p-adic L-Functions of Automorphic Forms and Exceptional Zeros

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Received: May 13, 2015

Communicated by Otmar Venjakob

Abstract. We construct p-adic L-functions for automorphic representations of GL_2 of a number field F, and show that the corresponding p-adic L-function of a modular elliptic curve E over F has an extra zero at the central point for each prime above p at which E has split multiplicative reduction, a part of the exceptional zero conjecture.

2010 Mathematics Subject Classification: 11F41, 11F67, 11F70, 11G40

Keywords and Phrases: p-adic L-function, automorphic forms, exceptional zero conjecture, Mazur-Tate-Teitelbaum conjecture

Introduction

Let F be a number field (with adele ring $\mathbb{A}_F$), and p a prime number. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Attached to π is the complex L-function $L(s,\pi)$, $s \in \mathbb{C}$, of Jacquet-Langlands [JL70]. Under certain conditions on π, we can also define a p-adic L-function $L_p(s,\pi)$ of π, with $s \in \mathbb{Z}_p$. It is related to $L(s,\pi)$ by the *interpolation property*: For every character $\chi : \mathcal{G}_p \to \mathbb{C}^*$ of finite order we have

$$L_p(0,\pi\otimes\chi) = \tau(\chi)\prod_{\mathfrak{p}|p} e(\pi_\mathfrak{p},\chi_\mathfrak{p})\cdot L(\tfrac{1}{2},\pi\otimes\chi),$$

where $e(\pi_\mathfrak{p},\chi_\mathfrak{p})$ is a certain Euler factor (see theorem 4.12 for its definition) and $\tau(\chi)$ is the Gauss sum of χ.

$L_p(s,\pi)$ was defined by Haran [Har87] in the case where π has trivial central character and $\pi_\mathfrak{p}$ is an ordinary spherical principal series representation for all $\mathfrak{p}|p$. For a totally real field F, Spieß [Sp14] has given a new construction of $L_p(s,\pi)$ that also allows for $\pi_\mathfrak{p}$ to be a special (Steinberg) representation for some $\mathfrak{p}|p$. In this article, we generalize Spieß' construction of $L_p(s,\pi)$ to

automorphic representations π of GL_2 over any number field, with arbitrary central character, and show that L_p has the conjectured number of exceptional zeros at the central point. We assume that π is ordinary at all primes $\mathfrak{p}|p$ (cf. definition 2.3), that π_v is discrete of weight 2 at all real infinite places v, and is the principal series representation $\sigma(|\cdot|^{1/2}, |\cdot|^{-1/2})$ at the complex places.
We define a p-adic measure μ_π, which heuristically is the image under the global reciprocity map of a product of certain local distributions $\mu_{\pi_\mathfrak{p}}$ on $F_\mathfrak{p}^*$ attached to $\pi_\mathfrak{p}$ for $\mathfrak{p}|p$ and a Whittaker function times the Haar measure on the group of p-ideles $\mathbb{I}^p = \prod'_{v\nmid p} F_v^*$.
Then we can define the p-adic L-function of π as an integral with respect to μ_π over the Galois group $\mathcal{G}_p$ of the maximal abelian extension that is unramified outside p and ∞; it is naturally a t-variable function, where t is the $\mathbb{Z}_p$-rank of $\mathcal{G}_p$:

$$L_p(\underline{s}, \pi) := L_p(s_1, \ldots, s_t, \pi) := \int_{\mathcal{G}_p} \prod_{i=1}^{t} \exp_p(s_i \ell_i(\gamma)) \mu_\pi(d\gamma)$$

for $s_1, \ldots, s_t \in \mathbb{Z}_p$, where the ℓ_i are $\mathbb{Z}_p$-valued homomorphisms corresponding to the t independent $\mathbb{Z}_p$-extensions of F (cf. section 4.7 for their definition).
For a modular elliptic curve E over F corresponding to π (i.e. the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places v of F), our construction allows us to define the p-adic L-function of E as $L_p(E, \underline{s}) := L_p(\underline{s}, \pi)$. The condition that π be ordinary at all $\mathfrak{p}|p$ means that E must have good ordinary or multiplicative reduction at all places $\mathfrak{p}|p$ of F.
The *exceptional zero conjecture* (formulated by Mazur, Tate and Teitelbaum [MTT86] for $F = \mathbb{Q}$, and by Hida [Hi09] for totally real F) states that

$$\mathrm{ord}_{s=0} L_p(E, s) \geq n, \tag{1}$$

where n is the number of $\mathfrak{p}|p$ at which E has split multiplicative reduction, and gives an explicit formula for the value of the n-th derivative $L_p^{(n)}(E, 0)$ as a multiple of certain L-invariants times $L(E, 1)$. The conjecture was proved in the case $F = \mathbb{Q}$ by Greenberg and Stevens [GS93] and independently by Kato, Kurihara and Tsuji, and for totally real fields F by Spieß [Sp14]. In this article, we prove (1) for all number fields F.

The structure of this article is as follows: In chapter 2, we describe the local distributions $\mu_{\pi_\mathfrak{p}}$ on $F_\mathfrak{p}^*$; they are the image of a Whittaker functional under a map δ on the dual of $\pi_\mathfrak{p}$. For constructing δ, we describe $\pi_\mathfrak{p}$ in terms of what we call the "Bruhat-Tits graph" of $F_\mathfrak{p}^2$: the directed graph whose vertices (resp. edges) are the lattices of $F_\mathfrak{p}^2$ (resp. inclusions between lattices). Roughly speaking, it is a covering of the (directed) Bruhat-Tits tree of $\mathrm{GL}_2(F_\mathfrak{p})$ with fibres $\cong \mathbb{Z}$. When $\pi_\mathfrak{p}$ is the Steinberg representation, $\mu_\mathfrak{p}$ can actually be extended to all of $F_\mathfrak{p}$.
In chapter 3, we attach a p-adic distribution μ_ϕ to any map $\phi(U, x^p)$ of an open compact subset $U \subseteq F_p^* := \prod_{\mathfrak{p}|p} F_\mathfrak{p}^*$ and an idele $x^p \in \mathbb{I}^p$ (satisfying certain

conditions). Integrating ϕ over all the infinite places, we get a cohomology class $\kappa_\phi \in H^d(F^{*\prime}, \mathcal{D}_f(\mathbb{C}))$ (where $d = r + s - 1$ is the rank of the group of units of F, $F^{*\prime} \cong F^*/\mu_F$ is a maximal torsion-free subgroup of F^*, and $\mathcal{D}_f(\mathbb{C})$ is a space of distributions on the finite ideles of F). We show that μ_ϕ can be described solely in terms of κ_ϕ, and μ_ϕ is a (vector-valued) p-adic measure if κ_ϕ is "integral", i.e. if it lies in the image of $H^d(F^{*\prime}, \mathcal{D}_f(R))$, for a Dedekind ring R consisting of "p-adic integers".
In chapter 4, we define a map ϕ_π by

$$\phi_\pi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}$$

($U \subseteq F_p^*$ compact open, $x^p \in \mathbb{I}^p$). ϕ_π satisfies the conditions of chapter 3, and we show that $\kappa_\pi := \kappa_{\phi_\pi}$ is integral by "lifting" the map $\phi_\pi \mapsto \kappa_\pi$ to a function mapping an automorphic form to a cohomology class in $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)$, for a certain space of functions $\mathcal{A}_f$. (Here $\mathrm{GL}_2(F)^+$ is the subgroup of $M \in \mathrm{GL}_2(F)$ with totally positive determinant.) For this, we associate to each automorphic form φ a harmonic form ω_φ on a generalized upper-half space $\mathcal{H}_\infty$, which we can integrate between any two cusps in $\mathbb{P}^1(F)$.
Then we can define the p-adic L-function $L_p(\underline{s}, \pi) := L_p(\underline{s}, \kappa_\pi)$ as above, with $\mu_\pi := \mu_{\phi_\pi}$. By a result of Harder [Ha87], $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)_\pi$ is one-dimensional, which implies that $L_p(\underline{s}, \pi)$ has values in a one-dimensional $\mathbb{C}_p$-vector space. Finally, we formulate an exceptional zero conjecture (conjecture 4.15) for all number fields F, and show that $L_p(\underline{s}, \pi)$ satisfies (1).

Acknowledgements

This paper is based on my Ph.D. thesis "p-adic L-functions of automorphic forms" [De13], submitted at Bielefeld University in August 2013.
I would like to thank Michael Spieß for suggesting and advising the thesis, and for many helpful discussions. I am also thankful to Werner Hoffmann for a useful discussion, to the referee for pointing out a few mistakes in an earlier version of this paper, and to the CRC 701, 'Spectral Structures and Topological Methods in Mathematics', for providing financial support during most of my studies.

CONTENTS

1 PRELIMINARIES

Let $\mathcal{X}$ be a totally disconnected locally compact topological space, R a topological Hausdorff ring. We denote by $C(\mathcal{X}, R)$ the ring of continuous maps $\mathcal{X} \to R$, and let $C_c(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)$ be the subring of compactly supported maps. When R has the discrete topology, we also write $C^0(\mathcal{X}, R) := C(\mathcal{X}, R)$, $C_c^0(\mathcal{X}, R) := C_c(\mathcal{X}, R)$.

We denote by $\mathfrak{Co}(\mathcal{X})$ the set of all compact open subsets of $\mathcal{X}$, and for an R-module M we denote by $\mathrm{Dist}(\mathcal{X}, M)$ the R-module of M-valued distributions on $\mathcal{X}$, i.e. the set of maps $\mu : \mathfrak{Co}(\mathcal{X}) \to M$ such that $\mu(\bigcup_{i=1}^n U_i) = \sum_{i=1}^n \mu(U_i)$ for any pairwise disjoint sets $U_i \in \mathfrak{Co}(\mathcal{X})$.

For an open set $H \subseteq \mathcal{X}$, we let $1_H \in C(\mathcal{X}, R)$ be the R-valued indicator function of H on $\mathcal{X}$.

Throughout this paper, we fix a prime p and embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\overline{\mathcal{O}}$ denote the valuation ring of $\overline{\mathbb{Q}}$ with respect to the p-adic valuation induced by ι_p.

We write $G := \mathrm{GL}_2$ throughout the article, and let B denote the Borel subgroup of upper triangular matrices, T the maximal torus (consisting of all diagonal matrices), and Z the center of G.
For a number field F, we let $G(F)^+ \subseteq G(F)$ and $B(F)^+ \subseteq B(F)$ denote the corresponding subgroups of matrices with totally positive determinant, i.e. $\sigma(\det(g))$ is positive for each real embedding $\sigma : F \hookrightarrow \mathbb{R}$. (If F is totally complex, this is an empty condition, so we have $G(F)^+ = G(F)$, $B(F)^+ = B(F)$ in this case.) Similarly, we define $G(\mathbb{R})^+$ and $G(\mathbb{C})^+ = G(\mathbb{C})$.

1.1 p-ADIC MEASURES

DEFINITION 1.1. Let $\mathcal{X}$ be a compact totally disconnected topological space. For a distribution $\mu : \mathfrak{Co}(\mathcal{X}) \to \mathbb{C}$, consider the extension of μ to the $\mathbb{C}_p$-linear map $C^0(\mathcal{X}, \mathbb{C}_p) \to \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, $f \mapsto \int f d\mu$. If its image is a finitely-generated $\mathbb{C}_p$-vector space, μ is called a *p-adic measure.*

We denote the space of p-adic measures on $\mathcal{X}$ by $\mathrm{Dist}^b(\mathcal{X}, \mathbb{C}) \subseteq \mathrm{Dist}(\mathcal{X}, \mathbb{C})$. It is easily seen that μ is a p-adic measure if and only if the image of μ, considered as a map $C^0(\mathcal{X}, \mathbb{Z}) \to \mathbb{C}$, is contained in a finitely generated $\overline{\mathcal{O}}$-module. A p-adic measure can be integrated against any continuous function $f \in C(\mathcal{X}, \mathbb{C}_p)$.

2 LOCAL RESULTS

For this chapter, let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ its ring of integers, ϖ its uniformizer and $\mathfrak{p} = (\varpi)$ the maximal ideal. Let q be the cardinality of $\mathcal{O}_F/\mathfrak{p}$, and set $U := U^{(0)} := \mathcal{O}_F^\times$, $U^{(n)} := 1 + \mathfrak{p}^n \subseteq U$ for $n \geq 1$.
We fix an additive character $\psi : F \to \overline{\mathbb{Q}}^*$ with $\ker\psi \supseteq \mathcal{O}_F$ and $\mathfrak{p}^{-1} \not\subseteq \ker\psi$. [1]
We let $|\cdot|$ be the absolute value on F^* (normalized by $|\varpi| = q^{-1}$), $\mathrm{ord} = \mathrm{ord}_\varpi$ the additive valuation, and dx the Haar measure on F normalized by $\int_{\mathcal{O}_F} dx = 1$. We define a (Haar) measure on F^* by $d^\times x := \frac{q}{q-1}\frac{dx}{|x|}$ (so $\int_{\mathcal{O}_F^\times} d^\times x = 1$).

2.1 GAUSS SUMS

Recall that the *conductor* of a character $\chi : F^* \to \mathbb{C}^*$ is by definition the largest ideal $\mathfrak{p}^n$, $n \geq 0$, such that $\ker\chi \supseteq U^{(n)}$, and that χ is *unramified* if its conductor is $\mathfrak{p}^0 = \mathcal{O}_F$.

DEFINITION 2.1. Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$. The *Gauss sum* of χ (with respect to ψ) is defined by

$$\tau(\chi) := [U : U^{(f)}] \int_{\varpi^{-f}U} \psi(x)\chi(x) d^\times x.$$

[1] Note that there is in general no ψ such that $\ker(\psi) = \mathcal{O}_F$, since $\mathfrak{p}^{-1}/\mathcal{O}_F$ has more than p points of order p if $F|\mathbb{Q}_p$ has inertia index > 1.

For a locally constant function $g : F^* \to \mathbb{C}$, we define

$$\int_{F^*} g(x)dx := \lim_{n\to\infty} \int_{x\in F^*, -n\leq \mathrm{ord}(x)\leq n} g(x)dx,$$

whenever that limit exists.

Lemma 2.2. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$. For $f = 0$, assume $|\chi(\varpi)| < q$. Then we have*

$$\int_{F^*} \chi(x)\psi(x)dx = \begin{cases} \frac{1-\chi(\varpi)^{-1}}{1-\chi(\varpi)q^{-1}} & \text{if } f = 0 \\ \tau(\chi) & \text{if } f > 0. \end{cases}$$

(Cf. [Sp14], lemma 3.4.)

2.2 Tamely ramified representations of $\mathrm{GL}_2(F)$

For an ideal $\mathfrak{a} \subset \mathcal{O}_F$, let $K_0(\mathfrak{a}) \subseteq G(\mathcal{O}_F)$ be the subgroup of matrices congruent to an upper triangular matrix modulo $\mathfrak{a}$.
Let $\pi : \mathrm{GL}_2(F) \to \mathrm{GL}(V)$ be an irreducible admissible infinite-dimensional representation on a $\mathbb{C}$-vector space V, with central quasicharacter χ. It is well-known (e.g [Ge75], Thm. 4.24) that there exists a maximal ideal $\mathfrak{c}(\pi) = \mathfrak{c} \subset \mathcal{O}_F$, the *conductor* of π, such that the space $V^{K_0(\mathfrak{c}),\chi} = \{v \in V | \pi(g)v = \chi(a)v \ \forall g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(\mathfrak{c})\}$ is non-zero (and in fact one-dimensional). A representation π is called *tamely ramified* if its conductor divides $\mathfrak{p}$.
If π is tamely ramified, then π is the spherical resp. special representation $\pi(\chi_1, \chi_2)$ (in the notation of [Ge75] or [Sp14]):
If the conductor is $\mathcal{O}_F$, π is (by definition) spherical and thus a principal series representation $\pi(\chi_1, \chi_2)$ for two unramified quasi-characters χ_1 and χ_2 with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$ ([Bu98], Thm. 4.6.4).
If the conductor is $\mathfrak{p}$, then $\pi = \pi(\chi_1, \chi_2)$ with $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.
For $\alpha \in \mathbb{C}^*$, we define a character $\chi_\alpha : F^* \to \mathbb{C}^*$ by $\chi_\alpha(x) := \alpha^{\mathrm{ord}(x)}$.
So let now $\pi = \pi(\chi_1, \chi_2)$ be a tamely ramified irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$; in the special case, we assume χ_1 and χ_2 to be ordered such that $\chi_1 = |\cdot|\chi_2$.
Set $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. (We also write $\pi = \pi_{\alpha_1,\alpha_2}$ sometimes.)
Set $a := \alpha_1 + \alpha_2$, $\nu := \alpha_1\alpha_2/q$. Define a distribution $\mu_{\alpha_1,\nu} := \mu_{\alpha_1/\nu} := \psi(x)\chi_{\alpha_1/\nu}(x)dx$ on F^*.
For later use, we will need the following condition on the α_i:

Definition 2.3. Let $\pi = \pi_{\alpha_1,\alpha_2}$ be tamely ramified. π is called *ordinary* if a and ν both lie in $\overline{\mathcal{O}}^*$ (i.e. they are p-adic units in $\overline{\mathbb{Q}}$). Equivalently, this means that either $\alpha_1 \in \overline{\mathcal{O}}^*$ and $\alpha_2 \in q\overline{\mathcal{O}}^*$, or vice versa.

Proposition 2.4. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasi-character with conductor $\mathfrak{p}^f$; for $f = 0$, assume $|\chi(\varpi)| < |\alpha_2|$. Then the integral $\int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx)$ converges* *and we have*

$$\int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx) = e(\alpha_1, \alpha_2, \chi)\tau(\chi)L(\tfrac{1}{2}, \pi \otimes \chi),$$

where

$$e(\alpha_1, \alpha_2, \chi) = \begin{cases} \frac{(1-\alpha_1\chi(\varpi)q^{-1})(1-\alpha_2\chi(\varpi)^{-1}q^{-1})(1-\alpha_2\chi(\varpi)q^{-1})}{(1-\chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ spherical,} \\ \frac{(1-\alpha_1\chi(\varpi)q^{-1})(1-\alpha_2\chi(\varpi)^{-1}q^{-1})}{(1-\chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ special,} \\ \left(\frac{\alpha_1}{\nu}\right)^{-f} = \left(\frac{\alpha_2}{q}\right)^{f}, & f > 0, \end{cases}$$

and where we assume the right-hand side to be continuously extended to the potential removable singularities at $\chi(\varpi) = q/\alpha_1$ or $= q/\alpha_2$.

Proof. This follows immediately from lemma 2.2 and the definition of the (Jacquet-Langlands) L-function. □

2.3 The Bruhat-Tits graph

Let $\tilde{\mathcal{V}}$ denote the set of lattices (i.e. submodules isomorphic to $\mathcal{O}_F^2$) in F^2, and let $\tilde{\mathcal{E}}$ be the set of all inclusion maps between two lattices; for such a map $e : v_1 \hookrightarrow v_2$ in $\tilde{\mathcal{E}}$, we define $o(e) := v_1$, $t(e) := v_2$. Then the pair $\tilde{\mathcal{T}} := (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ is naturally a directed graph, connected, with no directed cycles (specifically, $\tilde{\mathcal{E}}$ induces a partial ordering on $\tilde{\mathcal{V}}$). For each $v \in \tilde{\mathcal{V}}$, there are exactly $q+1$ edges beginning (resp. ending) in v, each.

Recall that the *Bruhat-Tits tree* $\mathcal{T} = (\mathcal{V}, \vec{\mathcal{E}})$ of $G(F)$ is the directed graph whose vertices are homothety classes of lattices of F^2 (i.e. $\mathcal{V} = \tilde{\mathcal{V}}/\sim$, where $v \sim \varpi^i v$ for all $i \in \mathbb{Z}$), and the directed edges $\overline{e} \in \vec{\mathcal{E}}$ are homothety classes of inclusions of lattices. We can define maps $o, t : \vec{\mathcal{E}} \to \mathcal{V}$ analogously. For each edge $\overline{e} \in \vec{\mathcal{E}}$, there is an opposite edge $\overline{e}' \in \vec{\mathcal{E}}$ with $o(\overline{e}') = t(\overline{e})$, $t(\overline{e}') = o(\overline{e})$; and the undirected graph underlying $\mathcal{T}$ is simply connected. We have a natural "projection map" $\pi : \tilde{\mathcal{T}} \to \mathcal{T}$, mapping each lattice and each homomorphism to its homothety class. Choosing a (set-theoretic) section $s : \mathcal{V} \to \tilde{\mathcal{V}}$, we get a bijection $\mathcal{V} \times \mathbb{Z} \xrightarrow{\cong} \tilde{\mathcal{V}}$ via $(v, i) \mapsto \varpi^i s(v)$.

