Decomposable Cycles and Noether-Lefschetz Loci

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Abstract. We prove that there exist smooth surfaces of degree d in $\mathbb{P}^3$ whose group of rational equivalence classes of decomposable 0-cycles has rank at least $\lfloor \frac{d-1}{3} \rfloor$.

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0. Introduction

Let X be a smooth complex surface: a rational equivalence class of 0-cycles on X is *decomposable* if it is the intersection of two divisor classes. Let $\mathrm{DCH}_0(X) \subset \mathrm{CH}_0(X)$ be the subgroup generated by decomposable 0-cycles. Beaville and Voisin [1] proved that if X is a $K3$ surface then $\mathrm{DCH}_0(X) \cong \mathbb{Z}$. What can be said of the group $\mathrm{DCH}_0(X)$ in general? An irregular surface X with non-zero map $\bigwedge^2 H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ provides an example with group of decomposable 0-cycles that is not finitely generated, even after tensorization with $\mathbb{Q}$. Let us assume that X is a regular surface: then $\mathrm{DCH}_0(X)$ is finitely generated because $\mathrm{CH}^1(X)$ is finitely generated, and we may ask for its rank. Blowing up regular surfaces with non-zero geometric genus at $(r-1)$ very general points, one gets examples of regular surfaces with $\mathrm{DCH}_0(X)$ of rank at least r (see Example 1.3 b) of [2]). What about a less artificial class of surfaces, such as (smooth) surfaces in $\mathbb{P}^3$? If the rank of $\mathrm{DCH}_0(X)$ is to be larger than 1 then the rank of $\mathrm{CH}^1(X)$ must be larger than 1, but the latter condition is not sufficient, for example curves on X whose canonical line-bundle is a (fractional) power of the hyperplane bundle do not increase the rank of $\mathrm{DCH}_0(X)$, see Subsection 1.2. The papers [13, 4] provide examples of smooth surfaces in $\mathbb{P}^3$ with Picard group of large rank and generated by lines: it follows that the group spanned

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by decomposable 0-cycles of such surfaces has rank 1. On the other hand Lie Fu proved that there exist degree-8 surfaces $X \subset \mathbb{P}^3$ such that $\mathrm{DCH}_0(X)$ has rank at least 2, see 1.4 of [6]. In the present paper we will prove the result below.

Theorem 0.1. *There exist smooth surfaces $X \subset \mathbb{P}^3$ of degree d such that the rank of* $\mathrm{DCH}_0(X)$ *is at least* $\lfloor \frac{d-1}{3} \rfloor$.

In particular the rank of the group of decomposable 0-cycles of a smooth surface in $\mathbb{P}^3$ can be arbitrarily large.

Let us explain the main ideas that go into the proof of Theorem 0.1. Let $C = C_1 \cup \ldots \cup C_n$ be the disjoint union of smooth irreducible curves $C_j \subset \mathbb{P}^3$. Suppose that $d \gg 0$, and that the curves C_j are not rationally canonical, i.e. there exists $e \in \mathbb{Z}$ such that $K_{C_j}^{\otimes m} \cong \mathscr{O}_{C_j}(e)$ only for $m = 0$; we prove that for a very general smooth $X \in |\mathscr{I}_C(d)|$, the classes $c_1(\mathscr{O}_X(1))^2, C_1 \cdot C_1, \ldots, C_n \cdot C_n$ in $\mathrm{CH}_0(X)$ are linearly independent. We argue as follows. Assume that they are *not* linearly independent for X very general; then there exists a non-zero $(a, r_1, \ldots, r_n) \in \mathbb{Z}^{n+1}$ such that

$$ac_1(\mathscr{O}_X(1))^2 + r_1 c_1(\mathscr{O}_X(C_1))^2 + \ldots + r_n c_1(\mathscr{O}_X(C_n))^2 = 0 \tag{0.1}$$

for all smooth $X \in |\mathscr{I}_C(d)|$. Now let $\pi\colon W \to \mathbb{P}^3$ be the blow up of C, let E be the exceptional divisor of π, and E_j be the component of E mapping to C_j. Let $\Lambda(d) := |\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)|$, and let $\mathscr{S} \subset W \times \Lambda(d)$ be the universal surface parametrized by $\Lambda(d)$. We let $p_W\colon \mathscr{S} \to W$ and $p_{\Lambda(d)}\colon \mathscr{S} \to \Lambda(d)$ be the projection maps. There is a natural identification $\Lambda(d) = |\mathscr{I}_C(d)|$, and the generic $S \in \Lambda(d)$ is isomorphic to the corresponding $X \in |\mathscr{I}_C(d)|$. Since (0.1) holds for all smooth X, an application of the spreading principle shows that the class

$$p_W^*(a\pi^* c_1(\mathscr{O}_{\mathbb{P}}^3(1))^2 + r_1 c_1(\mathscr{O}_W(E_1))^2 + \ldots + r_n c_1(\mathscr{O}_W(E_n))^2) \in \mathrm{CH}^2(\mathscr{S}) \tag{0.2}$$

is *vertical*, i.e. is represented by a linear combination of codimension-2 subvarieties $\Gamma_i \subset \mathscr{S}$ such that

$$\dim p_{\Lambda(d)}(\Gamma_i) < \dim \Gamma_i. \tag{0.3}$$

We prove that if the class in (0.2) is vertical, then $0 = a = r_1 = \ldots = r_n$. The key result that one needs is a Noether-Lefschetz Theorem for surfaces belonging to an integral codimension-1 closed subset $A \in \Lambda(d)$. More precisely one needs to prove that the following hold:

(1) If the generic $S \in A$ is isomorphic to $\pi(S) \subset \mathbb{P}^3$, i.e. S contains no fiber of $\pi\colon W \to \mathbb{P}^3$ over C, then $\mathrm{CH}^1(S)$ is generated (over $\mathbb{Q}$) by $\pi^* c_1(\mathscr{O}_{\mathbb{P}^3}(1))|_S, c_1(\mathscr{O}_S(E_1)), \ldots, c_1(\mathscr{O}_S(E_n))$.

(2) If the generic $S \in A$ contains a fiber R of $\pi\colon W \to \mathbb{P}^3$ over C, necessarily unique by genericity of S, then $\mathrm{CH}^1(S)$ is generated (over $\mathbb{Q}$) by the classes listed in Item (1), together with $c_1(\mathscr{O}_S(R))$.

The reason why such a Noether-Lefschetz Theorem is needed is the following. Let $\Gamma_i \subset \mathscr{S}$ be a codimension-2 subvariety such that (0.3) holds, and assume that the generic fiber of $\Gamma_i \to p_{\Lambda(d)}(\Gamma_i)$ has dimension 1; then $A := p_{\Lambda(d)}(\Gamma_i)$ is an integral closed codimension-1 subset of $\Lambda(d)$, and the restriction of Γ_i to the surface S_t parametrized by $t \in A$ is a divisor on S_t. Thus we are lead to prove the above Noether-Lefschetz result. There is a substantial literature on Noether-Lefschetz, but we have not found a result taylor made for our needs. A criterion of K. Joshi [9] is very efficient in disposing of "most" choices of a codimension-1 closed subset $A \in \Lambda(d)$. We deal with the remaining cases by appealing to the Griffiths-Harris approach to Noether-Lefschetz [8], as further developed by Lopez [12] and Brevik-Nollet [5].

The paper is organized as follows. In SECTION 1 we consider a smooth 3-fold V with trivial Chow groups, an ample divisor H on V and surfaces in the linear system $|\mathscr{I}_C(H)|$, where $C = C_1 \cup \ldots \cup C_n$ is the disjoint union of a fixed collection of smooth irreducible curves $C_i \subset V$. We prove that if the curves C_i are not rationally canonical, and a suitable Noether-Lefschetz Theorem holds, then the classes of $C_1^2, \ldots, C_n^2$ on a very general $X \in |\mathscr{I}_C(H)|$ are linearly independent, and they span a subgroup intersecting trivially the image of $\mathrm{CH}^2(V) \to \mathrm{CH}^2(X)$. In SECTION 2 we prove the required Noether-Lefschetz Theorem for $V = \mathbb{P}^3_{\mathbb{C}}$. In SECTION 3 we prove THEOREM 0.1 by combining the main results of SECTION 1 and SECTION 2.

CONVENTIONS AND NOTATION: We work over $\mathbb{C}$. Points are closed points.
Let X be a variety: "If x is a generic point of X, then..." is shorthand for "There exists an open dense $U \subset X$ such that if $x \in U$ then...". Similarly the expression "If x is a very general point of X, then..." is shorthand for "There exists a countable collection of closed nowhere dense $Y_i \in X$ such that if $x \in (X \setminus \bigcup_i Y_i)$ then...".

From now on we will denote by $\mathrm{CH}(X)$ the group of rational equivalence classes of cycles with *rational* coefficients. Thus if Z_1, Z_2 are cycles on X then $Z_1 \equiv Z_2$ means that for some non-zero integer ℓ the cycles ℓZ_1, ℓZ_2 are integral and rationally equivalent. If Z is a cycle on X we will often use the same symbol (i.e. Z) for the rational equivalence class represented by Z.

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1. THE FAMILY OF SURFACES CONTAINING GIVEN CURVES

1.1. THREEFOLDS WITH TRIVIAL CHOW GROUPS. Throughout the paper V is an integral smooth projective threefold.

HYPOTHESIS 1.1. *The cycle class map* $cl\colon \mathrm{CH}(V) \longrightarrow H(V;\mathbb{Q})$ *is an isomorphism.*

The archetypal such V is $\mathbb{P}^3$. A larger class of examples is given by 3-folds with an algebraic cellular decomposition (see Ex. 1.9.1 of [7]), and conjecturally the above assumption is equivalent to vanishing of $H^{p,q}(V)$ for $p \neq q$. An integral smooth projective threefold has *trivial* Chow group if HYPOTHESIS 1.1 holds.

Claim 1.2. *Let V be as above, in particular it has trivial Chow group. The natural map*

$$\mathrm{S}^2\,\mathrm{CH}^1(V) \longrightarrow \mathrm{CH}^2(V) \tag{1.1}$$

is surjective.

Proof. The natural map $\mathrm{S}^2\,H^2(V;\mathbb{Q}) \to H^4(V;\mathbb{Q})$ is surjective by Hard Lefschetz. The claim follows because of Hypothesis 1.1. □

1.2. Standard relations. Let V be an integral smooth projective 3-fold with trivial Chow group. Let $X \subset V$ be a closed surface, and $i\colon X \hookrightarrow V$ be the inclusion map. Let $\mathscr{R}^s(X) \subset \mathrm{CH}^s(X)$ be the image of the restriction map

$$\begin{array}{ccc} \mathrm{CH}^s(V) & \longrightarrow & \mathrm{CH}^s(X) \\ \xi & \mapsto & i^*\xi \end{array} \tag{1.2}$$

Notice that $\mathscr{R}^2(X) \subset \mathrm{DCH}_0(X)$ by Claim 1.2. Suppose that $C \subset X$ is an integral smooth curve. We will assume that $C \cdot C$ makes sense in $\mathrm{CH}_0(X)$, for example that will be the case if X is $\mathbb{Q}$-factorial. We will list elements of the kernel of the map

$$\begin{array}{ccc} \mathscr{R}^2(X) \oplus \mathscr{R}^1(X) \oplus \mathscr{R}^0(X) & \longrightarrow & \mathrm{DCH}_0(X) \\ (\alpha,\beta,\gamma) & \mapsto & \alpha + C\cdot\beta + \gamma\cdot C\cdot C \end{array} \tag{1.3}$$

Let $j\colon C \hookrightarrow V$ be the inclusion map. By Cor. 8.1.1 of [7] the following relation holds in $\mathrm{CH}_0(X)$:

$$i^*(j_*[C]) = C\cdot c_1(\mathscr{N}_{X/V}) = C\cdot i^*\mathscr{O}_V(X). \tag{1.4}$$

Thus

$$\alpha_C - C\cdot i^*\mathscr{O}_V(X) = 0, \tag{1.5}$$

where $\alpha_C := i^*(j_*C) \in \mathscr{R}^2(X)$. Equation (1.5) is the *first standard relation.* Now suppose that there exists $\xi \in \mathrm{CH}^1(V)$ such that

$$c_1(K_C) = \xi|_C. \tag{1.6}$$

(Recall that Chow groups are with $\mathbb{Q}$-coefficients, thus (1.6) means that there exists an integer $n > 0$ such that $K_C^{\otimes n}$ is the pull-back of a line-bundle on V.) By adjunction for $X \subset V$ and for $C \subset X$,

$$C\cdot C + C\cdot(i^*K_V + i^*\mathscr{O}_X(X)) \equiv C\cdot i^*\xi. \tag{1.7}$$

Thus there exists $\beta_C \in \mathscr{R}^1(X)$ such that

$$\beta_C\cdot C - C\cdot C = 0. \tag{1.8}$$

The above is the *second standard relation* (it holds assuming (1.6)).

Example 1.3. Let $V = \mathbb{P}^3$, let $X \subset \mathbb{P}^3$ be a smooth surface of degree d, and let $C \subset X$ be a smooth curve. The subgroup of $\mathrm{CH}_0(X)$ spanned by intersections of linear combinations of $H := c_1(\mathscr{O}_X(1))$ and C has rank at most 2. In fact the first standard relation reads $dC\cdot H = (\deg C)H\cdot H$. Suppose that $c_1(K_C) = mC\cdot H$, where $m \in \mathbb{Q}$. With this hypothesis, the second standard relation reads $C\cdot C = (m+4-d)C\cdot H$, and hence $C\cdot C, C\cdot H, H\cdot H$ span a rank-1 subgroup. In particular a curve of genus 0 or 1 does not add anything to the rank of $\mathrm{DCH}_0(X)$.

