $a_i = 1$ for $1 \leqslant i \leqslant n$, the isomorphism class of $\Theta(C)$ is k^{stab}.*

2.4 Periodicity

Following [Dyc11], given a commutative algebra Q over a field k and an element f of Q, we define $\mathrm{MF}^\infty(Q,f)$ to be the dg category of possibly infinitely-generated matrix factorizations; that is, objects of $\mathrm{MF}^\infty(Q,f)$ are defined in the same way as $\mathrm{MF}(Q,f)$, except the projective $\mathbb{Z}/2\mathbb{Z}$-graded Q-module P need not be finitely generated.

A version of Knörrer periodicity (Theorem 1.1) for isolated hypersurface singularities may be deduced from the following proposition:

Proposition 2.20. *Suppose Q and Q' are commutative algebras over a field k. Let $f \in Q$ and $f' \in Q'$, and suppose $Q/(f)$ and $Q'/(f')$ are* IHS. *If there exists an object X in $\mathrm{MF}(Q',f')$ such that*

(a) X is a compact generator of $[\mathrm{MF}^\infty(Q',f')]$, and

(b) the inclusion $k \hookrightarrow \mathrm{End}_{\mathrm{MF}(\widehat{Q'},f')}(X)$ is a quasi-isomorphism

then the dg functor

$$K_X : \mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q \otimes_k Q'}, f \oplus f')$$

given by

$$P \mapsto P \otimes_{\mathrm{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \mathrm{id}_X$$

on morphisms is a quasi-equivalence.

Proof. By Theorems 4.11, 5.1, and 5.7 in [Dyc11], the inclusion

$$\mathrm{End}_{\mathrm{MF}(\widehat{Q'}, f')}(X) \hookrightarrow \mathrm{MF}(\widehat{Q'}, f')$$

is a Morita equivalence. We have a chain of Morita equivalences

$$\mathrm{MF}(\widehat{Q}, f) \otimes_k k \hookrightarrow \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{End}_{\mathrm{MF}(\widehat{Q'}, f')}(X) \hookrightarrow \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f').$$

Composing with $\widehat{\mathrm{ST}}_{\mathrm{MF}}$, Proposition 2.14 yields a Morita equivalence

$$\mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q \otimes_k Q'}, f \oplus f').$$

This composition is clearly the functor K_X; thus, K_X is a Morita equivalence. Since both $\mathrm{MF}(\widehat{Q}, f)$ and $\mathrm{MF}(\widehat{Q \otimes_k Q'}, f \oplus f')$ are triangulated in the dg sense by Lemma 5.6 in [Dyc11], we may apply Theorem 3.2.1 in [Toë11] and Theorem 1.2.10 in [Hov07] to conclude that K_X is a quasi-equivalence. $\square$

To deduce a version of Knörrer periodicity for isolated hypersurface singularities, assume k to be an algebraically closed field such that $\mathrm{char}(k) \neq 2$, set $Q' = k[u, v]$ and $f' = u^2 + v^2$, and take X to be the matrix factorization

$$k[u, v] \underset{u-iv}{\overset{u+iv}{\rightleftarrows}} k[u, v].$$

This is the approach taken in Section 5.3 of [Dyc11].

We point out that k is not assumed to be algebraically closed in Proposition 2.20, and no assumptions on the characteristic of k are made, either. In particular, we may use Proposition 2.20 to prove an 8-periodic version of Knörrer periodicity over $\mathbb{R}$ (this result implies Theorem 1.2 from the introduction):

Theorem 2.21. *Suppose Q is an $\mathbb{R}$-algebra. Let $f \in Q$, and suppose $Q/(f)$ is* IHS. *Set $Q' := \mathbb{R}[u_1, \dots, u_8]$. Then there exists a matrix factorization X of $-u_1^2 - \cdots - u_8^2$ over Q' such that the dg functor*

$$\mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q \otimes_{\mathbb{R}} Q'}, f - u_1^2 - \cdots - u_8^2)$$

given by

$$P \mapsto P \otimes_{\mathrm{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \mathrm{id}_X$$

on morphisms is a quasi-equivalence.

Remark 2.22. One may replace $-u_1^2 - \cdots - u_8^2$ with $u_1^2 + \cdots + u_8^2$ and obtain a similar result; the proof is the same.

Proof. Set $q := -u_1^2 - \cdots - u_8^2 \in Q'$. We equip the matrix algebra $\mathrm{Mat}_{16}(\mathbb{R})$ of 16×16 of matrices over $\mathbb{R}$ with a $\mathbb{Z}/2\mathbb{Z}$-grading in the following way: $A = (a_{ij})$ is homogeneous of even degree if $a_{ij} = 0$ whenever $i + j$ is odd, and A is homogeneous of odd degree if $a_{ij} = 0$ whenever $i + j$ is even. By Proposition V.4.2 in [Lam05],

$$\mathrm{Cliff}_{\mathbb{R}}(q) \cong \mathrm{Mat}_{16}(\mathbb{R})$$

as $\mathbb{Z}/2\mathbb{Z}$-graded algebras. In particular, by Theorem 2.17,

$$[\mathrm{MF}(\widehat{Q'}, q)] \cong \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Mat}_{16}(\mathbb{R})),$$

where the right hand side is the category of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left $\mathrm{Mat}_{16}(\mathbb{R})$-modules. Let $M \in \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Mat}_{16}(\mathbb{R}))$ be the module consisting of elements of $\mathrm{Mat}_{16}(\mathbb{R})$ with nonzero entries only in the first column. Recall that, by Remark 2.18, the canonical map

$$[\mathrm{MF}(Q', q)] \to [\mathrm{MF}(\widehat{Q'}, q)]$$

is an equivalence; let X be an object of $[\mathrm{MF}(Q', q)]$ corresponding to M.
Let $\mathfrak{m} := (u_1, \ldots, u_8) \subseteq Q'$, and let $E_q \in \mathrm{MF}(Q'_{\mathfrak{m}}, q)$ be as in Section 2.1.3. Notice that, by Proposition 2.19, $(X \oplus X[1])^{\oplus 8} \cong E_q$ in $[\mathrm{MF}(Q'_{\mathfrak{m}}, q)]$. In particular, it follows from Theorems 4.1 and 4.11 of [Dyc11] that X is a compact generator of $[\mathrm{MF}^{\infty}(Q', q)]$.
Since $\mathrm{End}_{\mathrm{Mat}_{16}(\mathbb{R})}(M) \cong \mathbb{R}$ as $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-algebras, where $\mathbb{R}$ is concentrated in even degree, we have $H^0(\mathrm{End}_{\mathrm{MF}}(X)) \cong \mathbb{R}$. We now show $H^1(\mathrm{End}_{\mathrm{MF}}(X)) = 0$. By Section 5.5 of [Dyc11], $H^0(\mathrm{End}_{\mathrm{MF}}(E_q)) \oplus H^1(\mathrm{End}_{\mathrm{MF}}(E_q))$ is isomorphic, as a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-vector space, to $\mathrm{Cliff}_{\mathbb{R}}(q)$, and so $H^1(\mathrm{End}_{\mathrm{MF}}(E_q))$ has rank 128. Also, we have isomorphisms

$$\begin{aligned} H^1(\mathrm{End}_{\mathrm{MF}}(E_q)) &\cong H^1(\mathrm{End}_{\mathrm{MF}}((X \oplus X[1])^{\oplus 8})) \\ &\cong H^0(\mathrm{End}_{\mathrm{MF}}(X))^{128} \oplus H^1(\mathrm{End}_{\mathrm{MF}}(X))^{128}. \end{aligned}$$

Thus, $H^1(\mathrm{End}_{\mathrm{MF}}(X)) = 0$, and so the inclusion

$$\mathbb{R} \hookrightarrow \mathrm{End}_{\mathrm{MF}}(X)$$

is a quasi-isomorphism. Now apply Proposition 2.20. □

Remark 2.23. Theorem 2.21 implies the existence of a Knörrer-type periodicity for matrix factorizations over $\mathbb{R}$ of period at most 8. We point out that the period is exactly 8, since the Brauer-Wall group of $\mathbb{R}$ is the cyclic group $\mathbb{Z}/8\mathbb{Z}$ generated by the class of $\mathrm{Cliff}_{\mathbb{R}}(x^2)$.

3 Matrix factorizations and the topological K-theory of the Milnor fiber

We have demonstrated that matrix factorization categories exhibit 2- and 8-periodic versions of Knörrer periodicity over $\mathbb{C}$ and $\mathbb{R}$, respectively. This pattern resembles Bott periodicity in topological K-theory; the goal of this section is to explain this resemblance.

We give a rough sketch of our approach. The classical link between the periodicity of Clifford algebras up to $\mathbb{Z}/2\mathbb{Z}$-graded Morita equivalence and Bott periodicity in topological K-theory is the *Atiyah-Bott-Shapiro construction*, which first appeared in Part III of [ABS64] (and, in fact, a *proof* of Bott periodicity using Clifford algebras is provided by Wood in [Woo66]). Loosely speaking, the Atiyah-Bott-Shapiro construction is a way of mapping a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded module over a real or complex Clifford algebra to a class in the K-theory of a sphere.

