Remarks on L^p-Boundedness of Wave Operators for Schrödinger Operators with Threshold Singularities

K. Yajima [1]

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Abstract. We consider the continuity property in Lebesgue spaces $L^p(\mathbb{R}^m)$ of the wave operators $W_\pm$ of scattering theory for Schrödinger operators $H = -\Delta + V$ on $\mathbb{R}^m$, $|V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 2$ when H is of exceptional type, i.e. $\mathcal{N} = \{u \in \langle x\rangle^s L^2(\mathbb{R}^m) \colon (1 + (-\Delta)^{-1}V)u = 0\} \neq \{0\}$ for some $1/2 < s < \delta - 1/2$. It has recently been proved by Goldberg and Green for $m \geq 5$ that $W_\pm$ are in general bounded in $L^p(\mathbb{R}^m)$ for $1 \leq p < m/2$, for $1 \leq p < m$ if all $\phi \in \mathcal{N}$ satisfy $\int_{\mathbb{R}^m} V\phi dx = 0$ and, for $1 \leq p < \infty$ if $\int_{\mathbb{R}^m} x_i V\phi dx = 0$, $i = 1, \dots, m$ in addition. We make the results for $p > m/2$ more precise and prove in particular that these conditions are also necessary for the stated properties of $W_\pm$. We also prove that, for $m = 3$, $W_\pm$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$ and that the same holds for $1 < p < \infty$ if and only if all $\phi \in \mathcal{N}$ satisfy $\int_{\mathbb{R}^3} V\phi dx = 0$ and $\int_{\mathbb{R}^3} x_i V\phi dx = 0$, $i = 1, 2, 3$, simultaneously.

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1 Introduction

Let $H_0 = -\Delta$ be the free Schrödinger operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ with domain $D(H_0) = \{u \in \mathcal{H} \colon -\Delta u \in \mathcal{H}\}$ and $H = H_0 + V$, V being the multiplication operator with the real measurable function $V(x)$ which satisfies

$$|V(x)| \leq C\langle x\rangle^{-\delta} \ \text{ for some } \ \delta > 2, \ \langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}. \tag{1.1}$$

Then, H is selfadjoint in $\mathcal{H}$ with a core $C_0^\infty(\mathbb{R}^m)$ and it satisfies the following properties (see e.g. [18, 19, 21, 22, 23]):

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(i) The spectrum $\sigma(H)$ of H consists of the absolutely continuous (AC for short) part $[0,\infty)$ and a finite number of non-positive eigenvalues of finite multiplicities.

We write $\mathcal{H}_{ac}(H)$ for the AC spectral subspace of $\mathcal{H}$ for H, H_{ac} for the part of H in $\mathcal{H}_{ac}(H)$ and $P_{ac}(H)$ for the orthogonal projection onto $\mathcal{H}_{ac}(H)$.

(ii) Wave operators $W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ defined by strong limits exist and are complete, viz. $\operatorname{Image} W_\pm = \mathcal{H}_{ac}(H)$. They are unitary from $\mathcal{H}$ onto $\mathcal{H}_{ac}(H)$ and intertwine H_{ac} and H_0. Hence, for Borel functions f,

$$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^*. \tag{1.2}$$

If follows that various mapping properties of $f(H)P_{ac}$ may be deduced from those of $f(H_0)$ if the corresponding ones of $W_\pm$ are known. In particular, if $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^m))$ for $1 \le p_1 \le p \le p_2 < \infty$, then $W_\pm^* \in \mathbf{B}(L^q(\mathbb{R}^m))$ for $q_2 \le q \le q_1$, $1/p_j + 1/q_j = 1$, $j = 1, 2$, and

$$\|f(H)P_{ac}(H)\|_{\mathbf{B}(L^q,L^p)} \le C_{pq}\|f(H_0)\|_{\mathbf{B}(L^q,L^p)}, \tag{1.3}$$

for these p and q with C_{pq} which are *independent of* f. We define the Fourier and the conjugate Fourier transforms $\mathcal{F}u(\xi)$ and $\mathcal{F}^*u(\xi)$ respectively by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^m} e^{-ix\xi}u(x)dx \text{ and } \mathcal{F}^*u(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{ix\xi}u(x)dx.$$

We also write $\hat{u}(\xi)$ for $\mathcal{F}u(\xi)$.

The intertwining property (1.2) may be made more precise. Wave operators $W_\pm$ are transplantations ([24]) of the complete set of (generalized) eigenfunctions $\{e^{ix\xi}\colon \xi \in \mathbb{R}^m\}$ of $-\Delta$ by those of out-going and in-coming scattering eigenfunctions $\{\varphi_\pm(x,\xi)\colon \xi\in\mathbb{R}^m\}$ of $H = -\Delta + V$ ([19]):

$$W_\pm u(x) = \mathcal{F}_\pm^*\mathcal{F}u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \varphi_\pm(x,\xi)\hat{u}(\xi)d\xi,$$

where $\mathcal{F}_\pm$ and $\mathcal{F}_\pm^*$ are the generalized Fourier transforms associated with $\{\varphi_\pm(x,\xi)\colon \xi\in\mathbb{R}^m\}$ and the conjugate ones defined respectively by

$$\mathcal{F}_\pm u(\xi) = \int_{\mathbb{R}^d} \overline{\varphi_\pm(x,\xi)}u(x)dx, \quad \mathcal{F}_\pm^* u(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^d} \varphi_\pm(x,\xi)u(x)dx.$$

They satisfy $\mathcal{F}_\pm^*\mathcal{F}_\pm u = u$ for $u \in \mathcal{H}_{ac}(H)$ and, $\mathcal{F}_\pm\mathcal{F}_\pm^* u = u$ for $u \in L^2(\mathbb{R}^m)$. We define $F(D) \equiv \mathcal{F}^*M_F\mathcal{F}$ and $F(D_\pm) \equiv \mathcal{F}_\pm^*M_F\mathcal{F}_\pm$ for Borel functions F on $\mathbb{R}^m$ where M_F is the multiplication with $F(\xi)$. Then,

$$F(D_\pm) = W_\pm F(D)W_\pm^* u, \quad u \in \mathcal{H}_{ac}(H)$$

and $W_\pm$ transplant estimates for $F(D)$ in L^p-spaces to $F(D_\pm)$.

In this paper we are interested in the problem whether or not $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$. This will almost automatically imply the same property in Sobolev spaces $W^{k,p}(\mathbb{R}^m) = \{u \in L^p(\mathbb{R}^m)\colon \partial^\alpha u \in L^p(\mathbb{R}^m)\}$ for integers $0 \le k \le 2$ (see Section 7 of [8]).

There is now a large literature on this problem ([3, 4, 6, 8, 27, 31, 15, 16, 29, 33]) and it is well known that the answer depends on the spectral properties of H at 0, the bottom of the AC spectrum of H. We define

$$\mathcal{E} = \{u \in H^2(\mathbb{R}^m)\colon (-\Delta + V)u = 0\}, \tag{1.4}$$

the eigenspace of H with eigenvalue 0 and, for $1/2 < s < \delta - 1/2$,

$$\mathcal{N} = \{u \in \langle x\rangle^s L^2(\mathbb{R}^m) : (1 + (-\Delta)^{-1}V)u = 0\} = 0. \tag{1.5}$$

Functions ϕ in $\mathcal{N}$ satisfy $-\Delta\phi + V\phi = 0$ for $x \in \mathbb{R}^m$. The space $\mathcal{N}$ is finite dimensional, independent of $1/2 < s < \delta - 1/2$, $\mathcal{E} \subset \mathcal{N}$ and, if $m \ge 5$, $\mathcal{E} = \mathcal{N}$ ([14]). The operator H is said be of *generic type* if $\mathcal{N} = \{0\}$ and of *exceptional type* otherwise. When H is of *generic type*, we have rather satisfactory results (though there is much space for improving conditions on V) and it has been proved that $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for all $1 \le p \le \infty$ if $m \ge 3$ and, for all $1 < p < \infty$ if $m = 1$ and $m = 2$ under various smoothness and decay at infinity assumptions on V (see [4] for the best result when $m = 3$); but they are in general *not* bounded in $L^1(\mathbb{R}^1)$ or $L^\infty(\mathbb{R}^1)$ when $m = 1$ ([27]).

When H is of exceptional type, it is long known that the same results hold when $m = 1$ (see [27, 3, 6]). For higher dimensions $m \ge 3$, it is first shown ([33, 8]) that $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for $3/2 < p < 3$ if $m = 3$ and for $\frac{m}{m-2} < p < \frac{m}{2}$ if $m \ge 5$, which is subsequently extended to $1 < p < 3$ for $m = 3$ and $1 < p < m/2$ for $m \ge 5$ ([34]). Then, recently, Goldberg and Green ([10]) have substantially improved these results by proving the following theorem for $m \ge 5$. In what follows in this paper, we assume $m \ge 3$ and V satisfies the following assumption. The constant m_* is defined by

$$m_* = (m-1)/(m-2).$$

Assumption 1.1. *V is a real valued measurable function such that*

(1) $\mathcal{F}(\langle x\rangle^{2\sigma} V) \in L^{m_*}$ *for some* $\sigma > 1/m_*$.

(2) $|V(x)| \le C\langle x\rangle^{-\delta}$ *for some* $\delta > \begin{cases} m+4, & \text{if } 3 \le m \le 7, \\ m+3, & \text{if } m \ge 8 \end{cases}$ *and* $C > 0$.

The condition (1) requires certain smoothness on V.

We write $\langle u, v\rangle = \int_{\mathbb{R}^m} \overline{u(x)} v(x) dx$ and define subspaces $\mathcal{E}_1 \subset \mathcal{E}_0 \subset \mathcal{N}$ respectively by

$$\mathcal{E}_0 = \{\phi \in \mathcal{N}\colon \langle V, \phi\rangle = 0\}, \quad \mathcal{E}_1 = \{\phi \in \mathcal{E}_0\colon \langle xV, \phi\rangle = 0\}, \tag{1.6}$$

where $\langle xV, \phi\rangle = 0$ means $\langle x_i V, \phi\rangle = 0$ for all $1 \le i \le m$. We have $\dim \mathcal{N}/\mathcal{E}_0 \le 1$, $\mathcal{E}_0 = \mathcal{E}$ if $m = 3$ and $\mathcal{N} = \mathcal{E}$ if $m \ge 5$.

Theorem 1.2 (Goldberg-Green). *Suppose that V satisfies* Assumption 1.1 *and that H is of exceptional type. Then, if $m \geq 5$, $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ for $1 \leq p < m/2$. They are bounded in $L^p(\mathbb{R}^m)$ also for $1 \leq p < m$ if $\mathcal{N} = \mathcal{E}_0$ and for $1 \leq p < \infty$ if $\mathcal{N} = \mathcal{E}_1$.*

In this paper, we show following theorems which in particular prove the corresponding result for $m = 3$ and that the conditions $\mathcal{N} = \mathcal{E}_0$ and $\mathcal{N} = \mathcal{E}_1$ of Theorem 1.2 are also necessary for the stated properties of $W_{\pm}$ respectively. We write P, P_0 and P_1 for the orthogonal projections onto $\mathcal{E}$, $\mathcal{E}_0$ and $\mathcal{E}_1$ respectively. Because $(-\Delta)^{-1}V$ is a real operator, we may take the bases of $\mathcal{N}$, $\mathcal{E}_0$ and $\mathcal{E}_1$ which consist of real functions and P, P_0 and P_1 are real operators: For the conjugation $(\mathcal{C}u)(x) = \overline{u(x)}$,

$$\mathcal{C}^{-1}P\mathcal{C} = P, \quad \mathcal{C}^{-1}P_0\mathcal{C} = P, \quad \mathcal{C}^{-1}P_1\mathcal{C} = P_1. \tag{1.7}$$

We state results for $m = 3$, $m = 5$ and $m \geq 6$ separately.

Theorem 1.3. *Let $m = 3$. Suppose that V satisfies* Assumption 1.1 *and that H is of exceptional type. Then:*

(1) *$W_{\pm}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$.*

(2) *For $3 < p < \infty$, there exists a constant C such that*

$$\|(W_{\pm} \pm a\varphi \otimes |D|^{-1}V\varphi + P)u\|_{L^p} \leq C\|u\|_{L^p}, \tag{1.8}$$

where φ is the real function defined by (3.13) *(the canonical resonance), $a = 4\pi i|\langle V, \varphi\rangle|^{-2}$ and P may be replaced by $P \ominus P_1$.*

(3) *If $W_{\pm}$ are bounded in $L^p(\mathbb{R}^3)$ for some $3 < p < \infty$, then $\mathcal{N} = \mathcal{E}_1$. In this case they are bounded in $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$.*

Theorem 1.4. *Let $m = 5$. Suppose that V satisfies* Assumption 1.1 *and that H is of exceptional type. Then:*

(1) *$W_{\pm}$ are bounded in $L^p(\mathbb{R}^5)$ for $1 < p < 5/2$.*

(2) *For $5/2 < p < 5$, there exists a constant C such that*

$$\left\| \left(W_{\pm} \pm a_0(|D|^{-1}V\varphi) \otimes \varphi + \frac{P}{2} \right) u \right\|_{L^p} \leq C\|u\|_{L^p}, \tag{1.9}$$

where $\varphi = PV$, V being considered as a function, $a_0 = i/(24\pi^2)$ and P may be replaced by $P \ominus P_0$. If $W_{\pm}$ are bounded in $L^p(\mathbb{R}^5)$ for some $\frac{5}{2} < p < 5$, then $\mathcal{N} = \mathcal{E}_0$. In this case they are bounded in $L^p(\mathbb{R}^5)$ for all $1 < p < 5$.

(3) *By virtue of* (1) *and* (2), *the condition $\mathcal{E} = \mathcal{E}_0$ is necessary for $W_{\pm}$ to be bounded in $L^p(\mathbb{R}^5)$ for some $p > 5$. Suppose $\mathcal{E} = \mathcal{E}_0$. Then,*

$$\|(W_{\pm} + P)\, u\|_{L^p} \leq C\|u\|_{L^p} \tag{1.10}$$

for a constant C, where $P = P_0$ may be replaced by $P_0 \ominus P_1$. If $W_\pm$ are bounded in $L^p(\mathbb{R}^5)$ for some $p > m$, then $\mathcal{N} = \mathcal{E}_1$. In this case they are bounded in $L^p(\mathbb{R}^5)$ for all $1 < p < \infty$.

THEOREM 1.5. *Let $m \geq 6$. Suppose that V satisfies* Assumption 1.1 *and that H is of exceptional type. Then:*

(1) *$W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for $1 < p < m/2$.*

(2) *For $\frac{m}{2} < p < m$, there exists a constant $C > 0$ such that*

$$\|(W_\pm + D_m P)u\|_{L^p} \leq C_p \|u\|_{L^p}, \tag{1.11}$$

where P may be replaced by $P \ominus P_0$ and

$$D_m = \begin{cases} \dfrac{\Gamma\left(\frac{m-2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)}, & m \text{ is odd}, \quad (1.12) \\ \dfrac{2^m \Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \displaystyle\int_1^\infty (x^2+1)^{-(m-1)} dx, & m \text{ is even}. \quad (1.13) \end{cases}$$

If $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for some $m/2 < p < m$ then, $\mathcal{E} = \mathcal{E}_0$. In this case they are bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < m$

(3) *Suppose $\mathcal{E} = \mathcal{E}_0$. Let $m < p < \infty$. Then, for a constant C_p,*

$$\|(W_\pm + P)u\| \leq C_p \|u\|_{L^p}, \tag{1.14}$$

where P may be replaced by $P_0 \ominus P_1$. If $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for some $p > m$, then $\mathcal{E} = \mathcal{E}_1$. In this case they are bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < \infty$.

REMARK 1.6. (1) The integral in (1.13) may be computed explicitly:

$$\int_1^\infty (x^2+1)^{-(m-1)} dx = \frac{\Gamma\left(m - \frac{3}{2}\right)}{4\Gamma(m-1)} \left(\sqrt{\pi} - \sum_{j=1}^{m-2} \frac{\Gamma(j) 2^{-j+1}}{\Gamma\left(j + \frac{1}{2}\right)} \right). \tag{1.15}$$

(2) There are examples of V such that $\mathcal{E}_1 = \mathcal{E}_0 \subsetneq \mathcal{N}$, $\mathcal{E}_1 \subsetneq \mathcal{E}_0 = \mathcal{N}$ and $\mathcal{E}_1 \subsetneq \mathcal{E}_0 \subsetneq \mathcal{N}$ (see Example 8.4 of [13]).
(3) Murata's result (Theorem 1.2 of [20]) also implies that, if $\mathcal{N} \neq 0$, $W_\pm$ are not in general bounded in $L^p(\mathbb{R}^m)$ for $p > 3$ if $m = 3$ and for $p > \frac{m}{2}$ if $m \geq 5$.

The rest of the paper is devoted to the proof of Theorems. In spite that substantial part of Theorems 1.4 and 1.5 overlaps with Theorem 1.2 and that they miss critically important L^1-boundedness, we present the proof of Theorems which is very different from the one by Goldberg and Green ([10]). Our proof heavily uses harmonic analysis machinery, which produces sharper results for larger p's, however, at the same time, prevents us from reaching end points

$p = 1$ and $p = \infty$. We prove the theorems only for W_- since conjugation changes the direction of time, viz. $\mathcal{C}^{-1}e^{-itH}\mathcal{C} = e^{itH}$, and

$$W_+ = \mathcal{C}^{-1}W_-\mathcal{C}. \tag{1.16}$$

We use the following notation and conventions: The ℓ-th derivative of $f(x)$, $x \in \mathbb{R}$ is denoted by $f^{(\ell)}(x)$. $\Sigma = \mathbb{S}^{m-1} = \{x \colon x_1^2 + \cdots + x_m^2 = 1\}$ is the unit sphere in $\mathbb{R}^m$ and $\omega_{m-1} = 2\pi^{\frac{m}{2}}/\Gamma\left(\frac{m}{2}\right)$ is its area. The coupling and the inner product are anti-linear with respect to the first component,

$$(u, v) = \langle u, v\rangle = \int_{\mathbb{R}^n} \overline{u(x)}v(x)dx,$$

in accordance with the interchangeable notaion for the rank 1 operator

$$|u\rangle\langle v| = u \otimes v \ : \ \phi \mapsto u\langle v, \phi\rangle.$$

This notation is used also when v is in a certain function space and u in its dual space.

$$f \le_{|\cdot|} g \text{ means } |f| \le |g|.$$

For Banach spaces X and Y, $\mathbf{B}(X, Y)$ is the Banach space of bounded operators from X to Y and $\mathbf{B}(X) = \mathbf{B}(X, X)$; $\mathbf{B}_\infty(X, Y)$ and $\mathbf{B}_\infty(X)$ are spaces of compact operators; and the dual space $\mathbf{B}(X, \mathbb{C})$ of X is denoted by X^*. The identity operators in various Banach spaces are indistinguishably denoted by 1. For $1 \le p \le \infty$, $\|u\|_p = \|u\|_{L^p}$ is the norm of $L^p(\mathbb{R}^m)$ and p' is its dual exponent, $1/p + 1/p' = 1$. When $p = 2$, we often omit p and write $\|u\|$ for $\|u\|_2$. We interchangeably write $L^p_w(\mathbb{R}^m)$ or $L^{p,\infty}(\mathbb{R}^m)$ for weak-L^p spaces and $\|u\|_{p,w}$ or $\|u\|_{p,\infty}$ for their norms. For $s \in \mathbb{R}$,

$$L^2_s = \langle x\rangle^{-s}L^2 = L^2(\mathbb{R}^m, \langle x\rangle^{2s}dx), \quad H^s(\mathbb{R}^m) = \mathcal{F}L^2_s(\mathbb{R}^m)$$

are the weighted L^2 spaces and Sobolev spaces. The space of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^m)$.

We denote the resolvents of H and H_0 respectively by

$$R(z) = (H - z)^{-1}, \quad R_0(z) = (H_0 - z)^{-1}.$$

We parameterize $z \in \mathbb{C} \setminus [0, \infty)$ as $z = \lambda^2$ by $\lambda \in \mathbb{C}^+$, the open upper half plane of $\mathbb{C}$, so that the positive and the negative parts of the boundary $\{\lambda \colon \pm\lambda \in (0, \infty)\}$ are mapped onto the upper and the lower edges of the positive half line $\{z \in \mathbb{C} \colon z > 0\}$. We define

$$G(\lambda) = R(\lambda^2), \quad G_0(\lambda) = R_0(\lambda^2), \quad \lambda \in \mathbb{C}^+.$$

These are $\mathbf{B}(\mathcal{H})$-valued meromorphic functions of $\lambda \in \mathbb{C}^+$ and the limiting absorption principle [19] (LAP for short) asserts that, when considered as $\mathbf{B}(\langle x\rangle^{-s}L^2, \langle x\rangle^{t}L^2)$-valued functions for $s, t > \frac{1}{2}$ and $s + t > 2$, $G_0(\lambda)$ has

Hölder continuous extensions to its closure $\overline{\mathbb{C}}^+ = \{z : \Im z \geq 0\}$. The same is true also for $G(\lambda)$, but, if H is of exceptional type, it has singularities at $\lambda = 0$. In what follows $z^{\frac{1}{2}}$ is the branch of square root of z cut along the negative real axis such that $z^{\frac{1}{2}} > 0$ when $z > 0$.

The plan of the paper is as follows: In section 2, we record some results most of which are well known and which we use in the sequel. They include:

- Formulas for the integral kernel of $G_0(\lambda)$ as exponential-polynomials in odd dimensions or their superpositions in even dimensions.
- Representation of $\langle \psi | (G_0(\lambda) - G_0(-\lambda))u \rangle$ as the linear combination of Fourier transforms of $r^{j+1}M(r, \overline{\psi} * \check{u})$, $M(r, f)$ being the average of f over the sphere of radius r centered at the origin.
- The Muckenhaupt weighted inequality and examples of A_p-weights.

In section 3, we recall and improve results of [33] and [8] on the behavior as $\lambda \to 0$ of $(1 + G_0(\lambda)V)^{-1}$ and reduce the problem to the L^p-boundedness of

$$Z_s u = -\frac{1}{\pi i} \int_0^\infty G_0(\lambda) V S(\lambda)(G_0(\lambda) - G_0(-\lambda)) u \lambda F(\lambda) d\lambda \qquad (1.17)$$

where $S(\lambda)$ is the singular part of the expansion of $(1 + G_0(\lambda)V)^{-1}$ at $\lambda = 0$ and $F \in C_0^\infty(\mathbb{R})$ is such that $F(\lambda) = 1$ near $\lambda = 0$.

We prove Theorem 1.3 in Section 4, Theorem 1.4 for odd dimensions $m \geq 5$ in Section 5 and for even dimensions in Section 6. We explain the basic strategy of the proof at the end of §4.1 after most of basic ideas appears in the simplest form.

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2 Preliminaries

In this section we record some well known results which we use in what follows.

2.1 Integral kernel of the free resolvent

For $m \geq 2$, resolvent $G_0(\lambda)$ for $\Im \lambda \geq 0$ is the convolution with

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right) |x|^{m-2}} \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left(\frac{t}{2} - i\lambda|x|\right)^{\frac{m-3}{2}} dt \quad (2.1)$$

([28]). When $m \geq 3$ is odd, it is an exponential polynomial like function.

