Product Systems over Ore Monoids

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Abstract. We interpret the Cuntz–Pimsner covariance condition as a nondegeneracy condition for representations of product systems. We show that Cuntz–Pimsner algebras over Ore monoids are constructed through inductive limits and section algebras of Fell bundles over groups. We construct a groupoid model for the Cuntz–Pimsner algebra coming from an action of an Ore monoid on a space by topological correspondences. We characterise when this groupoid is effective or locally contracting and describe its invariant subsets and invariant measures.

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1. Introduction

Let A and B be C*-algebras. A *correspondence* from A to B is a Hilbert B-module $\mathcal{E}$ with a nondegenerate *-homomorphism from A to the C*-algebra of adjointable operators on $\mathcal{E}$. It is called *proper* if the left A-action is by compact operators, $A \to \mathbb{K}(\mathcal{E})$. If $\mathcal{E}_{AB}$ and $\mathcal{E}_{BC}$ are correspondences from A to B and from B to C, respectively, then $\mathcal{E}_{AB} \otimes_B \mathcal{E}_{BC}$ is a correspondence from A to C.

A triangle of correspondences consists of three C*-algebras A, B, C, correspondences $\mathcal{E}_{AB}$, $\mathcal{E}_{AC}$ and $\mathcal{E}_{BC}$ between them, and an isomorphism of correspondences $u\colon \mathcal{E}_{AB} \otimes_B \mathcal{E}_{BC} \to \mathcal{E}_{AC}$; that is, u is a unitary operator of Hilbert C-modules that also intertwines the left A-module structures. Such triangles appear naturally if we study the correspondence bicategory of C*-algebras introduced in [12].

This article started with the observation that a correspondence triangle with $A = B$ and $\mathcal{E}_{BC} = \mathcal{E}_{AC}$ is the same as a *Cuntz–Pimsner covariant* representation of the correspondence $\mathcal{E} := \mathcal{E}_{AB}$ by adjointable operators on $\mathcal{F} := \mathcal{E}_{BC} = \mathcal{E}_{AC}$, provided $\mathcal{E}_{AB}$ is proper. Thus we get to the Cuntz–Pimsner algebra directly, without going through the Cuntz–Toeplitz algebra.

This is limited, however, to proper correspondences and the absolute Cuntz–Pimsner algebra; that is, we cannot treat the relative Cuntz–Pimsner algebras introduced by Muhly and Solel [59] and Katsura [42]. The relative versions are most relevant if the left action map $A \to \mathbb{K}(\mathcal{E})$ is not faithful. Then the map from A to the Cuntz–Pimsner algebra is not faithful, and the latter may even be zero.

Our observation about the Cuntz–Pimsner algebra of a single proper correspondence has great conceptional value because it exhibits these (absolute) Cuntz–Pimsner algebras as a special case of a general construction, namely, colimits in the correspondence bicategory, see [2]. Other examples of such colimits are crossed products for group and crossed module actions, inductive limits for chains of *-homomorphisms, and Cuntz–Pimsner algebras for proper essential product systems.

In this article, we apply our observation on the Cuntz–Pimsner covariance condition to the case of Cuntz–Pimsner algebras for proper essential product systems over monoids. Much less is known about their structure. Following Fowler [33], they are always defined and treated through the corresponding Cuntz–Toeplitz algebra. Many articles never get farther than the Nica–Toeplitz algebra. We shall prove strong results about the structure of Cuntz–Pimsner algebras of proper essential product systems over Ore monoids. Commutative monoids and groups are Ore. So are extensions of commutative monoids by groups such as the monoid $\mathbb{M}_n(\mathbb{Z})^\times$ of integer matrices with non-zero determinant, with multiplication as group structure: this is an extension of the group $\mathrm{Gl}_n(\mathbb{Z})$ by the commutative monoid $(\mathbb{N}_{\geq 1}, \cdot)$. Thus most of the semigroups currently being treated in the operator algebras literature are Ore monoids. The main exception are free monoids, which are not Ore.

Let P be a cancellative Ore monoid and let G be its group completion. Let A be a C*-algebra and let $(\mathcal{E}_g)_{p\in P}$ be a proper, essential product system over P with unit fibre $\mathcal{E}_1 = A$; that is, the left action of A on each $\mathcal{E}_p$ for $p \in P$ is by a nondegenerate *-homomorphism $A \to \mathbb{K}(\mathcal{E}_p)$. The Ore conditions for P ensure that the diagram formed by the C*-algebras $\mathbb{K}(\mathcal{E}_p)$ for $p \in P$ with the maps

$$\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_p \otimes_A \mathcal{E}_q) \cong \mathbb{K}(\mathcal{E}_{pq})$$

for $q, p \in P$ is indexed by a directed set. Hence the colimit for this diagram behaves like an inductive limit; it may indeed be rewritten as an inductive limit of a chain of maps $\mathbb{K}(\mathcal{E}_{p_i}) \to \mathbb{K}(\mathcal{E}_{p_{i+1}})$ for a suitable map $\mathbb{N} \to P$ if P is countable. Let $\mathcal{O}_1$ be the inductive limit of this diagram of C*-algebras. We construct a Fell bundle $(\mathcal{O}_g)_{g\in G}$ over G with $\mathcal{O}_1$ as its unit fibre, such that its section algebra is the Cuntz–Pimsner algebra $\mathcal{O}$ of the given product system. Thus the construction of the Cuntz–Pimsner algebra of a product system over P has two steps: inductive limits and Fell bundle section algebras.

After putting this article on the arxiv, we learnt of the preprint [49] by Kwaśniewski and Szymański, which proves essentially the same result about the structure of Cuntz–Pimsner algebras over Ore monoids.

Assume now that the correspondences $\mathcal{E}_p$ are full as Hilbert A-modules. Then the C*-algebras $\mathbb{K}(\mathcal{E}_p)$ for $p \in P$ are all Morita–Rieffel equivalent to A and the Fell bundle $(\mathcal{O}_g)_{g\in G}$ is saturated. Let $\mathbb{K}$ be the C*-algebra of compact operators. By the Brown–Green–Rieffel Theorem, $\mathcal{E}_p \otimes \mathbb{K} \cong A \otimes \mathbb{K}$ as a Hilbert $A \otimes \mathbb{K}$-module, so we may replace the proper correspondence $\mathcal{E}_p$ by an endomorphism $\varphi_p\colon A\otimes\mathbb{K} \to A\otimes\mathbb{K}$. Choose a cofinal sequence $(p_i)_{i\geq 0}$ in P as above with $p_0 := 1$, and let $q_i \in P$ be such that $p_i = p_{i-1}q_i$. Then $\mathcal{O}_1 \otimes \mathbb{K}$ is the inductive limit of the inductive system

$$A \otimes \mathbb{K} \xrightarrow{\varphi_{q_1}} A \otimes \mathbb{K} \xrightarrow{\varphi_{q_2}} A \otimes \mathbb{K} \xrightarrow{\varphi_{q_3}} A \otimes \mathbb{K} \to \cdots ;$$

this inductive limit carries a natural G-action with $G \ltimes \mathcal{O}_1 \cong \mathcal{O} \otimes \mathbb{K}$.

Thus the K-theory of $\mathcal{O}_1$ is an inductive limit of copies of the K-theory of A; the maps are induced by the proper correspondences $\mathcal{E}_q$ or, equivalently, the endomorphisms φ_q of $A\otimes\mathbb{K}$. Roughly speaking, we have reduced the problem of computing the K-theory for Cuntz–Pimsner algebras of proper product systems over an Ore monoid P to the problem of computing the K-theory for crossed products with the group G. This latter problem may be difficult, but is much studied. We cannot hope for more because crossed products for G-actions are special cases of Cuntz–Pimsner algebras over P.

Lots of C*-algebras are or could be defined as Cuntz–Pimsner algebras of product systems over Ore monoids. Thus our structure theory for them has lots of potential applications. As a sample of how K-theory computations in this context might work, we consider certain higher-rank analogues of the Doplicher–Roberts algebras that motivated the introduction of graph algebras. (Our higher-rank analogues, however, need not be higher-rank graph algebras.)

Many Cuntz–Pimsner algebras are constructed from generalised dynamical systems, such as higher-rank topological graphs. The appropriate topological analogue of a product system over P is given by locally compact spaces X and M_p for $p \in P$ with continuous maps $r_p, s_p\colon M_p \to X$ and $\sigma_{p,q}\colon M_{pq} \xrightarrow{\sim} M_p \times_{s_p,X,r_q} M_q$. We assume r_p to be proper and s_p to be local homeomorphisms to turn (M_p, s_p, r_p) into proper correspondences over $\mathrm{C}_0(X)$. These form a product system over P with unit fibre $\mathrm{C}_0(X)$. The data above may be called a topological higher-rank graph over P; we prefer to call it an action of P on X by topological correspondences.

In the above situation, we construct a groupoid model for the Cuntz–Pimsner algebra of our product system. This model is a Hausdorff, locally compact, étale groupoid. We translate what it means for this groupoid to be effective, locally contracting, or minimal into the original data $(X, M_p, s_p, r_p, \sigma_{p,q})$. We also describe invariant subsets and invariant measures for the object space of our groupoid model. This gives criteria when the Cuntz–Pimsner algebra of an action by topological correspondences is simple or purely infinite and often describes its traces and KMS-states for certain one-parameter groups of automorphisms.

Our results are interesting already for the commutative Ore monoids $(\mathbb{N}^k, +)$. Several authors have considered examples of product systems over these and other commutative cancellative monoids ([10, 31, 32, 51, 79]). Commutativity seems to be a red herring: what is relevant are Ore conditions. Commutativity is hidden also in Exel's idea in [29] to extend a semigroup action to an "interaction semigroup." Examples in [31] show that interaction semigroups for product systems over $\mathbb{N}^2$ only exist under some commutativity assumptions about certain conditional expectations. Our approach shows that a deep study of these examples is possible without such technical commutativity assumptions (see Section 5.1).

Topological higher-rank graphs are already very close to our situation, so we compare our constructions to the existing ones for this class as we go along. As an example involving noncommutative Ore monoids, we discuss how the semigroup C*-algebras of Xin Li [55] fit into our approach.

2. The Cuntz–Pimsner covariance condition

We first reinterpret the Cuntz–Pimsner covariance condition for a single correspondence as a nondegeneracy condition.

Definition 2.1. A *correspondence* from A to B is a Hilbert B-module $\mathcal{F}$ with a *nondegenerate* left action of A by adjointable operators. A correspondence is *proper* if A acts by compact operators. We often write the left action multiplicatively as $a \cdot \xi$ for $a \in A$, $\xi \in \mathcal{F}$. An *isomorphism* between two correspondences from A to B is an A,B-bimodule map that is unitary for the B-valued inner products.

Definition 2.2. Let A and B be C*-algebras and let $\mathcal{E}$ be a correspondence from A to itself. A *transformation* from $(A, \mathcal{E})$ to B is a correspondence $\mathcal{F}$ from A to B with an isomorphism of correspondences $u\colon \mathcal{E} \otimes_B \mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

This definition is a special case of the standard notion of a "transformation" between two "morphisms" between two "bicategories" (see [52]). This point of view is developed further in [2]. Here it will not play any role besides guiding our choice of notation.

We want to relate transformations to certain Toeplitz representations of correspondences. The next proposition is already implicit in [58, §5] and has also been used by other authors before.

Proposition 2.3. *Let A, B_1, B_2 be* C*-*algebras. Let $\mathcal{E}\colon A \to B_1$, $\mathcal{F}_1\colon B_1 \to B_2$ and $\mathcal{F}_2\colon A \to B_2$ be correspondences. Isomorphisms $\mathcal{E} \otimes_{B_1} \mathcal{F}_1 \to \mathcal{F}_2$ of correspondences are in natural bijection with linear maps $S\colon \mathcal{E} \to \mathbb{B}(\mathcal{F}_1, \mathcal{F}_2)$ that satisfy*

(1) $S(a\xi) = aS(\xi)$ *for all* $a \in A$, $\xi \in \mathcal{E}$*;*
(2) $S(\xi_1)^* S(\xi_2) = \langle \xi_1, \xi_2 \rangle_{B_1}$ *for all* $\xi_1, \xi_2 \in \mathcal{E}$*;*
(3) $S(\mathcal{E}) \cdot \mathcal{F}_1$ *spans a dense subspace of* $\mathcal{F}_2$*.*

Furthermore, (2) *implies*

(4) $S(\xi b) = S(\xi)b$ *for all* $b \in B_1$*.*

Proof. First let $u\colon \mathcal{E} \otimes_{B_1} \mathcal{F}_1 \to \mathcal{F}_2$ be an isomorphism of correspondences. Define $S(\xi)(\eta) := u(\xi \otimes \eta)$ for $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}_1$. For fixed ξ, this is an adjointable operator $S(\xi)\colon \mathcal{F}_1 \to \mathcal{F}_2$ because u and the operator $\mathcal{F}_1 \to \mathcal{E} \otimes_{B_1} \mathcal{F}_1$, $\eta \mapsto \xi \otimes \eta$, are adjointable. This map S clearly satisfies (1). Since u is isometric,

$$\langle \eta_1, S(\xi_1)^* S(\xi_2)\eta_2\rangle = \langle S(\xi_1)\eta_1, S(\xi_2)\eta_2\rangle = \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \eta_1, \langle \xi_1, \xi_2\rangle \eta_2\rangle$$

for all $\xi_1, \xi_2 \in \mathcal{E}$, $\eta_1, \eta_2 \in \mathcal{F}_1$. This is equivalent to (2). Since u is unitary, it has dense range, which gives (3).

Conversely, let $S\colon \mathcal{E} \to \mathbb{B}(\mathcal{F}_1, \mathcal{F}_2)$ be given. Define u on the algebraic tensor product of $\mathcal{E}$ and $\mathcal{F}_1$ by linear extension of $u(\xi \otimes \eta) := S(\xi)(\eta)$. Condition (2) ensures that this is an isometry and hence extends to the completion $\mathcal{E} \otimes_{B_1} \mathcal{F}_1$. Hence S satisfies $S(\xi b)(\eta) = S(\xi)(b\eta)$ for all $\xi \in \mathcal{E}$, $b \in B_1$, $\eta \in \mathcal{F}_1$, which is equivalent to (4). Condition (1) says that u is A-linear, and (3) says that it has dense range. Being isometric, this means that u is unitary. The two constructions $u \leftrightarrow S$ are inverse to each other because u is determined by its values on the monomials $\xi \otimes \eta$. $\square$

A (Toeplitz) *representation* of $\mathcal{E}$ is usually defined as a map S satisfying (1) and (2) in the case $\mathcal{F}_1 = \mathcal{F}_2$; we also allow the case $\mathcal{F}_1 \neq \mathcal{F}_2$ for a while because this is used in [2] and in Proposition 2.8.

Definition 2.4. A representationn is *nondegenerate* if it satisfies (3).

By Proposition 2.3, a transformation from $(A, \mathcal{E})$ to B is equivalent to a correspondence $\mathcal{F}\colon A \to B$ with a nondegenerate representation of $\mathcal{E}$ by operators on $\mathcal{F}$. We now relate nondegeneracy to the Cuntz–Pimsner covariance condition:

Proposition 2.5. *Nondegenerate representations of a correspondence $\mathcal{E}$ are Cuntz–Pimsner covariant. The converse holds if $\mathcal{E}$ is proper.*

Proof. Let A, B_1 and B_2 be C*-algebras and let $\mathcal{E}\colon A \to B_1$, $\mathcal{F}_1\colon B_1 \to B_2$ and $\mathcal{F}_2\colon A \to B_2$ be correspondences as in Proposition 2.3. The left actions of A in our correspondences are nondegenerate *-homomorphisms

$$\varphi_{\mathcal{E}}\colon A \to \mathbb{B}(\mathcal{E}), \qquad \varphi_{\mathcal{F}_2}\colon A \to \mathbb{B}(\mathcal{F}_2).$$

Let $u\colon \mathcal{E} \otimes_{B_1} \mathcal{F}_1 \to \mathcal{F}_2$ be an isomorphism of correspondences. The map

$$\vartheta\colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{F}_2), \qquad T \mapsto u(T \otimes 1)u^*,$$

is a strictly continuous, unital *-homomorphism. It satisfies $\vartheta \circ \varphi_{\mathcal{E}} = \varphi_{\mathcal{F}_2}$ because u intertwines the left actions of A. If $\xi_1, \xi_2 \in \mathcal{E}$ and $|\xi_1\rangle\langle\xi_2|$ is the corresponding rank-one compact operator on $\mathcal{E}$, then

$$\vartheta(|\xi_1\rangle\langle\xi_2|) = S(\xi_1)S(\xi_2)^*.$$

This formula still defines a (possibly degenerate) *-homomorphism $\vartheta\colon \mathbb{K}(\mathcal{E}) \to \mathbb{B}(\mathcal{F}_2)$ for any representation $S\colon \mathcal{E} \to \mathbb{B}(\mathcal{F}_1, \mathcal{F}_2)$, see [65, p. 202].

Definition 2.6. A representation S is *Cuntz–Pimsner covariant* if $\vartheta(\varphi_{\mathcal{E}}(a)) = \varphi_{\mathcal{F}_2}(a)$ for all $a \in A$ with $\varphi_{\mathcal{E}}(a) \in \mathbb{K}(\mathcal{E})$.

By Proposition 2.3, a nondegenerate representation comes from an isomorphism of correspondences. We have already seen $\vartheta(\varphi_{\mathcal{E}}(a)) = \varphi_{\mathcal{F}_2}(a)$ for all $a \in A$ in that case. So nondegenerate representations are Cuntz–Pimsner covariant.
Conversely, let S be Cuntz–Pimsner covariant and assume that $\mathcal{E}$ is proper, that is, $\varphi_{\mathcal{E}}(A) \subseteq \mathbb{K}(\mathcal{E})$. Let $\langle X\rangle$ denote the closed linear span of X. We have

$$\begin{aligned}\langle S(\mathcal{E})\mathcal{F}_1\rangle \supseteq \langle S(\mathcal{E})S(\mathcal{E})^*\mathcal{F}_2\rangle = \langle \vartheta(\mathbb{K}(\mathcal{E}))\mathcal{F}_2\rangle \\ \supseteq \langle \vartheta(\varphi_{\mathcal{E}}(A))\mathcal{F}_2\rangle = \langle \varphi_{\mathcal{F}_2}(A)\mathcal{F}_2\rangle = \langle \mathcal{F}_2\rangle\end{aligned}$$

because $\varphi_{\mathcal{F}_2}$ is nondegenerate. Thus S is nondegenerate. □

For a proper correspondence $\mathcal{E}$, we may now reformulate the universal property that defines its Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$: it is the universal C*-algebra for *nondegenerate* representations of $\mathcal{E}$. Equivalently, $\mathcal{O}_{\mathcal{E}}$ is the universal target for transformations from $(A, \mathcal{E})$ to C*-algebras. The Cuntz–Pimsner algebra comes with a nondegenerate *-homomorphism $\varphi_0\colon A \to \mathcal{O}_{\mathcal{E}}$ and a representation $S_0\colon \mathcal{E} \to \mathcal{O}_{\mathcal{E}}$, which is Cuntz–Pimsner covariant and thus nondegenerate. This is equivalent to a transformation from $(A, \mathcal{E})$ to $\mathcal{O}_{\mathcal{E}}$; the underlying correspondence is $\mathcal{O}_{\mathcal{E}}$ itself as a Hilbert $\mathcal{O}_{\mathcal{E}}$-module, with A acting via φ_0. The isomorphism $u_0\colon \mathcal{E} \otimes_A \mathcal{O}_{\mathcal{E}} \cong \mathcal{O}_{\mathcal{E}}$ is the unitary that corresponds to S_0 by Proposition 2.3.
The transformation $(\mathcal{O}_{\mathcal{E}}, u_0)$ has the following universal property: if $(\mathcal{F}, u)$ is another transformation from $(A, \mathcal{E})$ to a C*-algebra B, then there is a unique representation $\psi\colon \mathcal{O}_{\mathcal{E}} \to \mathbb{B}(\mathcal{F})$ for which $u = u_0 \otimes_\psi \mathrm{id}_{\mathcal{F}}$. Conversely, a representation $\psi\colon \mathcal{O}_{\mathcal{E}} \to \mathbb{B}(\mathcal{F})$ provides a unitary $u = u_0 \otimes_\psi \mathrm{id}_{\mathcal{F}}$ from $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{E} \otimes_A \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{F}$ to $\mathcal{F} \cong \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{F}$. The pair $(\mathcal{F}, \psi)$ is the same as a correspondence from $\mathcal{O}_{\mathcal{E}}$ to B. Thus transformations from $(A, \mathcal{E})$ to B are the same as correspondences from $\mathcal{O}_{\mathcal{E}}$ to B.
What happens for a representation $S\colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ that does not satisfy the Cuntz–Pimsner covariance condition? The construction in the proof of Proposition 2.3 still gives a map $u\colon \mathcal{E} \otimes_A \mathcal{F} \to \mathcal{F}$, which is an A,B-bimodule map and isometric for the B-valued inner product. But this isometry u need not be unitary, not even adjointable. Thus allowing all Toeplitz representations replaces the unitary in the definition of a transformation by a possibly non-adjointable isometry.

Example 2.7. What goes wrong if $\mathcal{E}$ is not proper? Let us consider the simplest case, $A = \mathbb{C}$ and $\mathcal{E} = \ell^2(\mathbb{N})$. In this case, no non-zero element of A acts by a compact operator, so there is no difference between the Cuntz–Pimsner and the Cuntz–Toeplitz algebra. A correspondence from A to B is the same as a Hilbert B-module. The Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ is the famous Cuntz algebra $\mathcal{O}_\infty$. The identity map on $\mathcal{O}_\infty$ corresponds to a Cuntz–Pimsner covariant representation $S_0\colon \ell^2(\mathbb{N}) \to \mathcal{O}_\infty$, which maps the basis vector δ_i to the generating isometry S_i. The induced *-homomorphism $\mathbb{K}(\ell^2\mathbb{N}) \to \mathcal{O}_\infty$ is degenerate, however, because $\mathcal{O}_\infty$ is unital. It corresponds to the isometry of Hilbert $\mathcal{O}_\infty$-modules $\ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty \hookrightarrow \mathcal{O}_\infty$, $E_{ij} \otimes x \mapsto S_i x S_j^*$. If this were adjointable, its range would be of the form $p\mathcal{O}_\infty$ for a projection $p \in \mathcal{O}_\infty$ because $\mathcal{O}_\infty$ is unital. Then $[1]+p = p$ in $\mathrm{K}_0(\mathcal{O}_\infty)$ because $\mathcal{O}_\infty \oplus (\ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty) \cong \ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty$, giving $[1] = 0$ in $\mathrm{K}_0(\mathcal{O}_\infty)$, which is false.

Katsura's definition of a relative Cuntz–Pimsner algebra only requires the Cuntz–Pimsner covariance condition on a certain ideal $K \triangleleft A$ that acts on $\mathcal{E}$ by compact operators (see [41] or [42, Definition 3.4]). We may reformulate this as a *partial nondegeneracy* condition:

Proposition 2.8. *Let A and B be* C*-*algebras, let $\mathcal{E}$ and $\mathcal{F}$ be correspondences from A to A and from A to B, respectively. Let K be an ideal in A that acts on $\mathcal{E}$ by compact operators. A representation $S\colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ satisfies the Cuntz–Pimsner covariance condition on K if and only if $K \cdot S(\mathcal{E})\mathcal{F} = K \cdot \mathcal{F}$. Equivalently, the isometry $\mathcal{E} \otimes_A \mathcal{F} \to \mathcal{F}$ induced by S restricts to an isomorphism of correspondences $K\mathcal{E} \otimes_A \mathcal{F} \to K\mathcal{F}$.*

Proof. Proposition 2.3 says that an isomorphism $K\mathcal{E} \otimes_A \mathcal{F} \to K\mathcal{F}$ is equivalent to a nondegenerate representation $K\mathcal{E} \to \mathbb{B}(\mathcal{F}, K\mathcal{F})$. Now apply Proposition 2.5 to the correspondences $K\mathcal{E}\colon K \to A$, $\mathcal{F}\colon A \to B$, and $K\mathcal{F}\colon K \to B$, so substitute $K, A, B, K\mathcal{E}, \mathcal{F}, K\mathcal{F}$ for $A, B_1, B_2, \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2$. Since we assume K to act by compact operators on $\mathcal{E}$, the correspondence $K\mathcal{E}\colon K \to A$ is always proper. So the nondegeneracy condition $K\mathcal{E} \cdot \mathcal{F} = K\mathcal{F}$ is equivalent to the Cuntz–Pimsner covariance condition for the restriction of the left action of K to $K\mathcal{F}$. That is, $\vartheta(\varphi_\mathcal{E}(k))\xi = \varphi_\mathcal{F}(k)\xi$ for all $k \in K$ and $\xi \in K\mathcal{F}$, with $\vartheta\colon \mathbb{K}(\mathcal{E}) \to \mathbb{B}(\mathcal{F})$ as in the proof of Proposition 2.5. It remains to show that this equality for all $\xi \in K\mathcal{F}$ implies the same equality for all $\xi \in \mathcal{F}$: the latter is the usual coisometry condition for the ideal K. Let $T_k := \vartheta(\varphi_\mathcal{E}(k)) - \varphi_\mathcal{F}(k)$ for $k \in K$. Both T_k and $T_k^* = T_{k^*}$ map $\mathcal{F}$ to $K\mathcal{F} = K\mathcal{E} \cdot \mathcal{F}$, and they vanish on $K\mathcal{F}$ by the above computation. Therefore, $\langle T_k\xi, T_k\xi\rangle = \langle \xi, T_k^*T_k\xi\rangle = 0$ for all $\xi \in \mathcal{F}$. □

3. Cuntz–Pimsner algebras of product systems over Ore monoids

Product systems over a monoid P were introduced by Fowler [33], inspired by previous definitions by Arveson [6] and Dinh [21]. The following data is equivalent to a product system in Fowler's sense with the mild extra condition that each fibre be an essential left module over the unit fibre:

- a C*-algebra A;
- correspondences $\mathcal{E}_p$ from A to itself for all $p \in P \setminus \{1\}$;
- isomorphisms of correspondences $\mu_{p,q}\colon \mathcal{E}_p \otimes_A \mathcal{E}_q \to \mathcal{E}_{pq}$ for all $p, q \in P \setminus \{1\}$, which are *associative*, that is, the following diagram commutes for all $p, q, t \in P$:

$$\begin{tikzcd}
\mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A \mathcal{E}_t \arrow[r, "\mathrm{id}_{\mathcal{E}_p} \otimes_A \mu_{q,t}"] \arrow[d, "\mu_{p,q} \otimes_A \mathrm{id}_{\mathcal{E}_t}"'] & \mathcal{E}_p \otimes_A \mathcal{E}_{qt} \arrow[d, "\mu_{p,qt}"] \\
\mathcal{E}_{pq} \otimes_A \mathcal{E}_t \arrow[r, "\mu_{pq,t}"'] & \mathcal{E}_{pqt}
\end{tikzcd}$$

here we let $\mathcal{E}_1 = A$, and we let $\mu_{1,q}$ and $\mu_{p,1}$ be the isomorphisms $A \otimes_A \mathcal{E}_q \cong \mathcal{E}_q$ and $\mathcal{E}_p \otimes_A A \cong \mathcal{E}_p$ from the left and right A-module structures, respectively; this is needed to write down $\mu_{p,q}$ if $p \cdot q = 1$ and to formulate the associativity condition for $\mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A \mathcal{E}_t \to \mathcal{E}_{pqt}$ if $p \cdot q = 1$ or $q \cdot t = 1$.
Our main theorems will only hold if all correspondences $\mathcal{E}_p$ are proper. Then we speak of a *proper product system over* P.