Composing the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.17) with the Atiyah-Bott-Shapiro construction, we have a way of assigning a class in the topological K-theory of a sphere to a matrix factorization of a non-degenerate quadratic form over $\mathbb{R}$ or $\mathbb{C}$:

$$\text{mf's of real/complex quadratics} \xrightarrow{\text{ABS} \circ \text{BEH}} K\text{-theory of spheres}$$

The idea is to lift this composition; that is, we wish to associate a space X_f to a real or complex polynomial f and construct a map from matrix factorizations of f to the topological K-theory of X_f so that the diagram

$$\begin{tikzcd}
\text{mf's of real/complex quadratics} \arrow[r, "\text{ABS} \circ \text{BEH}"] \arrow[d, hook] & K\text{-theory of spheres} \arrow[d, hook] \\
\text{mf's of real/complex polynomials} \arrow[r, dotted] & K\text{-theory of spaces of the form } X_f
\end{tikzcd}$$

commutes.

It turns out that the right choice of X_f is the *Milnor fiber* (*positive or negative Milnor fiber*) associated to the complex (real) polynomial.

We begin this section with discussions of known results concerning the Milnor fiber and relative topological K-theory. Then, using the work of Atiyah-Bott-Shapiro in [ABS64] as a guide, we will complete the above diagram, and we will use the bottom arrow to explain a precise sense in which Knörrer periodicity and Bott periodicity are compatible phenomena.

3.1 The real and complex Milnor fibers

Let $f \in \mathbb{C}[x_1, \dots, x_n]$, and suppose $f(0) = 0$. We begin this section by describing the construction of the Milnor fiber associated to f, following the exposition in Section 1 of [BvS12]. We then discuss various properties of the Milnor fiber that we will make use of later on.

3.1.1 Construction of the Milnor fibration and some properties of the Milnor fiber

For $\epsilon > 0$, define B_ϵ to be the closed ball centered at the origin of radius ϵ in $\mathbb{C}^n$, and for $\delta > 0$, set D^*_δ to be the open punctured disk centered at the origin in $\mathbb{C}$ of radius δ.

Choose $\epsilon > 0$ so that, for $0 < \epsilon' \leqslant \epsilon$, $\partial B_{\epsilon'}$ intersects $f^{-1}(0)$ transversely. Upon choosing such a number ϵ, choose $\delta \in (0, \epsilon)$ such that $f^{-1}(t)$ intersects ∂B_ϵ transversely for all $t \in D^*_\delta$. Then the map

$$\psi : B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

given by $\psi(x) = f(x)$ is a locally trivial fibration.

The map ψ depends, of course, on our choices of ϵ and δ. However, if ϵ', δ' is another pair of positive numbers satisfying the above conditions, the fibration associated to these choices is fiber homotopy equivalent to the one above (see Definition 1.5 in Chapter 3, §1 of [Dim92] for the definition of a fiber homotopy equivalence). We are thus justified in calling ψ the *Milnor fibration* associated to f.

Remark 3.1. The Milnor fibration was originally introduced in [Mil68]. The above construction is not the same as the construction of the Milnor fibration in [Mil68] and is due to Lê ([Lê76]). The two constructions yield fiber homotopy equivalent fibrations ([Dim92] Chapter 3, §1).

Choose $t \in D^*_\delta$. The fiber of ψ over t is called the *Milnor fiber of f over t*; we will denote it by F_f. F_f is independent of our choices of ϵ, δ, and t up to homotopy equivalence, so we suppress these choices in the notation, and we will often refer to F_f as just the Milnor fiber of f. However, these choices will be significant at several points later on.

If $\mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/(f)$ is IHS (see Definition 2.9), set

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_n]]}{(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})} < \infty.$$

Theorem 3.2 ([Mil68] Theorem 6.5). *If $\mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/(f)$ is IHS, F_f is homotopy equivalent to a wedge sum of μ copies of S^{n-1}.*

Remark 3.3. Since ψ restricts to a fibration over a circle, F_f comes equipped with a monodromy homeomorphism

$$h : F_f \xrightarrow{\cong} F_f.$$

3.1.2 The Sebastiani-Thom homotopy equivalence

We recall the definition of the join of two topological spaces:

Definition 3.4. Let X and Y be compact Hausdorff spaces. The *join* of X and Y, denoted $X * Y$, is the quotient of $X \times Y \times I$ by the relations

$$(x_1, y, 0) \sim (x_2, y, 0)$$

$$(x, y_1, 1) \sim (x, y_2, 1)$$

equipped with the quotient topology.

Remark 3.5. We express the cone CX over a compact Hausdorff space X explicitly as the quotient of

$$X \times [0, 1]$$

by the relation

$$(x_1, 0) \sim (x_2, 0)$$

for all $x_1, x_2 \in X$. When X and Y are compact Hausdorff, $X * Y$ is homeomorphic to $(CX \times Y) \cup (X \times CY) \subseteq CX \times CY$; here, we identify X and Y with the subsets $X \times \{1\}$ and $Y \times \{1\}$ of CX and CY, respectively. By [Bro06] 5.7.4, an explicit homeomorphism

$$CX \times CY \xrightarrow{\cong} C(X * Y)$$

is given by

$$(x, t, y, t') \mapsto ((x, y, \frac{t}{2t'}), t'), \text{ if } t' \geqslant t,\ t' \neq 0$$

$$(x, t, y, t') \mapsto ((x, y, 1 - \frac{t'}{2t}), t), \text{ if } t \geqslant t',\ t \neq 0$$

$$(x, 0, y, 0) \mapsto ((x, y, 0), 0),$$

and this map restricts to a homeomorphism

$$w : (CX \times Y) \cup (X \times CY) \xrightarrow{\cong} X * Y.$$

Now, suppose $f \in \mathbb{C}[x_1, \dots, x_n]$, $f' \in \mathbb{C}[y_1, \dots, y_m]$, and $f(0) = 0 = f'(0)$. Assume $R := \mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/(f)$ and $R' := \mathbb{C}[y_1, \dots, y_m]_{(y_1,\dots,y_m)}/(f')$ are IHS (see Definition 2.9). Let $f \oplus f'$ denote the sum of f and f' thought of as an element of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$. The following theorem of Sebastiani-Thom relates the Milnor fibers of f, f', and $f \oplus f'$:

Theorem 3.6 ([ST71]). *There is a homotopy equivalence*

$$\mathrm{ST} : F_f * F_{f'} \to F_{f \oplus f'}$$

that is compatible with monodromy; that is, the square

$$\begin{array}{ccc} F_f * F_{f'} & \xrightarrow{\mathrm{ST}} & F_{f\oplus f'} \\ {\scriptstyle h*h}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ F_f * F_{f'} & \xrightarrow{\mathrm{ST}} & F_{f\oplus f'} \end{array}$$

commutes up to homotopy.

Remark 3.7. By results of Oka in [Oka73], the assumption in Theorem 3.6 that R and R' are IHS is not necessary if f and f' are quasi-homogeneous.

We refer the reader to Section 2.7 of [AGZV12] and §3 of Chapter 3 in [Dim92] for discussions related to Theorem 3.6. We now exhibit an explicit map realizing the homotopy equivalence in Theorem 3.6, following Section 2.7 of [AGZV12]. Choose real numbers ϵ'', δ'', such that the map

$$B_{\epsilon''} \cap (f \oplus f')^{-1}(D^*_{\delta''}) \to D^*_{\delta''}$$

given by $x \mapsto (f \oplus f')(x)$ is a locally trivial fibration, as above.
Similarly, choose ϵ, δ and ϵ', δ', as well as $t'' \in D^*_{\delta''}$, so that the analogous maps

$$B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

$$B_{\epsilon'} \cap (f')^{-1}(D^*_{\delta'}) \to D^*_{\delta'}$$

are locally trivial fibrations, and also so that

(a) ϵ, ϵ' are sufficiently small so that $B_\epsilon \times B_{\epsilon'} \subseteq B_{\epsilon''}$.

(b) $|t''| < \min\{\delta, \delta'\}$.

Set F_f, $F_{f'}$, and $F_{f\oplus f'}$ to be the Milnor fibers of f, f', and $f \oplus f'$ over t''. Applying Lemma 2.10 in [AGZV12], choose a continuous map

$$H : CF_f \to B_\epsilon$$

such that

- $H(x, 1) = x \in F_f \subseteq B_\epsilon$,
- $H(-, s) : F_f \to B_\epsilon$ maps into the Milnor fiber $B_\epsilon \cap f^{-1}(st'')$ for $s \in (0, 1)$, and
- $H(x, 0) = 0$ for all $x \in F_f$

Example 3.8. If f is quasi-homogeneous of degree d with weights $w_1, \dots, w_d$, such a map H may be given by

$$(x, s) \mapsto (s^{\frac{w_1}{d}} x_1, \dots, s^{\frac{w_n}{d}} x_n).$$

Notice our assumption that R is IHS is not needed here.

Choose H' similarly for the Milnor fiber $F_{f'}$. By the discussion on pages 54-55 of [AGZV12] and Remark 3.5, there is a homotopy equivalence

$$g : CF_f \times F_{f'} \cup F_f \times CF_{f'} \to F_{f \oplus f'}$$

given by

$$(x, s, y, s') \mapsto (H(x, \frac{1+s-s'}{2}), H'(y, \frac{1-s+s'}{2})).$$

Composing, one has a homotopy equivalence

$$g \circ w^{-1} : F_f * F_{f'} \to F_{f \oplus f'},$$

where w is the homeomorphism in Remark 3.5. The homotopy equivalence $g \circ w^{-1}$ enjoys the same properties as the map ST in Theorem 2.14.