The group $G(F)$ operates on $\tilde{\mathcal{V}}$ via its standard action on F^2, i.e. $gv = \{gx | x \in v\}$ for $g \in G(F)$, and on $\tilde{\mathcal{E}}$ by mapping $e : v_1 \to v_2$ to the inclusion map $ge : gv_1 \to gv_2$. The stabilizer of the standard vertex $v_0 := \mathcal{O}_F^2$ is $G(\mathcal{O}_F)$.

For a directed edge $\overline{e} \in \vec{\mathcal{E}}$ of the Bruhat-Tits tree $\mathcal{T}$, we define $U(\overline{e})$ to be the set of ends of $\overline{e}$ (cf. [Se80]/[Sp14]); it is a compact open subset of $\mathbb{P}^1(F)$, and we have $gU(\overline{e}) = U(g\overline{e})$ for all $g \in G(F)$.

For $n \in \mathbb{Z}$, we set $v_n := \mathcal{O}_F \oplus \mathfrak{p}^n \in \tilde{\mathcal{V}}$, and denote by e_n the edge from v_{n+1} to v_n; the "decreasing" sequence $(\pi(e_{-n}))_{n\in\mathbb{Z}}$ is the geodesic from ∞ to 0. (The geodesic from 0 to ∞ traverses the $\pi(v_n)$ in the natural order of $n \in \mathbb{Z}$.) We have $U(\pi(e_n)) = \mathfrak{p}^{-n}$ for each n.

On $\mathcal{T}$, we have the height function $h : \mathcal{V} \to \mathbb{Z}$ (cf. [BL95]) defined as follows: The geodesic ray from $v \in \mathcal{V}$ to ∞ must contain some $\pi(v_n)$ ($n \in \mathbb{Z}$), since it has non-empty intersection with $A := \{\pi(v_n)|n \in \mathbb{Z}\}$; we define $h(v) := n - d(v, \pi(v_n))$ for any such v_n. This is easily seen to be well-defined, and satisfies $h(\pi(v_n)) = n$ for all $n \in \mathbb{Z}$. We have the following lemma:

LEMMA 2.5. *(a) For all $\overline{e} \in \mathcal{E}$, we have*

$$h(t(\overline{e})) = \begin{cases} h(o(\overline{e})) + 1 & \textit{if } \infty \in U(\overline{e}), \\ h(o(\overline{e})) - 1 & \textit{otherwise.} \end{cases}$$

(b) For $a \in F^*$, $b \in F$, $\overline{v} \in \mathcal{V}$ we have

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \overline{v}\right) = h(\overline{v}) - \mathrm{ord}_\varpi(a).$$

(Cf. [Sp14], lemma 3.6)
Let R be a ring, M an R-module. We let $C(\tilde{\mathcal{V}}, M)$ be the R-module of maps $\phi : \tilde{\mathcal{V}} \to M$, and $C(\tilde{\mathcal{E}}, M)$ the R-module of maps $\tilde{\mathcal{E}} \to M$. Both are $G(F)$-modules via $(g\phi)(v) := \phi(g^{-1}v)$, $(gc)(e) := c(g^{-1}e)$.
We let $\mathcal{C}_c(\tilde{\mathcal{V}}, M) \subseteq C(\tilde{\mathcal{V}}, M)$ and $\mathcal{C}_c(\tilde{\mathcal{E}}, M) \subseteq C(\tilde{\mathcal{E}}, M)$ be the ($G(F)$-stable) submodules of maps with compact support, i.e. maps that are zero outside a finite set. We get pairings

$$\langle \text{-}, \text{-} \rangle : C_c(\tilde{\mathcal{V}}, R) \times C(\tilde{\mathcal{V}}, M) \to M, \quad \langle \phi_1, \phi_2 \rangle := \sum_{v \in \tilde{\mathcal{V}}} \phi_1(v)\phi_2(v) \tag{2}$$

and

$$\langle \text{-}, \text{-} \rangle : C_c(\tilde{\mathcal{E}}, R) \times C(\tilde{\mathcal{E}}, M) \to M, \quad \langle c_1, c_2 \rangle := \sum_{e \in \tilde{\mathcal{E}}} c_1(v)c_2(v). \tag{3}$$

We define Hecke operators $T, N : \mathcal{C}(\tilde{\mathcal{V}}, M) \to \mathcal{C}(\tilde{\mathcal{V}}, M)$ by

$$T\phi(v) = \sum_{t(e)=v} \phi(o(e)) \quad \text{and} \quad N\phi := \varpi\phi \text{ (i.e. } N\phi(v) = \phi(\varpi^{-1}v))$$

for all $v \in \tilde{\mathcal{V}}$. These restrict to operators on $C_c(\tilde{\mathcal{V}}, R)$, which we sometimes denote by T_c and N_c for emphasis. With respect to (2), T_c is adjoint to TN, and N_c is adjoint to its inverse operator $N^{-1} : \mathcal{C}_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R)$.

T and N obviously commute, and we have the following Hecke structure theorem on compactly supported functions on $\tilde{\mathcal{V}}$ (an analogue of [BL95], Thm. 10):

THEOREM 2.6. *$C_c(\tilde{\mathcal{V}}, R)$ is a free $R[T, N^{\pm 1}]$-module (where $R[T, N^{\pm 1}]$ is the ring of Laurent polynomials in N over the polynomial ring $R[T]$, with N and T commuting).*

Proof. Fix a vertex $v_0 \in \tilde{\mathcal{V}}$. For each $n \geq 0$, let C_n be the set of vertices $v \in \tilde{\mathcal{V}}$ such that there is a directed path of length n from v_0 to v in $\tilde{\mathcal{V}}$, and such that $d(\pi(v_0), \pi(v)) = n$ in the Bruhat-Tits tree $\mathcal{T}$. So $C_0 = \{v_0\}$, and C_n is a lift of the "circle of radius n around v_0" in $\mathcal{T}$, in the parlance of [BL95].
One easily sees that $\bigcup_{n=0}^{\infty} C_n$ is a complete set of representatives for the projection map $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$; specifically, for $n > 1$ and a given $v \in C_{n-1}$, C_n contains exactly q elements adjacent to v in $\tilde{\mathcal{V}}$; and we can write $\tilde{\mathcal{V}}$ as a disjoint union $\bigcup_{j\in\mathbb{Z}} \bigcup_{n=0}^{\infty} N^j(C_n)$.
We further define $V_0 := \{v_0\}$ and choose subsets $V_n \subseteq C_n$ as follows: We let V_1 be any subset of cardinality q. For $n > 1$, we choose $q-1$ out of the q elements of C_n adjacent to v', for every $v' \in C_{n-1}$, and let V_n be the union of these elements for all $v' \in C_{n-1}$. Finally, we set

$$H_{n,j} := \{\phi \in C_c(\tilde{\mathcal{V}}, R) |\, \mathrm{Supp}(\phi) \subseteq \bigcup_{i=0}^{n} N^j(C_i)\} \quad \text{for each } n \geq 0, j \in \mathbb{Z},$$

$H_n := \bigcup_{j\in\mathbb{Z}} H_{n,j}$, and $H_{-1} := H_{-1,j} := \{0\}$. (For ease of notation, we identify $v \in \tilde{\mathcal{V}}$ with its indicator function $1_{\{v\}} \in C_c(\tilde{\mathcal{V}}, R)$ in this proof.)
Define $T' : C_c(\tilde{\mathcal{V}}, R) \to C_c(\tilde{\mathcal{V}}, R)$ by

$$T'(\phi)(v) := \sum_{\substack{t(e)=(v),\\ o(e)\in N^j(C_n)}} \phi(o(e)) \qquad \text{for each } v \in N^j(C_{n-1}), j \in \mathbb{Z};$$

T' can be seen as the "restriction to one layer" $\bigcup_{n=0}^{\infty} N^j(C_n)$ of T. We have $T'(v) \equiv T(v) \mod H_{n-1}$ for each $v \in H_n$, since the "missing summand" of T' lies in H_{n-1}.
We claim that for each $n \geq 0$, the set $X_{n,j} := \bigcup_{i=0}^{n} N^j T^{n-i}(V_i)$ is an R-basis for $H_{n,j}/H_{n-1,j}$. By the above congruence, we can replace T by T' in the definition of $X_{n,j}$.
The claim is clear for $n = 0$. So let $n \geq 1$, and assume the claim to be true for all $n' \leq n$. For each $v \in C_{n-1}$, the q points in C_n adjacent to v are generated by the $q-1$ of these points lying in V_n, plus $T'v$ (which just sums up these q points). By induction hypothesis, v is generated by $X_{n-1,0}$, and thus (taking the union over all v), C_n is generated by $T'(X_{n-1,0}) \cup V_n = X_{n,0}$. Since the cardinality of $X_{n,0}$ equals the R-rank of $H_{n,0}/H_{n-1,0}$ (both are equal to $(q+1)q^{n-1}$), $X_{n,0}$ is in fact an R-basis.
Analoguously, we see that $H_{n,j}/H_{n-1,j}$ has $N^j(X_{n,0}) = X_{n,j}$ as a basis, for each $j \in \mathbb{Z}$.
From the claim, it follows that $\bigcup_{j\in\mathbb{Z}} X_{n,j}$ is an R-basis of H_n/H_{n-1} for each n, and that $V := \bigcup_{n=0}^{\infty} V_n$ is an $R[T, N^{\pm 1}]$-basis of $C_c(\tilde{\mathcal{V}}, R)$.

□

For $a \in R$ and $\nu \in R^*$, we let $\tilde{\mathcal{B}}_{a,\nu}(F, R)$ be the "common cokernel" of $T - a$ and $N - \nu$ in $C_c(\tilde{\mathcal{V}}, R)$, namely $\tilde{\mathcal{B}}_{a,\nu}(F, R) := C_c(\tilde{\mathcal{V}}, R)/(\mathrm{Im}(T-a) + \mathrm{Im}(N-\nu))$;

dually, we define $\tilde{\mathcal{B}}^{a,\nu}(F,M) := \ker(T-a) \cap \ker(N-\nu) \subseteq C(\tilde{\mathcal{V}},M)$.

For a lattice $v \in \tilde{\mathcal{V}}$, we define a valuation ord_v on F as follows: For $w \in F^2$, the set $\{x \in F | xw \in v\}$ is some fractional ideal $\varpi^m \mathcal{O}_F \subseteq F$ ($m \in \mathbb{Z}$); we set $\mathrm{ord}_v(w) := m$. This map can also be given explicitly as follows: Let λ_1, λ_2 be a basis of v. We can write any $w \in F^2$ as $w = x_1\lambda_1 + x_2\lambda_2$; then we have $\mathrm{ord}_v(w) = \min\{\mathrm{ord}_\varpi(x_1), \mathrm{ord}_\varpi(x_2)\}$. This gives a "valuation" map on F^2, as one easily checks. We restrict it to $F \cong F \times \{0\} \hookrightarrow F^2$ to get a valuation ord_v on F, and consider especially the value at $e_1 := (1,0)$.

Lemma 2.7. *Let $\alpha, \nu \in R^*$, and put $a := \alpha + q\nu/\alpha$. Define a map $\varrho = \varrho_{\alpha,\nu} : \tilde{\mathcal{V}} \to R$ by $\varrho(v) := \alpha^{h(\pi(v))}\nu^{-\mathrm{ord}_v(e_1)}$. Then $\varrho \in \tilde{\mathcal{B}}^{a,\nu}(F,R)$.*

Proof. One easily sees that $\left(v \mapsto \nu^{-\mathrm{ord}_v(e_1)}\right) \in \ker(N-\nu)$. It remains to show that $\varrho \in \ker(T-a)$:
We have the Iwasawa decomposition $G(F) = B(F)G(\mathcal{O}_F) = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}Z(F)G(\mathcal{O}_F)$; thus every vertex in $\tilde{\mathcal{V}}$ can be written as $\varpi^i v$ with $v = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) v_0$, with $i \in \mathbb{Z}$, $a \in F^*, b \in F$.
Now the lattice $v = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) v_0$ is generated by the vectors $\lambda_1 = \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$ and $\lambda_2 = \left(\begin{smallmatrix} b \\ 1 \end{smallmatrix}\right) \in \mathcal{O}_F^2$, so $e_1 = a^{-1}\lambda_1$ and thus $\mathrm{ord}_v(e_1) = \mathrm{ord}_\varpi(a^{-1}) = -\mathrm{ord}_\varpi(a)$. The $q+1$ neighbouring vertices v' for which there exists an $e \in \tilde{\mathcal{E}}$ with $o(e) = v', t(e) = v$ are given by $N_i v$, $i \in \{\infty\} \cup \mathcal{O}_F/\mathfrak{p}$, with $N_\infty := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, and $N_i := \left(\begin{smallmatrix} \varpi & i \\ 0 & 1 \end{smallmatrix}\right)$ where $i \in \mathcal{O}_F$ runs through a complete set of representatives mod ϖ. By lemma 2.5, $h(\pi(N_\infty v)) = h(\pi(v)) + 1$ and $h(\pi(N_i v)) = h(\pi(v)) - 1$ for $i \neq \infty$. By considering the basis $\{N_i\lambda_1, N_i\lambda_2\}$ of $N_i v$ for each N_i, we see that $\mathrm{ord}_{N_\infty v}(e_1) = \mathrm{ord}_v(e_1)$ and $\mathrm{ord}_{N_i v}(e_1) = \mathrm{ord}_v(e_1) - 1$ for $i \neq \infty$. Thus we have

$$\begin{aligned}
(T\varrho)(v) &= \sum_{t(e)=v} \alpha^{h(\pi(o(e)))}\nu^{-\mathrm{ord}_{o(e)}(e_1)} \\
&= \alpha^{h(\pi(v))+1}\nu^{-\mathrm{ord}_v e_1} + q \cdot \alpha^{h(\pi(v))-1}\nu^{1-\mathrm{ord}_v(e_1)} \\
&= (\alpha + q\alpha^{-1}\nu)\alpha^{h(\pi(v))}\nu^{-\mathrm{ord}_v e_1} = a\varrho(v),
\end{aligned}$$

and also $(T\varrho)(\varpi^i v) = (TN^{-i}\varrho)(v) = N^{-i}(a\varrho)(v) = a\varrho(\varpi^i v)$ for a general $\varpi^i v \in \tilde{\mathcal{V}}$, which shows that $\varrho \in \ker(T-a)$. $\square$

If $a^2 \neq \nu(q+1)^2$ (the "spherical case"), we put $\mathcal{B}_{a,\nu}(F,R) := \tilde{\mathcal{B}}_{a,\nu}(F,R)$ and $\mathcal{B}^{a,\nu}(F,M) := \tilde{\mathcal{B}}^{a,\nu}(F,M)$.

In the "special case" $a^2 = \nu(q+1)^2$, we need to assume that the polynomial $X^2 - a\nu X + q\nu^{-1} \in R[X]$ has a zero $\alpha' \in R$. Then the map $\varrho := \varrho_{\alpha',\nu} \in C(\tilde{\mathcal{V}},R)$ defined as above lies in $\tilde{\mathcal{B}}^{a\nu,\nu^{-1}}(F,R) = \ker(TN-a) \cap \ker(N^{-1}-\nu)$ by Lemma 2.7, since $a\nu = \alpha' + q\nu^{-1}/\alpha'$. In other words, the kernel of the map

$\langle\cdot,\varrho\rangle : C_c(\tilde{\mathcal{V}},R)\to R$ contains $\mathrm{Im}(T-a)+\mathrm{Im}(N-\nu)$; and we define

$$\mathcal{B}_{a,\nu}(F,R) := \ker\left(\langle\cdot,\varrho\rangle\right) / \left(\mathrm{Im}(T-a)+\mathrm{Im}(N-\nu)\right)$$

to be the quotient; evidently, it is an R-submodule of codimension 1 of $\tilde{\mathcal{B}}_{a,\nu}(F,R)$. Dually, $T-a$ and $N-\nu$ both map the submodule $\varrho M=\{\varrho\cdot m, m\in M\}$ of $C(\tilde{\mathcal{V}},M)$ to zero and thus induce endomorphisms on $C(\tilde{\mathcal{V}},M)/\varrho M$; we define $\mathcal{B}^{a,\nu}(F,M)$ to be the intersection of their kernels.
In the special case, since $\nu=\alpha^2$, Lemma 2.7 states that $\varrho(gv_0) = \chi_\alpha(ad)\varrho(v_0) = \chi_\alpha(\det g)\varrho(v_0)$ for all $g=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in B(F)$, and thus for all $g\in G(F)$ by the Iwasawa decomposition, since $G(\mathcal{O}_F)$ fixes v_0 and lies in the kernel of $\chi_\alpha\circ\det$. By the multiplicity of det, we have $(g^{-1}\varrho)(v) = \varrho(gv) = \chi_\alpha(\det g)\varrho(v)$ for all $g\in G(F)$, $v\in\tilde{\mathcal{V}}$. So $\phi\in\ker\langle\cdot,\varrho\rangle$ implies $\langle g\phi,\varrho\rangle = \langle\phi, g^{-1}\varrho\rangle = \chi_\alpha(\det g)\langle\phi,\varrho\rangle = 0$, i.e. $\ker\langle\cdot,\varrho\rangle$ and thus $\mathcal{B}_{a,\nu}(F,R)$ are $G(F)$-modules.
By the adjointness properties of the Hecke operators T and N, we have pairings $\mathrm{coker}(T_c-a)\times\ker(TN-a)\to M$ and $\mathrm{coker}(N_c-\nu)\times\ker(N^{-1}-\nu)\to M$, which "combine" to give a pairing

$$\langle\text{-}\,,\text{-}\rangle : \mathcal{B}_{a,\nu}(F,R)\times\mathcal{B}^{a\nu,\nu^{-1}}(F,M)\to M$$

(since $\ker(TN-a)\cap\ker(N^{-1}-\nu) = \ker(T-a\nu)\cap\ker(N-\nu^{-1})$), and a corresponding isomorphism $\mathcal{B}^{a\nu,\nu^{-1}}(F,M)\xrightarrow{\cong}\mathrm{Hom}(\mathcal{B}_{a,\nu}(F,R),M)$.

Definition 2.8. Let G be a totally disconnected locally compact group, $H\subseteq G$ an open subgroup. For a smooth $R[H]$-module M, we define the *(compactly) induced G-representation* of M, denoted $\mathrm{Ind}_H^G M$, to be the space of maps $f: G\to M$ such that $f(hg)=f(g)$ for all $g\in G, h\in H$, and such that f has compact support modulo H. We let G act on $\mathrm{Ind}_H^G M$ via $g\cdot f(x) := f(xg)$. (We can also write $\mathrm{Ind}_H^G M = R[G]\otimes_{R[H]} M$, cf. [Br82], III.5.)
We further define $\mathrm{Coind}_H^G M := \mathrm{Hom}_{R[H]}(R[G],M)$. Finally, for an $R[G]$-module N, we write $\mathrm{res}_H^G N$ for its underlying $R[H]$-module ("restriction").

By Theorem 2.6, T_c-a (as well as $N_c-\nu$) is injective, and the induced map

$$N_c-\nu : \mathrm{coker}(T_c-a) = C_c(\tilde{\mathcal{V}},R)/\mathrm{Im}(T_c-a)\to\mathrm{coker}(T_c-a)$$

(of $R[T,N^{\pm1}]/(T-a) = R[N^{\pm1}]$-modules) is also injective. Now since $G(F)$ acts transitively on $\tilde{\mathcal{V}}$, with the stabilizer of $v_0 := \mathcal{O}_F^2$ being $K := G(\mathcal{O}_F)$, we have an isomorphism $C_c(\tilde{\mathcal{V}},R)\cong\mathrm{Ind}_K^{G(F)} R$. Thus we have exact sequences

$$0\to\mathrm{Ind}_K^{G(F)} R\xrightarrow{T-a}\mathrm{Ind}_K^{G(F)} R\to\mathrm{coker}(T_c-a)\to 0 \tag{4}$$

and (for a,ν in the spherical case)

$$0\to\mathrm{coker}(T_c-a)\xrightarrow{N-\nu}\mathrm{coker}(T_c-a)\to\mathcal{B}_{a,\nu}(F,R)\to 0, \tag{5}$$

with all entries being free R-modules. Applying $\mathrm{Hom}_R(\cdot,M)$ to them, we get:

Lemma 2.9. *We have exact sequences of R-modules*

$$0 \to \ker(TN-a) \to \operatorname{Coind}_K^{G(F)} M \xrightarrow{T-a} \operatorname{Coind}_K^{G(F)} M \to 0$$

and, if $\mathcal{B}_{a,\nu}(F,M)$ is spherical (i.e. $a^2 \neq \nu(q+1)^2$),

$$0 \to \mathcal{B}^{a\nu,\nu^{-1}}(F,M) \to \ker(TN-a) \xrightarrow{N-\nu} \ker(TN-a) \to 0.$$

For the special case, we have to work a bit more to get similar exact sequences: By [Sp14], eq. (22), for the representation $St^-(F,R) := \mathcal{B}_{-(q+1),1}(F,R)$ (i.e. $\nu = 1$, $\alpha = -1$) with trivial central character, we have an exact sequence of G-modules

$$0 \to \operatorname{Ind}_{KZ}^G R \to \operatorname{Ind}_{K'Z}^G R \to St^-(F,R) \to 0, \tag{6}$$

where $K' = \langle W \rangle K_0(\mathfrak{p})$ is the subgroup of KZ generated by $W := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ and the subgroup $K_0(\mathfrak{p}) \subseteq K$ of matrices that are upper-triangular modulo $\mathfrak{p}$. (Since $W^2 \in Z$, $K_0(\mathfrak{p})Z$ is a subgroup of K' of order 2.) Now aany special representation (π, V) can be written as $\pi = \chi \otimes St^-$ for some character $\chi = \chi_Z$ (cf. the proof of lemma 2.12 below), and is obviously G-isomorphic to the representation $\pi \otimes (\chi \circ \det)$ acting on the space $V \otimes_R R(\chi \circ \det)$, where $R(\chi \circ \det)$ is the ring R with G-module structure given via $gr = \chi(\det(g))r$ for $g \in G, r \in R$. Tensoring (6) with $R(\chi \circ \det)$ over R gives an exact sequence of G-modules

$$0 \to \operatorname{Ind}_{KZ}^G \chi \to \operatorname{Ind}_{K'Z}^G \chi \to V \to 0. \tag{7}$$

It is easily seen that $R(\chi \circ \det)$ fits into another exact sequence of G-modules

$$0 \to \operatorname{Ind}_H^G R \xrightarrow{\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) - \chi(\varpi)\,\mathrm{id}} \operatorname{Ind}_H^G R \xrightarrow{\psi} R(\chi \circ \det) \to 0,$$

where $H := \{g \in G | \det g \in \mathcal{O}_F^\times\}$ is a normal subgroup containing K, $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)(f)(g) := f(\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} g)$ for $f \in \operatorname{Ind}_H^G R = \{f : G \to R | f(Hg) = f(g)$ for all $g \in G\}$, $g \in G$, is the natural operation of G, and where ψ is the G-equivariant map defined by $1_U \mapsto 1$.