1.3. SURFACES CONTAINING DISJOINT CURVES. Let V be a smooth projective 3-fold with trivial Chow group and $C_1, \ldots, C_n \subset V$ be pairwise disjoint integral smooth projective curves. Let $C := C_1 \cup \ldots \cup C_n$ and let $\pi\colon W \to V$ be the blow-up of C. Let E be the exceptional divisor of π, and let E_j, for $j \in \{1, \ldots, n\}$, be the irreducible component of E mapping to C_j. Let H be an ample divisor on V. For $j \in \{1, \ldots, n\}$ we let

(1.9)

$$\Sigma_j := \{S \in |\pi^*H - E| \mid \pi(S) \text{ is singular at some point of } C_j\}, \quad \Sigma := \cup_{j=1}^n \Sigma_j.$$

Let $S \in |\pi^*H - E|$, and let $X := \pi(S)$. Then $S \in \Sigma_j$ if and only if S contains one (at least) of the fibers of $E_j \to C_j$, or, equivalently, the map $S \to X$ given by restriction of π is *not* an isomorphism over C_j. We will always assume that $(\pi^*H - E)$ is very ample on W; with this hypothesis Σ_j is irreducible of codimension 1, or empty (compute the codimension of the loci of $S \in |\pi^*(H) - E|$ which contain one or two fixed fibers of $E_k \to C_k$). Suppose that H is sufficiently ample: then, in addition, if $S \in \Sigma_k$ is generic the surface $X = \pi(S)$ is smooth except for one ODP (ordinary double point) belonging to C_k, and the set of reducible $S \in |\pi^*H - E|$ is of large codimension in $|\pi^*H - E|$. We will assume that both of these facts hold (but we do not assume that H is "sufficiently ample", because we want to prove effective results).

HYPOTHESIS 1.4. *Let $C_1, \ldots, C_n \subset V$ and H be as above, in particular H is ample on V, and $(\pi^*H - E)$ is very ample on W. Suppose that*

(1) *for $j \in \{1, \ldots, n\}$, and $S \in \Sigma_j$ generic, the surface $\pi(S)$ is smooth except for one ODP (ordinary double point) belonging to C_j, and*
(2) *the set of reducible $S \in |\pi^*H - E|$ has codimension at least 3 in $|\pi^*H - E|$.*

Assume that HYPOTHESIS 1.4 holds, and let $S \in \Sigma_j$ be generic. Then there is a unique singular point of $\pi(S)$, call it x, and the line $\pi^{-1}(x)$ is contained in S.

HYPOTHESIS 1.5. *Let $C_1, \ldots, C_n \subset V$ and H be as above. Suppose that* HYPOTHESIS 1.4 *holds, and that in addition the following hold:*

(1) *If $A \subset |\pi^*H - E|$ is an integral closed codimension-1 subset, not equal to one of $\Sigma_1, \ldots, \Sigma_n$, and $S \in A$ is very general, the restriction map $\mathrm{CH}^1(W) \to \mathrm{CH}^1(S)$ is surjective.*
(2) *For $j \in \{1, \ldots, n\}$, $S \in \Sigma_j$ very general, and x the unique singular point of $\pi(S)$ (an ODP belonging to C_j, by* HYPOTHESIS 1.4*), $\mathrm{CH}^1(S)$ is generated by the image of the restriction map $\mathrm{CH}^1(W) \to \mathrm{CH}^1(S)$ together with the class of $\pi^{-1}(x)$.*

Remark 1.6. Let $V = \mathbb{P}^3$, and fix $C_1, \ldots, C_n \subset \mathbb{P}^3$. Let $d \gg 0$, and $H \in |\mathscr{O}_{\mathbb{P}^3}(d)|$. If $S \in \Sigma_j$ is generic, then $\pi^{-1}(x)$ does *not* belong to the image of the restriction map $\mathrm{CH}^1(W) \to \mathrm{CH}^1(S)$.

In the present section we will prove the following result.

Proposition 1.7. *Let $C_1, \ldots, C_n \subset V$ and H be as above, and assume that* Hypothesis 1.5 *holds. Suppose also that for $j \in \{1, \ldots, n\}$ there does* not *exist $\xi \in \mathrm{CH}^1(V)$ such that $c_1(K_{C_j}) = \xi|_{C_j}$. (Recall that Chow groups are with coefficients in $\mathbb{Q}$.) Then for very general smooth $X \in |\mathscr{I}_C(H)|$ the following hold:*

1. *The map $\mathrm{CH}^2(V) \to \mathrm{CH}_0(X)$ is injective.*
2. *Let $\{\zeta_1, \ldots, \zeta_m\}$ be a basis of $\mathrm{CH}^1(V)$ (as $\mathbb{Q}$-vector space). Suppose that for very general smooth $X \in |\mathscr{I}_C(H)|$*
$$0 = P(\zeta_1|X, \ldots, \zeta_m|X) + r_1 C_1^2 + \ldots + r_n C_n^2,$$
where $P \in \mathbb{Q}[x_1, \ldots, x_m]_2$ is a homogeneous quadratic polynomial. Then $0 = P(\zeta_1, \ldots, \zeta_m) = r_1 = \ldots = r_n$.

The proof of Proposition 1.7 will be given in Subsection 1.7. Throughout the present section we let V, C, W, E and H be as above.

1.4. The universal surface. Assume that Hypothesis 1.4 holds. Let

$$\Lambda := |\pi^*(H) - E| \tag{1.10}$$
$$\mathscr{S} := \{(x, S) \in W \times \Lambda \mid x \in S\}. \tag{1.11}$$

Let $p_W : \mathscr{S} \to W$ and $p_\Lambda : \mathscr{S} \to \Lambda$ be the forgetful maps. Thus we have

$$\begin{tikzcd} & & \mathscr{S} \arrow[dl, "p_W"'] \arrow[dr, "p_\Lambda"] & \\ V & W \arrow[l, "\pi"'] & & \Lambda \end{tikzcd} \tag{1.12}$$

Let $N := \dim \Lambda$. Since $(\pi^*(H) - E)$ is very ample it is globally generated and hence the map p_W is a $\mathbb{P}^{N-1}$-fibration. It follows that $\mathscr{S}$ is smooth and

$$\dim \mathscr{S} = (N + 2). \tag{1.13}$$

Definition 1.8. Let $\mathrm{Vert}^q(\mathscr{S}/\Lambda) \subset \mathrm{CH}^q(\mathscr{S})$ be the subspace spanned by rational equivalence classes of codimension-q integral closed subsets $Z \subset \mathscr{S}$ such that the dimension of $p_\Lambda(Z)$ is strictly smaller than the dimension of Z.

The result below is an instance of the spreading principle.

Claim 1.9. *Keep notation and assumptions as above, in particular* Hypothesis 1.4 *holds. Let $Q \in \mathbb{Q}[x_1, \ldots, x_m, y_1, \ldots, y_n]_2$ be a homogeneous polynomial of degree 2 and let $\zeta_1, \ldots, \zeta_m \in \mathrm{CH}^1(V)$. Then*

$$Q(\zeta_1|_X, \ldots, \zeta_m|_X, c_1(\mathscr{O}_X(C_1)), \ldots, c_1(\mathscr{O}_X(C_n))) = 0 \tag{1.14}$$

for all smooth $X \in |\mathscr{I}_C(H)|$ if and only if

$$p_W^* Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n) \in \mathrm{Vert}^2(\mathscr{S}/\Lambda). \tag{1.15}$$

Proof. Suppose that (1.14) holds for all smooth $X \in |\mathscr{I}_C(H)|$. Let $S \in \Lambda$ be generic, $X := \pi(S)$. Then X is smooth and the restriction of π to S defines an isomorphism $\varphi\colon S \xrightarrow{\sim} X$, thus by our assumption

$$p_W^* Q(\pi^*\zeta_1, \dots, \pi^*\zeta_m, E_1, \dots, E_n)|_S = 0.$$

Since S is generic in Λ it follows (see [3, 14]) that there exists an open dense subset $\mathscr{U} \subset \Lambda$ such that

$$p_W^* Q(\pi^*\zeta_1, \dots, \pi^*\zeta_m, E_1, \dots, E_n)|_{p_\Lambda^{-1}\mathscr{U}} = 0. \tag{1.16}$$

(We recall that Chow groups are with rational coefficients, if we consider integer coefficients then (1.16) holds only up to torsion.) Let $B := (\Lambda \setminus \mathscr{U})$. By the localization exact sequence

$$\mathrm{CH}_N(p_\Lambda^{-1}B) \longrightarrow \mathrm{CH}_N(\mathscr{S}) \longrightarrow \mathrm{CH}_N(p_\Lambda^{-1}\mathscr{U}) \longrightarrow 0$$

$p_W^* Q(\pi^*\zeta_1, \dots, \pi^*\zeta_m, E_1, \dots, E_n)$ is represented by an N-cycle supported on $p_\Lambda^{-1}B$, and hence (1.15) holds because $\dim B < N$. Next, suppose that (1.15) holds. Then, by definition, the left-hand side of (1.15) is represented by an N-cycle whose support is mapped by p_Λ to a proper closed subset $B \subset \Lambda$. Thus there exists an open dense $\mathscr{U} \subset \Lambda$ such that the restriction of $p_W^* Q(\pi^*\zeta_1, \dots, \pi^*\zeta_m, E_1, \dots, E_n)$ to $p_\Lambda^{-1}\mathscr{U}$ vanishes, e.g. $\mathscr{U} = \Lambda \setminus B$. By shrinking $\mathscr{U}$ we may assume that for $S \in \mathscr{U}$ the surface $X := \pi(S)$ is smooth. Let $S \in \mathscr{U}$: then $0 = p_W^* Q(\pi^*\zeta_1, \dots, \pi^*\zeta_m, E_1, \dots, E_n)|_S$, and since $X \cong S$ it follows that (1.14) holds for $X = \pi(S)$. On the other hand the locus of smooth $X \in |\mathscr{I}_C(H)|$ such that (1.14) holds is a countable union of closed subsets of Λ_{sm} (the open dense subset of Λ parametrizing smooth surfaces); since it contains an open dense subset of Λ_{sm} it is equal to Λ_{sm}. □

1.5. The Chow groups of $\mathscr{S}$ and W. Assume that Hypothesis 1.4 holds. Let $\xi \in \mathrm{CH}^1(\mathscr{S})$ be the pull-back of the hyperplane class on Λ via the map p_Λ of (1.12). Since p_W is the projectivization of a rank-N vector-bundle on W and ξ restricts to the hyperplane class on each fiber of p_W the Chow ring $\mathrm{CH}(\mathscr{S})$ is the $\mathbb{Q}$-algebra generated by $p_W^*\,\mathrm{CH}(W)$ and ξ, with ideal of relations generated by a single relation in codimension N. We have $N \geq 3$ because$(\pi^*H - E)$ is very ample by Hypothesis 1.4; thus

$$\begin{array}{ccc} \mathbb{Q} \oplus \mathrm{CH}^1(W) \oplus \mathrm{CH}^2(W) & \xrightarrow{\sim} & \mathrm{CH}^2(\mathscr{S}) \\ (a_0, a_1, a_2) & \mapsto & a_0\xi^2 + p_W^*(a_1)\cdot\xi + p_W^*(a_2) \end{array} \tag{1.17}$$

is an isomorphism. The Chow groups $\mathrm{CH}_q(W)$ are computed by first describing $\mathrm{CH}_q(E_j)$ for $j \in \{1, \dots, n\}$, and then considering the localization exact sequence

$$\bigoplus_j \mathrm{CH}_q(E_j) \longrightarrow \mathrm{CH}_q(W) \longrightarrow \mathrm{CH}_q(W \setminus (E_1 \cup \ldots \cup E_n)) \longrightarrow 0.$$

One gets an isomorphism

$$\begin{array}{ccc} \mathrm{CH}^1(V) \oplus \mathbb{Q}^n & \xrightarrow{\sim} & \mathrm{CH}^1(W) \\ (a, t_1, \dots, t_n) & \mapsto & \pi^*a + \sum_{j=1}^n t_j E_j \end{array} \tag{1.18}$$

and an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(W)_{\mathrm{hom}} \longrightarrow \mathrm{CH}^2(W) \xrightarrow{cl} H^4(W;\mathbb{Q}) \longrightarrow 0 \tag{1.19}$$

where $\mathrm{CH}^2(W)_{\mathrm{hom}}$ is described as follows. Let $\rho_j\colon E_j \to C_j$ be the restriction of the blow-up map π, and $\sigma_j\colon E_j \hookrightarrow W$ be the inclusion map; then we have an Abel-Jacobi isomorphism

$$\begin{array}{cccc} AJ\colon & \mathrm{CH}^2(W)_{\mathrm{hom}} & \xrightarrow{\sim} & \bigoplus_{j=1}^n \mathrm{CH}_0(C_j)_{\mathrm{hom}} \\ & \alpha & \mapsto & (\rho_{1,*}(\sigma_1^*\alpha),\ldots,\rho_{n,*}(\sigma_n^*\alpha)) \end{array} \tag{1.20}$$

Let AJ_j be the j-th component of the map AJ.