Remark 3.9. g extends to a homotopy equivalence of pairs

$$G : (CF_f \times CF_{f'}, CF_f \times F_{f'} \cup F_f \times CF_{f'}) \to (B_{\epsilon''}, F_{f \oplus f'})$$

that maps a point (x, s, y, s') to

$$(H(x, \frac{s}{2}), H'(y, \frac{2s'-s}{2}), \text{ if } s \leqslant s',\ s' \neq 0$$

$$(H(x, \frac{2s-s'}{2}), H'(y, \frac{s'}{2}), \text{ if } s' \leqslant s,\ s \neq 0$$

$$0, \text{ if } s = 0 = s'.$$

Remark 3.10. When f and f' are quasi-homogeneous (and R, R' are not necessarily IHS), we may use Example 3.8 to build a homotopy equivalence $g : CF_f \times F_{f'} \cup F_f \times CF_{f'} \to F_{f \oplus f'}$ in the same way as above ([Dim92] Chapter 3, Remark 3.19').

3.1.3 An analogue of the Milnor fibration over $\mathbb{R}$

Now, suppose $f \in \mathbb{R}[x_1, \dots, x_n]$ and $f(0) = 0$. One may construct a locally trivial fibration

$$\psi : B_\epsilon \cap f^{-1}((-\delta, 0) \cup (0, \delta)) \to (-\delta, 0) \cup (0, \delta)$$

for some $\epsilon > 0$ and δ such that $0 < \delta << \epsilon$ in the same way as above, where B_ϵ is now the closed ball of radius ϵ centered at the origin in $\mathbb{R}^n$.

But now, fibers over $(-\delta, 0)$ and $(0, \delta)$ need not be homotopy equivalent. For instance, if $f = x_1^2 + \cdots + x_n^2$, the positive fibers of ψ are homeomorphic to S^{n-1}, while the negative fibers are empty.

Choose $t \in (0, \delta)$ and $t' \in (-\delta, 0)$. The fiber of ψ over t is called the *positive Milnor fiber of f over t*, denoted by F_f^+, and the fiber of ψ over t' is called the *negative Milnor fiber of f over t'*, denoted F_f^-. As in the complex case,

F_f^+ and F_f^- are independent of our choices of ϵ, δ, t, and t' up to homotopy equivalence, so we suppress these choices in our notation, and we will often refer to F_f^+ and F_f^- as just the positive and negative Milnor fibers of f.
The topology of the real Milnor fibers is more complicated than that of the complex Milnor fiber. However, there is a version of Theorem 3.6 for real Milnor fibers of quasi-homogeneous polynomials. Suppose $f \in \mathbb{R}[x_1, \dots, x_n]$, $f' \in \mathbb{R}[y_1, \dots, y_m]$ are quasi-homogeneous and nonconstant. If F_f^+ and $F_{f'}^+$ are nonempty, there is a homotopy equivalence

$$F_f^+ * F_{f'}^+ \to F_{f \oplus f'}^+$$

([DP92] Remark 11). Moreover, the homotopy equivalence may be constructed as in Remark 3.10; that is, one has a homotopy equivalence of pairs

$$G : (CF_f^+ \times CF_{f'}^+, CF_f^+ \times F_{f'}^+ \cup F_f^+ \times CF_{f'}^+) \to (B_{\epsilon''}, F_{f \oplus f'}^+).$$

Since $F_f^- = F_{-f}^+$, one has a similar result for negative Milnor fibers.

3.2 Relative topological K-theory

We introduce some facts concerning relative topological K-theory. All of the results in this section are essentially due to Atiyah-Bott-Shapiro in [ABS64], but we modify their exposition at several points to suit our purposes.
Let X be a compact topological space, and let Y be a closed subspace of X such that there exists a homotopy equivalence of pairs between (X, Y) and a finite CW pair; we construct a category $\mathcal{C}_1(X, Y)$ from (X, Y) in the following way:
• An object of $\mathcal{C}_1(X, Y)$ is a pair of real vector bundles V_1, V_0 over X equipped with an isomorphism

$$V_1|_Y \xrightarrow{\sigma} V_0|_Y.$$

Denote objects of $\mathcal{C}_1(X, Y)$ by $(V_1, V_0; \sigma)$.
• Morphisms in $\mathcal{C}_1(X, Y)$ are pairs of morphisms of vector bundles over X

$$\alpha_1 : V_1 \to V_1',\ \alpha_0 : V_0 \to V_0'$$

such that the following diagram of maps of vector bundles over Y commutes:

$$\begin{tikzcd} V_1|_Y \arrow[r, "\sigma"] \arrow[d, "\alpha_1|_Y"'] & V_0|_Y \arrow[d, "\alpha_0|_Y"'] \\ V_1'|_Y \arrow[r, "\sigma'"] & V_0'|_Y \end{tikzcd}$$

We write morphisms in $\mathcal{C}_1(X, Y)$ as ordered pairs (α_1, α_0).
Remark 3.11. The reason for the subscript in the notation $\mathcal{C}_1(X, Y)$ is that, for any $n \geqslant 1$, one may similarly build a category $\mathcal{C}_n(X, Y)$ with objects given by ordered $(n+1)$-tuples of vector spaces on X whose restrictions to Y fit into an exact sequence (cf. [ABS64] §7).

Remark 3.12. We will work with real vector bundles throughout this section; however, there is an analogous version of every result in this section for complex vector bundles.

The following facts about $\mathcal{C}_1(X, Y)$ are easily verified:

- If $(V_1, V_0; \sigma)$ and $(V_1', V_0'; \sigma')$ are objects in $\mathcal{C}_1(X, Y)$, $(V_1 \oplus V_1', V_0 \oplus V_0', \sigma \oplus \sigma')$ is their coproduct.
- $\mathcal{C}_1(X, Y)$ is an additive category.
- A map $g : (X_1, Y_1) \to (X_2, Y_2)$ of pairs of spaces as above induces a functor
$$g^* : \mathcal{C}_1(X_2, Y_2) \to \mathcal{C}_1(X_1, Y_1)$$
via pullback.
- A morphism (α_1, α_0) in $\mathcal{C}_1(X, Y)$ is an isomorphism (resp. monomorphism, epimorphism) if and only if α_1 and α_0 are isomorphisms (resp. monomorphisms, epimorphisms) of vector bundles over X.

We shall call an object of $\mathcal{C}_1(X, Y)$ *elementary* if it is isomorphic to an object of the form $(V, V; \mathrm{id}_{V|_Y})$. It is easy to check that $(V_1, V_0; \sigma)$ is elementary if and only if σ can be extended to an isomorphism $\widetilde{\sigma} : V_1 \to V_0$.
If V and V' are objects in $\mathcal{C}_1(X, Y)$, we will say $V \sim V'$ if and only if there exist elementary objects E, E' such that

$$V \oplus E \cong V' \oplus E'.$$

The relation $\sim$ is an equivalence relation. Let $L_1(X, Y)$ denote the commutative monoid of equivalence classes under $\sim$ with operation $\oplus$. We shall denote by $[V_1, V_0; \sigma]$ the class in $L_1(X, Y)$ represented by (V_1, V_0, σ).

Remark 3.13. Let (X_1, Y_1), (X_2, Y_2) be pairs of spaces as above, and let $g : (X_1, Y_1) \to (X_2, Y_2)$ be a map of pairs. Then the functor

$$g^* : \mathcal{C}_1(X_2, Y_2) \to \mathcal{C}_1(X_1, Y_1)$$

applied to an elementary object is again elementary. Hence, g^* induces a map of monoids

$$L_1(X_2, Y_2) \to L_1(X_1, Y_1).$$

The main reason we are interested in the monoid $L_1(X, Y)$ is the following result:

PROPOSITION 3.14 (Atiyah-Bott-Shapiro, [ABS64]). *There exists a unique natural homomorphism*

$$\chi : L_1(X, Y) \to KO^0(X, Y)$$

which, when $Y = \emptyset$, is given by

$$\chi(E) = [V_0] - [V_1].$$

Moreover, χ is an isomorphism.

In particular, $L_1(X,Y)$ is an abelian group.
Let (X,Y), (X',Y') be pairs as above. We conclude this section by exhibiting a product map

$$L_1(X,Y) \otimes L_1(X',Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

that agrees, via χ, with the usual product on relative K-theory.
Let $V = (V_1, V_0; \sigma) \in \mathrm{Ob}(\mathcal{C}_1(X,Y))$ and $V' = (V_1', V_0'; \sigma') \in \mathrm{Ob}(\mathcal{C}_1(X',Y'))$. By Proposition 10.1 in [ABS64], we may lift σ, σ' to maps $\widetilde{\sigma}, \widetilde{\sigma}'$ of bundles over X and X', respectively.
Thinking of

$$0 \to V_1 \xrightarrow{\widetilde{\sigma}} V_0 \to 0$$

$$0 \to V_1' \xrightarrow{\widetilde{\sigma}'} V_0' \to 0$$

as complexes of bundles with V_1, V_1' in degree 1 and V_0, V_0' in degree 0, we may take their tensor product

$$0 \to V_1 \otimes V_1' \xrightarrow{\tau_2} (V_1 \otimes V_0') \oplus (V_0 \otimes V_1') \xrightarrow{\tau_1} V_0 \otimes V_0' \to 0,$$

where

$$\tau_1 = \begin{pmatrix} \widetilde{\sigma} \otimes \mathrm{id}_{V_0'} & \mathrm{id}_{V_0} \otimes \widetilde{\sigma}' \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} -\mathrm{id}_{V_1} \otimes \widetilde{\sigma}' \\ \widetilde{\sigma} \otimes \mathrm{id}_{V_1'} \end{pmatrix}$$

The result is a complex of vector bundles over $X \times X'$ that is exact upon restriction to $X \times Y' \cup Y \times X'$.
Choose a splitting π of $\tau_2|_{X \times Y' \cup Y \times X'}$. Then,