Now since $H \subseteq G$ is a normal subgroup, we have $\operatorname{Ind}_H^G R \cong R[G/H]$ as G-modules (in fact $G/H \cong \mathbb{Z}$ as an abstract group). Let $X \subseteq G$ be a subgroup such that the natural inclusion $X/(X \cap H) \hookrightarrow G/H$ has finite cokernel; let $g_i H$, $i = 1, \dots n$ be a set of representatives of that cokernel. Then we have a (non-canonical) X-isomorphism $\bigoplus_{i=0}^n \operatorname{Ind}_{X \cap H}^X \to \operatorname{Ind}_H^G R$ defined via $(1_{(X\cap H)x})_i \mapsto 1_{Hxg_i}$ for each $i = 1, \dots, n$ (cf. [Br82], III (5.4)).

Using this isomorphism and the "tensor identity" $\operatorname{Ind}_H^G M \otimes N \cong \operatorname{Ind}_H^G(M \otimes \operatorname{res}_H^G N)$ for any groups $H \subseteq G$, H-module M and G-module N ([Br82] III.5, Ex. 2), we have

$$\begin{aligned} \operatorname{Ind}_{KZ}^G R \otimes_R \operatorname{Ind}_H^G R &\cong \operatorname{Ind}_{KZ}^G(\operatorname{res}_{KZ}^G(\operatorname{Ind}_H^G R)) \\ &= \operatorname{Ind}_{KZ}^G((\operatorname{Ind}_{KZ\cap H}^{KZ} R)^2) \\ &= (\operatorname{Ind}_{KZ}^G(\operatorname{Ind}_K^{KZ} R))^2 = (\operatorname{Ind}_K^G R)^2 \end{aligned}$$

(since $KZ/KZ \cap H \hookrightarrow G/H$ has index 2), and similarly

$$\operatorname{Ind}_{K'Z}^{G} R \otimes_R \operatorname{Ind}_{H}^{G} R \cong (\operatorname{Ind}_{K'}^{G} R)^2.$$

Thus, we can resolve the first and second term of (7) into exact sequences

$$0 \to (\operatorname{Ind}_{K}^{G} R)^2 \to (\operatorname{Ind}_{K}^{G} R)^2 \to \operatorname{Ind}_{KZ}^{G} \chi \to 0,$$

$$0 \to (\operatorname{Ind}_{K'}^{G} R)^2 \to (\operatorname{Ind}_{K'}^{G} R)^2 \to \operatorname{Ind}_{\langle W\rangle K_0(\mathfrak{p})Z}^{G} \chi \to 0.$$

Dualizing (7) and these by taking $\operatorname{Hom}(\cdot, M)$ for an R-module M, we get a "resolution" of $\mathcal{B}^{a\nu,\nu^{-1}}(F, M)$ in terms of coinduced modules:

Lemma 2.10. *We have exact sequences*

$$\begin{aligned}
&0 \to \mathcal{B}^{a\nu,\nu^{-1}}(F, M) \to \operatorname{Coind}_{K'Z}^{G} M(\chi) \to \operatorname{Coind}_{KZ}^{G} M(\chi) \to 0,\\
&0 \to \operatorname{Coind}_{KZ}^{G} M(\chi) \to (\operatorname{Coind}_{K}^{G} R)^2 \to (\operatorname{Coind}_{K}^{G} R)^2 \to 0,\\
&0 \to \operatorname{Coind}_{K'Z}^{G} M(\chi) \to (\operatorname{Coind}_{K'}^{G} R)^2 \to (\operatorname{Coind}_{K'}^{G} R)^2 \to 0
\end{aligned}$$

for all special $\mathcal{B}_{a,\nu}(F, R)$ (i.e. $a^2 = \nu(q+1)^2$), where $\chi = \chi_Z$ is the central character.

It is easily seen that the above arguments could be modified to get a similar set of exact sequences in the spherical case as well (replacing K' by K everywhere), in addition to that given in lemma 2.9; but we will not need this.

2.4 Distributions on the Bruhat-Tits graph

For $\varrho \in C(\tilde{\mathcal{V}}, R)$ we define R-linear maps

$$\tilde{\delta}_\varrho : C(\tilde{\mathcal{E}}, M) \to C(\tilde{\mathcal{V}}, M), \quad \tilde{\delta}_\varrho(c)(v) := \sum_{v=t(e)} \varrho(o(e))c(e) - \sum_{v=o(e)} \varrho(t(e))c(e),$$

$$\tilde{\delta}^\varrho : C(\tilde{\mathcal{V}}, M) \to C(\tilde{\mathcal{E}}, M), \quad \tilde{\delta}^\varrho(\phi)(e) := \varrho(o(e))\phi(t(e)) - \varrho(t(e))\phi(o(e)).$$

One easily checks that these are adjoint with respect to the pairings (2) and (3), i.e. we have $\langle \tilde{\delta}_\varrho(c), \phi\rangle = \langle c, \tilde{\delta}^\varrho(\phi)\rangle$ for all $c \in C_c(\tilde{\mathcal{E}}, R)$, $\phi \in C(\tilde{\mathcal{V}}, M)$. We denote the maps corresponding to $\varrho \equiv 1$ by $\delta := \tilde{\delta}_1$, $\delta^* := \tilde{\delta}^1$.
For each ϱ, the map $\tilde{\delta}_\varrho$ fits into an exact sequence

$$C_c(\tilde{\mathcal{E}}, R) \xrightarrow{\tilde{\delta}_\varrho} C_c(\tilde{\mathcal{V}}, R) \xrightarrow{\langle \cdot, \varrho\rangle} R \to 0$$

but it is not injective in general: e.g. for $\varrho \equiv 1$, the map $\tilde{\mathcal{E}} \to R$ symbolized by

$$\begin{tikzcd}
\cdot \arrow[r, "-1"] \arrow[d, "1"] & \cdot \arrow[d, "-1"] \\
\cdot \arrow[r, "1"] & \cdot
\end{tikzcd}$$

(and zero outside the square) lies in $\ker\delta$.
The restriction $\delta^*|_{C_c(\tilde{\mathcal{V}},R)}$ to compactly supported maps is injective since $\tilde{\mathcal{T}}$ has no directed circles, and we have a surjective map

$$\operatorname{coker}\left(\delta^* : C_c(\tilde{\mathcal{V}},R) \to C_c(\tilde{\mathcal{E}},R)\right) \to C^0(\mathbb{P}^1(F),R)/R, \quad c \mapsto \sum_{e\in\tilde{\mathcal{E}}} c(e)1_{U(\pi(e))}$$

(which corresponds to an isomorphism of the similar map on the Bruhat-Tits tree $\mathcal{T}$). Its kernel is generated by the functions $1_{\{e\}} - 1_{\{e'\}}$ for $e, e' \in \tilde{\mathcal{E}}$ with $\pi(e) = \pi(e')$.
For $\varrho_1, \varrho_2 \in C(\tilde{\mathcal{V}},R)$ and $\phi \in C(\tilde{\mathcal{V}},M)$ it is easily checked that

$$\left(\tilde{\delta}_{\varrho_1} \circ \tilde{\delta}^{\varrho_2}\right)(\phi) = (T+TN)(\varrho_1\cdot\varrho_2)\cdot\phi - \varrho_2\cdot(T+TN)(\varrho_1\cdot\phi).$$

For $a' \in R$ and $\varrho \in \ker(T+TN-a')$, applying this equality for $\varrho_1 = \varrho$ and $\varrho_2 = 1$ shows that $\tilde{\delta}_\varrho$ maps $\operatorname{Im}\delta^*$ into $\operatorname{Im}(T+TN-a')$, so we get an R-linear map

$$\tilde{\delta}_\varrho : \operatorname{coker}\left(\delta^* : C_c(\tilde{\mathcal{V}},R) \to C_c(\tilde{\mathcal{E}},R)\right) \to \operatorname{coker}(T_c + T_cN_c - a').$$

Let now again $\alpha, \nu \in R^*$, and $a := \alpha + q\nu/\alpha$. We let $\varrho := \varrho_{\alpha,\nu} \in \tilde{\mathcal{B}}^{a,\nu}(F,R)$ as defined in lemma 2.7, and write $\tilde{\delta}_{\alpha,\nu} := \tilde{\delta}_\varrho$. Since $\tilde{\delta}_{\alpha,\nu}$ maps $1_{\{e\}} - 1_{\{\varpi e\}}$ into $\operatorname{Im}(R-\nu)$, it induces a map

$$\tilde{\delta}_{\alpha,\nu} : C^0(\mathbb{P}^1(F),R)/R \to \mathcal{B}_{a,\nu}(F,R)$$

(same name by abuse of notation) via the commutative diagram

$$\begin{tikzcd}
\operatorname{coker}\delta^* \arrow[r, "\tilde{\delta}_{\alpha,\nu}"] \arrow[d] & \operatorname{coker}(T_c + T_cN_c - a') \arrow[d, "\mathrm{mod}\ (N-\nu)"] \\
C^0(\mathbb{P}^1(F),R)/R \arrow[r, "\tilde{\delta}_{\alpha,\nu}"] & \mathcal{B}_{a,\nu}(F,R)
\end{tikzcd}$$

with $a' := a(1+\nu)$, since $\varrho \in \mathcal{B}^{a,\nu}(F,R) \subseteq \ker(T+TN-a')$.

Lemma 2.11. *We have* $\varrho(gv) = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$, *and thus*

$$\tilde{\delta}_{\alpha,\nu}(gf) = \chi_\alpha(d/a')\chi_\nu(a')g\tilde{\delta}_{\alpha,\nu}(f),$$

for all $v \in \tilde{\mathcal{V}}$, $f \in C^0(\mathbb{P}^1(F),R)/R$ *and* $g = \begin{pmatrix} a' & b \\ 0 & d\end{pmatrix} \in B(F)$.

Proof. (a) Using lemma 2.5(b) and the fact that $\operatorname{ord}_{gv}(e_1) = -\operatorname{ord}_\varpi(a') + \operatorname{ord}_v(e_1)$, we have

$$\varrho\left(\begin{pmatrix} a' & b \\ 0 & d\end{pmatrix} v\right) = \alpha^{h(v)-\operatorname{ord}_\varpi(a'/d)}\nu^{\operatorname{ord}_\varpi(a')-\operatorname{ord}_v(e_1)} = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$$

for all $v \in \tilde{\mathcal{V}}$. For f and g as in the assertion, we thus have

$$\begin{aligned}
\tilde{\delta}_{\alpha,\nu}(gf)(v) &= \sum_{v=t(e)} \varrho(o(e))f(g^{-1}e) - \sum_{v=o(e)} \varrho(t(e))f(g^{-1}e) \\
&= \sum_{g^{-1}v=t(e)} \varrho(o(ge))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(ge))f(e) \\
&= \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)\Big(\sum_{g^{-1}v=t(e)} \varrho(o(e))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(e))f(e)\Big) \\
&= \chi_\alpha(d/a')\chi_\nu(a')g\tilde{\delta}_{\alpha,\nu}(f)(v).
\end{aligned}$$

□

We define a function $\delta_{\alpha,\nu} : C_c(F^*, R) \to \mathcal{B}_{a,\nu}(F, R)$ as follows: For $f \in C_c(F^*, R)$, we let $\psi_0(f) \in C_c(\mathbb{P}^1(F), R)$ be the extension of $x \mapsto \chi_\alpha(x)\chi_\nu(x)^{-1}f(x)$ by zero to $\mathbb{P}^1(F)$. We set $\delta_{\alpha,\nu} := \tilde{\delta}_{\alpha,\nu} \circ \psi_0$. If $\alpha = \nu$, we can define $\delta_{\alpha,\nu}$ on all functions in $C_c(F, R)$.

We let F^* operate on $C_c(F, R)$ by $(tf)(x) := f(t^{-1}x)$; this induces an action of the group $T^1(F) := \{\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) | t \in F^*\}$, which we identify with F^* in the obvious way. With respect to it, we have

$$\psi_0(tf)(x) = \chi_\alpha(t)\chi_\nu(t)^{-1}t\psi_0(f)(x)$$

and

$$\tilde{\delta}_{\alpha,\nu}(tf) = \chi_\alpha^{-1}(t)\chi_\nu(t)t\tilde{\delta}_{\alpha,\nu}(f),$$

so $\delta_{\alpha,\nu}$ is $T^1(F)$-equivariant.

For an R-module M, we define an F^*-action on $\mathrm{Dist}(F^*, M)$ by $\int f d(t\mu) := t\int(t^{-1}f)d\mu$. Let $H \subseteq G(F)$ be a subgroup, and M an $R[H]$-module. We define an H-action on $\mathcal{B}^{a\nu,\nu^{-1}}(F, M)$ by requiring $\langle \phi, h\lambda \rangle = h \cdot \langle h^{-1}\phi, \lambda \rangle$ for all $\phi \in \mathcal{B}_{a,\nu}(F, M)$, $\lambda \in \mathcal{B}^{a\nu,\nu^{-1}}(F, M)$, $h \in H$. With respect to these two actions, we get a $T^1(F) \cap H$-equivariant mapping

$$\delta^{\alpha,\nu} : \mathcal{B}^{a\nu,\nu^{-1}}(F, M) \to \mathrm{Dist}(F^*, M), \quad \delta^{\alpha,\nu}(\lambda) := \langle \delta_{\alpha,\nu}(\cdot), \lambda \rangle$$

dual to $\delta_{\alpha,\nu}$.

2.5 Local distributions

Now consider the case $R = \mathbb{C}$. Let $\chi_1, \chi_2 : F^* \to \mathbb{C}^*$ be two unramified characters. We consider (χ_1, χ_2) as a character on the torus $T(F)$ of $\mathrm{GL}_2(F)$, which induces a character χ on $B(F)$ by

$$\chi\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} := \chi_1(t_1)\chi_2(t_2).$$

Put $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. Set $\nu := \chi_1(\varpi)\chi_2(\varpi) = \alpha_1\alpha_2 q^{-1} \in \mathbb{C}^*$, and $a := \alpha_1 + \alpha_2 = \alpha_i + q\nu/\alpha_i$ for either i. When a and ν are given by the α_i

like this, we will often write $\mathcal{B}_{\alpha_1,\alpha_2}(F,R) := \mathcal{B}_{a,\nu}(F,R)$ and $\mathcal{B}^{\alpha_1,\alpha_2}(F,M) := \mathcal{B}^{a\nu,\nu^{-1}}(F,M)$ (!) for its dual. In the special case $a^2 = \nu(q+1)^2$, we assume the χ_i to be sorted such that $\chi_1 = |\cdot|\chi_2$.

Let $\mathcal{B}(\chi_1,\chi_2)$ denote the space of continuous maps $\phi : G(F) \to \mathbb{C}$ such that

$$\phi\left(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} g\right) = \chi_{\alpha_1}(t_1)\chi_{\alpha_2}(t_2)|t_1|\phi(g) \tag{8}$$

for all $t_1, t_2 \in F^*$, $u \in F$, $g \in G(F)$. $G(F)$ operates canonically on $\mathcal{B}(\chi_1,\chi_2)$ by right translation (cf. [Bu98], Ch. 4.5). If $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, $\mathcal{B}(\chi_1,\chi_2)$ is a model of the spherical representation $\pi(\chi_1,\chi_2)$; if $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, the special representation $\pi(\chi_1,\chi_2)$ can be given as an irreducible subquotient of codimension 1 of $\mathcal{B}(\chi_1,\chi_2)$.[2]

Lemma 2.12. *We have a G-equivariant isomorphism $\tilde{\mathcal{B}}_{a,\nu}(F,\mathbb{C}) \cong \mathcal{B}(\chi_1,\chi_2)$. It induces an isomorphism $\mathcal{B}_{a,\nu}(F,\mathbb{C}) \cong \pi(\chi_1,\chi_2)$ both for spherical and special representations.*

Proof. We choose a "central" unramified character $\chi_Z : F^* \to \mathbb{C}$ satisfying $\chi_Z^2(\varpi) = \nu$; then we have $\chi_1 = \chi_Z{\chi_0}^{-1}, \chi_2 = \chi_Z\chi_0$ for some unramified character χ_0. We set $a' := \sqrt{q}\left(\chi_0(\varpi)^{-1} + \chi_0(\varpi)\right)$, which satisfies $a = \chi_Z(\varpi)a'$.

For a representation (π, V) of $G(F)$, by [Bu98], Ex. 4.5.9, we can define another representation $\chi_Z \otimes \pi$ on V via

$$(g,v) \mapsto \chi_Z(\det(g))\pi(g)v \qquad \text{for all } g \in G(F), v \in V,$$

and with this definition we have $\mathcal{B}(\chi_1,\chi_2) \cong \chi_Z \otimes \mathcal{B}(\chi_0^{-1},\chi_0)$. Since $\mathcal{B}(\chi_0^{-1},\chi_0)$ has trivial central character, [BL95], Thm. 20 (as quoted in [Sp14]) states that $\mathcal{B}(\chi_0^{-1},\chi_0) \cong \mathcal{B}_{a',1}(F,\mathbb{C}) \cong \operatorname{Ind}_{KZ}^{G(F)} R/\operatorname{Im}(T-a')$.

Define a G-linear map $\phi : \operatorname{Ind}_K^G R \to \chi_Z \otimes \operatorname{Ind}_{KZ}^G R$ by $1_K \mapsto (\chi_Z \circ \det)\cdot 1_{KZ}$. Since 1_K (resp. $(\chi_Z \circ \det)\cdot 1_{KZ}$) generates $\operatorname{Ind}_K^G R$ (resp. $\chi_Z \otimes \operatorname{Ind}_{KZ}^G R$) as a $\mathbb{C}[G]$-module, ϕ is well-defined and surjective.

ϕ maps $N1_K = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right) 1_K$ to

$$\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)((\chi_Z \circ \det)\cdot 1_{KZ}) = \chi_Z(\varpi)^2 \cdot ((\chi_Z \circ \det)\cdot 1_{KZ}) = \nu \cdot \phi(1_K).$$

Thus $\operatorname{Im}(N-\nu) \subseteq \ker\phi$, and in fact the two are equal, since the preimage of the space of functions of support in a coset KZg ($g \in G(F)$) under ϕ is exactly the space generated by the 1_{Kzg}, $z \in Z(F) = Z(\mathcal{O}_F)\{\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)\}^{\mathbb{Z}}$.

Furthermore, ϕ maps $T1_K = \sum_{i \in \mathcal{O}_F/(\varpi)\cup\{\infty\}} N_i 1_K$ (with the N_i as in Lemma 2.7) to

$$\sum_i \chi_Z(\det(N_i))\cdot((\chi_Z \circ \det)\cdot N_i 1_{KZ}) = \chi_Z(\varpi)\cdot(\chi_Z \circ \det)T1_{KZ}$$

(since $\det(N_i) = \varpi$ for all i),and thus $\operatorname{Im}(T-a)$ is mapped to $\operatorname{Im}\big(\chi_Z(\varpi)T - a\big) = \operatorname{Im}\big(\chi_Z(\varpi)(T-a')\big) = \operatorname{Im}(T-a')$.