Lemma 2.1. *Let $m \geq 3$ be odd. Then:*

$$G_0(\lambda, x) = \sum_{j=0}^{(m-3)/2} C_j \frac{(\lambda|x|)^j e^{i\lambda|x|}}{|x|^{m-2}} \ \textit{with}\ C_j = \frac{(-i)^j (m-3-j)!}{2^{m-1-j}\pi^{\frac{m-1}{2}} j! (\frac{m-3}{2}-j)!}. \tag{2.2}$$

The constant C_0 may also be written as $C_0 = (m-2)^{-1}\omega_{m-1}^{-1}$ and

$$iC_0 + C_1 = 0, \quad \textit{when } m \geq 5. \tag{2.3}$$

If m is even, the structure of $G_0(\lambda, x)$ is more complex and this makes the analysis harder. For partly circumventing the difficulty we express $G_0(\lambda, x)$ as a superposition of exponential-polynomial like functions of the form (2.2). This will allow a part of the proof for even dimensions to go in parallel with the odd dimensional cases. We set

$$\nu = \frac{m-2}{2}.$$

Define operators $T_j^{(a)}$, $j = 0, \dots, \nu$ for superposing over parameter $a > 0$ by

$$T_j^{(a)}[f(x,a)] = C_{m,j}\omega_{m-1} \int_0^\infty (1+a)^{-(2\nu - j + \frac{1}{2})} f(x,a) \frac{da}{\sqrt{a}}, \tag{2.4}$$

$$C_{m,j}\omega_{m-1} = (-2i)^j \frac{\Gamma\left(2\nu - j + \frac{1}{2}\right)}{(m-2)!\sqrt{\pi}} \binom{\nu}{j}. \tag{2.5}$$

The factor ω_{m-1} is added for shorting some formulas below (see (2.18)).

Lemma 2.2. *If $m \geq 4$ is even, then we have*

$$G_0(\lambda, x) = \sum_{j=0}^{\nu} \omega_{m-1}^{-1} T_j^{(a)} \left[e^{i\lambda|x|(1+2a)} \frac{(\lambda|x|)^j}{|x|^{m-2}} \right]. \tag{2.6}$$

Proof. Let $C_{m*} = 2^{m-1}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)$. In the formula (2.1):

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{C_{m*}|x|^{m-2}} \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left(t - 2i\lambda|x|\right)^{\frac{m-3}{2}} dt, \tag{2.7}$$

write $(t - 2i\lambda|x|)^{\frac{m-3}{2}} = (t - 2i\lambda|x|)^{\nu}(t - 2i\lambda|x|)^{-\frac{1}{2}}$, expand $(t - 2i\lambda|x|)^{\nu}$ via the binomial formula and use the identity

$$z^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-az} a^{-\frac{1}{2}} da, \quad \Re z > 0 \tag{2.8}$$

for $(t - 2i\lambda|x|)^{-\frac{1}{2}}$. The right hand side of (2.7) becomes

$$\sum_{j=0}^{\nu} \frac{(-2i)^j}{\sqrt{\pi}C_{m*}} \binom{\nu}{j} \iint_{\mathbb{R}_+^2} e^{-(1+a)t} t^{2\nu - j} \left(e^{i\lambda|x|(1+2a)} \frac{(\lambda|x|)^j}{|x|^{m-2}} \right) \frac{dt}{\sqrt{t}} \frac{da}{\sqrt{a}}.$$

The integral converges absolutely if $m \geq 4$ and we obtain (2.6) after performing the integral with respect to t. $\square$

2.2 Spectral measure of H_0

. The spectral measure of $H_0 = -\Delta$ is AC and Stone's theorem implies that the spectral projection $E_0(d\mu)$ is given for $\mu = \lambda^2$, $\lambda > 0$ by

$$E_0(d\mu) = \frac{1}{2\pi i}(R_0(\mu + i0) - R_0(\mu - i0))d\mu = \frac{1}{i\pi}(G_0(\lambda) - G_0(-\lambda))\lambda d\lambda.$$

Lemma 2.3. *Let $m \geq 3$ and $u, v \in (L^1 \cap L^2)(\mathbb{R}^m)$. Then, both sides of the following equation can be continuously extended to $\lambda = 0$ and*

$$\lambda^{-1}\langle v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \langle |D|^{-1}v, (G_0(\lambda) - G_0(-\lambda))u\rangle, \quad \lambda \geq 0. \tag{2.9}$$

For bounded continuous functions f on $\mathbb{R}$ we have for $\lambda \geq 0$,

$$f(\lambda)\langle v, (G_0(\lambda)u - G_0(-\lambda))u\rangle = \langle v, (G_0(\lambda)u - G_0(-\lambda))f(|D|)u\rangle. \tag{2.10}$$

Proof. For $u, v \in (L^1 \cap L^2)(\mathbb{R}^m)$ we have

$$\langle v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \frac{\lambda^{m-2}i}{2(2\pi)^{m-1}}\int_\Sigma \overline{\hat{v}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega, \tag{2.11}$$

where $\Sigma = \mathbb{S}^{m-1}$. It follows, since $\widehat{|D|^{-1}v}(\lambda\omega) = \lambda^{-1}\hat{v}(\lambda\omega)$, $\lambda > 0$, that

$$\langle |D|^{-1}v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \frac{\lambda^{m-3}i}{2(2\pi)^{m-1}}\int_\Sigma \overline{\hat{v}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega. \tag{2.12}$$

The right side extends to a continuous function of $\lambda \geq 0$ when $m \geq 3$ and (2.9) follows by comparing (2.11) and (2.12). Eqn. (2.10) likewise follows. ☐

We define the spherical average of a function f on $\mathbb{R}^m$ by

$$M(r, f) = \frac{1}{\omega_{m-1}}\int_\Sigma f(r\omega)d\omega, \quad \text{for all } r \in \mathbb{R}. \tag{2.13}$$

We often write $M_f(r) = M(r, f)$. We have $M_f(-r) = M_f(r)$ and Hölder's inequality implies

$$\left(\frac{1}{\omega_{m-1}}\int_0^\infty |M_f(r)|^p r^{m-1}dr\right)^{1/p} \leq \|f\|_p, \quad 1 \leq p \leq \infty. \tag{2.14}$$

For an even function $M(r)$ of $r \in \mathbb{R}$, define $\tilde{M}(\rho)$ by

$$\tilde{M}(\rho) = \int_\rho^\infty rM(r)dr \left(= -\int_{-\infty}^\rho rM(r)dr\right). \tag{2.15}$$

Lemma 2.4. *Suppose $M(r) = M(-r)$ and $\langle r\rangle^2 M(r)$ is integrable. Then,*

$$\int_{\mathbb{R}} e^{-ir\lambda}rM(r)dr = \frac{\lambda}{i}\int_{\mathbb{R}} e^{-ir\lambda}\tilde{M}(r)dr, \int_{\mathbb{R}}\tilde{M}(r)dr = \int_{\mathbb{R}} r^2M(r)dr. \tag{2.16}$$

Proof. Since $rM(r) = -\tilde{M}(r)'$, integration by parts gives the first equation. We differentiate both sides of the first and set $\lambda = 0$. The second follows. ☐

We denote $\breve{u}(x) = u(-x)$, $x \in \mathbb{R}^m$. (The sign $\breve{u}$ will be reserved for this purpose and will not be used to denote the conjugate Fourier transform.)

Representation formula for odd dimensions.

Lemma 2.5. *Let $m \geq 3$ be odd and $u, \psi \in C_0^\infty(\mathbb{R}^m)$. Define $c_j = \omega_{m-1} C_j$, $1 \leq j \leq \frac{m-3}{2}$, where C_j are the constants in* (2.2)*. Then, for $\lambda > 0$ we have*

$$\langle \psi, (G_0(\lambda) - G_0(-\lambda))u \rangle = \sum_{j=0}^{\frac{m-3}{2}} c_j (-1)^{j+1} \lambda^j \int_{\mathbb{R}} e^{-i\lambda r} r^{1+j} M_{\overline{\psi} * \check{u}}(r) dr. \quad (2.17)$$

Proof. We compute $\langle \psi, G_0(\lambda)u \rangle$ by using the integral kernel (2.2) of $G_0(\lambda)$. Change the order of integration and use polar coordinates. Then,

$$\langle \psi, G_0(\lambda)u \rangle = \sum_{j=0}^{\frac{m-3}{2}} C_j \int_{\mathbb{R}^m} \overline{\psi(x)} \left(\int_{\mathbb{R}^m} \frac{\lambda^j e^{i\lambda|y|} u(x-y)}{|y|^{m-2-j}} dy \right) dx$$

$$= \sum_{j=0}^{\frac{m-3}{2}} C_j \int_{\mathbb{R}^m} \frac{\lambda^j e^{i\lambda|y|} (\overline{\psi} * \check{u})(y)}{|y|^{m-2-j}} dy = \sum_{j=0}^{\frac{m-3}{2}} c_j \int_0^\infty \lambda^j e^{i\lambda r} r^{1+j} M_{\overline{\psi} * \check{u}}(r) dr.$$

Since $M_{\overline{\psi} * \check{u}}(r)$ is even, change of variable r by $-r$ yields

$$-\langle \psi, G_0(-\lambda)u \rangle = \sum_{j=0}^{\frac{m-3}{2}} c_j \int_{-\infty}^0 \lambda^j e^{i\lambda r} r^{1+j} M_{\overline{\psi} * \check{u}}(r) dr.$$

Add both sides of last two equations and change r by $-r$. □

Representation formula for even dimensions. If m is even, we have the analogue of (2.17). For a function $M(r)$ on $\mathbb{R}$ and $a > 0$, define

$$M^a(r) = M((1+2a)^{-1} r).$$

Lemma 2.6. *Let $m \geq 2$. Let $u, \psi \in C_0^\infty(\mathbb{R}^m)$. Then*

$$\langle \psi, (G_0(\lambda) - G_0(-\lambda))u \rangle = \sum_{j=0}^{\nu} (-1)^{j+1} T_j^{(a)} \left[\frac{\lambda^j \mathcal{F}(r^{j+1} M^{a}_{\overline{\psi} * \check{u}})(\lambda)}{(1+2a)^{j+2}} \right], \quad (2.18)$$

$$\textit{For } j = 0, \ -T_0^{(a)} \left[\frac{\mathcal{F}(r M^{a}_{\overline{\psi} * \check{u}})(\lambda)}{(1+2a)^2} \right] = i T_0^{(a)} \left[\frac{\lambda (\mathcal{F} \widetilde{M^{a}_{\overline{\psi} * \check{u}}})(\lambda)}{(1+2a)^2} \right]. \quad (2.19)$$

Proof. Define $B_j(\lambda, r, a) = e^{i\lambda r(1+2a)} (\lambda r)^j r^{-(m-2)}$ and

$$B_j(\lambda, a)u(x) = \int_{\mathbb{R}^m} B_j(\lambda, |y|, a) u(x-y) dy, \quad j = 0, \ldots, \nu.$$

Then, (2.6) and change of the order of integrations imply

$$\langle \psi, (G_0(\lambda) - G_0(-\lambda))u \rangle = \sum_{j=0}^{\nu} \frac{T_j^{(a)}}{\omega_{m-1}} \left[\langle \psi, (B_j(\lambda, a) - B_j(-\lambda, a))u \rangle \right]. \quad (2.20)$$

We have, as in odd dimensions, that for $u \in \mathcal{S}(\mathbb{R}^m)$ and $\psi \in L^1(\mathbb{R}^m)$

$$\begin{aligned}\langle \psi, B_j(\lambda, a)u\rangle &= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \overline{\psi(x)} B_j(\lambda, |y|, a)u(x-y)dy\right)dx \\ &= \int_{\mathbb{R}^m} B_j(\lambda, |y|, a)(\overline{\psi} * \check{u})(y)dy = \omega_{m-1}\int_0^\infty e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr.\end{aligned}$$

Replacing λ to $-\lambda$ and changing the variable r to $-r$, we have

$$-\langle \psi, B_j(-\lambda, a)u\rangle = \omega_{m-1}\int_{-\infty}^0 e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr,$$

where we used that $M_{\overline{\psi}*\check{u}}(-r) = M_{\overline{\psi}*\check{u}}(r)$. Adding these two yields

$$\langle \psi, (B_j(\lambda, a) - B_j(-\lambda, a))u\rangle = \omega_{m-1}\int_{\mathbb{R}} e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr. \quad (2.21)$$

Change r to $-r$ in the right of (2.21), plug the result with (2.20) and, at the end, change the variable r to $-r/(1+2a)$. Then, (2.21) becomes

$$\frac{(-1)^{j+1}\omega_{m-1}}{(1+2a)^{j+2}}\int_{\mathbb{R}} e^{-i\lambda r}\lambda^j r^{j+1} M^{a}_{\overline{\psi}*\check{u}}(r)dr = \frac{(-1)^{j+1}\omega_{m-1}\lambda^j}{(1+2a)^{j+2}}\mathcal{F}(r^{j+1}M^{a}_{\overline{\psi}*\check{u}})(\lambda)$$

and (2.18) follows. If we use the first of (2.16), the right of the last equation for $j=0$ becomes

$$\frac{i\lambda(\widetilde{\mathcal{F}M^{a}_{\overline{\psi}*\check{u}}})(\lambda)}{(1+2a)^2}\omega_{m-1}$$

and we obtain (2.19). ☐

2.3 Some results from harmonic analysis.

The following lemma on weighted inequality (cf. [11], Chapter 9) plays crucial role in this paper. We let $1 < p < \infty$ in this subsection.

Lemma 2.7. *The weight function $|r|^a$ is an A_p weight on $\mathbb{R}$ if and only if $-1 < a < p-1$. The Hilbert transform $\tilde{\mathcal{H}}$ and the Hardy-Littlewood maximal operator $\mathcal{M}$ are bounded in $L^p(\mathbb{R}, w(r)dr)$ for A_p weights $w(r)$.*

Modifying the Hilbert transform $\tilde{\mathcal{H}}$, we define

$$\mathcal{H}u(\rho) = \frac{(1+\tilde{\mathcal{H}})u(\rho)}{2} = \frac{1}{2\pi}\int_0^\infty e^{ir\rho}\hat{u}(r)dr. \quad (2.22)$$

We shall repeatedly use following A_p weights on $\mathbb{R}^1$ to the operator $\mathcal{M}\mathcal{H}$:

$$|r|^{m-1-p(m-1)}, \ |r|^{m-1-2p}, \ |r|^{m-1-p} \quad \text{and} \quad |r|^{m-1}, \quad (2.23)$$

respectively for $1 < p < \frac{m}{m-1}$, $\frac{m}{3} < p < \frac{m}{2}$, $\frac{m}{2} < p < m$ and $m < p$.
For a function $F(x)$ on $\mathbb{R}^m$, we say $G(|x|) \in L^1(\mathbb{R}^m)$ is a radially decreasing integrable majorant (RDIM for short) of F if $G(r) > 0$ is decreasing and $|F(x)| \le G(|x|)$ for a.e. $x \in \mathbb{R}^m$. The following lemma is well known (see e.g. [26], p.57).

Lemma 2.8. (1) *A rapidly decreasing function $F \in \mathcal{S}(\mathbb{R}^m)$ has a RDIM.*

(2) *If F has a RDIM. then there is a constant $C > 0$ such that*

$$|(F * u)(t)| \le C(\mathcal{M}u)(t), \quad t \in \mathbb{R}. \tag{2.24}$$

Lemma 2.9. *For u and $F \in L^1(\mathbb{R})$ such that $\hat{u}$, $\hat{F} \in L^1(\mathbb{R})$ we have*

$$\frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho} F(\lambda)\hat{u}(\lambda)d\lambda = (\mathcal{F}^* F * \mathcal{H}u)(\rho). \tag{2.25}$$

Proof. Let $\Theta(\lambda) = \begin{cases} 1, & \text{for } \lambda > 0 \\ 0, & \text{for } \lambda \le 0 \end{cases}$. Then, the left side of (2.25) equals

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\rho}F(\lambda)\Theta(\lambda)\hat{u}(\lambda)d\lambda &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{i\lambda(\rho-\xi)}\mathcal{F}^*F(\xi)d\xi\right)\Theta(\lambda)\hat{u}(\lambda)d\lambda \\ &= \int_{\mathbb{R}} \mathcal{F}^*F(\xi)\mathcal{F}^*\{\Theta(\lambda)\hat{u}(\lambda)\}(\rho-\xi)d\xi = (\mathcal{F}^*F * \mathcal{H}u)(\rho) \end{aligned}$$

as desired. □

3 Reduction to the low energy analysis

We write $W_- = W$ in the sequel. When $u \in \langle x\rangle^{-s}L^2$, $s > 1/2$, Wu may be expressed via the boundary values of resolvents (e.g. [19]):

$$Wu = u - \lim_{\varepsilon\downarrow 0, N\uparrow\infty}\frac{1}{\pi i}\int_\varepsilon^N G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda \tag{3.1}$$

$$= u - \frac{1}{\pi i}\int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda \tag{3.2}$$

Here the right of (3.1) is the Riemann integral of an $\langle x\rangle^t L^2$-valued continuous function where $t > 1/2$ is such that $s + t > 2$, the result belongs to $L^2(\mathbb{R}^m)$ and the limit exists in $L^2(\mathbb{R}^m)$, which we symbolically write as (3.2).
We decompose W into the high and the low energy parts

$$W = W_> + W_< \equiv W\Psi(H_0) + W\Phi(H_0), \tag{3.3}$$

by using cut off functions $\Phi \in C_0^\infty(\mathbb{R})$ and $\Psi \in C^\infty(\mathbb{R})$ such that

$$\Phi(\lambda^2) + \Psi(\lambda^2) \equiv 1, \quad \Phi(\lambda^2) = 1 \text{ near } \lambda = 0 \text{ and } \Phi(\lambda^2) = 0 \text{ for } |\lambda| > \lambda_0$$

for a small constant $\lambda_0 > 0$. We have proven in previous papers [33, 8] that, under Assumption 1.1, $W_>$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ if $m \geq 3$ and we only need to study $W_< = \Phi(H_0) + Z$ where

$$Z = -\frac{1}{\pi i}\int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)d\lambda. \tag{3.4}$$

Evidently $\Phi(H_0) \in \mathbf{B}(L^p(\mathbb{R}^m))$ for all $1 \leq p \leq \infty$ and we have only to study the operator Z defined by (3.4). Since $\delta > 2$, the LAP (cf. Lemma 2.2 of [33]) implies that $G_0(\lambda)V$ is a Hölder continuous function of $\lambda \in \mathbb{R}$ with values in $\mathbf{B}_\infty(L^{-s})$ for any $\frac{1}{2} < s < \delta - \frac{1}{2}$ and, the absence of positive eigenvalues ([17]) implies that $1 + G_0(\lambda)V$ is invertible for $\lambda > 0$ (cf. [1]). It follows from the resolvent equation $G(\lambda) = G_0(\lambda) - G_0(\lambda)VG(\lambda)$ that $G(\lambda)V$ may be expressed in terms of $G_0(\lambda)V$:

$$G(\lambda)V = G_0(\lambda)V(1 + G_0(\lambda)V)^{-1} \text{ for } \lambda \neq 0 \tag{3.5}$$

and it is *locally* Hölder continuous for $\lambda \in \mathbb{R} \setminus \{0\}$ with values in $\mathbf{B}_\infty(L^2_{-s})$. Thus, we have the expression of Z in terms of the free resolvent $G_0(\lambda)$:

$$Zu = -\frac{1}{\pi i}\int_0^\infty G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}(G_0(\lambda) - G_0(-\lambda))\lambda F(\lambda)ud\lambda, \tag{3.6}$$

where $F(\lambda) = \Phi(\lambda^2)$. If H is of generic type, $\mathrm{Ker}_{L^2_{-s}}(1 + G_0(0)V) = \mathcal{N} = \{0\}$ for any $\frac{1}{2} < s < \delta - \frac{1}{2}$ and $1 + G_0(\lambda)V$ is invertible for λ in a neighborhood $\lambda = 0$ and both sides of (3.5) become Hölder continuous. We then have shown in [33, 8] that Z is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ under Assumption 1.1.

3.1 Low energy behavior of $(1 + G_0(\lambda)V)^{-1}$.

If H is of exceptional type, $(1 + G_0(\lambda)V)^{-1}$ becomes singular at $\lambda = 0$ and we describe its singularities here. Before doing so we recall some properties of functions in $\mathcal{N}$. Recall ([25]) that for $0 < s < m$:

$$|D|^{-s}u(x) = \mathcal{F}^*(|\xi|^{-s}\hat{u})(x) = \frac{\Gamma\left(\frac{m-s}{2}\right)}{2^s\pi^{\frac{m}{2}}\Gamma\left(\frac{s}{2}\right)}\int_{\mathbb{R}^m}\frac{u(y)}{|x-y|^{m-s}}dy. \tag{3.7}$$

When $s = 1$ and $s = 2$, the constants in front of the integral respectively equal to $\pi^{-1}\omega_{m-2}^{-1}$ and $C_0 = (m-2)^{-1}\omega_{m-1}^{-1}$ of (2.2).

Lemma 3.1. (1) *Functions ϕ in $\mathcal{N}$ satisfy $\langle x\rangle^{-s}\phi \in H^2(\mathbb{R}^m) \cap C^1(\mathbb{R}^m)$ for any $s > 1/2$ and $\nabla\phi$ is Hölder continuous. They satisfy the following asymptotic expansion as $|x| \to \infty$:*

$$\phi(x) = -\frac{C_0}{|x|^{m-2}}\int_{\mathbb{R}^m}(V\phi)(y)dy - \frac{1}{\omega_{m-1}}\sum_{j=1}^m\frac{x_j}{|x|^m}\int_{\mathbb{R}^m}y_j(V\phi)(y)dy + O(|x|^{-m}). \tag{3.8}$$

(2) *For* $\phi \in \mathcal{N} \setminus \mathcal{E}_0$, $\phi \otimes \phi \notin \mathbf{B}(L^p(\mathbb{R}^m))$ *for any* $1 \le p \le \infty$ *if* $m = 3$ *or* $m = 4$ *and, if* $m \ge 5$, $\phi \otimes \phi \in \mathbf{B}(L^p(\mathbb{R}^m))$ *if and only if* $\frac{m}{m-2} < p < \frac{m}{2}$. *If* $\phi \in \mathcal{E}_0 \setminus \mathcal{E}_1$, *then* $\phi \otimes \phi \in \mathbf{B}(L^p(\mathbb{R}^m))$ *if and only if* $\frac{m}{m-1} < p < m$ *for any* $m \ge 3$ *and, if* $\phi \in \mathcal{E}_1$, *then* $\phi \otimes \phi \in \mathbf{B}(L^p(\mathbb{R}^m))$ *for all* $1 < p < \infty$.

(3) *If* $\langle x \rangle^2 u \in L^1(\mathbb{R}^m)$, $|D|^{-1}u(x)$ *has the following expansion as* $|x| \to \infty$*:*

$$\frac{\int_{\mathbb{R}^m} u dx}{\pi\omega_{m-2}|x|^{m-1}} + \sum_{j=1}^{m} \frac{(m-1)x_j}{\pi\omega_{m-2}|x|^{m+1}} \int_{\mathbb{R}^m} x_j u dx + O(|x|^{-m-1}). \tag{3.9}$$

Proof. (1) The smoothness property of ϕ is well known (see e.g. Corollary 2.6 of [2]). We have from (3.7) that

$$\phi(x) = -C_0 \int_{\mathbb{R}^m} \frac{V(y)\phi(y)}{|x-y|^{m-2}} dy. \tag{3.10}$$

Taylor's formula implies that

$$\left| \frac{1}{|x-y|^{m-2}} - \frac{1}{|x|^{m-2}} - \frac{(m-2)x \cdot y}{|x|^{m-1}} \right| \le C \frac{\langle y \rangle^2}{\langle x \rangle^m}, \quad |x-y| \ge 1$$

and (3.8) follows. Statement (2) follows from (3.8). We omit the proof of (3) which is similar to that of (3.9). □

3.1.1 Odd dimensional cases

The structure of singularities depends on m. For odd dimensions $m \ge 3$ we have the following results (see, e.g. Theorem 2.12 of [33]). We state it separately for $m = 3$ and $m \ge 5$. In the following Theorems 3.2 and 3.3 for odd $m \ge 3$ and Theorem 3.4 for even $m \ge 6$, we will indiscriminately write $E(\lambda)$ for the operator valued function of λ defined near $\lambda = 0$ which, when inserted in (3.6) for $(1 + G_0(\lambda)V)^{-1}$, produces the operator which is bounded in $L^p(\mathbb{R}^m)$ for all $1 \le p \le \infty$.