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}]$$

is an element of $L_1(X \times X', X \times Y' \cup Y \times X')$.
One may define monoids $L_n(X,Y)$ involving longer sequences of bundles; see [ABS64] Definition 7.1 for details. Denote elements of $L_n(X,Y)$ by

$$[V_n, \dots, V_0; \sigma_n, \dots, \sigma_1].$$

There is a map

$$j_n : L_1(X,Y) \to L_n(X,Y)$$

given by

$$[V_1, V_0; \sigma] \mapsto [0, \dots, 0, V_1, V_0; 0, \dots, 0, \sigma],$$

and, by Proposition 7.4 in [ABS64], j_n is an isomorphism for all n.
We will need the following technical lemma:

LEMMA 3.15. *Let (X,Y) be a pair as above, and let $[V_2, V_1, V_0; \sigma_2, \sigma_1] \in L_2(X,Y)$. If π is a splitting of σ_2,*

$$j_2([V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]) = [V_2, V_1, V_0; \sigma_2, \sigma_1].$$

Proof. First, suppose $\dim(V_1) > \dim(V_2) + \dim(X)$. Apply Lemma 7.2 in [ABS64] to construct a monomorphism

$$h : V_2 \to V_1$$

that extends σ_2. By the proof of Lemma 7.3 in [ABS64],

$$j_2([\mathrm{coker}(h), V_0; \overline{\sigma_1}]) = [V_2, V_1, V_0; \sigma_2, \sigma_1],$$

and so

$$j_2([\mathrm{coker}(h) \oplus V_2, V_0 \oplus V_2; A]) = [V_2, V_1, V_0; \sigma_2, \sigma_1],$$

where

$$A = \begin{pmatrix} \overline{\sigma_1} & 0 \\ 0 & \mathrm{id}_{V_2}|_Y \end{pmatrix}.$$

Hence, it suffices to show

$$[\mathrm{coker}(h) \oplus V_2, V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$$

Choose a splitting s of h, and let

$$p : V_1 \to \mathrm{coker}(h)$$

denote the canonical map. Then we have an isomorphism

$$\begin{pmatrix} p \\ s \end{pmatrix} : V_1 \to \mathrm{coker}(h) \oplus V_2.$$

Since $s|_Y$ is a splitting of σ_2, we also have an isomorphism

$$\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} : V_1|_Y \to V_0|_Y \oplus V_2|_Y.$$

We have a commutative square

$$\begin{tikzcd}
V_1|_Y \arrow[r, "\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} p|_Y \\ s|_Y \end{pmatrix}"] & V_0|_Y \oplus V_2|_Y \arrow[d, "\mathrm{id}_{V_0|_Y \oplus V_2|_Y}"] \\
\mathrm{coker}(h)|_Y \oplus V_2|_Y \arrow[r, "A"] & V_0|_Y \oplus V_2|_Y
\end{tikzcd}$$

Thus,

$$[\mathrm{coker}(h) \oplus V_2, V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}].$$

Notice that we have an object

$$[V_1 \times I, (V_0 \oplus V_2) \times I; t \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} + (1-t) \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$$

in $\mathcal{C}_1(X \times I, Y \times I)$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are $[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$ and $[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}]$, respectively. It now follows from Proposition 9.2 in [ABS64] that

$$[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}].$$

This finishes the case where $\dim(V_1) > \dim(V_2) + \dim(X)$.
For the general case, choose a bundle E such that

$$\dim(E) + \dim(V_1) > \dim(V_2) + \dim(X).$$

Define

$$U := [V_2, V_1 \oplus E, V_0 \oplus E; \begin{pmatrix} \sigma_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 & \mathrm{id}_E|_Y \end{pmatrix}],$$

$$U' := [V_1 \oplus E, V_0 \oplus E \oplus V_2; \begin{pmatrix} \sigma_1 & 0 \\ 0 & \mathrm{id}_E|_Y \\ \pi & 0 \end{pmatrix}]$$

Notice that

$$[V_2, V_1, V_0; \sigma_2, \sigma_1] = U,$$

and

$$[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] = U',$$

so that it suffices to show that $j(U') = U$. Since $\begin{pmatrix} \pi & 0 \end{pmatrix}$ is a splitting of $\begin{pmatrix} \sigma_2 \\ 0 \end{pmatrix}$, this follows from the case we have already considered. $\square$

Now, the pairing

$$L_1(X, Y) \otimes L_1(X', Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

described in Proposition 10.4 of [ABS64] is given by sending a simple tensor

$$[V_1, V_0; \sigma] \otimes [V_1', V_0'; \sigma']$$

to

$$j_2^{-1}([V_1 \otimes V_1', (V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), V_0 \otimes V_0'; \tau_2|_{X \times Y' \cup Y \times X'}, \tau_1|_{X \times Y' \cup Y \times X'}]);$$

this follows from the proof of Proposition 10.4.
Thus, by Lemma 3.15, the map

$$\mathrm{Ob}(\mathcal{C}_1(X,Y)) \times \mathrm{Ob}(\mathcal{C}_1(X',Y')) \to L_1(X \times X', X \times Y' \cup Y \times X')$$

given by

$$(V, V') \mapsto [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}]$$

determines

(a) a well-defined pairing on $\mathrm{Ob}(\mathcal{C}_1(X,Y)) \times \mathrm{Ob}(\mathcal{C}_1(X',Y'))$ up to our choices of liftings $\widetilde{\sigma}$, $\widetilde{\sigma'}$ and splitting π, and

(b) a pairing

$$L_1(X,Y) \otimes L_1(X',Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

that coincides with the pairing in Proposition 10.4 of [ABS64].

Let $[V]$,$[V']$ denote the classes represented by V and V' in $L_1(X,Y)$ and $L_1(X',Y')$. Define

$$[V] \otimes_{L_1} [V'] := [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}].$$

Remark 3.16. By Proposition 10.4 in [ABS64] and the above remarks,

$$\chi([V]) \otimes \chi([V']) = \chi([V] \otimes_{L_1} [V']).$$

3.3 A generalized Atiyah-Bott-Shapiro construction applied to matrix factorizations

In this section, we construct the maps $\phi_f^{\mathbb{C}}$ and $\phi_f^{\mathbb{R}}$ described in the introduction. We begin with a discussion of the Atiyah-Bott-Shapiro construction ([ABS64] Part III). Following Atiyah-Bott-Shapiro, we work with real Clifford algebras and KO-theory, and we point out that one may perform a similar construction involving complex Clifford algebras and KU-theory.

3.3.1 The Atiyah-Bott-Shapiro construction

Define

$$q_n := -x_1^2 - \cdots - x_n^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

for all $n \geqslant 1$, and set $C_n := \mathrm{Cliff}_{\mathbb{R}}(q_n)$. We also set $C_0 := \mathbb{R}$; we will think of C_0 as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra concentrated in degree 0.

Let $M(C_n)$ denote the free abelian group generated by isomorphism classes of finitely-generated, indecomposable $\mathbb{Z}/2\mathbb{Z}$-graded left C_n-modules. There are evident injective maps

$$i_n : C_n \to C_{n+1}$$

for all $n \geqslant 0$; these injections induce homomorphisms

$$i_n^* : M(C_{n+1}) \to M(C_n)$$

via restriction of scalars. Set

$$A_n := M(C_n)/i_n^*(M(C_{n+1})).$$

Define D^n to be the closed disk of radius 1 in $\mathbb{R}^n$. An important special case of the classical Atiyah-Bott-Shapiro construction is the group isomorphism

$$\alpha_n : A_n \xrightarrow{\cong} L_1(D^n, \partial D^n)$$

that appears in [ABS64] Theorem 11.5. α_n is defined as follows: let $M = M_1 \oplus M_0$ be a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left C_n-module. We use the $\mathbb{R}$-vector spaces M_1 and M_0 to construct real vector bundles over D^n:

$$V_1 := D^n \times M_1$$

$$V_0 := D^n \times M_0$$

and we define a map

$$\sigma : V_1 \to V_0$$

given by $(x, m) \mapsto (x, x \cdot m)$, where $\cdot$ denotes the action of C_n on M. Here, we are thinking of $D^n \subseteq \mathbb{R}^n$ as a subset of C_n. Notice that σ restricts to an isomorphism of bundles over ∂D^n. Thus, we have constructed an element $[V_1, V_0; \sigma] \in L_1(D^n, \partial D^n)$. Define

$$\alpha_n([M]) = [V_1, V_0; \sigma].$$

We refer the reader to [ABS64] for verification that the mapping

$$[M] \mapsto [V_1, V_0; \sigma]$$

is well-defined on the quotient A_n and determines an isomorphism.

3.3.2 A more general construction

Let $f \in (x_1, \dots, x_n) \subseteq Q := \mathbb{R}[x_1, \dots, x_n]$. Choose real numbers ϵ, δ, and t such that $\epsilon > 0$, $0 < \delta << \epsilon$, and $t \in (-\delta, 0)$ in such a way that we may construct a negative Milnor fiber F_f^- as in Section 3.1.3.

Denote by B_ϵ the closed ball of radius ϵ in $\mathbb{R}^n$ centered at the origin. We now construct a map

$$\mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_f^-)$$

that

(a) recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.17) when $f = q_n$, and

(b) descends to a group homomorphism
$$K_0[\mathrm{MF}(Q, f)] \to L_1(B_\epsilon, F_f^-).$$

We emphasize that a similar construction involving complex polynomials and their Milnor fibers may be performed *mutatis mutandis*. One may also perform the following construction using the positive Milnor fiber F_f^+ of f.