[2]Note that [Bu98] denotes this special representation by $\sigma(\chi_1,\chi_2)$, not by $\pi(\chi_1,\chi_2)$.

Putting everything together, we thus have G-isomorphisms

$$\begin{aligned} C_c(\tilde{\mathcal{V}},\mathbb{C})/\big(\operatorname{Im}(T-a)+\operatorname{Im}(N-\nu)\big) &\cong \operatorname{Ind}_K^G R/\big(\operatorname{Im}(T-a)+\operatorname{Im}(N-\nu)\big)\\ &\cong \chi_Z\otimes\big(\operatorname{Ind}_{KZ}^G R/\operatorname{Im}(T-a')\big)\\ &\cong \chi_Z\otimes\mathcal{B}(\chi_0^{-1},\chi_0)\cong\mathcal{B}(\chi_1,\chi_2). \end{aligned}$$

Thus, $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is isomorphic to the spherical principal series representation $\pi(\chi_1,\chi_2)$ for $a^2\neq\nu(q+1)^2$.
In the special case, $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is a G-invariant subspace of $\tilde{\mathcal{B}}_{a,\nu}(F,\mathbb{C})$ of codimension 1, so it must be mapped under the isomorphism to the unique G-invariant subspace of $\mathcal{B}(\chi_1,\chi_2)$ of codimension 1 (in fact, the unique infinite-dimensional irreducible G-invariant subspace, by [Bu98], Thm. 4.5.1), which is the special representation $\pi(\chi_1,\chi_2)$. ☐

By [Bu98], section 4.4, there exists thus for all pairs a,ν a *Whittaker functional* λ on $\mathcal{B}_{a,\nu}(F,\mathbb{C})$, i.e. a nontrivial linear map $\lambda:\mathcal{B}_{a,\nu}(F,\mathbb{C})\to\mathbb{C}$ such that $\lambda\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\phi\right)=\psi(x)\lambda(\phi)$. It is unique up to scalar multiples.
From it, we furthermore get a *Whittaker model* $\mathcal{W}_{a,\nu}$ of $\mathcal{B}_{a,\nu}(F,\mathbb{C})$:

$$\mathcal{W}_{a,\nu}:=\{W_\xi:GL_2(F)\to\mathbb{C}\,|\,\xi\in\mathcal{B}_{a,\nu}(F,\mathbb{C})\},$$

where $W_\xi(g):=\lambda(g\cdot\xi)$ for all $g\in GL_2(F)$. (see e.g. [Bu98], Ch. 3, eq. (5.6).)
Now write $\alpha:=\alpha_1$ for short. Recall the distribution $\mu_{\alpha,\nu}=\psi(x)\chi_{\alpha/\nu}(x)dx\in\operatorname{Dist}(F^*,\mathbb{C})$. For $\alpha=\nu$, it extends to a distribution on F. We have the following generalization of [Sp14], Prop. 3.10:

Proposition 2.13. *(a) There exists a unique Whittaker functional $\lambda=\lambda_{a,\nu}$ on $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ such that $\delta^{\alpha,\nu}(\lambda)=\mu_{\alpha,\nu}$.*
(b) For every $f\in C_c(F^*,\mathbb{C})$, there exists $W=W_f\in\mathcal{W}_{a,\nu}$ such that

$$\int_{F^*}(af)(x)\mu_{\alpha,\nu}(dx)=W_f\begin{pmatrix}a&0\\0&1\end{pmatrix}.$$

If $\alpha=\nu$, then for every $f\in C_c(F,\mathbb{C})$, there exists $W_f\in\mathcal{W}_{a,\nu}$ such that

$$\int_{F}(af)(x)\mu_{\alpha,\nu}(dx)=W_f\begin{pmatrix}a&0\\0&1\end{pmatrix}.$$

(c) Let $H\subseteq U=\mathcal{O}_F^\times$ be an open subgroup, and write $W_H:=W_{1_H}$. For every $f\in C_c^0(F^*,\mathbb{C})^H$ we have

$$\int_{F^*}f(x)\mu_{\alpha,\nu}(dx)=[U:H]\int_{F^*}f(x)W_H\begin{pmatrix}x&0\\0&1\end{pmatrix}d^\times x.$$

Proof. (a) By [Sp14], we have a Whittaker functional of the Steinberg representation given by the composite

$$St(F,\mathbb{C}):=C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C}\xrightarrow{\cong}C_c(F,\mathbb{C})\xrightarrow{\Lambda}\mathbb{C},\tag{9}$$

where the first map is the F-equivariant isomorphism

$$C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C} \to C_c(F,\mathbb{C}), \quad \phi \mapsto f(x) := \phi(x) - \phi(\infty),$$

(with F acting on $C_c(F,\mathbb{C})$ by $(x \cdot f)(y) := f(y-x)$, and on $C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C}$ by $x\phi := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\phi$), and the second is

$$\Lambda : C_c(F,\mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_F f(x)\psi(x)dx.$$

Let now $\lambda : \mathcal{B}_{a,\nu}(F,\mathbb{C}) \to \mathbb{C}$ be a Whittaker functional of $\mathcal{B}_{a,\nu}(F,\mathbb{C})$. By lemma 2.11, for $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in B(F)$,

$$(\lambda \circ \tilde{\delta}_{\alpha,\nu})(u\phi) = \lambda(u\tilde{\delta}_{\alpha,\nu}(\phi)) = \psi(x)\lambda(\tilde{\delta}_{\alpha,\nu}(\phi)),$$

so $\lambda \circ \tilde{\delta}_{\alpha,\nu}$ is a Whittaker functional if it is not zero.
To describe the image of $\tilde{\delta}_{\alpha,\nu}$, consider the commutative diagram

$$\begin{tikzcd}
C_c(\tilde{\mathcal{E}},R) \arrow[r, "\tilde{\delta}_{\alpha,\nu}"] \arrow[d, "(10)"] & C_c(\tilde{\mathcal{V}},R) \arrow[d, "\phi\mapsto\phi\cdot\varrho"] & & \\
C_c(\tilde{\mathcal{E}},R) \arrow[r, "\delta"] & C_c(\tilde{\mathcal{V}},R) \arrow[r, "\langle\cdot,1\rangle"] & R \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are defined by

$$C_c(\tilde{\mathcal{E}},R) \to C_c(\tilde{\mathcal{E}},R), \quad c \mapsto \big(e \mapsto c(e)\varrho(o(e))\varrho(t(e))\big) \tag{10}$$

resp. by mapping ϕ to $v \mapsto \phi(v)\varrho(v)$; both are obviously isomorphisms.
Since the lower row is exact, we have $\operatorname{Im}\delta = \ker\langle\cdot,1\rangle =: C_c^0(\tilde{\mathcal{V}},R)$ and thus $\operatorname{Im}\tilde{\delta}_{\alpha,\nu} = \varrho^{-1}\cdot C_c^0(\tilde{\mathcal{V}},R)$.
Since $\lambda \neq 0$ and $\mathcal{B}_{a,\nu}(F,\mathbb{C})$ is generated by (the equivalence classes of) the $1_{\{v\}}$, $v \in \tilde{\mathcal{V}}$, there exists a $v \in \tilde{\mathcal{V}}$ such that $\lambda(1_{\{v\}}) \neq 0$. Let ϕ be this $1_{\{v\}}$, and let $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in B(F)$ such that $x \notin \ker\psi$. Then

$$\varrho\cdot(u\phi-\phi) = \varrho\cdot(1_{\{u^{-1}v\}} - 1_{\{v\}}) = \varrho(v)(1_{\{u^{-1}v\}} - 1_{\{v\}}) \in C_c^0(\tilde{\mathcal{V}},R)$$

by lemma 2.11, so $0 \neq u\phi-\phi \in \operatorname{Im}\tilde{\delta}_{\alpha,\nu}$, but $\lambda(u\phi-\phi) = \psi(x)\lambda(\phi)-\lambda(\phi) \neq 0$.
So $\lambda\circ\tilde{\delta}_{\alpha,\nu} \neq 0$ is indeed a Whittaker functional. By replacing λ by a scalar multiple, we can assume $\lambda\circ\tilde{\delta}_{\alpha,\nu} = (9)$.
Considering λ as an element of $\mathcal{B}^{a\nu,\nu^{-1}}(F,\mathbb{C}) \cong \operatorname{Hom}(\mathcal{B}_{a,\nu}(F,\mathbb{C}),\mathbb{C})$, we have

$$\begin{aligned}
\delta^{\alpha,\nu}(\lambda)(f) &= \langle\delta_{\alpha,\nu}(f),\lambda\rangle \\
&= \Lambda(\chi_\alpha\chi_\nu^{-1}f) \\
&= \int_{F^*}\chi_\alpha(x)\chi_\nu^{-1}(x)f(x)\psi(x)dx \\
&= \mu_{\alpha,\nu}(f).
\end{aligned}$$

(b), (c) follow from (a) as in [Sp14]. □

2.6 SEMI-LOCAL THEORY

We can generalize many of the previous constructions to the semi-local case, considering all primes $\mathfrak{p}|p$ at once.
So let $F_1, \dots, F_m$ be finite extensions of $\mathbb{Q}_p$, and for each i, let q_i be the number of elements of the residue field of F_i. We put $\underline{F} := F_1 \times \cdots \times F_m$.
Let R again be a ring, and $a_i \in R, \nu_i \in R^*$ for each $i \in \{1, \dots, m\}$. Put $\underline{a} := (a_1, \dots, a_m)$, $\underline{\nu} := (\nu_1, \dots, \nu_m)$. We define $\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R)$ as the tensor product

$$\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R) := \bigotimes_{i=1}^{m} \mathcal{B}_{a_i,\nu_i}(F_i, R).$$

For an R-module M, we define $\mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(\underline{F}, M) := \mathrm{Hom}_R(\mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R), M)$; let

$$\langle \cdot, \cdot \rangle : \mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R) \times \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(\underline{F}, M) \to M \tag{11}$$

denote the evaluation pairing.
We have an obvious isomorphism

$$\bigotimes_{i=1}^{m} C_c^0(F_i^*, R) \to C_c^0(\underline{F}^*, R), \quad \bigotimes_i f_i \mapsto \left((x_i)_{i=1,\dots,m} \mapsto \prod_{i=1}^{m} f_i(x_i) \right). \tag{12}$$

Now when we have $\alpha_{i,1}, \alpha_{i,2} \in R^*$ such that $a_i = \alpha_{i,1} + \alpha_{i,2}$ and $\nu_i = \alpha_{i,1}\alpha_{i,2}q_i^{-1}$, we can define the $T^1(\underline{F})$-equivariant map

$$\delta_{\underline{\alpha}_{1,2}} := \delta_{\underline{\alpha_1},\underline{\nu}} : C_c^0(\underline{F}, R) \to \mathcal{B}_{\underline{a},\underline{\nu}}(\underline{F}, R)$$

as the inverse of (12) composed with $\bigotimes_{i=1}^m \delta_{\alpha_{i,1},\nu_i}$.
Again, we will often write $\mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F, R) := \mathcal{B}_{\underline{a\nu},\underline{\nu}^{-1}}(F, R)$ and $\mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F, M) := \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$.
If $H \subseteq G(F)$ is a subgroup, and M an $R[H]$-module, we define an H-action on $\mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$ by requiring $\langle \phi, h\lambda \rangle = h \cdot \langle h^{-1}\phi, \lambda \rangle$ for all $\phi \in \mathcal{B}_{\underline{a},\underline{\nu}}(F, M)$, $\lambda \in \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M)$, $h \in H$, and get a $T^1(\underline{F}) \cap H$-equivariant mapping

$$\delta^{\underline{\alpha_1},\underline{\alpha_2}} : \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, M) \to \mathrm{Dist}(\underline{F}^*, M), \quad \delta^{\underline{\alpha_1},\underline{\alpha_2}}(\lambda) := \langle \delta_{\underline{\alpha_1},\underline{\alpha_2}}(\cdot), \lambda \rangle.$$

Finally, we have a homomorphism

$$\begin{aligned} \bigotimes_{i=1}^{m} \mathcal{B}^{a_i\nu_i,\nu_i^{-1}}(F_i, R) &\xrightarrow{\cong} \bigotimes_{i=1}^{m} \mathrm{Hom}_R(\mathcal{B}_{a_i\nu_i,\nu_i^{-1}}(F_i, R), R) \\ &\to \mathrm{Hom}(\mathcal{B}_{a_1,\nu_1}(F_1, R), \mathrm{Hom}(\mathcal{B}_{a_2,\nu_2}(F_2, R), \mathrm{Hom}(\dots, R))\dots) \\ &\xrightarrow{\cong} \mathcal{B}^{\underline{a\nu},\underline{\nu}^{-1}}(F, R). \end{aligned} \tag{13}$$

where the second map is given by $\otimes_i f_i \mapsto (x_1 \mapsto (x_2 \mapsto (\dots \mapsto \prod_i f_i(x_i))\dots)$, and the last map by iterating the adjunction formula of the tensor product.

3 Cohomology classes and global measures

3.1 Definitions

From now on, let F denote a number field, with ring of integers $\mathcal{O}_F$. For each finite prime v, let $U_v := \mathcal{O}_v^*$. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adeles of F, and $\mathbb{I} = \mathbb{I}_F$ the group of ideles of F. For a finite subset S of the set of places of F, we denote by $\mathbb{A}^S := \{x \in \mathbb{A}_F \,|\, x_v = 0 \ \forall v \in S\}$ the S-adeles and by $\mathbb{I}^S$ the S-ideles, and put $F_S := \prod_{v\in S} F_v$, $U_S := \prod_{v\in S} U_v$, $U^S := \prod_{v\notin S} U_v$ (if S contains all infinite places of F), and similarly for other global groups.
For ℓ a prime number or ∞, we write S_ℓ for the set of places of F above ℓ, and abbreviate the above notations to $\mathbb{A}^\ell := \mathbb{A}^{S_\ell}$, $\mathbb{A}^{p,\infty} := \mathbb{A}^{S_p\cup S_\infty}$, and similarly write $\mathbb{I}^p$, $\mathbb{I}^\infty$, F_p, F_∞, U^∞, U_p, $U^{p,\infty}$, $\mathbb{I}_\infty$ etc.
Let F have r real embeddings and s pairs of complex embeddings. Set $d := r + s - 1$. Let $\{\sigma_0, \dots, \sigma_{r-1}, \sigma_r, \dots, \sigma_d\}$ be a set of representatives of these embeddings (i.e. for $i \geq r$, choose one from each pair of complex embeddings), and denote by $\infty_0, \dots, \infty_d$ the corresponding archimedian primes of F. We let $S_\infty^0 := \{\infty_1, \dots, \infty_d\} \subseteq S_\infty$.
For each place v, let dx_v denote the associated self-dual Haar measure on F_v, and $dx := \prod_v dx_v$ the associated Haar measure on $\mathbb{A}_F$. We define Haar measures $d^\times x_v$ on F_v^* by $d^\times x_v := c_v \frac{dx_v}{|x_v|_v}$, where $c_v = (1 - \frac{1}{q_v})^{-1}$ for v finite, $c_v = 1$ for $v|\infty$. For $v|\infty$ complex, we use the decomposition $\mathbb{C}^* = \mathbb{R}_+^* \times S^1$ (with $S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$) to write $d^\times x_v = d^\times r_v \, d\vartheta_v$ for variables r_v, ϑ_v with $r_v \in \mathbb{R}_+^*$, $\vartheta_v \in S^1$.
Let $S_1 \subseteq S_p$ be a set of primes of F lying above p, $S_2 := S_p - S_1$. Let R be a topological Hausdorff ring.

Definition 3.1. We define the module of continuous functions

$$\mathcal{C}(S_1, R) := C(F_{S_1} \times F_{S_2}^* \times \mathbb{I}^{p,\infty}/U^{p,\infty}, R);$$

and let $\mathcal{C}_c(S_1, R)$ be the submodule of all compactly supported $f \in \mathcal{C}(S_1, R)$. We write $\mathcal{C}^0(S_1, R)$, $\mathcal{C}_c^0(S_1, R)$ for the submodules of locally constant maps (or of continuous maps where R is assumed to have the discrete topology).We further define

$$\mathcal{C}_c^\flat(S_1, R) := \mathcal{C}_c(\varnothing, R) + \mathcal{C}_c^\flat(S_1, R) \subseteq \mathcal{C}_c^\flat(S_1, R)$$

to be the module of continuous compactly supported maps that are "constant near $(0_\mathfrak{p}, x^\mathfrak{p})$" for each $\mathfrak{p} \in S_1$.

Definition 3.2. For an R-module M, let $\mathcal{D}_f(S_1, M)$ denote the R-module of maps

$$\phi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}_F^{p,\infty} \to M$$

that are $U^{p,\infty}$-invariant and such that $\phi(\cdot, x^{p,\infty})$ is a distribution for each $x^{p,\infty} \in \mathbb{I}_F^{p,\infty}$.

Since $\mathbb{I}_F^{p,\infty}/U^{p,\infty}$ is a discrete topological group, $\mathcal{D}_f(S_1, M)$ naturally identifies with the space of M-valued distributions on $F_{S_1} \times F_{S_2}^* \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$. So there exists a canonical R-bilinear map

$$\mathcal{D}_f(S_1, M) \times \mathcal{C}_c^0(S_1, R) \to M, \quad (\phi, f) \mapsto \int f \, d\phi, \tag{14}$$

which is easily seen to induce an isomorphism $\mathcal{D}_f(S_1, M) \cong \mathrm{Hom}_R(\mathcal{C}_c^0(S_1, R), M)$.

For a subgroup $E \subseteq F^*$ and an $R[E]$-module M, we let E operate on $\mathcal{D}_f(S_1, M)$ and $\mathcal{C}_c^0(S_1, R)$ by $(a\phi)(U, x^{p,\infty}) := a\phi(a^{-1}U, a^{-1}x^{p,\infty})$ and $(af)(x^\infty) := f(a^{-1}x^\infty)$ for $a \in E$, $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, $x^{\cdot} \in \mathbb{I}_F^{\cdot}$; thus we have $\int (af) \, d(a\phi) = a \int f \, d\phi$ for all a, f, ϕ.

When $M = V$ is a finite-dimensional vector space over a p-adic field, we write $\mathcal{D}_f^b(S_1, V)$ for the subset of $\phi \in \mathcal{D}_f(S_1, V)$ such that ϕ is even a measure on $F_{S_1} \times F_{S_2} \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$.

Definition 3.3. For a $\mathbb{C}$-vector space V, define $\mathcal{D}(S_1, V)$ to be the set of all maps $\phi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^p \to V$ such that:

(i) ϕ is invariant under $F^\times$ and $U^{p,\infty}$.

(ii) For $x^p \in \mathbb{I}^p$, $\phi(\cdot, x^p)$ is a distribution of $F_{S_1} \times F_{S_2}$.

(iii) For all $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$, the map $\phi_U : \mathbb{I} = F_p^\times \times \mathbb{I}^p \to V, (x_p, x^p) \mapsto \phi(x_p U, x^p)$ is smooth, and rapidly decreasing as $|x| \to \infty$ and $|x| \to 0$.

We will need a variant of this last set: Let $\mathcal{D}'(S_1, V)$ be the set of all maps $\phi \in \mathcal{D}(S_1, V)$ that are "$(S^1)^s$-invariant", i.e. such that for all complex primes ∞_j of F and all $\zeta \in S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$, we have

$$\phi(U, x^{p,\infty_j}, \zeta x_{\infty_j}) = \phi(U, x^{p,\infty_j}, x_{\infty_j}) \text{ for all } x^p = (x^{p,\infty_j}, x_{\infty_j}) \in \mathbb{I}^p.$$

There is an obvious surjective map

$$\mathcal{D}(S_1, V) \to \mathcal{D}'(S_1, V), \quad \phi \mapsto \left((U, x) \mapsto \int_{(S^1)^s} \phi(U, x) d\vartheta_r \cdots d\vartheta_{r+s-1} \right)$$

given by integrating over $(S^1)^s \subseteq (\mathbb{C}^*)^s \hookrightarrow \mathbb{I}_\infty$.

Let F_+^* denote the set of all $x \in F*$ that are totally positive, i.e. positive with respect to every real embedding of F. (For F totally imaginary, we have $F^* = F_+^*$.) Let $F^{*\prime} \subseteq F_+^*$ be a maximal torsion-free subgroup of F_+^*. If F has at least one real embedding, we obviously have $F^{*\prime} = F_+^*$; for totally imaginary F, $F^{*\prime}$ is a subgroup of finite index of F^* with $F/F^{*\prime} \cong \mu_F$, the roots of unity of F.