Remark 3.1. A nondegenerate *-homomorphism $f\colon A \to B$ gives a proper correspondence $\mathcal{E}_f$ from A to B: take $\mathcal{E}_f = B$ with A acting through f. For two composable nondegenerate *-homomorphisms, we have a natural isomorphism $\mathcal{E}_f \otimes_B \mathcal{E}_g \cong \mathcal{E}_{gf}$. Due to this change in the order of products, an action of the opposite monoid P^{op} by ordinary nondegenerate *-homomorphisms gives a product system over P. The Cuntz–Pimsner algebra of this product system is the same as the crossed product for the original action by endomorphisms, see [33, Section 3].
The change from P to P^{op} also explains why left Ore conditions are needed in [16, 50] to study actions of P by endomorphisms, while we will need right Ore conditions to study product systems over P.

Definition 3.2. Let $(A, \mathcal{E}_p, \mu_{p,q})$ be a product system over P. A *transformation* from it to a C*-algebra B consists of a correspondence $\mathcal{F}$ from A to B and isomorphisms of correspondences $V_p\colon \mathcal{E}_p \otimes_A \mathcal{F} \to \mathcal{F}$ for $p \in P \setminus \{1\}$, such that for all $p, q \in P \setminus \{1\}$, the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
\mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A \mathcal{F} \arrow[r, "\mathrm{id}_{\mathcal{E}_p} \otimes_A V_q"] \arrow[d, "\mu_{p,q} \otimes_A \mathrm{id}_{\mathcal{F}}"'] & \mathcal{E}_p \otimes_A \mathcal{F} \arrow[d, "V_p"] \\
\mathcal{E}_{pq} \otimes_A \mathcal{F} \arrow[r, "V_{pq}"'] & \mathcal{F}
\end{tikzcd} \tag{3.3}$$

We let V_1 be the canonical isomorphism $A \otimes_A \mathcal{F} \cong \mathcal{F}$ and use this in (3.3) if $p \cdot q = 1$.

By Proposition 2.3, each isomorphism V_p corresponds to a nondegenerate representation $S_p\colon \mathcal{E}_p \to \mathbb{B}(\mathcal{F})$ of the correspondence $\mathcal{E}_p$. By convention, $S_1\colon \mathcal{E}_1 = A \to \mathbb{B}(\mathcal{F})$ is the representation of A that is part of the correspondence $\mathcal{F}$. Equation (3.3) means that both maps around the square agree on all monomials $\xi_p \otimes \xi_q \otimes \eta \in \mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A \mathcal{F}$. This amounts to the condition

$$S_p(\xi_p) \cdot S_q(\xi_q) = S_{pq}(\mu_{p,q}(\xi_p \otimes \xi_q)) \qquad \text{for all } \xi_p \in \mathcal{E}_p, \xi_q \in \mathcal{E}_q,$$

which is standard for representations of product systems.
Example 2.7 shows that we cannot expect enough transformations to exist unless our product system is proper. We assume this from now on. By Proposition 2.5, the nondegeneracy of the representations S_p is equivalent to the Cuntz–Pimsner covariance condition for all of them. Hence the universal property that defines the Cuntz–Pimsner algebra gives a natural bijection between

correspondences from it to a C*-algebra B and transformations from the product system to B; this bijection leaves the underlying Hilbert module $\mathcal{F}$ unchanged.

A transformation $(\mathcal{F}, V_p)$ gives unital, strictly continuous *-homomorphisms

$$\vartheta_p\colon \mathbb{B}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{E}_p \otimes_A \mathcal{F}) \xrightarrow{\cong} \mathbb{B}(\mathcal{F}), \qquad T \mapsto V_p(T \otimes_A \mathrm{id}_{\mathcal{F}})V_p^*,$$

for all $p \in P$. Similarly, the isomorphisms $\mu_{p,q}\colon \mathcal{E}_p \otimes_A \mathcal{E}_q \to \mathcal{E}_{pq}$ induce nondegenerate *-homomorphisms

$$\varphi_{p,q}\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq}), \qquad T \mapsto \mu_{p,q}(T \otimes_A \mathrm{id}_{\mathcal{E}_q})\mu_{p,q}^*. \tag{3.4}$$

Since $\mathcal{E}_q$ is proper, $\varphi_{p,q}(\mathbb{K}(\mathcal{E}_p))$ is contained in $\mathbb{K}(\mathcal{E}_{pq})$. The commuting diagram (3.3) gives $\vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p$ for all $p, q \in P$.

This situation invites us to take a colimit (or inductive limit) of the C*-algebras $\mathbb{K}(\mathcal{E}_p)$ along the maps $\varphi_{p,q}$. More precisely, let $\mathcal{C}_P$ be the category with object set P and arrow set $P \times P$, where (p, q) is an arrow from p to pq, and where $(pq, t) \cdot (p, q) := (p, qt)$ for all $p, q, t \in P$.

Lemma 3.5. *The maps $p \mapsto \mathbb{K}(\mathcal{E}_p)$ and $(p, q) \mapsto \varphi_{p,q}$ form a functor from $\mathcal{C}_P$ to the category of* C*-*algebras and nondegenerate* *-*homomorphisms.*

Proof. Functoriality means that $\varphi_{pq,t} \circ \varphi_{p,q} = \varphi_{p,qt}$ for all $p, q, t \in P$. This is equivalent to the associativity of μ. □

The diagram in Lemma 3.5 has a colimit in the category of C*-algebras and nondegenerate *-homomorphisms. This colimit will act nondegenerately on $\mathcal{F}$ by its universal property. Therefore, it is part of the Cuntz–Pimsner algebra of the product system. In general, the colimit involves amalgamated free products, which make it rather intractable. To get a well-behaved Cuntz–Pimsner algebra, we assume that $\mathcal{C}_P$ is a *filtered* category in the following sense:

Definition 3.6 ([57, Section IX.1]). A category $\mathcal{C}$ is *filtered* if it is nonempty and

(F1) for any two objects $x, y \in \mathcal{C}_0$, there are an object $z \in \mathcal{C}_0$ and arrows $g \in \mathcal{C}(x, z)$ and $h \in \mathcal{C}(y, z)$;
(F2) for any two parallel arrows $g, h \in \mathcal{C}(x, y)$, there are $z \in \mathcal{C}_0$ and $k \in \mathcal{C}(y, z)$ with $kg = kh$.

These conditions for $\mathcal{C}_P$ are equivalent to the following *Ore conditions* for P:

(O1) for all $x_1, x_2 \in P$, there are $y_1, y_2 \in P$ with $x_1y_1 = x_2y_2$;
(O2) if $xy_1 = xy_2$ for $y_1, y_2, x \in P$, then there is $z \in P$ with $y_1z = y_2z$.

Definition 3.7. We call P a *right Ore monoid* if it has these two properties or, equivalently, $\mathcal{C}_P$ is filtered. We call P a *left Ore monoid* if P^{op} is a *right Ore monoid*.

Condition (O2) follows if P has cancellation. Both hold if $P \subseteq G$ for a group G with $PP^{-1} = G$. Cancellative Ore monoids have already been considered by C*-algebraists; see, for instance, [16, 50]. We do not expect product systems over non-cancellative monoids to be very interesting (Lemma 4.16 hints strongly towards this), but it costs little extra effort to work in this greater generality. The monoid P is cancellative if and only if there is at most one arrow between any two objects in $\mathcal{C}_P$. Then $\mathcal{C}_P$ is the category associated to a directed set, and colimits over $\mathcal{C}_P$ are the same as colimits over this directed set. Directed sets are familiar to analysts as the indexing sets for nets.

Let P be a right Ore monoid. We may construct a group out of P by taking equivalence classes of formal quotients $pq^{-1} := (p,q)$ for $p,q \in P$, where $(p_1,q_1) \sim (p_2,q_2)$ if there are $t_1,t_2 \in P$ with $(p_1t_1,q_1t_1) = (p_2t_2,q_2t_2)$ (see also [13]). Condition (O1) implies that this relation is transitive and that products $p_1q_1^{-1} \cdot p_2q_2^{-1}$ may be rewritten as pq^{-1} by finding a common multiple of q_1 and p_2: if $q_1t_1 = p_2t_2$, then

$$p_1q_1^{-1} \cdot p_2q_2^{-1} = (p_1t_1)(q_1t_1)^{-1} \cdot (p_2t_2)(q_2t_2)^{-1} = (p_1t_1)(q_2t_2)^{-1}.$$

Hence we define the multiplication by $[p_1,q_1]\cdot[p_2,q_2] := [p_1t_1,q_2t_2]$ for $t_1,t_2 \in P$ with $q_1t_1 = p_2t_2$. The conditions (O1) and (O2) imply that this is a well-defined group structure on $G := P/{\sim}$.

Example 3.8. All commutative monoids are Ore: we may take $y_1 = x_2$ and $y_2 = x_1$ in (O1) and $z = x$ in (O2). Groups are also clearly Ore monoids. We may combine both classes as follows.

Assume that $G \subseteq P$ is a group such that $xg_1 = xg_2$ for $x \in P$, $g_1,g_2 \in G$ implies $g_1 = g_2$; assume further that $pG = Gp$ for all $p \in P$ and $p_1p_2G = p_2p_1G$ for all $p_1,p_2 \in P$; roughly speaking, G is a normal subgroup of P such that P/G is commutative. We claim that such a monoid P is Ore. In (O1), given $x_1,x_2 \in P$, we first take $y_1 = x_2$ and $y_2^0 = x_1$ as in the commutative case; then $x_1y_1G = x_2y_2^0G$, so there is $g \in G$ with $x_1y_1 = x_2y_2^0g$, so $y_2 = y_2^0g$ will do. In (O2), assume $xy_1 = xy_2$ for some $x,y_1,y_2 \in P$. Then $y_1xG = xy_1G = xy_2G = y_2xG$, so there is $g \in G$ with $y_1x = y_2xg$. Then $xy_1x = xy_2xg = xy_1xg$, which implies $g = 1$ by one of our assumptions. Thus $y_1x = y_2x$, so $z = x$ works in (O2).

Example 3.9. Let R be a commutative, unital ring. Let $R^\times \subseteq R$ be the subset of all elements that are not zero divisors; this is the largest submonoid of $(R,\cdot)$ that does not contain 0. The "$ax+b$-monoid" of R is the monoid $P = R^\times \ltimes R$, consisting of pairs $(a,b) \in R^\times \ltimes R$ with the multiplication $(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2, a_1b_2+b_1)$. This monoid is cancellative because we have taken away the zero divisors. It acts on R by $(a,b)\cdot x := ax+b$. It contains the additive group $(R,+)$ as a normal subgroup, and the quotient P/R is the commutative monoid $(R^\times,\cdot)$. Hence it is a special case of Example 3.8. C*-algebras associated to semigroups of this form have recently been studied by several authors, following Cuntz [15] and Li [54].

Example 3.10. Assume that P is cancellative and has a "norm" homomorphism $N\colon P \to (\mathbb{N}^\times,\cdot)$ such that $G := \ker N$ is a subgroup. This is a special case of the situation in Example 3.8, with $P/G \subseteq (\mathbb{N}^\times,\cdot)$. One example of this type is the monoid $\mathbb{M}_n(\mathbb{Z})^\times$, the ring of integer matrices with non-zero determinant, with the determinant as norm: the kernel of the norm is the group $\mathrm{Gl}_n(\mathbb{Z})$. Another example is the monoid $\mathbb{M}_n(\mathbb{Z})^\times \ltimes \mathbb{Z}^n$, with the determinant of the matrix part as norm and the semidirect product group $\mathrm{Gl}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ as the kernel of the norm. More generally, we may replace $\mathbb{M}_n(\mathbb{Z})$ by the ring of integers in a simple algebra over $\mathbb{Q}$; this also includes integer quaternion algebras. Semigroups of this form appear in generalisations of the Bost–Connes dynamical system, see [8, 14].

Example 3.11. The matrices of the form

$$h(a,b,c) := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for $a,b,c \in \mathbb{N}$ form a noncommutative, cancellative monoid $H_\mathbb{N}$ under matrix multiplication:

$$h(a_1,b_1,c_1) \cdot h(a_2,b_2,c_2) = h(a_1+a_2, b_1+b_2, c_1+c_2+a_1b_2).$$

This is an Ore monoid. To check the Ore condition (O1), pick $h(a_1,b_1,c_1)$ and $h(a_2,b_2,c_2)$ in $H_\mathbb{N}$. Let

$$a := \max(a_1,a_2), \quad b := \max(b_1,b_2), \quad c := \max(c_1+a_1(b-b_1), c_2+a_2(b-b_2)).$$

For $i = 1,2$, let $a_i^\perp := a - a_i$, $b_i^\perp := b - b_i$, and $c_i^\perp := c - c_i - a_i b_i^\perp$; then $h(a_i,b_i,c_i) \cdot h(a_i^\perp, b_i^\perp, c_i^\perp) = h(a,b,c)$ for $i = 1,2$, so we have found the desired common multiple. A similar formula works for the opposite monoid, so $H_\mathbb{N}$ is both left and right Ore.

An inductive limit in the usual sense is the same as a colimit over the category associated to the poset $(\mathbb{N}, \le)$, which is easily seen to be filtered. Colimits over general filtered categories behave very much like inductive limits. This is well-known to category theorists. For the operator algebraists, we now assume that P is a countable Ore monoid, so that $\mathcal{C}_P$ is a countable filtered category. Then we may replace a colimit over $\mathcal{C}_P$ by an inductive limit over $(\mathbb{N}, \le)$:

Lemma 3.12. *Let $\mathcal{C}$ be a countable filtered category. Then there is a sequence of objects $(x_n)_{n\in\mathbb{N}}$ and maps $f_n \in \mathcal{C}(x_{n-1}, x_n)$ such that for any object y of $\mathcal{C}$ there is $n \in \mathbb{N}$ and an arrow $y \to x_n$. Furthermore, if $y \to x_n$ and $y \to x_m$ are two such arrows, they become equal by composing with $f_{N-1} \circ \dots \circ f_n \colon x_n \to x_{n+1} \to \dots \to x_N$ and $f_{N-1} \circ \dots \circ f_m \colon x_m \to x_{m+1} \to \dots \to x_N$ for sufficiently large N.*

Such a sequence of objects and maps is called *cofinal* or final. More precisely, the functor $(\mathbb{N}, \le) \to \mathcal{C}$ given by the objects x_n and the maps f_n is called final in [57].

Proof. It is shown in [5] that any filtered category receives a cofinal functor from a directed (partially ordered) set. A partially ordered set is viewed as a category by putting a unique arrow $x \to y$ if $x \le y$, and no arrow otherwise. A category is of this form if and only if for any two objects there is at most one arrow between them. To simplify the proof, we first use [5] to reduce to a countable, directed set. The category $\mathcal{C}_P$ comes from a directed set if and only if P has cancellation.

Let $(y_n)_{n\in\mathbb{N}}$ be an enumeration of the objects of $\mathcal{C}$. We construct x_n for $n \in \mathbb{N}$ inductively so that it receives maps from $y_1, \dots, y_n$. We start with $x_0 = y_0$. Assume x_i and f_i have been constructed for $i < n$. Since $\mathcal{C}$ is filtered, there is an object x_n that receives maps from y_n and x_{n-1}. Let f_n be the arrow $x_{n-1} \to x_n$. Since already x_{n-1} receives maps from y_i for $i < n$, so does x_n by composing with f_n. Thus every object y has a map to some x_n. Our simplifying assumption makes the second part of the lemma trivial. □

We now describe the colimit of the inductive system on $\mathcal{C}_P$ given by the C*-algebras $\mathbb{K}(\mathcal{E}_p)$ for $p \in P$ and the maps $\varphi_{p,q}$ for $p, q \in P$ defined in (3.4).

We first do this quickly in the countable case. Then Lemma 3.12 allows us to choose a cofinal functor $(\mathbb{N}, \le)$ to $\mathcal{C}_P$, that is, we get a pair of sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ in P with $p_{n+1} = p_n q_n$ for all $n \in \mathbb{N}$ that is "cofinal" in $\mathcal{C}_P$. The C*-algebras $\mathbb{K}(\mathcal{E}_{p_n})$ and the nondegenerate *-homomorphisms $\varphi_{p_n,q_n}\colon \mathbb{K}(\mathcal{E}_{p_n}) \to \mathbb{K}(\mathcal{E}_{p_n q_n}) = \mathbb{K}(\mathcal{E}_{p_{n+1}})$ form an inductive system in the usual sense. Let $\mathcal{O}_1$ be its inductive limit C*-algebra. Cofinality implies that this inductive limit is also a colimit of the whole diagram on $\mathcal{C}_P$.

Now we give the more complicated construction without using Lemma 3.12, which also works in the uncountable case. Let

$$\mathcal{O}_\sqcup := \bigsqcup_{p\in P} \mathbb{K}(\mathcal{E}_p).$$

Let $\mathcal{O}_\sim$ be the set of equivalence classes for the equivalence relation on $\mathcal{O}_\sqcup$ generated by the relations $(x, p) \sim (\varphi_{p,q}(x), pq)$ for all $p, q \in P$, $x \in \mathbb{K}(\mathcal{E}_p)$.

Lemma 3.13. *There is a unique *-algebra structure for which all maps $\mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_\sim$ are *-homomorphisms, and the maximal* C*-*seminorm on $\mathcal{O}_\sim$ exists. Let $\mathcal{O}_1$ be the resulting* C*-*completion of $\mathcal{O}_\sim$. If $(\mathcal{F}, V_p)$ is a transformation from $(A, \mathcal{E}_p, \mu_{p,q})$ to a* C*-*algebra B, then the resulting maps $\vartheta_p\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{F})$ factor through a unique nondegenerate *-homomorphism $\Theta\colon \mathcal{O}_1 \to \mathbb{B}(\mathcal{F})$.*

Proof. Let $x \in \mathbb{K}(\mathcal{E}_p)$, $y \in \mathbb{K}(\mathcal{E}_q)$. There are $t_1, t_2 \in P$ with $pt_1 = qt_2$. Then $(x, p) \sim (\varphi_{p,t_1}(x), pt_1)$ and $(y, q) \sim (\varphi_{q,t_2}(y), qt_2)$ both belong to the C*-algebra $\mathbb{K}(\mathcal{E}_{pt_1}) = \mathbb{K}(\mathcal{E}_{qt_2})$; this dictates what their sum or product should be in $\mathcal{O}_\sim$. If we choose $t_1', t_2' \in P$ with $pt_1' = qt_2'$ instead, then we may find $m_1, m_2 \in P$ with $pt_1'm_1 = pt_1m_2$ and hence $qt_2'm_1 = qt_2m_2$. If not yet $t_2'm_1 = t_2m_2$, then we find $n \in P$ with $t_2'm_1n = t_2m_2n$ and replace m_1, m_2 by m_1n, m_2n. Similarly, we achieve $t_1'm_1 = t_1m_2$. Then multiplication with m_2 and m_1 will map our two choices of the sum or product to the same sum or product in $\mathbb{K}(\mathcal{E}_{pt_1m_2})$, respectively. Thus the multiplication and addition on $\mathcal{O}_\sim$ are well-defined.

A similar argument shows that any finite subset of $\mathcal{O}_\sim$ belongs to the image of $\mathbb{K}(\mathcal{E}_p)$ in $\mathcal{O}_\sim$ for some $p \in P$. Since the algebraic operations are defined using those in $\mathbb{K}(\mathcal{E}_p)$, $\mathcal{O}_\sim$ is a *-algebra. By construction, this is the only *-algebra structure for which all maps $\mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_\sim$ are *-homomorphisms.

The kernel of the map $\mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_\sim$ is the union of the kernels of the *-homomorphisms $\varphi_{q,p}\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq})$. Thus the image of $\mathbb{K}(\mathcal{E}_p)$ in $\mathcal{O}_\sim$ is the quotient by a union of closed *-ideals. We equip it with the quotient seminorm, which is a C*-seminorm (there may be a nullspace because the union of ideals need not be closed). All these C*-seminorms on subalgebras of $\mathcal{O}_\sim$ together are compatible with each other and thus define a C*-seminorm on $\mathcal{O}_\sim$. Since any *-homomorphism between C*-algebras is contractive, this is the maximal C*-seminorm on the *-algebra $\mathcal{O}_\sim$. Let $\mathcal{O}_1$ be the (Hausdorff) completion of $\mathcal{O}_\sim$ for this C*-seminorm. This is a C*-algebra with *-homomorphisms $\vartheta_p^0\colon \mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_1$ for all $p \in P$ that satisfy $\vartheta_{pq}^0 \circ \varphi_{p,q} = \vartheta_p^0$ for all $p, q \in P$.

Now take a transformation to B as above. The resulting maps $\vartheta_p\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{F})$ satisfy $\vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p$. Hence the map $\bigsqcup_{p\in G} \vartheta_p\colon \mathcal{O}_\sqcup \to \mathbb{B}(\mathcal{F})$ descends to a map $f\colon \mathcal{O}_\sim \to \mathbb{B}(\mathcal{F})$. Since all ϑ_p are *-homomorphisms, so is f. Since we took the maximal C*-seminorm on $\mathcal{O}_\sim$, we may extend f uniquely to a *-homomorphism $\Theta\colon \mathcal{O}_1 \to \mathbb{B}(\mathcal{F})$ with $\Theta \circ \vartheta_p^0 = \vartheta_p$ for all $p \in P$. □

Any functor $(p_n, q_n)\colon (\mathbb{N}, \le) \to \mathcal{C}_P$ induces a *-homomorphism from the inductive limit C*-algebra of the inductive system $(\mathbb{K}(\mathcal{E}_{p_n}), \varphi_{p_n,q_n})$ described above Lemma 3.13 to $\mathcal{O}_1$. If the functor is cofinal, then this map is an isomorphism. Hence the simplified construction for countable P gives the same C*-algebra $\mathcal{O}_1$.

So far, we have described only a part of the Cuntz–Pimsner algebra of the product system. For a single endomorphism, this is the fixed-point subalgebra of the gauge action. We now describe the whole Cuntz–Pimsner algebra through a Fell bundle over the group completion G of P.

Elements of G are equivalence classes of formal fractions $p_1 p_2^{-1}$ for $p_1, p_2 \in P$, with $p_1 p_2^{-1} \sim (p_1 q)(p_2 q)^{-1}$. The fibre of the desired Fell bundle over G at $1 \in G$ is the C*-algebra $\mathcal{O}_1$ described above.

Definition 3.14. Fix $g \in G$. Let

$$R_g = \{(p_1, p_2) \in P \times P \mid p_1 p_2^{-1} = g \text{ in } G\}$$

be its set of representatives. Let $\mathcal{C}_P^g$ be the category with object set R_g and arrow set $R_g \times P$, where (p_1, p_2, q) is an arrow $(p_1, p_2) \to (p_1 q, p_2 q)$; the multiplication is $(p_1 q, p_2 q, t) \cdot (p_1, p_2, q) = (p_1, p_2, tq)$.