Let $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$ be a matrix factorization of f over Q. Denote by $C(B_\epsilon)$ the ring of $\mathbb{R}$-valued continuous functions on B_ϵ. Applying extension of scalars along the inclusion
$$Q \hookrightarrow C(B_\epsilon),$$
we obtain a map
$$P_1 \otimes_Q C(B_\epsilon) \xrightarrow{d_1 \otimes \mathrm{id}} P_0 \otimes_Q C(B_\epsilon)$$
of finitely generated projective $C(B_\epsilon)$-modules.

The category of real vector bundles over B_ϵ is equivalent to the category of finitely generated projective $C(B_\epsilon)$-modules; on objects, the equivalence sends a bundle to its space of sections. Let
$$V_1 \xrightarrow{d_1} V_0$$
be a map of real vector bundles over B_ϵ corresponding to the above map $d_1 \otimes \mathrm{id}$ under this equivalence. Since $d_1 \circ d_0 = f \cdot \mathrm{id}_{P_0}$ and $d_0 \circ d_1 = f \cdot \mathrm{id}_{P_1}$, and since the restriction of the polynomial f, thought of as a map $\mathbb{R}^n \to \mathbb{R}$, to $F_f^- = B_\epsilon \cap f^{-1}(t)$ is constant with value $t \neq 0$, $d_1|_{F_f^-}$ is an isomorphism of vector bundles on F_f^-. Its inverse is the restriction to F_f^- of the map $V_0 \to V_1$ determined by
$$P_0 \otimes_Q C(B_\epsilon) \xrightarrow{\frac{1}{t}(d_0 \otimes \mathrm{id})} P_1 \otimes_Q C(B_\epsilon).$$
Define $\Phi_f^{\mathbb{R}}(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) = (V_1, V_0; d_1|_{F_f^-}) \in \mathrm{Ob}(\mathcal{C}_1(B_\epsilon, F_f^-))$.

Remark 3.17. The map analogous to $\Phi_f^{\mathbb{R}}$ in the setting of polynomials over $\mathbb{C}$ and KU-theory appears in [BvS12]; we discuss this in detail in Section 3.3.3.

A morphism in $Z^0\mathrm{MF}(Q, f)$ determines a morphism in $\mathcal{C}_1(B_\epsilon, F_f^-)$ in an obvious way (see Section 2.1.1 for the definition of the category $Z^0\mathrm{MF}(Q, f)$). Hence, we have shown:

PROPOSITION 3.18. *There is an additive functor*
$$\Phi_f^{\mathbb{R}} : Z^0\mathrm{MF}(Q, f) \to \mathcal{C}_1(B_\epsilon, F_f^-)$$
given, on objects, by
$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \mapsto [V_1, V_0; d_1|_{F_f^-}].$$

In particular, we have a map

$$\mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_f^-).$$

Suppose $f = q_n$. Then ϵ can be chosen to be 1 in the construction of the negative Milnor fiber F_f^-, and the fiber can be chosen to be exactly $S^{n-1} \subseteq \mathbb{R}^n$. Let $\mathrm{Iso}([\mathrm{MF}(Q, f)])$ and $\mathrm{Iso}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_{\mathbb{R}}(q_n)))$ denote the sets of isomorphism classes of objects in $[\mathrm{MF}(Q, f)]$ and $\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_{\mathbb{R}}(q_n))$. It is easy to check that one has a commutative triangle

$$\begin{tikzcd}
\mathrm{Iso}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_{\mathbb{R}}(q_n))) \arrow[r, "\mathrm{ABS}"] \arrow[d, "{[\Theta]}"] & L_1(B_1, F_f^-) \\
\mathrm{Iso}([\mathrm{MF}(Q, f)]) \arrow[ur, "\Phi_f^{\mathbb{R}}"] &
\end{tikzcd}$$

where $[\Theta]$ denotes the bijection on isomorphism classes induced by the explicit construction Θ of the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.17) provided in the proof of Theorem 14.7 of [Yos90], and ABS denotes the Atiyah-Bott-Shapiro construction. Hence, our construction recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog equivalence when $f = q_n$.

Our next goal is to show that $\Phi_f^{\mathbb{R}}$ induces a map on K-theory:

Proposition 3.19. *$\Phi_f^{\mathbb{R}}$ induces a group homomorphism*

$$\phi_f^{\mathbb{R}} : K_0[\mathrm{MF}(Q, f)] \to L_1(B_\epsilon, F_f^-).$$

We will adopt the following notational conventions for the purposes of the proof of Proposition 3.19:

(1) A pair (ϵ, t) is a *good pair* if $\epsilon > 0$, $t < 0$, and the map

$$\psi : B_\epsilon \cap f^{-1}((-\delta, 0) \cup (0, \delta)) \to (-\delta, 0) \cup (0, \delta)$$

from Section 3.1.3 is a locally trivial fibration for some $\delta > 0$ such that

$$0 < |t| < \delta << \epsilon.$$

(2) If (ϵ, t) is a good pair, we denote the negative Milnor fiber $B_\epsilon \cap f^{-1}(t)$ by F_t^-.

We will need the following technical lemma:

Lemma 3.20. *Let $(\epsilon_1, t_1), (\epsilon_2, t_2)$ be good pairs. Then there is an isomorphism*

$$g : L_1(B_{\epsilon_1}, F^-_{t_1}) \xrightarrow{\cong} L_1(B_{\epsilon_2}, F^-_{t_2})$$

yielding a commutative triangle

$$\begin{tikzcd}
\mathrm{Ob}(\mathrm{MF}(Q,f)) \arrow[r, "\Phi^{\mathbb{R}}_f"] \arrow[d, "\Phi^{\mathbb{R}}_f"] & L_1(B_{\epsilon_2}, F^-_{t_2}) \\
L_1(B_{\epsilon_1}, F^-_{t_1}) \arrow[ur, "g"] &
\end{tikzcd}$$

Proof. The case where $t_1 = t_2$ is immediate, so we may assume $t_1 \neq t_2$. First, suppose $\epsilon_1 = \epsilon_2$. Without loss, assume $t_2 < t_1$.
Set $F^-_{[t_2,t_1]} := f^{-1}([t_2, t_1])$. Since the inclusions

$$F^-_{t_1} \hookrightarrow F^-_{[t_2,t_1]}, \; F^-_{t_2} \hookrightarrow F^-_{[t_2,t_1]}$$

are homotopy equivalences, the pullback maps

$$L_1(B_{\epsilon_1}, F_{[t_2,t_1]}) \to L_1(B_{\epsilon_1}, F_{t_1}), \; L_1(B_{\epsilon_1}, F_{[t_2,t_1]}) \to L_1(B_{\epsilon_1}, F_{t_2})$$

are isomorphisms.
We have commuting triangles

$$\begin{tikzcd}
\mathrm{Ob}(\mathrm{MF}(Q,f)) \arrow[r, "\Phi^{\mathbb{R}}_f"] \arrow[d, "\Phi^{\mathbb{R}}_f"] & L_1(B_{\epsilon_1}, F^-_{t_1}) \\
L_1(B_{\epsilon_1}, F^-_{[t_2,t_1]}) \arrow[ur, "\cong"] &
\end{tikzcd}$$

for $i = 1, 2$. It follows that the result holds when $\epsilon_1 = \epsilon_2$.
For the general case, assume, without loss, that $|t_2| < |t_1|$. Then (ϵ_1, t_2) is also a good pair. By the cases we've already considered, the result holds for the pairs (ϵ_1, t_1) and (ϵ_1, t_2), and also for the pairs (ϵ_1, t_2) and (ϵ_2, t_2). Hence, the result holds for the pairs (ϵ_1, t_1), (ϵ_2, t_2). $\square$

We now prove Proposition 3.19:

Proof. It is not hard to see that $\Phi^{\mathbb{R}}_f(P \oplus P') = \Phi^{\mathbb{R}}_f(P) \oplus \Phi^{\mathbb{R}}_f(P')$; we need only show that $\phi^{\mathbb{R}}_f$ is well-defined. First, suppose $P \cong 0$ in $[\mathrm{MF}(Q,f)]$. Then id_P is a boundary in $\mathrm{MF}(Q,f)$, and so id_P factors through a trivial matrix factorization, by Proposition 2.4.
Write $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$. Since P is a summand of a trivial matrix factorization, $\mathrm{coker}(d_1)$ is a projective $Q/(f)$ module. Choose $g \in Q$ such

that $g(0) \neq 0$ and $\mathrm{coker}(d_1)_g$ is free over $Q_g/(f)$, and choose $\epsilon' \in (0, \epsilon)$ such that $B_{\epsilon'} \cap g^{-1}(0) = \emptyset$. The inclusion $Q \hookrightarrow Q_g$ induces a functor $\mathrm{MF}(Q, f) \to \mathrm{MF}(Q_g, f)$. Choose t' such that (ϵ', t') is a good pair. Applying Lemma 3.20, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ob}(\mathrm{MF}(Q,f)) \arrow[r] \arrow[d, "\Phi_f^{\mathbb{R}}"] \arrow[dr, "\Phi_f^{\mathbb{R}}"] & \mathrm{Ob}(\mathrm{MF}(Q_g,f)) \arrow[d, "\Phi_f^{\mathbb{R}}"] \\ L_1(B_\epsilon, F_t^-) \arrow[r, "\cong"] & L_1(B_{\epsilon'}, F_{t'}^-) \end{tikzcd}$$

It is easy to see that the $\phi_f^{\mathbb{R}}$ is well-defined when $f = 0$, so assume $f \neq 0$. Then f is a non-zero-divisor in Q, so we may apply Proposition 2.5 to conclude that the image of P in $\mathrm{Ob}(\mathrm{MF}(Q_g, f))$ maps to 0 via $\Phi_f^{\mathbb{R}}$. Hence, the map $\Phi_f^{\mathbb{R}} : \mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_t^-)$ sends P to 0, as required.