The case $m = 3$. By virtue of (3.8), we have for $\phi \in \mathcal{N}$ that

$$\phi(x) = \frac{L(\phi)}{|x|} + O(|x|^{-2}) \text{ as } |x| \to \infty, \quad L(\phi) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} V(x)\phi(x) dx. \tag{3.11}$$

Thus, $\mathcal{E} = \{\phi \in \mathcal{N} \setminus \{0\} \colon L(\phi) = 0\} (= \mathcal{E}_0)$ and, as $\mathcal{N} \ni \phi \mapsto L(\phi) \in \mathbb{C}$ is continuous, $\dim \mathcal{N}/\mathcal{E} \le 1$. Any $\varphi \in \mathcal{N} \setminus \mathcal{E}$ is called *threshold resonance* of H. We say that H is of exceptional type of *the first kind* if $\mathcal{E} = \{0\}$, *the second* if $\mathcal{E} = \mathcal{N}$ and *the third kind* if $\{0\} \subsetneq \mathcal{E} \subsetneq \mathcal{N}$. We let $D_0, D_1, \dots$ be integral operators defined by

$$D_j u(x) = \frac{1}{4\pi j!} \int_{\mathbb{R}^3} |x-y|^{j-1} u(y) dy, \quad j = 0, 1, \dots.$$

so that we have the formal Taylor expansion

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy = \sum_{j=1}^{\infty}(i\lambda)^j D_j u.$$

If H is of exceptional type of the third kind, $-(V\phi,\phi)$ defines inner product on $\mathcal{N}$ and there is a unique real $\psi \in \mathcal{N}$ such that

$$-(V\psi,\phi) = 0, \quad \forall\phi \in \mathcal{E}, \quad -(V\psi,\psi) = 1 \text{ and } L(\psi) > 0. \tag{3.12}$$

We define *the canonical resonance* by

$$\varphi = \psi + PVD_2V\psi \in \mathcal{N}. \tag{3.13}$$

If H is of exceptional type of the first kind, then $\dim \mathcal{N} = 1$ and there is a unique $\varphi \in \mathcal{N}$ such that $-(V\varphi,\varphi) = 1$ and $L(\varphi) > 0$ and we call this the canonical resonance. We have the following result for $m = 3$ (see e.g. [33]).

THEOREM 3.2. *Let $m = 3$ and let V satisfy $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 3$. Suppose that H is of exceptional type of the third kind and let φ be the canonical resonance and $a = 4\pi i|\langle V,\varphi\rangle|^{-2}$. Then:*

$$(I + G_0(\lambda)V)^{-1} = \frac{PV}{\lambda^2} + i\frac{PVD_3VPV}{\lambda} - \frac{a}{\lambda}|\varphi\rangle\langle\varphi|V + E(\lambda). \tag{3.14}$$

If H is of exceptional type of the first or the second kind, (3.14) *holds with $P = 0$ or $\varphi = 0$ respectively.*

THE CASE $m \ge 5$. If $m \ge 5$, (3.8) implies $\mathcal{N} = \mathcal{E}$.

THEOREM 3.3. *Let $m \ge 5$ be odd and $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > m+3$. Suppose H is of exceptional type. Then:*

(1) *If $m = 5$ then, with $\varphi = PV$, V being considered as a function,*

$$(I + G_0(\lambda)V)^{-1} = \frac{PV}{\lambda^2} - \frac{a_0}{\lambda}|\varphi\rangle\langle\varphi|V + E(\lambda), \quad a_0 = \frac{i}{24\pi^2}. \tag{3.15}$$

(2) *If $m \ge 7$ then*

$$(I + G_0(\lambda)V)^{-1} = \frac{PV}{\lambda^2} + E(\lambda). \tag{3.16}$$

Define $S(\lambda) = (I + G_0(\lambda)V)^{-1} - E(\lambda)$ and

$$Z_s = \frac{i}{\pi}\int_0^{\infty} G_0(\lambda)VS(\lambda)(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda d\lambda. \tag{3.17}$$

Then, it follows from Theorems 3.2 and 3.3 that $Z - Z_s \in \mathbf{B}(L^p(\mathbb{R}^m))$ for all $1 \le p \le \infty$ and we have only to study Z_s in what follows.

3.2 Even dimensional case

When m is even, singular terms of $(1+G_0(\lambda)V)^{-1}$ may contain logarithmic factors. The following is the improvement of Proposition 3.6 of [8]. We let $\dim \mathcal{E} = d$ and $\{\phi_1, \dots, \phi_d\}$ be the real orthonormal basis of $\mathcal{E}$. For making the expression simpler, we state the theorem for $V(1+G_0(\lambda)V)^{-1}$.

Theorem 3.4. *Let $m \geq 6$ be even. Suppose $|V(x)| \leq C\langle x\rangle^{-\delta}$ for $\delta > m+4$ if $m=6$ and for $\delta > m+3$ if $m \geq 8$. Let $\varphi = PV$ with V being considered as a function. Then, we have the following statements for $\Im\lambda \geq 0$ and $\log\lambda$ such that $\log\lambda \in \mathbb{R}$ for $\lambda > 0$:*

(1) *If $m=6$ then, we have that*

$$V(1+G_0(\lambda)V)^{-1} = \frac{VPV}{\lambda^2} + \frac{\omega_5}{(2\pi)^6}\log\lambda(V\varphi\otimes V\varphi) + \left(\frac{\omega_5\|\varphi\|}{(2\pi)^6}\right)^2 \lambda^2\log^2\lambda(V\varphi\otimes V\varphi) + \lambda^2\log\lambda F_2 + VE(\lambda), \quad (3.18)$$

where F_2 is an operator of rank at most 8 such that

$$F_2 = \sum_{a,b=1}^{8} \varphi_a \otimes \psi_b, \quad \varphi_a,\ \psi_b \in (L^1\cap L^\infty)(\mathbb{R}^6). \quad (3.19)$$

(2) *If $m \geq 8$, then we have with a constant c_m that*

$$V(1+G_0(\lambda)V)^{-1} = \frac{VPV}{\lambda^2} + c_m(V\varphi\otimes V\varphi)\lambda^{m-6}\log\lambda + VE(\lambda). \quad (3.20)$$

(3) *If $m \geq 12$, then $c_m(V\varphi\otimes V\varphi)\lambda^{m-6}\log\lambda$ of (3.20) may be included in $VE(\lambda)$.*

Proof. We prove (1) only, using the notation of the proof of subsection 3.2.1 of [8]. A slightly more careful look at the argument there shows that, in spite of Eqn.(3.5) of [8], $V(1+G_0(\lambda)V)^{-1}$ is actually given by

$$\frac{VPV}{\lambda^2} + VD_{01}\log\lambda + VD_{21}\lambda^2\log\lambda + VD_{22}\lambda^2\log^2\lambda + VE(\lambda). \quad (3.21)$$

Here, with $F_{jk} = F_{jk}(0)$, $F_{jk}(\lambda)$ being defined by (3.16) of [8], and $A(0) = (2\pi)^{-6}\omega_{m-1}(1\otimes 1)$, VD_{01} and VD_{22} are rank 1 operators given by

$$VD_{01} = VPVF_{01}PV = VPVA(0)VPV = \frac{\omega_{m-1}}{(2\pi)^6}(V\varphi\otimes V\varphi), \quad (3.22)$$

$$VD_{22} = V(PVF_{01})^2PV = V(PVA(0)VP)^2V = \frac{\omega_{m-1}^2}{(2\pi)^{12}}\|\varphi\|^2(V\varphi\otimes V\varphi),$$

where we have used $PVQ = PV$ and $VQP = VP$ and,

$$VD_{21} = VPV(F_{21} + F_{00}PVF_{01} + F_{01}PVF_{00})PV \tag{3.23}$$
$$- VX(0)\overline{Q}D_2VPVF_{01}PV - VX(0)\overline{Q}A(0)VPV \tag{3.24}$$
$$- VPVF_{01}PVQD_2V\overline{Q}X(0) - PVQA(0)V\overline{Q}X(0). \tag{3.25}$$

It is obvious that the first line (3.23) is of rank at most 4 and of the form $\sum \alpha_{jk}(V\phi_j \otimes V\phi_k)$; four other operators are of rank one and of the form $f \otimes g$ with $f \in (L^1 \cap L^\infty)(\mathbb{R}^6)$. We check this for $VX(0)\overline{Q}D_2VPVF_{01}PV$ as a prototype. We have $D_2 = D_0^2$ and $D_0V\varphi = -\varphi$. Thus, (3.22) implies

$$VX(0)\overline{Q}D_2VPVF_{01}PV = -(2\pi)^{-6}\omega_{m-1}(VX(0)\overline{Q}D_0\varphi) \otimes (V\varphi).$$

Here $D_0\varphi \in C^2(\mathbb{R}^6)$ and satisfies $D_0\varphi \leq_{|\,\cdot\,|} C\langle x\rangle^{-2}$ by virtue of Lemma 3.1. Hence, a fortiori $D_0\varphi \in C_0(\mathbb{R}^6)$, the Banach space of continuous functions which converge to 0 as $|x| \to \infty$. It is obvious that $\mathcal{X} \equiv \overline{Q}C_0(\mathbb{R}^6) \subset C_0(\mathbb{R}^6)$ and $X(0) = N^{-1}(0) = [\overline{Q}(1 + D_0V)\overline{Q}]^{-1}$ is an isomorphism of $\mathcal{X}$. This is because $T = \overline{Q}D_0V\overline{Q}$ is compact both in $\mathcal{X} = \overline{Q}C_0(\mathbb{R}^6)$ and $\mathcal{Y} = \overline{Q}L^2_{-\delta+2}(\mathbb{R}^6)$, $\mathcal{X} \cap \mathcal{Y}$ is dense in $\mathcal{Y}$ and $\mathrm{Ker}_{\mathcal{Y}}(1 + T) = \{0\}$ (see e.g. Lemma 2. 11 of [9]). Thus, $VX(0)\overline{Q}D_0\varphi(x) \leq_{|\,\cdot\,|} C\langle x\rangle^{-\delta}$. □

It follows from Theorem 3.4 that $Zu = Z_su + Z_{\log}u$ modulo the operator which is bounded in L^p for all $1 \leq p \leq \infty$ and we need study

$$Z_{se} = \frac{i}{\pi}\int_0^\infty G_0(\lambda)VPV(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda^{-1}d\lambda, \tag{3.26}$$

$$Z_{\log} = \sum_{j,k}\frac{i}{\pi}\int_0^\infty G_0(\lambda)\lambda^{2j}(\log\lambda)^k D_{jk}(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda d\lambda, \tag{3.27}$$

for even $m \geq 6$, where the sum and D_{jk} are as in Theorem 3.4.

4 Proof of Theorem 1.3

The proof of Theorem 1.3 for $m = 3$ is the simplest and is the prototype for other dimensions and, most of the basic ideas already appear here.

4.1 The case of exceptional type of the first kind

We begin with the case that H is of exceptional type of the first kind and, we let φ be the canonical resonance, $a = 4\pi i|\langle V, \varphi\rangle|^{-2} \neq 0$ and

$$\psi(x) = |D|^{-1}(V\varphi)(x) = \frac{1}{2\pi^2}\int \frac{V(y)\varphi(y)}{|x - y|^2}dy. \tag{4.1}$$

The following lemma proves Theorem 1.3 when H is of exceptional type of the first kind.

Lemma 4.1. (1) *For $1 < p < 3$, there exists a constant C_p such that*

$$\|Z_s u\|_p \le C_p \|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^3). \tag{4.2}$$

(2) *For $3 < p < \infty$, there exists a constant C_p such that*

$$\|(Z_s + a\varphi \otimes \psi)u\|_p \le C_p \|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^3). \tag{4.3}$$

(3) *For $p \ge 3$, Z_s is unbounded in $L^p(\mathbb{R}^3)$.*

Proof. Recall $c_0 = C_0\omega_2 = 1$. We have $S(\lambda) = -\frac{a}{\lambda}|\varphi\rangle\langle\varphi|V$ and

$$Z_s u = -\frac{ia}{\pi}\int_0^\infty G_0(\lambda)V\varphi\rangle\langle V\varphi|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda. \tag{4.4}$$

Defining $M(r) = M(r, (V\varphi) * \check{u})$, we substitute (2.2) and (2.17) respectively for $G_0(\lambda)$ and $\langle V\varphi|(G_0(\lambda) - G_0(-\lambda))u\rangle$. Then,

$$Z_s u = \frac{ai}{\pi}\int_0^\infty \left(\int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}V(y)\varphi(y)}{4\pi|x-y|}dy\right)\left(\int_{\mathbb{R}} e^{-i\lambda r} rM(r)dr\right) F(\lambda)d\lambda.$$

If we change the order of integrations,

$$Z_s u = \frac{ai}{2\pi}\int_{\mathbb{R}^3} \frac{K_0(|x-y|)V(y)\varphi(y)}{|x-y|}dy, \tag{4.5}$$

$$K_0(\rho) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}F(\lambda)\left(\int_{\mathbb{R}} e^{-ir\lambda} rM(r)dr\right)d\lambda. \tag{4.6}$$

Since $\mathcal{F}^*F \in \mathcal{S}(\mathbb{R})$, it follows by virtue of Lemmas 2.8 and 2.9 that

$$K_0(\rho) = \{(\mathcal{F}^*F) * \mathcal{H}(rM(r))\}(\rho) \le_{|\,\cdot\,|} C\mathcal{M}\mathcal{H}(rM)(\rho). \tag{4.7}$$

Function $K_0(\rho)$ may also be expressed as

$$K_0(\rho) = \frac{i}{2\pi\rho}\int_0^\infty e^{i\lambda\rho}\left(F(\lambda)\int_{\mathbb{R}} e^{-ir\lambda} rM(r)dr\right)' d\lambda. \tag{4.8}$$

and, after integration by parts, we see that $K_0(\rho)$ satisfies also

$$K_0(\rho) \le_{|\,\cdot\,|} C\rho^{-1}(\mathcal{M}\mathcal{H}(r^2M)(\rho) + \mathcal{M}\mathcal{H}(rM)(\rho)). \tag{4.9}$$

The boundary term does not appear in (4.8) since $\int_{\mathbb{R}} rM(r)dr = 0$.
(1a) Let $3/2 < p < 3$. By virtue of Young's inequality

$$\|Z_s u\|_p \le \frac{|a|(4\pi)^{1/p}}{2\pi}\|V\varphi\|_1 \left(\int_0^\infty \left|\frac{K_0(\rho)}{\rho}\right|^p \rho^2 d\rho\right)^{1/p}. \tag{4.10}$$

We estimate $K_0(\rho)$ by (4.7) and use that ρ^{2-p} is an A_p weight on $\mathbb{R}$. Lemma 2.7 and Young's inequality imply

$$\left(\int_0^\infty \left|\frac{K_0(\rho)}{\rho}\right|^p \rho^2 d\rho\right)^{1/p} \le C\left(\int_0^\infty |\mathcal{M}\mathcal{H}(rM)(\rho)|^p \rho^{2-p} d\rho\right)^{1/p}$$
$$\le C_p\left(\int_0^\infty M(r)^p r^2 dr\right)^{1/p} \le C_p\|V\varphi * u\|_p \le C_p\|V\varphi\|_1\|u\|_p. \tag{4.11}$$

and $\|Z_s u\|_p \le C_p\|V\varphi\|_1^2\|u\|_p$.
(1b) For $1 < p < \frac{3}{2}$, we use estimate (4.9) and that ρ^{2-2p} is an A_p weight on $\mathbb{R}$ and obtain that

$$\left(\int_0^\infty \left|\frac{K_0(\rho)}{\rho}\right|^p \rho^2 d\rho\right)^{\frac{1}{p}} \le \left(\int_0^\infty |(\mathcal{M}\mathcal{H}(r^2M) + \mathcal{M}\mathcal{H}(rM))(\rho)|^p \rho^{2-2p} d\rho\right)^{\frac{1}{p}}$$
$$\le C\left(\int_0^\infty |M(r)|^p \max(r^2, r^{2-p}) dr\right)^{\frac{1}{p}} \le C(\|V\varphi\|_1 + \|V\varphi\|_{p'})\|u\|_p, \tag{4.12}$$

where we estimated the integral over $0 \le r \le 1$ by using that

$$\sup|M(r)| \le \|V\varphi * u\|_\infty \le \|V\varphi\|_{p'}\|u\|_p. \tag{4.13}$$

Thus, we have $\|Z_s u\|_p \le C(\|V\varphi\|_1 + \|V\varphi\|_{p'})\|V\varphi\|_1\|u\|_p$ for $1 < p < 3/2$. Combining (1a) and (1b), we obtain (4.2) for $1 < p < 3$ by interpolation([5]).
(2) Let $p > 3$. Writing $\int_{\mathbb{R}} re^{-ir\lambda}M(r)dr = i\left(\int_{\mathbb{R}} e^{-ir\lambda}M(r)dr\right)'$ in (4.6), we apply integration by parts and obtain yet another expression of $K_0(\rho)$:

$$K_0(\rho) = \frac{-i}{2\pi}\int_{\mathbb{R}} M(r)dr - \frac{i}{2\pi}\int_0^\infty \left(e^{i\lambda\rho}F(\lambda)\right)'\left(\int_{\mathbb{R}} e^{-ir\lambda}M(r)dr\right)d\lambda. \tag{4.14}$$

Denote the second term by $\tilde{K}_0(\rho)$. By virtue of Lemmas 2.8 and 2.9,

$$\tilde{K}_0(\rho) \le_{|\cdot|} C(\rho+1)\mathcal{M}\mathcal{H}(M)(\rho). \tag{4.15}$$

Substituting (4.14) for $K_0(\rho)$ in (4.5), we obtain $Z_s u = Z_b u + Z_i u$, where Z_b and Z_i are operators produced by $\frac{-i}{2\pi}\int_{\mathbb{R}} M(r)dr$ and $\tilde{K}_0(\rho)$, respectively. Because

$$\frac{1}{\pi}\int_{\mathbb{R}} M(r)dr = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{(V\varphi)(x+y)}{|x|^2}dx\right)u(y)dy = \langle\psi, u\rangle \tag{4.16}$$

by the definition (4.1), we have by using (3.10) for $m = 3$ that

$$Z_b u(x) = \frac{a}{4\pi^2}\int_{\mathbb{R}} M(r)dr \cdot \int_{\mathbb{R}^3}\frac{V(y)\varphi(y)}{|x-y|}dy = -a|\varphi\rangle\langle\psi|u\rangle. \tag{4.17}$$

We splite the integral as

$$Z_i u(x) = \frac{ai}{2\pi}\left(\int_{|y|\le 1} + \int_{|y|>1}\right)\frac{\tilde{K}_0(|y|)(V\varphi)(x-y)}{|y|}dy = I_1(x) + I_2(x). \tag{4.18}$$

For estimating I_2 we use (4.15) for $\rho \geq 1$: $|\tilde{K}_0(\rho)| \leq C\rho\mathcal{M}\mathcal{H}(M)(\rho)$. Since ρ^2 is an A_p-weight on $\mathbb{R}$ for $p > 3$, we have by using Young's and Hölder's inequalities and Lemma 2.7 that

$$\|I_2\|_p \leq C\|V\varphi\|_1 \left(\int_0^\infty |\mathcal{M}\mathcal{H}(M)(\rho)|^p \rho^2 d\rho\right)^{\frac{1}{p}}$$
$$\leq C\|V\varphi\|_1 \left(\int_0^\infty |M(r)|^p r^2 dr\right)^{\frac{1}{p}} \leq C\|V\varphi\|_1^2 \|u\|_p. \tag{4.19}$$

Hölder's inequality implies, with $p' = p/p-1$, that

$$|I_1(x)| \leq C \left(\int_{|y|\leq 1} \left|\frac{(V\varphi)(x-y)}{|y|}\right|^{p'} dy\right)^{1/p'} \left(\int_0^1 |\tilde{K}_0(\rho)|^p \rho^2 d\rho\right)^{1/p}.$$

Since $\tilde{K}_0(\rho) \leq_{|\,\cdot\,|} C\mathcal{M}\mathcal{H}(M)(\rho)$ for $0 < \rho < 1$ by virtue of (4.15) and since ρ^2 is an A_p-weight, we obtain as in (4.19) that

$$\left(\int_0^1 |\tilde{K}_0(\rho)|^p \rho^2 d\rho\right)^{1/p} \leq C\left(\int_0^\infty |\mathcal{M}\mathcal{H}(M)(\rho)|^p \rho^2 d\rho\right)^{1/p} \leq C\|u\|_p. \tag{4.20}$$

It follows by virtue of Minkowski's inequality that

$$\|I_1\|_p \leq C\|u\|_p \left\|\left(\int_{|y|\leq 1} \left|\frac{(V\varphi)(x-y)}{|y|}\right|^{p'} dy\right)^{1/p'}\right\|_p \leq C\|u\|_p\|V\varphi\|_p \tag{4.21}$$

because $1 < p' < 3/2 < 3 < p < \infty$. Thus,

$$\left\|\int_{\mathbb{R}^3} \frac{\tilde{K}_0(|x-y|)V(y)\varphi(y)}{|x-y|} dy\right\|_p \leq C(\|V\varphi\|_p + \|V\varphi\|_1)\|u\|_p.$$

With (4.17) this proves (4.3).
(3) Since $\int_{\mathbb{R}^3} V\varphi dx \neq 0$, Lemma 3.1 implies that $\varphi \notin L^p(\mathbb{R}^3)$ for $1 \leq p \leq 3$ and that $\psi \in L^p(\mathbb{R}^3)^*$ if and only if $p > 3$. Hence, $\varphi \otimes \psi$ is unbounded in $L^p(\mathbb{R}^3)$ for any $1 \leq p \leq \infty$. Thus, statement (2) implies that Z_s is unbounded in $L^p(\mathbb{R}^3)$ for $p \geq 3$. This completes the proof of the lemma. □

We review here the basic strategy of this subsection as it will be repeatedly employed in the following (sub)sections. We express $Z_s u$ as the convolution (4.5) of $V\varphi$ and $K_0(\rho)$ of (4.6). By applying integration by parts if necessary we represent and estimate $K_0(\rho)$ as in (4.7), (4.9) or (4.15) by using $\mathcal{M}\mathcal{H}$. These estimates are used for proving

$$\left(\int_0^\infty |K_0(\rho)|^p \rho^{2-p} d\rho\right)^{\frac{1}{p}} \left(= \omega_2^{-\frac{1}{p}} \left\|\frac{K_0(|x|)}{|x|}\right\|_p\right) \leq C\|u\|_p \tag{4.22}$$

via the weighted inequality for $\frac{3}{2} < p < 3$, $1 < p < \frac{3}{2}$ and $p > 3$ respectively. Desired estimates are then obtained by combining (4.22) and Young's inequality. However, the boundary term appears in the integration by parts for large values of $p > 3$ which obstructs the L^p-boundedness. We represent the obstruction explicitly in terms of functions of $\mathcal{N}$ and show that L^p-boundedness depends on the properties of functions in $\mathcal{N}$. Suitable modifications, improvements and additional arguments will be of course necessary in what follows.