LEMMA 1.10. *Assume that* HYPOTHESIS 1.4 *holds. Let*

$$\omega := \pi^*\alpha + \sum_{j=1}^n E_j \cdot \pi^*\beta_j + \sum_{j=1}^n \gamma_j E_j \cdot E_j,$$

where $\alpha \in \mathrm{CH}^2(V)$, $\beta_j \in \mathrm{CH}^1(V)$, *and* $\gamma_j \in \mathbb{Q}$ *for* $j \in \{1,\ldots,n\}$. *Then the following hold:*

(1) *The cohomology class of* ω *vanishes if and only if*

$$\alpha = \sum_{j=1}^n \gamma_j C_j, \tag{1.21}$$

and for all $j \in \{1,\ldots,n\}$

$$\deg(\beta_j \cdot C_j) = -\gamma_j \deg(\mathscr{N}_{C_j/V}). \tag{1.22}$$

(2) *Suppose that* (1.21) *and* (1.22) *hold. Then for* $j \in \{1,\ldots,n\}$

$$AJ_j(\omega) = -\gamma_j c_1(\mathscr{N}_{C_j/V}) - c_1(\beta_j|_{C_j}). \tag{1.23}$$

Proof. Since the cohomology class map $cl\colon \mathrm{CH}^1(V) \to H^2(V;\mathbb{Q})$ is a surjection (by hypothesis), also the cohomology class map $cl\colon \mathrm{CH}^1(W) \to H^2(W;\mathbb{Q})$ is surjective. By Poincarè duality it follows that $cl(\omega) = 0$ if and only if $\deg(\omega \cdot \xi) = 0$ for all $\xi \in \mathrm{CH}^1(W)$. By (1.18) we must test $\xi = \pi^*\zeta$ with $\zeta \in \mathrm{CH}^1(V)$ and $\xi = E_i$ for $i \in \{1,\ldots,n\}$. We have

$$\deg(\omega \cdot \pi^*\zeta) = \deg\left(\left(\alpha - \sum_{j=1}^n \gamma_j C_j\right) \cdot \zeta\right). \tag{1.24}$$

Since the cycle map $\mathrm{CH}^2(V) \to H^4(V;\mathbb{Q})$ is an isomorphism, it follows that $\deg(\omega \cdot \pi^*\zeta) = 0$ for all $\zeta \in \mathrm{CH}^1(V)$ if and only if (1.21) holds. Next, we test $\xi = E_i$. In $\mathrm{CH}_0(C_i)$

$$\rho_{i,*} c_1(\mathscr{O}_{E_i}(E_i))^2 = -c_1(\mathscr{N}_{C_i/V}), \tag{1.25}$$

and hence

$$\deg(\omega \cdot E_i) = -\deg(\beta_i \cdot C_i) - \gamma_i \deg(\mathscr{N}_{C_i/V}). \tag{1.26}$$

This proves Item (1). Item (2) follows from Equation (1.25). $\square$

Remark 1.11. By Lemma 1.10 the kernel of the map
(1.27)

$$\begin{array}{ccc} \mathrm{CH}^2(V)\oplus\bigoplus_{k=1}^{n}\mathrm{CH}^1(V)\oplus\bigoplus_{k=1}^{n}\mathbb{Q} & \longrightarrow & \mathrm{CH}^2(W) \\ (\alpha,\beta_1,\ldots\beta_n,\gamma_1,\ldots,\gamma_n) & \mapsto & \pi^*\alpha+\sum_{j=1}^n E_j\cdot\pi^*\beta_j+\sum_{j=1}^n\gamma_j E_j\cdot E_j \end{array}$$

is generated over $\mathbb{Q}$ by the classes $E_j\cdot\pi^*\beta$, where $\beta\in\mathrm{CH}^1(V)$ and $\beta|_{C_j}=0$, together with the classes

$$\pi^*[C_j]+E_j\cdot\pi^*\beta+E_j\cdot E_j, \tag{1.28}$$

where $\beta\in\mathrm{CH}^1(V)$, $\deg(\beta\cdot C_j)=-\deg(\mathscr{N}_{C_j/V})$, and

$$-c_1(\mathscr{N}_{C_j/V})-c_1(\beta|_{C_j})=0. \tag{1.29}$$

Next notice that (1.29) holds if and only if $c_1(K_{C_j})$ is equal to the restriction of a class in $\mathrm{CH}^1(V)$ i.e. (1.6) holds. Assume that this is the case, and that $X\in|\mathscr{I}_C(H)|$ is a surface smooth at all points of C_j. Let $S\in|\pi^*H-E|$ be the strict transform of S. Then S is isomorphic to X over C_j, and restricting to S the equation $\pi^*[C_j]+E_j\cdot\pi^*\beta+E_j\cdot E_j=0$ we get the second standard relation (1.8).

1.6. A vertical cycle on $\mathscr{S}$. According to Claim 1.9, for every codimension-2 relation that holds between $\mathscr{O}_X(C_1),\ldots,\mathscr{O}_X(C_n)$ and restrictions to X of divisors on V, where X is an arbitrary smooth member of $\in|\mathscr{I}_C(H)|$, there is a polynomial in classes of $\pi^*\mathrm{CH}^1(V)$ and the classes of the exceptional divisors of π which is "responsible" for the relation, i.e. when we pull-it back to $\mathscr{S}$ it is a vertical class. We have shown that $\pi^*[C_j]+E_j\cdot\pi^*\beta+E_j\cdot E_j$ is the class responsible for the second standard relation (1.8), see Remark 1.11, and in fact this class vanishes. In the present subsection we will write out a cycle responsible for the first standard relation (1.5), this time the pull-back to $\mathrm{CH}^2(\mathscr{S})$ is a non-zero vertical class. We record for later use the following formulae:

$$\sigma_{j,*}\rho_j^*c_1(\mathscr{N}_{C_j/V}) = \pi^*C_j+E_j\cdot E_j, \tag{1.30}$$

$$p_{W,*}(\xi^N) = (\pi^*H-E). \tag{1.31}$$

The first formula follows from the "Key formula" for π^*C_j, see Prop. 6.7 of [7]. The second formula is immediate (recall that $N=\dim\Lambda$). Let $j\in\{1,\ldots,n\}$. By Hypothesis 1.4 there exists an open dense $U\subset\Sigma_j$ such that, if $S\in U$, then $S\cdot E_j=\mathbf{L}_x+Z$, where $x\in C_j$ is the unique singular point of $\pi(S)$, $\mathbf{L}_x:=\pi^{-1}(x)$, and Z is the residual divisor (whose support does not contain $\mathbf{L}_x$). It follows that

$$E_j\cap p_\Lambda^{-1}(U)=\mathscr{V}_j+\mathscr{Z}_j, \tag{1.32}$$

where, for every $S\in U$, the restrictions to $E_j\cap S$ of $\mathscr{V}_j$, $\mathscr{Z}_j$ are equal to $\mathbf{L}_x$ and Z, respectively. We let

$$\Theta_j:=\overline{\mathscr{V}}_j. \tag{1.33}$$

Thus $p_\Lambda(\Theta_j) = \Sigma_j$, and the generic fiber of $\Theta_j \to \Sigma_j$ is a projective line. By Hypothesis 1.4 Θ_j is of pure codimension 2 in $\mathscr{S}$ (or empty), and hence

$$\Theta_j \in \mathrm{Vert}^2(\mathscr{S}/\Lambda). \tag{1.34}$$

The result below will be instrumental in writing out the class of Θ_j in $\mathrm{CH}^2(\mathscr{S})$ according to Decomposition (1.17).

Proposition 1.12. *Let $j \in \{1, \ldots, n\}$. Then*

$$p_{W,*}(\Theta_j \cdot \xi^{N-1}) = 2E_j \cdot \pi^* H - E_j \cdot E_j - \pi^* C_j. \tag{1.35}$$

Proof. Let $\alpha, \beta \in H^0(W, \pi^*(H) - E)$ be generic. Then $\mathrm{div}(\alpha|_{E_j})$ and $\mathrm{div}(\beta|_{E_j})$ are smooth divisors intersecting transversely at points $p_1, \ldots, p_s$. Let $q_i := \pi(p_i)$ for $i \in \{1, \ldots, s\}$. Let $R = \mathbb{P}(\langle \alpha, \beta \rangle) \subset \Lambda$; thus $p_\Lambda^{-1} R$ represents ξ^{N-1}. Given p_i, there exists $[\lambda_i, \mu_i] \in \mathbb{P}^1$ such that $\mathrm{div}(\lambda_i \alpha + \mu_i \beta)$ contains $\pi^{-1}(q_i)$, and hence $[\lambda_i \alpha + \mu_i \beta] \in R \cap \Sigma_j$. Conversely, every point of $R \cap \Sigma_j$ is of this type. The line R intersects transversely Σ_j because it is generic, and hence

$$p_{W,*}(\Theta_j \cdot \xi^{N-1}) = \sigma_{j,*} \rho_j^*(q_1 + \ldots + q_s). \tag{1.36}$$

Thus in order to compute $p_{W,*}(\Theta_j \cdot \xi^{N-1})$ we must determine the class of the 0-cycle $q_1 + \ldots + q_s$. Let $\phi\colon C_j \times R \to C_j$ and $\psi\colon C_j \times R \to R$ be the projections and $\mathscr{F}$ the rank-2 vector-bundle on $C_j \times R$ defined by

$$\mathscr{F} := \phi^*(\mathscr{N}_{C_j/V}^\vee \otimes \mathscr{O}_{C_j}(H)) \otimes \psi^* \mathscr{O}_R(1).$$

The composition of the natural maps

$$\langle \alpha, \beta \rangle \hookrightarrow H^0(W, \pi^* H - E) \longrightarrow H^0(E_j, \mathscr{O}_{E_j}(\pi^* H - E)) \longrightarrow H^0(C_j, \mathscr{N}_{C_j/V}^\vee \otimes \mathscr{O}_{C_j}(H)) \tag{1.37}$$

defines a section $\tau \in H^0(\mathscr{F})$ whose zero-locus consists of points $p_1', \ldots, p_s'$ such that $\pi(p_i') = q_i$. Now, the zero-locus of τ represents $c_2(\mathscr{F})$, and hence

$$p_{W,*}(\Theta_j \cdot \xi^{N-1}) = \sigma_{j,*}(\rho_j^*(\phi_* c_2(\mathscr{F})))$$

by (1.36). The formula

$$c_2(\mathscr{F}) = \phi^*(2c_1(\mathscr{O}_C(H)) - c_1(\mathscr{N}_{C/\mathbb{P}^3})) \cdot \psi^* c_1(\mathscr{O}_R(1)).$$

gives

$$p_{W,*}(\Theta_j \cdot \xi^{N-1}) = 2E_j \cdot \pi^* H - \sigma_{j,*}\left(\rho_j^* c_1(\mathscr{N}_{C_j/V})\right). \tag{1.38}$$

Then (1.35) follows from the above equality together with (1.30). □

Corollary 1.13. *Let $j \in \{1, \ldots, n\}$. Then*

$$\Theta_j = \xi \cdot p_W^* E_j + p_W^*(E_j \cdot \pi^* H - \pi^* C_j). \tag{1.39}$$

Proof. By (1.17) there exist $\beta_h \in \mathrm{CH}^h(W)$ for $h = 0, 1, 2$ such that

$$\Theta_j = \xi^2 \cdot p_W^* \beta_0 + \xi \cdot p_W^* \beta_1 + p_W^* \beta_2.$$

Restricting p_W to Θ_j we get a $\mathbb{P}^{N-2}$-fibration $\Theta_j \to E_j$: it follows that $\beta_0 = 0$ and $\beta_1 = E_j$. By (1.31)

$$p_{W,*}(\Theta_j \cdot \xi^{N-1}) = p_{W,*}(\xi^N \cdot p_W^* E_j + \xi^{N-1} \cdot p_W^* \beta_2) = (E_j \cdot \pi^* H - E_j \cdot E_j + \beta_2). \tag{1.40}$$

On the other hand $p_{W,*}(\Theta_j \cdot \xi^{N-1})$ is equal to the right-hand side of (1.35): equating that expression and the right-hand side of (1.40) we get $\beta_2 = (E_j \cdot \pi^* H - \pi^* C_j)$. $\square$

Corollary 1.14. *Let* $j \in \{1, \ldots, n\}$. *Then* $p_W^*(E_j \cdot \pi^* H - \pi^* C_j) \in \mathrm{Vert}^2(\mathscr{S}/\Lambda)$.

Proof. By Corollary 1.13 we have

$$p_W^*(E_j \cdot \pi^* H - \pi^* C_j) = \Theta_j - \xi \cdot p_W^* E_j.$$

Now $\Theta_j \in \mathrm{Vert}^2(\mathscr{S}/\Lambda)$ (see (1.34)) and $\xi \cdot p_W^* E_j \in \mathrm{Vert}^2(\mathscr{S}/\Lambda)$ because it is supported on the inverse image of a hyperplane via p_Λ; thus $p_W^*(E_j \cdot \pi^* H - \pi^* C_j) \in \mathrm{Vert}^2(\mathscr{S}/\Lambda)$. $\square$

By Claim 1.9 the relation $p_W^*(E_j \cdot \pi^* H - \pi^* C_j) \in \mathrm{Vert}^2(\mathscr{S}/\Lambda)$ gives a relation in $\mathrm{CH}(X)$ for an arbitrary smooth $X \in |\mathscr{I}_C(H)|$. In fact it gives the first standard relation (1.5).