Lemma 3.15. *The categories $\mathcal{C}_P^g$ for $g \in G$ are filtered if P is an Ore monoid. The functor $\mathcal{C}_P \to \mathcal{C}_P^1$ that maps an object p in $\mathcal{C}_P$ to (p, p) and an arrow (p, q) in $\mathcal{C}_P$ to (p, p, q) is cofinal.*

Proof. First we check that $\mathcal{C}_P^g$ is filtered. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be elements of R_g. We must prove two things. First, there should be arrows $h\colon p \to t$ and $k\colon q \to t$ with the same target $t \in R_g$. Secondly, if $h, k\colon p \rightrightarrows q$ are two parallel arrows, there is an arrow $l\colon q \to t$ for some object t such that $l \circ h = l \circ k$. Since p and q both represent $g \in G$, there are $h, k \in P$ with $p_1 h = q_1 k$ and $p_2 h = q_2 k$. Hence $h\colon (p_1, p_2) \to (p_1 h, p_2 h)$ and $k\colon (q_1, q_2) \to (q_1 k, q_2 k)$ have the same target, as desired. The second claim above is immediate from (O2): we may simply forget p_2 and q_2.

The functor $\mathcal{C}_P \to \mathcal{C}_P^1$ is fully faithful. If $(p_1, p_2) \in R_1$, then there are $q, h \in P$ with $(p_1 h, p_2 h) = (q, q)$. Hence the functor $\mathcal{C}_P \to \mathcal{C}_P^1$ is cofinal. □

For $(p_1, p_2) \in R_g$, let $\mathcal{O}_{p_1,p_2} := \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$; for now, we view this as a Banach space. For $q \in P$, $(p_1, p_2) \in R_g$, we define a contraction

$$\varphi_{p_1,p_2,q} \colon \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{K}(\mathcal{E}_{p_2 q}, \mathcal{E}_{p_1 q}), \qquad T \mapsto \mu_{p_1,q}(T \otimes_A \mathrm{id}_{\mathcal{E}_q})\mu^*_{p_2,q}.$$

These maps form a functor from $\mathcal{C}_P^g$ to the category of Banach spaces with linear contractions. Since $\mathcal{C}_P^g$ is filtered by Lemma 3.15, the colimit $\mathcal{O}_g$ of this diagram may be constructed as in Lemma 3.13: first take the disjoint union of the Banach spaces $\mathcal{O}_{p_1,p_2}$ for all $(p_1, p_2) \in R_g$; then divide out the relations given by the maps $\varphi_{p_1,p_2,q}$; this gives a vector space, and it inherits a canonical seminorm by taking the quotient seminorms on the images of $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$; finally, take the completion to get $\mathcal{O}_g$. If P is countable, then we may also use a cofinal sequence in $\mathcal{C}_P^g$ to describe the colimit as an inductive limit over $(\mathbb{N}, \le)$.
Since the functor $\mathcal{C}_P \to \mathcal{C}_P^1$ is cofinal, the colimit of a diagram over $\mathcal{C}_P^1$ is the same as the colimit of its restriction to $\mathcal{C}_P$. Hence the construction of $\mathcal{O}_g$ for $g = 1$ gives the same C*-algebra $\mathcal{O}_1$ as in Lemma 3.13, as suggested by our notation.
If $g_1, g_2 \in G$, $(p_1, p_2) \in R_{g_1}$, $(p_2, p_3) \in R_{g_2}$, then $p_1 p_3^{-1} = p_1 p_2^{-1} \cdot p_2 p_3^{-1} = g_1 \cdot g_2$, that is, $(p_1, p_3) \in R_{g_1 \cdot g_2}$. The composition of compact operators gives a bounded bilinear map $\mathcal{O}_{p_1,p_2} \times \mathcal{O}_{p_2,p_3} \to \mathcal{O}_{p_1,p_3}$. These maps define a bounded bilinear map

$$\mathcal{O}_{g_1} \times \mathcal{O}_{g_2} \to \mathcal{O}_{g_1 g_2}$$

because for any (p_1', p_2'), (p_2, p_3) in $R_{g_1} \times R_{g_2}$ there are $h, k \in P$ with $p_2 h = p_2' k$, so that the composition is defined on $\mathcal{O}_{p_1' k, p_2' k} \times \mathcal{O}_{p_2 h, p_3 h}$, and these composition maps are compatible with the structure maps of the inductive systems. Similarly, taking adjoints gives maps $\mathcal{O}_{p_1,p_2} \to \mathcal{O}_{p_2,p_1}$, $T \mapsto T^*$, for all $p_1, p_2 \in R_g$; these maps induce an involution $\mathcal{O}_g \to \mathcal{O}^*_{g^{-1}}$. These multiplication maps and involutions on $(\mathcal{O}_g)_{g\in G}$ give a Fell bundle over the group G. The resulting C*-algebra structure on its unit fibre $\mathcal{O}_1$ is the one already described in Lemma 3.13.

THEOREM 3.16. *Let P be an Ore monoid and let $(A, \mathcal{E}_p, \mu_{p,q})$ be a proper, nondegenerate product system over P. Its Cuntz–Pimsner algebra is isomorphic to the full sectional* C*-*algebra of the Fell bundle $(\mathcal{O}_g)_{g\in G}$ described above.*

Proof. Let C denote the Cuntz–Pimsner algebra of our product system. By construction, a nondegenerate *-homomorphism $C \to \mathcal{M}(B)$ for a C*-algebra B is the same as a Cuntz–Pimsner covariant representation of our product system on B that is nondegenerate on the unit fibre A. The Cuntz–Pimsner covariance condition is equivalent to the nondegeneracy condition $\mathcal{E}_p \cdot B = B$ for all $p \in P$ by Proposition 2.5 because we assume all $\mathcal{E}_p$ to be proper and nondegenerate left A-modules, and the left A-action on B is nondegenerate as well.
We are going to find a natural bijection between representations of the product system with $\mathcal{E}_p \cdot B = B$ for all $p \in P$ and representations of the

Fell bundle $(\mathcal{O}_g)_{g\in G}$ in $\mathcal{M}(B)$. By the universal property of the sectional C*-algebra of a Fell bundle, this gives a natural bijection between nondegenerate *-homomorphisms $C \to \mathcal{M}(B)$ and $\mathrm{C}^*((\mathcal{O}_g)_{g\in G}) \to \mathcal{M}(B)$, and this implies $C \cong \mathrm{C}^*((\mathcal{O}_g)_{g\in G})$.
By Proposition 2.3, a representation of the product system that is nondegenerate in the above sense is equivalent to a transformation from $(A, \mathcal{E}_p, \mu_{p,q})$ to B with underlying Hilbert B-module B. We write $\mathcal{F} = B$ to be consistent with our previous notation. We already constructed *-homomorphisms $\vartheta_p : \mathbb{K}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{F})$ with $\vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p$ for all $p, q \in P$. The same recipe gives linear contractions

$$\vartheta_{p_1,p_2} : \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{B}(\mathcal{F}), \qquad T \mapsto V_{p_1}(T \otimes \mathrm{id}_{\mathcal{F}})V_{p_2}^*.$$

These satisfy $\vartheta_{p_1q,p_2q} \circ \varphi_{p_1,p_2,q} = \vartheta_{p_1,p_2}$ for all $p_1, p_2, q \in P$. Hence they induce maps $\Theta_g : \mathcal{O}_g \to \mathbb{B}(\mathcal{F})$ on the Banach space inductive limits. Routine computations show that

$$\vartheta_{p_2,p_1}(T)^* = \vartheta_{p_1,p_2}(T^*), \qquad \vartheta_{p_1,p_2}(T) \circ \vartheta_{p_2,p_3}(T_2) = \vartheta_{p_1,p_3}(T \circ T_2) \tag{3.17}$$

for all $p_1, p_2, p_3 \in P$, $T \in \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$, $T_2 \in \mathbb{K}(\mathcal{E}_{p_3}, \mathcal{E}_{p_2})$. Hence the maps Θ_g form a representation of the Fell bundle $(\mathcal{O}_g)_{g\in G}$.
Conversely, a representation of the Fell bundle $(\mathcal{O}_g)_{g\in G}$ gives maps

$$\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{B}(\mathcal{F})$$

that satisfy (3.17). For $p_2 = 1$, there is a canonical isomorphism $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \cong \mathcal{E}_{p_1}$ because $\mathcal{E}_1 = A$. Hence the Fell bundle representation gives maps $S_p : \mathcal{E}_p \to \mathbb{B}(\mathcal{F})$. Since $A = \mathbb{K}(\mathcal{E}_1) \subseteq \mathcal{O}_1$, the conditions of a Fell bundle representation imply that the maps S_p form a representation of the product system. Since the maps $\mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_1 \to \mathbb{B}(\mathcal{F})$ are nondegenerate, $S_p(\mathcal{E}_p)\mathcal{F} \supseteq \mathbb{K}(\mathcal{E}_p)\mathcal{F} = \mathcal{F}$. This gives the desired bijection between Fell bundle representations and Cuntz–Pimsner covariant representations of the product system and finishes the proof. □

Theorem 3.16 is similar to [49, Theorem 3.8], only the assumptions on the product system differ slightly. Unlike in [49], we assume the product system to be nondegenerate, but allow the left action to be non-injective. Similar ideas are used in [16, Section 4] to dilate certain actions of Ore semigroups by endomorphisms to group actions on a larger algebra. In fact, actions by endomorphisms are a special case of product systems by Remark 3.1. To reduce the dilation results in [16] or [50] to Theorem 3.16, another step is missing: checking when the Fell bundle in Theorem 3.16 comes from an ordinary group action by automorphisms. We refrain from doing this because, from our point of view, saturated Fell bundles are already actions of the underlying group.

Proposition 3.18. *The Fell bundle $(\mathcal{O}_g)_{g\in G}$ is saturated if $\mathcal{E}_p$ is a full Hilbert A-module for each $p \in P$.*

Proof. Let $g \in G$ and let $p \in P$. We want to show that the image of $\mathbb{K}(\mathcal{E}_p)$ in $\mathcal{O}_1$ is contained in the space of right inner products from $\mathcal{O}_g$. There is

$(p_1, p_2) \in R_g$ and $q \in P$ with $pq = p_1$. The image of $\mathbb{K}(\mathcal{E}_p)$ in $\mathcal{O}_1$ is contained in the image of $\mathbb{K}(\mathcal{E}_{pq}) = \mathbb{K}(\mathcal{E}_{p_1})$.

Since $\mathcal{E}_{p_1}$ and $\mathcal{E}_{p_2}$ are full, both $\mathbb{K}(\mathcal{E}_{p_1})$ and $\mathbb{K}(\mathcal{E}_{p_2})$ are Morita–Rieffel equivalent to A and hence equivalent to each other. The equivalence between them is $\mathbb{K}(A, \mathcal{E}_{p_1}) \otimes_A \mathbb{K}(\mathcal{E}_{p_2}, A) \cong \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$. Hence the latter is a full Hilbert bimodule over $\mathbb{K}(\mathcal{E}_{p_2})$ and $\mathbb{K}(\mathcal{E}_{p_1})$. Since $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \subseteq \mathcal{O}_g$, it follows that the right inner products from $\mathcal{O}_g$ give a dense subspace of $\mathbb{K}(\mathcal{E}_{p_1})$ in $\mathcal{O}_1$. Thus the Fell bundle $(\mathcal{O}_g)_{g\in G}$ is saturated. □

Remark 3.19. The criterion in Proposition 3.18 is not necessary for rather trivial reasons. If the left A-actions on $\mathcal{E}_p$ are not faithful, then it may happen that $\mathcal{O}_1 = 0$. Since this has nothing to do with $\mathcal{E}_p$ being full as a right Hilbert module, the Fell bundle $(\mathcal{O}_g)$ may be saturated although not all $\mathcal{E}_p$ are full.

Saturated Fell bundles over a group G are interpreted as actions of G by correspondences in [12]. Long before, it was known that one may replace a saturated Fell bundle $(\mathcal{O}_g)_{g\in G}$ with unit fibre $\mathcal{O}_1$ by an action of G by automorphisms on a C*-algebra $\tilde{\mathcal{O}}_1$ that is Morita–Rieffel equivalent to $\mathcal{O}_1$: this is the Packer–Raeburn Stabilisation Trick. Non-saturated Fell bundles over G are interpreted in [11] as actions of G by Hilbert bimodules, that is, *partial Morita–Rieffel equivalences*. The analogue of the Packer–Raeburn Stabilisation Trick says that any Fell bundle, saturated or not, is equivalent to an action of G by partial *-isomorphisms.

A saturated Fell bundle over G may, of course, be restricted to a product system over P. Which product systems are of this form?

Proposition 3.20. *A proper product system over P is the restriction of a saturated Fell bundle over G if and only if each $\mathcal{E}_p$ is an A,A-imprimitivity bimodule, that is, each $\mathcal{E}_p$ is a full right Hilbert A-module and the left action is by an* isomorphism $A \cong \mathbb{K}(\mathcal{E}_p)$. *The saturated Fell bundle over G is unique up to isomorphism.*

Proof. In a saturated Fell bundle over G, each $\mathcal{E}_g$ is an imprimitivity bimodule. Conversely, assume that $\mathcal{E}_p$ is an imprimitivity bimodule for each $p \in P$. Then all the maps $\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq})$ in our inductive system are isomorphisms, so that the inductive limit $\mathcal{O}_1$ is isomorphic to $A = \mathbb{K}(\mathcal{E}_1)$. Similarly, $\mathcal{O}_p \cong \mathcal{E}_p$ for all $p \in P$. Thus our product system is the restriction to P of a Fell bundle over G. Since all $\mathcal{E}_p$ are assumed to be full, this Fell bundle is saturated by Proposition 3.18.

Now start with a saturated Fell bundle $(\mathcal{O}_g)_{g\in G}$, restrict it to P, and then go back to a Fell bundle over G. The maps $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{K}(\mathcal{E}_{p_2 q}, \mathcal{E}_{p_1 q})$ are isomorphisms for all $p_1, p_2, q \in P$, so the inductive systems that give the fibres of the new Fell bundle are constant. Thus the colimit $\mathcal{O}_g$ is canonically isomorphic to $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$ for any $(p_1, p_2) \in R_g$, and our construction of a Fell bundle from $(\mathcal{E}_p)_{p\in P}$ reproduces the original Fell bundle up to isomorphism. Hence the product system on P determines the saturated Fell bundle over G uniquely up to isomorphism. □

A non-saturated Fell bundle over G need not give a *proper* product system on P: this requires $\mathcal{E}_p$ to be full as a left Hilbert A-module for each $p \in P$.

Theorem 3.21. *If A is nuclear or exact, then so is $\mathcal{O}_1$. If A is nuclear and the group G generated by P is amenable, then the Cuntz–Pimsner algebra $\mathcal{O}$ is nuclear. If A is exact and G is amenable, then $\mathcal{O}$ is exact.*

Proof. The first claim follows because $\mathcal{O}_1$ is an inductive limit of C*-algebras Morita–Rieffel equivalent to A and because nuclearity and exactness are hereditary under Morita–Rieffel equivalence and filtered inductive limits.
The other statements follow from Theorem 3.16 and general results about nuclearity and exactness of Fell bundle C*-algebras. First, if the group is amenable, then any Fell bundle over it has the approximation property, which implies that the full and reduced sectional C*-algebras coincide (see [24]). The exactness of the reduced sectional C*-algebra is proved in [25], assuming exact unit fibre and an exact group. The nuclearity of the full sectional C*-algebra is proved in [1], assuming nuclear unit fibre and an amenable group. □

Next we describe the K-theory of the unit fibre $\mathcal{O}_1$ of our Fell bundle. Since $\mathcal{E}_p$ is a proper correspondence from A to A, it gives an element $[\mathcal{E}_p] \in \mathrm{KK}_0(A, A)$ with zero operator F. This gives a map

$$(\mathcal{E}_p)_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(A);$$

here $\mathrm{K}_*(A)$ denotes the $\mathbb{Z}/2$-graded K-theory of A comprising both K_0 and K_1. The Kasparov product of $[\mathcal{E}_p]$ and $[\mathcal{E}_q]$ is $[\mathcal{E}_p \otimes_A \mathcal{E}_q]$ with zero operator; since the Fredholm operator is irrelevant, this case of the Kasparov product is easy. The isomorphisms $\mu_{p,q}$ now show that $[\mathcal{E}_p] \otimes_A [\mathcal{E}_q] = [\mathcal{E}_{pq}]$ and hence $(\mathcal{E}_q)_* \circ (\mathcal{E}_p)_* = (\mathcal{E}_{pq})_*$ for all $p, q \in P$. The order of p and q is changed here because $\otimes_A$ is the composition product in KK in reverse order. Hence our product system over P gives an action of P^{op} on $\mathrm{K}_*(A)$. We view this as a right module structure over the monoid ring $\mathbb{Z}[P]$. The group ring $\mathbb{Z}[G]$ is a left module over $\mathbb{Z}[P]$.

Theorem 3.22. *Let P be an Ore monoid and let $(A, \mathcal{E}_p, \mu_{p,q})$ give a proper, nondegenerate product system over P. Assume also that all $\mathcal{E}_p$ are full right Hilbert A-modules. Then the* K*-theory of $\mathcal{O}_1$ is isomorphic to $\mathrm{K}_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$ as a right $\mathbb{Z}[G]$-module; here we use the canonical right module structure on $\mathrm{K}_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$ by right multiplication and the module structure on $\mathrm{K}_*(\mathcal{O}_1)$ induced by the saturated Fell bundle $(\mathcal{O}_g)_{g \in G}$.*

Proof. It is well-known that K-theory is compatible with inductive limits. This extends to colimits over countable filtered categories by Lemma 3.12. We leave it to the reader interested in uncountable monoids to check that the result remains true for arbitrary filtered colimits. Hence $\mathrm{K}_*(\mathcal{O}_1)$ is the colimit of the diagram over $\mathcal{C}_P$ that maps $p \in P$ to $\mathrm{K}_*(\mathbb{K}(\mathcal{E}_p))$ and $(p, q)\colon p \to pq$ to $(\varphi_{p,q})_*\colon \mathrm{K}_*(\mathbb{K}(\mathcal{E}_p)) \to \mathrm{K}_*(\mathbb{K}(\mathcal{E}_{pq}))$.
Since $\mathcal{E}_p$ is a full Hilbert bimodule, it gives a Morita–Rieffel equivalence from $\mathbb{K}(\mathcal{E}_p)$ to A. This correspondence with zero operator F is a cycle for

$\mathrm{KK}_0(\mathbb{K}(\mathcal{E}_p), A)$. This is a KK-equivalence: the inverse is the inverse imprimitivity bimodule $\mathbb{K}(\mathcal{E}_p, A)$ with zero operator F. We use this KK-equivalence to identify $\mathrm{K}_*(\mathbb{K}(\mathcal{E}_p)) \cong \mathrm{K}_*(A)$ for all $p \in P$. Composing the maps

$$\mathrm{K}_*(A) \underset{\cong}{\to} \mathrm{K}_*(\mathbb{K}(\mathcal{E}_p)) \xrightarrow{(\varphi_{p,q})_*} \mathrm{K}_*(\mathbb{K}(\mathcal{E}_{pq})) \underset{\cong}{\to} \mathrm{K}_*(A)$$

requires composing three KK_0-cycles with zero operator F, which amounts to tensoring the underlying correspondences. Identifying $\mathcal{E}_{pq} \cong \mathcal{E}_p \otimes_A \mathcal{E}_q$ as in the definition of $\varphi_{p,q}$, we see that this composite is $[\mathcal{E}_q]$. Thus the inductive system with colimit $\mathrm{K}_*(\mathcal{O}_1)$ is isomorphic to the inductive system with entries $\mathrm{K}_*(A)$ at all $p \in P$, where the arrow $(p,q)\colon p \to pq$ in $\mathcal{C}_P$ induces the map $(\mathcal{E}_q)_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(A)$.

Define a diagram of left $\mathbb{Z}[P]$-modules over $\mathcal{C}_P$ by taking the free module $\mathbb{Z}[P]$ at all objects and letting $q\colon p \to pq$ act by $\delta_x \mapsto \delta_{xq}$ for all $x,p,q \in P$. The colimit of this diagram of modules is isomorphic to $\mathbb{Z}[G]$ by mapping $\mathbb{Z}[P] \ni \delta_x$ at the object p of $\mathcal{C}_P$ to $\delta_{xp^{-1}} \in \mathbb{Z}[G]$. Hence $M \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$ for a right $\mathbb{Z}[P]$-module M is the colimit of the diagram over $\mathcal{C}_P$ with entries $M \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P] \cong M$ and with $q\colon p \to pq$ acting by $m \mapsto m \cdot q$ for all $m \in M$, $p,q \in P$. Now compare this with our description of the inductive system that computes $\mathrm{K}_*(\mathcal{O}_1)$ to get $\mathrm{K}_*(\mathcal{O}_1) \cong \mathrm{K}_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$.

Since the Hilbert modules $\mathcal{E}_p$ are full, the Fell bundle $(\mathcal{O}_g)_{g\in G}$ is saturated by Proposition 3.18. Then each $\mathcal{O}_g$ is a proper correspondence from $\mathcal{O}_1$ to itself and hence gives a class $[\mathcal{O}_g]$ in $\mathrm{KK}_0(\mathcal{O}_1, \mathcal{O}_1)$. This induces maps $(\mathcal{O}_g)_*\colon \mathrm{K}_*(\mathcal{O}_1) \to \mathrm{K}_*(\mathcal{O}_1)$. Since we have a saturated Fell bundle, we have $\mathcal{O}_g \otimes_{\mathcal{O}_1} \mathcal{O}_h \cong \mathcal{O}_{gh}$. Therefore, $g \mapsto (\mathcal{O}_g)_*$ defines a representation of G^{op} on the Abelian group $\mathrm{K}_*(\mathcal{O}_1)$; we view this as a right $\mathbb{Z}[G]$-module structure.

To describe this action, it suffices to compute, for $p \in P$, how $(\mathcal{O}_g)_*$ acts on the image of $\mathrm{K}_*(\mathbb{K}(\mathcal{E}_p))$ in $\mathrm{K}_*(\mathcal{O}_1)$ under the map $\mathrm{K}_*(\vartheta^0_p)$ induced by $\vartheta^0_p\colon \mathbb{K}(\mathcal{E}_p) \to \mathcal{O}_1$. First choose $(p_1,p_2) \in R_g$ and then $q \in P$ with $pq = p_1$. Then $\mathrm{K}_*(\vartheta^0_p) = \mathrm{K}_*(\vartheta^0_{p_1}) \circ \mathrm{K}_*(\varphi_{p,q})$, so it suffices to describe how $(\mathcal{O}_g)_*$ acts on the image of $\mathrm{K}_*(\mathbb{K}(\mathcal{E}_{p_1}))$. The map $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathcal{O}_g$ shows that $(\vartheta^0_{p_1})^*(\mathcal{O}_g) \cong \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \otimes_{\vartheta^0_{p_2}} \mathcal{O}_1$ as correspondences from $\mathbb{K}(\mathcal{E}_{p_1})$ to $\mathcal{O}_1$. Now compose these correspondences with the KK-equivalences between $\mathbb{K}(\mathcal{E}_{p_1})$ and A. Then we see that $(\mathcal{O}_g)_*$ acts on the entry $\mathrm{K}_*(A)$ at p_1 in the inductive system describing $\mathrm{K}_*(\mathcal{O}_1)$ by sending it to the same entry at p_2. Right multiplication by $g = p_1p_2^{-1}$ in $\mathrm{K}_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$ has the same effect. Thus the action of G on $\mathrm{K}_*(\mathcal{O}_1)$ induced by the Fell bundle corresponds to the one by right multiplication on $\mathrm{K}_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$. □

By the Packer–Raeburn Stabilisation Trick, there is a G-action by automorphisms on the stabilisation $\tilde{\mathcal{O}}_1 := \mathcal{O}_1 \otimes \mathbb{K}(L^2G)$ such that the (full) crossed product $G \ltimes \tilde{\mathcal{O}}_1$ is Morita–Rieffel equivalent to the Cuntz-Pimsner algebra $\mathcal{O}$ (this follows from [12, Corollary 5.5]). Thus computing the K-theory of the Cuntz–Pimsner algebra becomes a matter for the (full) Baum–Connes conjecture for G with certain coefficients.

For a-T-menable groups, the Baum–Connes assembly map is known to be an isomorphism for all coefficients, also for the full crossed product (see [36]). The meaning of the Baum–Connes conjecture here is that we may compute $\mathrm{K}_*(\mathcal{O})$ by topological means from $\mathrm{K}_*(\mathcal{O}|_H)$, the section algebras for restrictions of $(\mathcal{O}_g)_{g\in G}$ to all finite subgroups H. These topological means may be expressed as a spectral sequence, and it can be quite hard to perform this computation in practice. At least, the results above show that the computation for a Cuntz–Pimsner algebra over P is not more difficult than in the special case of an action of G by automorphisms.

For instance, let $P = (\mathbb{N}^k, +)$ for $k \in \mathbb{N}$. Then $G = \mathbb{Z}^k$, and the computation of $\mathrm{K}_*(\mathcal{O})$ is a matter of iterating the Pismner–Voiculescu sequence k times. We will consider a concrete case where a K-theory computation along these lines is feasible in Section 3.3. Already two iterations may be very hard because the boundary maps for the second iteration are not determined by the original data.

Remark 3.23. The iteration of the Pimsner–Voiculescu sequence that we get is equivalent to one by Deaconu in [20]. This is because the Pimsner–Voiculescu sequence for $\mathbb{Z}$-actions *can* be obtained from the Cuntz–Toeplitz algebra of the product system over $\mathbb{N}$ associated to the $\mathbb{Z}$-action.

Our theory contains the case of semigroup crossed products for actions of left Ore monoids by endomorphisms by Remark 3.1. In that case, $(\mathcal{E}_p)_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(A)$ is simply the map induced by the underlying endomorphism.