4.2 The cases of the second and third kinds

Let H be of exceptional type of the second kind. Then,

$$S(\lambda) = \frac{PV}{\lambda^2} + i\frac{PVD_3VPV}{\lambda}, \tag{4.23}$$

where D_3 is the integral operator with kernel $|x-y|^2/4\pi$. We take the real orthonormal basis $\{\phi_1, \dots, \phi_n\}$ of $\mathcal{E}$ and define $a_{jk} = \pi^{-1}\langle \phi_j | VD_3V | \phi_k\rangle \in \mathbb{R}$. We have $\langle V, \phi_j\rangle = 0$, $1 \le j \le n$. Substituting (4.23) for $S(\lambda)$ in (3.17), we have

$$Z_s u = Z_{s0}u + Z_{s1}u = \sum_{j,k=1}^{n} Z_{s0,jk}u + \sum_{j=1}^{n} Z_{s1,j}, \tag{4.24}$$

$$Z_{s0,jk}u = ia_{jk}\int_0^\infty G_0(\lambda)V\phi_j\rangle\langle V\phi_k|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda, \tag{4.25}$$

$$Z_{s1,j}u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)V\phi_j\rangle\langle V\phi_j|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)\frac{d\lambda}{\lambda}. \tag{4.26}$$

Lemma 4.2. *For any $1 < p < \infty$, there exists a constant C_p such that*

$$\|Z_{s0}u\|_p \le C_p\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^3). \tag{4.27}$$

Proof. The operator $Z_{s0,jk}$ is equal to Z_s of (4.4) with two $\varphi \in \mathcal{N}$'s being replaced by ϕ_j and $\phi_k \in \mathcal{E}$ and a by $-\pi a_{jk}$. Thus, the proof of Lemma 4.1 implies that $Z_{s0,jk} \in \mathbf{B}(L^p(\mathbb{R}^3))$ for $1 < p < 3$ and that

$$Z_{s0,jk} - \pi a_{jk}\phi_j \otimes |D|^{-1}(V\phi_k) \in \mathbf{B}(L^p(\mathbb{R}^3)), \quad p > 3. \tag{4.28}$$

Here $\phi_j \otimes |D|^{-1}(V\phi_k)$ is bounded in $L^p(\mathbb{R}^3)$ for $p > 3$ because $\phi_j \in L^p(\mathbb{R}^3)$ and $|D|^{-1}(V\phi_k) \in (L^p(\mathbb{R}^3))^*$ by virtue of (3.8) and (3.9). Thus $Z_{s0,jk} \in \mathbf{B}(L^p(\mathbb{R}^3))$ for $3 < p$ and, hence, for $1 < p < \infty$ by interpolation. This proves the lemma. □

Lemma 4.3. (1) *Let $1 < p < 3$. Then, for a constant C_p, we have*

$$\|Z_{s1}u\|_p \le C_p\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^3). \tag{4.29}$$

(2) *Let $3 < p < \infty$. Then, for a constant C_p, we have*

$$\|(Z_{s1} + P)u\|_p \le C\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^3). \tag{4.30}$$

In (4.30) *P may be replaced by $P \ominus P_1$ by virtue of* Lemma 3.1.

(3) *The operator Z_{s1} is bounded in $L^p(\mathbb{R}^3)$ for some $p > 3$ if and only if $\mathcal{E} = \mathcal{E}_1$. In this case Z_{s1} is bounded in $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$.*

Proof. Define $\psi_j(x) = |D|^{-1}(V\phi_j)(x)$, $j = 1, \dots, n$. Then Lemma 2.3 implies

$$Z_{s1,j}u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)|V\phi_j\rangle\langle\psi_j|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda \tag{4.31}$$

which can be obtained from $Z_s u$ of (4.4) by replacing a by -1, the first $V\varphi$ by $V\phi_j$ and the second by ψ_j. Thus, it may be expressed by using $K_{0,j}(\rho)$ of (4.6) with $M(r)$ being replaced by $M_j(r) = M(r, \psi_j * \check{u})$:

$$Z_{s1,j}u = \frac{1}{2\pi i}\int_{\mathbb{R}^3} \frac{K_{0,j}(|x-y|)V(y)\phi_j(y)}{|x-y|}dy. \tag{4.32}$$

(1) The argument of (1a) in the proof of Lemma 4.1 implies

$$\|Z_{s1,j}u\|_p \le C\|V\phi_j\|_1\|\psi_j * u\|_p, \quad 3/2 < p < 3 \tag{4.33}$$

(see (4.11)) and the one of (1b) does

$$\|Z_{s1,j}u\|_p \le C\|V\phi_j\|_1(\|\psi_j * u\|_p + \|\psi_j * u\|_\infty), \quad 1 < p < 3/2 \tag{4.34}$$

(see (4.12)). Since $\int V\phi_j dx = 0$, (3.9) implies that $\psi_j = |D|^{-1}\phi_j \in L^q(\mathbb{R}^3)$ for all $1 < q \le \infty$ and that the convolution operator with $\psi_j(x)$ is bounded in L^p for any $1 < p < \infty$ via Calderón-Zygmund theory (see e.g. [26], pp. 30-36). Thus, $\|\psi_j * u\|_p \le C\|u\|_p$, $\|\psi_j * u\|_\infty \le \|\psi_j\|_{p'}\|u\|_p$ and $Z_{s1,j}$ is bounded in $L^p(\mathbb{R}^3)$ for all $1 < p < 3$, $j = 1, \dots, n$. Statement (1) follows.
(2) Integration by parts as in (4.14) by using the identity $\int_{\mathbb{R}} e^{-ir\lambda} r M_j(r)dr = i\left(\int_{\mathbb{R}} e^{-ir\lambda}M_j(r)dr\right)'$ implies that $K_{0,j}(\rho)$ may be written as

$$-\frac{i}{2\pi}\int_{\mathbb{R}} M_j(r)dr - \frac{i}{2\pi}\int_0^\infty \left(e^{i\lambda\rho}F(\lambda)\right)'\left(\int_{\mathbb{R}} e^{-ir\lambda}M_j(r)dr\right)d\lambda, \tag{4.35}$$

which we insert into (4.32). Since $|D|^{-1}\psi_j = (-\Delta)^{-1}(V\phi_j) = -\phi_j$, (4.16) with $\psi_j \in \mathcal{E}$ in place of $V\varphi$ produces $-\langle\phi_j|u\rangle$. It follows that the boundary term of (4.35) produces

$$\frac{-1}{4\pi}\int_{\mathbb{R}^3}\frac{V(y)\phi_j(y)}{|x-y|}dy \cdot \frac{1}{\pi}\int_{\mathbb{R}} M_j(r)dr = -|\phi_j\rangle\langle\phi_j|u\rangle \tag{4.36}$$

as in (4.17). Denote by $\tilde{K}_{0j}(\rho)$ and $\tilde{Z}_{s1,j}$ the second term of (4.35) and the operator it produces via (4.32). They can respectively be obtained from $\tilde{K}_0(\rho)$ of (4.14) and Z_i of (4.18) by replacing $M(r)$ and $\tilde{K}_0(\rho)$ by $M_j(r)$ and $\tilde{K}_{0,j}(\rho)$. Thus, the argument of step (2) of the proof of Lemma 4.1, (4.19) and (4.21) in particular, implies that

$$\|\tilde{Z}_{s1,j}u\|_p \le C(\|V\phi_j\|_p + \|V\phi_j\|_1)\|\psi_j * u\|_p, \quad 3 < p < \infty. \tag{4.37}$$

The Calderón-Zygmund theory with (3.9) once more implies $\left\|\tilde{Z}_{s1,j}u\right\|_p \le C\|u\|_p$. Since $\phi \otimes \phi \in \mathbf{B}(L^p)$ for all $1 < p < \infty$ if $\phi \in \mathcal{E}_1$ by virtue of (3.8), this together with (4.36) proves statement (2).
(3) It is obvious from (1) and (2) that $Z_{s1} \in \mathbf{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$ if $\mathcal{E} = \mathcal{E}_1$. Suppose then that $Z_{s1} \in \mathbf{B}(L^p(\mathbb{R}^3))$ for some $p > 3$ then $P \ominus P_1$ must be bounded in $L^p(\mathbb{R}^3)$ by virtue of (2). Take the orthonormal basis $\{\phi_1, \ldots, \phi_d\}$ of $\mathcal{E} \ominus \mathcal{E}_1$ and $\{\rho_1, \ldots, \rho_d\} \subset C_0^\infty(\mathbb{R}^3)$ such that $\{(\rho_j, \phi_k)\}$ becomes the unit matrix. Then, $(P \ominus P_1)\rho_j = \phi_j$, $j = 1, \ldots, n$ and, if $P \ominus P_1$ is bounded in $L^p(\mathbb{R}^3)$ for some $p \ge 3$, there must exist a constant $C > 0$ such that

$$|(u, \phi_j)| = |((P \ominus P_1)u, \rho_j)| \le C_j \|u\|_p, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^3).$$

Then, ϕ_j has to be in $L^{p'}(\mathbb{R}^3)$ for $p' \le 3/2$ for all $j = 1, \ldots, n$. This implies $\phi_j = 0$ by virtue of (3.8). Thus, $\mathcal{E} = \mathcal{E}_1$ must hold. This completes the proof. □

Lemma 4.2 and Lemma 4.3 prove Theorem 1.3 when H is of exceptional type of the second kind. The following lemma completes the proof of Theorem 1.3.

Lemma 4.4. *Suppose that H is of exceptional type of the third kind. Then:*

(1) *W is bounded in $L^p(\mathbb{R}^3)$ for all $1 < p < 3$.*

(2) *$W + a\varphi \otimes (|D|^{-1}V\varphi) + P$ is bounded in $L^p(\mathbb{R}^3)$ for all $p > 3$.*

(3) *W is unbounded in $L^p(\mathbb{R}^3)$ for any $p > 3$ and $p = 1$.*

Proof. The combination of Lemmas 4.1, 4.2 and 4.3 proves statements (1) and (2). Suppose that W is bounded in $L^p(\mathbb{R}^3)$ for some $3 < p < \infty$. Then, so is $a(\varphi \otimes (|D|^{-1}V\varphi)) + P$. Let $\psi \in \mathcal{N}$ be the function which defines the canonical resonance φ by (3.13) and which satisfies (3.12). Then,

$$(V\psi, a(\varphi \otimes (|D|^{-1}V\varphi))u + Pu) = -a(|D|^{-1}V\varphi, u), \quad u \in C_0^\infty(\mathbb{R}^3)$$

and this must be extended to a bounded functional of $u \in L^p(\mathbb{R}^3)$. Hence, $|D|^{-1}V\varphi \in L^q(\mathbb{R}^3)$ for $q = (p-1)/p < 3/2$. This contradicts (3.8) because $\int_{\mathbb{R}^3} V(x)\varphi(x)dx \ne 0$ and (3) is proved. □

5 Proof of Theorems 1.4 and 1.5 for odd m

If $m \ge 5$, then $\mathcal{N} = \mathcal{E}$ and we let $\{\phi_1, \ldots, \phi_d\}$ be the real orthonormal basis of $\mathcal{E}$. Theorem 3.3 implies that, with $a_0 = i/(24\pi^2)$,

$$S(\lambda) = \begin{cases} \lambda^{-2}PV - a_0\lambda^{-1}(\varphi \otimes V\varphi), & \text{if } m = 5, \\ \lambda^{-2}PV, & \text{if } m \ge 7. \end{cases} \tag{5.1}$$

Note that $\varphi \neq 0$ if and only if $\mathcal{E}_1 \neq \mathcal{E}$. We substitute (5.1) for $S(\lambda)$ in (3.17) and apply (2.2) and (2.17) as previously. Let C_j, c_k, $1 \leq j,k \leq \frac{m-3}{2}$ respectively be constants of (2.2) and (2.17). Then, we have

$$Z_s u = Z_{s0}u + Z_{s1}u, \tag{5.2}$$

where $Z_{s0} = 0$ for $m \geq 7$ and, for $m = 5$, with $M(r) = M(r, V\varphi * \check{u})$

$$Z_{s0}u = -2ia_0 \sum_{j,k=0,1} (-1)^{j+1} C_k c_j Z_{s0}^{jk} u, \tag{5.3}$$

$$Z_{s0}^{jk}u(x) = \int_{\mathbb{R}^5} \frac{V\varphi(y)}{|x-y|^{3-k}} K_0^{(j,k)}(|x-y|)dy, \tag{5.4}$$

$$K_0^{(j,k)}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}\lambda^{j+k}\left(\int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right) F(\lambda)d\lambda, \tag{5.5}$$

and $Z_{s1}u$ is defined for all $m \geq 5$ by

$$Z_{s1}u = \sum_{l=1}^{d} Z_{s1}(\phi_l)u \tag{5.6}$$

where, for $\phi \in \mathcal{E}$, with $M(r) = M(r, V\phi * \check{u})$,

$$Z_{s1}(\phi)u = 2i \sum_{j,k=0}^{\frac{m-3}{2}} (-1)^{j+1} C_k c_j Z_{s1}^{jk}(\phi), \tag{5.7}$$

$$Z_{s1}^{jk}(\phi)u(x) = \int_{\mathbb{R}^m} \frac{V\phi(y)}{|x-y|^{m-2-k}} K^{(j,k)}(|x-y|)dy, \tag{5.8}$$

$$K^{(j,k)}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}\lambda^{j+k-1}\left(\int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right) F(\lambda)d\lambda. \tag{5.9}$$

Note that $Z_{s0}^{jk}u$ and $K_0^{(j,k)}(\rho)$ are obtained from $Z_{s1}^{jk}u$ and $K^{(j,k)}(\rho)$ by changing ϕ by φ and λ^{j+k-1} by λ^{j+k} in (5.9).

We shall prove the last statements of (2) and (3) of Theorems 1.4 and 1.5 only for $Z_{s1}(\phi)$ since the proof of Lemma 4.3 (3) can easily be adapted for proving the same statements for Z_{s0}.

5.1 Estimate of Z_{s0} for $m = 5$

We begin by proving the following lemma for Z_{s0}, assuming $\varphi \neq 0$.

Lemma 5.1. (1) *Z_{s0} is bounded in $L^p(\mathbb{R}^5)$ for $1 < p < 5$.*

(2) *$Z_{s0} + a_0|\varphi\rangle\langle|D|^{-1}(V\varphi)|$ is bounded in $L^p(\mathbb{R}^5)$ for $5/2 < p < \infty$.*

(3) *Z_{s0} is not bounded in $L^p(\mathbb{R}^5)$ if $p \geq 5$.*

Proof. For $\varphi = PV$, we have $\int_{\mathbb{R}^5} V\varphi dx = \|\varphi\|^2 > 0$ and, by virtue of (3.8) and (3.9), $\varphi \otimes |D|^{-1}(V\varphi) \in \mathbf{B}(L^p(\mathbb{R}^5))$ if and only if $5/3 < p < 5$. Hence, statement (3) follows (2). Using that $e^{i\rho\lambda} = (i\rho)^{-(k+1)}\partial_\lambda^{k+1} e^{i\rho\lambda}$ and $\int_{\mathbb{R}} \lambda^{j+1} e^{-i\lambda r} M(r)dr = i^j \left(\int_{\mathbb{R}} e^{-i\lambda r} M(r)dr\right)^{(j)}$, we apply integration by parts to (5.5) and write $K_0^{(j,k)}(\rho)$ in two ways

$$K_0^{(j,k)}(\rho) = \frac{i^{k+1}}{2\pi\rho^{k+1}} \int_0^\infty e^{i\rho\lambda} \left(\lambda^{j+k} F(\lambda) \int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right)^{(k+1)} d\lambda \tag{5.10}$$

$$= \frac{(-i)^j}{2\pi} \int_0^\infty \left(e^{i\lambda\rho} \lambda^{j+k} F(\lambda)\right)^{(j)} \left(\int_{\mathbb{R}} e^{-i\lambda r} rM(r)dr\right) d\lambda. \tag{5.11}$$

Note that boundary terms do not appear in (5.10) since $\int_{\mathbb{R}} rM(r)dr = 0$ and, if $k = 1$, we may apply further integration by parts to (5.11) without having boundary term and

$$K_0^{(j,k)}(\rho) = \frac{(-i)^{j+1}}{2\pi} \int_0^\infty \left(e^{i\lambda\rho} \lambda^{j+k} F(\lambda)\right)^{(j+1)} \left(\int_{\mathbb{R}} e^{-i\lambda r} M(r)dr\right) d\lambda. \tag{5.12}$$

We then apply Lemmas 2.8 and 2.9 to the right sides and obtain the following estimates for $j, k = 0, 1$:

$$K_0^{(j,k)}(\rho) \le_{|\cdot|} \begin{cases} C\rho^{-(k+1)} \sum_{l=0}^{k+1} \mathcal{M}\mathcal{H}(r^{j+l+1}M)(\rho), & (5.13) \\ C(1+\rho^{j+k})\mathcal{M}\mathcal{H}(r^{1-k}M)(\rho). & (5.14) \end{cases}$$

(a) Let $1 < p < 5/4$. Since $|r|^{-4(p-1)}$ is an A_p weight on $\mathbb{R}$ and $3p - 4 > -1$, we have by using (5.13) and (4.13) that, for any $j, k = 0, 1$,

$$\left\| \frac{K_0^{(j,k)}(|y|)}{|y|^{3-k}} \right\|_p \le C \sum_{l=0}^{k+1} \left(\int_0^\infty \frac{|\mathcal{M}\mathcal{H}(r^{j+l+1}M)(\rho)|^p}{\rho^{4(p-1)}} d\rho \right)^{1/p}$$
$$\le C \left(\int_0^1 \frac{|M(r)|^p dr}{r^{3p-4}} + \int_1^\infty |M(r)|^p r^4 dr \right)^{\frac{1}{p}} \le C(\|V\varphi\|_{p'} + \|V\varphi\|_1)\|u\|_p. \tag{5.15}$$

Young's inequality then implies $\|Z_{s0}^{jk} u\|_p \le C\|V\varphi\|_1 (\|V\varphi\|_{p'} + \|V\varphi\|_1)\|u\|_p$.
(b) We next show that $\|Z_{s0}^{j1} u\|_p \le C\|u\|_p$ for $p > 5$ and $j = 0, 1$. Interpolating this with the result of (a), we then have the same for all $1 < p < \infty$. We split the integral as in (4.18) and repeat the argument after it:

$$|Z_{0s}^{j1} u(x)| \le C \left(\int_{|y|\le 1} + \int_{|y|>1} \right) \frac{|V\varphi(x-y)|}{|y|^2} |K_0^{(j,1)}(|y|)| dy = I_1(x) + I_2(x).$$

For $\rho \geq 1$, we have $K_0^{(j,1)}(\rho) \leq_{|\cdot|} C\rho^2 \mathcal{M}\mathcal{H}(M(r))(\rho)$ by virtue of (5.14) and since r^4 is A_p weight on $\mathbb{R}$ if $p > 5$. It follows that

$$\|I_2\|_p \leq C\|V\varphi\|_1 \left\| \frac{K_0^{(j,1)}}{|x|^2} \right\|_{L^p(|x|\geq 1)} \leq C\|V\varphi\|_1 \left(\int_0^\infty |\mathcal{M}\mathcal{H}(M)(\rho)|^p \rho^4 d\rho \right)^{\frac{1}{p}}$$
$$\leq C\|V\varphi\|_1 \left(\int_0^\infty |M(r)|^p r^4 dr \right)^{1/p} \leq C\|V\varphi\|_1^2 \|u\|_p. \quad (5.16)$$

Hölder's inequality and (5.14) for $0 \leq \rho \leq 1$, $K_0^{(j,1)}(\rho) \leq_{|\cdot|} C\mathcal{M}\mathcal{H}(M)(\rho)$, imply

$$|I_1(x)| \leq C \left(\int_{|y|\leq 1} \left| \frac{|V\varphi(x-y)|}{|y|^2} \right|^{p'} dy \right)^{\frac{1}{p'}} \left(\int_0^1 |\mathcal{M}\mathcal{H}(M(r))(\rho)|^p \rho^4 d\rho \right)^{\frac{1}{p}}.$$

Since $p' \leq \frac{5}{4}$ if $p > 5$, Minkowski's inequality and (5.16) imply

$$\|I_1\|_p \leq C\|V\varphi\|_1 \|V\varphi\|_p \|u\|_p. \quad (5.17)$$

(c) We finally prove $-2ia_0C_0(c_1Z_{s0}^{10} - c_0Z_{s0}^{00}) + a_0|\varphi\rangle\langle|D|^{-1}(V\varphi)| \in \mathbf{B}(L^p(\mathbb{R}^5))$ for $p > 5/2$. This will complete the proof of the lemma because this and (b) imply statement (2) by virtue of (5.3); since $|\varphi\rangle\langle|D|^{-1}(V\varphi)|$ is bounded in $L^p(\mathbb{R}^5)$ for $5/3 < p < 5$ as remarked previously, this also implies $-2ia_0C_0(c_1Z_{s0}^{10} - c_0Z_{s0}^{00}) \in \mathbf{B}(L^p(\mathbb{R}^5))$ for $5/3 < p < 5$ and, hence, for $1 < p < 5$ by virtue of result (a) and interpolation. Then, (b) yields statement (1). If $k = 0$, further integration by parts to (5.11) produces boundary term:

$$K_0^{(j,0)}(\rho) = \frac{(-i)^{j+1}}{2\pi} j! \int_{\mathbb{R}} M(r)dr$$
$$+ \frac{(-i)^{j+1}}{2\pi} \int_0^\infty \left(e^{i\lambda\rho}\lambda^j F(\lambda)\right)^{(j+1)} \left(\int_{\mathbb{R}} e^{-i\lambda r} M(r)dr \right) d\lambda. \quad (5.18)$$

The second integral, which we denote by $\tilde{K}_0^{(j,0)}(\rho)$, satisfies

$$\tilde{K}_0^{(j,0)}(\rho) \leq_{|\cdot|} C(1+\rho^{j+1})\mathcal{M}\mathcal{H}(M)(\rho) \leq C(1+\rho^{j+2})\mathcal{M}\mathcal{H}(M)(\rho) \quad (5.19)$$

and we estimate the operator $\tilde{Z}^{j0}$ obtained by replacing $K_0^{(j,0)}(\rho)$ by $\tilde{K}_0^{(j,0)}(\rho)$ in (5.4) by repeating the argument of step (b): Split $\tilde{Z}^{j0}u(x)$ as in step (b) and obtain $\|I_2\|_p \leq C\|u\|_p$ for $5/2 < p < 5$ (resp. $p > 5$) by using the first (resp. second) estimate of (5.19) and that r^{4-p} (resp. r^4) is an A_p-weight on $\mathbb{R}$. Likewise we obtain $\|I_1\|_p \leq C\|u\|_p$ for $5/2 < p < 5$ (resp. $p > 5$) by first applying Hölder's inequality by considering the integrand as $(|V\varphi(x-y)|/|y|^2) \cdot (|\tilde{K}_0^{(j,0)}(|y|)|/|y|)$ (resp. $|V\varphi(x-y)|/|y|^3 \cdot |\tilde{K}_0^{(j,0)}(|y|)|$) and then using Minkowski's inequality. Thus, we have for $j = 0, 1$ that

$$\|\tilde{Z}^{j0}u\|_p \leq C\|u\|_p, \quad 5/2 < p < \infty. \quad (5.20)$$

The contribution of boundary terms of (5.18) to $c_0K_0^{(00)} - c_1K_0^{(10)}$ is given by virtue of (2.3) and (3.9) by

$$(c_1 - ic_0) \times \frac{1}{2\pi}\int_{\mathbb{R}} M(r)dr = \frac{c_0}{\pi i}\int_{\mathbb{R}} M(r)dr = -4\pi^2 C_0 i\langle |D|^{-1}(V\varphi), u\rangle$$

and this contributes to $2a_0iC_0(c_0Z_{s0}^{00} - c_1Z_{s0}^{10})u(x)$ by

$$8\pi^2 a_0 C_0^2 \int_{\mathbb{R}^5} \frac{V\varphi(y)}{|x-y|^3}dy \cdot (\langle |D|^{-1}(V\varphi), u\rangle) = -a_0\varphi(x)\langle |D|^{-1}(V\varphi), u\rangle,$$

where we used $8\pi^2 C_0 = 1$ when $m = 5$. This proves the lemma. $\square$

5.2 Estimates of Z_{s1} for $m \geq 5$.

We next study $Z_{s1}u$ for all $m \geq 7$. By virtue of (5.6) and (5.7) and the remark at the beginning of section 5, it suffices to study $Z_{1s}^{jk}(\phi)u$ defined by (5.8) for $\phi \in \mathcal{E}$. For simplifying notation, we often omit ϕ from $Z_{1s}^{jk}(\phi)$. Define

$$M_*(r) = M(r, |D|^{-1}(V\phi) * \check{u}). \tag{5.21}$$

Then, by virtue of (2.9), $K^{(j,k)}(\rho)$ may also be expressed as

$$K^{(j,k)}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}\lambda^{j+k}F(\lambda)\left(\int_{\mathbb{R}} e^{-i\lambda r}r^{j+1}M_*(r)dr\right)d\lambda \tag{5.22}$$

which has the larger factor λ^{k+j} than λ^{k+j-1} of (5.9). We omit the proof of the following lemma which is essentially the same as that of (5.13, 5.14)

Lemma 5.2. *$K^{(j,k)}(\rho)$ satisfies the following estimates:*

$$K^{(j,k)}(\rho) \le_{|\cdot|} \begin{cases} C\rho^{-k-1}\sum_{l=0}^{k+1}\mathcal{MH}(r^{j+1+l}M)(\rho), & j \geq 2. \quad (5.23)\\ C(1+\rho^{j-1})\mathcal{MH}(r^2M)(\rho), & j \geq 1. \quad (5.24)\\ C(1+\rho^{j})\mathcal{MH}(rM)(\rho), & k+j \geq 1. \quad (5.25)\\ C(1+\rho^{j+1})\mathcal{MH}(M)(\rho), & k \geq 2. \quad (5.26)\\ C(1+\rho^{j})\mathcal{MH}(rM_*)(\rho), & k \geq 0. \quad (5.27)\end{cases}$$

Lemma 5.3. *Suppose $m \geq 5$ and $\phi \in \mathcal{E}$. Then:*

(1) *If $j \geq 2$, $Z_{1s}^{jk}(\phi)$, $k = 0, \dots, \frac{m-3}{2}$, are bounded in $L^p(\mathbb{R}^m)$ for $1 < p < \frac{m}{2}$.*

(2) *For $k \geq 2$, $Z_{1s}^{jk}(\phi)$, $j = 0, \dots, \frac{m-3}{2}$, are bounded in $L^p(\mathbb{R}^m)$ for $\frac{m}{3} < p$.*

(3) *For all j and k, $Z_{1s}^{jk}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for $\frac{m}{3} < p < \frac{m}{2}$.*

If both $j,k \geq 2$, $Z^{jk}_{1s}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < \infty$.