1.7. Proof of the main result of the section.

Lemma 1.15. *Assume that* Hypothesis 1.5 *holds. Then the projection* $\mathrm{CH}^2(\mathscr{S}) \to \mathrm{CH}^2(W)$ *determined by* (1.17) *maps* $\mathrm{Vert}^2(\mathscr{S}/\Lambda)$ *to the subspace spanned by*

$$(E_1 \cdot \pi^* H - \pi^* C_1), \ldots, (E_j \cdot \pi^* H - \pi^* C_j), \ldots, (E_n \cdot \pi^* H - \pi^* C_n). \tag{1.41}$$

Proof. Let $Z \subset \mathscr{S}$ be an irreducible closed codimension-2 subset of $\mathscr{S}$ such that

$$\dim p_\Lambda(Z) < \dim Z = N. \tag{1.42}$$

Since the fibers of p_Λ are surfaces,

$$\dim p_\Lambda(Z) = \begin{cases} N-2, & \text{or} \\ N-1. \end{cases} \tag{1.43}$$

Suppose that $\dim p_\Lambda(Z) = N - 2$. We claim that

$$Z = p_\Lambda^{-1}(p_\Lambda(Z)). \tag{1.44}$$

Since $Z \subset p_\Lambda^{-1}(p_\Lambda(Z))$, it will suffice to prove that $p_\Lambda^{-1}(p_\Lambda(Z))$ is irreducible of dimension N. First we notice that every irreducible component of $p_\Lambda^{-1}(p_\Lambda(Z))$ has dimension at least N. In fact, letting $\iota \colon p_\Lambda(Z) \hookrightarrow \Lambda$ be the inclusion and $\Delta_\Lambda \subset \Lambda \times \Lambda$ the diagonal, $p_\Lambda^{-1}(p_\Lambda(Z))$ is identified with $(\iota, p_\Lambda)^{-1}\Delta_\Lambda$, and the claim follows because Δ_Λ is a l.c.i. of codimension N. Since every fiber of p_Λ has dimension 2, it follows that every irreducible component of $p_\Lambda^{-1}(p_\Lambda(Z))$ dominates $p_\Lambda(Z)$. On the other hand, since $\mathrm{cod}(p_\Lambda(Z), \Lambda) = 2$, there exists an open dense $U \subset p_\Lambda(Z)$ such that $p_\Lambda^{-1}(t)$ is irreducible for all $t \in U$ by Hypothesis 1.4, and hence $p_\Lambda^{-1}(U)$ is irreducible of dimension N. It follows that there is a single irreducible component of $p_\Lambda^{-1}(p_\Lambda(Z))$ dominating $p_\Lambda(Z)$, and hence $p_\Lambda^{-1}(p_\Lambda(Z))$ is irreducible (of dimension N). We have proved (1.44). Since Λ is a projective space, $p_\Lambda([Z])$ is a multiple of $c_1(\mathscr{O}_\Lambda(1))^2$. It follows that the class of Z is a multiple of ξ^2 and hence the projection $\mathrm{CH}^2(\mathscr{S}) \to \mathrm{CH}^2(W)$ maps it to 0. Now assume that $\dim p_\Lambda(Z) = N-1$. Let $Y := p_\Lambda(Z)$. For $t \in \Lambda$, we let $S_t := p_\Lambda^{-1}(t)$. We distinguish between the two cases:

(1) $p_\Lambda(Z) \notin \{\Sigma_1, \ldots, \Sigma_n\}$.

(2) There exists $j \in \{1, \ldots, n\}$ such that $p_\Lambda(Z) = \Sigma_j$.

Suppose that (1) holds. Let $Y^{sm} \subset Y$ be the subset of smooth points. If $t \in Y^{sm}$, we may intersect the cycles Z and S_t in $p_\Lambda^{-1}(Y)$ (because S_t is a l.c.i.), and the resulting cycle class $Z \cdot S_t$ belongs to $\mathrm{CH}^1(S_t)$. By Hypothesis 1.5 there exists $\Gamma \in \mathrm{CH}^1(W)$ such that $\Gamma|_{S_t} = Z \cdot S_t$ for $t \in Y^{sm}$. It follows that there exists an open dense $U \subset Y^{sm}$ such that

$$\Gamma|_{p_\Lambda^{-1}(U)} \equiv Z|_{p_\Lambda^{-1}(U)}.$$

(Recall that Chow groups are with $\mathbb{Q}$-coefficients.) By the localization sequence applied to $p_\Lambda^{-1}(U) \subset p_\Lambda^{-1}(Y)$, it follows that there exists a cycle $\Xi \in \mathrm{CH}_N(p_\Lambda^{-1}(Y \setminus U))$ such that

$$[Z] = \Xi + p_W^*(\Gamma) \cdot p_\Lambda^*([Y]).$$

Here, by abuse of notation, we mean cycle classes in $\mathrm{CH}_N(\mathscr{S})$: thus $[Z]$ and Ξ are actually the push-forwards of the corresponding classes in $\mathrm{CH}_N(p_\Lambda^{-1}(Y))$ and $\mathrm{CH}_N(p_\Lambda^{-1}(Y \setminus U))$ via the obvious closed embeddings. By (1.44) Ξ is represented by a linear combination of varieties $p_\Lambda^{-1}(B_i)$, where $B_1, \ldots, B_m$ are the irreducible components of $Y \setminus U$; it follows that $\Xi = a\xi^2$ for some $a \in \mathbb{Q}$. On the other hand $[Y] \in \mathrm{CH}^1(\Lambda) = \mathbb{Q}c_1(\mathscr{O}_\Lambda(1))$, and hence $p_W^*(\Gamma) \cdot p_\Lambda^*([Y]) = bp_W^*(\Gamma)\xi$ for some $b \in \mathbb{Q}$. It follows that the projection $\mathrm{CH}^2(\mathscr{S}) \to \mathrm{CH}^2(W)$ maps Z to 0. Lastly suppose that Item (2) holds. Arguing as above, one shows that there exist $\Gamma \in \mathrm{CH}^1(W)$, an open dense $U \subset Y$, a cycle $\Xi \in \mathrm{CH}_N(p_\Lambda^{-1}(Y \setminus U))$, and $a \in \mathbb{Q}$ such that

$$[Z] = \Xi + p_W^*(\Gamma) \cdot p_\Lambda^*([Y]) + a\Theta_j.$$

By Corollary 1.13 the projection $\mathrm{CH}^2(\mathscr{S}) \to \mathrm{CH}^2(W)$ maps $[Z]$ to $a(E_j \cdot \pi^*H - \pi^*C_j)$. This proves that $\mathrm{Vert}^2(\mathscr{S}/\Lambda)$ is mapped into the subspace spanned by the elements of (1.41). Since $[\Theta_j]$ is a vertical class and is mapped to $(E_j \cdot \pi^*H - \pi^*C_j)$, we have proved the lemma. □

Proof of Proposition 1.7. Let $P \in \mathbb{Q}[x_1, \ldots, x_m]$ be homogeneous of degree 2 and $r_1, \ldots, r_n \in \mathbb{Q}$. The set of smooth $X \in |\mathscr{I}_C(H)|$ such that

$$0 = P(\zeta_1|X, \ldots, \zeta_m|X) + r_1C_1^2 + \ldots + r_nC_n^2 \tag{1.45}$$

is a countable union of closed subsets of the open dense subset of $|\mathscr{I}_C(H)|$ parametrizing smooth surfaces. It follows that if the proposition is false then there exist P and $r_1, \ldots, r_n$, not all zero, such that (1.45) holds for all smooth $X \in |\mathscr{I}_C(H)|$. Now we argue by contradiction. By Claim 1.9

$$p_W^*(P(\pi^*\zeta_1, \ldots, \pi^*\zeta_m) + \sum_{j=1}^n r_jE_j^2) \in \mathrm{Vert}^2(\mathscr{S}/\Lambda). \tag{1.46}$$

By LEMMA 1.15 it follows that there exist rationals $s_1, \ldots, s_n$ such that

$$P(\pi^*\zeta_1, \ldots, \pi^*\zeta_m) + \sum_{j=1}^{n} r_j E_j^2 = \sum_{j=1}^{n} s_j (E_j \cdot \pi^* H - \pi^* C_j),$$

i.e.,

$$0 = \pi^*(P(\zeta_1, \ldots, \zeta_m) + \sum_{j=1}^{n} s_j C_j) - \sum_{j=1}^{n} s_j E_j \cdot \pi^* H + \sum_{j=1}^{n} r_j E_j^2. \tag{1.47}$$

Let ω be the right hand side of (1.47); then the homology class of ω vanishes, and also the Abel-Jacobi image $AJ(\omega)$, notation as in (1.20). Item (2) of LEMMA 1.10, together with our hypothesis that there does *not* exist $\xi \in \mathrm{CH}^1(V)$ such that $c_1(K_{C_j}) = \xi|_{C_j}$, gives $r_j = 0$ for $j \in \{1, \ldots, n\}$. By (1.21)

$$P(\zeta_1, \ldots, \zeta_m) + \sum_{j=1}^{n} s_j C_j = 0, \tag{1.48}$$

and hence $\sum_{j=1}^{n} s_j E_j \cdot \pi^* H = 0$. Thus

$$0 = E_i \cdot \left(\sum_{j=1}^{n} s_j E_j \cdot \pi^* H \right) = -s_i \deg(C_i \cdot H). \tag{1.49}$$

for $i \in \{1, \ldots, n\}$. By hypothesis H is ample, and hence $s_i = 0$ follows from (1.49). Thus $P(\zeta_1, \ldots, \zeta_m) = 0$ by (1.48). □

2. NOETHER-LEFSCHETZ LOCI FOR LINEAR SYSTEMS OF SURFACES IN $\mathbb{P}^3$ WITH BASE-LOCUS

2.1. THE MAIN RESULT. In the present section we let $V = \mathbb{P}^3$. Thus $C_1, \ldots, C_n \subset \mathbb{P}^3$, and $\pi \colon W \to \mathbb{P}^3$. We let $\Lambda(d) := |\pi^* \mathscr{O}_{\mathbb{P}^3}(d)(-E)|$. For $j \in \{1, \ldots, n\}$ let $\Sigma_j(d) \subset \Lambda(d)$ be the subset Σ_j considered in SECTION 1; thus $\Sigma_j(d)$ parametrizes surfaces $S \in \Lambda(d)$ such that $\pi(S)$ is singular at some point of C_j. Let $\Sigma(d) := \Sigma_1(d) \cup \ldots \cup \Sigma_n(d)$. We denote the tangent sheaf of a smooth variety X by T_X. Below is the main result of the present section.

THEOREM 2.1. *Suppose that $d \geq 5$, and that the following hold:*

(1) $\pi^* \mathscr{O}_{\mathbb{P}^3}(d-3)(-E)$ *is very ample.*
(2) $H^1(C, T_C(d-4)) = 0$.
(3) *The sheaf $\mathscr{I}_C$ is $(d-2)$-regular.*
(4) *The curves $C_1, \ldots, C_n$ are not planar.*

Then HYPOTHESIS 1.5 *holds for $H \in |\mathscr{O}_{\mathbb{P}^3}(d)|$.*

Recall that HYPOTHESIS 1.5 states that HYPOTHESIS 1.4 holds, and that Items (1) and (2) (our Noether-Lefschetz hypotheses) of HYPOTHESIS 1.5 hold. The proof that HYPOTHESIS 1.4 holds is elementary, and will be given in SUBSECTION 2.2. We will prove that Items (1) and (2) of HYPOTHESIS 1.5 hold by applying Joshi's main criterion (Prop. 3.1 of [9]), and the main idea in

Griffiths-Harris' proof of the classical Noether-Lefschetz Theorem [8], as further developed by Lopez [12] and Brevik-Nollet [5]. The proof will be outlined in SUBSECTION 2.3, details are in the remaining subsections.

Remark 2.2. Choose disjoint integral smooth curves $C_1, \ldots, C_n \subset \mathbb{P}^3$ such that for each $j \in \{1, \ldots, n\}$ there does *not* exist $r \in \mathbb{Q}$ such that $c_1(K_{C_j}) = rc_1(\mathscr{O}_{C_j}(1))$. Let $d \gg 0$. Then the hypotheses of THEOREM 2.1 are satisfied, and hence by PROPOSITION 1.7 the following holds: if $X \in |\mathscr{I}_C(d)|$ is very general, then the 0-cycle classes $c_1(\mathscr{O}_X(1))^2, c_1(\mathscr{O}_X(C_1))^2, \ldots, c_1(\mathscr{O}_X(C_n))^2$ are linearly independent. Thus the group of decomposable 0-cycles of X has rank at least $n+1$. The proof of THEOREM 0.1 is achieved by making the above argument effective, see SECTION 3.

2.2. DIMENSION COUNTS. We will prove that, if the hypotheses of THEOREM 2.1 are satisfied, then HYPOTHESIS 1.4 holds for $H \in |\mathscr{O}_{\mathbb{P}^3}(d)|$. First, H is ample on $\mathbb{P}^3$, and $\pi^*(H)-E$ is very ample on W because it is the tensor product of the line-bundle $\pi^*\mathscr{O}_{\mathbb{P}^3}(d-3)(-E)$, which is very ample by hypothesis, and the base-point free line-bundle $\pi^*\mathscr{O}_{\mathbb{P}^3}(3)$. Let $\Delta(r) \subset \Lambda(r)$ be the closed subset parametrizing singular surfaces.