The K-theory computation for semigroup C*-algebras in [16] also uses the Baum–Connes isomorphism for the group G. In the situation of [16], a dilation of the semigroup action to an action of G on a larger C*-algebra is easy to write down by hand, giving a direct route to $\mathrm{K}_*(\mathcal{O}_1)$. But then extra assumptions on the action of G on $\mathrm{K}_*(\mathcal{O}_1)$ are needed to compute $\mathrm{K}_*(\mathcal{O})$.

3.1. Making left actions faithful. Let $(A, \mathcal{E}_p, \mu_{p,q})$ be a proper product system over an Ore monoid P. Taking suitable quotients of A and $\mathcal{E}_p$, we are going to construct another product system $(A', \mathcal{E}'_p, \mu'_{p,q})$ with the same nondegenerate representations and hence the same Cuntz–Pimsner algebra, such that the left actions $\varphi'_p\colon A' \to \mathbb{B}(\mathcal{E}'_p)$ are injective for all $p \in P$.

For $p \in P$, let $\varphi_p\colon A \to \mathbb{K}(\mathcal{E}_p)$ denote the left action map and let $I_p := \ker \varphi_p$. Recall the maps $\varphi_{p,q}\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq})$ for $p, q \in P$. Since $\varphi_{p,q} \circ \varphi_p = \varphi_{pq}$, we have $I_p \subseteq I_{pq}$ for all $p, q \in P$. Since $\mathcal{C}_P$ is filtered, this implies that the ideals I_p form a directed set of ideals in A. Thus $I := \overline{\bigcup_{p\in P} I_p}$ is another ideal in A. We let $A' := A/I$ and $\mathcal{E}'_p := \mathcal{E}_p \otimes_A A' \cong \mathcal{E}_p \,/\, (\mathcal{E}_p \cdot I)$.

Lemma 3.24. *The induced left action $A \to \mathbb{K}(\mathcal{E}'_p)$ factors through A', and the isomorphism $\mu_{p,q}\colon \mathcal{E}_p \otimes_A \mathcal{E}_q \xrightarrow{\sim} \mathcal{E}_{pq}$ descends to an isomorphism of correspondences $\mu'_{p,q}\colon \mathcal{E}'_p \otimes_A \mathcal{E}'_q \xrightarrow{\sim} \mathcal{E}'_{pq}$. This gives a product system $(A', \mathcal{E}'_p, \mu'_{p,q})$.*

Proof. Let $p \in P$. To prove that the induced left A-module structure on $\mathcal{E}'_p$ descends to A', we must show that $I\mathcal{E}_p \subseteq \mathcal{E}_p I$. Since P is an Ore monoid, the subset pP is cofinal in P, so $\bigcup_{q\in P} I_{pq}$ is still dense in I. Thus it suffices to prove $I_{pq}\mathcal{E}_p \subseteq \mathcal{E}_p I$ for all $p,q \in P$. We will prove the following more precise result:

$$I_{pq} = \{a \in A \mid a\mathcal{E}_p \subseteq \mathcal{E}_p I_q\}. \tag{3.25}$$

Let $\xi \in \mathcal{E}_p$. We have $\xi \otimes_A \eta = 0$ in $\mathcal{E}_p \otimes_A \mathcal{E}_q$ for all $\eta \in \mathcal{E}_q$ if and only if

$$0 = \langle \xi \otimes \eta_1, \xi \otimes \eta_2\rangle = \langle \eta_1, \varphi_q(\langle \xi, \xi\rangle_A)\eta_2\rangle$$

for all $\eta_1, \eta_2 \in \mathcal{E}_q$, if and only $\varphi_q(\langle \xi,\xi\rangle_A) = 0$, if and only if $\langle \xi,\xi\rangle_A \in I_q$. We claim that this is equivalent to $\xi \in \mathcal{E}_p \cdot I_q$. Since I_q is an ideal, we have $\langle \xi,\xi\rangle_A \in I_q$ for $\xi \in \mathcal{E}_p \cdot I_q$. Conversely, if $\langle \xi,\xi\rangle_A \in I_q$, then the closure of $\xi \cdot A$ in $\mathcal{E}_p$ is a Hilbert I_q-module containing ξ, and thus it is nondegenerate as a right I_q-module, so that $\xi \in \mathcal{E}_p I_q$. Hence $\xi \otimes_A \eta = 0$ in $\mathcal{E}_p \otimes_A \mathcal{E}_q$ for all $\eta \in \mathcal{E}_q$ if and only $\xi \in \mathcal{E}_p I_q$.

Now let $a \in A$. Then $a\xi \in \mathcal{E}_p I_q$ for all $\xi \in \mathcal{E}_p$ if and only if $a\xi \otimes_A \eta = 0$ for all $\xi \in \mathcal{E}_p$, $\eta \in \mathcal{E}_q$, if and only if the left action by a vanishes on $\mathcal{E}_p \otimes_A \mathcal{E}_q \cong \mathcal{E}_{pq}$. This is equivalent to $a \in I_{pq}$. This finishes the proof of (3.25). In turn, this implies that the left A-module structure on $\mathcal{E}'_p$ descends to A'. Now

$$\mathcal{E}'_p \otimes_{A'} \mathcal{E}'_q = \mathcal{E}_p \otimes_A A' \otimes_{A'} \mathcal{E}_q \otimes_A A' = \mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A A' \cong \mathcal{E}_{pq} \otimes_A A' = \mathcal{E}'_{pq}.$$

This gives the multiplication maps $\mu'_{p,q}$. From another point of view, $\mu'_{p,q}$ is the map on the quotient spaces $\mathcal{E}'_p$ and $\mathcal{E}'_q$ induced by $\mu_{p,q}$. Hence these maps inherit associativity from the maps $\mu_{p,q}$, so we have constructed a product system. □

Theorem 3.26. *The product system $(A', \mathcal{E}'_p, \mu'_{p,q})$ has faithful left action maps $A' \to \mathbb{K}(\mathcal{E}'_p)$, and it has the same nondegenerate representations as the original system. Hence it also has the same Cuntz–Pimsner algebra.*

Proof. Fix $p \in P$. An operator on $\mathcal{E}_p$ induces the zero operator on $\mathcal{E}_p/\mathcal{E}_p I_q \cong \mathcal{E}_p \otimes_A (A/I_q)$ if and only if it maps $\mathcal{E}_p$ into $\mathcal{E}_p I_q$. Thus (3.25) shows that the map $\varphi_p\colon A \to \mathbb{K}(\mathcal{E}_p)$ descends to an injective *-homomorphism $A/I_{pq} \hookrightarrow \mathbb{K}(\mathcal{E}_p/\mathcal{E}_p I_q)$. The C*-algebras A/I_{pq} and $\mathbb{K}(\mathcal{E}_p/\mathcal{E}_p I_q)$ for $q \in P$ form inductive systems indexed by the filtered category $\mathcal{C}_P$, and the maps $A/I_{pq} \hookrightarrow \mathbb{K}(\mathcal{E}_p/\mathcal{E}_p I_q)$ form a morphism of inductive systems, consisting of injective maps. It follows that the induced map between the inductive limits $\varinjlim A/I_{pq} = A/\overline{\bigcup_{q\in P} I_{pq}} = A'$ and $\varinjlim \mathbb{K}(\mathcal{E}_p/\mathcal{E}_p I_q) = \mathbb{K}(\mathcal{E}'_p)$ is injective as well. That is, the left action $A' \to \mathbb{K}(\mathcal{E}'_p)$ is faithful.

Now let $\vartheta'\colon A' \to \mathbb{B}(\mathcal{F})$ and $S'_p\colon \mathcal{E}'_p \to \mathbb{B}(\mathcal{F})$ for $p \in P$ give a nondegenerate representation of the product system $(A', \mathcal{E}'_p, \mu'_{p,q})$. Composing with the quotient maps $A \to A'$ and $\mathcal{E}_p \to \mathcal{E}'_p$ then gives a nondegenerate representation of $(A, \mathcal{E}_p, \mu_{p,q})$. We claim that any nondegenerate representation (ϑ, S_p) of $(A, \mathcal{E}_p, \mu_{p,q})$ factors through the quotient maps $A \to A'$ and $\mathcal{E}_p \to \mathcal{E}'_p$ and thus comes from a unique representation (ϑ', S'_p). This gives a bijection on the level of nondegenerate representations and thus an isomorphism of Cuntz–Pimsner algebras because they are universal for nondegenerate representations by Proposition 2.5.

Recall the maps $\vartheta_p\colon \mathbb{K}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{F})$ with $\vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p$ for all $p, q \in P$. In particular, $\vartheta = \vartheta_p \circ \varphi_p\colon A \to \mathbb{B}(\mathcal{F})$, so ϑ must vanish on I_p. Since this holds for all $p \in P$, we get $\vartheta|_I = 0$, so ϑ factors through the quotient map $A \to A'$. Since $S_p(\xi)^* S_p(\xi) = \vartheta(\langle \xi, \xi\rangle_A)$ for $\xi \in \mathcal{E}_p$ and $\langle \xi, \xi\rangle \in I$ for $\xi \in \mathcal{E}_p \cdot I$, we also get $S_p(\xi) = 0$ for $\xi \in \mathcal{E}_p \cdot I$. Thus S_p factors through $\mathcal{E}'_p$. □

Proposition 3.27. *If the maps $A \to \mathbb{K}(\mathcal{E}_p)$ are injective for all $p \in P$, then so are the induced maps $\varphi_{p,q,t}\colon \mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{qt}, \mathcal{E}_{pt})$ for $p, q, t \in P$ and the maps $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathcal{O}$ to the Cuntz–Pimsner algebra.*

Proof. We assume that $I_p = \{0\}$ for all $p \in P$. The proof of (3.25) shows that $\xi \in \mathcal{E}_p$ satisfies $\xi \otimes_A \eta = 0$ in $\mathcal{E}_p \otimes_A \mathcal{E}_q$ for all $\eta \in \mathcal{E}_q$ if and only if $\xi = 0$. Hence the maps $\varphi_{p,q,t}$ are injective. Since $\mathcal{O}_g \subseteq \mathcal{O}$ is the filtered colimit of the spaces $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p)$, this implies the same for the maps $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathcal{O}_g \subseteq \mathcal{O}$. □

3.2. What happens without the Ore conditions? We now consider an example of a monoid without the Ore conditions where we can, nevertheless, describe the Cuntz–Pimsner algebra by hand. Let F_n^+ be the free monoid on n generators, $n \geq 2$. Elements in F_n^+ are finite words in the letters $a_1, \dots, a_n$, including the empty word. This monoid violates the Ore conditions: there are no words $w_1, w_2 \in F_n^+$ with $a_1 w_1 = a_2 w_2$. A proper product system over F_n^+ is equivalent to a C*-algebra A with proper correspondences $\mathcal{E}_i$ from A to itself for $i = 1, \dots, n$, without any further data or conditions: given this data, we may define $\mathcal{E}_w$ for a word w by composing the correspondences for the letters in w, and we use the canonical multiplication maps between them.

Proposition 3.28. *Let A be a* C**-algebra and let $\mathcal{E}_i$ for $1 \leq i \leq n$ be proper correspondences from A to A. Let $\mathcal{O}_i$ be the Cuntz–Pimsner algebra of $\mathcal{E}_i$ for $1 \leq i \leq n$. The Cuntz–Pimsner algebra of the resulting product system over F_n^+ is the amalgamated free product of the Cuntz–Pimsner algebras $\mathcal{O}_i$ over A.*

Proof. Let D be another C*-algebra and let $\mathcal{G}$ be a Hilbert module over D. A nondegenerate representation of our product system over F_n^+ on $\mathcal{G}$ is already determined by what it does on the correspondences $\mathcal{E}_i$, and $\mathcal{E}_i$ may act by arbitrary nondegenerate representations because F_n^+ is a free monoid. A nondegenerate representation of $\mathcal{E}_i$ is equivalent to a representation of the Cuntz–Pimsner algebra $\mathcal{O}_i$ by Proposition 2.5. Since all these representations give the same representation when we compose with the canonical map $A \to \mathcal{O}_i$, we get a representation of the amalgamated free product of the C*-algebras $\mathcal{O}_i$ over A. Conversely, a representation of this free product gives nondegenerate representations of the correspondences $\mathcal{E}_i$ and thus of A, and it gives the same representation on A for each i. This data may be extended to a nondegenerate representation of the product system over F_n^+. □

Free products with amalgamation are, unfortunately, rather large and complicated. In particular, they are almost never nuclear or exact. Thus we view Proposition 3.28 as a negative result: it tells us that we should not expect Cuntz–Pimsner algebras for proper product systems over F_n^+ to have a nice structure. Standard assumptions in the theory of Cuntz–Toeplitz and Cuntz–Pimsner algebras are that the underlying semigroup be "quasi-lattice-ordered" and the product system "compactly aligned," see [33]. Both assumptions are satisfied for proper product systems over F_n^+. If two elements in F_n^+ have an upper bound, they have a least upper bound because two elements in F_n^+ only have an upper bound if one of them is a subword of the other, and then the longer of the two is a least upper bound. Hence Cuntz–Pimsner algebras of compactly aligned product systems over quasi-lattice-ordered monoids need not be tractable.

3.3. Higher-rank Doplicher–Roberts algebras.

In this section, we consider higher-rank analogues of the C*-algebras introduced by Doplicher and Roberts in [22]. The Doplicher–Roberts C*-algebras were an important motivation for Kumjian, Pask, Raeburn and Renault when they defined graph C*-algebras in [48].

Our higher-rank analogue is constructed from a compact Lie group G and finite-dimensional representations $\pi_1, \dots, \pi_k$ of G; in addition, we need a representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ that contains each irreducible representation of G. Different choices for ρ will, however, give Morita–Rieffel equivalent C*-algebras, so we consider ρ to be auxiliary data only. From the above data, we are going to construct a product system over the commutative monoid $(\mathbb{N}^k, +)$ and then take its Cuntz–Pimsner algebra. The case $k = 1$ is considered in [48].

For $m = (m_1, \dots, m_k) \in \mathbb{N}^k$, we form the representation

$$\pi^m := \pi_1^{\otimes m_1} \otimes \dots \otimes \pi_k^{\otimes m_k} \colon G \to \mathcal{U}(V^m);$$

here V^m denotes the finite-dimensional Hilbert space on which π^m acts. There are canonical unitary operators $\mu_{m_1,m_2}\colon V^{m_1} \otimes V^{m_2} \cong V^{m_1+m_2}$ that intertwine the representations $\pi^{m_1} \otimes \pi^{m_2}$ and $\pi^{m_1+m_2}$, and which satisfy the properties of a symmetric monoidal category.

Let $\mathcal{E}_m \subseteq \mathbb{K}(\mathcal{H}, V^m \otimes \mathcal{H})$ be the space of all compact intertwining operators between the representations ρ and $\pi^m \otimes \rho$. Define multiplication maps $\mathcal{E}_{m_1} \times \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$ by mapping (T_1, T_2) to the composite intertwining operator

$$\rho \xrightarrow{T_2} \pi^{m_2} \otimes \rho \xrightarrow{1\otimes T_1} \pi^{m_2} \otimes \pi^{m_1} \otimes \rho \xrightarrow{\mu_{m_2,m_1}\otimes 1} \pi^{m_1+m_2} \otimes \rho;$$

this composite is compact because T_2 is compact.

Lemma 3.29. *The multiplication above is associative.*

Proof. Let $m_1, m_2, m_3 \in \mathbb{N}^k$ and let $T_i \in \mathcal{E}_{m_i}$ for $i = 1, 2, 3$. Then the products $(T_1T_2)T_3$ and $T_1(T_2T_3)$ are equal to the composite operators

$$\begin{tikzcd}
\rho \arrow[r, "T_3"] & \pi^{m_3} \otimes \rho \arrow[r, "1\otimes T_2"] & \pi^{m_3} \otimes \pi^{m_2} \otimes \rho \arrow[r, "1\otimes 1\otimes T_1"] \arrow[d, "\mu_{m_3,m_2}\otimes 1"'] & \pi^{m_3} \otimes \pi^{m_2} \otimes \pi^{m_1} \otimes \rho \arrow[d, "\mu_{m_3,m_2}\otimes 1"] \\
& & \pi^{m_3+m_2} \otimes \rho \arrow[r, "1\otimes 1\otimes T_1"] & \pi^{m_3+m_2} \otimes \pi^{m_1} \otimes \rho \arrow[d, "\mu_{m_3+m_2,m_1}\otimes 1"] \\
& & & \pi^{m_3+m_2+m_1} \otimes \rho
\end{tikzcd}$$

because $\mu_{m_3+m_2,m_1} \circ (\mu_{m_3,m_2} \otimes 1) = \mu_{m_3,m_2+m_1} \circ (1 \otimes \mu_{m_2,m_1})$. □

The unit fibre $\mathcal{E}_0$ is the C*-algebra of all compact intertwining operators of ρ. The multiplication maps above turn each $\mathcal{E}_m$ into an $\mathcal{E}_0$-bimodule. We define an $\mathcal{E}_0$-valued right inner product on $\mathcal{E}_m$ by $\langle T_1, T_2\rangle := T_1^*T_2$ for $T_1, T_2 \in \mathcal{E}_m$. This turns each $\mathcal{E}_m$ into a correspondence from the C*-algebra $\mathcal{E}_0$ to itself.

Lemma 3.30. *The* C**-algebra* $\mathbb{K}(\mathcal{E}_m)$ *is isomorphic to the* C**-algebra of compact intertwiners of the representation* $\pi^m \otimes \rho$*, acting on* $\mathcal{E}_m$ *by left multiplication. More generally,* $\mathbb{K}(\mathcal{E}_{m_1}, \mathcal{E}_{m_2})$ *is isomorphic to the space of compact intertwiners* $\pi^{m_1} \otimes \rho \to \pi^{m_2} \otimes \rho$.

Proof. The map sending $|T_1\rangle\langle T_2| \in \mathbb{K}(\mathcal{E}_m)$ for $T_1, T_2 \in \mathcal{E}_m$ to the intertwiner $T_1T_2^*\colon \pi^m \otimes \rho \to \pi^m \otimes \rho$ extends to a *-homomorphism from $\mathbb{K}(\mathcal{E}_m)$ to the C*-algebra of compact intertwiners of $\pi^m \otimes \rho$. Since $|T_1\rangle\langle T_2|T_3 = T_1T_2^*T_3$, this representation is faithful. It remains to show that it is surjective.
Any compact intertwiner on $\pi^m \otimes \rho$ may be approximated by linear combinations of intertwiners with irreducible range because the representation $\pi^m \otimes \rho$, like any representation of G, is a direct sum of irreducible representations. Since any irreducible representation of G occurs in ρ, any intertwiner with irreducible range factors through the representation ρ. Thus we may write it as $T_1T_2^*$ for $T_1, T_2 \in \mathcal{E}_m$. This shows that $\mathbb{K}(\mathcal{E}_m)$ is mapped onto the C*-algebra of compact intertwiners of $\pi^m \otimes \rho$.
The same argument still works in the more general case of $\mathbb{K}(\mathcal{E}_{m_1}, \mathcal{E}_{m_2})$. □

If $T \in \mathcal{E}_0$, then the induced operator $1 \otimes T\colon \pi^m \otimes \rho \to \pi^m \otimes \rho$ is compact as well because the representation π^m has finite dimension. Thus the correspondence $\mathcal{E}_m$ is proper by Lemma 3.30.

Lemma 3.31. *The multiplication maps induce unitary operators* $\mathcal{E}_{m_1} \otimes_{\mathcal{E}_0} \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$.

Proof. It is routine to check that the map $\mathcal{E}_{m_1} \times \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$ defined above preserves the inner products, so it gives an isometry $\mathcal{E}_{m_1} \otimes_{\mathcal{E}_0} \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$. This induces a *-homomorphism $\mathbb{K}(\mathcal{E}_{m_1}) \to \mathbb{K}(\mathcal{E}_{m_1+m_2})$, $T \mapsto T \otimes 1$. In terms of Lemma 3.30, this is given by the map $T \mapsto T \otimes 1$ from compact intertwiners of $\pi^{m_1} \otimes \rho$ to compact intertwiners of $\pi^{m_1} \otimes \pi^{m_2} \otimes \rho \cong \pi^{m_1+m_2} \otimes \rho$. This

*-homomorphism on compact operators is nondegenerate. Hence the underlying isometry $\mathcal{E}_{m_1} \otimes_{\mathcal{E}_0} \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$ must be surjective. □

Lemma 3.31 says that the correspondences $\mathcal{E}_m$ with the above multiplication maps form an essential product system over the commutative monoid $(\mathbb{N}^k, +)$. As remarked above, Lemma 3.30 implies that this product system is proper. Since any irreducible representation occurs in ρ, the Hilbert $\mathcal{E}_0$-module $\mathcal{E}_m$ is full and carries a faithful left $\mathcal{E}_0$-action.

Definition 3.32. The Cuntz–Pimsner algebra of the product system $(\mathcal{E}_m)_{m\in\mathbb{N}^k}$ over $(\mathbb{N}^k, +)$ is the *higher-rank Doplicher–Roberts algebra* for the representations $\pi_1, \dots, \pi_n$ of G, relative to ρ.

Lemma 3.33. *The higher-rank Doplicher–Roberts algebras for different choices of ρ are canonically Morita–Rieffel equivalent.*

Proof. Let ρ and ρ' be two representations of G that contain all irreducible representations. Lemma 3.30 identifies $\mathcal{E}_0^\rho$ and $\mathcal{E}_0^{\rho'}$ with the C*-algebras of compact intertwiners on the representations ρ and ρ', respectively. Let $\mathcal{F}_{\rho\rho'}$ be the space of all compact intertwining operators $\rho' \to \rho$. This is a full Hilbert bimodule for $\mathcal{E}_0^\rho$ and $\mathcal{E}_0^{\rho'}$. Furthermore, we may naturally identify both $\mathcal{F}_{\rho\rho'} \otimes_{\mathcal{E}_0^{\rho'}} \mathcal{E}_m^{\rho'}$ and $\mathcal{E}_m^\rho \otimes_{\mathcal{E}_0^\rho} \mathcal{F}_{\rho\rho'}$ with the space of compact intertwiners from ρ' to $\pi^m \otimes \rho$. These identifications provide a Morita equivalence between the product systems for ρ and ρ' and thus induce a Morita–Rieffel equivalence between their Cuntz–Pimsner algebras. □

To clarify the link to previous constructions, take $k = 1$ and let ρ be the direct sum of all irreducible representations with multiplicity 1. Then the C*-algebra $\mathcal{E}_0$ of compact intertwiners of ρ is $\mathrm{C}_0(\hat{G})$. Since $k = 1$, our product system is determined by the single self-correspondence of $\mathrm{C}_0(\hat{G})$ given by $\mathcal{E}_1$. Such a self-correspondence is equivalent to a graph with vertex set $\hat{G}$. Since the left action on $\mathcal{E}_1$ is faithful and $\mathcal{E}_1$ is proper and full, our graph has neither sources nor sinks and no infinite emitters. Hence our absolute Cuntz–Pimsner algebra agrees with the relative one used to define graph C*-algebras. Our Doplicher–Roberts algebra is exactly the graph C*-algebra considered in [48, Section 7]. As shown there, the C*-algebra defined by Doplicher and Roberts in [22] is isomorphic to a full corner in this graph C*-algebra (assuming that each irreducible representation of G occurs in π^m for some $m \in \mathbb{N}$).

Remark 3.34. For $k > 1$, it seems unlikely that our higher-rank Doplicher–Roberts algebras are higher-rank graph C*-algebras. For $k = 1$, any product system over $(\mathbb{N}^k, +)$ with unit fibre of the form $\mathrm{C}_0(V)$ for a discrete set V ("vertices") gives a higher-rank graph C*-algebra. For $k > 1$, this fails: we also need the multiplication isomorphisms in the product system to be given by permutation matrices in some chosen bases for our self-correspondences.

Let $\mathcal{D}$ denote our higher-rank Doplicher–Roberts algebra. The general theory above applies here and shows that $\mathcal{D} = \bigoplus_{m\in\mathbb{Z}^k} \mathcal{D}_m$ is the section C*-algebra of a Fell bundle $(\mathcal{D}_m)_{m\in\mathbb{Z}^k}$ over $\mathbb{Z}^k$, where $\mathcal{D}_m$ is the filtered colimit of the Banach spaces $\mathbb{K}(\mathcal{E}_a, \mathcal{E}_{a+m})$ for $a \in \mathbb{N}^k$ with $a + m \in \mathbb{N}^k$. Here Lemma 3.30 identifies this Banach space with the space of compact intertwiners $\pi^a \otimes \rho \to \pi^{a+m} \otimes \rho$. In particular, the zero fibre $\mathcal{D}_0$ is the inductive limit of the system of C*-algebras $\mathbb{K}(\mathcal{E}_a)$ for $a \in \mathbb{N}^k$; the sequence given by $a = (a_1, a_1, \dots, a_1)$ for $a_1 \in \mathbb{N}$ is cofinal in $\mathbb{N}^k$, so we may as well take this sequence.

For each $a \in \mathbb{N}^k$, $\mathbb{K}(\mathcal{E}_a)$ is a C*-algebra of compact operators, hence a direct sum of matrix algebras. The summands are in bijection with $\hat{G}$ because all irreducible representations of G occur in ρ and hence in $\pi^a \otimes \rho$. In particular, $\mathbb{K}(\mathcal{E}_a)$ is Morita–Rieffel equivalent to $\mathrm{C}^*(G)$ for each $a \in \mathbb{N}^k$, and it is an AF-algebra whose K-theory is equal to the K-theory $\mathrm{K}_0(\mathrm{C}^*(G))$ of the group C*-algebra of G and hence to the representation ring of G. Concretely, its elements are functions $\hat{G} \to (\mathbb{Z}, +)$, and the positive cone in $\mathrm{K}_0(\mathbb{K}(\mathcal{E}_a))$ consists of all functions $\hat{G} \to (\mathbb{N}, +)$.