Proof. (a) We first prove (1) for $1 < p < \frac{m}{m-1}$. General case follows from this and (3) by interpolation. We use (5.23) and that $r^{-(m-1)(p-1)}$ is an A_p weight on $\mathbb{R}$ for $1 < p < \frac{m}{m-1}$. Then, estimating as in (5.15), we obtain

$$\begin{aligned} \|Z^{jk}_{s1}u\|_p &\leq C\|V\phi\|_1 \left(\int_0^\infty |M(r)|^p r^{m-1}dr + \int_0^1 \frac{|M(r)|^p}{r^{(m-4)p}} r^{m-1}dr\right)^{1/p} \\ &\leq C\|V\phi\|_1(\|V\phi\|_1 + \|V\phi\|_{p'})\|u\|_p. \end{aligned} \tag{5.28}$$

(b) We next prove (2) for $p > m$. General case then follows from this and (3) by interpolation. We split the integral as in (4.18):

$$Z^{jk}_{s1}u(x) \leq_{|\cdot|} \left(\int_{|y|\leq 1} + \int_{|y|\geq 1}\right) \frac{|V\phi(x-y)|}{|y|^{m-2-k}} |K^{(j,k)}(|y|)|dy = I_1(x) + I_2(x).$$

Using (5.26) for $\rho \geq 1$ and that r^{m-1} is A_p weight on $\mathbb{R}$ if $p > m$, we obtain

$$\|I_2\|_p \leq C\|V\phi\|_1 \left(\int_1^\infty |\mathcal{M}\mathcal{H}(M)(\rho)|^p \rho^{m-1}d\rho\right)^{\frac{1}{p}} \leq C\|V\phi\|_1^2\|u\|_p. \tag{5.29}$$

Hölder's inequality and (5.26) for $0 \leq \rho \leq 1$ imply that

$$|I_1(x)| \leq \left(\int_{|y|\leq 1} \left|\frac{V\phi(x-y)|}{|y|^{m-2-k}}\right|^{p'} dy\right)^{1/p'} \left(\int_0^1 |\mathcal{M}\mathcal{H}(M)(\rho)|^p \rho^{m-1}d\rho\right)^{1/p}. \tag{5.30}$$

Then, Minkowski's inequality and the estimate as in (5.29) yield

$$\|I_1\|_p \leq C\|V\phi\|_1\|u\|_p \left(\int_{|x|<1} \frac{\|V\phi\|_p^{p'}dx}{|x|^{(m-2-k)p'}}\right)^{1/p'} \leq C\|V\phi\|_1\|V\phi\|_p\|u\|_p$$

because $p' \leq \frac{m}{m-1}$ if $p > m$ and $|y|^{-(m-2-k)p'}$ is integrable over $|y| \leq 1$. Thus, statement (2) for $p > m$ follows.

(c) We prove statement (3) by modifying the argument in step (b). Let $\frac{m}{3} < p < \frac{m}{2}$. Then, r^{m-1-2p} is an A_p weight on $\mathbb{R}$. We split the integral of $Z^{jk}_{s1}u(x)$ as in step (b).

(i) Let $j \geq 1$. Estimate (5.24) for $\rho \geq 1$ and Lemma 2.7 yield

$$\|I_2\|_p \leq C\|V\phi\|_1 \left(\int_0^\infty |\mathcal{M}\mathcal{H}(r^2M)(\rho)|^p \rho^{m-1-2p}d\rho\right)^{\frac{1}{p}} \leq C\|V\phi\|_1^2\|u\|_p. \tag{5.31}$$

Estimate (5.24) for $\rho \leq 1$ and Hölder's inequality imply

$$|I_1(x)| \leq \left(\int_{|y|\leq 1} \left|\frac{|V\phi(x-y)|}{|y|^{m-4-k}}\right|^{p'} dy\right)^{\frac{1}{p'}} \left(\int_0^1 |\mathcal{M}\mathcal{H}(r^2M)(\rho)|^p \rho^{m-1-2p}d\rho\right)^{\frac{1}{p}}.$$

Minkowski's inequality and the second estimate of (5.31) imply $\|I_1\|_p \leq C\|V\phi\|_p\|V\phi\|_1\|u\|_p$ as previously and, hence, $\|Z_{s1}^{jk}u\|_p \leq C\|u\|_p$.
(b) Let $j=0$. Express $K^{(0,k)}(\rho)$ by using $\tilde{M}(r)$ of (2.15) and estimate as

$$K^{(0,k)}(\rho) = \frac{1}{2i\pi}\int_0^\infty e^{i\lambda\rho}\lambda^k\left(\int_{\mathbb{R}} e^{-i\lambda r}\tilde{M}(r)dr\right)F(\lambda)d\lambda \leq_{|\cdot|} C\mathcal{M}\mathcal{H}(\tilde{M})(\rho). \tag{5.32}$$

Since $\rho^{-(m-2-k)} \leq \rho^{-2}$ for $\rho \geq 1$, Young's inequality, Lemma 2.7 and Hardy's inequality yield

$$\begin{aligned}\|I_2\|_p &\leq C\|V\phi\|_1\left(\int_0^\infty |\tilde{M}(r)|^p r^{m-1-2p}dr\right)^{1/p} \\ &\leq C\|V\phi\|_1\left(\int_0^\infty |M(r)|^p r^{m-1}dr\right)^{1/p} \leq C\|V\phi\|_1\|V\phi\|_p\|u\|_p. \end{aligned}\tag{5.33}$$

Hölder's inequality and (5.32) imply

$$|I_1(x)| \leq \left(\int_{|y|\leq 1}\left|\frac{|V\phi(x-y)|}{|y|^{m-4-k}}\right|^{p'}dy\right)^{1/p'}\left(\int_0^1 |\mathcal{M}\mathcal{H}(\tilde{M})(\rho)|^p\rho^{m-1-2p}d\rho\right)^{1/p}$$

Estimate the second factor by (5.33) and use Minkowski's equality. This yields $\|I_1\|_p \leq C\|V\phi\|_p\|V\phi\|_1\|u\|_p$. The last statement follows from (1) and (2) by interpolation. $\square$

LEMMA 5.4. *Let $m \geq 5$ and $\phi \in \mathcal{E}$. Then:*

(1) *For $1 < p < \frac{m}{2}$, $\|(c_0 Z_{s1}^{(0,k)} - c_1 Z_{s1}^{(1,k)})u\|_p \leq C\|u\|_p$ for all $0 \leq k \leq \frac{m-3}{2}$.*

(2) *The operator $Z_{s1}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for $1 < p < \frac{m}{2}$.*

Proof. It suffices to prove the estimate of (1) for $1 < p < \frac{m}{m-1}$ since that for $1 < p < \frac{m}{2}$ follows from this and Lemma 5.3 (3) by interpolation and since statement (2) follows from this and statement (1) of Lemma 5.3. Using the identity $e^{i\lambda\rho} = (i\rho)^{-k-1}\partial_\lambda^{k+1}e^{i\lambda\rho}$, we apply integration by parts $k+1$ times to the integral of (5.32) and use the identity (2.16). We obtain

$$\begin{aligned}K^{(0,k)}(\rho) &= \frac{i^k}{2\pi\rho^{k+1}}\left(k!\int_{\mathbb{R}} r^2 M(r)dr\right. \\ &\left. + \sum_{l=0}^{k+1}\binom{k+1}{l}\int_0^\infty e^{i\lambda\rho}(\lambda^k F)^{(k+1-l)}\int_{\mathbb{R}} e^{-i\lambda r}(-ir)^l\tilde{M}drd\lambda\right).\end{aligned}\tag{5.34}$$

Integration by parts $k+1$ times to $K^{(1,k)}(\rho)$ of (5.9) likewise yields

$$\begin{aligned}K^{(1,k)}(\rho) &= \frac{i^k}{2\pi i\rho^{k+1}}\left(-k!\int_{\mathbb{R}} r^2 M(r)dr\right. \\ &\left. - \sum_{l=0}^{k+1}\binom{k+1}{l}\int_0^\infty e^{i\lambda\rho}(\lambda^k F)^{(k+1-l)}\int_{\mathbb{R}} e^{-i\lambda r}(-ir)^l r^2 Mdrd\lambda\right).\end{aligned}\tag{5.35}$$

Since $c_0 - ic_1 = 0$, the boundary terms of (5.34) and (5.35) cancel out and

$$\frac{c_0 K^{(0,k)}(\rho) - c_1 K^{(1,k)}(\rho)}{\rho^{m-2-k}} \le_{|\cdot|} \frac{C}{\rho^{m-1}} \sum_{l=0}^{k+1} (\mathcal{M}\mathcal{H}(r^l \tilde{M})(\rho) + \mathcal{M}\mathcal{H}(r^{l+2}M)(\rho)).$$

For $1 < p < \frac{m}{m-1}$, $\rho^{-(m-1)(p-1)}$ is an A_p-weight on $\mathbb{R}$. It follows by Young's inequality, Lemma 2.7 and Hardy's inequality that $\|(c_0 Z^{(0,k)} - c_1 Z^{(1,k)})u\|_p$ is bounded by $C\|V\phi\|_1$ times

$$\sum_{l=0}^{k+1} \left(\int_0^\infty (|\tilde{M}(r)|^p r^{pl} + |M(r)|^p r^{p(l+2)}) r^{m-1-p(m-1)} dr \right)^{1/p} \tag{5.36}$$

$$\le C \left(\int_0^1 \frac{|M(r)|^p}{r^{p(m-3)}} r^{m-1} dr + \int_0^\infty |M(r)|^p r^{m-1} dr \right)^{1/p} \tag{5.37}$$

$$\le C(\|V\phi\|_{p'} + \|V\phi\|_p)\|u\|_p. \tag{5.38}$$

Here we used $k+3 \le m-1$ for $m \ge 5$ in the first step and $p(m-1) < m$ in the last. This proves the estimate of (1) for $1 < p < \frac{m}{m-1}$. □

Lemma 5.1 and the second statement of Lemma 5.4 prove statement (1) of Theorems 1.4 and 1.5 for odd m. The following lemma (and Lemma 5.1 for the case $m = 5$) proves statement (2) of these theorems for odd m.

Lemma 5.5. *Let $m \ge 5$, $\phi \in \mathcal{E}$ and $\frac{m}{2} < p < m$. Then, for a constant $C > 0$,*

$$\left\| Z_{s1}(\phi)u + \frac{\Gamma\left(\frac{m-2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \langle u, \phi\rangle \phi \right\|_p \le C\|u\|_p. \tag{5.39}$$

If $Z_{s1}(\phi) \in \mathbf{B}(L^p)$ for some $\frac{m}{2} < p < m$, then $\phi \in \mathcal{E}_0$ and $Z_{s1}(\phi) \in \mathbf{B}(L^p)$ for all $1 < p < m$.

Proof. Let $j + k \ge 1$. Since $m - 2 - (k+j) \ge 1$, we have from (5.25) that

$$\frac{K^{(j,k)}(\rho)}{\rho^{m-2-k}} \le_{|\cdot|} C \left(\frac{1}{\rho^{m-2-k}} + \frac{1}{\rho} \right) \mathcal{M}\mathcal{H}(rM)(\rho).$$

Using that r^{m-1-p} is A_p weight and $(m-2)p' < m$ for $m/2 < p < m$, we repeat the argument of the step (b) or (c) of the proof of Lemma 5.3 and obtain

$$\|Z_{s1}^{jk} u\|_p \le C\|u\|_p, \quad j + k \ge 1. \tag{5.40}$$

It remains to consider $-2iC_0 c_0 Z_{s1}^{00}$, see (5.7). We apply integration by parts to the right of (5.22) with $j = k = 0$:

$$K^{(0,0)}(\rho) = \frac{i}{2\pi} \int_0^\infty e^{i\lambda\rho} F(\lambda) \partial_\lambda \left(\int_{\mathbb{R}} e^{-i\lambda r} M_*(r) dr \right) d\lambda$$

$$= \frac{-i}{2\pi} \int_{\mathbb{R}} M_*(r) dr - \frac{i}{2\pi} \int_0^\infty (e^{i\lambda\rho} F(\lambda))' \left(\int_{\mathbb{R}} e^{-i\lambda r} M_*(r) dr \right) d\lambda. \tag{5.41}$$

We denote the second integral of (5.41) by $K_*^{(0,0)}(\rho)$ and, by Z_*^{00} the operator produced by substituting $K_*^{(0,0)}(\rho)$ for $K^{(0,0)}(\rho)$ in (5.8). We have $|K_*^{(0,0)}(\rho)| \leq C(1+\rho)\mathcal{MH}(M_*)(\rho)$. Decompose

$$Z_*^{00}u(x) \leq_{|\,\cdot\,|} \left(\int_{|y|\leq 1} + \int_{|y|\geq 1}\right) |(V\phi)(x-y)| \frac{|K_*^{(0,0)}(|y|)|}{|y|^{m-2}} dy = I_1(x) + I_2(x)$$

as previously. For estimating $\|I_2\|_p$, define $1/q = 1/p - 1/m$ and apply Young's inequality, Hölder's inequality, Lemma 2.7 noticing that $q > m$ and r^{m-1} is A_q weight and, Hardy-Littlewood-Soblev inequality recalling that $|D|^{-1}(V\phi) * (x) \leq_{|\,\cdot\,|} C\langle x\rangle^{1-m}$. We obtain

$$\begin{aligned} \|I_2\|_p &\leq C\|V\phi\|_1 \left(\int_0^\infty |\mathcal{MH}(M_*)(\rho)|^q \rho^{m-1} d\rho\right)^{1/q} \left\|\frac{1}{|y|^{m-3}}\right\|_{L^m(|y|>1)} \\ &\leq C\|V\phi\|_1 \||D|^{-1}(V\phi) * \check{u}\|_q \leq C\|V\phi\|_1 \||D|^{-1}(V\phi)\|_{\frac{m}{m-1},w} \|u\|_p. \end{aligned} \tag{5.42}$$

For $I_1(x)$, Hölder's inequality implies

$$|I_1(x)| \leq C\left(\int_{|y|\leq 1} \left|\frac{(V\phi)(x-y)}{|y|^{m-2}}\right|^{q'} dy\right)^{1/q'} \left(\int_{|y|\leq 1} |\mathcal{MH}(M_*)(|y|)|^q dy\right)^{1/q}.$$

The second factor on the right is bounded by $C\||D|^{-1}(V\phi)\|_{\frac{m}{m-1},w}\|u\|_p$ as in (5.42) and $q' < \frac{m}{m-1} < \frac{m}{2} < p$. It follows by Minkowski's inequality that

$$\|I_1\|_p \leq C\|V\phi\|_p \|u\|_p \left(\int_{|y|\leq 1} \frac{dy}{|y|^{(m-2)q'}}\right)^{1/q'} \leq C\|V\phi\|_p\|u\|_p.$$

Thus, we have $\|Z_*^{00}u\|_p \leq C\|u\|_p$ for $\frac{m}{2} < p < m$. The boundary term of (5.41) is, by virtue of (3.7) and that $c_0 = (m-2)^{-1}$, equal to

$$\begin{aligned} \frac{-i}{2\pi}\int_{\mathbb{R}} M_*(r)dr &= \frac{-i}{\pi\omega_{m-1}} \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \frac{|D|^{-1}(V\phi)(y)}{|x-y|^{m-1}} dy\right) u(x)dx \\ &= \frac{-i\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \int_{\mathbb{R}^m} |D|^{-2}(V\phi)(x)u(x)dx = \frac{i\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \langle \phi, u\rangle. \end{aligned} \tag{5.43}$$

Inserting this into the right of (5.8) for $j = k = 0$, we see the contribution of the boundary term to $Z_{s1}(\phi)u$ is given by

$$\frac{2c_0C_0\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \int_{\mathbb{R}^m} \frac{V\phi(y)}{|x-y|^{m-2}} dy \langle\phi, u\rangle = -\frac{\Gamma\left(\frac{m-2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} |\phi\rangle\langle\phi, u\rangle.$$

This proves the first statement. If $Z_{s1} \in \mathbf{B}(L^p)$ for some $\frac{m}{2} < p < m$, (5.39) implies $\phi \otimes \phi \in \mathbf{B}(L^p)$ for this p. Then, (3.8) implies that ϕ must satisfy $\langle \phi, V\rangle = 0$ and $\phi\otimes\phi \in \mathbf{B}(L^p)$ for all $\frac{m}{m-1} < p < m$. Then, $Z_{s1} \in \mathbf{B}(L^p)$ must be satisfied for all $\frac{m}{2} < p < m$ and, hence, for all $1 < p < m$ by Lemma 5.4 and interpolation. ☐

We finally study $Z_{1s}(\phi)$ in $L^p(\mathbb{R}^m)$ for $p > m$. If $Z_{1s}(\phi) \in \mathbf{B}(L^p(\mathbb{R}^m))$ for some $p > m$, then Lemma 5.5 implies $\phi \in \mathcal{E}_0$. Thus, assume $\phi \in \mathcal{E}_0$ in the following lemma. The following lemma proves statements (3) of Theorem 1.4 and Theorem 1.5 for odd $m \geq 7$.

Lemma 5.6. *Let $m \geq 5$ be odd, $p > m$ and $\phi \in \mathcal{E}_0$. Then:*

(1) *For a constant $C_p > 0$, $\|Z_{s1}(\phi)u + |\phi\rangle\langle\phi|)u\|_p \leq C\|u\|_p$.*

(2) *If $Z_{1s}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for some $p > m$, then $\phi \in \mathcal{E}_1$. In this case Z_{1s} is bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < \infty$.*

Proof. Considering that $\int_{\mathbb{R}} r^{j+1} e^{-i\lambda r} M_*(r)dr = i^{j+1}\left(\int_{\mathbb{R}} e^{-i\lambda r} M_*(r)dr\right)^{(j+1)}$, we apply integration by parts to (5.22). Then, for $k \geq 1$, we have

$$K^{(j,k)}(\rho) = \frac{(-i)^{j+1}}{2\pi}\int_0^\infty \left(e^{i\lambda\rho}\lambda^{j+k}F(\lambda)\right)^{(j+1)}\left(\int_{\mathbb{R}} e^{-i\lambda r}M_*(r)dr\right)d\lambda \quad (5.44)$$

and, if $k = 0$, additional boundary term which is given by virtue of (5.43) by

$$\frac{(-i)^{j+1}j!}{2\pi}\int_{\mathbb{R}} M_*(r)dr = \frac{i(-i)^j j!\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)}\langle\phi, u\rangle, \quad j = 0, \ldots, \frac{m-3}{2}. \quad (5.45)$$

Denote the right of (5.44) by $\tilde{K}^{(j,0)}(\rho)$ when $k = 0$. Then,

$$\frac{K^{(j,k)}(\rho)}{\rho^{m-2-k}} \leq_{|\,\cdot\,|} C\left(1 + \frac{1}{\rho^{m-2}}\right)\mathcal{MH}(M_*)(\rho), \quad 0 \leq j, k \leq \frac{m-3}{2} \quad (5.46)$$

and the same for $\tilde{K}^{(j,0)}(\rho)$. We split $Z_{s1}^{jk}u$ as previously:

$$Z_{s1}^{jk}u(x) = \left(\int_{|x-y|\leq 1} + \int_{|x-y|>1}\right)\frac{V\phi(y)K^{(j,k)}(|x-y|)}{|x-y|^{m-2-k}}dy = I_1(x) + I_2(x).$$

We estimate $I_2(x)$ by using (5.46) for $\rho \geq 1$, that ρ^{m-1} is A_p weight for $p > m$, (3.8) for $\phi \in \mathcal{E}_0$ and the Calderón-Zygmund theory. This yields

$$\begin{aligned}\|I_2\|_p &\leq \|V\phi\|_1\left(\int_1^\infty |\mathcal{MH}(M_*)(\rho)|^p\rho^{m-1}d\rho\right)^{1/p} \\ &\leq \|V\phi\|_1\||D|^{-1}(V\phi) * u\|_p \leq C\|V\phi\|_1\|u\|_p. \quad (5.47)\end{aligned}$$

Hölder's inequality and (5.46) for $\rho \leq 1$ imply

$$|I_1(x)| \leq C\left(\int_{|y|\leq 1}\left|\frac{(V\phi)(x-y)}{|y|^{m-2}}\right|^{p'}dy\right)^{1/p'}\left(\int_{|y|\leq 1}|\mathcal{MH}(M_*)(|y|)|^p dy\right)^{1/p}.$$

The second factor on the right is bounded by $C\|u\|_p$ as in (5.47). Since $p' < \frac{m}{m-1} < m < p$, it follows by Minkowski's inequality that

$$\|I_1\|_p \le C\|V\phi\|_p\|u\|_p \left(\int_{|y|\le 1} \frac{dy}{|y|^{(m-2)p'}}\right)^{1/p'} \le C\|V\phi\|_p\|u\|_p.$$

Thus, $Z_{s1}^{jk} \in \mathbf{B}(L^p(\mathbb{R}^m))$ for $p > m$ if $k \ge 1$ and the same for the operator $\tilde{Z}_{s1}^{j0}$ produced by $\tilde{K}^{(j,0)}(\rho)$. The contribution of boundary terms (5.45) to $Z_{s1}(\phi)$ is given by using the constants C_j of (2.2) by

$$2i\sum_{j=0}^{\frac{m-3}{2}} C_0C_j(-1)^{j+1}\omega_{m-1}\left(\int_{\mathbb{R}^d} \frac{(V\phi)(y)}{|x-y|^{m-2}}dy\right)\frac{i(-i)^j j!\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)}\langle\phi, u\rangle$$

$$= -\tilde{D}_m|\phi\rangle\langle\phi,u\rangle, \quad \tilde{D}_m = \sum_{j=0}^{\frac{m-3}{2}} \frac{(m-3-j)!}{2^{m-3-j}\left(\frac{m-3}{2}\right)!\left(\frac{m-3}{2}-j\right)!}. \qquad (5.48)$$

The constant $\tilde{D}_m$ can be elementarily computed and with $n = \frac{m-3}{2}$

$$\tilde{D}_m = \sum_{k=0}^{n} \frac{1}{2^{2n-k}}\binom{2n-k}{n-k} = \sum_{k=0}^{n}\frac{1}{2^{n+k}}\binom{n+k}{k} = 1.$$

(see also page 167 of [12].) This proves statement (1). We omit the proof of (2) which is similar to the corresponding statement of Lemma 5.5. □

Since $Z_{s1}u = \sum_{i=1}^n Z_{s1}(\phi_j)$ for the orthonormal basis of $\mathcal{E}$, the combination of lemmas in this section proves Theorems 1.4 and 1.5 for odd m.