PROPOSITION 2.3. *Suppose that $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-1)(-E)$ is very ample. Then the following hold:*

1. *Let $x \in C$. The linear system $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ has base locus equal to C, and codimension 2 in $|\mathscr{I}_C(r)|$. If X is generic in $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ then it has an ODP at x and no other singularity.*
2. *Given $x \in W \setminus E$ there exists $S \in \Delta(r)$ which has an ODP at x and is smooth away from x.*
3. *The closed subset $\Delta(r)$ is irreducible of codimension 1 in $\Lambda(r)$, and the generic $S \in \Delta(r)$ has a unique singular point, which is an ODP.*
4. *Let $j \in \{1, \ldots, n\}$. If S is a generic element of $\Sigma_j(r)$, then $\pi(S)$ has a unique singular point x, which is an ODP (notice that S is smooth).*

Proof. Let $q \in \mathbb{P}^3 \setminus C$. Since $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-1)(-E)$ is very ample there exists $X \in |\mathscr{I}_C(r-1)|$ such that $q \notin X$. Let $P \subset \mathbb{P}^3$ be a plane containing x but not q: then $X+P \in |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ does not pass through q, and this proves that $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ has base locus equal to C. Since $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-1)(-E)$ is very ample there exist $F, G \in H^0(\mathbb{P}^3, \mathscr{I}_C(r-1))$ and $q_1, \ldots, q_m \in (C \setminus \{x\})$ such that $V(F), V(G)$ are smooth and transverse at each point of $C \setminus \{q_1, \ldots, q_m\}$. Let $P \subset \mathbb{P}^3$ be a plane not passing through x: the pencil in $|\mathscr{I}_C(r)|$ spanned by $V(F)+P$ and $V(G)+P$ does not intersect $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$, and hence $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ has codimension at least 2 in $|\mathscr{I}_C(r)|$. The codimension is equal to 2 because imposing on $X \in |\mathscr{I}_C(r)|$ that it be singular at $x \in C$ is equivalent to 2 linear equations being satisfied. In order to show that the singularities of a generic element of $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ are as claimed we consider the embedding

$$\begin{array}{ccc} \mathbb{P}(H^0(\mathbb{P}^3, \mathscr{I}_x(1)) \oplus H^0(\mathbb{P}^3, \mathscr{I}_x(1))) & \longrightarrow & \Sigma_j(r) \\ [A, B] & \mapsto & V(AF+BG) \end{array} \tag{2.1}$$

where F, G are as above. The image is a sublinear system of $|\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ whose base locus is C, hence the generic $V(A \cdot F + B \cdot G)$ is smooth away from C by Bertini's Theorem. A local computation shows that the projectivized tangent cone of $V(AF + BG)$ at x is a smooth conic for generic A, B. Lastly let $q \in C \setminus \{x\}$. The set of $[A, B]$ such that $V(AF + BG)$ is singular at q has codimension 2 if $q \notin \{q_1, \ldots, q_m\}$, codimension 1 if $q \in \{q_1, \ldots, q_m\}$: it follows that for generic $[A, B]$ the surface $V(AF + BG)$ is smooth at all points of $C \setminus \{x\}$. This proves Item (1). The remaining items are proved similarly. □

Remark 2.4. Let $x \in C$. The proof of Proposition 2.3 shows that the projectivized tangent cone at x of the generic $X \in |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ is the generic conic in $\mathbb{P}(T_x\mathbb{P}^3)$ containing the point $\mathbb{P}(T_xC)$.

Proposition 2.5. *Suppose that $\pi^*\mathscr{O}_{\mathbb{P}^3}(r)(-E)$ is very ample and that $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-3)(-E)$ is base point free. Then the locus of non-integral surfaces $S \in |\Lambda(r)|$ has codimension at least* 4.

Proof. Let $\mathrm{Dec}(r) \subset \Lambda(r)$ be the (closed) subset of non-integral surfaces, and $\mathrm{Dec}(r)_1, \ldots, \mathrm{Dec}(r)_m$ be its irreducible components. Let $j \in \{1, \ldots, m\}$; we will prove that the locus of non-integral surfaces $S \in \mathrm{Dec}(r)_j$ has codimension at least 4. Suppose first that the generic $S \in \mathrm{Dec}(r)_j$ contains one (at least) of the components of E, say E_k. Since $\pi^*\mathscr{O}_{\mathbb{P}^3}(r)(-E)$ is very ample, and E_k is a $\mathbb{P}^1$-bundle, the image of the restriction map

$$H^0(W, \pi^*\mathscr{O}_{\mathbb{P}^3}(r)(-E)) \to H^0(E_k, \pi^*\mathscr{O}_{\mathbb{P}^3}(r)(-E)|_{E_k})$$

has dimension at least 4, and hence the locus of $S \in |\pi^*\mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ which contain E_k has codimension at least 4.
Next, suppose that the generic $S \in \mathrm{Dec}(r)_j$ does not contain any of the components of E. Let $\mathrm{Dec}(r)'_j \subset |\mathscr{I}_C(r)|$ be the image of $\mathrm{Dec}(r)_j$ under the natural isomorphism $\Lambda(r) \xrightarrow{\sim} |\mathscr{I}_C(r)|$. Let $X \in \mathrm{Dec}(r)'_j$ be generic; we claim that

$$\dim(\operatorname{sing} X \setminus C) \geq 1. \tag{2.2}$$

In fact $X = \pi(S)$, where $S \in \mathrm{Dec}(r)_j$ is generic, and since S is non-integral we may write $S = S_1 + S_2$ where S_1, S_2 are effective non-zero divisors on W (we will identify effective divisors and pure codimension-1 subschemes of W and $\mathbb{P}^3$). Thus $X = X_1 + X_2$, where $X_i := \pi(S_i)$. Since X_1, X_2 are effective non-zero divisors on $\mathbb{P}^3$ (non-zero because neither S_1 nor S_2 contains a component of E), their intersection has dimension at least 1. Now $X_1 \cap X_2 \subset \operatorname{sing} X$, hence in order to prove (2.2) it suffices to show that $X_1 \cap X_2$ is not contained in C. Suppose that $X_1 \cap X_2$ is contained in C; then, since it has dimension at least 1, there exists $k \in \{1, \ldots, n\}$ such that $X_1 \cap X_2$ contains C_k, and this implies that S contains E_k, contradicting our assumption. We have proved (2.2).
Next, let $p \neq q \in (\mathbb{P}^3 \setminus C)$, and let $\Omega_{p,q}(r) \subset |\mathscr{I}_C(r)|$ be the subset of divisors X which are singular at p, q, with degenerate quadratic terms. If $X \in \mathrm{Dec}(r)'_j$, then by (2.2) there exists a couple of distinct $p, q \in (X \setminus C)$ such that X is singular at p and q, with degenerate quadratic terms (in fact the set of such

couples is infinite). Thus, if Item (2) holds, then

$$\mathrm{Dec}(r)'_j \subset \bigcup_{p\neq q\in(\mathbb{P}^3\setminus C)} \Omega_{p,q}(r). \tag{2.3}$$

Hence it suffices to prove that the codimension of $\Omega_{p,q}$ in $|\mathscr{I}_C(r)|$ is 10 (as expected) for each $p \neq q \in (\mathbb{P}^3 \setminus C)$. Let $Y \in |\mathscr{I}_C(r-3)|$ be a surface not containing p nor q (it exists because $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-3)(-E)$ is base point free), and consider the subset

$$P_Y := \{Y + Z \mid Z \in |\mathscr{O}_{\mathbb{P}^3}(3)|\}.$$

An explicit computation shows that the codimension of the set of $Z \in |\mathscr{O}_{\mathbb{P}^3}(3)|$ singular at p, q, with degenerate quadratic terms, has codimension 10: it follows that $\Omega_{p,q}(r) \cap P_Y$ has codimension 10, and hence $\Omega_{p,q}(r)$ has codimension 10 in $|\mathscr{I}_C(r)|$. □

PROPOSITION 2.3 and PROPOSITION 2.5 prove that, if the hypotheses of THEOREM 2.1 are satisfied, then HYPOTHESIS 1.4 holds for $H \in |\mathscr{O}_{\mathbb{P}^3}(d)|$.

2.3. OUTLINE OF THE PROOF THAT THE NOETHER-LEFSCHETZ HYPOTHESIS HOLDS. Let A be an integral closed codimension-1 subset of $\Lambda(d)$. Let $A^\vee \subset \Lambda(d)^\vee$ be the projective dual of A, i.e. the closure of the locus of projective tangent hyperplanes $\mathbf{T}_S A$ for S a point in the smooth locus A^{sm} of A. Since $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$ is very ample we have the natural embedding $W \hookrightarrow \Lambda(d)^\vee$, and hence it makes sense to distinguish between the following two cases:

(I) $A^\vee$ is not contained in W.
(II) $A^\vee$ is contained in W.

Thus (I) holds if and only if, for the generic $S \in A^{sm}$, the projective tangent hyperplane $\mathbf{T}_S A$ is a base point free linear subsystem of $\Lambda(d)$. On the other hand, examples of codimension-1 subsets of $\Lambda(d)$ for which (II) holds are given by $\Delta(d)$ and by $\Sigma_j(d)$ for $j \in \{1,\ldots,n\}$. In fact $\Delta(d)^\vee = W$ and $\Sigma_j(d)^\vee = E_j$. The last equality holds because $S \in \Lambda(d)$ belongs to $\Sigma_j(d)$ if and only if it is tangent to E_j, thus $\Sigma_j(d) = E_j^\vee$, and hence $\Sigma_j(d)^\vee = E_j$ by projective duality. Let $\mathrm{NL}(\Lambda(d)\setminus\Delta(d))$ be the *Noether-Lefschetz locus*, i.e. the set of those smooth surfaces $S \in \Lambda(d)$ such that the restriction map $\mathrm{Pic}(W)_{\mathbb{Q}} \to \mathrm{Pic}(S)_{\mathbb{Q}}$ is *not* surjective. As is well-known $\mathrm{NL}(\Lambda(d) \setminus \Delta(d))$ is a countable union of closed subsets of $\Lambda(d) \setminus \Delta(d)$. In SUBSECTION 2.5 we will apply Joshi's criterion (Proposition 3.1 of [9]) in order to prove the following result.

PROPOSITION 2.6. *Suppose that $d \geq 5$ and that the following hold:*

(1) $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$ *is ample.*
(2) $H^1(C, T_C(d-4)) = 0$.
(3) *The sheaf $\mathscr{I}_C$ (on $\mathbb{P}^3$) is $(d-2)$-regular.*

Let $A \subset \Lambda(d)$ be an integral closed subset of codimension 1, and suppose that there exists $S \in (A\setminus\Delta(d))$ such that A is smooth at S, and the projective tangent space $\mathbf{T}_S A$ is a base-point free linear subsystem of $\Lambda(d)$. Then $A \setminus \Delta(d)$ does not belong to the Noether-Lefschetz locus $NL(\Lambda(d) \setminus \Delta(d))$.

The above result deals with codimension-1 subsets $A \subset \Lambda(d)$ for which (I) above holds. Thus, in order to finish the proof of THEOREM 2.1, it remains to deal with those A such that (II) above holds.

DEFINITION 2.7. Given $p \in W$ and $F \subset T_pW$ a vector subspace, we let

$$\Lambda_{p,F}(d) := \{S \in |\mathscr{I}_p \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)| : F \subset T_pS\}. \tag{2.4}$$

Let $\Gamma(d) := |\mathscr{I}_C(d)|$. We have a tautological identification $\Lambda(d) \xrightarrow{\sim} \Gamma(d)$: we let $\Gamma_{p,F}(d) \subset \Gamma(d)$ be the image of $\Lambda_{p,F}(d)$, and for $j \in \{1, \ldots, n\}$ we let $\Pi_j(d) \subset \Gamma(d)$ be the image of $\Sigma_j(d)$.

Notice that $\Lambda_{p,F}(d)$ and $\Gamma_{p,F}(d)$ are linear subsystems of $\Lambda(d)$ and $\Gamma(d)$ respectively. In SUBSECTION 2.6 we will prove the result below by applying an idea of Griffiths-Harris [8] as further developed by Lopez [12] and Brevik-Nollet [5].

PROPOSITION 2.8. *Suppose that the following hold:*

(1) $d \geq 4$ *and* $\pi^*\mathscr{O}_{\mathbb{P}^3}(d-3)(-E)$ *is very ample.*
(2) *None of the curves* $C_1, \ldots, C_n$ *is planar.*

Let X *be a very general element*

(a) *of* $\Gamma_{p,F}(d)$, *where either* $p \notin E$, *or else* $p \in E$ *and*

$$T_p(\pi^{-1}(\pi(p))) \not\subset F \subsetneq T_pE, \tag{2.5}$$

(b) *or of* $\Pi_j(d)$ *for some* $j \in \{1, \ldots, n\}$.

Then the Chow group $\mathrm{CH}^1(X)_{\mathbb{Q}}$ *is generated by* $c_1(\mathscr{O}_X(1))$ *and the classes of* $C_1, \ldots, C_n$.

Granting PROPOSITION 2.8, let us prove that the statement of THEOREM 2.1 holds for A such that $A^\vee$ is contained in W. We will distinguish between the following two cases:

(IIa) $A \notin \{\Sigma_1(d), \ldots, \Sigma_n(d)\}$.
(IIb) $A \in \{\Sigma_1(d), \ldots, \Sigma_n(d)\}$.