A countable inductive limit of AF-algebras remains an AF-algebra, so $\mathcal{D}_0$ is AF. We compute its K-theory. The group $\mathrm{K}_0(\mathrm{C}^*(G))$ is a commutative ring through the tensor product of representations: the representation ring of G. The map

$$\mathrm{K}_0(\mathrm{C}^*(G)) \cong \mathrm{K}_0(\mathbb{K}(\mathcal{E}_a)) \to \mathrm{K}_0(\mathbb{K}(\mathcal{E}_{a+m})) \cong \mathrm{K}_0(\mathrm{C}^*(G))$$

induced by the canonical *-homomorphism $\mathbb{K}(\mathcal{E}_a) \to \mathbb{K}(\mathcal{E}_{a+m})$ is the multiplication with $[\pi^m]$ in the ring structure on $\mathrm{K}_0(\mathrm{C}^*(G))$. Hence

$$\mathrm{K}_0(\mathcal{D}_0) = \varinjlim\bigl(\mathrm{K}_0(\mathrm{C}^*(G)) \xrightarrow{[\pi^{1_k}]} \mathrm{K}_0(\mathrm{C}^*(G)) \xrightarrow{[\pi^{1_k}]} \mathrm{K}_0(\mathrm{C}^*(G)) \to \cdots\bigr),$$

where $[\pi^{1_k}]$ denotes multiplication with the class of the representation $\pi_1 \otimes \cdots \otimes \pi_k$ in the representation ring. This inductive limit is the localisation of the representation ring $\mathrm{K}_0(\mathrm{C}^*(G))$ of G in which we invert $[\pi^{1_k}]$. Our K-theory computation determines $\mathcal{D}_0$ uniquely up to Morita–Rieffel equivalence by Elliott's classification of AF-algebras.

Each $\mathcal{D}_m$ is a Hilbert bimodule from $\mathcal{D}_0$ to itself, which acts on K_0 by multiplication with the class of π^m. From this information, it is sometimes possible to compute the K-theory of $\mathcal{D}$. We do not pursue this in general but merely consider one special case where we can completely describe the higher-rank Doplicher–Roberts algebra.

Theorem 3.35. *Let $G = \mathrm{SU}(n)$ for $n \geq 2$ and let $\pi_i = \Lambda^i(\mathbb{C}^n) \in \hat{G}$ for $i = 1, \dots, n-1$ be the exterior powers of the standard representation on $\mathbb{C}^n$. The associated rank-$n-1$ Doplicher–Roberts algebra $\mathcal{D}$ is purely infinite, simple, separable, nuclear and in the bootstrap class, and has $\mathrm{K}_0(\mathcal{D}) = \mathbb{Z}$, $\mathrm{K}_1(\mathcal{D}) = 0$.*

Kirchberg's Classification Theorem implies that $\mathcal{D}$ is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$, but we would not expect this isomorphism to be constructible.

Proof. The representations $\pi_1, \dots, \pi_{n-1} \in \hat{G}$ are the fundamental representations of $\mathrm{SU}(n)$, that is, they are irreducible and generate a ring isomorphism $\mathbb{Z}[x_1, \dots, x_{n-1}] \mapsto R(G)$, $x_i \mapsto [\pi_i]$.

Our Fell bundle description of $\mathcal{D}$ shows that it is stably isomorphic to a crossed product of an action of $\mathbb{Z}^{n-1}$ on an AF-algebra. Hence it is separable, nuclear and in the bootstrap class. We compute the K-theory of $\mathcal{D}$ by iterating the Pimsner–Voiculescu exact sequence $n-1$ times. We write $\mathcal{D}\rtimes\mathbb{Z}^i$ for the crossed product with $\mathbb{Z}^i \subseteq \mathbb{Z}^{n-1}$, although this is really a Fell bundle section algebra; the action by automorphisms only occurs on a stably isomorphic C*-algebra.
The AF-algebra $\mathcal{D}_0$ has $\mathrm{K}_0(\mathcal{D}_0) = \mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]$ because the representation ring of G is isomorphic to the polynomial algebra $\mathbb{Z}[x_1,\dots,x_{n-1}]$ with $x_i = [\pi_i]$ and localising at the elements $x_1,\dots,x_{n-1}$ simply adjoins their inverses. The elements $1-x_1,\dots,1-x_{n-1}$ form a regular sequence in this algebra; that is, for each i, multiplication by $1-x_{i+1}$ is injective on the quotient $\mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]/(1-[x_1],\dots,1-[x_i])$ by the ideal generated by $1-[x_1],\dots,1-[x_i]$. This is what allows us to compute the K-theory by repeated application of the Pimsner–Voiculescu exact sequence.
In each step, we are supposed to consider the kernel and cokernel of the map $1-\alpha_i$ on $\mathrm{K}_*(\mathcal{D}_0\rtimes\mathbb{Z}^{i-1})$, where α_i is induced by the action of the ith factor of $\mathbb{Z}$ on $\mathcal{D}_0\rtimes\mathbb{Z}^{i-1}$. By induction, we show that $\mathrm{K}_1(\mathcal{D}_0\rtimes\mathbb{Z}^i)$ vanishes and that $\mathrm{K}_0(\mathcal{D}_0\rtimes\mathbb{Z}^i)$ is the quotient ring $\mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]/(1-x_1,\dots,1-x_i) \cong \mathbb{Z}[x_{i+1}^{\pm1},\dots,x_{n-1}^{\pm1}]$. This is clear for $i=0$. In each induction step, we use the Pimsner–Voiculescu exact sequence. Since we have a regular sequence, multiplication by $1-x_i$ is injective on $\mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]/(1-x_1,\dots,1-x_{i-1})$. Thus $\mathrm{K}_1(\mathcal{D}_0\rtimes\mathbb{Z}^i)$ vanishes and $\mathrm{K}_0(\mathcal{D}_0\rtimes\mathbb{Z}^i)$ is $\mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]/(1-x_1,\dots,1-x_i)$. After $n-1$ steps, we get $\mathrm{K}_1(\mathcal{D})=0$ and

$$\mathrm{K}_0(\mathcal{D}) \cong \mathbb{Z}[x_1^{\pm1},\dots,x_{n-1}^{\pm1}]/(1-x_1,\dots,1-x_{n-1}) \cong \mathbb{Z}.$$

Thus $\mathcal{D}$ has the asserted K-theory.
Next we prove that $\mathcal{D}$ is simple. We do not claim that the AF-algebra $\mathcal{D}_0$ is simple. The crossed product $\mathcal{D}_0\rtimes\mathbb{Z}^1$, however, is simple by [48, Corollary 7.3] because the representation π_1 of G on $\mathbb{C}^n$ is faithful. This crossed product is just a rank-1 Doplicher–Roberts algebra, hence stably isomorphic to a graph algebra. The graph algebra description of $\mathcal{D}_0\rtimes\mathbb{Z}^1$ shows also that it is purely infinite.
The K-theory computation above shows that $\mathcal{D}_0\rtimes\mathbb{Z}^1$ has K-theory isomorphic to $\mathbb{Z}[x_2^{\pm1},\dots,x_{n-1}^{\pm1}]$, where the automorphism associated to $\pi_2^{m_2}\cdots\pi_{n-1}^{m_{n-1}}$ acts by multiplication with $x_2^{m_2}\cdots x_{n-1}^{m_{n-1}}$. Since this is never the identity map, none of these automorphisms can be inner. Since $\mathcal{D}_0\rtimes\mathbb{Z}^1$ is simple, separable and purely infinite, the (reduced) crossed product by the group $\mathbb{Z}^{n-2}$ remains simple and purely infinite by [46, Lemma 10]. Since the stabilisation $(\mathcal{D}_0\rtimes\mathbb{Z}^1)\rtimes\mathbb{Z}^{n-2}$ of $\mathcal{D}$ is simple and purely infinite, so is $\mathcal{D}$ itself. □

There is another way to construct higher-rank Doplicher–Roberts algebras using the comultiplication $\Delta\colon \mathrm{C}^*(G)\to\mathrm{C}^*(G)\otimes\mathrm{C}^*(G)$, which is defined by $\lambda_g\mapsto\lambda_g\otimes\lambda_g$ for the standard multipliers λ_g of $\mathrm{C}^*(G)$ for $g\in G$. This comultiplication turns $\mathrm{C}^*(G)$ into a discrete quantum group. Our comultiplication is

only a morphism, that is, its image is only in the multiplier algebra of $\mathrm{C}^*(G)$. We know, however, that $(\mathrm{C}^*(G)\otimes 1)\cdot\Delta(\mathrm{C}^*(G)) = \mathrm{C}^*(G)\otimes\mathrm{C}^*(G)$.
Let $\pi\colon \mathrm{C}^*(G)\to\mathbb{M}_n(\mathbb{C})$ be some finite-dimensional representation of $\mathrm{C}^*(G)$ or, equivalently, of G. We get a morphism

$$\mathrm{C}^*(G)\xrightarrow{\Delta}\mathrm{C}^*(G)\otimes\mathrm{C}^*(G)\xrightarrow{\pi\otimes\mathrm{id}}\mathbb{M}_n\otimes\mathrm{C}^*(G)\cong\mathbb{K}(\mathrm{C}^*(G)^n).$$

Since $(\mathrm{C}^*(G)\otimes 1)\cdot\Delta(\mathrm{C}^*(G)) = \mathrm{C}^*(G)\otimes\mathrm{C}^*(G)$, this morphism has values in $\mathbb{K}(\mathrm{C}^*(G)^n)$. Thus any finite-dimensional representation π of $\mathrm{C}^*(G)$ induces a *proper* correspondence $\mathcal{E}(\pi)$ from $\mathrm{C}^*(G)$ to itself.
Let $\pi_i\colon \mathrm{C}^*(G)\to\mathbb{M}_{n_i}(\mathbb{C})$ for $i=1,2$ be two finite-dimensional representations and let $\pi_1\otimes\pi_2\colon \mathrm{C}^*(G)\to\mathbb{M}_{n_1n_2}(\mathbb{C})$ be their tensor product representation. The coassociativity of Δ gives an isomorphism of correspondences

$$\mathcal{E}(\pi_1)\otimes_{\mathrm{C}^*(G)}\mathcal{E}(\pi_2)\cong\mathcal{E}(\pi_1\otimes\pi_2).$$

Hence the obvious intertwining unitary $\pi_1\otimes\pi_2\cong\pi_2\otimes\pi_1$ gives canonical isomorphisms $\mathcal{E}(\pi_1)\otimes_{\mathrm{C}^*(G)}\mathcal{E}(\pi_2)\cong\mathcal{E}(\pi_2)\otimes_{\mathrm{C}^*(G)}\mathcal{E}(\pi_1)$. The category of representations of G with the tensor product of representations and the obvious associators and commuters $\pi_1\otimes\pi_2\cong\pi_2\otimes\pi_1$ is a symmetric monoidal category. Therefore, k representations $\pi_1,\dots,\pi_k$ of G give a product system over the monoid $(\mathbb{N}^k,+)$ with fibres

$$\mathcal{E}(m_1,\dots,m_k) := \mathcal{E}(\pi_1^{\otimes m_1}\otimes\cdots\otimes\pi_k^{\otimes m_k})$$

and with the canonical isomorphisms

$$\mathcal{E}(m_1,\dots,m_k)\otimes_{\mathrm{C}^*(G)}\mathcal{E}(m_1',\dots,m_k')\cong\mathcal{E}(m_1+m_1',\dots,m_k+m_k').$$

We claim that this product system is the same as the one constructed above if ρ is the regular representation. A first point is that the C^*-algebra of compact intertwiners of ρ is canonically isomorphic to $\mathrm{C}^*(G)$, acting by the right regular representation. Hence $\mathcal{E}_0^\rho\cong\mathrm{C}^*(G)$. Furthermore, ρ absorbs every other representation by the Fell absorption principle: $\pi\otimes\rho$ is *canonically* isomorphic to a sum of n copies of ρ if π has dimension n; the intertwiners $L^2(G,\mathbb{C}^n)\leftrightarrow L^2(G,\mathbb{C}^n)$ are given by pointwise multiplication with the matrix π_g at $g\in G$. Hence we may identify $\mathrm{C}^*(G)^n$ canonically with the Hilbert $\mathcal{E}_0^\rho$-module of compact intertwiners $\rho\to\pi\otimes\rho$. These identifications provide an isomorphism between our product systems because the tensor product of representations of G is induced by the comultiplication Δ.

4. Actions of Ore monoids on spaces

Now let X be a locally compact, Hausdorff space and let $A=\mathrm{C}_0(X)$. Since any automorphism of A comes from a homeomorphism on X, we may turn an action of a group G on A by automorphisms into an action of G on the space X and form a transformation groupoid $G\ltimes X$. The crossed product $G\ltimes\mathrm{C}_0(X)$ is canonically isomorphic to the groupoid C^*-algebra of $G\ltimes X$. When is there such a groupoid model for the Cuntz–Pimsner algebra of a self-correspondence on A?

As a counterexample, consider a Hermitian vector bundle over X. It gives a proper self-correspondence from A to itself by taking the Hilbert module of sections with its usual inner product and the left action by pointwise multiplication. The resulting Cuntz–Pimsner algebra is a locally trivial field of C^*-algebras over X with Cuntz algebras as fibres. Such C^*-algebras are classified by Dădărlat in [18] in terms of certain cohomology groups. Unless the field of C^*-algebras over X is particularly simple, it seems to have no natural groupoid model.

Therefore, we restrict attention to self-correspondences of $\mathrm{C}_0(X)$ that are induced by topological correspondences (see [43]). We define product systems of such topological correspondences in the obvious fashion, so that they induce a product system of C^*-correspondences. We will build a "transformation groupoid" for a *proper* product system of topological correspondences and show that its groupoid C^*-algebra is isomorphic to the Cuntz–Pimsner algebra of the product system. Our transformation groupoid construction is similar in spirit to the boundary path groupoid of Yeend [79] for a higher-rank topological graph, that is, for the case $P = \mathbb{N}^k$ for some $k \geq 1$. Yeend's construction, however, depends on special features of $\mathbb{N}^k$. In contrast, our construction depends on the properness of the product systems.

A topological correspondence between two spaces X and Y is given by a third space M with two maps $r\colon M \to X$ and $s\colon M \to Y$. We want to turn this into a C^*-correspondence from $\mathrm{C}_0(X)$ to $\mathrm{C}_0(Y)$. There are two ways to do this. First, we may assume that s is a local homeomorphism; this is Katsura's definition of a *topological correspondence* in [43]. Other names for this are *continuous graphs* (see [19]) or *polymorphisms* (see [7, 17]). Secondly, we may add extra data, namely, a family of measures $(\lambda_x)_{x\in X}$ on the fibres of s; this is what Muhly and Tomforde call a *topological quiver* in [60]. The family of measures (λ_x) is equivalent to a *transfer operator* for s in the notation of Exel [26], or a Markov operator in the notation of [39]. A topological correspondence gives a topological quiver when combined with a suitably normalised family of counting measures on the (discrete) fibres of s.

A topological quiver (M, r, s, λ_x) gives a C^*-correspondence $\mathcal{E}_{r,M,s}$ over $\mathrm{C}_0(X)$: complete $\mathrm{C_c}(M)$ with respect to the $\mathrm{C}_0(X)$-valued inner product

$$\langle \xi_1, \xi_2\rangle(x) := \int_{s^{-1}(x)} \bigl(\overline{\xi_1}\xi_2\bigr)(y)\,\mathrm{d}\lambda_x(y)$$

for $\xi_1, \xi_2 \in \mathrm{C_c}(M)$, $x \in X$; the left and right module structures are $(f\xi)(m) := f(r(m))\xi(m)$ and $(\xi f)(m) := \xi(m)f(s(m))$ for all $m \in M$, $f \in \mathrm{C}_0(X)$, $\xi \in \mathrm{C_c}(M)$. In particular, this construction applies to topological correspondences, where we always take the family of counting measures.

Proposition 4.1. *The* C^**-correspondence* $\mathcal{E}_{r,M,s}$ *is proper if and only if* r *is proper and* s *is a local homeomorphism. In that case, the isomorphism class of* $\mathcal{E}_{r,M,s}$ *does not depend on* (λ_x)*, so we may always use the family of counting measures. The* C^**-correspondence* $\mathcal{E}_{r,M,s}$ *is full if and only if* s *is surjective.*

Proof. The C*-correspondence $\mathcal{E}_{r,M,s}$ is proper if and only if $\varphi^{-1}(\mathbb{K}(\mathcal{E}_{r,M,s})) = \mathrm{C}_0(X)$. In the notation of [60, Definition 3.14], all vertices are finite emitters. [60, Corollary 3.12] shows that this happens if and only if r is proper and s is a local homeomorphism.

Let (λ_x) and (λ'_x) be two families of measures that make (M, r, s) into a topological quiver. Since both λ_x and λ'_x have the same discrete subset $s^{-1}(x)$ as support, they are equivalent, say, $\lambda'_x = f_x \cdot \lambda_x$ for a unique function $f_x \colon s^{-1}(x) \to (0, \infty)$. The functions f_x may be pieced together to a function $f \colon M \to (0, \infty)$. The continuity of (λ_x) and (λ'_x) implies that f is a continuous function. Hence multiplication with $\sqrt{f}$ is a unitary operator between the Hilbert modules over $\mathrm{C}_0(X)$ associated to the two families of measures. This unitary also intertwines the left actions, which are by multiplication operators. It is routine to check that $\mathcal{E}_{r,M,s}$ is full if and only if s is surjective. □

Definition 4.2. A topological correspondence is called *proper* if r is proper and s is a local homeomorphism.

Lemma 4.3. *Consider two topological correspondences*

$$X \xleftarrow{r_1} M_1 \xrightarrow{s_1} X \xleftarrow{r_2} M_2 \xrightarrow{s_2} X.$$

Define $M := M_1 \times_{s_1,X,r_2} M_2$, $r \colon M \to X$, $(m_1, m_2) \mapsto r_1(m_1)$, $s \colon M \to X$, $(m_1, m_2) \mapsto s_2(m_2)$. *Then*

$$\mathcal{E}_{r_1,M_1,s_1} \otimes_{\mathrm{C}_0(X)} \mathcal{E}_{r_2,M_2,s_2} \cong \mathcal{E}_{r,M,s}.$$

If r_1 and r_2 are proper, so is r. If s_1 and s_2 are surjective, so is s.

Proof. The first part is routine to prove and holds even for topological quivers, see [60, Lemmas 6.1–4]. The statements about proper and surjective maps are easy as well; they amount to the statement that tensor products of proper or full C*-correspondences are again proper or full, respectively. □

Proposition 4.1 says that the C*-correspondence associated to a topological quiver is proper if and only if we are dealing with a proper topological correspondence; the family of measures does not matter. We restrict attention to proper topological correspondences from now on.

The notion of a "topological graph algebra" interprets a topological correspondence as a "topological graph," where vertices and (oriented) edges form topological spaces. This interpretation, however, fails to elucidate the lack of symmetry between r and s in the construction of the C*-correspondence. Another interpretation is that a topological correspondence (r, M, s) is a multi-valued map from Y to X, where $r(m) \in X$ for $m \in s^{-1}(y)$ are the possible values at $y \in Y$. If s is a local homeomorphism and r is proper, then the subset of values $r(s^{-1}(y))$ of y is discrete. The interpretation as a multivalued map breaks down, however, if there are different $m, m' \in M$ with $s(m) = s(m')$ and $r(m) = r(m')$. We suggest the following more dynamical interpretation of a (proper) topological correspondence.

We consider points in M as possible *developments* or, briefly, *stories*. Each story $m \in M$ assumes a certain *initial situation* $s(m) \in Y$ and leads to a

certain *ending* $r(m) \in X$. Several stories may have the same initial situation and ending.
How does this interpretation account for the assumptions that s be a local homeomorphism and r be proper? That s is a local homeomorphism means the following: if we modify the initial situation $s(m)$ of a story m a little bit, then there is a unique story m_x close to m with initial situation x. Roughly speaking, m_x describes how "the same" story would go in a slightly different initial situation, and fits our intuition of story-telling. That r is proper means that, given a compact set of possible endings, the set of stories with such an ending is also compact. This is a rather technical finiteness condition on the space of possible stories. It ensures that the space of complete histories defined below is locally compact.

Definition 4.4. Let P be a monoid. An *action of P on X by proper topological correspondences* consists of the following data:

- proper topological correspondences (M_p, r_p, s_p) from X to X for $p \in P \setminus \{1\}$;
- homeomorphisms $\sigma_{p,q}\colon M_{pq} \to M_p \times_{s_p,X,r_q} M_q$ for $p,q \in P \setminus \{1\}$.

Let $M_1 = X$ and $r_1 = s_1 = \mathrm{id}_X$, and let $\sigma_{p,1}$ and $\sigma_{1,q}$ be the canonical homeomorphisms $M_p \cong M_p \times_{s_p,X,\mathrm{id}_X} X$ and $M_q \cong X \times_{\mathrm{id}_X,X,r_q} M_q$ for $p,q \in P$. For an action of P, we require the diagram

$$\begin{tikzcd} M_p \times_X M_q \times_X M_t & M_{pq} \times_X M_t \arrow[l, "\sigma_{p,q} \times_X \mathrm{id}_{M_t}"'] \\ M_p \times_X M_{qt} \arrow[u, "\mathrm{id}_{M_p} \times_X \sigma_{q,t}"] & M_{pqt} \arrow[l, "\sigma_{p,qt}"] \arrow[u, "\sigma_{pq,t}"'] \end{tikzcd} \tag{4.5}$$

to commute for all $p,q,t \in P \setminus \{1\}$ (since $pq = 1$ or $qt = 1$ is possible, we have to define (M_1, s_1, r_1), $\sigma_{1,q}$ and $\sigma_{p,1}$ for this condition to make sense). This diagram commutes automatically if $p = 1$, $q = 1$ or $t = 1$, so our assumption implies that it commutes for all $p,q,t \in P$.

Example 4.6. An action of $\mathbb{N}^k$ on a countable discrete set X by proper topological correspondences is equivalent to a row-finite rank-k graph by [34]. The Cuntz–Pimsner algebra that we shall attach to this data is *not* always the higher-rank graph C*-algebra, however, because we do not incorporate Katsura's modification of the Cuntz–Pimsner algebra into our definition. See also [67].

We fix an action of P on X by proper topological correspondences as above. The proper topological correspondences (M_p, r_p, s_p) induce proper C*-correspondences $\mathcal{E}_p$ from $\mathrm{C}_0(X)$ to itself for $p \in P \setminus \{1\}$, and we let $\mathcal{E}_1 := \mathrm{C}_0(X)$. The homeomorphisms $\sigma_{p,q}$ induce isomorphisms of C*-correspondences

$$\mu_{p,q}\colon \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{E}_q \to \mathcal{E}_{pq}$$

for $p, q \in P \setminus \{1\}$ by Lemma 4.3, and we let $\mu_{1,q}$ and $\mu_{p,1}$ be the canonical isomorphisms. The diagram (4.5) ensures the associativity of these multiplication maps $\mu_{p,q}$ for all $p, q, t \in P \setminus \{1\}$ (even if $pq = 1$ or $qt = 1$); associativity is automatic if $p = 1$, $q = 1$ or $t = 1$. So an action of P on X by topological correspondences induces a proper product system over P with unit fibre $\mathrm{C}_0(X)$, as expected.

The defining property of the fibre product means that $\sigma_{p,q} = (r_{p,q}, s_{p,q})$ for two continuous maps

$$r_{p,q}\colon M_{pq} \to M_p, \qquad s_{p,q}\colon M_{pq} \to M_q$$

with $s_p \circ r_{p,q} = r_q \circ s_{p,q}$. Since $\sigma_{p,1}$ and $\sigma_{q,1}$ are the canonical maps,

$$s_{p,1} = s_p, \quad r_{p,1} = \mathrm{id}_{M_p}, \qquad s_{1,q} = \mathrm{id}_{M_q}, \quad r_{1,q} = r_q$$

for all $p, q \in P$. The associativity condition (4.5) is equivalent to

$$r_{p,q} \circ r_{pq,t} = r_{p,qt}, \quad s_{p,q} \circ r_{pq,t} = r_{q,t} \circ s_{p,qt}, \quad s_{q,t} \circ s_{p,qt} = s_{pq,t}. \tag{4.7}$$

Lemma 4.8. *The maps $r_{p,q}$ are proper and the maps $s_{p,q}$ are local homeomorphisms. If all s_p are surjective, then so are the maps $s_{p,q}$.*

Proof. The map $r_{p,q}$ is the composite of the homeomorphism $\sigma_{p,q}$ and the coordinate projection $M_p \times_{s_p,X,r_q} M_q \to M_p$. This coordinate projection is proper if r_q is proper because properness is hereditary under this type of fibre products. Similarly, the map $s_{p,q}$ is the composite of the homeomorphism $\sigma_{p,q}$ and the coordinate projection $M_p \times_{s_p,X,r_q} M_q \to M_q$; the latter inherits the property of being surjective or a local homeomorphism from s_p. □

We interpret elements of P as a (multi-dimensional) kind of *time*, and elements of M_p as *stories of length* $p \in P$; a story $m \in M_p$ *starts* in the situation $s_p(m)$ and *ends* in $r_p(m)$. The maps $r_{p,q}\colon M_{pq} \to M_p$ and $s_{p,q}\colon M_{pq} \to M_q$ cut a story m of length pq into two stories of length p and q: its *ending* $m_1 = r_{p,q}(m) \in M_p$ and its *beginning* $m_2 = s_{p,q}(m) \in M_q$. These satisfy $s_p(m_1) = r_q(m_2)$, that is, the story m_1 starts where m_2 ends. The inverse of $\sigma_{p,q}$ combines a pair $m_1 \in M_p$, $m_2 \in M_q$ of stories of lengths p and q to a story $m_1 \circ m_2$ of length pq, provided m_1 starts where m_2 ends. The assumption that $\sigma_{p,q}$ be a homeomorphism says that $m \in M_{pq}$ and $(m_1, m_2) \in M_p \times M_q$ with $s_p(m_1) = r_q(m_2)$ determine each other uniquely and continuously.