6 Proof of Theorem 1.5 for even $m \ge 6$

For proving Theorem 1.5 for even dimensions $m \ge 6$ we need study Z_s and $Z_{\log}$ of (3.26) and (3.27). Since $Z_{\log}$ may be studied in a way similar to but simpler than that for Z_s, we shall be mostly concentrated on Z_s and only briefly comment on $Z_{\log}$ at the end of the section. As in odd dimensions we take the real orthonormal basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{E}$ and define, for $\phi \in \mathcal{E}$,

$$Z_s(\phi)u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)|V\phi\rangle\langle\phi V|(G_0(\lambda)-G_0(-\lambda))F(\lambda)\lambda^{-1}d\lambda. \qquad (6.1)$$

Then, we have

$$Z_s u = \sum_{j=1}^d Z_s(\phi_j)u$$

and we study $Z_s(\phi)$ for $\phi \in \mathcal{E}$. In this section we choose and fix a $\phi \in \mathcal{E}$ arbitrarily and write $M(r) = M(r, V\phi * \check{u})$.

We wish to apply the argument for odd dimensions also to even dimensions as much as possible and, we express $Z_s(\phi)$ as a superposition of operators which are of the same form as those studied in odd dimensions except scaling. We set $\nu = (m-2)/2$. Define for $a > 0$

$$M^a(r) = M(r/(1+2a)) \tag{6.2}$$

and, for $j, k = 0, \ldots, \nu$ and $a, b > 0$,

$$Q^{a,b}_{jk}(\rho) = \frac{(-1)^{j+1}}{2\pi(1+2a)^{j+2}} \int_0^\infty \lambda^{j+k-1} e^{i\lambda(1+2b)\rho} \mathcal{F}(r^{j+1}M^a)(\lambda)F(\lambda)d\lambda. \tag{6.3}$$

As in (5.22), we may express $Q^{a,b}_{jk}(\rho)$ by using $M_*(r)$ and increase the factor λ^{j+k-1} of (6.3) to λ^{j+k}:

$$Q^{a,b}_{jk}(\rho) = \frac{(-1)^{j+1}}{2\pi(1+2a)^{j+2}} \int_0^\infty \lambda^{j+k} e^{i\lambda(1+2b)\rho} \mathcal{F}(r^{j+1}M^a_*)(\lambda)F(\lambda)d\lambda. \tag{6.4}$$

When $j = 0$, we also use $\tilde{M}(r)$ of (2.15) to express $Q^{a,b}_{0k}(\rho)$ as follows:

$$Q^{a,b}_{0k}(\rho) = \frac{i}{2\pi(1+2a)^2} \int_0^\infty \lambda^k e^{i\lambda(1+2b)\rho} \mathcal{F}(\widetilde{M^a})(\lambda)F(\lambda)d\lambda. \tag{6.5}$$

Lemma 6.1. *Let $Q^{a,b}_{jk}(\rho)$ be defined by* (6.3), (6.4) *or* (6.5). *Then,*

$$Z_s(\phi)u(x) = \frac{2i}{\omega_{m-1}} \sum_{j,k=0}^{\nu} T^{(a)}_j T^{(b)}_k \left[\int_{\mathbb{R}^m} \frac{(V\phi)(x-y)Q^{a,b}_{jk}(|y|)}{|y|^{m-2-k}} dy\right]. \tag{6.6}$$

Proof. We apply (2.18) for $\langle V\phi, (G_0(\lambda) - G_0(-\lambda))u\rangle$ and (2.6) for $G_0(\lambda)$ in (6.1). We see that $Z_s(\phi)u(x)$ is the integral with respect to $\lambda \in (0, \infty)$ of

$$\frac{i}{\pi} \sum_{j,k=0}^{\nu} T^{(a)}_j T^{(b)}_k \left[\frac{(-1)^{j+1}\lambda^{j+k-1}}{(1+2a)^{j+2}\omega_{m-1}} \left(\frac{e^{i\lambda(1+2b)|y|}}{|y|^{m-2-k}} * V\phi\right) \mathcal{F}(r^{j+1}M^a)(\lambda)\right] F(\lambda).$$

Integrating with respect to λ first yields (6.6). □

We define, for $0 \le j, k \le \nu$ and $a, b > 0$, that

$$Z^{jk}(\phi)u(x) = \frac{2i}{\omega_{m-1}} T^{(a)}_j T^{(b)}_k \left[Z^{jk}_{a,b}(\phi)u(x)\right], \tag{6.7}$$

$$Z^{jk}_{a,b}(\phi)u(x) = \int_{\mathbb{R}^m} \frac{(V\phi)(x-y)Q^{a,b}_{jk}(|y|)}{|y|^{m-2-k}} dy. \tag{6.8}$$

Lemma 6.1 implies $Z_s(\phi)u = \sum Z^{jk}(\phi)u$. In what follows we often write $Z^{jk}u$ and $Z^{jk}_{a,b}$ respectively for $Z^{jk}(\phi)u$ and $Z^{jk}_{a,b}(\phi)$.

6.1 Estimate of $\|Z^{jk}u\|_p$ for $(j,k) \neq (\nu,\nu)$.

We estimate Z^{jk} for the case $(j,k) \neq (\nu,\nu)$ first, postponing the case $(j,k) = (\nu,\nu)$ to the next subsection. As we shall see, the argument used for odd dimensions applies to Z^{jk} if $(j,k) \neq (\nu,\nu)$ modulo superpositions and scalings.

Lemma 6.2. *With suitable constants $C > 0$, followings are majorants of $Q^{a,b}_{jk}(\rho)$ for $0 \le k, j \le \nu$ which satisfy the attached conditions respectively:*

$$(1) \quad C\frac{\{\mathcal{MH}(r^{j+1}M^a)\}((1+2b)\rho)}{(1+2a)^{j+2}}, \quad \textit{if} \ \ j+k \ge 1. \tag{6.9}$$

$$(2) \quad C\frac{\mathcal{MH}(\widetilde{M}^a)((1+2b)\rho)}{(1+2a)^2}, \quad \textit{if} \ \ j=0. \tag{6.10}$$

$$(3) \quad C\sum_{l=0}^{k+1}\frac{\mathcal{MH}(r^{j+l+1}M^a)((1+2b)\rho)}{(1+2a)^{j+2}(1+2b)^{k+1}\rho^{k+1}}, \quad \textit{if} \ \ 2 \le j \le \nu. \tag{6.11}$$

$$(4) \quad C\frac{\mathcal{MH}(r^2M^a)((1+2b)\rho)}{(1+2a)^{j+2}}\{(1+2b)^{j-1}\rho^{j-1}+1\}, \quad \textit{if} \ \ 1 \le j. \tag{6.12}$$

$$(5) \quad C\frac{\mathcal{MH}(rM^a)((1+2b)\rho)}{(1+2a)^{j+2}}\{(2b+1)^j\rho^j+1\}, \quad \textit{for all} \ \ j,k. \tag{6.13}$$

Proof. Define $\Phi_{jk}(\lambda) = \lambda^{j+k-1}F(\lambda)$. If $j+k \ge 1$, $\Phi_{jk} \in C_0^\infty(\mathbb{R})$ and Lemma 2.9 implies $Q^{a,b}_{jk}(\rho) = (-1)^{j+1}(1+2a)^{-(j+2)}\{(\mathcal{F}\Phi_{jk}) * \mathcal{H}(r^{j+1}M^a)\}((1+2b)\rho)$. Then, (6.9) follows by applying (2.24). Likewise we have (6.10) from (6.5). If $j \ge 2$, we apply integration by parts $k+1$ times to (6.3) using that $e^{i\lambda(1+2b)\rho} = (i(1+2b)\rho)^{-(k+1)}\partial_\lambda^{k+1}e^{i\lambda(1+2b)\rho}$ then, without boundary terms,

$$\begin{aligned} Q^{a,b}_{jk}(\rho) &= \sum_{l=0}^{k+1}\frac{(-1)^{j+1}}{2\pi(1+2a)^{j+2}}\left(\frac{1}{-i(1+2b)\rho}\right)^{k+1}\binom{k+1}{l} \\ &\times \int_0^\infty e^{i\lambda(1+2b)\rho}\Phi_{jk}(\lambda)^{(k+1-l)}\mathcal{F}((-i)^l r^{j+l+1}M^a)(\lambda)d\lambda \end{aligned} \tag{6.14}$$

and (6.11) follows as previously. If $j \ge 1$, we may apply integration by parts to (6.3) by using that $\mathcal{F}(r^{j+1}M^a)(\lambda) = i^{j-1}\{\mathcal{F}(r^2M^a)(\lambda)\}^{(j-1)}$. Then

$$Q^{a,b}_{jk}(\rho) = i^{j-1}\int_0^\infty \frac{\left(\lambda^{j+k-1}F(\lambda)e^{i\lambda(1+2b)\rho}\right)^{(j-1)}\mathcal{F}(r^2M^a)(\lambda)}{2\pi(1+2a)^{j+2}}d\lambda \tag{6.15}$$

and (6.12) follows. Apply another integration by parts in (6.15). No boundary term appears as $\mathcal{F}(rM^a)(0) = 0$, and we obtain (6.13). ∎

6.1.1 Estimate for $1 < p < \frac{m}{m-1}$

Define for $0 \le \sigma \le m-1$ and $1 < p < \frac{m}{m-1}$:

$$N^{a,b}_\sigma(u) = \left(\int_0^\infty |\mathcal{MH}(r^\sigma M^a)((1+2b)\rho)|^p\rho^{m-1-p(m-1)}d\rho\right)^{1/p}. \tag{6.16}$$

Lemma 6.3. *For any* $\frac{m}{1+\sigma} \leq q \leq \infty$, *we have*

$$N_\sigma^{a,b} \leq C \frac{(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^{m-1-\frac{m}{p}-\sigma}} (\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p. \tag{6.17}$$

Proof. Change variable ρ by $(1+2b)^{-1}\rho$ first. Since $\rho^{m-1-p(m-1)}$ is an A_p-weight,

$$\begin{aligned} N_\sigma^{a,b} &= (1+2b)^{m-1-\frac{m}{p}} \left(\int_0^\infty |\mathcal{M}\mathcal{H}(r^\sigma M^a)(\rho)|^p \rho^{m-1-p(m-1)} d\rho \right)^{1/p} \\ &\leq C \frac{(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^{m-1-\frac{m}{p}-\sigma}} \left(\int_0^\infty |M(r)|^p r^{m-1-p(m-1-\sigma)} dr \right)^{1/p}. \end{aligned} \tag{6.18}$$

Denote by I the integral on (6.18). Let $\kappa = m-1-\sigma$. If $\kappa = 0$, then $I \leq C\|V\phi * u\|_p \leq C\|V\phi\|_1\|u\|_p$ and (6.17) follows. Let $0 < \kappa \leq m-1$. Split I into integral over $0 < r < 1$ and $r > 1$ and use $r^{m-1-p\kappa} \leq r^{m-1}$ for $r \geq 1$. Then, we have $I \leq C(\||x|^{-\kappa}(V\phi * u)(x)\|_{L^p(|x|<1)} + \|V\phi\|_1\|u\|_p)$. Take κ' such that $\kappa < \kappa' < m$ and apply Hölder's and Young's inequalities for the integral over $|x| \leq 1$. We obtain with $q = \frac{m}{m-\kappa'} \in \left[\frac{m}{1+\sigma}, \infty\right]$ that

$$I \leq C(\||x|^{-\kappa}\|_{L^{\frac{m}{\kappa'}}(|x|\leq 1)}\|V\phi\|_q + \|V\phi\|_1)\|u\|_p. \tag{6.19}$$

This completes the proof. □

Lemma 6.4. *Suppose* $1 < p < \frac{m}{m-1}$. *Then, for* $2 \leq j \leq \nu$ *and* $0 \leq k \leq \nu$ *such that* $(j,k) \neq (\nu,\nu)$,

$$\|Z^{jk}u\|_p \leq C\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^m). \tag{6.20}$$

Proof. Minkowski's and Young's inequality imply

$$\|Z^{jk}u\|_p \leq 2\omega_{m-1}^{-1}\|V\phi\|_1 \cdot T_j^{(a)} T_k^{(b)} \left[\left\| |x|^{2+k-m} Q_{jk}^{a,b} \right\|_p \right]. \tag{6.21}$$

We apply (6.11) to estimate $Q_{jk}^{a,b}(|x|)$. Then, since $\sigma \equiv j+l+1 \leq m-1$ for $(j,k) \neq (\nu,\nu)$, Lemma 6.3 implies

$$\left\| |x|^{2+k-m} Q_{jk}^{a,b} \right\|_p \leq C(1+2a)^{\frac{m}{p}-(m-k-1)}(1+2b)^{m-2-\frac{m}{p}-k}\|u\|_p. \tag{6.22}$$

We plug this to (6.21) and use $m-k-1 \geq j+2$. Then,

$$\begin{aligned} &\|Z^{jk}u\|_p \leq C_{mjk} T_j^{(a)} T_k^{(b)} [(1+2a)^{\frac{m}{p}-(j+2)}(1+2b)^{m-2-\frac{m}{p}-k}]\|u\|_p \\ &\leq C\|u\|_p \left(\int_0^\infty \frac{(1+2a)^{\frac{m}{p}-(j+2)}}{(1+a)^{(2\nu-j+\frac{1}{2})}} \frac{da}{\sqrt{a}} \right) \left(\int_0^\infty \frac{(1+2b)^{m-2-\frac{m}{p}-k}}{(1+b)^{(2\nu-k+\frac{1}{2})}} \frac{db}{\sqrt{b}} \right). \end{aligned}$$

Counting powers show that the integrals are finite and the lemma follows. □

As in odd dimensions we use the cancellation in

$$
\begin{aligned}
&Z^{0k}u + Z^{1k}u \\
&\quad = \frac{2i}{\omega_{m-1}} \int_{\mathbb{R}^m} \frac{(V\phi)(x-y)}{|y|^{m-2-k}} T_k^{(b)} (T_0^{(a)} Q_{0k}^{a,b}(|y|) + T_1^{(a)} Q_{1k}^{a,b}(|y|)) dy \qquad (6.23)
\end{aligned}
$$

and obtain the following lemma.

Lemma 6.5. *For $1 < p < \frac{m}{m-1}$, there exists a constant $C > 0$ such that*

$$
\|(Z^{(0,k)} + Z^{(1,k)})u\|_p \le C\|u\|_p, \quad k = 0, \dots, \nu. \qquad (6.24)
$$

Proof. We apply integration by parts $k+1$ times to (6.5) and (6.3) as in the proof of (6.11). This produces

$$
Q_{0k}^{a,b}(\rho) = \frac{-i^k k! (\mathcal{F}\widetilde{M^a})(0)\omega_{m-1}}{2\pi(1+2a)^2(1+2b)^{k+1}\rho^{k+1}} - \frac{i^k \omega_{m-1}}{2\pi} \sum_{l=0}^{k+1} C_{k+1,l} Q_{0k,l}^{a,b}(\rho), \qquad (6.25)
$$

$$
Q_{1k}^{a,b}(\rho) = \frac{i^{k+1} k! \mathcal{F}(r^2 M^a)(0)\omega_{m-1}}{2\pi(1+2a)^3(1+2b)^{k+1}\rho^{k+1}} + \frac{i^{k+1} \omega_{m-1}}{2\pi} \sum_{l=0}^{k+1} C_{k+1,l} Q_{1k,l}^{a,b}(\rho), \qquad (6.26)
$$

where $Q_{0k,l}^{a,b}(\rho)$ and $Q_{1k,l}^{a,b}(\rho)$ are given and estimated as follows:

$$
\begin{aligned}
Q_{0k,l}^{a,b}(\rho) &= \int_0^\infty \frac{e^{i\lambda(1+2b)\rho} (\lambda^k F(\lambda))^{(k+1-l)} (\mathcal{F}((-ir)^l \widetilde{M^a})(\lambda))}{(1+2a)^2(1+2b)^{k+1}\rho^{k+1}} d\lambda \\
&\le_{|\cdot|} C \frac{\mathcal{M}\mathcal{H}(r^l \widetilde{M^a})((1+2b)\rho))}{(1+2a)^2(1+2b)^{k+1}\rho^{k+1}}, \qquad (6.27) \\
Q_{1k,l}^{a,b}(\rho) &= (-i)^l \int_0^\infty \frac{e^{i\lambda(1+2b)\rho} (\lambda^k F(\lambda))^{(k+1-l)} \mathcal{F}(r^{2+l} M^a)(\lambda))}{(1+2a)^3(1+2b)^{k+1}\rho^{k+1}} d\lambda \\
&\le_{|\cdot|} C \frac{\mathcal{M}\mathcal{H}(r^{2+l} M^a)((1+2b)\rho)}{(1+2a)^3(1+2b)^{k+1}\rho^{k+1}}. \qquad (6.28)
\end{aligned}
$$

Eqn.(2.16) shows $\mathcal{F}(\widetilde{M^a})(0) = \mathcal{F}(r^2 M^a)(0) = (1+2a)^3 \int_0^\infty r^2 M(r) dr$ and

$$
T_1^{(a)}[i] = T_0^{(a)}[(1+2a)] = (m-3)^{-1}
$$

It follows that the sum of the superposition via $T_0^{(a)}$ of the boundary term of (6.25) and that via $T_1^{(a)}$ of (6.26) vanishes:

$$
\frac{i^k k!}{(1+2b)^{k+1}\rho^{k+1}} \left(\int_0^\infty r^2 M(r) dr \right) (T_1^{(a)}[i] - T_0^{(a)}[(1+2a)]) = 0. \qquad (6.29)
$$

For $1 < p < \frac{m}{m-1}$, $\rho^{m-1-p(m-1)}$ is an A_p weight on $\mathbb{R}$ and we have the identity:

$$
\widetilde{M^a}(r) = \int_r^\infty s M^a(s) ds = (1+2a)^2 \tilde{M}((1+2a)^{-1} r). \qquad (6.30)
$$

Then, Lemma 2.7, (6.30), change of variable and Hardy's inequality imply

$$\left\| \frac{Q_{0k,l}^{a,b}(|x|)}{|x|^{m-k-2}} \right\|_p \leq \frac{C(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^2(1+2b)^{k+1}} \left(\int_0^\infty |r^l \widetilde{M^a}(r)|^p r^{m-1-p(m-1)} dr \right)^{1/p}$$
$$\leq \frac{C(1+2a)^{\frac{m}{p}-(m-1-l)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}} \left(\int_0^\infty |M(r)|^p r^{m-1-p(m-3-l)} dr \right)^{1/p}. \tag{6.31}$$

The integral is similar to the integral which appeared in (6.18) and we remark $m-3-l \geq 0$ for $m \geq 6$. Thus, applying (6.19) with $\sigma = l+2$, we obtain

$$(6.31) \leq \frac{C(1+2a)^{\frac{m}{p}-(m-k-2)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}} (\|V\phi\|_1 + \|V\phi\|_{\frac{m}{3}}) \|u\|_p, \ 0 \leq l \leq k+1. \tag{6.32}$$

Counting the powers of a and b, we thus have from (6.32) that

$$T_0^{(a)} T_k^{(b)} \left[\left\| |x|^{2+k-m} Q_{0k,l}^{a,b} \right\|_p \right] \leq C\|u\|_p. \quad 0 \leq l \leq k+1. \tag{6.33}$$

Entirely similarly, starting from (6.28), we obtain

$$\left\| \frac{Q_{1k,l}^{a,b}(|x|)}{|x|^{m-k-2}} \right\|_p \leq \frac{C(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^3(1+2b)^{k+1}} \left(\int_0^\infty |r^{2+l} M^a(r)|^p r^{m-1-p(m-1)} dr \right)^{1/p}$$
$$\leq \frac{C(1+2a)^{\frac{m}{p}-(m-k-1)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}} (\|V\phi\|_1 + \|V\phi\|_{\frac{m}{3}}) \|u\|_p, \ 0 \leq l \leq k+1. \tag{6.34}$$

The extra decaying factor $(1+2a)^{-1}$ of (6.34) compared to (6.32) cancels the extra increasing factor $(1+a)$ of $T_1^{(a)}$ compared to $T_0^{(a)}$ and we have

$$T_1^{(a)} T_k^{(b)} \left[\left\| |x|^{k+2-m} Q_{0k,l}^{a,b}(|x|) \right\|_p \right] \leq C\|u\|_p, \quad 0 \leq l \leq k+1. \tag{6.35}$$

In view of (6.23), (6.25), (6.26) and (6.29), (6.33) and (6.35) with the help of Young's and Minkowski's inequalities imply the lemma. □

6.1.2 Estimate for $\frac{m}{3} < p < \frac{m}{2}$

The following lemma together with Lemma 6.4 and Lemma 6.5 will prove that $\sum_{(j,k)\neq(\nu,\nu)} Z^{jk}$ is bounded in $L^p(\mathbb{R}^m)$ for $1 < p < \frac{m}{2}$.

Lemma 6.6. *Let $\frac{m}{3} < p < \frac{m}{2}$. Then, for $(j,k) \neq (\nu,\nu)$,*

$$\|Z^{jk} u\|_p \leq C_p \|u\|_p \tag{6.36}$$

for a constant $C_p > 0$ independent of $u \in C_0^\infty(\mathbb{R}^m)$.