Suppose that (IIa) holds. By projective duality A is the closure of

$$\bigcup_{p \in (A^\vee)^{sm}} \Lambda_{p,T_pA^\vee} \tag{2.6}$$

Let $p \in (A^\vee)^{sm}$ be generic. We claim that Item (a) of PROPOSITION 2.8 hold for p and $F = T_pA^\vee$. In fact if $A^\vee \not\subset E$ then $p \notin E$ by genericity. If $A^\vee \subset E$ then $A^\vee$ is contained in E_j for a certain $j \in \{1, \ldots, n\}$. We claim that $A^\vee$ is a proper subset of E_j, and it is not equal to a fiber of the restriction of π to E_j. In fact, if $A^\vee = E_j$, then $A = E_j^\vee = \Sigma_j(d)$, and that contradicts the assumption that(IIa) holds. Now suppose that $A^\vee = \pi^{-1}(q)$ for a certain $q \in C_j$. Let $S \in A$ be generic. Since A is the closure of (2.6), S is tangent to $\pi^{-1}(q)$, and hence contains $\pi^{-1}(q)$ because S has degree 1 on every fiber of $E_j \to C_j$. It follows that S is tangent to E_j, and hence $A \subset E_j^\vee = \Sigma_j(d)$, contradicting the hypothesis that (IIa) holds.

Thus Item (a) of Proposition 2.8 hold for $p \in (A^\vee)^{sm}$ generic and $F = T_p A^\vee$, and hence if $S \in \Lambda_{p,T_pA^\vee}(d)$ is very general, then $\mathrm{CH}^1(X)_{\mathbb{Q}}$ is generated by $c_1(\mathscr{O}_X(1))$ and the classes of $C_1, \ldots, C_n$.
On the other hand, since $A \not\subset \Sigma(d)$, S intersects transversely E, and hence the restriction of π to S is an isomorphism $S \xrightarrow{\sim} X$. It follows that $\mathrm{CH}^1(S)_{\mathbb{Q}}$ is equal to the image of $\mathrm{CH}^1(W)_{\mathbb{Q}} \to \mathrm{CH}^1(S)_{\mathbb{Q}}$. This proves that there exists $S \in A$ such that $\mathrm{CH}^1(S)_{\mathbb{Q}}$ is equal to the image of $\mathrm{CH}^1(W)_{\mathbb{Q}} \to \mathrm{CH}^1(S)_{\mathbb{Q}}$. Actually our argument proves that there exists such an S which is smooth if $A \neq \Delta(d)$, and that if $A = \Delta(d)$ there exists such an S whose singular set consists of a single ODP. If the former holds, then we are done because $\mathrm{NL}(A \setminus \Delta(d))$ is a countable union of closed subsets of $A \setminus \Delta(d)$, and we have shown that the complement is non-empty. If the latter holds, let $\Delta(d)^0 \subset \Delta(d)$ be the open dense subset parametrizing surfaces with an ODP and no other singular point, then the set of $S \in \Delta(d)^0$ such that $\mathrm{CH}^1(W) \to \mathrm{CH}^1(S)$ is *not* surjective is a countable union of closed subsets of $\Delta(d)^0$ (take a simultaneous resolution), and we are done because we have shown that the complement is non empty.
Lastly suppose that (IIb) holds, i.e. $A = \Sigma_j(d)$ for a certain $j \in \{1, \ldots, n\}$. By Proposition 2.3 there exists an open dense subset $\Sigma_j(d)^0 \subset \Sigma_j(d)$ with the following property. If $S \in \Sigma_j(d)^0$ and $X = \pi(S)$, then X has a unique singular point, call it x (obviously $x \in C_j$), which is an ordinary node, and the restriction of π to S is the blow-up of X with center x (in particular S is smooth). Now suppose that $S \in \Sigma_j(d)^0$ is very general. Then by Proposition 2.8 the Chow group $\mathrm{CH}^1(S)_{\mathbb{Q}}$ is generated by the image of $\mathrm{CH}^1(W)_{\mathbb{Q}} \to \mathrm{CH}^1(S)_{\mathbb{Q}}$ and the class of $\pi^{-1}(x)$. Now notice that the set of $S \in \Sigma_j(d)^0$ such that $\mathrm{CH}^1(S)$ is *not* generated by the image of $\mathrm{CH}^1(W)_{\mathbb{Q}}$ together with $\pi^{-1}(x)$ is a countable union of closed subsets of $\Sigma_j(d)^0$; since the complement is not empty, we are done. □
Summing up: we have shown that in order to prove Theorem 2.1 it suffices to prove Proposition 2.6 and Proposition 2.8. The proofs are in the following subsections.

2.4. Infinitesimal Noether-Lefschetz results. We will recall a key result of K. Joshi. Let $U \subset H^0(W, \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))$ be a subspace and $\sigma \in U$ be non-zero. We let $S := V(\sigma)$, and we assume that S is smooth. Let $\mathfrak{m}_{\sigma,U} \subset \mathscr{O}_{\mathbb{P}(U),[\sigma]}$ be the maximal ideal and $\mathscr{T}_{\sigma,U} := \mathrm{Spec}(\mathscr{O}_{\mathbb{P}(U),[\sigma]}/\mathfrak{m}_\sigma^2)$ be the first-order infinitesimal neighborhood of $[\sigma]$ in $\mathbb{P}(U)$. We let $\mathscr{S}_{\sigma,U} \to \mathscr{T}_{\sigma,U}$ be the restriction of the family $\mathscr{S}_\Lambda \to \Lambda$ to $\mathscr{T}_{\sigma,U}$. The Infinitesimal Noether Lefschetz (INL) Theorem is valid at (U, σ) (see Section 2 of [9]) if the group of line-bundles on $\mathscr{S}_{\sigma,U}$ is equal to the image of the composition

$$\mathrm{Pic}(W) \longrightarrow \mathrm{Pic}(W \times_{\mathbb{C}} \mathscr{T}_{\sigma,U}) \longrightarrow \mathrm{Pic}(\mathscr{S}_{\sigma,U}). \tag{2.7}$$

Let $A \subset \Lambda(d)$ be an integral closed subset. Let $[\sigma]$ be a smooth point of A, and suppose that $S = V(\sigma)$ is smooth. Let $\mathbb{P}(U)$ be the projective tangent space

to A at $[\sigma]$. If the INL Theorem holds for (U,σ) then $A\setminus\Delta(d)$ does *not* belong to the Noether-Lefschetz locus $NL(\Lambda(d)\setminus\Delta(d))$.
Joshi [9] gave a cohomological condition which suffices for the validity of the INL Theorem. Suppose that $U\subset H^0(W,\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))$ generates $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$; we let $M(U)$ be the locally-free sheaf on W fitting into the exact sequence

$$0\longrightarrow M(U)\longrightarrow U\otimes\mathscr{O}_W\longrightarrow\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)\longrightarrow 0. \tag{2.8}$$

Proposition 2.9 (K. Joshi, Prop. 3.1 of [9]). *Let $U\subset H^0(W,\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))$ be a subspace which generates $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$. Let $0\neq\sigma\in U$. Suppose that $S=V(\sigma)$ is smooth, and that*

(a) $H^1(W,\Omega^2_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))=0$.
(b) $H^1(W,M(U)\otimes K_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))=0$.

Then the INL Theorem holds at (U,σ).

2.5. The generic tangent space is a base-point free linear system. We will prove Proposition 2.6 by applying Proposition 2.9.

Lemma 2.10. *Suppose that*

$$0=H^1(\mathbb{P}^3,\mathscr{I}_C\otimes T_{\mathbb{P}^3}(d-4))=H^1(C,T_C(d-4)). \tag{2.9}$$

Then $H^1(W,\Omega^2_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))=0$.

Proof. Since $\Omega^2_W\cong T_W\otimes K_W$ it is equivalent to prove that

$$0=H^1(W,T_W\otimes K_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E))=H^1(W,T_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)). \tag{2.10}$$

Let $\rho\colon E\to C$ be the restriction of π. Restricting the differential of π to E, one gets an exact sequence

$$0\longrightarrow\mathscr{O}_W(E)|_E\longrightarrow\rho^*\mathscr{N}_{C/\mathbb{P}^3}\longrightarrow\xi\longrightarrow 0 \tag{2.11}$$

of sheaves on E, where ξ is an invertible sheaf. Let $\iota\colon E\hookrightarrow W$ be the inclusion map. The differential of π gives the exact sequence

$$0\longrightarrow T_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)\longrightarrow\pi^*T_{\mathbb{P}^3}(d-4)\longrightarrow\iota_*(\xi\otimes\rho^*\mathscr{O}_C(d-4))\longrightarrow 0. \tag{2.12}$$

Below is a piece of the associated long exact sequence of cohomology:

$$\begin{aligned} &H^0(W,\pi^*T_{\mathbb{P}^3}(d-4))\to H^0(W,\iota_*(\xi\otimes\rho^*\mathscr{O}_C(d-4)))\to\\ &\to H^1(W,T_W\otimes\pi^*\mathscr{O}_{\mathbb{P}^3}(d-4))\to H^1(W,\pi^*T_{\mathbb{P}^3}(d-4)).\end{aligned} \tag{2.13}$$

We claim that $H^1(W,\pi^*T_{\mathbb{P}^3}(d-4))=0$. In fact the spectral sequence associated to π and abutting to the cohomology $H^q(W,\pi^*T_{\mathbb{P}^3}(d-4))$ gives an exact sequence

$$\begin{aligned} 0\to H^1(\mathbb{P}^3,\pi_*\pi^*T_{\mathbb{P}^3}(d-4))\to H^1(W,\pi^*T_{\mathbb{P}^3}(d-4))\to\\ \to H^0(\mathbb{P}^3,R^1\pi_*\pi^*T_{\mathbb{P}^3}(d-4))\to 0.\end{aligned} \tag{2.14}$$

Now $\pi_*\pi^*T_{\mathbb{P}^3}(d-4)\cong T_{\mathbb{P}^3}(d-4)$ and hence $H^1(\mathbb{P}^3,\pi_*\pi^*T_{\mathbb{P}^3}(d-4))=0$. Moreover $R^1\pi_*\pi^*T_{\mathbb{P}^3}(d-4)=0$ because $R^1\pi_*\mathscr{O}_W=0$, and hence

$H^1(W, \pi^* T_{\mathbb{P}^3}(d-4)) = 0$. By (2.13), in order to complete the proof it suffices to show that the map

$$H^0(W, \pi^* T_{\mathbb{P}^3}(d-4)) \to H^0(W, \iota_*(\xi \otimes \rho^* \mathscr{O}_C(d-4))) \tag{2.15}$$

is surjective. The long exact cohomology sequence associated to (2.11) gives an isomorphism

$$H^0(C, \mathscr{N}_{C/\mathbb{P}^3}(d-4)) \xrightarrow{\sim} H^0(W, \iota_*(\xi \otimes \rho^* \mathscr{O}_C(d-4))),$$

and moreover the map of (2.15) is identified with the composition

$$H^0(\mathbb{P}^3, T_{\mathbb{P}^3}(d-4)) \xrightarrow{\alpha} H^0(C, T_{\mathbb{P}^3}(d-4)|_C) \xrightarrow{\beta} H^0(C, \mathscr{N}_{C/\mathbb{P}^3}(d-4)). \tag{2.16}$$

The map α is surjective by the first vanishing in (2.9), while β is surjective by the second vanishing in (2.9). □

Let $U \subset H^0(\mathbb{P}^3, \mathscr{I}_C(d))$ be a subspace which generates $\mathscr{I}_C(d)$; we let $\overline{M}(U)$ be the sheaf on $\mathbb{P}^3$ fitting into the exact sequence

$$0 \longrightarrow \overline{M}(U) \longrightarrow U \otimes \mathscr{O}_{\mathbb{P}^3} \longrightarrow \mathscr{I}_C(d) \longrightarrow 0. \tag{2.17}$$

The curve C is a local complete intersection because C is smooth, and hence $\overline{M}(U)$ is locally-free.

LEMMA 2.11. *Suppose that the hypotheses of* LEMMA 2.10 *hold and that in addition the sheaf $\mathscr{I}_C$ is d-regular. Let $U \subset H^0(\mathbb{P}^3, \mathscr{I}_C(d))$ be a subspace which generates $\mathscr{I}_C(d)$, and let c be its codimension. Then $\bigwedge^p \overline{M}(U)$ is $(p+c)$-regular.*

Proof. Let $\overline{M} := \overline{M}(H^0(\mathscr{I}_C(d)))$. Then $\overline{M}$ is 1-regular: in fact $H^1(\mathbb{P}^3, \overline{M}) = 0$ because the exact sequence induced by (2.17) on H^0 is exact by definition, and $H^i(\mathbb{P}^3, \overline{M}(1-i)) = 0$ for $i \geq 2$ because $\mathscr{I}_C$ is d-regular. It follows that $\bigwedge^p \overline{M}$ is p-regular (Corollary 1.8.10 of [11]). Then, arguing as in the proof of the Lemma on p. 371 of [10] (see also Example 1.8.15 of [11]) one gets that $\bigwedge^p \overline{M}(U)$ is $(p+c)$-regular. □

Proof of PROPOSITION 2.6. By hypothesis there exists a smooth point $[\sigma]$ of $(A \setminus \Delta(d))$, such that the projective tangent space $\mathbf{T}_S A$ is a base-point free codimension-1 linear subsystem of Λ. We have $\mathbf{T}_S A = \mathbb{P}(U)$, where $U \subset H^0(\pi^* \mathscr{O}_{\mathbb{P}^3}(d)(-E))$ is a codimension-1 subspace which generates $\mathscr{O}_{\mathbb{P}^3}(d)(-E)$. We will prove that the INL Theorem holds for (U, σ), and PROPOSITION 2.6 will follow. By Joshi's PROPOSITION 2.9 it suffices to prove that the following hold:

(a) $H^1(W, \Omega^2_W \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(d)(-E)) = 0$.
(b) $H^1(W, M(U) \otimes K_W \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(d)(-E)) = 0$.