We set

$$E' := F^{*\prime} \cap O_F^\times \subseteq O_F^\times,$$

so E' is a torsion-free $\mathbb{Z}$-module of rank d. E' operates freely and discretely on the space

$$\mathbb{R}_0^{d+1} := \left\{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} | \sum_{i=0}^{d} x_i = 0\right\}$$

via the embedding

$$\begin{array}{rcl} E' & \hookrightarrow & \mathbb{R}_0^{d+1} \\ a & \mapsto & (\log|\sigma_i(a)|)_{i \in S_\infty} \end{array}$$

(cf. proof of Dirichlet's unit theorem, e.g. in [Neu92], Ch. 1), and the quotient $\mathbb{R}_0^{d+1}/E'$ is compact. We choose the orientation on $\mathbb{R}_0^{d+1}$ induced by the natural orientation on $\mathbb{R}^d$ via the isomorphism $\mathbb{R}^d \cong \mathbb{R}_0^{d+1}$, $(x_1, \ldots, x_d) \mapsto (-\sum_{i=1}^d x_i, x_1, \ldots, x_d)$. So $\mathbb{R}_0^{d+1}/E'$ becomes an oriented compact d-dimensional manifold.

Let $\mathcal{G}_p$ be the Galois group of the maximal abelian extension of F which is unramified outside p and ∞; for a $\mathbb{C}$-vector space V, let $\mathrm{Dist}(\mathcal{G}_p, V)$ be the set of V-valued distributions of $\mathcal{G}_p$. Denote by $\varrho : \mathbb{I}_F/F^* \to \mathcal{G}_p$ the projection given by global reciprocity.

3.2 Global measures

Now let $V = \mathbb{C}$, equipped with the trivial $F^{*\prime}$-action. We want to construct a commutative diagram

$$\begin{tikzcd} \mathcal{D}(S_1, \mathbb{C}) \arrow[rr, "\phi \mapsto \kappa_\phi"] \arrow[dr, "\phi \mapsto \mu_\phi"'] & & H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})\big) \arrow[dl, "\kappa \mapsto \mu_\kappa = \kappa \cap \partial(\cdot)"] \\ & \mathrm{Dist}(\mathcal{G}_p, \mathbb{C}) & \end{tikzcd} \tag{15}$$

First, let R be any topological Hausdorff ring. Let $\overline{E'}$ denote the closure of E' in U_p. The projection map $\mathrm{pr} : \mathbb{I}^\infty/U^{p,\infty} \to \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty})$ induces an isomorphism

$$\mathrm{pr}^* : C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R) \to H^0(E', C_c(\mathbb{I}^\infty/U^{p,\infty}, R)),$$

and the reciprocity map induces a surjective map $\overline{\varrho} : \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}) \to \mathcal{G}_p$. Now we can define a map

$$\begin{aligned} \varrho^\sharp : H_0(F^{*\prime}/E', C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R)) &\to C(\mathcal{G}_p, R), \\ [f] &\mapsto \Big(\overline{\varrho}(x) \mapsto \sum_{\zeta \in F^{*\prime}/E'} f(\zeta x) \text{ for } x \in \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty})\Big). \end{aligned}$$

This is an isomorphism, with inverse map $f \mapsto [(f \circ \overline{\varrho}) \cdot 1_{\mathcal{F}}]$, where $1_{\mathcal{F}}$ is the characteristic function of a fundamental domain $\mathcal{F}$ of the action of $F^{*\prime}/E'$ on $\mathbb{I}^\infty/U^\infty$.

We get a composite map

$$\begin{aligned} C(\mathcal{G}_p, R) &\xrightarrow{(\varrho^\sharp)^{-1}} H_0\big(F^{*\prime}/E', C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R)\big) \\ &\xrightarrow{\mathrm{pr}^*} H_0\big(F^{*\prime}/E', H^0(E', C_c(\mathbb{I}^\infty/U^{p,\infty}, R))\big) \\ &\longrightarrow H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c(S_1, R))\big), \end{aligned} \tag{16}$$

where the last arrow is induced by the "extension by zero" from $C_c(\mathbb{I}^\infty/U^{p,\infty}, R)$ to $\mathcal{C}_c(S_1, R)$.
Now let $\eta \in H_d(E', \mathbb{Z}) \cong \mathbb{Z}$ be the generator that corresponds to the given orientation of $\mathbb{R}_0^{d+1}$. This gives us, for every R-module A, a homomorphism

$$H_0\big(F^{*\prime}/E', H^0(E', A)\big) \xrightarrow{\cap\eta} H_0\big(F^{*\prime}/E', H_d(E', A)\big)$$

Composing this with the edge morphism

$$H_0\big(F^{*\prime}/E', H_d(E', A)\big) \to H_d(F^{*\prime}, A) \tag{17}$$

(and setting $A = \mathcal{C}_c(S_1, R)$) gives a map

$$H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c(S_1, R))\big) \to H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big) \tag{18}$$

We define

$$\partial : C(\mathcal{G}_p, R) \to H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big)$$

as the composition of (16) with this map.
Now, letting M be an R-module equipped with the trivial $F^{*\prime}$-action, the bilinear form (14)

$$\begin{array}{rcl} \mathcal{D}_f(S_1, M) \times \mathcal{C}_c(S_1, R) & \to & M \\ (\phi, f) & \mapsto & \displaystyle\int f \, d\phi \end{array}$$

induces a cap product

$$\cap : H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, M)\big) \times H_d\big(F^{*\prime}, \mathcal{C}_c(S_1, R)\big) \to H_0(F^{*\prime}, M) = M. \tag{19}$$

Thus for each $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, M))$, we get a distribution μ_κ on $\mathcal{G}_p$ by defining

$$\int_{\mathcal{G}_p} f(\gamma) \, \mu_\kappa(d\gamma) := \kappa \cap \partial(f) \tag{20}$$

for all continuous maps $f : \mathcal{G}_p \to R$.
Now let $M = V$ be a finite-dimensional vector space over a p-adic field K, and let $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V))$. We identify κ with its image in $H^d(F^{*\prime}, \mathcal{D}_f(S_1, V))$; then it is easily seen that μ_κ is also a measure, i.e. we have a map

$$H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V)) \to \mathrm{Dist}^b(\mathcal{G}_p, V), \quad \kappa \mapsto \mu_\kappa. \tag{21}$$

Let $L|F$ be a $\mathbb{Z}_p$-extension of F. Since it is unramified outside p, it gives rise to a continuous homomorphism $\mathcal{G}_p \to \mathrm{Gal}(L|F)$ via $\sigma \mapsto \sigma|_L$. Fixing an isomorphism $\mathrm{Gal}(L|F) \cong p^{\varepsilon_p}\mathbb{Z}_p$ (where $\varepsilon_p = 2$ for $p = 2$, $\varepsilon_p = 1$ for p odd), we obtain a surjective homomorphism $\ell : \mathcal{G}_p \to p^{\varepsilon_p}\mathbb{Z}_p$. (Note that $p^{\varepsilon_p}\mathbb{Z}_p$ is the space of definition of the p-adic exponential function $\exp_p$.)

Example 3.4. Let L be the cyclotomic $\mathbb{Z}_p$-extension of F. Then we can take $\ell = \log_p \circ \mathcal{N}$, where $\mathcal{N} : \mathcal{G}_p \to \mathbb{Z}_p^*$ is the p-adic cyclotomic character, defined by requiring $\gamma\zeta = \zeta^{\mathcal{N}(\gamma)}$ for all $\gamma \in \mathcal{G}_p$ and all p-power roots of unity ζ. It is well-known (cf. [Wa82], par. 5) that $\log_p(\mathbb{Z}_p^*) = p^{\varepsilon_p}\mathbb{Z}_p$.

It is well-known that F has t independent $\mathbb{Z}_p$-extensions, where $s+1 \leq t \leq [F:\mathbb{Q}]$; the Leopoldt conjecture implies $t = s+1$. μ_κ defines a t-variable p-adic L-function as follows:

DEFINITION 3.5. Let K be a p-adic field, V a finite-dimensional K-vector space, $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V))$. Let $\ell_1, \ldots, \ell_t : \mathcal{G}_p \to p^{\varepsilon_p}\mathbb{Z}_p$ be continuous homomorphisms.The *p-adic L-function* of κ is given by

$$L_p(\underline{s}, \kappa) := L_p(s_1, \ldots, s_t, \kappa) := \int_{\mathcal{G}_p} \left(\prod_{i=1}^{t} \exp_p(s_i \ell_i(\gamma)) \right) \mu_\kappa(d\gamma)$$

for all $s_1, \ldots, s_t \in \mathbb{Z}_p$.

Remark 3.6. Let $\Sigma := \{\pm 1\}^r$, where r is the number of real embeddings of F. The group isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}, \varepsilon \mapsto (-1)^\varepsilon$, induces a pairing

$$\langle \cdot, \cdot \rangle : \Sigma \to \{\pm 1\}, \quad \langle ((-1)^{\varepsilon_i})_i, ((-1)^{\varepsilon_i'})_i \rangle := (-1)^{\sum_i \varepsilon_i \varepsilon_i'}.$$

For a field k of characteristic zero, a $k[\Sigma]$-module V and $\underline{\mu} = (\mu_0, \ldots, \mu_{r-1}) \in \Sigma$, we put $V_{\underline{\mu}} := \{v \in V \mid \langle \underline{\mu}, \underline{\nu} \rangle v = \underline{\nu} v \ \forall \underline{\nu} \in \Sigma\}$, so that we have $V = \bigoplus_{\underline{\mu} \in \Sigma} V_{\underline{\mu}}$. We write $v_{\underline{\mu}}$ for the projection of $v \in V$ to $V_{\underline{\mu}}$, and $v_+ := v_{(1,\ldots,1)}$.
For $r > 0$, we identify Σ with $F^*/F^{*\prime}$ via the isomorphism $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* \cong F^*/F^{*\prime} = F^*/F_+^*$. Then for each F^*-module M, Σ acts on $H^d(F^{*\prime}, \mathcal{D}_f(S_1, M))$ and on $H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, M))$. For $r = 0$, we let the trivial group Σ act on these groups as well for ease of notation. The exact sequence $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* = \mathbb{I}_\infty/\mathbb{I}_\infty^0 \to \mathcal{G}_p \to \mathcal{G}_p^+ \to 0$ of class field theory (where $\mathbb{I}_\infty^0$ is the maximal connected subgroup of $\mathbb{I}_\infty$) yields an action of Σ on $\mathcal{G}_p$. We easily check that (21) is Σ-equivariant, and that the maps $\gamma \mapsto \exp_p(s\ell_i(\gamma))$ factor over $\mathcal{G}_p \to \mathcal{G}_p^+$ (since $\mathbb{Z}_p$-extensions are unramified at ∞). Therefore we have $L_p(\underline{s}, \kappa) = L_p(\underline{s}, \kappa_+)$.

For $\phi \in \mathcal{D}(S_1, V)$ and $f \in C^0(\mathbb{I}/F^*, \mathbb{C})$, let

$$\int_{\mathbb{I}/F^*} f(x)\phi(d^\times x_p, x^p)\, d^\times x^p := [U_p : U] \int_{\mathbb{I}/F^*} f(x)\phi_U(x)\, d^\times x,$$

where we choose an open set $U \subseteq U_p$ such that $f(x_p u, x^p) = f(x_p, x^p)$ for all $(x_p, x^p) \in \mathbb{I}$ and $u \in U$; such a U exists by lemma 3.7 below. Since this integral is additive in f, there exists a unique V-valued distribution μ_ϕ on $\mathcal{G}_p$ such that

$$\int_{\mathcal{G}_p} f \, d\mu_\phi = \int_{\mathbb{I}/F^*} f(\varrho(x))\phi(d^\times x_p, x^p) \, d^\times x^p \tag{22}$$

for all functions $f \in C^0(\mathcal{G}_p, V)$.

Lemma 3.7. *Let $F : \mathbb{I}/F^* \to X$ be a locally constant map to a set X. Then there exists an open subgroup $U \subseteq \mathbb{I}$ such that f factors over $\mathbb{I}/F^*U$.*

Proof. $\mathbb{I}_\infty = \prod_{v|\infty} F_v$ is connected, thus f factors over $\overline{f} : \mathbb{I}/F^*\mathbb{I}_\infty \to X$. Since $\mathbb{I}/F^*\mathbb{I}_\infty$ is profinite, $\overline{f}$ further factors over a subgroup $U' \subseteq \mathbb{I}^\infty$ of finite index, which is open. □

Let $U_\infty^0 := \prod_{v \in S_\infty^0} \mathbb{R}_+^*$; the isomorphisms $U_\infty^0 \cong \mathbb{R}^d$, $(r_v)_v \mapsto (\log r_v)_v$, and $\mathbb{R}^d \cong \mathbb{R}_0^{d+1}$ give it the structure of a d-dimensional oriented manifold (with the natural orientation). It has the d-form $d^\times r_1 \cdot \ldots \cdot d^\times r_d$, where (by slight abuse of notation) we choose $d^\times r_i$ on F_{∞_i} corresponding to the Haar measure $d^\times x_i$ resp. $d^\times r_i$ on $\mathbb{R}_+^* \subseteq F_{\infty_i}^*$. E' operates on U_∞^0 via $a \mapsto (|\sigma_i(a)|)_{i \in S_\infty^0}$, so the isomorphism $U_\infty^0 \cong \mathbb{R}_0^{d+1}$ is E'-equivariant.
For $\phi \in \mathcal{D}'(S_1, V)$, set

$$\begin{aligned} \int_0^\infty \phi \, d^\times r_0 \colon \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^{p,\infty_0} &\to \mathbb{C} \\ (U, x^{p,\infty_0}) &\mapsto \int_0^\infty \phi(U, r_0, x^{p,\infty_0}) \, d^\times r_0, \end{aligned}$$

where we let $r_0 \in F_{\infty_0}$ run through the positive real line $\mathbb{R}_+^*$ in F_{∞_0}. Composing this with the projection $\mathcal{D}(S_1, V) \to \mathcal{D}'(S_1, V)$ gives us a map

$$\begin{aligned} \mathcal{D}(S_1, V) &\to H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))\big), \\ \phi &\mapsto \int_{(S^1)^s} \left(\int_0^\infty \phi \, d^\times r_0 \right) d\vartheta_r \, d\vartheta_{r+1} \ldots d\vartheta_{r+s-1} \end{aligned} \tag{23}$$

(where $C^\infty(U_\infty^0, V)$ denotes the space of smooth V-valued functions on U_∞^0), since one easily checks that $\int_0^\infty \phi \, d^\times r_0$ is $F^{*\prime}$-invariant.
Define the complex $C^\bullet := \mathcal{D}_f(S_1, \Omega^\bullet(U_\infty^0, V))$. By the Poincare lemma, this is a resolution of $\mathcal{D}_f(S_1, V)$. We now define the map $\phi \mapsto \kappa_\phi$ as the composition of (23) with the composition

$$H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))\big) \to H^0(F^{*\prime}, C^d) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, V)), \tag{24}$$

where the first map is induced by

$$C^\infty(U_\infty^0, V) \to \Omega^d(U_\infty^0, V), \quad f \mapsto f(r_1, \ldots, r_d) d^\times r_1 \cdot \ldots \cdot d^\times r_d, \tag{25}$$

and the second is an edge morphism in the spectral sequence

$$H^q(F^{*\prime}, C^p) \Rightarrow H^{p+q}(F^{*\prime}, \mathcal{D}_f(S_1, V)). \tag{26}$$

Specializing to $V = \mathbb{C}$, we now have:

PROPOSITION 3.8. *The diagram* (15) *commutes, i.e., for each* $\phi \in \mathcal{D}(S_1, \mathbb{C})$, *we have*

$$\mu_\phi = \mu_{\kappa_\phi}.$$

Proof. Analoguously to [Sp14], proof of prop. 4.21, we define a pairing

$$\langle\ ,\ \rangle \colon \mathcal{D}(S_1, \mathbb{C}) \times C^0(\mathcal{G}_p, \mathbb{C}) \to \mathbb{C}$$

as the composite of (23) $\times$ (16) with

$$\begin{aligned} H^0\big(F^{*\prime}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, \mathbb{C}))\big) \times H_0\big(F^{*\prime}/E', H^0(E', \mathcal{C}_c^0(S_1, \mathbb{C}))\big) \\ \xrightarrow{\cap} H_0\big(F^{*\prime}/E', H^0(E', C^\infty(U_\infty^0, \mathbb{C}))\big) \to H_0(F^{*\prime}/E', \mathbb{C}) \cong \mathbb{C}, \end{aligned} \tag{27}$$

where $\cap$ is the cap product induced by (14), and the second map is induced by

$$H^0\big(E', \mathcal{C}^\infty(U_\infty^0, \mathbb{C})\big) \to \mathbb{C}, \quad f \mapsto \int_{U_\infty^0/E'} f(r_1, \ldots, r_d)\, d^\times r_1 \ldots d^\times r_d. \tag{28}$$

Then we can show that

$$\kappa_\phi \cap \partial(f) = \langle \phi, f \rangle = \int_{\mathcal{G}_p} f(\gamma)\, \mu_\phi(d\gamma) \quad \text{ for all } f \in C^0(\mathcal{G}_p, \mathbb{C}),$$

by copying the proof for the totally real case (replacing F_+^* by $F^{*\prime}$, E_+ by E'), using the fact that for a d-form on the d-dimensional oriented manifold $M := \mathbb{R}_0^{d+1}/E' \cong U_\infty^0/E'$, integration over M corresponds to taking the cap product with the fundamental class η of M under the canonical isomorphism $H_{dR}^d(M) \cong H_{sing}^d(M) = H^d(E', \mathbb{C})$. □

3.3 EXCEPTIONAL ZEROS

Now let $\ell_1, \ldots, \ell_t : \mathcal{G}_p \to \mathbb{Z}_p$ be continuous homomorphisms. Let again $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq S_p$ be a set of primes above p, of cardinality $n := \#S_1$.

PROPOSITION 3.9. *For each* $\underline{x} = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$ *set* $|\underline{x}| := \sum_{i=1}^t x_i$. *Then*

$$\partial\Big(\prod_{i=1}^t \ell_i^{x_i}\Big) = 0 \quad \textit{for all } \underline{x} \textit{ with } |\underline{x}| \le n-1.$$

Proof. We can readily generalize the proof of Spieß' result for the p-adic cyclotomic character ($\ell = \log_p \circ \mathcal{N}$) in the totally real case ([Sp14], Prop. 4.6(a), Lemmas 4.1 and 4.7) to show that $\partial(\ell^x) = 0$ for all $0 \leq x \leq n-1$, using the facts that we can write $F^{*\prime} = E' \times \mathcal{T}$ for some subgroup $\mathcal{T} \subseteq F^{*\prime}$ (since $F^{*\prime}/E' = F^*/\mathcal{O}_F^\times$ is a free $\mathbb{Z}$-module), and that for each homomorphism $\ell : \mathcal{G}_p \to \mathbb{Z}_p$, the composition

$$\tilde{\ell} : \mathbb{I}^\infty \xrightarrow{\varrho} \mathcal{G}_p \xrightarrow{\ell} \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p.$$

is zero on $\mathbb{I}^{\infty,p}$ (since the pro-q-part of $\mathcal{G}_p$ is finite for every prime $q \neq p$ and $\mathbb{Q}_p$ is torsion-free).

Now for a ring $R \supseteq \mathbb{Q}$, each monomial $\prod_{i=1}^t X_i^{n_i} \in R[X_1, \ldots, X_t]$ of degree $n = \sum_i n_i$ can be written as a linear combination of n-th powers $(X_i + r_{i,j} X_j)^n$. Therefore each product $\prod_{i=1}^t \ell_i^{x_i}$ of degree $x = |\underline{x}|$ is a linear combination of x-th powers of the homomorphisms $\ell_{i,j} := \ell_i + r_{i,j}\ell_j : \mathcal{G}_p \to \mathbb{Z}_p$. This proves the proposition. ☐

Definition 3.10. A t-variable p-adic analytic function $f(\underline{s}) = f(s_1, \ldots, s_t)$ ($s_i \in \mathbb{Z}_p$) *has vanishing order* $\geq n$ at the point $\underline{0} = (0, \ldots, 0)$ if all its partial derivatives of total order $\leq n-1$ vanish, i.e. if

$$\frac{\partial^k}{(\partial \underline{s})^{\underline{k}}} f(\underline{0}) := \frac{\partial^k}{\partial s_1^{k_1} \cdots \partial s_t^{k_t}} f(\underline{0}) = 0$$

for all $\underline{k} = (k_1, \ldots, k_t) \in \mathbb{N}_0^t$ with $k := |\underline{k}| \leq n-1$. We write $\operatorname{ord}_{\underline{s}=\underline{0}} f(\underline{s}) \geq n$ in this case.