The length $1 \in P$ is the neutral element, so nothing can happen in time 1, and adding a story of length 1 before or after another story does nothing. This means that $M_1 = X$ and that $\sigma_{p,1}$ and $\sigma_{1,q}$ are the canonical maps. The associativity conditions (4.7) say that the two ways of cutting a story of length pqt into three pieces of length p, q and t give the same results.

If P is a free monoid on n generators (which, however, is not Ore), then the situation above may be interpreted as describing a game where the players may do n different things in each time interval. If, say, the player has the three options a, b, c, then $p = baac$ means a time interval of length 4 in which the player first does c, then twice a, then b. If the game was in situation $x \in X$ initially, then the points in $s_p^{-1}(x) \subseteq M_p$ are the possible game developments in this length-4 time period, provided the player's actions are *baac*. And $r_p(m)$ for $m \in s_p^{-1}(x)$ is the situation after this time period. If $s_p^{-1}(x)$ has more than one point, then the game contains randomness. It makes sense to quantify this randomness by a transfer operator with $\sum_{s_p(m)=y} \mu_p(m) = 1$ for all $x \in X$, where $\mu_p(m)$ is the probability that the game develops as in story m, given the initial situation y. We do not add such probabilities to our setup because they are irrelevant for us by Proposition 4.1.

A relation in the monoid P means that certain actions of the player always and automatically have the same effect on the game. For instance, if P is the free *Abelian* monoid $\mathbb{N}^n$ on n generators, then the order in which the player does various things does not matter. I know no game with this property; so the interpretation through games works best for free monoids.

There are three simple cases of actions by proper topological correspondences:

(1) $M_p = X$ and $s_p = \mathrm{id}_X$ for all $x \in X$, $p \in P$; that is, a situation $x \in X$ *determines its future* uniquely;
(2) $M_p = X$ and $r_p = \mathrm{id}_X$ for all $x \in X$, $p \in P$; that is, a situation $x \in X$ *determines its past* uniquely;
(3) M_p is arbitrary, but $s_p = r_p$ for all $p \in P$; that is, *the situation never changes*; then $s_p = r_p$ must be both proper and a local homeomorphism; equivalently, it is a finite covering map.

From now on, we assume that P is a right Ore monoid. In this case, the Cuntz–Pimsner algebra of the product system $(\mathcal{E}_p, \mu_{p,q})$ over P is described more concretely in Section 3. We are going to identify this Cuntz–Pimsner algebra with the groupoid C*-algebra of an étale, locally compact groupoid H. We first describe the object space H^0 of this groupoid. The first associativity condition in (4.7) says that the spaces M_p for $p \in P$ and the continuous maps $r_{p,q}$ for $p, q \in P$ form a projective system of locally compact spaces indexed by the directed category $\mathcal{C}_P$. We let

$$H^0 := \varprojlim_{\mathcal{C}_P} (M_p, r_{p,q}).$$

Thus a point in H^0 consists of $m_p \in M_p$ for all $p \in P$ that satisfy $r_{p,q}(m_{pq}) = m_p$ for all $p, q \in P$. In other words, the m_p are stories that are consistent in the sense that m_p is the ending of m_{pq} for each $p, q \in P$. We call a point in H^0 a *complete history* and think of m_p as describing what happened in the last length-p time period.

Lemma 4.9. *The space H^0 is locally compact and Hausdorff. The projection maps $\pi_q \colon H^0 \to M_q$, $(m_p)_{p\in P} \mapsto m_q$, are proper for all $q \in P$.*

Proof. Fix $(m_p)_{p\in P} \in H^0$ and let $K \subset X = M_1$ be a compact neighbourhood of m_1. The preimage of K in H^0 is the subset of all $(m'_p)_{p\in P}$ with $m'_1 \in K$ and hence $r_p(m'_p) \in K$ for all $p \in P$. Since all the maps r_p are proper, the subsets $r_p^{-1}(K) \subseteq M_p$ are compact. Hence so is the product $L := \prod_{p\in I} r_p^{-1}(K)$ by Tychonov's Theorem. Thus the map $\pi_1 \colon H^0 \to X$ is proper. The same argument shows that all the maps π_q are proper. Since L is also a compact neighbourhood of $(m_p)_{p\in P}$ in H^0, the space H^0 is locally compact. If $(m'_p) \neq (m_p)$, then there is $p \in P$ with $m'_p \neq m_p$. There are open neighbourhoods in M_p that separate m'_p and m_p. These yield open neighbourhoods in H^0 that separate (m'_p) and (m_p), so H^0 is Hausdorff. □

Given a complete history $(m_p)_{p\in P}$ and $t \in P$, we may forget what happened in the last time period of length t; this gives another complete history, defined formally by $m'_p := s_{t,p}(m_{tp})$ for $p \in P$; the second condition in (4.7) implies $r_{p,q}(m'_{pq}) = m'_p$ for all $p, t \in P$, that is, $(m'_p)_{p\in P}$ is again a complete history as expected. Thus $(m_p) \mapsto (m'_p)$ defines a map $\tilde{s}_t \colon H^0 \to H^0$.

Lemma 4.10. *Let $t \in P$. The map $(\pi_t, \tilde{s}_t)\colon H^0 \to M_t \times_{s_t,X,\pi_1} H^0$ is a homeomorphism, and $\tilde{s}_t\colon H^0 \to H^0$ is a local homeomorphism. If s_t is surjective, so is $\tilde{s}_t$. If s_t is a homeomorphism, so is $\tilde{s}_t$.*

Proof. Let $(m_p)_{p\in P} \in H^0$. Then $\pi_t(m_p) = m_t$ and $\tilde{s}_t((m_p)) = (s_{t,p}(m_{tp}))_{p\in P}$. Since $m_{tp} = r_{t,p}(m_{tp}) \cdot s_{t,p}(m_{tp}) = m_t \cdot \tilde{s}_t((m_p))_p$, we have $\pi_1 \circ \tilde{s}_t = s_t \circ \pi_t$, that is, the image of $(\pi_t, \tilde{s}_t)$ is contained in the fibre product $M_t \times_{s_t,X,\pi_1} H^0$. Since the map $(\pi_t, \tilde{s}_t)$ is clearly continuous, we must prove that it is a bijection with a continuous inverse. So we let $(m'_p) \in H^0$ be a complete history and let $m_t \in M_t$ be a length-t story with $m'_1 = s_t(m_t) \in X$. We must show that there is a unique complete history (m_p) with given m_t and with $\tilde{s}_t((m_p)) = (m'_p)$, which depends continuously on (m'_p) and m_t.
First assume that $(m_p)_{p\in P}$ as above has been found. Let $q \in P$. Then we may write $qu = tp$ for some $u, p \in P$ because P is a right Ore monoid. The story m_{tp} is the concatenation $m_t \circ m'_p$ of $r_{t,p}(m_{pt}) = m_t$ and $s_{t,p}(m_{tp}) = m'_p$, which exists because $r_p(m'_p) = m'_1 = s_t(m_t)$. Thus $m_q = r_{q,u}(m_{qu}) = r_{q,u}(m_t \circ m'_p)$. So there is at most one possible solution $(m_p)_{p\in P}$, and it depends continuously on (m'_p) and m_t. We must show that the length-q stories $m_q := r_{q,u}(m_t \circ m'_p)$ with $u, p \in P$ as above form a complete history, that is, $r_{q,v}(m_{qv}) = m_q$ for all $q, v \in P$.
We have $m_{qv} = r_{qv,u_2}(m_t \circ m'_{p_2})$ for some $u_2, p_2 \in P$ with $qvu_2 = tp_2$. Since P is a right Ore monoid, there are $u_3, u_4 \in P$ with $vu_2u_3 = uu_4$. Then $tp_2u_3 = qvu_2u_3 = quu_4 = tpu_4$. Since P is a right Ore monoid, there is $u_5 \in P$ with $p_2u_3u_5 = pu_4u_5$. To simplify notation, we replace (u_3, u_4) by (u_3u_5, u_4u_5); thus $p_2u_3 = pu_4$.
Since $quu_4 = tpu_4$, we could also use (uu_4, pu_4) instead of (u, p) to define m_q. The associativity conditions in (4.7) show that this gives the same result:

$$\begin{aligned} r_{q,uu_4}(m_t \circ m'_{pu_4}) &= r_{q,u} r_{qu,u_4}(m_t \circ m'_{pu_4}) \\ &= r_{q,u} r_{qu,u_4}(m_t \circ m'_p \circ s_{p,u_4}(m'_{pu_4})) = r_{q,u}(m_t \circ m'_p). \end{aligned}$$

Similarly, we get the same result for m_{qv} if we use (u_2u_3, p_2u_3) instead of (u_2, p_2). Thus we may assume that $p_2 = p$ and $u = vu_2$. Then

$$r_{q,v}(m_{qv}) = r_{q,v}(r_{qv,u_2}(m_t \circ m'_p)) = r_{q,vu_2}(m_t \circ m'_p) = r_{q,u}(m_t \circ m'_p) = m_q.$$

This finishes the proof that $(\pi_t, \tilde{s}_t)$ is a homeomorphism. This homeomorphism transforms the map $\tilde{s}_t$ into the second coordinate projection $M_t \times_{s_t,X,\pi_1} H^0 \to H^0$, which is a (local) homeomorphism if s_t is one, and surjective if s_t is. □

First forgetting the last length-t time period and then the last length-u time period gives the same result as directly forgetting the last time period of length tu. That is, $\tilde{s}_u \circ \tilde{s}_t = \tilde{s}_{tu}$ for all $t, u \in P$. Formally, this follows from the third condition in (4.7). Thus the monoid P^{op} acts on H^0 by local homeomorphisms. Why do we get the opposite monoid here? The maps $\tilde{s}_t\colon H^0 \to H^0$ and $\tilde{r}_t := \mathrm{id}_{H^0}$ form an action of P by topological correspondences with the extra property that any situation determines its past uniquely: a "situation" in H^0 is a complete history, which simply contains its past. Thus we still have an action by topological correspondences, but one where the maps $\tilde{r}_p$ are all identity maps, so that we may forget about them. This gives an action of the opposite monoid P^{op} by local homeomorphisms because of the direction of the maps $\tilde{s}_p$.

Example 4.11. Suppose that we start with an action of P on X by proper maps r_p and let s_p be identity maps; that is, every situation determines its future. Then the maps $\tilde{s}_t$ are homeomorphisms by Lemma 4.10. Thus our action of P^{op} on H^0 extends to the group completion G of P^{op}. The groupoid model we are going to construct is the transformation groupoid of this group action.

Definition 4.12. The *transformation groupoid* $H := P^{\mathrm{op}} \ltimes H^0$ associated to the P^{op}-action $(\tilde{s}_p)$ on H^0 by local homeomorphisms has object space H^0, arrow set

$$H^1 := \{(x, g, y) \in H^0 \times G \times H^0 \mid \exists p_1, p_2 \in P,\ g = p_1 p_2^{-1},\ \tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)\},$$

range and source maps $r(x, g, y) := x$, $s(x, g, y) := y$, and multiplication

$$(x_1, g_1, y_1) \cdot (x_2, g_2, y_2) := (x_1, g_1 g_2, y_2)$$

if $y_1 = x_2$. The unit on $x \in H^0$ is $(x, 1, x)$, the inverse of (x, g, y) is (y, g^{-1}, x). We describe the topology on H^1. For $p_1, p_1 \in P$, let

$$H^1_{p_1,p_2} := H^0 \times_{\tilde{s}_{p_1},H^0,\tilde{s}_{p_2}} H^0 := \{(x, y) \in H^0 \times H^0 \mid \tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)\},$$

the fibre product of the diagram $H^0 \xrightarrow{\tilde{s}_{p_1}} H^0 \xleftarrow{\tilde{s}_{p_2}} H^0$. We give each $H^1_{p_1,p_2}$ the subspace topology from the product $H^0 \times H^0$, and $\bigsqcup_{p_1,p_2 \in P} H^1_{p_1,p_2}$ the disjoint union topology. We map $H^1_{p_1,p_2} \to H^1$ by $(x, y) \mapsto (x, p_1 p_2^{-1}, y) \in H^1$. This gives a surjection $\bigsqcup H^1_{p_1,p_2} \to H^1$. We give H^1 the quotient topology from this map.

To verify that the transformation groupoid has desirable properties, we rewrite it using filtered colimits. Let $H^1_g := \{(x, g, y) \in H^1\}$ for $g \in G$; so $H^1 = \bigsqcup_{g \in G} H^1_g$. We describe H^1_g for fixed $g \in G$ as a colimit over $\mathcal{C}^g_P$ (see Definition 3.14), which is a filtered category by Lemma 3.15. If $p_1 q = p_3$, $p_2 q = p_4$, then $\tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)$ implies $\tilde{s}_{p_3}(x) = \tilde{s}_{p_4}(y)$, so $H^1_{p_1,p_2} \subseteq H^1_{p_3,p_4} \subseteq H^0 \times H^0$.

Since right multiplication with q is locally injective, any $(x,y) \in H^1_{p_3,p_4}$ has a neighbourhood in $H^0 \times H^0$ so that for (x',y') in this neighbourhood, $\tilde{s}_{p_1}(x') \neq \tilde{s}_{p_2}(y')$ implies $\tilde{s}_{p_1q}(x') \neq \tilde{s}_{p_2q}(y')$. Thus the subset $H^1_{p_1,p_2}$ is relatively open in $H^1_{p_3,p_4}$; so the spaces $H^1_{p_1,p_2}$ for $(p_1,p_2) \in P^2$ and the inclusion maps $H^1_{p_1,p_2} \to H^1_{p_3,p_4}$ form a diagram of subsets of $H^0 \times H^0$ with open inclusion maps.

Lemma 4.13. *If $p_1p_2^{-1} = p_3p_4^{-1}$, then there are $p_5, p_6 \in P$ with $p_5p_6^{-1} = p_1p_2^{-1}$ and so that both $H^1_{p_1,p_2}$ and $H^1_{p_3,p_4}$ are open subsets of $H^1_{p_5,p_6}$; hence the subspace topologies from $H^1_{p_1,p_2}$ and $H^1_{p_3,p_4}$ coincide on $H^1_{p_1,p_2} \cap H^1_{p_3,p_4}$, and this subset is open both in $H^1_{p_1,p_2}$ and in $H^1_{p_3,p_4}$. Each $H^1_{p_1,p_2}$ is open in H^1_g, and the topology on H^1_g restricts to the given topology on each $H^1_{p_1,p_2}$.*

Proof. Since $\mathcal{C}_P^g$ is filtered, there is an object (p_5,p_6) that dominates both (p_1,p_2) and (p_3,p_4). This has all required properties. Thus all the embeddings $H^1_{p_1,p_2} \to H^1_{p_3,p_4}$ are open. This implies that the quotient topology on $H^1_g = \bigcup_{(p_1,p_2)\in R_g} H^1_{p_1,p_2}$ from $\bigsqcup_{(p_1,p_2)\in R_g} H^1_{p_1,p_2}$ restricts to the given topology on each subset $H^1_{p_1,p_2}$. □

Thus the subsets $H^1_{p_1,p_2}$ for $p_1,p_2 \in P$ form an *open* covering of H^1, and the topology on H^1 restricts to the usual topology on each $H^1_{p_1,p_2}$. In the following, we identify $H^1_{p_1,p_2}$ with its image in H^1, which is an open subset.

Proposition 4.14. *The groupoid H is étale, locally compact and Hausdorff. The decomposition $H^1 = \bigsqcup_{g\in G} H^1_g$ satisfies $H^1_g \cdot H^1_h \subseteq H^1_{gh}$ and $(H^1_g)^{-1} = H^1_{g^{-1}}$. If the maps s_p for $p \in P$ are surjective, then $H^1_g \cdot H^1_h = H^1_{gh}$ for all $g,h \in G$. The groupoid $H^1_1 \subseteq H^1$ is an increasing union of open subgroupoids that are proper and étale and describe equivalence relations on H^0.*

Proof. The space H^0 is locally compact and Hausdorff by Lemma 4.9. The coordinate projections $H^1_{p_1,p_2} \rightrightarrows H^0$ are étale because P acts by local homeomorphisms. Since the subsets $H^1_{p_1,p_2}$ for $p_1,p_2 \in P$ form an open covering of H^1, the coordinate projections $H^1 \rightrightarrows H^0$ are étale. Any two points of H^1_g are contained in the same Hausdorff, locally compact, open subset $H^1_{p_1,p_2}$ for suitable p_1,p_2, so they may be separated by open subsets of H^1_g; since the subsets H^1_g are open, it is also possible to separate points in H^1_g and H^1_h for $g \neq h$. Thus H^1 is Hausdorff.

If the maps s_p for $p \in P$ are surjective, then so are the maps $\tilde{s}_p$ for $p \in P$ by Lemma 4.10. Now let $(x,gh,y) \in H^1_{gh}$. Hence there are $p_1,p_2 \in P$ with $gh = p_1p_2^{-1}$ and $\tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)$. Write $g = p_3p_4^{-1}$, $h = p_5p_6^{-1}$. Then also $g = p_3q(p_4q)^{-1}$ and $h = (p_5t)(p_6t)^{-1}$ for all $q,t \in P$. The Ore condition (O1) allows us to choose q and t such that $p_4q = p_5t$. Hence we may assume without loss of generality that $p_4 = p_5$. Then

$$p_1p_2^{-1} = gh = p_3p_4^{-1}p_5p_6^{-1} = p_3p_6^{-1}.$$

If $\tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)$, then also $\tilde{s}_{p_1 t}(x) = \tilde{s}_{p_2 t}(y)$ for any $t \in P$, so we may rewrite $gh = (p_1 t)(p_2 t)^{-1}$. We may also replace (p_3, p_4, p_5, p_6) by $(p_3 q, p_4 q, p_5 q, p_6 q)$ for any $q \in P$. Choosing q and t by condition (O1), we may achieve $p_3 q = p_1 t$, $p_6 q = p_2 t$, by the definition of the group G. Hence we find $p_1, p_2, p_3 \in P$ with $gh = p_1 p_2^{-1}$, $g = p_1 p_3^{-1}$, $h = p_3 p_2^{-1}$ and $\tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)$. Since $\tilde{s}_{p_3}$ is surjective, we may choose $z \in H^0$ with $\tilde{s}_{p_3}(z) = \tilde{s}_{p_1}(x) = \tilde{s}_{p_2}(y)$. Then $(x, g, z) \in H^1_g$ and $(z, h, y) \in H^1_h$ satisfy $(x, g, z) \cdot (z, h, y) = (x, gh, y)$.

The open subgroupoid H^1_1 is defined as the union of $H^1_{p_1,p_2}$ with $(p_1, p_2) \in R_1$. Since $\mathcal{C}_P$ is cofinal in $\mathcal{C}^1_P$ by Lemma 3.15, $H^1_1 = \bigcup_{p\in P} H^1_{p,p}$. Here $H^1_{p,p}$ is the set of all $(x, y) \in H^0 \times H^0$ with $\tilde{s}_p(x) = \tilde{s}_p(y)$, and it carries the subspace topology from $H^0 \times H^0$. So $H^1_{p,p}$ is a proper equivalence relation on H^0, and H^1_1 is the union of these open subgroupoids. □

If P is countable, then we may choose a cofinal sequence in $\mathcal{C}^1_P$ and write H^1_1 as an increasing union of a sequence of proper étale equivalence relations. Hence H^1_1 is an *approximately proper equivalence relation* in the notation of [71]. These are called *hyperfinite relations* in [47]. We allow ourselves to call H^1_1 approximately proper also if P is uncountable, replacing a sequence of proper (finite) open subrelations by a directed set of such subrelations.

If the maps s_p for $p \in P$ are surjective, then the subsets H^1_g for $g \in G$ form a G-grading in the notation of [11]. This is equivalent to an action of G on the groupoid H^1_1 by Morita equivalences with transformation groupoid H. Thus we may think of H as the transformation groupoid associated to an action of G on the noncommutative orbit space H^1/H^1_1. Points in this orbit space are equivalence classes of complete histories, where two complete histories are identified if they coincide in the distant past, that is, $\tilde{s}_p(x) = \tilde{s}_p(y)$ for some $p \in P$. The group G acts on this by “time translations.”

If the maps s_p are not surjective, then the G-action on H^1 is only a *partial* action because time translations $x \mapsto px$ into the future are not everywhere defined. A partial G-action is the same as an action of a certain inverse semigroup associated to G, see [23].

Definition 4.15. A situation $x \in X$ is (historically) *possible* if $x \in r_p(M_p)$ for all $p \in P$.

Let $X' \subseteq X$ be the subset of possible situations. We have $X' = X$ if and only if all the maps r_p are surjective. Let $M'_p = s_p^{-1}(X')$ and let r'_p and s'_p be the restrictions of r_p and s_p to M'_p. Any situation that occurs at some point in a complete history is possible, so we have $m_p \in M'_p$ for any $(m_p)_{p\in P} \in H^0$. Conversely, a situation that is possible is the endpoint m_1 of some complete history $(m_p)_{p\in P}$ because the maps r_p are proper (Lemma 4.9) and a projective limit of non-empty compact spaces is non-empty. Thus $\pi_p^{-1}(H^0) = M'_p$, $r'_p(M'_p) = X'$ for all $p \in P$, and $s'_p(M'_p) \subseteq X'$ by associativity: if a situation has a possible past, then it is itself possible because we may concatenate stories. We still have isomorphisms $M'_p \times_{X'} M'_q \cong M'_{pq}$, so restricting to the possible situations gives a new action by topological correspondences. By construction, both systems $(X, M, s_p, r_p, \sigma_{p,q})$ and $(X', M'_p, s'_p, r'_p, \sigma'_{p,q})$ have the same complete histories and thus the same transformation groupoid H.

Lemma 4.16. *Let $q, a_1, a_2 \in P$ satisfy $qa_1 = qa_2$. Then $r_{q,a_1}|_{M'_{qa_1}} = r_{q,a_2}|_{M'_{qa_2}}$.*

Proof. Condition (O2) gives us $b \in P$ with $a_1 b = a_2 b$. The associativity property (4.7) of the maps $r_{p,p}$ gives $r_{q,a_i} \circ r_{qa_i,b} = r_{q,a_i b}$. Hence r_{q,a_1} and r_{q,a_2} coincide on the range of $r_{q,a_1 b} = r_{q,a_2 b}$. The subspace M'_{qa_1} is contained in that range. □

Write $p \geq q$ for $p, q \in P$ if there is $a \in P$ with $p = qa$. Lemma 4.16 shows that after restricting to the possible situations, the truncation map $r_{q,a} \colon M'_p \to M'_q$ for $p \geq q$ does not depend on the choice of a.

Theorem 4.17. *The groupoid* C*-*algebra* $\mathrm{C}^*(H)$ *is canonically isomorphic to the Cuntz–Pimsner algebra of the product system $(\mathcal{E}_p)_{p\in P}$ over P described above.*

Proof. The proof has two parts. In the first part, we show that the original action on X has the same colimit as the induced action on H^0. So we are reduced to the special case of actions by correspondences of the special form where the maps $r_p \colon M_p \to X$ are all identity maps. In the second part, we do this case by hand.

Let D be a C*-algebra. A transformation from the product system $(\mathcal{E}_p)_{p\in P}$ to D consists of a correspondence $\mathcal{F}$ to D together with isomorphisms of correspondences $V_p \colon \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{F} \cong \mathcal{F}$. The left action of $\mathrm{C}_0(X)$ on $\mathcal{E}_p$ extends to an action of $\mathrm{C}_0(M_p)$ by pointwise multiplication. Thus we get a canonical left action of $\mathrm{C}_0(M_p)$ on $\mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{F} \cong \mathcal{F}$, where we use use the isomorphism V_p. Thus $\mathrm{C}_0(M_p)$ acts on $\mathcal{F}$ in a canonical way for each $p \in P$.

If $p, q \in P$, then the isomorphism $V_{pq} \colon \mathcal{E}_{pq} \otimes_{\mathrm{C}_0(X)} \mathcal{F} \cong \mathcal{F}$ is equal to the composite isomorphism where we first identify $\mathcal{E}_{pq} \cong \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{E}_q$ and then apply V_q and V_p. As a consequence, the action of $\mathrm{C}_0(M_p)$ on $\mathcal{F}$ is the composite of the action of $\mathrm{C}_0(M_{pq})$ on $\mathcal{F}$ and the *-homomorphism $r_{p,q} \colon \mathrm{C}_0(M_p) \to \mathrm{C}_0(M_{pq})$. So the left actions of $\mathrm{C}_0(M_p)$ fit together to an action of the inductive limit

$$\varinjlim_{\mathcal{C}_P} (\mathrm{C}_0(M_p), r^*_{p,q}) \cong \mathrm{C}_0\Bigl(\varprojlim_{\mathcal{C}_P} (M_p, r_{p,q})\Bigr) = \mathrm{C}_0(H^0).$$

Thus $\mathcal{F}$ carries a nondegenerate *-representation of $\mathrm{C}_0(H^0)$, turning it into a correspondence $\tilde{\mathcal{F}}$ from $\mathrm{C}_0(H^0)$ to D.