We start by noting that, since $T_{\mathbb{P}^3}$ is -1-regular, and by hypothesis $\mathscr{I}_C$ is $(d-2)$ regular, the sheaf $\mathscr{I}_C \otimes T_{\mathbb{P}^3}$ is $(d-3)$-regular, see Proposition 1.8.9 and Remark 1.8.11 of [11]. Thus the hypotheses of LEMMA 2.10 are satisfied, and hence Item (a) holds. Let us prove that Item (b) holds. Tensoring (2.8) by

$K_W \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E) \cong \pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)$ and taking cohomology we get an exact sequence

(2.18)
$$0 \to H^0(W, M(U) \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)) \to U \otimes H^0(W, \pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)) \xrightarrow{\alpha} H^0(W, \pi^*\mathscr{O}_{\mathbb{P}^3}(2d-4)(-E)) \to$$
$$\to H^1(W, M(U) \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d-4)) \to U \otimes H^1(W, K_W \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)).$$

Now $H^1(W, K_W \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)) = 0$ because by hypothesis $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$ is ample. Thus it suffices to prove that the map α is surjective. We have an identification $H^0(W, \pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)) = H^0(\mathbb{P}^3, \mathscr{I}_C(d))$, and hence U is identified with a codimension-1 subspace of $H^0(\mathbb{P}^3, \mathscr{I}_C(d))$ that we will denote by the same symbol. Clearly it suffices to prove that the natural map

$$U \otimes H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(d-4)) \longrightarrow H^0(\mathbb{P}^3, \mathscr{I}_C(2d-4)) \tag{2.19}$$

is surjective. Tensorize Exact Sequence (2.17) by $\mathscr{O}_{\mathbb{P}^3}(d-4)$ and take the associated long exact sequence of cohomology: then (2.19) appears in that exact sequence, and hence it suffices to prove that $H^1(\mathbb{P}^3, \overline{M}(U)(d-4)) = 0$. By Lemma 2.11 the sheaf $\overline{M}(U)$ is 2-regular, and by hypothesis $d \geq 5$: the required vanishing follows. □

2.6. All tangent spaces at smooth points are linear systems with a base-point. We will prove Proposition 2.8. We start with an elementary result.

Lemma 2.12. *Assume that $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-3)(-E)$ is very ample. Let $p \in W$ and $F \subset T_pW$ be a subspace such that one of the following holds:*

(1) *$p \notin E$ and $F \neq T_pW$,*
(2) *$p \notin E$ and $F = T_pW$,*
(3) *$p \in E$, and $T_p(\pi^{-1}(\pi(p))) \not\subset F \subsetneq T_pE$.*

Let $X \in \Gamma_{p,F}(r)$ (see Definition 2.7*) be generic. Then X is smooth if Item (1) or (3) holds, while X has an ODP at $q = \pi(p)$ and is smooth elsewhere if Item (2) holds.*

Proof. Suppose first that (1) or (2) holds, i.e. $p \notin E$, and let $q := \pi(p)$. The linear system $\Gamma_{p,F}(r)$ has base locus $C \cup \{q\}$. In fact, let $z \in (\mathbb{P}^3 \setminus C \setminus \{q\})$; then there exists $Y \in |\mathscr{I}_C(r-2)|$ not containing z because $\pi^*\mathscr{O}_{\mathbb{P}^3}(r-2)(-E)$ is very ample, and a quadric $Q \in \Gamma_{p,F}(2)$ not containing z. Thus $Y + Q$ is an element of $\Gamma_{p,F}(r)$ which does not contain z. Hence the generic $X \in \Gamma_{p,F}(r)$ is smooth away from $C \cup \{q\}$ by Bertini. Considering $Y + Q \in \Gamma_{p,F}(r)$ as above we also get that the behaviour in q of the generic element of $\Gamma_{p,F}(r)$ is as claimed. It remains to prove that the generic $X \in \Gamma_{p,F}(r)$ is smooth at every point of C, i.e. that $\Gamma_{p,F}(r)$ is not a subset of $\Sigma(r)$. The proof that $\Gamma_{p,F}(r)$ has base locus $C \cup \{q\}$ proves also that

$$\dim \Gamma_{p,F}(r) = \dim |\mathscr{I}_C(r)| - \dim F - 1. \tag{2.20}$$

Thus in order to prove that $\Gamma_{p,F}(r)$ is not a subset of $\Sigma(r)$, it suffices to prove that for $x \in C$

$$\dim |\mathscr{I}_x^2(r)| \cap \Gamma_{p,F}(r) \leq \dim |\mathscr{I}_C(r)| - \dim F - 3. \tag{2.21}$$

By Item (1) of Proposition 2.3, $\dim |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)| = \dim |\mathscr{I}_C(r)| - 2$, and hence (2.21) is equivalent to

$$\operatorname{cod}(|\mathscr{I}_x^2(r)| \cap \Gamma_{p,F}(r), |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|) = \dim F + 1. \tag{2.22}$$

We must check that imposing to $X \in |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ that it contains q and that $d\pi(p)(F) \subset T_q X$, gives $\dim F + 1$ linearly independent conditions. By Item (1) of Proposition 2.3, there exists $Y \in |\mathscr{I}_x^2(r-2)| \cap |\mathscr{I}_C(r-2)|$ not containing q. Consider the linear subsystem $A \subset |\mathscr{I}_x^2(r)| \cap |\mathscr{I}_C(r)|$ whose elements are $Y + Q$, where $Q \in |\mathscr{O}_{\mathbb{P}^3}(2)|$; imposing to $X \in A$ that it contains q and that $d\pi(p)(F) \subset T_q X$, gives $\dim F + 1$ linearly independent conditions, and hence (2.22) follows. This finishes the proof that if (1) or (2) holds, then the conclusion of the lemma holds.

Now suppose that (3) holds. Suppose that $F = \{0\}$, and let $S \in \Lambda_{p,F}(r) = |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ be generic. Then S is smooth at p because $\pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)$ is very ample, and by Bertini's Theorem it is smooth away from p as well. In order to prove that $X = \pi(S)$ is smooth we must check that S does not contain any of the lines $\mathbf{L}_x := \pi^{-1}(x)$ for $x \in C$. Since $\pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)$ is very ample,

$$\operatorname{cod}(|\mathscr{I}_{\mathbf{L}_x} \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)| \cap |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|, |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|) = \begin{cases} 1 & \text{if } x = q, \\ 2 & \text{if } x \neq q. \end{cases} \tag{2.23}$$

It follows that a generic $S \in |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ does not contain any $\mathbf{L}_x$.

We are left to deal with the case of a 1-dimensional $F \subset T_p E$ transverse to $T_p(\pi^{-1}(q))$. Let $Z \subset W$ be the 0-dimensional scheme of length 2 supported at p, with tangent space F; thus $Z \subset E$. We must prove that a generic $S \in |\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ is smooth and contains no line $\mathbf{L}_x$ where $x \in C$.

We claim that the (reduced) base-locus of $|\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ is p. In fact no $z \in (\mathbf{L}_q \setminus \{p\})$ is in the base-locus of $|\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ because $\mathbf{L}_q$ is a line and there exists $S \in |\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ which is not tangent to $\mathbf{L}_q$ at p. Moreover no $z \in (W \setminus \mathbf{L}_q)$ is in the base-locus of $|\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ because of the following argument. There exist $T \in |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r-1)(-E)|$ not containing z, and a plane $P \subset \mathbb{P}^3$ containing q and not containing $\pi(z)$; then $(T+P) \in |\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ does not contain z. This proves that the (reduced) base-locus of $|\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ is p; it follows that the generic $S \in |\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ is smooth.

We finish by showing that (2.23) holds with $\mathscr{I}_p$ replaced by $\mathscr{I}_Z$. The case $x = q$ is immediate. If $x \in C \setminus \{q\}$ we get the result by considering elements $(T+P) \in |\mathscr{I}_Z \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r)(-E)|$ where P is a fixed plane containing q and not containing x, and $T \in |\mathscr{I}_p \otimes \pi^* \mathscr{O}_{\mathbb{P}^3}(r-1)(-E)|$. $\square$

Remark 2.13. The proof of Lemma 2.12 shows that, if Item (2) holds, the projectivized tangent cone at q of the generic $X \in \Gamma_{p,F}(r)$ is the generic conic in $\mathbb{P}(T_q\mathbb{P}^3)$.

Proof of Proposition 2.8. Let $r \in \{d-1, d\}$. Suppose that $p \in W$, $F \subset T_pW$, and either $p \notin E$, or else $p \in E$ and (2.5) holds. By Lemma 2.12 there exists an open dense subset $\mathscr{U}_{p,F}(r) \subset \Gamma_{p,F}(r)$ such that for $X \in \mathscr{U}_{p,F}(r)$ the following holds:

(1) X is smooth if $p \notin E$ and $F \neq T_pW$, or $p \in E$.
(2) X has an ODP at $q = \pi(p)$, and is smooth elsewhere, if $p \notin E$ and $F = T_pW$.

Similary, let $j \in \{1, \ldots, n\}$, and $q \in C_j$. By Proposition 2.3 there exists an open dense subset $\mathscr{U}_{q,j}(r) \subset |\mathscr{I}_q^2(r)| \cap \Sigma_j(r)$ such that every $X \in \mathscr{U}_{q,j}(r)$ has an ODP at q and is smooth elsewhere. It will suffice to prove that if X is a very general element of $\mathscr{U}_{p,F}(r)$ or of $\mathscr{U}_{q,j}(r)$, then $\mathrm{CH}^1(X)_{\mathbb{Q}}$ is generated by $c_1(\mathscr{O}_X(1))$ and the classes of $C_1, \ldots, C_n$. Notice that if X is an element of $\mathscr{U}_{p,F}(r)$ or of $\mathscr{U}_{q,j}(r)$, then X is $\mathbb{Q}$-factorial. More precisely: if D is a Weil divisor on X then $2D$ is a Cartier divisor. Let $\mathrm{NL}(\mathscr{U}_{p,F}(d)) \subset \mathscr{U}_{p,F}(d)$ be the subset of X such that $\mathrm{Pic}(X) \otimes \mathbb{Q}$ is not generated by $\mathscr{O}_X(1)$ and $\mathscr{O}_X(2C_1), \ldots, \mathscr{O}_X(2C_n)$, and define similarly $\mathrm{NL}(\mathscr{U}_{q,j}(d)) \subset \mathscr{U}_{q,j}(d)$. Then $\mathrm{NL}(\mathscr{U}_{p,F}(d))$ is a countable union of closed subsets of $\mathscr{U}_{p,F}(d)$ (there exists a simultaneous resolution if the surfaces in $\mathscr{U}_{p,F}(d)$ are not smooth), and similarly for $\mathrm{NL}(\mathscr{U}_{q,j}(d))$. Hence it suffices to prove that $\mathscr{U}_{p,F}(d) \setminus \mathrm{NL}(\mathscr{U}_{p,F})(d)$ and $\mathscr{U}_{q,j}(d) \setminus \mathrm{NL}(\mathscr{U}_{q,j}(d))$ are not empty.

Let Y be an element of $\mathscr{U}_{p,F}(d-1)$ or of $\mathscr{U}_{q,j}(d-1)$, and let X be a generic element of $\mathscr{U}_{p,F}(d)$ or of $\mathscr{U}_{q,j}(d)$. Since $\pi^*\mathscr{O}_{\mathbb{P}^3}(d)(-E)$ is very ample and X is generic, the intersection of X and Y is reduced, and there exists an integral curve $C_0 \subset \mathbb{P}^3$ such that its irreducible decomposition is

$$X \cap Y = C_0 \cup C_1 \cup \ldots \cup C_n. \tag{2.24}$$

Now let $P \subset \mathbb{P}^3$ be a generic plane, in particular transverse to $C_0 \cup C_1 \cup \ldots \cup C_n$. Let $X = V(f)$, $Y = V(g)$ and $P = V(l)$. Let

$$\mathscr{X} := V(g \cdot l + tf) \subset \mathbb{P}^3 \times \mathbb{A}^1. \tag{2.25}$$

The projection $\mathscr{X} \to \mathbb{A}^1$ is a family of degree-d surfaces, with central fiber $Y + P$. The 3-fold $\mathscr{X}$ is singular. First $\mathscr{X}$ is singular at the points $(x, 0)$ such that $x \in X \cap Y \cap P$, and it has an ODP at each of these points because P is transverse to $C_0 \cup C_1 \cup \ldots \cup C_n$. Moreover

(I) $\mathscr{X}$ has no other singularities if we are dealing with $\mathscr{U}_{p,F}(d)$ and $F \neq T_pW$,
(II) $\mathscr{X}$ is also singular at $\{q\} \times \mathbb{A}^1$ if we are dealing with $\mathscr{U}_{p,F}(d)$ and $F = T_pW$, or if we are dealing with $\mathscr{U}_{q,j}(d)$.

We desingularize $\mathscr{X}$ as follows. The ODP's are eliminated by a small resolution (we follow p. 35 of [8], and choose a specific small resolution among the many possible ones), while to desingularize $\{q\} \times \mathbb{A}^1$ we blow-up that curve: let

$\widehat{\mathscr{X}} \to \mathscr{X}$ be the birational morphism. Then $\widehat{\mathscr{X}}$ is smooth (if $p \notin E$ and $F = T_pW$, or if we are dealing with $\mathscr{U}_{q,k}(d)$, then $\widehat{\mathscr{X}}$ is smooth over $\{q\} \times \mathbb{A}^1$ by REMARK 2.4 and REMARK 2.13).