We now show that, if $\alpha : P \to P'$ is a morphism in $Z^0\mathrm{MF}(Q, f)$, $\Phi_f^{\mathbb{R}}(P) \oplus \Phi_f^{\mathbb{R}}(\mathrm{cone}(\alpha))$ and $\Phi_f^{\mathbb{R}}(P')$ represent the same class in $L_1(B_\epsilon, F_t^-)$. We start by showing $\Phi_f^{\mathbb{R}}(P[1]) = -\Phi_f^{\mathbb{R}}(P)$ in $L_1(B_\epsilon, F_t^-)$. Write $\Phi_f^{\mathbb{R}}(P) = (V_1, V_0; d_1|_{F_t^-})$, so that $\Phi_f^{\mathbb{R}}(P[1]) = (V_0, V_1; -d_0|_{F_t^-})$. Since $\mathrm{cone}(\mathrm{id}_P)$ is contractible, the class represented by

$$\Phi_f^{\mathbb{R}}(\mathrm{cone}(\mathrm{id}_P)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_t^-} & \mathrm{id} \\ 0 & -d_1|_{F_t^-} \end{pmatrix})$$

in $L_1(B_\epsilon, F_t^-)$ is 0. The object

$$((V_0 \oplus V_1) \times I, (V_1 \oplus V_0) \times I; \begin{pmatrix} d_0|_{F_t^-} & s \cdot \mathrm{id} \\ 0 & -d_1|_{F_t^-} \end{pmatrix})$$

of $\mathcal{C}_1(B_\epsilon \times I, F_t^- \times I)$ restricts to $\Phi_f^{\mathbb{R}}(\mathrm{cone}(\mathrm{id}_P))$ at $s = 1$ and $\Phi_f^{\mathbb{R}}((P \oplus P[1])[1])$ at $s = 0$. Since $(P \oplus P[1])[1] \cong P \oplus P[1]$, we may use Proposition 9.2 in [ABS64] to conclude that $\Phi_f^{\mathbb{R}}(P[1]) = -\Phi_f^{\mathbb{R}}(P)$ in $L_1(B_\epsilon, F_t^-)$.

Now, we have

$$\Phi_f^{\mathbb{R}}(\mathrm{cone}(\alpha)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_t^-} & \alpha_1 \\ 0 & -d_1'|_{F_t^-} \end{pmatrix}).$$

Using Proposition 9.2 in [ABS64] in the same manner as above, we may conclude that $\Phi_f^{\mathbb{R}}(\mathrm{cone}(\alpha))$ and $\Phi_f^{\mathbb{R}}(P') \oplus \Phi_f^{\mathbb{R}}(P[1])$ represent the same class in $L_1(B_\epsilon, F_t^-)$.

Finally, suppose $\alpha : P \xrightarrow{\cong} P'$ is an isomorphism in $[\mathrm{MF}(Q, f)]$. Then $\mathrm{cone}(\alpha)$ is contractible, and so the results we just established imply that $\Phi_f^{\mathbb{R}}(P) = \Phi_f^{\mathbb{R}}(P')$.

Since every distinguished triangle in $[\mathrm{MF}(Q,f)]$ is isomorphic to one of the form

$$P \xrightarrow{\alpha} P' \to \mathrm{cone}(\alpha) \to P[1],$$

and we have shown that $\Phi_f^{\mathbb{R}}$ preserves such triangles, we are done. $\square$

3.3.3 The kernel and image of $\phi_f^{\mathbb{C}}$

Let $Q := \mathbb{C}[x_1, \ldots, x_n]$, and set $\mathfrak{m} := (x_1, \ldots, x_n) \subseteq Q$. Fix $f \in \mathfrak{m}$, and define $R := Q/(f)$. Assume the hypersurface R has an isolated singularity at the origin in the sense of Definition 2.9. Choose $\epsilon, \delta > 0$ so that the map

$$B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

given by $x \mapsto f(x)$ is a locally trivial fibration, as in Section 3.1; let F_f denote the Milnor fiber of f over some value $t \in D_\delta^*$. We wish to examine the kernel and image of the map

$$\phi_f^{\mathbb{C}} : K_0[\mathrm{MF}(Q,f)] \to L_1(B_\epsilon, F_f).$$

Recall that, by Theorem 3.2, F_f is homotopy equivalent to a wedge sum of μ copies of S^{n-1}, where μ is the Milnor number of f. Thus,

$$L_1(B_\epsilon, F_f) \cong KU^0(B_\epsilon, F_f) \cong KU^{-1}(F_f)$$

$$\cong \bigoplus_\mu KU^{-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}^\mu & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

In particular, when n is odd, $\phi_f^{\mathbb{C}} = 0$.
As we noted in Section 3.1, F_f is equipped with a monodromy homeomorphism

$$h : F_f \xrightarrow{\cong} F_f.$$

Let $S \subseteq D_\delta^*$ denote the circle of radius $|t|$ centered at the origin, and set $E := B_\epsilon \cap f^{-1}(S)$. One has a long exact sequence, the *Wang exact sequence* ([Dim92] page 74)

$$\cdots \to H^i(E) \xrightarrow{j^*} H^i(F_f) \xrightarrow{h^*-1} H^i(F_f) \to H^{i+1}(E) \to \cdots$$

where $j : F_f \hookrightarrow E$ is the inclusion. One also has an automorphism $T : L_1(B_\epsilon, F_f) \xrightarrow{\cong} L_1(B_\epsilon, F_f)$ induced by h.
We have the following result regarding the image of $\phi_f^{\mathbb{C}}$:

Proposition 3.21. $\phi_f^{\mathbb{C}}(K_0[\mathrm{MF}(Q,f)]) \subseteq \ker(T-1)$.

Proof. The result is obvious when n is odd, since $L_1(B_\epsilon, F_f) = 0$ in this case. Suppose n is even. Notice that $\phi_f^{\mathbb{C}}(K_0[\mathrm{MF}(Q,f)]) \subseteq l^*(L_1(B_\epsilon, E))$, where $l : (B_\epsilon, F_f) \hookrightarrow (B_\epsilon, E)$ is the inclusion of pairs. Thus, the result follows from the commutative diagram

$$\begin{tikzcd}
L_1(B_\epsilon, E) \otimes \mathbb{Q} \arrow[r, "l^*"] \arrow[d, "\cong"] & L_1(B_\epsilon, F_f) \otimes \mathbb{Q} \arrow[r, "T-1"] \arrow[d, "\cong"] & L_1(B_\epsilon, F_f) \otimes \mathbb{Q} \arrow[d, "\cong"] \\
KU^{-1}(E) \otimes \mathbb{Q} \arrow[r, "j^*"] \arrow[d] & KU^{-1}(F_f) \otimes \mathbb{Q} \arrow[r, "h^*-1"] \arrow[d, "\cong"] & KU^{-1}(F_f) \otimes \mathbb{Q} \arrow[d, "\cong"] \\
H^{n-1}(E;\mathbb{Q}) \arrow[r, "j^*"] & H^{n-1}(F_f;\mathbb{Q}) \arrow[r, "h^*-1"] & H^{n-1}(F_f;\mathbb{Q})
\end{tikzcd}$$

and the Wang exact sequence. The bottom-most vertical arrows are Chern class maps; the bottom-middle and bottom-right vertical maps are isomorphisms because F_f has nonzero odd cohomology only in degree $n-1$. ☐

The map $\Phi_f^{\mathbb{C}} : \mathrm{Ob}(\mathrm{MF}(Q,f)) \to L_1(B_\epsilon, F_f)$ is used in [BvS12] to study the Hochster theta pairing. We recall the definition of this pairing:

Definition 3.22. The *Hochster theta pairing*

$$\theta : K_0[\mathrm{MF}(Q_\mathfrak{m}, f)] \times K_0[\mathrm{MF}(Q_\mathfrak{m}, f)] \to \mathbb{Z}$$

sends a pair $([P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0], [P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0'])$ to

$$l(\mathrm{Tor}_2^{R_\mathfrak{m}}(\mathrm{coker}(d_1), \mathrm{coker}(d_1'))) - l(\mathrm{Tor}_1^{R_\mathfrak{m}}(\mathrm{coker}(d_1), \mathrm{coker}(d_1'))),$$

where l denotes length as an $R_\mathfrak{m}$-module.

Remark 3.23. Our assumption that R is IHS guarantees that the lengths in Definition 3.22 are finite. The pairing θ was introduced in [Hoc81]; for more detailed discussions related to this pairing, we refer the reader to [BvS12], [Dao13], and [MPSW11].

Remark 3.24. Under our assumptions, by Theorem 4.11 of [Dyc11], the map

$$K_0[\mathrm{MF}(Q,f)] \to K_0[\mathrm{MF}(Q_\mathfrak{m}, f)]$$

induced by inclusion is an isomorphism, so we may think of θ as a pairing on $K_0[\mathrm{MF}(Q,f)]$.