Proof. Except the superposition the proof is virtually the repetition of that of statement (2) of Lemma 5.3.
(1) Let $j \geq 1$ first. Since ρ^{m-1-2p} is A_p weight for $\frac{m}{3} < p < \frac{m}{2}$, we have

$$\left(\int_0^\infty |\{\mathcal{M}\mathcal{H}(r^2 M^a)\}(\rho)|^p \rho^{m-1-2p} d\rho\right)^{\frac{1}{p}} \leq C(1+2a)^{\frac{m}{p}} \|V\phi\|_1 \|u\|_p. \quad (6.37)$$

Splitting the integral of (6.8) we define

$$Z_{jk}^{a,b} u(x) = \left(\int_{|y|<\frac{1}{1+2b}} + \int_{|y|>\frac{1}{1+2b}}\right) \frac{(V\phi)(x-y) Q_{jk}^{a,b}(|y|)}{|y|^{m-2-k}} dy = I_1(x) + I_2(x). \quad (6.38)$$

For $I_1(x)$, we estimate $|y|^{-(m-k-2)} \leq |y|^{-(m-2)}$ for $|y| \leq 1$ and apply Hölder's inequality. Then

$$\|I_1\|_p \leq \left\|\int_{|y|\leq\frac{1}{2b+1}} \frac{|(V\phi)(x-y)|^{p'} dy}{|y|^{p'(m-4)}}\right\|_{p/p'}^{1/p'} \left(\int_{|y|\leq\frac{1}{2b+1}} \left|\frac{Q_{jk}^{a,b}(|y|)}{|y|^2}\right|^p dy\right)^{1/p}$$

Minkowski's inequality implies that the first factor on the right is bounded by $C\|V\phi\|_p(1+2b)^{m-4-\frac{m}{p'}}$ and $\frac{m}{p'} - (m-4) > 1$. For the second factor, we apply (6.12) for $(1+2b)\rho < 1$ and then (6.37). We obtain

$$\|I_1\|_p \leq C(1+2a)^{\frac{m}{p}-j-2}(1+2b)^{1-\frac{m}{p}} \|V\phi\|_1 \|V\phi\|_p \|u\|_p. \quad (6.39)$$

By Young's inequality $\|I_2\|_p \leq C\|V\phi\|_1 \||x|^{2+k-m} Q_{jk}^{a,b}(|x|)\|_{L^p((1+2b)|x|>1)}$. For the second factor, we use (6.12) for $(1+2b)\rho \geq 1$ and, after changing the variables $\rho \to (1+2b)^{-1}\rho$, we estimate $\rho^{-(m-2-k-(j-1))} \leq \rho^{-2}$ for $\rho \geq 1$ (here we used $(j,k) \neq (\nu,\nu)$) and apply (6.37) once more. Then,

$$\begin{aligned}\|I_2\|_p &\leq C\|V\phi\|_1 \frac{(1+2b)^{m-2-k-\frac{m}{p}}}{(1+2a)^{j+2}} \left(\int_1^\infty |\{\mathcal{M}\mathcal{H}(r^2 M^a)\}(\rho)|^p \rho^{m-1-2p} d\rho\right)^{\frac{1}{p}} \\ &\leq C(1+2a)^{\frac{m}{p}-j-2}(1+2b)^{m-2-k-\frac{m}{p}} \|V\phi\|_1^2 \|u\|_p. \quad (6.40)\end{aligned}$$

Since $m-2-k \geq 1$ and $(1+2a)^{\frac{m}{p}-j-2}(1+2b)^{m-2-k-\frac{m}{p}}$ is summable by $T_j^{(a)} T_k^{(b)}$, (6.39) and (6.40) imply

$$\|Z^{jk}u\|_p \leq C\|V\phi\|_1(\|V\phi\|_1 + \|V\phi\|_p)\|u\|_p. \quad (6.41)$$

(2) When $j = 0$, we apply the argument in the proof in (1) for estimating $Q_{0k}^{a,b}$ but by using (6.10) in stead of (6.12). Then, by the help of (6.30) and Hardy's inequality, it leads to estimates (6.39) and (6.40) and, hence, to the desired (6.36) for Z^{0k}. This completes the proof of the lemma. ∎

6.1.3 Estimate for $m/2 < p < m$ and for $p > m$.

We now estimate Z^{jk}, $(j,k) \neq (\nu,\nu)$, in $L^p(\mathbb{R}^m)$ for $\frac{m}{2} < p < m$ and for $p > m$. As in odd dimensions, Z^{00} will not in general be bounded in $L^p(\mathbb{R}^m)$ when $\frac{m}{2} < p < m$ and likewise for all Z^{0k}, $k = 0, \dots, \frac{m-2}{2}$ when $p > m$. Elementary computations using

$$z^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-zt} t^{k-1} dt, \quad \Re z > 0, \quad k > 0$$

and the formula (2.5) for $C_{m,j}\omega_{m-1}$ we obtain the following lemma.

Lemma 6.7.

(1) *We have* $T_j^{(a)}[1] = (m-3-j)!/(m-2)!$.

(2) *For* $k \geq 1$ *and* $j = 0, \cdots, \nu$, $T_j^{(a)}[(1+2a)^{-k}]$ *is given by*

$$\frac{(-i)^j 2^{m-1} \Gamma(2\nu - j + k)}{(m-2)!\Gamma(k)} \binom{\nu}{j} \cdot \int_1^\infty \frac{(x^2-1)^{k-1}}{(x^2+1)^{2\nu-j+k}} dx \tag{6.42}$$

Lemma 6.8. *Let* $\frac{m}{2} < p < m$ *and* $\phi \in \mathcal{E}$. *Then:*

(1) *If* $(j,k) \neq (0,0)$ *or* $(j,k) \neq (\nu,\nu)$, Z^{jk} *is bounded in* $L^p(\mathbb{R}^m)$:

$$\|Z^{jk}u\|_p \leq C\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^m) \tag{6.43}$$

(2) *There exists a constant* $C > 0$ *such that for* $u \in C_0^\infty(\mathbb{R}^m)$, *we have*

$$\left\| Z^{00}u + D_m|\phi\rangle\langle\phi, u\rangle \right\|_p \leq C\|u\|_p, \tag{6.44}$$

$$D_m = \frac{2^m \Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)} \int_1^\infty (1+x^2)^{m-1} dx. \tag{6.45}$$

If $Z^{00}(\phi)$ *is bounded in* $L^p(\mathbb{R}^m)$ *for some* $\frac{m}{2} < p < m$ *then* $\phi \in \mathcal{E}_0$. *In this case* $Z^{00}(\phi)$ *is bounded in* $L^p(\mathbb{R}^m)$ *for all* $\frac{m}{2} < p < m$.

Proof. (1) Split $Z_{a,b}^{jk}u(x)$ as in (6.38) and apply the argument thereafter to $I_1(x)$ and $I_2(x)$ by using the estimate (6.13). Since $m-2-(k+j) \geq 1$ and ρ^{m-1-p} is an A_p weight for $\frac{m}{2} < p < m$, we have, as in (6.40),

$$\|I_2\|_p \leq C \frac{(1+2b)^{m-2-k-\frac{m}{p}}}{(1+2a)^{j+2-\frac{m}{p}}} \|V\phi\|_1^2 \|u\|_p. \tag{6.46}$$

For dealing with $I_1(x)$, we estimate $|y|^{-(m-2-k)} \leq |y|^{m-2}$ for $|y| \leq 1$ as previously but now decompose $|y|^{-(m-2)} = |y|^{-(m-3)} \cdot |y|^{-1}$, remarking that $(m-3)p' < m$ and $p/p' > 1$. Then, we obtain as in (6.39) that

$$\|I_1\|_p \leq \frac{(1+2a)^{\frac{m}{p}-j-2}}{(1+2b)^{\frac{m}{p}}} \|V\phi\|_1 \|V\phi\|_p \|u\|_p. \tag{6.47}$$

Summing up (6.46) and (6.47) by $T_j^{(a)}T_k^{(b)}$, we obtain (6.43).
(2) Let $j=k=0$. We apply integration by parts to (6.4).

$$
\begin{aligned}
Q_{00}^{a,b}(\rho) &= \frac{-i}{2\pi(1+2a)^2}\int_0^\infty e^{i\lambda(1+2b)\rho}\mathcal{F}(M_*^a)'(\lambda)F(\lambda)d\lambda \\
&= \frac{i}{2\pi}\int_{\mathbb{R}}\frac{M_*^a(r)}{(1+2a)^2}dr+\frac{i}{(1+2a)^2}\int_0^\infty (F(\lambda)e^{i\lambda(1+2b)\rho})'\mathcal{F}(M_*^a)(\lambda)d\lambda. \qquad (6.48)
\end{aligned}
$$

Denote the second term on (6.48) by $\tilde{Q}_{00}^{ab}(\rho)$ and by $\tilde{Z}^{00}$ the operator produced by inserting $\tilde{Q}_{00}^{ab}(\rho)$ for $Q_{00}^{ab}(\rho)$ in (6.7). We have

$$
\tilde{Q}_{00}^{a,b}(\rho)\le_{|\,\cdot\,|} C\frac{\mathcal{MH}(M_*^a)((1+2b)\rho)}{(1+2a)^2}(1+(1+2b)\rho). \qquad (6.49)
$$

Let $\frac{m}{2}<p<m$. We split as in (6.38) and estimate I_2 first:

$$
\tilde{Z}^{00}u(x)=\left(\int_{|y|<\frac{1}{1+2b}}+\int_{|y|\ge\frac{1}{1+2b}}\right)\frac{(V\phi)(x-y)\tilde{Q}_{00}^{a,b}(|y|)}{|y|^{m-2}}dy=I_1(x)+I_2(x).
$$

We obtain

$$
\begin{aligned}
\|I_2\|_p &\le C\|V\phi\|_1\frac{(1+2b)^{m-2-\frac{m}{p}}}{(1+2a)^2}\left\|\frac{\mathcal{MH}(M_*^a)(|y|)}{|y|^{m-3}}\right\|_{L^p(|y|>1)} \\
&\le C\|V\phi\|_1\frac{(1+2b)^{m-2-\frac{m}{p}}}{(1+2a)^2}\left\|\frac{1}{|y|^{m-3}}\right\|_{L^m(|y|>1)}\left(\int_0^\infty |M_*^a(r)|^q r^{m-1}dr\right)^{\frac{1}{q}} \\
&\le C\frac{\|V\phi\|_1(1+2b)^{m-2-\frac{m}{p}}}{(1+2a)^{2-\frac{m}{q}}}\||D|^{-1}(V\phi)\|_{\frac{m}{m-1},\infty}\|u\|_p, \qquad (6.50)
\end{aligned}
$$

where we used Young's inequality, (6.49) for $(1+2b)\rho\ge 1$ and the change of variable $(1+2b)\rho$ to ρ in the first step, Hölder's inequality considering $p^{-1}=m^{-1}+q^{-1}$ and that 1 is an A_q weight $q=mp/(m-p)>m$ in the second and finally weak-Young's inequality. For I_1, we apply Hölder's and Minkowski's inequalities and (6.50) and obtain

$$
\begin{aligned}
\|I_1\|_p &\le C\left\|\left(\int_{|y|\le\frac{1}{1+2b}}\left|\frac{(V\phi)(x-y)}{|y|^{m-2}}\right|^{q'}dy\right)^{\frac{1}{q'}}\right\|_p \\
&\quad\times(1+2b)^{-\frac{m}{p}}(1+2a)^{-2}\left(\int_{|y|\le 1}|\mathcal{MH}(M_*^a)(|y|)|^q dy\right)^{1/q} \\
&\le C(1+2b)^{-\frac{m}{p}}(1+2a)^{\frac{m}{p}-2}\|V\phi\|_p\||D|^{-1}(V\phi)\|_{\frac{m}{m-1},\infty}\|u\|_p. \qquad (6.51)
\end{aligned}
$$

Summing (6.50) and (6.51) by $T_0^{(a)}T_0^{(b)}$, we obtain $\|\tilde{Z}^{(0,0)}u\|_p\le C\|u\|_p$.

By virtue of (3.10) and (5.43), the contribution to $Z^{00}u$ of the boundary term of (6.48) is given by

$$\frac{2i}{\omega_{m-1}}T_0^{(a)}T_0^{(b)}\left[\int_{\mathbb{R}^m}\frac{(V\phi)(y)dy}{|x-y|^{m-2}}\cdot\frac{i}{2\pi}\int_{\mathbb{R}}\frac{M_*(r)}{(1+2a)}dr\right]$$
$$=-\frac{2}{C_0\omega_{m-1}}T_0^{(a)}[(1+2a)^{-1}]T_0^{(b)}[1]\frac{\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)}\langle\phi,u\rangle\phi. \tag{6.52}$$

By using Lemma 6.7 and $C_0\omega_{m-1}=(m-2)^{-1}$. we can simplify (6.52) to $-D_m\langle\phi,u\rangle\phi$ with D_m given by (6.45) and (6.44) follows. The last statement follows as in the odd dimensional case, see the remark after Lemma 5.5. □

Finally in this section we study $Z^{jk}(\phi)u$ for $(j,k)\neq(\nu,\nu)$ in $L^p(\mathbb{R}^m)$ when $p>m$, assuming $\phi\in\mathcal{E}_0$ by the same reason as in odd dimensions. We define

$$D_{m,j}=2^m\binom{\nu}{j}\frac{\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m-1}{2}\right)}\int_1^\infty\frac{(x^2-1)^j}{(x^2+1)^{m-1}}dx,\quad j=0,\dots,\nu. \tag{6.53}$$

Lemma 6.9. *Let $m\geq 6$ be even and $p>m$. Suppose that $\phi\in\mathcal{E}_0$. Then:*

(1) *For (j,k) such that $k\neq 0$ and $(j,k)\neq(\nu,\nu)$, Z^{jk} is bounded in $L^p(\mathbb{R}^m)$.*

(2) *There exists a constant $C>0$ such that*

$$\|Z^{j0}u+D_{j,m}\langle\phi,u\rangle\phi\|_p\leq C\|u\|_p,\quad j=0,\dots,\nu. \tag{6.54}$$

(3) *If $Z^{j0}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for some $0\leq j\leq\nu$ and some $m<p<\infty$, then $\phi\in\mathcal{E}_1$. In this case, $Z^{j0}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for all $1<p<\infty$ and $0\leq j\leq\nu$.*

Proof. We apply integration by parts $j+1$ times to (6.4):

$$Q_{jk}^{a,b}(\rho)=\int_0^\infty\frac{(-i)^{j+1}\lambda^{j+k}F(\lambda)e^{i\lambda(1+2b)\rho}\partial_\lambda^{j+1}\{\mathcal{F}(M_*^a)(\lambda)\}}{2\pi(1+2a)^{j+2}}d\lambda. \tag{6.55}$$

(1) If $k\geq 1$, then no boundary terms appear and we have

$$Q_{jk}^{a,b}(\rho)\leq_{|\cdot|}\frac{C\mathcal{MH}(M_*^a)((1+2b)\rho)}{(1+2a)^{j+2}}\{(1+2b)^{j+1}\rho^{j+1}+1\}. \tag{6.56}$$

Observing that $m-2-(k+j+1)\geq 0$ for $(j,k)\neq(\nu,\nu)$, that r^{m-1} is A_p weight on $\mathbb{R}$ for $p>m$ and that $(m-2-k)p'<m$, we apply the argument used for proving (6.50) and (6.51) in the proof of the previous lemma and obtain

$$\|Z_{a,b}^{jk}u\|_p\leq\frac{C\|V\phi\|_1(1+2b)^{m-2-k-\frac{m}{p}}}{(1+2a)^{j+2-\frac{m}{p}}}\||D|^{-1}(V\phi)*u\|_p$$
$$+\frac{C(1+2b)^{-\frac{m}{p}}\|V\phi\|_p}{(1+2a)^{j+2-\frac{m}{p}}}\left(\int_{|y|<\frac{1}{1+2b}}\frac{dy}{|y|^{(m-2-k)p'}}\right)^{\frac{1}{p'}}\||D|^{-1}(V\phi)*u\|_p. \tag{6.57}$$

Since $\int (V\phi)(x)dx = 0$, $\||D|^{-1}(V\phi)*u\|_p \le C\|u\|_p$ for any $1<p<\infty$ by virtue of (3.9) and the Calderón-Zygmund theory. It follows that

$$\|Z^{jk}u\|_p \le T_j^{(a)}T_k^{(b)}\|Z_{a,b}^{jk}u\|_p \le C\|u\|_p$$

for $k \ge 1$ and $(j,k) \ne (\nu,\nu)$ and statement (1) is proved.
(2) If $k=0$, then, $j+1$ times integration by parts in (6.55) produces the integral and boundary terms. The integral is bounded by (6.56) and the repetition of the argument in step (1) implies that its contribution to Z^{j0} is the operator which is bounded in $L^p(\mathbb{R}^m)$ for all $p>m$. The boundary term may be expressed as follows by using (5.43) once more,

$$\frac{i^{j+1}j!}{2\pi(1+2a)^{j+1}}\int_{\mathbb{R}} M_*(r)dr = \frac{-i^{j+1}j!}{(1+2a)^{j+1}}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\sqrt{\pi}}\langle\phi,u\rangle, \tag{6.58}$$

and its contribution $Z^{j0}u$ may be computed as follows:

$$\begin{aligned}&\frac{2i}{\omega_{m-1}}T_j^{(a)}T_0^{(b)}\left[\int_{\mathbb{R}^m}\frac{(V\phi)(y)dy}{|x-y|^{m-2}}\left(-\frac{i^{j+1}j!}{(1+2a)^{j+1}}\right)\right]\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\sqrt{\pi}}\langle\phi,u\rangle \\ &\qquad = 2i^{j+2}j!T_j^{(a)}[(1+2a)^{-(j+1)}]\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\sqrt{\pi}}\langle\phi,u\rangle\phi = -D_{m,j}\langle\phi,u\rangle\phi.\end{aligned}$$

where we used $C_0\omega_{m-1} = T_0^{(b)}[1] = (m-2)^{-1}$ and (6.42) with $k=j+1$ for $T_j^{(a)}[(1+2a)^{-(j+1)}]$. This proves statement (2). We omit the proof of statement (3) which is similar to the corresponding part of the previous lemma. □

Lemma 6.10. *Define* $\tilde{D}_m = \sum_{j=0}^{\nu} D_{m,j}$. *Then,* $\tilde{D}_m = 1$.

Proof. Use binomial formula for (6.53). We have

$$\tilde{D}_m = 2^m\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\sqrt{\pi}}\int_1^\infty \frac{x^{m-2}}{(x^2+1)^{m-1}}dx$$

Change of variable $x \to x^{-1}$ shows that the integral is equal to the same integral over the interval $0<x<1$. It follows after making the change of variable $x^2=t$ that the integral is equal to

$$\frac{1}{4}\int_0^\infty \frac{t^{\nu-\frac{1}{2}}}{(t+1)^{m-1}}dt = \frac{\Gamma\left(\frac{m-1}{2}\right)^2}{2^2\Gamma(m-1)}.$$

Thus, $\tilde{D}_m = 2^{m-2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)\Gamma(m-1)^{-1}\pi^{-\frac{1}{2}} = 1$. □

In the next two sections we prove that $Z^{\nu\nu}$ and $Z_{\log}$ are bounded in $L^p(\mathbb{R}^m)$ for all $1<p<\infty$. These will complete the proof of Theorem 1.5.

6.2 Estimate of $\|Z^{\nu\nu}u\|_p$ for $1<p<\infty$

In this section we prove

$$\|Z^{\nu\nu}u\|_p \le C\|u\|_p, \quad 1<p<\infty. \tag{6.59}$$

The method of previous subsection does not apply for proving this and we exploit more direct method. By virtue of interpolation, it suffices to prove (6.59) for arbitrarily small $p>1$ and large $p>m$.

6.2.1 The case for $1<p<\frac{2(m-1)}{m+1}$

We first show (6.59) for $1<p<\frac{2(m-1)}{m+1}$. After changing the variable r to $(1+2a)r$ in (6.3), we write $Q^{a,b}_{\nu\nu}(\rho)/\rho^\nu$ in the form

$$\frac{(-1)^{\nu+1}}{2\pi\rho^\nu}\int_0^\infty e^{i(1+2b)\rho\lambda}\lambda^{m-3}F(\lambda)\left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda}r^{\nu+1}M(r)dr\right)d\lambda. \tag{6.60}$$

Integration by parts implies that (6.60) is equal to

$$\begin{aligned}&\frac{i(-1)^{\nu+1}}{2\pi(1+2b)\rho^{\nu+1}}\int_0^\infty e^{i(1+2b)\rho\lambda}(\lambda^{m-3}F(\lambda))'\left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda}r^{\nu+1}M(r)dr\right)d\lambda\\ &+\frac{(-1)^{\nu+1}(1+2a)}{2\pi(1+2b)\rho^{\nu+1}}\int_0^\infty e^{i(1+2b)\rho\lambda}\lambda^{m-3}F(\lambda)\left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda}r^{\nu+2}Mdr\right)d\lambda.\end{aligned}$$

The first line becomes $i(1+2b)^{-1}Q^{a,b}_{\nu(\nu-1)}(\rho)/\rho^{m-2-(\nu-1)}$ if we replace $(m-3)F(\lambda)+\lambda F'(\lambda)$ by $F(\lambda)$ and the former function can play the same role as the latter does in the argument of previous sections and, $\nu-1\ge 1$ if $m\ge 6$. Thus, if we substutute it for $Q^{a,b}_{\nu\nu}(\rho)/\rho^\nu$ in (6.8) for $(j,k)=(\nu,\nu)$ and, then the resulting function for $Z^{\nu\nu}_{a,b}(\phi)u(x)$ in (6.7), it produces the operator which has the same L^p property as $Z^{\nu(\nu-1)}$ which is bounded in $L^p(\mathbb{R}^m)$ for $1<p<\infty$. Hence, we need study only the operator produced by the second line. Once again we substitute it for $Q^{a,b}_{\nu\nu}(\rho)/\rho^\nu$ in (6.8) and the result for $Z^{\nu\nu}_{a,b}(\phi)u(x)$ in (6.7). We denote the functiotn thus obtain by $Z^{\nu\nu}u(x)$, abusing notation. We want to show that this $Z^{\nu\nu}u(x)$ satisfies (6.59) for $1<p<\frac{m}{m-1}$. Integrating with respect to a,b first via Fubini's theorem shows

$$Z^{\nu\nu}u(x)=\frac{2i}{\omega_{m-1}}\int_{\mathbb{R}^m}(V\phi)(x-y)X_\nu(|y|)dy, \tag{6.61}$$

$$\begin{aligned}X_\nu(\rho)=&\frac{2iC^2_{m,\nu}\omega_{m-1}}{\rho^{\nu+1}}\int_0^\infty\left\{e^{i\lambda\rho}\lambda^{m-3}\left(\int_0^\infty\frac{(1+2b)^{-1}e^{2i\lambda\rho b}}{(1+b)^{\nu+\frac12}}\frac{db}{\sqrt{b}}\right)\right.\\ &\left.\times\int_{\mathbb{R}} e^{-i\lambda r}\left(\int_0^\infty\frac{(1+2a)e^{-2ia\lambda r}}{(1+a)^{\nu+\frac12}}\frac{da}{\sqrt{a}}\right)r^{\nu+2}M(r)dr\right\}F(\lambda)d\lambda.\end{aligned} \tag{6.62}$$

Let $\chi_\pm(r) = 1$ for $\pm r > 0$ and $\chi_\pm(r) = 0$ for $\pm r \le 0$. Define, for $t > 0$,

$$g_\pm(t) = \int_0^\infty \left(1 + \frac{a}{t}\right)\left(1 + \frac{a}{2t}\right)^{-\nu-\frac12} e^{\pm ia}\frac{da}{\sqrt{a}}, \tag{6.63}$$

$$h_\pm(t) = \int_0^\infty \left(1 + \frac{b}{t}\right)^{-1}\left(1 + \frac{b}{2t}\right)^{-\nu-\frac12} e^{\pm ib}\frac{db}{\sqrt{b}} \tag{6.64}$$

and, with $C = iC_{m,\nu}^2\omega_{m-1}$, write $X_\nu(\rho)$ as follows:

$$X_\nu(\rho) = \frac{C}{\rho^{\nu+\frac32}}\int_{\mathbb{R}} (L_+(\rho, r) + L_-(\rho, r))r^{\nu+2}|r|^{-\frac12}M(r)dr, \tag{6.65}$$

$$L_\pm(\rho, r) = \chi_\pm(r)\int_0^\infty e^{i\lambda(\rho-r)}\lambda^{m-4}h_+(\lambda\rho)g_\mp(\pm r\lambda)F(\lambda)d\lambda. \tag{6.66}$$

LEMMA 6.11. *Suppose that f is C^∞ on $[0,\infty)$ and satisfies $|f^{(j)}(c)| \le C_j c^{-(j+1)}$ for $c \ge 1$, $j = 0, 1, \dots$. Define*

$$\ell_\pm(t) = \int_0^\infty e^{\pm ic} f(c/t)\frac{dc}{\sqrt{c}}.$$

Then, $\ell_\pm(t)$ is C^∞ for $t > 0$ and satisfies the following properties.