Theorem 3.11. *Let $n := \#(S_1)$, $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V))$, V a finite-dimensional vector space over a p-adic field. Then $L_p(\underline{s}, \kappa)$ is a locally analytic function, and we have*

$$\operatorname{ord}_{\underline{s}=\underline{0}} L_p(\underline{s}, \kappa) \geq n.$$

Proof. We have

$$\frac{\partial^k}{(\partial \underline{s})^{\underline{k}}} L_p(\underline{0}, \kappa) = \int_{\mathcal{G}_p} \left(\prod_{i=1}^t \ell_i(\gamma)^{k_i} \right) \mu_\kappa(d\gamma) = \kappa \cap \partial \left(\prod_{i=1}^t \ell_i(\gamma)^{k_i} \right)$$

for all $\underline{k} = (k_1, \ldots, k_t) \in \mathbb{N}_0^t$. Thus the theorem follows from proposition 3.9. ☐

3.4 Integral cohomology classes

Definition 3.12. A nonzero cohomology class $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ is called *integral* if κ lies in the image of

$$H^d(F^{*\prime}, \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$$

for some Dedekind ring $R \subseteq \overline{\mathcal{O}}$. If, in addition, there exists a torsion-free R-submodule $M \subseteq H^d(F^{*\prime}, \mathcal{D}_f(S_1, R))$ of rank ≤ 1 (i.e. M can be embedded into R) such that κ lies in the image of $M \otimes_R \mathbb{C} \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$, then κ is *integral of rank* ≤ 1.

For κ as in def. 3.12 and $R \subseteq \mathbb{C}$, we let $L_{\kappa,R}$ be the image of

$$H_d(F^{*\prime}, \mathcal{C}_c^0(S_1, R)) \to H_0(F^{*\prime}, \mathbb{C}) = \mathbb{C}, \quad x \mapsto \kappa \cap x.$$

PROPOSITION 3.13. *Let $\kappa \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ be integral. Then*
(a) μ_κ is a p-adic measure.
(b) There exists a Dedekind ring $R \subseteq \overline{\mathcal{O}}$ such that $L_{\kappa,R}$ is a finitely generated R-module (resp. a torsion-free R-module of rank ≤ 1, if κ is integral of rank ≤ 1).
For each such R, the map $H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}} \to \mathcal{H}^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ is injective and κ lies in its image.

Proof. The proofs of the corresponding results for totally real F ([Sp14], prop. 4.17 and cor. 4.18) also work in the general case. ☐

Remark 3.14. Let κ be integral with Dedekind ring R as above. By (b) of the proposition, we can view κ as an element of $H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}}$. Put $V_\kappa := L_{\kappa,R} \otimes_R \mathbb{C}_p$; let $\overline{\kappa}$ be the image of κ under the composition

$$\begin{array}{rcl} H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \overline{\mathbb{Q}} & \to & H^d(F^{*\prime}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \mathbb{C}_p \\ & \to & H^d(F^{*\prime}, \mathcal{D}_f^b(S_1, V_\kappa)), \end{array}$$

where the second map is induced by $\mathcal{D}_f(S_1, L_{\kappa,R}) \otimes_R \mathbb{C}_p \to \mathcal{D}_f^b(S_1, V_\kappa)$. By [Sp14], lemma 4.15, $\overline{\kappa}$ does not depend on the choice of R.
Since μ_κ is a p-adic measure, $\mu_{\overline{\kappa}}$ allows integration of all continuous functions $f \in C(\mathcal{G}_p, \mathbb{C}_p)$, and by abuse of notation, we write $L_p(\underline{s}, \kappa) := L_p(\underline{s}, \overline{\kappa})$ (cf. remark 3.6). So $L_p(\underline{s}, \kappa)$ has values in the finite-dimensional $\mathbb{C}_p$-vector space V_κ.

4 p-ADIC L-FUNCTIONS OF AUTOMORPHIC FORMS

We keep the notations from chapter 3; so F is again a number field with r real embeddings and s pairs of complex embeddings.
For an ideal $0 \neq \mathfrak{m} \subseteq \mathcal{O}_F$, we let $K_0(\mathfrak{m})_v \subseteq G(\mathcal{O}_{F_v})$ be the subgroup of matrices congruent to an upper triangular matrix modulo $\mathfrak{m}$, and we set $K_0(\mathfrak{m}) := \prod_{v \nmid \infty} K_0(\mathfrak{m})_v$, $K_0(\mathfrak{m})^S := \prod_{v \nmid \infty, v \notin S} K_0(\mathfrak{m})_v$ for a finite set of primes S. For each $\mathfrak{p}|p$, let $q_\mathfrak{p} = N(\mathfrak{p})$ denote the number of elements of the residue class field of $F_\mathfrak{p}$.
We denote by $|\cdot|_\mathbb{C}$ the square of the usual absolute value on $\mathbb{C}$, i.e. $|z|_\mathbb{C} = z\overline{z}$ for all $z \in \mathbb{C}$, and write $|\cdot|_\mathbb{R}$ for the usual absolute value on $\mathbb{R}$ in context. We write $|\alpha| := |\alpha|_\mathbb{C}^{\frac{1}{2}}$ for the archimedean absolute value when α is given as a complex number in the context; whereas in the context of the p-adic characters, $|\cdot|$ denotes the p-adic absolute value, unless otherwise noted.

Definition 4.1. Let $\mathfrak{A}_0(G, \underline{2}, \chi_Z)$ denote the set of all *cuspidal automorphic representations* $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_F)$ with central character χ_Z such that $\pi_v \cong \sigma(|\cdot|_{F_v}^{1/2}, |\cdot|_{F_v}^{-1/2})$ at all archimedian primes v. Here we follow the notation of [JL70]; so $\sigma(|\cdot|_{F_v}^{1/2}, |\cdot|_{F_v}^{-1/2})$ is the discrete series of weight 2, $\mathcal{D}(2)$, if v is real, and is isomorphic to the principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1(z) = z^{1/2}\overline{z}^{\,-1/2}$, $\mu_2(z) = z^{-1/2}\overline{z}^{1/2}$ if v is complex (cf. section 4.5 below).

We will only consider automorphic representations that are *p-ordinary* , i.e $\pi_\mathfrak{p}$ is ordinary (in the sense of chapter 2) for every $\mathfrak{p}|p$.
Therefore, for each $\mathfrak{p}|p$ we fix two non-zero elements $\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2} \in \overline{\mathcal{O}} \subseteq \mathbb{C}$ such that $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}$ is an ordinary, unitary representation. By the classification of unitary representations (see e.g. [Ge75], Thm. 4.27), a spherical representation $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if either χ_1, χ_2 are both unitary characters (i.e. $|\alpha_{\mathfrak{p},1}| = |\alpha_{\mathfrak{p},2}| = \sqrt{q_\mathfrak{p}}$), or $\chi_{1,2} = \chi_0|\cdot|^{\pm s}$ with χ_0 unitary and $-\frac{1}{2} < s < \frac{1}{2}$. A special representation $\pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if the central character $\chi_1\chi_2$ is unitary. In all three cases, we have thus $\max\{|\alpha_{\mathfrak{p},1}|, |\alpha_{\mathfrak{p},2}|\} \geq \sqrt{q_\mathfrak{p}}$. Without loss of generality, we will assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$.
As in chapter 2, we define $a_\mathfrak{p} := \alpha_{\mathfrak{p},1} + \alpha_{\mathfrak{p},2}$, $\nu_\mathfrak{p} := \alpha_{\mathfrak{p},1}\alpha_{\mathfrak{p},2}/q_\mathfrak{p}$.
Let $\underline{\alpha_i} := (\alpha_{\mathfrak{p},i}, \mathfrak{p}|p)$, for $i = 1, 2$. We denote by $\mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$ the subset of all $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z)$ such that $\pi_\mathfrak{p} = \pi_{\alpha_{\mathfrak{p},1},\alpha_{\mathfrak{p},2}}$ for all $\mathfrak{p}|p$.
For later use we note that $\pi^\infty = \otimes_{v\nmid\infty}\pi_v$ is known to be defined over a finite extension of $\mathbb{Q}$, the smallest such field being the *field of definition* of π (cf. [Sp14]).

4.1 Upper half-space

For $k \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{H}_m := \mathcal{H}_k := k \times \mathbb{R}_+^*$ be the upper half-space of dimension $m := [k : \mathbb{R}] + 1$. Each $\mathcal{H}_m$ is a differentiable manifold of dimension m. If we write $x = (u, t) \in \mathcal{H}_m$ with $t \in \mathbb{R}_+^*$, u in $\mathbb{R}$ or $\mathbb{C}$, respectively, it has a Riemannian metric $ds^2 = \frac{dt^2 + du\, d\overline{u}}{t}$, which induces a hyperbolic geometry on $\mathcal{H}_m$, i.e. the geodesic lines on $\mathcal{H}_m$ are given by "vertical" lines $\{u\} \times \mathbb{R}_+^*$ and half-circles with center in the line or plane $t = 0$. $\mathcal{H}_\mathbb{R}$ is naturally isomorphic to the complex upper half-plane $\{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$.
We have the decompositions $\mathrm{GL}_2(\mathbb{C})^+ = B'_\mathbb{C} \cdot Z(\mathbb{C}) \cdot K_\mathbb{C}$ and $\mathrm{GL}_2(\mathbb{R})^+ = B'_\mathbb{R} \cdot Z(\mathbb{R}) \cdot K_\mathbb{R}$, where $B'_k \subseteq GL_2(k)$ is the subgroup of matrices $\left(\begin{smallmatrix} \mathbb{R}_+^* & k \\ 0 & 1 \end{smallmatrix}\right)$ for $k = \mathbb{R}, \mathbb{C}$, Z is the center, and $K_\mathbb{R} = \mathrm{SO}(2)$, $K_\mathbb{C} = \mathrm{SU}(2)$ (cf. [By98], Cor. 43). Identifying B'_k with $\mathcal{H}_k$ via $\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \mapsto (z, t)$ gives natural projections

$$\pi_\mathbb{R} : \mathrm{GL}_2(\mathbb{R})^+ \twoheadrightarrow \mathrm{GL}_2(\mathbb{R})^+/Z(\mathbb{R})\,\mathrm{SO}(2) \cong \mathcal{H}_2,$$

$$\pi_\mathbb{C} : \mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{C})/Z(\mathbb{C})K_\mathbb{C} \cong \mathcal{H}_\mathbb{C}$$

and corresponding left $\mathrm{GL}_2(k)$-actions on cosets.
A differential form ω on $\mathcal{H}_m$ is called *left-invariant* if it is invariant under the pullback L_g^* of left multiplication $L_g : x \mapsto gx$ on $\mathcal{H}_m$, for all $g \in G$.

Following [By98], eqs. (4.20), (4.24), we choose the following basis of left-invariant differential 1-forms on $\mathcal{H}_3$:

$$\beta_0 := -\frac{dz}{t}, \quad \beta_1 := \frac{dt}{t}, \quad \beta_2 := \frac{d\overline{z}}{t},$$

and on $\mathcal{H}_2$ (writing $z = x + iy \in \mathcal{H}_2 \subseteq \mathbb{C}$):

$$\beta_1 := \frac{dz}{y}, \quad \beta_2 := -\frac{d\overline{z}}{y}.$$

We note that a form $f_1\beta_1 + f_2\beta_2$ is harmonic on $\mathcal{H}_2$ if and only if f_1/y and f_2/y are holomorphic functions in z ([By98], lemma 60).
The Jacobian $J(g, (0,1))$ of left multiplication by g in $(0,1) \in \mathcal{H}_m$ with respect to the basis $(\beta_i)_i$ gives rise to a representation

$$\varrho = \varrho_k : Z(k) \cdot K_k \to \mathrm{SL}_m(\mathbb{C})$$

with $\varrho|_{Z(k)}$ trivial, which on K_k is explicitly given by

$$\varrho_{\mathbb{C}}(h) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\overline{v} & u\overline{u} - v\overline{v} & v\overline{u} \\ \overline{v}^2 & -2\overline{uv} & \overline{u}^2 \end{pmatrix} \quad \text{for } h = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in \mathrm{SU}(2),$$

resp.

$$\varrho_{\mathbb{R}} \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} = \begin{pmatrix} e^{2i\vartheta} & 0 \\ 0 & e^{-2i\vartheta} \end{pmatrix}$$

([By98], (4.27), (4.21)). In the real case, we will only consider harmonic forms on $\mathcal{H}_2$ that are multiples of β_1, thus we sometimes identify $\varrho_{\mathbb{R}}$ with its restriction $\varrho_{\mathbb{R}}^{(1)}$ to the first basis vector β_1,

$$\varrho_{\mathbb{R}}^{(1)} : \mathrm{SO}(2) \to S^1 \subseteq \mathbb{C}^*, \quad \kappa_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mapsto e^{2i\vartheta}.$$

For each i, let ω_i be the left-invariant differential 1-form on $\mathrm{GL}_2(k)$ which coincides with the pullback $(\pi_{\mathbb{C}})^*\beta_i$ at the identity. Write $\underline{\omega}$ (resp. $\underline{\beta}$) for the column vector of the ω_i (resp. β_i). Then we have the following lemma from [By98]:

Lemma 4.2. *For each i, the differential ω_i on G induces β_i on $\mathcal{H}_m$, by restriction to the subgroup $B'_k \cong \mathcal{H}_m$. For a function $\phi : G \to \mathbb{C}^m$, the form $\phi \cdot \underline{\omega}$ (with $\mathbb{C}^m$ considered as a row vector, so $\cdot$ is the scalar product of vectors) induces $f \cdot \underline{\beta}$, where $f : \mathcal{H}_m \to \mathbb{C}^m$ is given by*

$$f(z,t) := \phi\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right).$$

(See [By98], Lemma 57.)
To consider the infinite primes of F all at once, we define

$$\mathcal{H}_\infty := \prod_{i=0}^{d} \mathcal{H}_{m_i} = \prod_{i=0}^{r-1} \mathcal{H}_2 \times \prod_{i=r}^{d} \mathcal{H}_3$$

(where $m_i = 2$ if σ_i is a real embedding, and $m_i = 3$ if σ_i is complex), and let $\mathcal{H}_\infty^0 := \prod_{i=1}^{d} \mathcal{H}_{m_i}$ be the product with the zeroth factor removed. (The choice of the 0-th factor is for convenience; we could also choose any other infinite place, whether real or complex.)

For each embedding σ_i, the elements of $\mathbb{P}^1(F)$ are cusps of $\mathcal{H}_{m_i}$: for a given complex embedding $F \hookrightarrow \mathbb{C}$, we can identify F with $F \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ and define the "extended upper half-space" as $\overline{\mathcal{H}_3} := \mathcal{H}_3 \cup F \cup \{\infty\} \subseteq \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$; similarly for a given real embedding $F \hookrightarrow \mathbb{R}$, we get the extended upper half-plane $\overline{\mathcal{H}_2} := \mathcal{H}_2 \cup F \cup \{\infty\}$. A basis of neighbourhoods of the cusp ∞ is given by the sets $\{(u,t) \in \mathcal{H}_m | t > N\}$, $N \gg 0$, and of $x \in F$ by the open half-balls in $\mathcal{H}_m$ with center $(x, 0)$.
Let $G(F)^+ \subseteq G(F)$ denote the subgroup of matrices with totally positive determinant. It acts on $\mathcal{H}_\infty^0$ by composing the embedding

$$G(F)^+ \hookrightarrow \prod_{v|\infty, v \neq v_0} G(F_v)^+, \qquad g \mapsto (\sigma_1(g), \ldots, \sigma_d(g)),$$

with the actions of $G(\mathbb{C})^+ = G(\mathbb{C})$ on $\mathcal{H}_3$ and $G(\mathbb{R})^+$ on $\mathcal{H}_2$ as defined above, and on $\Omega^d_{\mathrm{harm}}(\mathcal{H}_\infty^0)$ by the inverse of the corresponding pullback, $\gamma \cdot \underline{\omega} := (\gamma^{-1})^* \underline{\omega}$. Both are left actions.
For each complex v, we write the codomain of ϱ_{F_v} as

$$\varrho_{F_v} : Z(F_v) \cdot K_{F_v} \to \mathrm{SL}_3(\mathbb{C}) =: \mathrm{SL}(V_v),$$

for a three-dimensional $\mathbb{C}$-vector space V_v. We denote the harmonic forms on $\mathrm{GL}_2(F_v)$, $\mathcal{H}_{F_v}$ defined above by $\underline{\omega_v}$, $\underline{\beta_v}$ etc.
Let $V = \bigotimes_{v \in S_\mathbb{C}} V_v \cong (\mathbb{C}^3)^{\otimes s}$, $Z_\infty = \prod_{v|\infty} Z(F_v)$, $K_\infty = \prod_{v|\infty} K_{F_v}$. Denoting by $S_\mathbb{C}$ (resp. $S_\mathbb{R}$) the set of complex (resp. real) archimedian primes of F, we can merge the representations ϱ_{F_v} for each $v|\infty$ into a representation

$$\varrho = \varrho_\infty := \bigotimes_{v \in S_\mathbb{C}} \varrho_\mathbb{C} \otimes \bigotimes_{v \in S_\mathbb{R}} \varrho_\mathbb{R}^{(1)} : Z_\infty \cdot K_\infty \to \mathrm{SL}(V),$$

and define V-valued vectors of differential forms

$$\underline{\omega} := \bigotimes_{v \in S_\mathbb{C}} \underline{\omega_v} \otimes \bigotimes_{v \in S_\mathbb{R}} \omega_v^1, \quad \underline{\beta} := \bigotimes_{v \in S_\mathbb{C}} \underline{\beta_v} \otimes \bigotimes_{v \in S_\mathbb{R}} (\beta_v)_1$$

on $\mathrm{GL}_2(F_\infty)$ and $\mathcal{H}_\infty$, respectively.

4.2 Automorphic forms

Let $\chi_Z : \mathbb{A}_F^*/F^* \to \mathbb{C}^*$ be a Hecke character that is trivial at the archimedian places. We also denote by χ_Z the corresponding character on $Z(\mathbb{A}_F)$ under the isomorphism $\mathbb{A}_F^* \to Z(\mathbb{A}_F)$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

Definition 4.3. An *automorphic cusp form of parallel weight $\underline{2}$ with central character χ_Z* is a map $\phi : G(\mathbb{A}_F) \to V$ such that

(i) $\phi(z\gamma g) = \chi_Z(z)\phi(g)$ for all $g \in G(\mathbb{A})$, $z \in Z(\mathbb{A})$, $\gamma \in G(F)$.

(ii) $\phi(gk_\infty) = \phi(g)\varrho(k_\infty)$ for all $k_\infty \in K_\infty$, $g \in G(\mathbb{A})$ (considering V as a row vector).

(iii) ϕ has "moderate growth" on $B'_{\mathbb{A}} := \{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G(\mathbb{A})\}$, i.e. $\exists C, \lambda\ \forall A \in B'_{\mathbb{A}} : \|\phi(A)\| \le C \cdot \sup(|y|^\lambda, |y|^{-\lambda})$ (for any fixed norm $\|\cdot\|$ on V);

and $\phi|_{G(\mathbb{A}_\infty)} \cdot \underline{\omega}$ is the pullback of a harmon ic form $\omega_\phi = f_\phi \cdot \underline{\beta}$ on $\mathcal{H}_\infty$.

(iv) There exists a compact open subgroup $K' \subseteq G(\mathbb{A}^\infty)$ such that $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K'$.

(v) For all $g \in G(\mathbb{A}_F)$,

$$\int_{\mathbb{A}_F/F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx = 0. \qquad \text{(“Cuspidality”)}$$

We denote by $\mathcal{A}_0(G, \text{harm}, \underline{2}, \chi_Z)$ the space of all such maps ϕ.

For each $g^\infty \in \mathbb{A}_F^\infty$, let $\omega_\phi(g^\infty)$ be the restriction of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ from $G(\mathbb{A}_F^\infty)$ to $\mathcal{H}_\infty$; it is a $(d+1)$-form on $\mathcal{H}_\infty$.
We want to integrate $\omega_\phi(g^\infty)$ between two cusps of the space $\mathcal{H}_{m_0}$. (We will identify each $x \in \mathbb{P}^1(F)$ with its corresponding cusp in $\overline{\mathcal{H}_{m_0}}$ in the following.) The geodesic between the cusps $x \in F$ and ∞ in $\overline{\mathcal{H}_{m_0}}$ is the line $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ and the integral of ω_ϕ along it is finite since ϕ is uniformly rapidly decreasing:

Theorem 4.4. *(Gelfand, Piatetski-Shapiro) An automorphic cusp form ϕ is rapidly decreasing modulo the center on a fundamental domain $\mathcal{F}$ of $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)$;*
i.e. there exists an integer r such that for all $N \in \mathbb{N}$ there exists a $C > 0$ such that

$$\phi(zg) \le C|z|^r\|g\|^{-N}$$

for all $z \in Z(\mathbb{A}_F)$, $g \in \mathcal{F} \cap \mathrm{SL}_2(\mathbb{A}_F)$. Here $\|g\| := \max\{|g_{i,j}|, |(g^{-1})_{i,j}|\}_{i,j\in\{1,2\}}$.