Let $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$ be a matrix factorization of f over Q. We observe that the image of $\phi_f^{\mathbb{C}}([P])$ under the isomorphism $L_1(B_\epsilon, F_f) \cong KU^{-1}(F_f)$ coincides with $\alpha(\mathrm{coker}(d_1)_\mathfrak{m})|_{F_f}$, where α is as in Section 4 of [BvS12]. Thus, Proposition 4.1 and Theorem 4.2 of [BvS12] immediately imply:

Proposition 3.25. *If* $X \in \ker(\phi_f^{\mathbb{C}})$*,* $\theta(X,-) : K_0[\mathrm{MF}(Q,f)] \to \mathbb{Z}$ *is the zero map.*

Set $K_0[\mathrm{MF}(Q,f)]_{\mathrm{tors}}$ to be the torsion subgroup of $K_0[\mathrm{MF}(Q,f)]$. We conclude this section with the following explicit description of $\ker(\phi_f^{\mathbb{C}})$ when $n=2$:

Proposition 3.26. *If* $f \in (x_1,x_2) \subseteq Q = \mathbb{C}[x_1,x_2]$*, and the hypersurface* $Q/(f)$ *has an isolated singularity at the origin in the sense of Definition 2.9,* $\ker(\phi_f^{\mathbb{C}}) = K_0[\mathrm{MF}(Q,f)]_{\mathrm{tors}}$.

Proof. $K_0[\mathrm{MF}(Q,f)]_{\mathrm{tors}} \subseteq \ker(\phi_f^{\mathbb{C}})$ is obvious. Suppose $[P] \in \ker(\phi_f^{\mathbb{C}})$. By Proposition 3.25, the map $\theta([P],-) : K_0[\mathrm{MF}(Q,f)] \to \mathbb{Z}$ is the zero map. Set $R = Q/(f)$. Since $K_0[\mathrm{MF}(Q_{(x_1,x_2)},f)] \cong G_0(R_{(x_1,x_2)})/[R_{(x_1,x_2)}]$, an application of Proposition 3.3 in [Dao13] finishes the proof. □

3.4 Knörrer periodicity and Bott periodicity

We now use our constructions $\phi_f^{\mathbb{R}}$ and $\phi_f^{\mathbb{C}}$ to exhibit a compatibility between Knörrer periodicity (Theorem 1.1) and Bott periodicity. Set

$$Q := \mathbb{R}[x_1,\dots,x_n],\ Q' := \mathbb{R}[y_1,\dots,y_m]$$

and let

$$f \in (x_1,\dots,x_n) \subseteq Q,\ f' \in (y_1,\dots,y_m) \subseteq Q'$$

be quasi-homogeneous polynomials.

Remark 3.27. We are assuming f and f' are quasi-homogeneous so that the version of the Sebastiani-Thom homotopy equivalence for real polynomials is available to us (see Section 3.1.3). Analogous versions of every result in this section hold over $\mathbb{C}$ when both $f \in \mathbb{C}[x_1,\dots,x_n]$ and $f' \in \mathbb{C}[y_1,\dots,y_m]$ are either quasi-homogeneous or IHS.

We now construct the negative Milnor fibers of f and f'. Choose real numbers ϵ'', δ'', such that the map

$$B_{\epsilon''} \cap (f \oplus f')^{-1}((-\delta'',0)) \to (-\delta'',0)$$

given by $x \mapsto (f \oplus f')(x)$ is a locally trivial fibration. Similarly, choose ϵ, δ and ϵ', δ', as well as $t'' \in (-\delta'',0)$, so that the analogous maps

$$B_\epsilon \cap f^{-1}((-\delta,0)) \to (-\delta,0)$$

$$B_{\epsilon'} \cap (f')^{-1}((-\delta',0)) \to (-\delta',0)$$

are locally trivial fibrations, and also so that

(a) ϵ, ϵ' are sufficiently small so that $B_\epsilon \times B_{\epsilon'} \subseteq B_{\epsilon''}$.

(b) $|t''| < \min\{\delta, \delta'\}$.

Set F^-_f, $F^-_{f'}$, and $F^-_{f\oplus f'}$ to be the negative Milnor fibers of f, f', and $f \oplus f'$ over t''. Assume they are nonempty.

Remark 3.28. We could proceed using positive Milnor fibers as well, but we use negative fibers to stay consistent with Section 3.3.2.

Recall from Remark 2.12 that we have a map

$$K_0[\mathrm{MF}(Q,f)] \otimes K_0[\mathrm{MF}(Q',f')] \to K_0[\mathrm{MF}(Q \otimes_{\mathbb{R}} Q', f \oplus f')]$$

given by $[P]\otimes[P'] \mapsto [P \otimes_{\mathrm{MF}} P']$. The following proposition is the key technical result in this section.

Proposition 3.29. *There exists a map*

$$\mathrm{ST}_{L_1} : L_1(B_\epsilon, F^-_f) \otimes L_1(B_{\epsilon'}, F^-_{f'}) \to L_1(B_{\epsilon''}, F^-_{f\oplus f'})$$

such that, given matrix factorizations P and P' of f and f', respectively,

$$\mathrm{ST}_{L_1}(\phi^{\mathbb{R}}_f([P]) \otimes \phi^{\mathbb{R}}_{f'}([P'])) = \phi^{\mathbb{R}}_{f\oplus f'}([P \otimes_{\mathrm{MF}} P']).$$

Proof. Write

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0),\ P' = (P'_1 \underset{d'_0}{\overset{d'_1}{\rightleftarrows}} P'_0)$$

and

$$\Phi^{\mathbb{R}}_f(P) = [V_1, V_0; d_1|_{F^-_f}],\ \Phi^{\mathbb{R}}_{f'}(P') = [V'_1, V'_0; d'_1|_{F^-_{f'}}].$$

We note that

$$\phi^{\mathbb{R}}_{f\oplus f'}([P \otimes_{\mathrm{MF}} P']) = [(V_1 \otimes V'_0) \oplus (V_0 \otimes V'_1), (V_0 \otimes V'_0) \oplus (V_1 \otimes V'_1); A],$$

where A is the restriction of the matrix

$$\begin{pmatrix} d_1 \otimes \mathrm{id} & \mathrm{id} \otimes d'_1 \\ -\mathrm{id} \otimes d'_0 & d_0 \otimes \mathrm{id} \end{pmatrix}$$

to $F^-_{f\oplus f'}$.

As in Section 3.1.2, choose a continuous injection $H : CF^-_f \to B_\epsilon$ such that

- $H(x,1) = x \in F^-_f \subseteq B_\epsilon$,
- $H(-,s) : F^-_f \to B_\epsilon$ maps into the Milnor fiber $B_\epsilon \cap f^{-1}(st'')$ for $s \in (0,1)$, and
- $H(x,0) = 0$ for all $x \in F^-_f$

Choose $H' : CF^-_{f'} \to B_{\epsilon'}$ similarly. The maps of pairs

$$l : (CF^-_f, F^-_f) \to (B_\epsilon, F^-_f),\ l' : (CF^-_{f'}, F^-_{f'}) \to (B_{\epsilon'}, F^-_{f'})$$

induced by H and H' yield isomorphisms on L_1 upon pullback; this is immediate from the long exact sequence in KO-theory and the naturality of the map χ from Section 3.2 with respect to maps of pairs.
Recall from Section 3.2 that we have a map

$$L_1(CF_f^-, F_f^-) \otimes L_1(CF_{f'}^-, F_{f'}^-) \to L_1(CF_f^- \times CF_{f'}^-, CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-)$$

denoted by

$$[V] \otimes [V'] \mapsto [V] \otimes_{L_1} [V'].$$

Define

$$\mathrm{ST}_{L_1} : L_1(B_\epsilon, F_f^-) \otimes L_1(B_{\epsilon'}, F_{f'}^-) \to L_1(B_{\epsilon''}, F_{f\oplus f'}^-)$$

to be given by

$$[V] \otimes [V'] \mapsto (G^*)^{-1}(l^*([V]) \otimes_{L_1} (l')^*([V'])),$$

where

$$G : (CF_f^- \times CF_{f'}^-, CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-) \to (B_{\epsilon''}, F_{f\oplus f'}^-)$$

is as in Section 3.1.3. Recall that G is an explicit formulation of the Sebastiani-Thom homotopy equivalence.
We now compute $l^*(\phi_f^{\mathbb{R}}(P)) \otimes_{L_1} (l')^*(\phi_f^{\mathbb{R}}(P'))$ explicitly. A splitting of the restriction of

$$\begin{pmatrix} -\mathrm{id} \otimes (H')^*(d_1') \\ H^*(d_1) \otimes \mathrm{id} \end{pmatrix}$$

to $CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-$ is given, on the fiber over (x, s, y, s'), by

$$\frac{1}{f(H(x,s)) + f'(H'(y,s'))} \begin{pmatrix} -\mathrm{id} \otimes (H')^*(d_0') & H^*(d_0) \otimes \mathrm{id} \end{pmatrix}$$

(notice that $f(H(x,s)) + f'(H'(y,s')) = (s+s')t'' \neq 0$ when $(x,s,y,s') \in CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-$, since either s or s' is equal to 1). Thus, by the discussion at the end of Section 3.2, the product

$$l^*([V_1, V_0; d|_{F_f^-}]) \otimes_{L_1} (l')^*([V_1', V_0'; d'|_{F_{f'}^-}])$$

is equal to

$$[(W_1 \otimes W_0') \oplus (W_0 \otimes W_1'), (W_0 \otimes W_0') \oplus (W_1 \otimes W_1'); B],$$

where $W_i = H^*(V_i)$ and $W_i' = (H')^*(V_i')$ for $i = 0, 1$, and B is given, on the fiber over $(x, s, y, s') \in CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-$, by the matrix

$$\begin{pmatrix} H^*(d_1) \otimes \mathrm{id} & \mathrm{id} \otimes (H')^*(d_1') \\ \frac{1}{f(H(x,s))+f'(H'(y,s'))}(-\mathrm{id} \otimes (H')^*(d_0')) & \frac{1}{f(H(x,s))+f'(H'(y,s'))}(H^*(d_0) \otimes \mathrm{id}) \end{pmatrix}.$$