(1) *$\ell_\pm(1/t)$ can be exteded to a C^∞ function on $[0, 1]$, hence, $\lim_{t\to\infty} \ell_\pm(t) = \alpha_\pm$ exists and for $t \ge 1$, $|\ell_\pm^{(j)}(t)| \le C_j t^{-j-1}$, $j = 1, 2, \dots$.*

(2) *For $0 < t < 1$, $|t^j \ell_\pm^{(j)}(t)| \le C_j\sqrt{t} \le C_j$, $j = 0, 1, \dots$.*

Proof. We prove the lemma for $\ell_+(t)$ only and omit the $+$-sign. It is evident that $\ell(t)$ is C^∞ for $t > 0$. Splitting the interval, we define

$$\ell(t) = \left(\int_0^1 + \int_1^\infty\right) f\left(\frac{c}{t}\right) e^{ic}\frac{dc}{\sqrt{c}} \equiv \ell_1(t) + \ell_2(t).$$

It is obvious that $\ell_1(1/t)$ is of $C^\infty[0, 1]$. To see the same for $\ell_2(1/t)$, we perform integration by parts n times for $t > 0$:

$$i^n\ell_2(1/t) = B_n(t) + (-1)^n\int_1^\infty \partial_c^n\left(\frac{f(ct)}{\sqrt{c}}\right)e^{ic}dc. \tag{6.67}$$

The boundary term $B_n(t)$ is a polynomial of order n and Leibniz' formula implies $\partial_c^n\left(\frac{f(ct)}{\sqrt{c}}\right) = \sum_{j=0}^n C_{nj}f^{(j)}(ct)(ct)^j c^{-\frac12-n}$. Since $\partial_y^k(f^{(j)}(y)y^j)$ is bounded for any $j, k = 0, 1, \dots$ and

$$\partial_t^k\left(\sum_{j=0}^n C_{nj}f^{(j)}(ct)(ct)^j c^{-\frac12-n}\right) = \sum_{j=0}^n C_{nj}\ \partial_y^k(f^{(j)}(y)y^j)\Big|_{y=ct} c^{-\frac12-n+k},$$

the integral of (6.67) is a function of class $C^{n-1}([0,1])$. Since n is arbitray, this proves (1). For proving (2), after changing the variable we decompose:

$$\ell(t) = \sqrt{t}\left(\int_0^1 + \int_1^\infty\right) f(c)e^{ict}\frac{dc}{\sqrt{c}} \equiv \sqrt{t}(\tilde{\ell}_1(t) + \tilde{\ell}_2(t))$$

We obseve that $\sqrt{t}$ satisfies the property (2) and that, if $\alpha(t)$ satisfies (2) and $|t^j\beta^{(j)}(t)| \le C_j$, then so does $\gamma(t) = \alpha(t)\beta(t)$. Hence, $\sqrt{t}\tilde{\ell}_1(t)$ satisfies (2) because $\tilde{\ell}_1(t)$ is entire. To prove the same for $\sqrt{t}(\tilde{\ell}_2(t)$, it suffices to show that $|(t^n\tilde{\ell}_2(t))^{(n)}| \le C_n$ for $0 < t < 1$, $n = 0,1,2,\ldots$. By integration by parts we have

$$\begin{aligned}(it)^n\tilde{\ell}_2(t) &= \int_1^\infty (\partial_c^n e^{itc}) f(c)\frac{dc}{\sqrt{c}} \\ &= \sum_{j=0}^{n-1}(-1)^{j+1}\partial_c^j\left(\frac{f(c)}{\sqrt{c}}\right)\partial_c^{n-j-1}(e^{itc})\big|_{c=1} + \int_1^\infty e^{itc}(f(c)c^{-\frac12})^{(n)}dc.\end{aligned}$$

The boundary term is a polynomial of t and the integral is n times continuously differentiable and a fortiori $(t^n\tilde{\ell}_2(t))^{(n)} \le C$ for $0 < t < 1$. □

We define $L_{\pm,\sigma}(\rho,r)$ for an integer $\sigma \ge 0$ and functions $g_\pm$ and h by

$$L_{\pm,\sigma}(\rho,r) = \chi_\pm(r)\int_0^\infty e^{i\lambda(\rho-r)}\lambda^\sigma h_+(\lambda\rho)g_\mp(\pm r\lambda)F(\lambda)d\lambda \tag{6.68}$$

so that we have $L_\pm(\rho,r) = L_{\pm,m-4}(\rho,r)$ (see (6.66)).

Lemma 6.12. *Suppose that $g_\pm(t)$ and $h_+(t)$ are C^∞ functions of $t > 0$ and they satisfy following properties replacing f:*

(a) *The limit* $\lim_{t\to\infty} f(t)$ *exists.*

(b) $|t^j f^{(j)}(t)| \le C_j\begin{cases} t^{-1}, & 1 < t, \quad j = 1,2,\ldots, \\ \sqrt{t}, & 0 < t < 1, \quad j = 0,1,\ldots. \end{cases}$.

Then, $L_{\pm,\sigma}$ is C^∞ with respect to $\rho > 0$ and $r > 0$ and, for a constant $C > 0$,

$$|L_{\pm,\sigma}(\rho,r)| \le C\langle\rho - r\rangle^{-(\sigma+1)} \tag{6.69}$$

Proof. We prove the lemma for $L_{+,\sigma}$. The proof for $L_{-,\sigma}$ is similar. It is obvious that $L_{+,\sigma}(\rho,r)$ is smooth and is bounded for $\rho, r > 0$ and, it suffices to prove (6.69) for $|\rho - r| \ge 1$. We apply integration by parts $\sigma + 1$ times to

$$L_{+,\sigma}(\rho,r) = \frac{(-i)^{\sigma+1}}{(\rho-r)^{\sigma+1}}\int_0^\infty \left(\partial_\lambda^{\sigma+1}e^{i\lambda(\rho-r)}\right)\lambda^\sigma h_+(\lambda\rho)g_-(r\lambda)F(\lambda)d\lambda.$$

By Leibniz' rule, derivatives $(\lambda^\sigma h_+(\lambda\rho)g_-(r\lambda)F(\lambda))^{(\kappa)}$ are linear combinations over indices (β,γ,δ) such that $\kappa - \sigma \le \beta + \gamma + \delta \le \kappa$ of

$$\lambda^{\sigma-\kappa+\delta}(\lambda\rho)^\beta h^{(\beta)}(\lambda\rho)(r\lambda)^\gamma g_-^{(\gamma)}(r\lambda)F^{(\delta)}(\lambda) \tag{6.70}$$

and they converge to 0 as $\lambda \to 0$ if $\kappa \le \sigma$. It follows that $(\rho-r)^{\sigma+1}L_{+,\sigma}(\rho,r)$ is the linear combination over the same set of (β,γ,δ) as above but with $\kappa=\sigma+1$ of

$$I_{\beta\gamma\delta}(\rho,r)=\int_0^\infty e^{i(\rho-r)\lambda}\lambda^{\delta-1}(\lambda\rho)^\beta h^{(\beta)}(\lambda\rho)(r\lambda)^\gamma g_-^{(\gamma)}(r\lambda)F^{(\delta)}(\lambda)d\lambda.$$

It suffices to show that $I_{\beta\gamma\delta}(\rho,r)$ is bounded. If $\delta\neq 0$, $F^{(\delta)}(\lambda)=0$ outside $0<c_0<\lambda<c_1<\infty$ and it is clear that $I_{\beta\gamma\delta}(\rho,r)\le_{|\,\cdot\,|}C$. Thus, we assume $\delta=0$ in what follows. We may also assume $0<r<\rho<\infty$ by symmetry. We split the interval of integration as $(0,\infty)=(0,1/\rho)\cup[1/\rho,1/r]\cup(1/r,\infty)$ and denote integrals over these intervals by I_1, I_2 and I_3 in this order so that $I_{\beta\gamma\delta}(\rho,r)=I_1+I_2+I_3$.
(1) If $0<\lambda<1/\rho$ then $0<r\lambda<\rho\lambda<1$ and $(\rho\lambda)^\beta h^{(\beta)}(\rho\lambda)\le_{|\,\cdot\,|}C\sqrt{\rho\lambda}$ and $(r\lambda)^\gamma g_-^{(\gamma)}(r\lambda)\le_{|\,\cdot\,|}C\sqrt{r\lambda}$. It follows that

$$I_1\le_{|\,\cdot\,|}C\int_0^{1/\rho}\sqrt{\rho r}d\lambda=C\sqrt{\frac{r}{\rho}}\le C \tag{6.71}$$

(2) If $1/\rho\le\lambda\le 1/r$, we have $0<r\lambda\le 1\le\rho\lambda$ and we estimate as $(\rho\lambda)^\beta h^{(\beta)}(\rho\lambda)\le_{|\,\cdot\,|}C$ and $(r\lambda)^\gamma g_-^{(\gamma)}(r\lambda)\le_{|\,\cdot\,|}C\sqrt{r\lambda}$. It follows that

$$I_2\le_{|\,\cdot\,|}C\int_{1/\rho}^{1/r}\lambda^{-\frac12}\sqrt{r}d\lambda=2C\sqrt{r}\left(\frac{1}{\sqrt{r}}-\frac{1}{\sqrt{\rho}}\right)\le 2C. \tag{6.72}$$

(3) Finally if $1<r\lambda<\rho\lambda$, then we likewise estimate

$$(\lambda\rho)^\beta h^{(\beta)}(\lambda\rho)(r\lambda)^\gamma g_-^{(\gamma)}(r\lambda)\le_{|\,\cdot\,|}C\begin{cases}(r\lambda)^{-1}, & \text{if } \beta=0,\ \gamma\neq 0\\ (\rho\lambda)^{-1}, & \text{if } \beta\neq 0,\ \gamma=0,\\ (\rho\lambda)^{-1}(r\lambda)^{-1}, & \text{if } \beta,\gamma\neq 0.\end{cases}$$

The right hand side is bounded by $Cr^{-1}\lambda^{-1}$ and

$$I_3\le_{|\,\cdot\,|}C\int_{1/r}^\infty\lambda^{-2}r^{-1}d\lambda=C.$$

This completes the proof. ∎

PROPOSITION 6.13. *Let $m\ge 6$ and $\phi\in\mathcal{E}$. For $1\le p\le\frac{2(m-1)}{m+1}$, we have*

$$\|Z^{\nu\nu}u\|_p\le C_p\|u\|_p. \tag{6.73}$$

Proof. We recall (6.61). Lemma 6.12 implies $L_\pm(\rho,r)\le_{|\,\cdot\,|}C\langle\rho-r\rangle^{-(m-3)}$. It follows by Young's inequality and (6.65) that

$$\|Z^{\nu\nu}u\|_p\le C\|V\phi\|_1\left(\int_0^\infty\left(\int_{\mathbb{R}}\frac{\rho^{\frac{m-1}{p}-\frac{m+1}{2}}|r^{\frac{m+1}{2}}M(r)|}{\langle\rho-r\rangle^{m-3}}dr\right)^p d\rho\right)^{\frac1p}. \tag{6.74}$$

Define $\kappa = \frac{m-1}{p} - \frac{m+1}{2}$, then $\kappa \geq 0$ for $1 \leq p \leq \frac{2(m-1)}{m+1}$ and $m-3-\kappa \geq \frac{3}{2}$ for any $1 \leq p < \infty$ if $m \geq 6$. Thus, we may estimate

$$\rho^\kappa \langle \rho - r\rangle^{-(m-3)} \leq C \begin{cases} \langle \rho - r\rangle^{-\frac{3}{2}} & \text{if } \ |r| \leq 1 \\ \langle \rho - r\rangle^{-\frac{3}{2}} |r^\kappa| & \text{if } \ |r| \geq 1 \end{cases}$$

and Young's inequality implies

$$\|Z^{\nu\nu}u\|_p \leq C\|V\phi\|_1 \left(\int_0^1 |r^{\frac{m+1}{2}} M(r)|^p dr + \int_1^\infty |r^{\frac{m-1}{p}} M(r)|^p dr \right)^{\frac{1}{p}},$$

which is bounded by $C(\|V\phi * u\|_\infty + \|V\phi * u\|_p) \leq (\|V\phi\|_{p'} + \|V\phi\|_1)\|u\|_p$. This completes the proof of the proposition. ☐

6.2.2 The case $\frac{2(m-1)}{m-3} \leq p < \infty$

Lemma 6.14. *Let $m \geq 6$ and $\phi \in \mathcal{E}$. Then, $Z^{\nu\nu}(\phi)$ is bounded in $L^p(\mathbb{R}^m)$ for any $\frac{2(m-1)}{m-3} \leq p < \infty$.*

Proof. we apply integration by parts to (6.60) by using the identity that $\int_{\mathbb{R}} e^{-i(1+2a)r\lambda} r^{\nu+1} M(r)dr = i(1+2a)^{-1}\partial_\lambda \left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda} r^\nu M(r)dr\right)'$. We see that $\rho^{-\nu} Q^{a,b}_{\nu\nu}(\rho)$ is equal to

$$\frac{(-1)^{\nu+1}}{2\pi\rho^\nu(i(1+2a))} \int_0^\infty e^{i(1+2b)\rho\lambda} (\lambda^{m-3}F(\lambda))' \left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda} r^\nu M(r)dr \right) d\lambda$$
$$+ \frac{(-1)^{\nu+1}(1+2b)}{2\pi\rho^{\nu-1}(1+2a)} \int_0^\infty e^{i(1+2b)\rho\lambda} \lambda^{m-3} F(\lambda) \left(\int_{\mathbb{R}} e^{-i(1+2a)r\lambda} r^\nu M(r)dr \right) d\lambda$$

The argument similar to the one at the beginning of the proof of Proposition 6.13 shows that the operator produced by the first line has the same L^p property as $Z^{(\nu-1)\nu}$ and, hence, is bounded in $L^p(\mathbb{R}^m)$ for any $1 < p < \infty$. Thus, we need consider the operator produced by the second line, which we substitute for $Q^{a,b}_{\nu\nu}(\rho)/\rho^\nu$ in (6.8) and the resulting function for $Z^{\nu\nu}_{a,b}(\phi)u(x)$ in (6.7). The result is given by (6.61) where $X_\nu(\rho)$ is replaced by

$$\tilde{X}_\nu(\rho) = \frac{C}{\rho^{\nu-1}} \int_0^\infty \left\{ e^{i\lambda\rho} \lambda^{m-3} \left(\int_0^\infty \frac{(1+2b)e^{2i\lambda\rho b}}{(1+b)^{\nu+\frac{1}{2}}} \frac{db}{\sqrt{b}} \right) \right.$$
$$\times \left. \int_{\mathbb{R}} e^{-i\lambda r} \left(\int_0^\infty \frac{(1+2a)^{-1} e^{-2ia\lambda r}}{(1+a)^{\nu+\frac{1}{2}}} \frac{da}{\sqrt{a}} \right) r^\nu M(r)dr \right\} F(\lambda)d\lambda, \qquad (6.75)$$

which can be simplified into the form (6.65), (6.66) with the roles of g and h being replaced and the factors $\rho^{-\left(\nu+\frac{3}{2}\right)}$ and $r^{\nu+2}|r|^{-\frac{1}{2}}$ being replaced by $\rho^{-\left(\nu-\frac{1}{2}\right)}$ and $r^\nu|r|^{-\frac{1}{2}}$ respectively. Then, Lemmas 6.11 and 6.12 imply

$$X_\nu(\rho) \leq_{|\cdot|} \frac{C}{\rho^{\nu-\frac{1}{2}}} \int_{\mathbb{R}} \langle \rho - r\rangle^{3-m} |r|^\nu |r|^{-\frac{1}{2}} M(r)dr.$$

We estimate $\|X_\nu(|y|)\|_{L^p(|y|\geq 1)}$ for $p \geq \frac{2(m-1)}{m-3}$ and $\|X_\nu(|y|)\|_{L^1(|y|<1)}$. Let $\kappa = \frac{m-1}{p} - \nu + \frac{1}{2}$. If $p \geq \frac{2(m-1)}{m-3}$, then $\kappa \leq 0$ and $m-3+\kappa \geq \frac{3}{2}$ for $m \geq 6$ and for $\rho \geq 1$

$$\rho^\kappa \langle \rho - r\rangle^{3-m}|r|^{\nu-\frac{1}{2}} \leq C\langle \rho - r\rangle^{-\frac{3}{2}}\langle r\rangle^\kappa |r|^{\nu-\frac{1}{2}} \leq C\langle \rho - r\rangle^{-\frac{3}{2}}|r|^{\frac{m-1}{p}}.$$

It follows by Young's inequality that for any $2 \leq p < \infty$,

$$\begin{aligned} \|X_\nu(|y|)\|_{L^p(|y|\geq 1)} &\leq C\left(\int_0^\infty \left|\int_{\mathbb{R}} \langle \rho - r\rangle^{-\frac{3}{2}}|r|^{\frac{m-1}{p}} M(r)dr\right|^p d\rho\right)^{\frac{1}{p}} \\ &\leq C\left(\int_0^\infty |M(r)|^p r^{m-1}dr\right)^{\frac{1}{p}} \leq C\|V\phi\|_1\|u\|_p. \qquad (6.76) \end{aligned}$$

When $\rho \leq 1$, we have $\rho^{m-1-\nu+\frac{3}{2}} \leq 1$ and $\langle \rho - r\rangle^{3-m} \leq C\langle r\rangle^{3-m}$. Hence,

$$\|X_\nu(|y|)\|_{L^1(|y|<1)} \leq C\int_{\mathbb{R}} \langle r\rangle^{3-m}|r|^{\nu-\frac{1}{2}}|M(r)|dr \leq C\|M\|_\infty \leq C\|V\phi\|_{p'}\|u\|_p.$$

We therefore obtain by using Young's inequality again after splitting the integral corresponding to (6.61) into the ones over $|y| < 1$ and $|y| \geq 1$ that

$$\|Z^{\nu\nu}u\|_p \leq C(\|V\phi\|_1^2 + \|V\phi\|_p\|V\phi\|_{p'})\|u\|_p.$$

This completes the proof. ∎

6.3 Estimate of $\|Z_{\log}u\|_p$

In this section we study $Z_{\log}$ and prove the following lemma. The combination of the lemma with results of the previous subsections proves Theorem 1.5 for even dimensions $m \geq 6$, the formal proof of which will be omitted.

Lemma 6.15. (1) *If $m = 6$, then $Z_{\log}$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < m$. If $\mathcal{E} = \mathcal{E}_0$, then so is $Z_{\log}$ for all $1 < p < \infty$.*

(2) *If $m \geq 8$, then $Z_{\log}$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 < p < \infty$.*

Proof. We prove the lemma for $m = 6$ only. The proof for $m \geq 8$ is similar and easier. Out of three operators on the right of (3.27) for $m = 6$, we first study

$$Z_{1,\log} = \int_0^\infty G_0(\lambda)(V\varphi \otimes V\varphi)\lambda \log\lambda(G_0(\lambda) - G_0(-\lambda))F(\lambda)d\lambda, \qquad (6.77)$$

where we have ignored the constant $\omega_{m-1}/\pi(2\pi)^m$ which is not important. Since $Z_{1,\log} = 0$, if $\mathcal{E} = \mathcal{E}_0$, it suffices to prove (1) for $1 < p < \frac{m}{m-1}$ and $\frac{m}{2} < p < m$. By using (2.6) and (2.18) as previously, we express $Z_{1,\log}$ as the sum over $0 \leq j, k \leq \nu$ of

$$Z_{1,\log}^{jk}u(x) = C_{jk}T_j^{(a)}T_k^{(b)}\left[\int_{\mathbb{R}^m} \frac{(V\phi)(x-y)Q_{jk,\log}^{a,b}(|y|)}{|y|^{m-2-k}}dy\right], \qquad (6.78)$$

where $Q^{a,b}_{jk,\log}(\rho)$ are defined by (6.3) or (6.5) (for the case $j=0$) by replacing λ^{j+k-1} or λ^k respectively by $\lambda^{j+k+1}\log\lambda$ or $\lambda^{k+2}\log\lambda$. We prove

$$\|Z^{jk}_{1,\log}u\|_p \le C\|u\|_p \tag{6.79}$$

separately for $(j,k)\neq(\nu,\nu)$ and $(j,k)=(\nu,\nu)$ by repeating the argument in corresponding subsections.

Let $(j,k)\neq(\nu,\nu)$. We first observe that, if $j\ge 1$, Fourier inverse transforms of the derivatives upto the order $k+1$ of $\lambda^{j+k+1}(\log\lambda)F$ have the RDIM

$$\mathcal{F}^*(\lambda^{j+k+1}(\log\lambda)F)^{(l)})(\rho)\le_{|\,\cdot\,|} C(1+\rho)^{-2}\langle\log(1+\rho)\rangle, \quad 0\le l\le k+1$$

and estimates corresponding to (6.11) and (6.27) are satisfied by $Q^{a,b}_{jk,\log}(\rho)$ respectively for $1\le j\le\nu$ and for $j=0$ (without producing the boundary term). Then, the argument in §6.1.1 goes through for $Z^{jk}_{1,\log}$ and produces estimate (6.79) for $1<p<\frac{m}{m-1}$. By the same reason the estimate corresponding (6.13) for $m/2<p<m$ is satisfied by $Q^{a,b}_{jk,\log}(\rho)$ for all j,k and we likewise have (6.79) for $m/2<p<m$ by using the argument of the first part of proof of Lemma 6.8. It is then obvious that the same holds for $Z_{2,\log}$ which is obtained from $Z_{1,\log}$ by replacing $\lambda\log\lambda$ by $\lambda^3(\log\lambda)$ and, that the operator

$$Z^{(a,b)}_{3,\log}=\int_0^\infty G_0(\lambda)(\varphi_a\otimes\psi_b)\lambda^3(\log\lambda)^2(G_0(\lambda)-G_0(-\lambda))F(\lambda)d\lambda. \tag{6.80}$$

produced by $\lambda^2\log\lambda F_2$ of (3.18) satisfies (6.79) for all $1<p<m$.

We next prove (6.79) when $(j,k)=(\nu,\nu)$. It suffices prove it for $1<p<p_0$ for some $p_0>1$ and $p\ge p_1$ for some $p_1>2$. The argument at the beginnings of §6.2.1 and §6.2.2 shows that respectively for $1<p<p_0$ and $p\ge p_1$, we have only to estimate operators obtained by replacing $Q^{a,b}_{jk,\log}(\rho)$ by

$$\frac{1+2a}{(1+2b)\rho^{\nu+1}}\int_0^\infty e^{i(1+2b)\rho\lambda}\lambda^{m-1}\log\lambda F(\lambda)\left(\int_{\mathbb{R}}e^{-i(1+2a)r\lambda}r^{\nu+2}Mdr\right)d\lambda$$

and

$$\frac{1+2b}{(1+2a)\rho^{\nu-1}}\int_0^\infty e^{i(1+2b)\rho\lambda}\lambda^{m-1}\log\lambda F(\lambda)\left(\int_{\mathbb{R}}e^{-i(1+2a)r\lambda}r^{\nu+2}Mdr\right)d\lambda$$

in (6.78). We then repeat the argument of §6.2. We have $\lambda^{m-2}\log\lambda$ in place of λ^{m-4} in (6.66). If we change λ^σ by $\lambda^{\sigma+2}\log\lambda$ in the definition (6.68) of $\tilde{L}_\pm(\rho,r)$, then (6.69) is satisfied with faster decaying factor $\langle\rho-r\rangle^{-(\sigma+2)}$ in place of $\langle\rho-r\rangle^{-(\sigma+1)}$. Thus, $\|Z^{\nu\nu}_{\log}u\|_p$ is bounded $C\|V\phi\|_1$ times (6.74) with $\langle\rho-r\rangle^{-(m-2)}$ in place of $\langle\rho-r\rangle^{-(m-3)}$ and this proves the lemma for $1<p<p_0$. The proof for $p\ge p_1$ is similar and we omit further details. □

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K. Yajima
Department of Mathematics
Gakushuin University
1-5-1 Mejiro
Toshima-ku
Tokyo 171-8588
Japan