Now let $\tilde{\mathcal{E}}_p := \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathrm{C}_0(H^0)$. Then

$$\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}} \cong \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathrm{C}_0(H^0) \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}} \cong \mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{F} \cong \mathcal{F},$$

where the last isomorphism is V_p and all other isomorphisms are trivial. By definition, the correspondence $\tilde{\mathcal{E}}_p$ from $\mathrm{C}_0(X)$ to $\mathrm{C}_0(H^0)$ is obtained from a topological correspondence as well, namely, we replace M_p by $\tilde{M}_p = M_p \times_{s_p, X, \pi_1} H^0$ and use the map $(m, \omega) \mapsto r_p(m)$ as range and the map $(m, \omega) \mapsto \omega$ as

source map. Lemma 4.10 describes a homeomorphism $\tilde{M}_p \cong H^0$ such that the second coordinate projection becomes $\tilde{s}_p\colon H^0 \to H^0$. We let $\tilde{r}_p$ be the identity map $H^0 \to H^0$ and thus turn $\tilde{\mathcal{E}}_p$ into a correspondence from $\mathrm{C}_0(H^0)$ to itself. Since $\tilde{s}_p \circ \tilde{s}_q = \tilde{s}_{qp}$, the usual composition of topological correspondences defines an action of P on H^0 by topological correspondences. Thus the associated C*-correspondences $\tilde{\mathcal{E}}_p$ form a product system over P with unit fibre $\mathrm{C}_0(H^0)$.
We claim that the isomorphism of Hilbert modules $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}} \cong \tilde{\mathcal{F}}$ constructed above is an isomorphism of correspondences from $\mathrm{C}_0(H^0)$ to D with this choice of left action of $\mathrm{C}_0(H^0)$ on $\tilde{\mathcal{E}}_p$. It suffices that $\mathrm{C}_0(M_q)$ acts in the same way on both sides for each $q \in P$. The Ore condition (O1) gives $t, u \in P$ with $pt = qu$. Since the action of $\mathrm{C}_0(M_q)$ factors through $\mathrm{C}_0(M_{qu})$ and $qu = pt$, we may assume $q = pt$.
The left action of $\mathrm{C}_0(M_{pt})$ on $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}}$ is obtained as follows. First, as Hilbert D-modules, we identify

$$\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}} \cong (\mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathrm{C}_0(M_t)) \otimes_{\mathrm{C}_0(M_t)} (\mathcal{E}_t \otimes_{\mathrm{C}_0(X)} \mathcal{F}).$$

Then we identify $\mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathrm{C}_0(M_t)$ with the C*-correspondence from $\mathrm{C}_0(M_{pt})$ to $\mathrm{C}_0(M_t)$ associated to the topological correspondence $M_{pt} \cong M_p \times_{s_p,X,r_t} M_t \to M_t$, where the range map is the homeomorphism $\sigma_{p,t}$ and the source map is pr_2. By Lemma 4.3, the composite of this with the C*-correspondence $\mathcal{E}_t$ from $\mathrm{C}_0(M_t)$ to $\mathrm{C}_0(X)$ is the C*-correspondence associated to the composite topological correspondence, $(M_p \times_{s_p,X,r_t} M_t) \times_{\mathrm{pr}_2,M_t,r_t} M_t$. This is $M_p \times_{s_p,X,r_t} M_t$ now viewed as a topological correspondence from M_{pt} to X. As such, it is isomorphic to M_{pt}. Thus we may also get the left action of $\mathrm{C}_0(M_{pt})$ on $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}}$ by identifying this in the canonical way with $\mathcal{E}_{pt} \otimes_{\mathrm{C}_0(X)} \mathcal{F}$ and then acting on the first tensor factor by pointwise multiplication. This, however, is exactly how the left action of $\mathrm{C}_0(M_{pt})$ on $\tilde{\mathcal{F}}$ is defined. Thus the left actions of $\mathrm{C}_0(M_t)$ on $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ coincide as expected.
Since $\mathrm{C}_0(X) \subseteq \mathrm{C}_0(H^0)$, we may view $\mathcal{E}_p$ as a subspace of $\tilde{\mathcal{E}}_p$. The map from the algebraic tensor product $\mathcal{E}_p \odot \mathcal{F}$ to $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \mathcal{F}$ has dense range because the target is isomorphic to $\mathcal{F}$, which is also isomorphic to the correspondence tensor product $\mathcal{E}_p \otimes_{\mathrm{C}_0(X)} \mathcal{F}$, and there $\mathcal{E}_p \odot \mathcal{F}$ is certainly dense. Hence the map from $\mathcal{E}_p \odot \mathcal{E}_q \odot \mathcal{F}$ to $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \tilde{\mathcal{E}}_q \otimes_{\mathrm{C}_0(H^0)} \mathcal{F}$ also has dense range. The coherence condition for a transformation of product systems (3.2) holds on this dense subspace by assumption, and hence it holds everywhere. Thus our isomorphisms of C*-correspondences $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \mathcal{F} \cong \mathcal{F}$ form a transformation as expected.
Thus a transformation from the product system $\mathcal{E}_p$ to D gives a transformation from the product system $\tilde{\mathcal{E}}_p$ to D, with the same underlying Hilbert module $\mathcal{F}$. Conversely, a transformation from the product system $\tilde{\mathcal{E}}_p$ to D gives one from the product system $\mathcal{E}_p$ to D because $\mathcal{E}_p \subseteq \tilde{\mathcal{E}}_p$. Since $\mathcal{E}_p \odot \mathcal{F}$ is dense in $\tilde{\mathcal{E}}_p \otimes_{\mathrm{C}_0(H^0)} \mathcal{F}$, these two constructions must be inverse to each other. Summing up, we have found a bijection between the transformations from our two product systems to D that does not change the underlying Hilbert module $\mathcal{F}$. The

results in Section 2 show that such transformations are in bijection with nondegenerate representations of the Cuntz–Pimsner algebras of the two product systems on $\mathcal{F}$, respectively. Having found bijections between the representations of both Cuntz–Pimsner algebras on any Hilbert module, we conclude by the Yoneda Lemma that they must be isomorphic.

We have now reduced the general case of an action by topological correspondences to the special case of an action with $M_p = X$ and $r_p = \mathrm{id}$ for all $p \in P$. This case is much easier because the groupoid model has $H^0 = X$, so we merely take a transformation groupoid and do not change the underlying object space.

By definition, $\mathrm{C}^*(H)$ is the C^*-completion of the dense *-subalgebra $\mathrm{C_c}(H^1)$ of compactly supported, continuous functions on H^1, equipped with the usual convolution and involution

$$f_1 * f_2(h) := \sum_{h_1 h_2 = h} f_1(h_1) f_2(h_2), \qquad f^*(h) := \overline{f(h^{-1})},$$

for $f_1, f_2, f \in \mathrm{C_c}(H^1)$, $h \in H^1$ (see [28, Section 3]). Here "C^*-completion" means that we complete in the largest C^*-seminorm on $\mathrm{C_c}(H^1)$. There is no need to assume boundedness for the I-norm. First, the argument in [28] shows that every Hilbert space representation and hence every C^*-seminorm is continuous for the inductive limit topology; secondly, [70, Corollaire 4.8] shows that such representations and C^*-seminorms are bounded for the I-norm.

The disjoint decomposition $H^1 = \bigsqcup_{g \in G} H^1_g$ gives $\mathrm{C_c}(H^1) = \bigoplus_{g \in G} \mathrm{C_c}(H^1_g)$. This is a non-saturated G-grading, that is, $\mathrm{C_c}(H^1_g) * \mathrm{C_c}(H^1_h) \subseteq \mathrm{C_c}(H^1_{gh})$ and $\mathrm{C_c}(H^1_g)^* = \mathrm{C_c}(H^1_{g^{-1}})$. This G-grading turns $\mathrm{C}^*(H)$ into the section algebra of a Fell bundle over G. Of course, our proof will show that this Fell bundle structure corresponds to the same structure on the Cuntz–Pimsner algebra.

The space H^1_g is an increasing union of the open subsets $H^1_{p_1,p_2}$. Thus any function in $\mathrm{C_c}(H^1_g)$ already belongs to $\mathrm{C_c}(H^1_{p,q})$ for some $p, q \in P$ with $pq^{-1} = g$:

$$\mathrm{C_c}(H^1_g) = \bigcup_{pq^{-1} = g} \mathrm{C_c}(H^1_{p,q}).$$

Since $X = H^1$, we have $H^1_{p,q} \cong X \times_{s_p, X, s_q} X$, the set of pairs (x, y) with $s_p(x) = s_q(x)$. We are now going to relate $\mathrm{C_c}(X \times_{s_p, X, s_q} X)$ to the space $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p)$ in the description of the Cuntz–Pimsner algebra in the proof of Theorem 3.16.

Given a function $k \in \mathrm{C_c}(X \times_{s_p, X, s_q} X)$, we define

$$T_k \colon \mathcal{E}_q \to \mathcal{E}_p, \qquad (T_k \xi)(m_1) := \sum_{s_p(m_1) = s_q(m_2)} k(m_1, m_2) \xi(m_2);$$

these sums are uniformly finite for m_1 in a compact subset because s_p and s_q are local homeomorphisms and the support of k is compact. The operator T_k is a rank-one operator if $k(m_1, m_2) = k_1(m_1) \cdot k_2(m_2)$ with $k_1 \in \mathrm{C_c}(X)$, $k_2 \in \mathrm{C_c}(X)$. Since functions k of this form are dense in $\mathrm{C_c}(X \times_{s_p, X, s_q} X)$ and the map $k \mapsto T_k$ is continuous, we have $T_k \in \mathbb{K}(\mathcal{E}_q, \mathcal{E}_p)$ for all $k \in \mathrm{C_c}(X \times_{s_p, X, s_q} X)$. Since compactly supported functions are dense in $\mathcal{E}_p$ and $\mathcal{E}_q$, operators of the form T_k for $k \in \mathrm{C_c}(X \times_{s_p, X, s_q} X)$ are dense in $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p)$. If $T_k = 0$, then $k = 0$.

We may express the product and involution on compact operators through kernel functions: if $k \in \mathrm{C_c}(X \times_{s_p,X,s_q} X)$ and $l \in \mathrm{C_c}(X \times_{s_q,X,s_t} X)$, then $T_k \circ T_l$ has the kernel $(m_1, m_2) \mapsto \sum_{s_q(m)=s_p(m_1)=s_t(m_2)} k(m_1, m)l(m, m_2)$, and the adjoint T_k^* has the kernel $(m_1, m_2) \mapsto \overline{k(m_2, m_1)}$. These formulas correspond to the convolution and involution in $\mathrm{C_c}(H^1)$. Therefore, the map $k \mapsto T_k$ is an injective *-homomorphism with dense range from $\mathrm{C_c}(H^1) = \sum_{p,q\in P} \mathrm{C_c}(H^1_{p,q})$ to the *-algebra $\sum_{g\in G} \mathcal{O}_g$ of compactly supported sections of the Fell bundle $(\mathcal{O}_g)_{g\in G}$.

It remains to show that this *-homomorphism extends to an isomorphism between the C^*-completions. It suffices to prove that the restriction of any *-representation of $\mathrm{C_c}(H^1)$ to $\mathrm{C_c}(H^1_g)$ is bounded with respect to the norm of $\mathcal{O}_g$. Since $\|\xi\|^2 = \|\xi^*\xi\|_{\mathcal{O}_1}$ for all $\xi \in \mathcal{O}_g$, this holds for all g once it holds for $g = 1$. Thus it remains to show that the unit fibre $\mathcal{O}_1$ of the Cuntz–Pimsner algebra is the C^*-completion of $\mathrm{C_c}(H^1_1)$ for the subgroupoid H^1_1.

If $p \in P$, then the subset $H^1_{p,p} = X \times_{s_p,X,s_p} X$ of H^1_1 is the groupoid describing the equivalence relation $\sim_p$ induced by the map s_p. This equivalence relation is proper, that is, the map $X \to X/\sim_p$ is proper, since s_p is a local homeomorphism. Hence the C^*-algebra of the groupoid $H^1_{p,p}$ is Morita–Rieffel equivalent to $\mathrm{C}_0(s_p(X))$. The Hilbert bimodule constructed in the proof of this Morita–Rieffel equivalent in [61] is exactly our correspondence $\mathcal{E}_p$. Hence $\mathrm{C}^*(H^1_{p,p}) = \mathbb{K}(\mathcal{E}_p)$. The C^*-algebras $\mathrm{C}^*(H^1)$ and $\mathcal{O}_1$ are the colimits of the diagrams of C^*-algebras $\mathrm{C}^*(H^1_{p,p})$ and $\mathbb{K}(\mathcal{E}_p)$ over the filtered category $\mathcal{C}_P$. Hence they are also canonically isomorphic. □

5. Some relations to previous work

The construction of groupoid models above is very general and contains many known constructions. We discuss some of them in this section.

First let $P = (\mathbb{N}, +)$. An action of $\mathbb{N}$ by topological correspondences is already determined by the single topological correspondence (M_1, r_1, s_1), where s_1 must be a local homeomorphism ($1 \in \mathbb{N}$ is not the unit element here, there is a conflict with our usual multiplicative notation). This topological correspondence is the same as a topological graph. We assume r_1 to be proper to get a proper topological correspondence; then the composite correspondences M_n for $n \in \mathbb{N}$ are automatically proper. We also assume r_1 to be surjective; equivalently, all maps r_n are surjective and $X = X'$. What we are dealing with is a row-finite topological graph without sources, which we simply call *regular*. The space X is its space of vertices, and M_1 is its space of edges, with $m \in M_1$ giving an edge from $s_1(x)$ to $r_1(x)$. A point in M_n is a path in the topological graph of length n, and the maps s_n and r_n send such a path to its initial and final point. The homeomorphism $M_n \times_{s_n,X,r_m} M_m \cong M_{n+m}$ builds a path of length $n + m$ by concatenating two paths of length n and m. If the space X of vertices is discrete, then so is the space of edges M_1 because s_1 is a local homeomorphism. Thus the case where X is discrete gives the ordinary graphs among the topological graphs.

For a regular topological graph, the topological graph C*-algebra of Katsura [43–45] is, by definition, the same absolute Cuntz–Pimsner algebra that we study. If there are sources, that is, r_1 is not surjective, then neither our (absolute) Cuntz–Pimsner algebra nor its groupoid model see a difference between the topological graph (X, M_1, r_1, s_1) and its restriction (X', M_1', r_1', s_1'), which now has surjective r_1'. Thus we get the topological graph C*-algebra of the regular graph (X', M_1', r_1', s_1'); this is quite different from Katsura's C*-algebra for the original topological graph.

Groupoid models played an important role in the definition of (discrete) graph C*-algebras by Kumjian, Pask, Raeburn and Renault in [48]. They are, however, not used by Katsura to develop the theory of topological graph C*-algebras. A pretty general construction of a groupoid model for topological graph C*-algebras is due to Deaconu [19], under the extra assumption that both maps s_1 and r_1 be surjective local homeomorphisms and both spaces X and M_1 be compact. These assumptions are removed by Yeend [78], who constructs groupoid models for the more general class of *higher-rank* topological graphs. His construction works for all rank-1 graphs, that is, he may allow r_1 to be any map. The construction simplifies, however, if r_1 is proper and surjective. Since points in M_n are paths of length n, our complete histories in H^0 are the same as infinite paths in the topological graph. The map $\pi_n\colon H^0 \to M_n$ gives the initial segment of a path of length n. The local homeomorphism $\tilde{s}_n$ truncates an infinite path by throwing away the initial segment of length n. The groupoid H is $\mathbb{Z}$-graded,

$$H = \bigsqcup_{n\in\mathbb{Z}} H_n,$$

and the unit fibre $H_0 \subseteq H$ describes the equivalence relation of *tail equivalence*: we identify two infinite paths if they eventually become equal (without shift). The whole groupoid H combines tail equivalence with the shift map on infinite paths: an arrow in H_n means that two infinite paths become eventually equal if we also shift one of them by n steps.

The construction above is exactly how the usual groupoid model for a regular topological graph is constructed. Thus our groupoid H is the same as Yeend's groupoid model for the C*-algebra of a regular topological graph, which in turn generalises the groupoid models in [19, 48]. In particular, we get the familiar groupoid model for the C*-algebra of a row-finite graph without sources. In the irregular case, Yeend adds certain finite paths to H^0, and he defines the topology on the resulting space of "boundary paths" carefully to get a locally compact space.

Next consider $P = (\mathbb{N}^k, +)$ for some $k \geq 2$. An action of the Ore monoid $\mathbb{N}^k$ by topological correspondences is the same, almost by definition, as a *topological rank-k graph*. The case where the underlying space X is discrete corresponds to an ordinary rank-k graph. Topological higher-rank graphs are introduced by Yeend, who also describes a groupoid model for them in [79]. He requires the source maps to be local homeomorphisms, but does not require the range maps to be proper; instead, he assumes a weaker condition called "compact alignment," which may be formulated for lattice-ordered semigroups. He constructs groupoid models for the Toeplitz C*-algebra and the relative Cuntz–Pimsner algebra of the product system over $\mathbb{N}^k$ associated to a compactly aligned topological higher-rank graph. The relative and absolute Cuntz–Pimsner algebras agree if and only if all range maps r_p are surjective ("no sources"). We call a topological higher-rank graph *regular* if the maps r_p are surjective and proper for all $p \in \mathbb{N}^k$. In the regular case, the groupoid model constructed by Yeend [79] is the same one that we have constructed above: the boundary paths that form the object space for Yeend's groupoid model are the same as our complete histories by [79, Lemma 6.6]. In the irregular case, the object space of Yeend's groupoid combines infinite paths with certain finite and partially infinite paths. It is unclear how to carry this over to actions of Ore monoids.

How about groupoid models for actions of semigroups other than $\mathbb{N}^k$? Here we are only aware of constructions in particular cases. We discuss two general situations. First, if the action is by local homeomorphisms, then we are already very close to an inverse semigroup action, which is easily translated to an étale groupoid (see [28,31,64]). Secondly, the construction of a semigroup C*-algebra by Xin Li in [16, 55] may also be based on an action of the semigroup by topological correspondences.

5.1. Semigroups of partial local homeomorphisms. Let P be a monoid, and let P act on a locally compact space X by *partial local homeomorphisms*, that is, by topological correspondences of the special form

$$X \xleftarrow{\text{inclusion}} U_p \xrightarrow{\alpha_p} X, \tag{5.1}$$

where $U_p \subseteq X$ is an open subset and α_p is a local homeomorphism. These topological correspondences are only proper if the domains U_p are also closed. But the following construction of a groupoid does not need this assumption, and neither does it require the monoid P to be Ore. We do not claim, however, that the groupoid C*-algebra of the resulting groupoid is isomorphic to the Cuntz–Pimsner algebra of the product system over P associated to our action: our proof only gives this if P is an Ore monoid and the correspondences are proper, that is, the subsets U_p are clopen.

First we make the multiplication maps explicit. The fibre product $U_p \times_{s_p,X,r_q} U_q$ consists of pairs (x, y) with $x \in U_p$, $y \in U_q$, and $\alpha_p(x) = y$, and the range and source maps on $U_p \times_{s_p,X,r_q} U_q$ take (x, y) to x and $\alpha_q(y)$, respectively. Since $y = \alpha_p(x)$, the map $(x, y) \mapsto x$ identifies $U_p \times_{s_p,X,r_q} U_q$ with $U_p \cap \alpha_p^{-1}(U_q)$; under this identification, the range and source maps become the inclusion map and the map $x \mapsto \alpha_q(\alpha_p(x))$, respectively. Thus we must have

$$U_{pq} = U_p \cap \alpha_p^{-1}(U_q), \qquad \alpha_{pq} = \alpha_q \circ \alpha_p. \tag{5.2}$$

Conversely, these conditions give a unique isomorphism of topological correspondences $U_p \times_{s_p,X,r_q} U_q \cong U_{pq}$. These isomorphisms automatically verify the associativity condition required for an action of P by topological correspondences. Thus an action of P by topological correspondences of the special form (5.1) is the same as a homomorphism from P^{op} to the monoid of partial local homeomorphisms of X, with the composition of partial local homeomorphisms defined in (5.2).

If $V \subseteq U_p$ is such that $\alpha_p|_V$ is injective, then $\alpha_p|_V$ is a partial homeomorphism on X. Since α_p is a local homeomorphism, any point in U_p has a neighbourhood V on which $\alpha_p|_V$ is injective, so that these partial homeomorphisms contained in α_p cover α_p. Hence we do not lose any information if we replace α_p by the set of all partial homeomorphisms $\alpha_p|_V$ for $V \subseteq U_p$ such that $\alpha_p|_V$ is injective. These partial homeomorphisms $\alpha_p|_V$ form a semigroup because $\alpha_q|_V \circ \alpha_p|_W = \alpha_{pq}|_{W\cap\alpha_p^{-1}(V)}$ and $W \cap \alpha_p^{-1}(V)$ is an open subset of U_{pq} on which α_{pq} is injective by (5.2).

We let S be the *inverse* semigroup of partial homeomorphisms generated by these partial homeomorphisms $\alpha_p|_V$. This inverse semigroup acts on X by construction, and this action has an associated transformation groupoid $X \rtimes S$, also called groupoid of germs; see [28, 64]. This groupoid is often the same as the groupoid model constructed in Section 4, but there are some "trivial" counterexamples. The issue is how to define the germ relation. To always get the groupoid constructed in Section 4, we do the following.

First, we assume now that P is an Ore monoid with group completion G. Since the construction in Section 4 is only for actions by proper correspondences, we also require that the domains U_p are clopen. We let S_0 be the free inverse semigroup on symbols (p, V) for $\alpha_p|_V$ as above. This comes with a canonical homomorphism $\gamma\colon S \to G$ by mapping $(p, V) \mapsto p^{-1}$, and with a canonical action on X by mapping $(p, V) \mapsto \alpha_p|_V$. Let S be the quotient of S_0 by the kernel of this map. That is, we consider two elements of S_0 equivalent if they give the same element of G and the same partial homeomorphism on X. Now we take the groupoid of germs of the action of S on X with the germ relation from [28], that is, two elements $s, t \in S$ have the same germ at $x \in X$ if there is an idempotent e in S defined at x so that $se = te$.

Lemma 5.3. *Assume that P is an Ore monoid and the subsets U_p are clopen. Then the groupoid $X \rtimes S$ above is canonically isomorphic to the groupoid model in Definition* 4.12.

Proof. We map the free inverse semigroup S_0 above to the inverse semigroup of bisections of the groupoid H in Definition 4.12 by mapping (p, V) to $H^1_{1,p} \cap s^{-1}(V)$; this is easily seen to be a bisection of H that acts on $X = H^0$ by the partial homeomorphism $\alpha_p|_V$ and has degree p^{-1}. Thus the action of S_0 on X and the homomorphism γ both factor through the inverse semigroup of bisections of H, where γ maps bisections contained in H^1_g to g and where the action of H on X is used to let bisections act on X.

By construction, $s \in S_0$ is annihilated by γ if and only if the corresponding bisection in H is contained in H_1. This groupoid comes from an equivalence relation, so a bisection is trivial if and only if it acts trivially on X. Hence

$s \in S_0$ becomes idempotent in S if and only if it is mapped to an idempotent bisection of H. This shows that the quotient S of S_0 is exactly the image of S_0 in the inverse semigroup of bisections of H.

The groupoid H is covered by bisections that belong to S and are of the form

$$(H^1_{1,p} \cap s^{-1}(V)) \circ (H^1_{1,q} \cap s^{-1}(W))^{-1}$$

for $p, q \in P$ and $V \subseteq U_p$, $W \subseteq U_q$ such that $\alpha_p|_V$ and $\alpha_q|_W$ are injective. It is not clear whether these bisections already form an inverse semigroup; but at least, since they cover the arrow space of H, any product of such bisections is again covered by bisections of H of this special form. Therefore, the inverse semigroup S and the inverse semigroup of bisections of H have the same germ groupoids attached to their actions on X, that is, $H \cong X \rtimes S$. $\square$

The examples considered by Exel and Renault in [31] are actions of $(\mathbb{N}^k, +)$ by (globally defined) local homeomorphisms, so they certainly fit into our framework. This is remarkable because [31] also contains counterexamples where Exel's interaction group approach to non-invertible dynamical systems does not apply. This leads Exel and Renault to speculate that something should go wrong in these counterexamples.

Exel defines *interactions* in [27] as a way to describe dynamical systems that are non-deterministic in both past and future time directions. A local homeomorphism with a transfer operator is a particular example of an interaction, and the dynamics generated by a single local homeomorphism may be studied quite well using interactions. Exel proposed the concept of an *interaction group* in [29] in order to extend this to more general dynamical systems. In [31], Exel and Renault give rather simple examples of commuting local homeomorphisms $S, T\colon X \to X$ that cannot be embedded in an interaction group over $\mathbb{Z}^2$. We are going to discuss this, assuming both S, T to be surjective because this happens in the counterexamples in [31].

The problem is the following. The local homeomorphism S generates an equivalence relation on X by $x \sim_S y$ if $S(x) = S(y)$, and similarly for T. If there is an interaction group, then these relations $\sim_S$ and $\sim_T$ must commute, that is, there is $z \in X$ with $x \sim_S z \sim_T y$ if and only if there is $w \in X$ with $x \sim_T w \sim_S y$ (see [31, Proposition 14.1]). There are, however, commuting endomorphisms S, T for which the relations $\sim_S$ and $\sim_T$ do not commute. So such S, T cannot be part of an interaction group. Why is this no problem for our groupoid model?