We wish to show that, upon applying $(G^*)^{-1}$ to this class, one obtains

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C],$$

where C is the restriction of the matrix

$$\begin{pmatrix} d_1 \otimes \mathrm{id} & \mathrm{id} \otimes d_1' \\ \frac{1}{t''}(-\mathrm{id} \otimes d_0') & \frac{1}{t''}(d_0 \otimes \mathrm{id}) \end{pmatrix}$$

to $F^-_{f \oplus f'}$. This will finish the proof, since the class

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C]$$

is clearly equal to

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); A].$$

Observe that we have an object

$$[((W_1 \otimes W_0') \oplus (W_0 \otimes W_1')) \times I, ((W_0 \otimes W_0') \oplus (W_1 \otimes W_1')) \times I; D]$$

in $\mathcal{C}_1(CF_f^- \times CF_{f'}^- \times I, (CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-) \times I)$, where D is given, on the fiber over

$$(x, s, y, s', T) \in (CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-) \times I,$$

by the matrix

$$\begin{pmatrix} H^*(d_1) \otimes \mathrm{id} & \mathrm{id} \otimes (H')^*(d_1') \\ \frac{1}{f(a(T))+f'(b(T))}(-\mathrm{id} \otimes (H')^*(d_0')) & \frac{1}{f(a(T))+f'(b(T))}(H^*(d_0) \otimes \mathrm{id}) \end{pmatrix}.$$

Here, f, f', and the entries of d_1, d_1', d_0, d_0' are evaluated at the point

$$(a(T), b(T)) := (H(x, \frac{T(1-s'-s)+2s}{2}), H'(y, \frac{T(1-s'-s)+2s'}{2})).$$

Notice that $f(a(T)) + f'(b(T)) \neq 0$ for all

$$(x, s, y, s', T) \in (CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-) \times I,$$

so this matrix is indeed an isomorphism on every fiber over $(CF_f^- \times F_{f'}^- \cup F_f^- \times CF_{f'}^-) \times I$.

Restricting to $T = 0$, one obtains the object

$$((W_1 \otimes W_0') \oplus (W_0 \otimes W_1'), (W_0 \otimes W_0') \oplus (W_1 \otimes W_1'); B).$$

Restricting to $T = 1$ and applying $(G^*)^{-1}$, one obtains

$$((V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C).$$

Now apply Proposition 9.2 in [ABS64]. □

Remark 3.30. It follows easily from the naturality of the map χ from Section 3.2 and Remark 3.16 that ST_{L_1} induces a map

$$\mathrm{ST}_{KO} : KO^0(B_\epsilon, F_f^-) \otimes KO^0(B_{\epsilon'}, F_{f'}^-) \to KO^0(B_{\epsilon''}, F_{f\oplus f'}^-).$$

Remark 3.31. We emphasize that the group homomorphism ST_{L_1} in Proposition 3.29 is given by the composition of the product in topological K-theory with the inverse of the pullback along a specific formulation of the Sebastiani-Thom homotopy equivalence. Hence, Proposition 3.29 yields a precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence.

Let us now consider the case where $Q' = \mathbb{R}[u_1, \dots, u_8]$ and $f' = -u_1^2 - \dots - u_8^2$. By Theorem 2.21 and Remark 2.18, $[\mathrm{MF}(Q', f')] \cong [\mathrm{MF}(\mathbb{R}, 0)]$. It follows that $K_0[\mathrm{MF}(\mathbb{R}[u_1, \dots, u_8], -u_1^2 - \dots - u_8^2)]$ is isomorphic to $\mathbb{Z}$, generated by the class represented by the matrix factorization X constructed in the proof of Theorem 2.21.

Also, $F^-_{-u_1^2-\dots-u_8^2}$ is homeomorphic to S^7, and so $L_1(B_{\epsilon'}, F^-_{-u_1^2-\dots-u_8^2})$ is isomorphic to $\mathbb{Z}$. This group is generated by $\phi^{\mathbb{R}}_{-u_1^2-\dots-u_8^2}(X)$; thus, $\phi^{\mathbb{R}}_{-u_1^2-\dots-u_8^2}([X])$ is a Bott element in the group $L_1(B_{\epsilon'}, F_{-u_1^2-\dots-u_8^2}) \cong \widetilde{KO}^0(S^8)$; we shall denote by $\beta_{\mathbb{R}}$ the map

$$KO^0(B_\epsilon, F_f^-) \to KO^0(B_\epsilon, F_f^-) \otimes KO^0(B_{\epsilon'}, F_{-u_1^2-\dots-u_8^2})$$

given by $(\chi \otimes \chi) \circ (- \otimes \phi^{\mathbb{R}}_{-u_1^2-\dots-u_8^2}([X])) \circ \chi^{-1}$. $\beta_{\mathbb{R}}$ is the Bott periodicity isomorphism.

Since real Knörrer periodicity may be induced by tensoring with the matrix factorization X, we will denote by $K_{\mathbb{R}}$ the map

$$K_0[\mathrm{MF}(Q, f)] \to K_0[\mathrm{MF}(Q[u_1, \dots, u_8], f - u_1^2 - \dots - u_8^2)]$$

given by $- \otimes_{\mathrm{MF}} [X]$.

The following result gives a precise sense in which Bott periodicity and Knörrer periodicity are compatible; it follows immediately from Proposition 3.29. We emphasize that a virtually identical proof yields a similar result involving positive Milnor fibers.

Theorem 3.32. *Let $f \in Q = \mathbb{R}[x_1, \dots, x_n]$ be a quasi-homogeneous polynomial such that $F_f^- \neq \emptyset$, and set $q = -u_1^2 - \dots - u_8^2$. Then the diagram*

$$\begin{tikzcd}
K_0[\mathrm{MF}(Q, f)] \arrow[r, "\chi\circ\phi_f^{\mathbb{R}}"] \arrow[dd, "K_{\mathbb{R}}"] & KO^0(B_\epsilon, F_f^-) \arrow[d, "\beta_{\mathbb{R}}"] \\
 & KO^0(B_\epsilon, F_f^-) \otimes KO^0(B_{\epsilon'}, F_q^-) \arrow[d, "\mathrm{ST}_{KO}"] \\
K_0[\mathrm{MF}(Q[u_1, \dots, u_8], f+q)] \arrow[r, "\chi\circ\phi_{f+q}^{\mathbb{R}}"] & KO^0(B_{\epsilon''}, F_{f+q}^-)
\end{tikzcd}$$

commutes.

We state the analogous version of this result over the complex numbers. Let Y denote the matrix factorization $\mathbb{C}[u,v] \underset{u-iv}{\overset{u+iv}{\rightleftarrows}} \mathbb{C}[u,v]$ of u^2+v^2, and let

$$K : K_0[\mathrm{MF}(Q,f)] \to K_0[\mathrm{MF}(Q[u,v], f+u^2+v^2)]$$

be given by $-\otimes_{\mathrm{MF}}[Y]$. $K_0[\mathrm{MF}(\mathbb{C}[u,v], u^2+v^2)] \cong \mathbb{Z}$, and the group is generated by $[Y]$. Also, by Theorem 3.2, $F_{u^2+v^2}$ is homotopy equivalent to S^1. Thus, $L_1(B_{\epsilon'}, F^-_{u^2+v^2})$ is isomorphic to $\mathbb{Z}$, generated by $\phi^{\mathbb{C}}_{u^2+v^2}([Y])$. $\phi^{\mathbb{C}}_{u^2+v^2}([Y])$ is a Bott element in the group $L_1(B_{\epsilon'}, F_{u^2+v^2}) \cong \widetilde{KU}^0(S^2)$; we shall denote by β the map

$$KU^0(B_\epsilon, F_f) \to KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2})$$

given by $(\chi \otimes \chi) \circ (- \otimes \phi^{\mathbb{C}}_{u^2+v^2}([Y])) \circ \chi^{-1}$. β is the complex Bott periodicity isomorphism. Let ST_{KU} denote the pairing on relative KU-theory induced by the complex version of the pairing ST_{L_1}. The following is a complex analogue of Theorem 3.32:

Theorem 3.33. *Let $f \in (x_1, \ldots, x_n) \subseteq Q = \mathbb{C}[x_1, \ldots, x_n]$ and suppose either*

- *The hypersurface $\mathbb{C}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/(f)$ is* IHS *(see Definition 2.9), or*
- *f is quasi-homogeneous.*

Then the diagram

$$\begin{tikzcd}
K_0[\mathrm{MF}(Q,f)] \arrow[r, "\chi\circ\phi^{\mathbb{C}}_f"] \arrow[dd, "K"] & KU^0(B_\epsilon, F_f) \arrow[d, "\beta"] \\
& KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2}) \arrow[d, "\mathrm{ST}_{KU}"] \\
K_0[\mathrm{MF}(Q[u,v], f+u^2+v^2)] \arrow[r, "\chi\circ\phi^{\mathbb{C}}_{f+u^2+v^2}"] & KU^0(B_{\epsilon''}, F_{f+u^2+v^2})
\end{tikzcd}$$

commutes.

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