The composition of $\widehat{\mathscr{X}} \to \mathscr{X}$ and the projection $\mathscr{X} \to \mathbb{A}^1$ is a flat family of surfaces $\varphi\colon \widehat{\mathscr{X}} \to \mathbb{A}^1$. The central fiber $\widehat{Z}_0 := \varphi^{-1}(0)$ has normal crossings, it is the union of Y and the blow-up $\widetilde{P}$ of P at the points of $X \cap Y \cap P$, the curve $Y \cap P$ being glued to its strict transform in $\widetilde{P}$. There will be an open dense $B \subset \mathbb{A}^1$ containing 0 such that $\widehat{Z}_t := \varphi^{-1}(t)$ is smooth for $t \in B \setminus \{0\}$, and it is isomorphic to $Z_t := V(g \cdot l + tf)$ in Case (I), while it is the blow-up of Z_t at q (an ODP) in Case (II). We replace $\widehat{\mathscr{X}}$ by $\varphi^{-1}(B)$ but we do not give it a new name.

One proves that if P is very general, then the following hold:

(I′) In Case (I), if t is very general in $B \setminus \{0\}$, then $\mathrm{Pic}(\widehat{Z}_t) \otimes \mathbb{Q}$ is generated by the classes of $\mathscr{O}_{\widehat{Z}_t}(1)$, $\mathscr{O}_{\widehat{Z}_t}(C_1), \ldots, \mathscr{O}_{\widehat{Z}_t}(C_n)$. (Notice that $\widehat{Z}_t = Z_t$ because we are in case (I).)

(II′) In Case (II), if t is very general in $B \setminus \{0\}$, letting $\mu_t\colon \widehat{Z}_t \to Z_t$ be the blow-up of q and $R_t \subset \widehat{Z}_t$ the exceptional curve, the group $\mathrm{Pic}(\widehat{Z}_t) \otimes \mathbb{Q}$ is generated by the classes of $\mu_t^*\mathscr{O}_{Z_t}(1)$, $\mu_t^*\mathscr{O}_{Z_t}(2C_1), \ldots, \mu_t^*\mathscr{O}_{Z_t}(2C_n)$ and $\mathscr{O}_{\widehat{Z}_t}(R_t)$.

One does this by controlling the Picard group of the degenerate fiber $\widehat{Z}_0$. As proved in [8, 12, 5] it suffices to show that the following hold:

(a) Let $\mathscr{V} \subset |\mathscr{O}_{\mathbb{P}^3}(1)|$ be the open subset of planes intersecting transversely $C_0 \cup \ldots \cup C_n$, let $I \subset (C_0 \cup \ldots \cup C_n) \times \mathscr{V}$ be the incidence subset and $\rho\colon I \to \mathscr{V}$ be the natural finite map: then the mododromy of ρ acts on a fiber $(D_0, \ldots, D_n, P)$ as the product of the symmetric groups $\mathfrak{S}_{\deg C_0} \times \ldots \times \mathfrak{S}_{\deg C_n}$.

(b) Let $j \in \{0, \ldots, n\}$, let $P \subset \mathbb{P}^3$ be a very general plane, and let $a, b \in C_j \cap P$ be distinct points; then the divisor class $a - b$ on the (smooth) curve $Y \cap P$ is not torsion.

Now Item (a) is Proposition II.2.6 of [12]. It remains to prove that (b) holds. To this end we will show that C_0 is not planar and we will control the set of planes P such that $P \cap Y$ is reducible (see the proof of Item (b) of Lemma 3.4 of [5]).

CLAIM 2.14. *The curve C_0 (see (2.24)) is not planar.*

Proof. By hypothesis $\pi^*\mathscr{O}_{\mathbb{P}^3}(d-3)(-E)$ is very ample, in particular it has a non-zero section, and hence there exists a non-zero $\tau \in H^0(\mathbb{P}^3, \mathscr{I}_C(d-3))$. Multiplying τ by sections of $\mathscr{O}_{\mathbb{P}^3}(3)$ we get that that $h^0(\mathbb{P}^3, \mathscr{I}_C(d)) \geq 20$. Now assume that C_0 is planar. Recall that $C = C_1 \cup \ldots \cup C_n$, and let

$$H^0(\mathbb{P}^3, \mathscr{I}_C(d)) \xrightarrow{\alpha} H^0(Y, \mathscr{O}_Y(d))$$

be the restriction map. Since $(C + C_0) \in |\mathscr{O}_Y(d)|$, the image of α is equal to $H^0(Y, \mathscr{O}_Y(C_0))$, and hence has dimension at most 4 because C_0 is planar.

The kernel of α has dimension 4 because Y has degree $(d-1)$. It follows that $h^0(\mathbb{P}^3, \mathscr{I}_C(d)) \leq 8$, contradicting the inequality $h^0(\mathbb{P}^3, \mathscr{I}_C(d)) \geq 20$. □

Thus none of the curves $C_0, C_1, \ldots, C_n$ is planar.

Lemma 2.15. *Let $Y \subset \mathbb{P}^3$ be a surface which is either smooth or has ODP's. The set of planes P such that $P \cap Y$ is reducible is the union of a finite set and the collection of pencils through lines of Y.*

Proof. Suppose the contrary. Then there exists a 1-dimensional family of planes P such that $P \cdot Y = C_1 + C_2$ with C_1, C_2 divisors which intersect properly, $\operatorname{supp} C_1$ is irreducible, and $\deg C_i > 1$. Next, we distinguish between the two cases:

(1) The generic P does not contain any singular point of Y.
(2) The generic P contains a single point $a \in \operatorname{sing} Y$, or two points $a, b \in \operatorname{sing} Y$.

Assume that (1) holds. Let $m_i := \deg C_i$ for $i = 1, 2$. Then

$$m_1 m_2 = (C_1 \cdot C_2)_P = (C_1 \cdot C_2)_Y = (C_1 \cdot (P - C_1))_Y = m_1 - (C_1 \cdot C_1)_Y \tag{2.26}$$

where $(C_1 \cdot C_2)_P$ is the intersection number of C_1, C_2 in the plane P, and $(C_1 \cdot C_2)_Y$ is the intersection number of C_1, C_2 in the surface Y (this makes sense because Y has ODP singularities, and hence is $\mathbb{Q}$-Cartier). The first equality of (2.26) holds by Bèzout, the second equality is proved by a local computation of the multiplicity of intersection at each point of $C_1 \cap C_2$ (one needs the hypothesis that Y is smooth at each such point). Thus (2.26) gives $(C_1 \cdot C_1)_Y = m_1(1 - m_2) < 0$, and this contradicts the hypothesis that C_1 moves in Y. If (2) holds one argues similarly. We go through the computations in the case that P contains two singular points. Let $\widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\{a, b\}$, and $\widetilde{Y}, \widetilde{P} \subset \widetilde{\mathbb{P}}^3$ be the strict transforms of Y and P respectively. By hypothesis Y has an ODP at each of its singular points and hence $\widetilde{Y}$ is smooth, and of course $\widetilde{P}$ is smooth. Let $\widetilde{C}_i$ be the strict transform of C_i in $\widetilde{\mathbb{P}}^3$. Let $r_i := \operatorname{mult}_a C_i$, $s_i := \operatorname{mult}_b C_i$. Then the equality

$$(\widetilde{C}_1 \cdot \widetilde{C}_2)_{\widetilde{P}} = (\widetilde{C}_1 \cdot \widetilde{C}_2)_{\widetilde{Y}} \tag{2.27}$$

gives

$$(\widetilde{C}_1 \cdot \widetilde{C}_2)_{\widetilde{Y}} = -(m_1 m_2 - m_1 - r_1 r_2 - s_1 s_2 + r_1 + s_1). \tag{2.28}$$

Now $r_i + s_i \leq m_i$ for $i = 1, 2$, because otherwise the line $\langle a, b\rangle$ would be contained in $Y \cap C_i$, and hence we would be considering curves residual to a line in Y, against the hypothesis. Since $r_i + s_i \leq m_i$ for $i = 1, 2$ the right-hand side of (2.28) is strictly negative, and this is a contradiction. □

Now we prove that Item (b) holds. Let $j \in \{0, \ldots, n\}$. Let $a, b \in C_j$ be generic, in particular they are smooth points of Y. By Lemma 2.15 every plane containing a, b intersects Y in an irreducible curve. Let $\widehat{Y} \to Y$ be the blow-up of the base-locus of the pencil of plane sections of Y containing a, b. Then $\widehat{Y}$ has at most A_n-singularities, and hence is $\mathbb{Q}$-factorial. Let E, F be the exceptional sets over a and b respectively, both have strictly negative self-intersection. Let $i > 0$ be such that iE and iF are Cartier. Let $\varphi\colon \widehat{Y} \to \mathbb{P}^1$ be the regular map defined by the pencil of plane sections of Y containing a, b; for $s \in \mathbb{P}^1$ we let $D_s := \varphi^{-1}(s)$. It suffices to prove that, given $r > 0$, the set of $s \in \mathbb{P}^1$ such that $\mathscr{O}_{\widehat{Y}}(riE - riF)|_{D_s}$ is trivial is finite. Assume the contrary: then, since every plane containing a, b intersects Y in an irreducible curve, there exists $\ell \in \mathbb{Q}$ such that $riE - riF \equiv \varphi^*(\ell p)$ in $\mathrm{Pic}(\widehat{Y})_{\mathbb{Q}}$, where $p \in \mathbb{P}^1$ (see the proof of Item (b) of Lemma 3.4 of [5]). It follows that the degrees of $\mathscr{O}_{\widehat{Y}}(riE - riF)$ on E and F are both equal to ℓ, and that is absurd because they have opposite signs. □

3. Proof of the main result

We will prove Theorem 0.1. Let $Q \subset \mathbb{P}^3$ be a smooth quadric and choose an isomorphism $\varphi\colon Q \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$: we let $\mathscr{O}_Q(a, b) := \varphi^*(\mathscr{O}_{\mathbb{P}^1}(a) \boxtimes \mathscr{O}_{\mathbb{P}^1}(b))$.

Proposition 3.1. *A curve in* $|\mathscr{O}_Q(2,3)|$ *is* 3*-regular.*

Proof. Let $D \in |\mathscr{O}_Q(2,3)|$. Considering the exact sequence $0 \to \mathscr{I}_D \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_D \to 0$ we see right away that if $i = 2, 3$, then $H^i(\mathbb{P}^3, \mathscr{I}_D(3-i)) = 0$. In order to prove that $H^1(\mathbb{P}^3, \mathscr{I}_D(2)) = 0$ we must show that $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \to H^0(D, \mathscr{O}_D(2))$ is surjective. The map $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \to H^0(Q, \mathscr{O}_Q(2,2))$ is surjective, hence it suffices to prove that $H^0(Q, \mathscr{O}_Q(2,2)) \to H^0(C, \mathscr{O}_D(2))$ is surjective. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_Q(0,-1) \longrightarrow \mathscr{O}_Q(2,2) \longrightarrow \mathscr{O}_D(2) \longrightarrow 0,$$

and since $H^1(Q, \mathscr{O}_Q(0,-1)) = 0$ the map $H^0(Q, \mathscr{O}_Q(2,2)) \to H^0(D, \mathscr{O}_D(2))$ is indeed surjective. □

Proof of Theorem 0.1. If $d \le 6$ there is nothing to prove, hence we may assume that $d \ge 7$. Let $n := \lfloor \frac{d-4}{3} \rfloor$. Choose disjoint smooth curves $C_1, \ldots, C_n$ such that each C_j is a $(2,3)$-curve on a smooth quadric, and let $C := C_1 \cup \ldots \cup C_n$. We may assume that for $j \in \{1, \ldots, n\}$ the degree-0 class in $\mathrm{CH}_0(C_j)$ given by $5c_1(K_{C_j}) - 2c_1(\mathscr{O}_{C_j}(1))$ is *not* zero. Let us show that the hypotheses of Theorem 2.1 are satisfied. Let $j \in \{1, \ldots, n\}$. We let $\pi_j \colon W_j \to \mathbb{P}^3$ be the blow-up of C_j, and $F_j \subset W_j$ be the exceptional divisor. Then $\pi_j^*\mathscr{O}_{\mathbb{P}^3}(3)(-F_j)$ is globally generated, and $\pi_j^*\mathscr{O}_{\mathbb{P}^3}(4)(-F_j)$ is very ample: since $d-3 \ge 3(n-1)+4$ it follows that $\pi^*\mathscr{O}_{\mathbb{P}^3}(d-3)(-E)$ is very ample. Let $j \in \{1, \ldots, n\}$: since $d \ge 7$ the cohomology group $H^1(C_j, T_{C_j}(d-4))$ vanishes, and hence $H^1(C, T_C(d-4)) = 0$. By Proposition 3.1 and Example 1.8.32 of [11] the curve C is $3n$-regular, and since $3n \le (d-4)$ the curve C is $(d-2)$-regular. Lastly, by construction no curve C_j is planar. We have shown that the hypotheses of Theorem 2.1 are satisfied, and hence Hypothesis 1.5 holds for $H \in |\mathscr{O}_{\mathbb{P}^3}(d)|$. Let $X \in |\mathscr{I}_C(d)|$ be smooth and very generic: since for $j \in \{1, \ldots, n\}$ the class $5c_1(K_{C_j}) - 2c_1(\mathscr{O}_{C_j}(1))$ is not zero, the decomposable classes $H^2, C_1^2, \ldots, C_n^2$ on X are linearly independent by Proposition 1.7. Thus $\mathrm{DCH}_0(X)$ has rank at least $n + 1 = \lfloor \frac{d-1}{3} \rfloor$. □

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