(See [CKM04], Thm. 2.2; or [Kur78], (6) for quadratic imaginary F.)
In fact, the integral of $\omega_\phi(g^\infty)$ along $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ equals the integral of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ along a path $g_t \in \mathrm{GL}_2(F_{\infty_0})$, $t \in \mathbb{R}_+^*$, where we can choose

$$g_t = \frac{1}{\sqrt{t}} \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{t}} & \frac{x}{\sqrt{t}} \\ 0 & \sqrt{t} \end{pmatrix},$$

and thus have $\|g_t\| = \sqrt{t}$ for all $t \gg 0$, $\|g_t\| = C\frac{1}{\sqrt{t}}$ for $t \ll 1$, so the integral $\int_x^\infty \omega_\phi(g^\infty) \in \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty)$ is well-defined by the theorem.
For any two cusps $a, b \in \mathbb{P}^1(F)$, we now define

$$\int_a^b \omega_\phi(g^\infty) := \int_a^\infty \omega_\phi(g^\infty) - \int_b^\infty \omega_\phi(g^\infty) \in \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty).$$

Since ϕ is *uniformly* rapidly decreasing ($\|g_t\|$ does not depend on x, for $t \gg 0$), this integral along the path $(a, 0) \to (a, \infty) = (b, \infty) \to (b, 0)$ in $\overline{\mathcal{H}}_{m_0}$ is the same as the limit (for $t \to \infty$) of the integral along $(a, 0) \to (a, t) \to (b, t) \to (b, 0)$; and since ω_ϕ is harmonic (and thus integration is path-independent within $\mathcal{H}_{m_0}$) the latter is in fact independent of t, so equality holds for each $t > 0$, or along any path from $(a, 0)$ to $(b, 0)$ in $\mathcal{H}_{m_0}$. Thus $\int_a^b \omega_\phi(g^\infty)$ equals the integral of $\omega_\phi(g^\infty)$ along the geodesic from a to b, and we have

$$\int_a^b \omega_\phi(g^\infty) + \int_b^c \omega_\phi(g^\infty) = \int_a^c \omega_\phi(g^\infty)$$

for any three cusps $a, b, c \in \mathbb{P}^1(F)$. Let $\mathrm{Div}(\mathbb{P}^1(F))$ denote the free abelian group of divisors of $\mathbb{P}^1(F)$, and let $\mathcal{M} := \mathrm{Div}_0(\mathbb{P}^1(F))$ be the subgroup of divisors of degree 0.
We can extend the definition of the integral linearly to get a homomorphism

$$\mathcal{M} \to \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty), \quad m \mapsto \int_m \omega_\phi(g^\infty),$$

and easily check that

$$\gamma^* \left(\int_{\gamma m} \omega_\phi(\gamma g) \right) = \int_m \omega_\phi(g). \tag{29}$$

for all $\gamma \in G(F)^+$, $g \in G(\mathbb{A}^\infty)$, $m \in \mathcal{M}$.
Now let $\mathfrak{m}$ be an ideal of F prime to p, let χ_Z be a Hecke character of conductor dividing $\mathfrak{m}$, and $\underline{\alpha_1}, \underline{\alpha_2}$ as above.

Definition 4.5. We define $S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2})$ to be the $\mathbb{C}$-vector space of all maps

$$\Phi : G(\mathbb{A}^p) \to \mathcal{B}^{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, V) = \mathrm{Hom}(\mathcal{B}_{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, \mathbb{C}), V)$$

such that:

(a) ϕ is "almost" $K_0(\mathfrak{m})$-invariant (in the notation of [Ge75]), i.e. $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A}^p)$ and $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$, and $\phi(gk) = \chi_Z(a)\phi(g)$ for all $v|\mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^p)$.

(b) For each $\psi \in \mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C})$, the map

$$\langle \Phi, \psi \rangle : G(\mathbb{A}) = G(F_p) \times G(\mathbb{A}^p) \to V,\ (g_p, g^p) \mapsto \Phi(g^p)(g_p\psi)$$

lies in $\mathcal{A}_0(G, \mathrm{harm}, \underline{2}, \chi_Z)$.

Note that (a) implies that ϕ is K'-invariant for some open subgroup $K' \subseteq K_0(\mathfrak{m})^p$ of finite index ([By98]/[We71]).

4.3 Cohomology of $\mathrm{GL}_2(F)$

Let M be a left $G(F)$-module and N an $R[H]$-module, for a ring R and a subgroup $H \subseteq G(F)$. Let $S \subseteq S_p$ be a set of primes of F dividing p; as above, let $\chi = \chi_Z$ be a Hecke character of conductor $\mathfrak{m}$ prime to p.

Definition 4.6. For a compact open subgroup $K \subseteq K_0(\mathfrak{m})^S \subseteq G(\mathbb{A}^{S,\infty})$, we denote by $\mathcal{A}_f(K, S, M; N)$ the R-module of all maps $\Phi : G(\mathbb{A}^{S,\infty}) \times M \to N$ such that

1. $\Phi(gk, m) = \Phi(g, m)$ for all $g \in G(\mathbb{A}^{S,\infty})$, $m \in M$, $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$;
2. $\Phi(gk) = \chi_Z(a)\Phi(g)$ for all $v|\mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^{S,\infty})$, $m \in M$.

We denote by $\mathcal{A}_f(S, M; N)$ the union of the $\mathcal{A}_f(K, S, M; N)$ over all compact open subgroups K.

$\mathcal{A}_f(S, M; N)$ is a left $G(\mathbb{A}^{S,\infty})$-module via $(\gamma \cdot \Phi)(g, m) := \Phi(\gamma^{-1}g, m)$ and has a left H-operation given by $(\gamma \cdot \Phi)(g, m) := \gamma\Phi(\gamma^{-1}g, \gamma^{-1}m)$, commuting with the $G(\mathbb{A}^{S,\infty})$-operation.
In contrast to our previous notation, we consider two subsets $S_1 \subseteq S_2 \subseteq S_p$ in this section. We put $(\underline{\alpha_1}, \underline{\alpha_2})_{S_1} := \{(\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}) | \mathfrak{p} \in S_1\}$, we set

$$\mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) = \mathcal{A}_f(S_2, M; \mathcal{B}^{(\underline{\alpha_1},\underline{\alpha_2})_{S_1}}(F_{S_1}, N));$$

we write $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) := \mathcal{A}_f(K_0(\mathfrak{m}), (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N)$. If $S_1 = S_2$, we will usually drop S_2 from all these notations.
We have a natural identification of $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N)$ with the space of maps $G(\mathbb{A}^{S,\infty}) \times M \times \mathcal{B}_{(\underline{\alpha_1},\underline{\alpha_2})_S}(F_S, R) \to N$ that are "almost" K-invariant.
Let $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_p$ be subsets. The pairing (11) induces a pairing

$$\mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \times \mathcal{B}_{(\underline{\alpha_1},\underline{\alpha_2})_{S_0}}(F_{S_0}, R) \to \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N)$$

which, when restricting to K-invariant elements, induces an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \cong \mathcal{B}^{(\underline{\alpha_1}, \underline{\alpha_2})_{S_1 - S_0}}(F_{S_1 - S_0}, \mathcal{A}_f(\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N).$$

Putting $S_0 := S_1 - \{\mathfrak{p}\}$ for a prime $\mathfrak{p} \in S_1$, we specifically get an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_{S_1}, S_2, M; N) \cong \mathcal{B}^{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}(F_{\mathfrak{p}}, \mathcal{A}_f(\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S_2, M; N).$$

Lemmas 2.9 and 2.10 now immediately imply the following:

LEMMA 4.7. *Let $S \subseteq S_p$, $\mathfrak{p} \in S$, $S_0 := S - \{\mathfrak{p}\}$. Let $K \subseteq G(\mathbb{A}^{S,\infty})$ be a compact open subgroup.*
(a) If $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is spherical, we have exact sequences

$$0 \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N) \to Z \xrightarrow{N - \nu_{\mathfrak{p}}} Z \to 0$$

and

$$0 \to Z \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N) \xrightarrow{T - a_{\mathfrak{p}}} \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N) \to 0$$

for a $G(\mathbb{A}^{S_0,\infty})$-module Z and a compact open subgroup $K_0 = K \times K_{\mathfrak{p}}$ of $G(\mathbb{A}^{S_0,\infty})$.
(b) If $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is special (with central character $\chi_{\mathfrak{p}}$), we have exact sequences

$$0 \to \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; N) \to Z' \to Z \to 0$$

and

$$\begin{aligned} 0 \to & Z \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to \mathcal{A}_f(K_0, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to 0, \\ 0 \to & Z' \to \mathcal{A}_f(K_0', (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to \mathcal{A}_f(K_0', (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, M; N)^2 \to 0, \end{aligned}$$

with $Z^{(')} := \mathcal{A}_f(K_0^{(')}, (\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, S, M; N(\chi_{\mathfrak{p}}))$, where $K_0^{(')} = K \times K_{\mathfrak{p}}^{(')}$ are compact open subgroups of $G(\mathbb{A}^{S_0,\infty})$.

PROPOSITION 4.8. *Let $S \subseteq S_p$ and let K be a compact open subgroup of $G(\mathbb{A}^{S,\infty})$.*
(a) For each flat R-module N (with trivial $G(F)$-action), the canonical map

$$H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)) \otimes_R N \to H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; N))$$

is an isomorphism for each $q \geq 0$.
(b) If R is finitely generated as a $\mathbb{Z}$-module, $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)$ is finitely generated over R.

Proof. We can copy the proof of [Sp14], Prop. 5.6, using lemma 4.7 instead of [Sp14], lemma 5.4 to reduce to the case $S = \varnothing$. ◻

We define

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; R)) := \varinjlim H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; R))$$

where the limit runs over all compact open subgroups $K \subseteq G(\mathbb{A}^{S,\infty})$; and similarly define $H^q_*(B(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)$. The proposition immediately implies

Corollary 4.9. *Let $R \to R'$ be a flat ring homomorphism. Then the canonical map*

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R)) \otimes_R R' \to H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; R')$$

is an isomorphism, for all $q \geq 0$.

If $R = k$ is a field of characteristic zero, $H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; R)$ is a smooth $G(\mathbb{A}^{S,\infty})$-module, and we have

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)^K = H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; k).$$

We identify $G(F)/G(F)^+$ with the group $\Sigma = \{\pm 1\}^r$ via the isomorphism

$$G(F)/G(F^+) \xrightarrow{\det} F^*/F^*_+ \cong \Sigma$$

(with all groups being trivial for $r = 0$). Then Σ acts on $H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)$ and $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha_1}, \underline{\alpha_2})_S, M; k)$ by conjugation.For $\pi \in \mathfrak{A}_0(G, \underline{2})$ and $\underline{\mu} \in \Sigma$, we write $H^q_*(G(F)^+, \cdot)_{\pi,\underline{\mu}} := \mathrm{Hom}_{G(\mathbb{A}^{S,\infty})}(\pi^S, H^q_*(G(F)^+, \cdot))_{\underline{\mu}}$.

Proposition 4.10. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$, $S \subseteq S_p$. Let k be a field which contains the field of definition of π. Then for every $\underline{\mu} \in \Sigma$, we have*

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; k)_{\pi,\underline{\mu}} = \begin{cases} k, & \text{if } q = d; \\ 0, & \text{if } q \in \{0, \dots, d-1\} \end{cases} \tag{30}$$

Proof. The case $S = \varnothing$ is proved analogously to [Sp14], prop. 5.8, using the results of Harder [Ha87]. For $S = S_0 \cup \{\mathfrak{p}\}$ and $\pi_\mathfrak{p}$ spherical, lemma 4.7(a) and the statement for S_0 give an isomorphism

$$H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_{S_0}, \mathcal{M}; k))_{\pi,\underline{\mu}} \cong H^q_*(G(F)^+, \mathcal{A}_f((\underline{\alpha_1}, \underline{\alpha_2})_S, \mathcal{M}; k))_{\pi,\underline{\mu}}$$

since the Hecke operators $T_\mathfrak{p}$ and $N_\mathfrak{p}$ act on the left-hand side by multiplication with $a_\mathfrak{p}$ and $\nu_\mathfrak{p}$, respectively. If $\pi_\mathfrak{p}$ is special, we can similarly deduce the statement for S from that for S_0, using the first exact sequence of lemma 4.7(b), since the results of [Ha87] also hold when twisting k by a (central) character. ☐

4.4 Eichler-Shimura map

From now on, let $S_1 \subseteq S_p$ be the set of places such that $\pi_\mathfrak{p}$ is the Steinberg representation (i.e. $\alpha_{\mathfrak{p},1} = \nu_\mathfrak{p} = 1$, $\alpha_{\mathfrak{p},2} = q$).
Given a subgroup $K_0(\mathfrak{m})^p \subseteq G(\mathbb{A}^{p,\infty})$ as above, there is a map

$$I_0 : S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \to H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}; \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty)))$$

given by

$$I_0(\Phi) : (\psi, (g, m)) \mapsto \int_m \omega_{\langle \Phi, \psi \rangle}(1_p, g),$$

for $\psi \in \mathcal{B}_{\underline{\alpha_1}, \underline{\alpha_2}}(F_p, \mathbb{C}), g \in G(\mathbb{A}^{p,\infty}), m \in \mathcal{M}$, where 1_p denotes the unity element in $G(F_p)$.
This is well-defined since both sides are "almost" $K_0(\mathfrak{m})$-invariant, and the $G(F)^+$-invariance of $I_0(\Phi)$ follows from a straightforward calculation, using (29).
From the complex

$$\mathcal{A}_f(m, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}; \mathbb{C}) \to C^\bullet := \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}; \Omega^\bullet_{\mathrm{harm}}(\mathcal{H}^0_\infty))$$

we get a map

$$S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \to H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}; \mathbb{C})) \tag{31}$$

by composing I_0 with an edge morphism of the spectral sequence

$$H^q(G(F)^+, C^p) \implies H^{p+q}(G(F)^+, C^\bullet).$$

Using the map $\delta^{\underline{\alpha_1}, \underline{\alpha_2}} : \mathcal{B}^{\underline{\alpha_1}, \underline{\alpha_2}}(F, V) \to \mathrm{Dist}(F_p^*, V)$ from section 2.6, we next define a map

$$\Delta_V^{\underline{\alpha_1}, \underline{\alpha_2}} : S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \to \mathcal{D}(S_1, V) \tag{32}$$

by

$$\Delta_V^{\underline{\alpha_1}, \underline{\alpha_2}}(\Phi)(U, x^p) = \delta^{\underline{\alpha_1}, \underline{\alpha_2}} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right) (U)$$

for $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}), x^p \in \mathbb{I}^p$, and we denote by $\Delta^{\underline{\alpha_1}, \underline{\alpha_2}} : S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \to \mathcal{D}(S_1, \mathbb{C})$ its (1,...,1)th coordinate function (i.e. corresponding to the harmonic forms $\bigotimes_{v|\infty}(\omega_v)_1$, $\bigotimes_{v|\infty}(\beta_v)_1$ in section 4.1):

$$\Delta^{\underline{\alpha_1}, \underline{\alpha_2}}(\Phi)(U, x^p) = \delta^{\underline{\alpha_1}, \underline{\alpha_2}} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right)_{(1,\ldots,1)} (U).$$

Since for each complex prime v, $S^1 \cong \mathrm{SU}(2) \cap T(\mathbb{C})$ operates on Φ via ϱ_v, $\Delta^{\underline{\alpha_1}, \underline{\alpha_2}}$ is easily seen to be S^1-invariant, i.e. it lies in $\mathcal{D}'(S_1, \mathbb{C})$.
We also have a natural (i.e. commuting with the complex maps of each $C^\bullet$) family of maps

$$\mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \Omega^i_{\mathrm{harm}}(\mathcal{H}^0_\infty)) \to \mathcal{D}_f(S_1, \Omega^i(U^0_\infty, \mathbb{C})) \tag{33}$$

for all $i \geq 0$, and

$$\mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C}) \to \mathcal{D}_f(S_1, \mathbb{C}) \tag{34}$$

(the $i = -1$-th term in the complexes), by mapping $\Phi \in \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \cdot)$ first to

$$(U, x^{p,\infty}) \mapsto \Phi\left(\begin{pmatrix} x^{p,\infty} & 0 \\ 0 & 1 \end{pmatrix}, \infty - 0\right)(\delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U)) \in \Omega^i_{\mathrm{harm}}(\mathcal{H}^0_\infty) \text{ resp. } \in \mathbb{C},$$

and then for $i \geq 0$ restricting the differential forms to $\Omega^i(U^0_\infty)$ via

$$U^0_\infty = \prod_{v \in S^0_\infty} \mathbb{R}^*_+ \hookrightarrow \prod_{v \in S^0_\infty} \mathcal{H}_v = \mathcal{H}^0_\infty.$$

One easily checks that (33) and (34) are compatible with the homomorphism of "acting groups" $F^{*\prime} \hookrightarrow G(F)^+, x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so we get induced maps in cohomology

$$H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty))) \to H^0(F^{*\prime}, \mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) \tag{35}$$

and

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C})) \to H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})), \tag{36}$$

which are linked by edge morphisms of the respective spectral sequences to give a commutative diagram (given in the proof below).

Proposition 4.11. *We have a commutative diagram:*

$$\begin{tikzcd}
S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \arrow[r, "(31)"] \arrow[d, "\Delta^{\underline{\alpha_1},\underline{\alpha_2}}"] & H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C})) \arrow[d, "(36)"] \\
\mathcal{D}'(S_1, \mathbb{C}) \arrow[r, "\phi \mapsto \kappa_\phi"] & H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})\big)
\end{tikzcd}$$

Proof. The given diagram factorizes as

$$\begin{tikzcd}
S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}) \arrow[r, "I_0"] \arrow[d, "\Delta^{\underline{\alpha_1},\underline{\alpha_2}}"] & H^0(G(F)^+, \mathcal{A}_f(\Omega^d_{\mathrm{harm}}(\mathcal{H}^0_\infty))) \arrow[r] \arrow[d, "(35)"] & H^d(G(F)^+, \mathcal{A}_f(\mathbb{C})) \arrow[d, "(36)"] \\
\mathcal{D}'(S_1, \mathbb{C}) \arrow[r] & H^0(F^{*\prime}, \mathcal{D}_f(S_1, \Omega^d(U^0_\infty, \mathbb{C}))) \arrow[r] & H^d\big(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C})\big)
\end{tikzcd}$$

(where we write $\mathcal{A}_f(\cdot)$ instead of $\mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \cdot)$ for brevity). The right-hand square is the naturally commutative square mentioned above; the commutativity of the left-hand square can easily be checked by hand. $\square$

4.5 Whittaker model

We now consider an automorphic representation $\pi = \otimes_\nu \pi_\nu \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$. Denote by $\mathfrak{c}(\pi) := \prod_{v \text{ finite}} \mathfrak{c}(\pi_v)$ the conductor of π.

Let $\chi : \mathbb{I}^\infty \to \mathbb{C}^*$ be a unitary character of the finite ideles; for each finite place v, set $\chi_v = \chi|_{F_v^*}$. For each prime v of F, let $\mathcal{W}_v$ denote the Whittaker model of π_v. For each finite and each real prime, we choose $W_v \in \mathcal{W}_v$ such that the local L-factor equals the local zeta function at $g = 1$, i.e. such that

$$L(s, \pi_v \otimes \chi_v) = \int_{F_v^*} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \chi_v(x) |x|^{s - \frac{1}{2}} \, d^\times x \tag{37}$$

for any unramified quasi-character $\chi_v : F_v^* \to \mathbb{C}^*$ and $\mathrm{Re}(s) \gg 0$.

This is possible by [Ge75], Thm. 6.12 (ii); and by loc.cit., Prop. 6.17, W_v can be chosen such that $\mathrm{SO}(2)$ operates on W_v via ϱ_v for real archimedian v, and is "almost" $K_0(\mathfrak{c}(\pi_v))$-invariant for finite v.

For complex primes v of F, we can also choose a W_v satisfying (37) and which behaves well with respect to the $\mathrm{SU}(2)$-action ϱ_v, as follows:

By [Kur77], there exists a function

$$\underline{W_v} = (W_v^0, W_v^1, W_v^2) : G(F_v) \to \mathbb{C}^3$$

such that $W_v^i \in \mathcal{W}_v$ for all i, and such that $\mathrm{SU}(2)$ operates by the right via ϱ_v on $\underline{W_v}$; i.e. for all $g \in G(F_v)$ and $h \in \mathrm{SU}(2)$, we have

$$\underline{W_v}(gh) = \underline{W_v}(g) \varrho_{\mathbb{C}}(h).$$

Note that W_v^1 is thus invariant under right multiplication by a diagonal matrix $\begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}$ with $u \in S^1 \subseteq \mathbb{C}$. Since π_v has trivial central character for archimedian v by our assumption, a function in $\mathcal{W}_v$ is also invariant under $Z(F_v)$. Thus we have

$$W_v^1 \left(g \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) = W_v^1(g) \quad \text{for all } g \in G(F_v), \ u \in S^1.$$

W_v^1 can be described explicitly in terms of a certain Bessel function, as follows. The modified Bessel differential equation of order $\alpha \in \mathbb{C}$ is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2) y = 0.$$

Its solution space (on $\{\mathrm{Re}\, z > 0\}$) is two-dimensional; we are only interested in the second standard solution K_v, which is characterised by the asymptotics

$$K_v(z) \sim \sqrt{\frac{\pi}{2z}} \, e^{-z}$$

(cf. [We71]). By [Kur77],[3] we have $W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{\pi} x^2 K_0(4\pi x)$.
(W_v^0 and W_v^2 can also be described in terms of Bessel functions; they are linearly dependent and scalar multiples of $x^2 K_1(4\pi x)$.)
By [JL70], Ch. 1, Thm. 6.2(vi), $\sigma(|\cdot|_{\mathbb{C}}^{1/2}, |\cdot|_{\mathbb{C}}^{-1/2}) \cong \pi(\mu_1, \mu_2)$ with

$$\mu_1(z) = z^{1/2}\overline{z}^{\,-1/2} = |z|_{\mathbb{C}}^{-1/2} z, \qquad \mu_2(z) = z^{-1/2}\overline{z}^{1/2} = |z|_{\mathbb{C}}^{-1/2}\overline{z};$$

and the L-series of the representation is the product of the L-factors of these two characters:

$$\begin{aligned} L_v(s, \pi_v) = L(s, \mu_1) L(s, \mu_2) &= 2\,(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\tfrac{1}{2}) \cdot 2\,(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\tfrac{1}{2}) \\ &= 4\,(2\pi)^{-(2s+1)}\Gamma(s+\tfrac{1}{2})^2. \end{aligned}$$

On the other hand, letting $d^\times x = \frac{dx}{|x|_{\mathbb{C}}} = \frac{dr}{r} d\vartheta$ (for $x = re^{i\vartheta}$), we have for $\mathrm{Re}(s) > -\frac{1}{2}$:

$$\begin{aligned} \int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, d^\times x &= \int_{S^1} \int_{\mathbb{R}_+} W_v^1 \begin{pmatrix} re^{i\vartheta} & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, \frac{dr}{r}\, d\vartheta \\ &= 4 \int_0^\infty x^2 K_0(4\pi x) x^{2s-1}\, \frac{dx}{x} \\ &\quad \text{(invariance under } \mathrm{SU}(2)\cdot Z(F_v) \text{ gives a constant integral w.r.t. } \vartheta) \\ &= 4\,(4\pi)^{-2s+1} \int_0^\infty K_0(x) x^{2s}\, dx \\ &= 4\,(4\pi)^{-2s+1}\, 2^{2s-1}\, \Gamma(s+\tfrac{1}{2})^2 \\ &= 4\,(2\pi)^{-2s+1}\, \Gamma(s+\tfrac{1}{2})^2 \end{aligned}$$

by ([DLMF] 10.43.19). Thus we have

$$\int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}}\, d^\times x = (2\pi)^2\, L_v(s, \pi_v)$$

for all $\mathrm{Re}(s) > -\frac{1}{2}$.We set $W_v := (2\pi)^{-2}\, W_v^1$; thus (37) holds also for complex primes.
Now that we have defined W_v for all primes v, we put $W^p(g) := \prod_{v \nmid p} W_v(g_v)$ for all $g = (g_v)_v \in G(\mathbb{A}^p)$. We will also need the vector-valued function $\underline{W}^p : G(\mathbb{A}_F) \to V$ given by

$$\underline{W}^p(g) := \prod_{v \nmid p \text{ finite or } v \text{ real}} W_v(g_v) \cdot \bigotimes_{v \text{ complex}} (2\pi)^{-2} \underline{W_v}(g_v).$$

[3] Note that [Kur77] uses a slightly different definition of the K_v, which is $\frac{2}{\pi}$ times our K_v.

4.6 p-ADIC MEASURES OF AUTOMORPHIC FORMS

Now return to our $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$. We fix an additive character $\psi : \mathbb{A} \to \mathbb{C}^*$ which is trivial on F, and let ψ_v denote the restriction of ψ to $F_v \hookrightarrow \mathbb{A}$, for all primes v. We further require that $\ker(\psi_\mathfrak{p}) \supseteq \mathcal{O}_\mathfrak{p}$ and $\mathfrak{p}^{-1} \not\subseteq \ker \psi_\mathfrak{p}$ for all $\mathfrak{p}|p$, so that we can apply the results of chapter 2.
As in chapter 2, let $\mu_{\pi_\mathfrak{p}} := \mu_{\alpha_{\mathfrak{p},1}/\nu_\mathfrak{p}} = \mu_{q_\mathfrak{p}/\alpha_{\mathfrak{p},2}}$ denote the distribution $\chi_{q_\mathfrak{p}/\alpha_{\mathfrak{p},2}}(x)\psi_\mathfrak{p}(x)dx$ on $F_\mathfrak{p}$, and let $\mu_{\pi_p} := \prod_{\mathfrak{p}|p} \mu_{\pi_\mathfrak{p}}$ be the product distribution on $F_p := \prod_{\mathfrak{p}|p} F_\mathfrak{p}$.
Define $\phi = \phi_\pi : \mathfrak{Co}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^p \to \mathbb{C}$ by

$$\phi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}.$$

By proposition 2.13(a), we have for each $U \in \mathfrak{Co}(F_{S_1} \times F_{S_2}^*)$:

$$\begin{aligned} \phi_U(x) := \phi(x_p U, x^p) &= \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta x_p U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\zeta \in F^*} W \begin{pmatrix} \zeta x & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $W(g) := W_U(g_p) W^p(g^p)$ lies in the global Whittaker model $\mathcal{W} = \mathcal{W}(\pi)$ for all $g = (g_p, g^p) \in G(\mathbb{A})$, putting $W_U := W_{1_U}$; so ϕ is well-defined and lies in $\mathcal{D}(S_1, \mathbb{C})$ (since W is smooth and rapidly decreasing; distribution property, F^*- and $U^{p,\infty}$-invariance being clear by the definitions of ϕ and W^p).
Let $\mu_\pi := \mu_{\phi_\pi}$ be the distribution on $\mathcal{G}_p$ corresponding to ϕ_π, as defined in (22), and let $\kappa_\pi := \kappa_{\phi_\pi} \in H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))$ be the cohomology class defined by (23) and (24).

THEOREM 4.12. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha_1}, \underline{\alpha_2})$; we assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$. (So $\chi_{\mathfrak{p},1} = |\cdot|\chi_{\mathfrak{p},2}$ for all special $\pi_\mathfrak{p}$.)*
(a) Let $\chi : \mathcal{G}_p \to \mathbb{C}^*$ be a character of finite order with conductor $\mathfrak{f}(\chi)$. Then we have the interpolation property

$$\int_{\mathcal{G}_p} \chi(\gamma) \mu_\pi(d\gamma) = \tau(\chi) \prod_{\mathfrak{p} \in S_p} e(\pi_\mathfrak{p}, \chi_\mathfrak{p}) \cdot L(\tfrac{1}{2}, \pi \otimes \chi),$$

where

$$e(\pi_\mathfrak{p}, \chi_\mathfrak{p}) = \begin{cases} \dfrac{(1 - \alpha_{\mathfrak{p},1} x_\mathfrak{p} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p}^{-1} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p} q_\mathfrak{p}^{-1})}{(1 - x_\mathfrak{p} \alpha_{\mathfrak{p},2}^{-1})}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0 \\ & \textit{and } \pi \textit{ spherical,} \\ \dfrac{(1 - \alpha_{\mathfrak{p},1} x_\mathfrak{p} q_\mathfrak{p}^{-1})(1 - \alpha_{\mathfrak{p},2} x_\mathfrak{p}^{-1} q_\mathfrak{p}^{-1})}{(1 - x_\mathfrak{p} \alpha_{\mathfrak{p},2}^{-1})}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0 \\ & \textit{and } \pi \textit{ special,} \\ (\alpha_{\mathfrak{p},2}/q_\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi))}, & \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) > 0 \end{cases}$$

and $x_{\mathfrak{p}} := \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})$.
(b) κ_π *is integral (cf. definition 3.12). For* $\underline{\mu} \in \Sigma$*, let* $\kappa_{\pi,\underline{\mu}}$ *be the projection of* κ_π *to* $H^d(F^{*\prime}, \mathcal{D}_f(S_1, \mathbb{C}))_{\pi,\underline{\mu}}$*. Then* $\kappa_{\pi,\underline{\mu}}$ *is integral of rank* ≤ 1*.*

Proof. (a) We consider χ as a character on $\mathbb{I}_F/F^*$, and choose a subgroup $V = \prod_{\mathfrak{p}|p} V_{\mathfrak{p}} \subseteq U_p$ such that $\chi_p|_V = 1$.
Since π is unitary, we have $|\alpha_{\mathfrak{p},2}| \geq \sqrt{q_{\mathfrak{p}}} > 1 = |\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})|$ for all $\mathfrak{p}$, thus $e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)$ is non-singular for all $s \geq 0$, and we will be able to apply proposition 2.4 locally below.
We have

$$\int_{\mathcal{G}_p} \chi(\gamma)\mu_\pi(d\gamma) = [U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x)\phi_V(x)d^\times x,$$

and therefore we have to show that the equality

$$[U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x)|x|^s\phi_V(x)d^\times x = N(\mathfrak{f}(\chi))^s\tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)\cdot L(s+\tfrac{1}{2}, \pi\otimes\chi)$$

holds for $s = 0$. Since both the left-hand side and $L(s+\frac{1}{2}, \pi\otimes\chi)$ are holomorphic in s (cf. [Ge75], Thm. 6.18), it suffices to show this for $\mathrm{Re}(s) \gg 0$. But for such s, we have

$$[U_p : V] \int_{\mathbb{I}_F/F^*} \chi(x)|x|^s\phi_V(x)d^\times x = \int_{\mathbb{I}_F} \chi(x)|x|^s W\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x$$

$$\begin{aligned}
&= [U_p : V] \int_{F_p^*} \chi_p(x)|x|^s W_V\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x \cdot \int_{\mathbb{I}_F^p} \chi^p(y)|y|^s W^p\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^\times y \\
&= \prod_{\mathfrak{p}|p} \int_{F_{\mathfrak{p}}^*} \chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^s \mu_{\pi_{\mathfrak{p}}}(dx) \cdot L_{S_p}(s+\tfrac{1}{2}, \pi\otimes\chi) \\
&= \prod_{\mathfrak{p}|p} \left(e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)\tau(\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)\right) \cdot L(s+\tfrac{1}{2}, \pi\otimes\chi) \\
&= N(\mathfrak{f}(\chi))^s\tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \cdot L(s+\tfrac{1}{2}, \pi\otimes\chi)
\end{aligned}$$

by propositions 2.13, 2.4 and equation (37).
(b) Let $\lambda_{\underline{\alpha_1},\underline{\alpha_2}} \in \mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C})$ be the image of $\otimes_{v|p}\lambda_{a_v,\nu_v}$ under the map (13). For each $\psi \in \mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F_p, \mathbb{C})$, define

$$\begin{aligned}
\langle\Phi_\pi, \psi\rangle(g^p, g_p) \;&:=\; \sum_{\zeta\in F^*} \lambda_{\underline{\alpha_1},\underline{\alpha_2}}\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g_p \cdot \psi\right) \underline{W}^p\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g^p\right) \\
&=: \sum_{\zeta\in F^*} \underline{W_\psi}\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g\right)
\end{aligned}$$

for a V-valued function $\underline{W_\psi}$ whose every coordinate function is in $\mathcal{W}(\pi)$.
This defines a map $\Phi_\pi : G(\mathbb{A}^p) \to \mathcal{B}^{\underline{\alpha_1},\underline{\alpha_2}}(F_p, V)$. In fact, Φ_π lies in $S_2(G, \mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2})$, where $\mathfrak{m}$ is the prime-to-p part of $\mathfrak{f}(\pi)$:
Condition (a) of definition 4.5 follows from the fact that the W_v are almost $K_0(\mathfrak{c}(\pi_v))$-invariant, for $v \nmid p, \infty$. For condition (b), we check that $\langle \Phi_\pi, \psi\rangle$ satisfies the conditions (i)-(v) in the definition of $\mathcal{A}_0(G, \mathrm{harm}, \underline{2}, \chi)$:
Each coordinate function of $\langle \Phi_\pi, \psi\rangle$ lies in (the underlying space of) π by [Bu98], Thm. 3.5.5, thus $\langle \Phi, \psi\rangle$ fulfills (i) and (v), and has moderate growth. (ii) and (iv) follow from the choice of the W_v and $\underline{W_v}$. Now since $\pi_v \cong \sigma(|\cdot|_v^{1/2}, |\cdot|_v^{-1/2})$ for $v|\infty$, $\langle \Phi, \psi\rangle|_{B'_{F_v}} \cdot \underline{\beta_v} = C \sum_{\zeta \in F^*} \underline{W_v} \left(\begin{smallmatrix} \zeta t & 0 \\ 0 & 1\end{smallmatrix}\right) \cdot \underline{\beta_v}$ is harmonic for each archimedean place v of F: for real v, it is well-known that $f(z)/y$ is holomorphic for $f \in \mathcal{D}(2)$, and thus $f \cdot (\beta_v)_1$ is harmonic; for complex v, harmonicity follows from the other conditions, see e.g. [Kur78], p. 546 or [We71].
An easy calculation shows that

$$\lambda_{\underline{\alpha_1},\underline{\alpha_2}}\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1\end{pmatrix} \delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U)\right) = \int_{\zeta U} \prod_{\mathfrak{p}|p} \chi_{\alpha_\mathfrak{p},2}(-x)\psi_\mathfrak{p}(-x)dx = \mu_{\pi_p}(\zeta U)$$

for all $\zeta \in F^*$, and therefore we have

$$\begin{aligned} \Delta^{\underline{\alpha_1},\underline{\alpha_2}}(\Phi_\pi)(U, x^p) &= \sum_{\zeta \in F^*} \lambda_{\underline{\alpha_1},\underline{\alpha_2}}\left(\begin{pmatrix} \zeta & 0 \\ 0 & 1\end{pmatrix} \delta_{\underline{\alpha_1},\underline{\alpha_2}}(1_U)\right) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1\end{pmatrix} \\ &= \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1\end{pmatrix} = \phi_\pi(U, x^p). \end{aligned}$$

Let R be the integral closure of $\mathbb{Z}[a_\mathfrak{p}, \nu_\mathfrak{p}; \mathfrak{p}|p]$ in its field of fractions; thus R is a Dedekind ring $\subseteq \overline{\mathcal{O}}$ for which $\mathcal{B}_{\underline{\alpha_1},\underline{\alpha_2}}(F, R)$ is defined. Since $\mathbb{C}$ is a flat R-module,

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, R)) \otimes \mathbb{C} \to H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, \mathbb{C}))$$

is an isomorphism by proposition 4.8. The map (36) can be described as the "R-valued" map

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha_1}, \underline{\alpha_2}, \mathcal{M}, R)) \to H^d\big(F^{*\prime}, \mathcal{D}_f(R)\big)$$

tensored with $\mathbb{C}$. By proposition 4.11, κ_π lies in its image, and thus in $H^d\big(F^{*\prime}, \mathcal{D}_f(R)\big) \otimes \mathbb{C}$; i.e. it is integral.
Similarly, it follows from propositions 4.8 and 4.10 that $\kappa_{\pi,\underline{\mu}}$ is integral of rank ≤ 1. □

Corollary 4.13. *μ_π is a p-adic measure.*

Proof. By proposition 3.8, $\mu_\pi = \mu_{\phi_\pi} = \mu_{\kappa_\pi}$. Since κ_π is integral, μ_{κ_π} is a p-adic measure by corollary 3.13. □

4.7 Vanishing order of the p-adic L-function

Let $L_1, \dots, L_t$ be independent $\mathbb{Z}_p$-extensions of F, and let $\ell_1, \dots, \ell_t : \mathcal{G}_p \to p^{\varepsilon_p}\mathbb{Z}_p$ be the homomorphisms corresponding to them (as in section 3.2). Then we have the *p-adic L-function*

$$L_p(\underline{s}, \pi) := L_p(\underline{s}, \kappa_\pi) := L_p(s_1, \dots, s_t, \kappa_{\pi,+}) := \int_{\mathcal{G}_p} \prod_{i=1}^{t} \exp_p(s_i \ell_i(\gamma))\mu_\pi(d\gamma)$$

of definition 3.5, with $s_1, \dots, s_t \in \mathbb{Z}_p$. $L_p(\underline{s}, \pi)$ is a locally analytic function with values in the one-dimensional $\mathbb{C}_p$-vector space $V_{\kappa_\pi,+} = L_{\kappa,\overline{\mathcal{O}},+} \otimes_{\overline{\mathcal{O}}} \mathbb{C}_p$. By theorem 3.11, we have

Theorem 4.14. *$L_p(\underline{s}, \pi)$ is a locally analytic (t-variabled) function, and all partial derivatives of order $\leq n := \#(S_1)$ vanish; i.e. we have*

$$\operatorname{ord}_{\underline{s}=\underline{0}} L_p(\underline{s}, \pi) \geq n.$$

Now let E be a modular elliptic curve over F, corresponding to an automorphic representation π; by this we mean that the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places v of F. From the definition of the respective L-factors (cf. [Si86] for the Hasse-Weil L-function, [Ge75] for the automorphic L-function) we know that π has trivial central character. Moreover, for $\mathfrak{p}|p$, $\pi_\mathfrak{p}$ is a principal series representation iff E has good reduction at $\mathfrak{p}$, and in this case $\pi_\mathfrak{p}$ is ordinary iff E is ordinary (i.e. not supersingular) at $\mathfrak{p}$; $\pi_\mathfrak{p}$ is a special (resp. Steinberg) representation iff E has multiplicative (resp. split multiplicative) reduction at $\mathfrak{p}$. For $v|\infty$, π_v is "of weight 2" as assumed before.
We say that E is *p-ordinary* if it has good ordinary or multiplicative reduction at all places $\mathfrak{p}|p$ of F. So E is p-ordinary iff π is ordinary at all $\mathfrak{p}|p$. In this case, we define the p-adic L-function of E by $L_p(E, \underline{s}) := L_p(\underline{s}, \pi)$.
For each $i \in \{1, \dots, t\}$ and each prime $\mathfrak{p}|p$ of F, we write $\ell_{\mathfrak{p},i}$ for the restriction of ℓ_i to $F_\mathfrak{p} \hookrightarrow \mathbb{I} \twoheadrightarrow \mathcal{G}_p$. Let $q_\mathfrak{p}$ be the Tate period of $E|F_\mathfrak{p}$ and $\operatorname{ord}_\mathfrak{p}$ the normalized valuation on $F_\mathfrak{p}^*$. Defining the *L-invariants* of $E|F_\mathfrak{p}$ with respect to L_i by

$$\mathcal{L}_{\mathfrak{p},i}(E) := \ell_{\mathfrak{p},i}(q_\mathfrak{p}) / \operatorname{ord}_\mathfrak{p}(q_\mathfrak{p}),$$

we can generalize Hida's exceptional zero conjecture to general number fields:

Conjecture 4.15. *Let S_1 be the set of $\mathfrak{p}|p$ at which E has split multiplicative reduction, $n := \#S_1$, $S_2 := S_p \setminus S_1$. Then*

$$\operatorname{ord}_{\underline{s}=\underline{0}} L_p(E, \underline{s}) \geq n, \tag{38}$$

and we have

$$\frac{\partial^n}{\partial s_i^n} L_p(E, \underline{s})|_{\underline{s}=\underline{0}} = n! \prod_{\mathfrak{p} \in S_1} \mathcal{L}_{\mathfrak{p},i}(E) \prod_{\mathfrak{p} \in S_2} e(\pi_\mathfrak{p}, 1) \cdot L(E, 1), \tag{39}$$

for all $i = 1, \ldots, t$, *where* $e(\pi_\mathfrak{p}, 1) = (1 - \alpha_{\mathfrak{p},1}{}^{-1})^2$ *if* E *has good ordinary reduction at* $\mathfrak{p}$, *and* $e(\pi_\mathfrak{p}, 1) = 2$ *if* E *has non-split multiplicative reduction at* $\mathfrak{p}$.

Note that the conjecture (when considered for all sets of independent $\mathbb{Z}_p$-extensions of F) also determines the "mixed" partial derivatives $\frac{\partial^k}{\partial^{\underline{n}} \underline{s}} L_p(E, \underline{0})$ of order n, since they can be written as $\mathbb{Q}$-linear combinations of n-th "pure" partial derivatives $\frac{\partial^n}{\partial {s'}_i^n} L_p(E, \underline{0})$ with respect to other choices of independent $\mathbb{Z}_p$-extensions of F (cf. the proof of proposition 3.9).
Theorem 4.14 immediately implies the first part (38) of the conjecture:

Corollary 4.16. *Let* E *be a* p*-ordinary modular elliptic curve over* F. *Let* n *be the number of places* $\mathfrak{p}|p$ *at which* E *has split multiplicative reduction. Then we have*

$$\mathrm{ord}_{\underline{s}=\underline{0}}\, L_p(E, \underline{s}) \geq n.$$

In future work, we hope to also establish formula (39) for a class of non-totally-real number fields.

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