Since our topological correspondences are already local homeomorphisms, our groupoid H has object space $H^0 = X$. The group completion of $(\mathbb{N}^2, +)$ is $(\mathbb{Z}^2, +)$, so the groupoid H is $\mathbb{Z}^2$-graded, $H = \bigsqcup_{g \in \mathbb{Z}^2} H_g$. A point in H_g is given by $(x, y) \in X$ and $n_1, n_2, m_1, m_2 \in \mathbb{N}$ with $S^{n_1} T^{n_2}(x) = S^{m_1} T^{m_2}(y)$ and $(m_1, m_2) - (n_1, n_2) = g$, and this is an arrow $x \leftarrow y$. Thus the range and source maps identify H_g with the union of the subsets

$$H_{n_1,n_2,m_1,m_2} = \{(x, y) \in X \times X \mid S^{n_1} T^{n_2}(x) = S^{m_1} T^{m_2}(y)\}.$$

We treat these as relations on X. The subsets H_{n_1,n_2,m_1,m_2} are closed in $X \times X$. If $k_1, k_2 \in \mathbb{N}$ then H_{n_1,n_2,m_1,m_2} is both open and closed in $H_{n_1+k_1,n_2+k_2,m_1+k_1,m_2+k_2}$ because the map $S^{k_1}T^{k_2}$ is a local homeomorphism, hence locally injective. The relation H_{n_1,n_2,m_1,m_2} may also be interpreted as the graph of the multi-valued map $S^{-n_1}T^{-n_2}S^{m_1}T^{m_2}$ on X.

What happens when we compose our relations? We have $(x, y) \in H_{k_1,k_2,l_1,l_2} \circ H_{l_1,l_2,m_1,m_2}$ if and only if there is $z \in X$ with $S^{k_1}T^{k_2}(x) = S^{l_1}T^{l_2}(z) = S^{m_1}T^{m_2}(y)$. Since $S^{l_1}T^{l_2}$ is surjective by assumption, the point z can always be found if $S^{k_1}T^{k_2}(x) = S^{m_1}T^{m_2}(y)$, so

$$H_{k_1,k_2,l_1,l_2} \circ H_{l_1,l_2,m_1,m_2} = H_{k_1,k_2,m_1,m_2}. \tag{5.4}$$

Exel and Renault say that S and T *star-commute* if for all $x, y \in X$ with $T(x) = S(y)$ there is a unique $z \in X$ with $S(z) = x$ and $T(z) = y$. Under this assumption, they construct an interaction group containing S and T. In our notation, S and T star-commute if and only if $H_{0,1,1,0} = H_{0,0,1,0} \circ H_{0,1,0,0}$; the inclusion $H_{0,1,1,0} \supseteq H_{0,0,1,0} \circ H_{0,1,0,0}$ is trivial. The relation $H_{0,1,1,0}$ describes the multi-valued map $T^{-1} \circ S$, whereas the relation $H_{0,0,1,0} \circ H_{0,1,0,0}$ describes the multi-valued map $S \circ T^{-1}$. So S and T *star-commute* if and only if $T^{-1} \circ S = S \circ T^{-1}$.

If this fails, there is no good way to define a topological correspondence or an interaction for the element $(1, -1) \in \mathbb{Z}^2$. The difference between these two relations is, however, always small in the sense that

$$T \circ (S \circ T^{-1}) = S \circ (T \circ T^{-1}) = S = T \circ (T^{-1} \circ S).$$

Since $x, y \in X$ with $T(x) = T(y)$ are equivalent in our groupoid H, the difference between the relations $S \circ T^{-1}$ and $T^{-1} \circ S$ does not matter once we take the whole groupoid into account.

Let us also examine this issue from the point of view of the Cuntz–Pimsner algebra of the resulting product system $(\mathcal{E}_p)_{p \in \mathbb{N}^2}$. The subspace H_{n_1,n_2,m_1,m_2} of the groupoid corresponds to the subspace $\mathbb{K}(\mathcal{E}_{(m_1,m_2)}, \mathcal{E}_{(n_1,n_2)})$ of the Cuntz–Pimsner algebra, compare the proof of Theorem 4.17. Since we assume the maps S, T to be surjective, the correspondences $\mathcal{E}_p$ are full for all $p \in \mathbb{N}^2$. Hence

$$\mathbb{K}(\mathcal{E}_{(m_1,m_2)}, \mathcal{E}_{(n_1,n_2)}) \cdot \mathbb{K}(\mathcal{E}_{(k_1,k_2)}, \mathcal{E}_{(m_1,m_2)}) = \mathbb{K}(\mathcal{E}_{(k_1,k_2)}, \mathcal{E}_{(n_1,n_2)})$$

for all $k_1, k_2, m_1, m_2, n_1, n_2 \in \mathbb{N}$. This corresponds to (5.4).

The zero fibre of the Cuntz–Pimsner algebra is the inductive limit of the C^*-subalgebras $\mathbb{K}(\mathcal{E}_{n_1,n_2})$ for $n_1, n_2 \to \infty$. In particular, $\mathbb{K}(\mathcal{E}_{0,1})$ and $\mathbb{K}(\mathcal{E}_{1,0})$ are contained in $\mathbb{K}(\mathcal{E}_{1,1})$. Although $\mathcal{E}_{1,1} = \mathcal{E}_{1,0} \otimes_{\mathrm{C}_0(X)} \mathcal{E}_{0,1} \cong \mathcal{E}_{0,1} \otimes_{\mathrm{C}_0(X)} \mathcal{E}_{1,0}$, we cannot expect in general that $\mathbb{K}(\mathcal{E}_{0,1}) \cdot \mathbb{K}(\mathcal{E}_{1,0})$ is equal to $\mathbb{K}(\mathcal{E}_{1,1})$: this goes wrong if S and T do not star-commute. It may also happen that $\mathbb{K}(\mathcal{E}_{0,1}) \cdot \mathbb{K}(\mathcal{E}_{1,0}) \neq \mathbb{K}(\mathcal{E}_{0,1}) \cdot \mathbb{K}(\mathcal{E}_{1,0})$ or, equivalently, that $\mathbb{K}(\mathcal{E}_{0,1}) \cdot \mathbb{K}(\mathcal{E}_{1,0})$ is not a C^*-algebra.

Summing up, we have seen that the commutative semigroup $\mathbb{N}^2$ may well generate some noncommutative phenomena both on the groupoid and C^*-algebra level. *So the reason why Cuntz–Pimsner algebras for proper product systems* *over $\mathbb{N}^2$ are tractable is not that $\mathbb{N}^2$ is commutative—it is that $\mathbb{N}^2$ satisfies Ore conditions.*

To make this clearer, consider now an arbitrary semigroup P. The Cuntz–Pimsner algebra of a proper product system $(\mathcal{E}_p)_{p\in P}$ over P must contain $\mathbb{K}(\mathcal{E}_p)$ for all $p \in P$. Given $p, q \in P$, we therefore need a C*-algebra containing both $\mathbb{K}(\mathcal{E}_p)$ and $\mathbb{K}(\mathcal{E}_q)$. If P is an Ore monoid, then there is $t \in P$ with $t \geq p, q$, and then $\mathbb{K}(\mathcal{E}_p)$ and $\mathbb{K}(\mathcal{E}_q)$ are both contained in $\mathbb{K}(\mathcal{E}_t)$. It is irrelevant for the construction how much of $\mathbb{K}(\mathcal{E}_t)$ is generated by $\mathbb{K}(\mathcal{E}_p)$ and $\mathbb{K}(\mathcal{E}_q)$ or whether $\mathbb{K}(\mathcal{E}_p)\cdot\mathbb{K}(\mathcal{E}_q)$ is a C*-algebra. Indeed, we would not even ask for any relationship unless t were chosen minimal, $t = p \vee q$, which only exists in lattice-ordered semigroups. The examples in [31] show that for commutative P the interaction group approach of Exel is trying implicitly to combine $\mathbb{K}(\mathcal{E}_p)$ and $\mathbb{K}(\mathcal{E}_q)$ in such a way that $\mathbb{K}(\mathcal{E}_p) \cdot \mathbb{K}(\mathcal{E}_q) = \mathbb{K}(\mathcal{E}_q) \cdot \mathbb{K}(\mathcal{E}_p)$ is again a C*-algebra. This led Exel in [30] to study when $A \cdot B = B \cdot A$ for two C*-subalgebras A and B of another C*-algebra.

Finally, there is one thing where an interaction group helps. If we only have an action of $\mathbb{N}^2$, then we can only restrict it to submonoids of $\mathbb{N}^2$. An interaction group on $\mathbb{Z}^2$ may also be restricted to subgroups of $\mathbb{Z}^2$; in the above notation, an interaction group gives well-defined topological correspondences $S^{n_1}T^{n_2}$ for all $n_1, n_2 \in \mathbb{Z}$. Without it, we only have this if n_1 and n_2 have the same sign. Examples of such restrictions are the polymorphisms in [17]. These may, however, be written directly as topological graphs.

5.2. Semigroup C*-algebras. How to define the C*-algebra of a monoid P? A satisfactory answer is given by Xin Li [55], assuming P to be left cancellative. If P is Ore, we shall describe Xin Li's C*-algebra as the Cuntz–Pimsner algebra of a product system. More precisely, we change the order of multiplication in P so as to get product systems over P instead of over P^{op}, compare Remark 3.1. Thus for us P is a right cancellative, right Ore monoid, and we describe the semigroup C*-algebra of P^{op} in the notation of [55]. The discussion below is closely related to the description of semigroup C*-algebras in [16].

Why is it non-trivial to construct semigroup C*-algebras? There is an obvious product system over any monoid: just take the complex numbers everywhere, with the obvious multiplication maps. A nondegenerate representation of this product system, however, is a representation of P by *unitaries*, not isometries. Hence the Cuntz–Pimsner algebra of this product system is the group C*-algebra of the group completion of P. To get an interesting Cuntz–Pimsner algebra, we need a non-trivial product system.

So why not take the universal C*-algebra for representations of P by isometries? This is generated by one isometry s_p for each $p \in P$, with the relations $s_p^* s_p = 1$ for all $p \in P$ and $s_p s_q = s_{qp}$ for all $p, q \in P$. This universal semigroup C*-algebra of P^{op} is introduced by Murphy [62]. It is, however, usually too "wild" to say much about it. It is rarely simple or exact.

The right way to "tame" Murphy's universal C*-algebra of a semigroup is to impose relations on the range projections of the isometries s_p. Xin Li [55] proposes a set of such relations modelled on properties of the regular representation of the semigroup. We are about to construct a product system $(\mathcal{E}_p)_{p\in P}$ that gives Xin Li's C*-algebra, and such that $s_p^* \in \mathcal{E}_p$. Thus $s_p s_p^* \in \mathcal{E}_1$. In our approach, the desired relations among the range projections $s_p s_p^*$ are encoded in the C*-algebra $D := \mathcal{E}_1$.

Let D be the C*-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions of left ideals of the form $Ppq^{-1} \cap P$ for $p, q \in P$, where

$$Ppq^{-1} \cap P = \{x \in P \mid \exists y \in P\colon xq = yp\}.$$

In the following, we will use the group completion G of P and write Pg for $g \in G$. Since $P \subseteq G$ is right cancellative, we have $xq \in Pp$ if and only if $xqt \in Ppt$ for some $t \in P$; thus Pg for $g \in G$ is well-defined, that is, does not depend on how we write $g = pq^{-1}$ for $p, q \in P$.

Since D is commutative and unital, it is of the form $\mathrm{C}(X)$ for a compact space X. Right translation by $p \in P$ maps $Pg \cap P$ to $Pgp \cap Pp$. The characteristic function of this intersection is the product of the characteristic functions of $Pgp \cap P$ and $Pp \cap P$ and hence belongs to D. Thus right translation by p gives an endomorphism $\varphi_p\colon D \to D$. These maps form an action of P^{op} on D by endomorphisms, that is, $\varphi_p \circ \varphi_q = \varphi_{qp}$ for all $p, q \in P$.

The endomorphism φ_p for $p \in P$ maps the constant function 1 to the characteristic function e_p of the subset $P \cdot p \subseteq P$. Actually, $\varphi_p\colon D \to e_p D$ is an isomorphism because the map

$$P \to P \cdot p, \qquad x \mapsto x \cdot p,$$

is bijective and this map and its inverse map the generators of D to the characteristic functions of subsets of the form $Pg \cap Pp$, which are exactly all the generators of $e_p D$.

We may turn the action of P^{op} by the endomorphisms $(\varphi_p)_{p\in P}$ into a product system $(\mathcal{E}_p)_{p\in P}$ over P as in Remark 3.1, with reversed order of multiplication in P. Explicitly, $\mathcal{E}_p = \varphi_p(D) \cdot D = \varphi_p(1_D) \cdot D = e_p D$ as a Hilbert D-module, with left action of D through φ_p, and the multiplication maps are $\mathcal{E}_p \otimes_D \mathcal{E}_q \xrightarrow{\sim} \mathcal{E}_{pq}$, $x \otimes y \mapsto \varphi_q(x) \cdot y$. The Cuntz–Pimsner C*-algebra of this product system is canonically isomorphic to the semigroup crossed product for the action $(\varphi_p)_{p\in P}$ of P^{op} on D (see [33, Section 3]), and this is isomorphic to Xin Li's semigroup C*-algebra of P^{op} by [55, Lemma 2.14].

Explicitly, the isomorphism looks as follows. The Cuntz–Pimsner algebra $\mathcal{O}$ of $(\mathcal{E}_p)_{p\in P}$ is generated by D and copies $S_p(\mathcal{E}_p)$ of $\mathcal{E}_p$ for each $p \in P$. Since $\mathcal{E}_p = e_p \cdot D$ and $S_p(e_p d) = S_p(e_p)d$ for all $d \in D$, $\mathcal{O}$ is already generated by D and the elements $s_p = S_p(e_p)^*$. Since $\langle e_p, e_p\rangle_{\mathcal{E}_p} = e_p$ and $|e_p\rangle\langle e_p| = \mathrm{id}_{\mathcal{E}_p}$, the element s_p is an isometry with range projection e_p. Furthermore, $s_p^* \cdot s_q^* = s_{pq}^*$ or, equivalently, $s_p \cdot s_q = s_{qp}$ for all $p, q \in P$. As it turns out, the relations of D and the relations $s_p s_p^* = e_p$, $s_p^* s_p = 1$, $s_p \cdot s_q = s_{qp}$ for $p, q \in P$ imply all relations for the Cuntz–Pimsner algebra of $(\mathcal{E}_p)_{p\in P}$. Thus the Cuntz–Pimsner algebra agrees with Xin Li's semigroup C*-algebra of P^{op}.

Now we describe a groupoid model for our Cuntz–Pimsner algebra. Such groupoid models are already constructed in [56], even under weaker assumptions on the semigroup P.

The projection e_p in $\mathrm{C}(X)$ corresponds to a clopen subset $V_p \subseteq X$. The isomorphism $\varphi_p\colon D \to e_pD$ corresponds to a homeomorphism $r_p\colon V_p \to X$. More precisely, $\varphi_p(x)|_{V_p} = x \circ r_p$ and $\varphi_p(x)|_{X\setminus V_p} = 0$ for all $x \in D$. Let $s_p\colon V_p \to X$ be the inclusion map; this is a homeomorphism onto a clopen subset and hence a local homeomorphism. Thus (V_p, r_p, s_p) is a proper topological correspondence on X. The resulting correspondence on $D = \mathrm{C}(X)$ is $\mathrm{C}(V_p) = e_pD$ with the obvious right Hilbert D-module structure and the left D-action $\varphi_p = r_p^*$. This is equal to the C^*-correspondence $\mathcal{E}_p$ associated to the endomorphism φ_p. Now identify $V_p \cong X$ through r_p and rewrite our topological correspondence in the form $(X, \mathrm{id}_X, \theta_p)$, where $\theta_p\colon X \to U_p \subseteq X$ applies the inverse r_p^{-1}; these topological correspondences are as in (5.1), where $U_p = X$ and θ_p is a homeomorphism onto a clopen subset of X. We have $\theta_p \circ \theta_q = \theta_{qp}$ for all $p, q \in P$ because $\varphi_p \circ \varphi_q = \varphi_{qp}$. Hence we get an action of P^{op} on X by partial homeomorphisms, which is an action of the type considered in Section 5.1. The resulting product system over P is canonically isomorphic to the product system $(\mathcal{E}_p)_{p\in P}$ associated to the action $(\varphi_p)_{p\in P}$ of P^{op} on $D = \mathrm{C}(X)$ by *-endomorphisms.

As in Section 5.1, we get a groupoid model for the Cuntz–Pimsner algebra of the product system $(\mathcal{E}_p)_{p\in P}$, and this groupoid model is the groupoid of germs for the pseudogroup of partial homeomorphisms of X generated by the partial homeomorphisms θ_p for $p \in P$.

Finally, we mention a quicker alternative definition of Xin Li's semigroup C^*-algebras, see also [56] for an extension of this approach to more general semigroups. The main result of [50] shows that any semigroup crossed product is a full corner in the crossed product for a group action on a larger C^*-algebra. In our case, this larger C^*-algebra is the C^*-subalgebra A of $\ell^\infty(G)$ generated by the characteristic functions of subsets of the form $Pg \subseteq G$ for $g \in G$. The right translation action of G on $\ell^\infty(G)$ restricts to an action of G on A by automorphisms. This also induces an action of G by homeomorphisms on the spectrum Y of the C^*-algebra A. The C^*-algebra D is the full corner in A corresponding to the projection 1_P, the characteristic function of $P \subseteq G$. The action (φ_p) on D above is the compression of the action of G on A. Hence the semigroup crossed product discussed above is canonically isomorphic to the full corner $1_P(A \rtimes G)1_P$ in the crossed product $A \rtimes G$. Similarly, the groupoid model for the action of P on X is the restriction of the transformation groupoid $Y \rtimes G$ to the compact-open subset $X \subseteq Y$.

6. Properties of the groupoid model

Let an Ore monoid P act on a locally compact space X by topological correspondences $(M_p, \sigma_{p,q})$. The Cuntz–Pimsner algebra of the resulting product system over P is identified with a groupoid C^*-algebra $\mathrm{C}^*(H)$ in Theorem 4.17.

Many properties of $\mathrm{C}^*(H)$ are equivalent or closely related to properties of the underlying groupoid H. We harvest some known results of this type regarding nuclearity, simplicity or ideal structure, tracial and KMS weights, and pure infiniteness.

One interesting aspect of our groupoid model is that it involves a "precompiler": given an action of an Ore monoid P by *topological correspondences* on a space X, we first construct an action of P on another space H^0 by *local homeomorphisms*, and then we take the Cuntz–Pimsner algebra of this new action. Hence any C^*-algebra that may be obtained as the Cuntz–Pimsner algebra of some action of P by topological correspondences may also be obtained from an action of P on another space by local homeomorphisms. Therefore, for some purposes we may assume without loss of generality that we are dealing with an action of P by local homeomorphism. In this section, however, the main point is to rewrite properties of the groupoid H in terms of the original action on X. For actions of P by local homeomorphisms, what we are going to do is already well-known. Our criteria simplify further if the space X is discrete; this happens, in particular, for higher-rank graphs.

We begin with quick criteria for separability, unitality and nuclearity.

Remark 6.1. The groupoid C^*-algebra of an étale locally compact groupoid is separable if and only if the underlying groupoid is second countable. This happens if and only if the closed subspace $X' \subseteq X$ is second countable and the group G is countable. This follows if X is second countable and P is countable; in the latter case, we can see directly that the Cuntz–Pimsner algebra is separable.

Remark 6.2. The C^*-algebra $\mathrm{C}^*(H)$ is unital if and only if H^0 is compact. This happens if and only if the closed subspace $X' \subseteq X$ is compact because the projection map $\pi_1 \colon H^0 \to X$ is a continuous, proper map with image X' (see Lemma 4.9 and the discussion after Definition 4.15).

Theorem 6.3. *The full groupoid* C^**-algebra* $\mathrm{C}^*(H)$ *is nuclear if and only if the groupoid H is topologically amenable. In that case,* $\mathrm{C}^*(H)$ *belongs to the bootstrap class. The groupoid $H_1 \subseteq H$ is always topologically amenable. If G is amenable, then the groupoid H is also amenable, and* $\mathrm{C}^*(H_1)$ *belongs to the bootstrap class.*

Proof. An étale, Hausdorff, locally compact groupoid is (topologically) amenable if and only if its reduced C^*-algebra is nuclear by [4, Corollary 6.2.14]. Furthermore, if H is amenable, then its reduced and full C^*-algebras coincide, so the full one is also nuclear. Conversely, if the full groupoid C^*-algebra is nuclear, then so is the reduced one because nuclearity is hereditary for quotients. Hence nuclearity of the full groupoid C^*-algebra is also equivalent to amenability of the groupoid.

Any amenable groupoid is "a-T-menable" by [76, Lemma 3.5]; that is, it acts properly and isometrically on a continuous field of affine Euclidean spaces. The proof of the Baum–Connes conjecture for a-T-menable groupoids also shows

that their groupoid C*-algebras belong to the bootstrap class, see [76, Proposition 10.7].
Since $\mathrm{C}_0(X)$ is nuclear, Theorem 3.21 shows that the unit fibre $\mathcal{O}_1$ in the associated Cuntz–Pimsner algebra is always nuclear; then $\mathcal{O}$ itself is nuclear if G is amenable. Theorem 4.17 and its proof identify $\mathcal{O}_1$ and $\mathcal{O}$ with $\mathrm{C}^*(H_1)$ and $\mathrm{C}^*(H)$, respectively. So the statements about amenability of H_1 and H follow from the first sentence in the theorem.
It is elementary to prove the topological amenability of H_1 directly. The open subgroupoids $H^1_{p,p}$ for $p \in P$ are proper equivalence relations. So we may normalise the counting measure on the fibres of $H^1_{p,p}$ to give an invariant mean on $H^1_{p,p}$. When we view these invariant means on $H^1_{p,p}$ as means on H_1 for p in the filtered category $\mathcal{C}_P$, we get an approximately invariant mean on H_1. □

We have not yet tried to characterise amenability of H in terms of the original action by topological correspondences.

6.1. Open invariant subsets and minimality.

Definition 6.4. A topological groupoid H is *minimal* if H^0 has no open, invariant subsets besides $\emptyset$ and H^0.

Being minimal is a necessary condition for $\mathrm{C}^*(H)$ to be simple because open invariant subsets of H^0 generate ideals in $\mathrm{C}^*(H)$. We are going to describe the open, invariant subsets of H^0 in terms of the original data $(M_p, \sigma_{p,q})$. The following lemma gives a base for the topology on H^0 and will also be used for other purposes.

Lemma 6.5. *For $p \in P$ and an open subset $U \subseteq M_p$, let*

$$\pi_p^{-1}(U) := \{(m_q)_{q\in P} \in H^0 \mid m_p \in U\} \subseteq H^0.$$

The family $\mathcal{B}$ of subsets of this form is a base for the topology on H^0 and, for each $x \in H^0$, the subsets $\pi_p^{-1}(U)$ with $x \in U$ form a neighbourhood base at x. This base for the topology is closed under finite unions, finite intersections, and under applying $\tilde{s}_t^{-1}$ for all $t \in P$; and

$$\tilde{s}_t\big(\pi_t^{-1}(U)\big) = \pi_1^{-1}(s_t(U)) \qquad \text{for all } t \in P \text{ and } U \subseteq M_t \text{ open.} \tag{6.6}$$

Proof. By definition of the product topology, *intersections* $\bigcap_{p\in F} \pi_p^{-1}(U_p)$ for finite subsets $F \subseteq P$ and open subsets $U_p \subseteq M_p$ for $p \in F$ form a base of the topology on H^0, and such intersections with $x \in \pi_p^{-1}(U_p)$ form a neighbourhood base for $x \in H^0$. If $\mathcal{B}$ is closed under finite intersections, then $\mathcal{B}$ itself is this canonical base of the topology, and similarly for neighbourhoods of x.
Let $p, q \in P$. Then $r_{p,q}(m_{pq}) = m_p$ for all $(m_t)_{t\in P} \in H^0$. Thus

$$\pi_{pq}^{-1}(r_{p,q}^{-1}(U)) = \pi_p^{-1}(U) \tag{6.7}$$

for each open subset $U \subseteq M_p$. Since the maps $r_{p,q}$ are continuous, $r_{p,q}^{-1}(U)$ is again open. Now we consider a finite intersection $\bigcap_{i=1}^{n} \pi_{p_i}^{-1}(U_i)$ for $F = \{p_1, \dots, p_n\} \subseteq P$ and $U_i = U_{p_i} \subseteq M_{p_i}$. Since P is a right Ore monoid, there are $p \in P$ and $q_i \in P$ with $p_i q_i = p$ for $i = 1, \dots, n$. Then $\pi_{p_i}^{-1}(U_i) = \pi_p^{-1}(r_{p_i,q_i}^{-1}(U_i))$. Thus

$$\bigcap_{i=1}^{n} \pi_{p_i}^{-1}(U_i) = \pi_p^{-1}\Bigl(\bigcap_{i=1}^{n} r_{p_i,q_i}^{-1}(U_i)\Bigr),$$
$$\bigcup_{i=1}^{n} \pi_{p_i}^{-1}(U_i) = \pi_p^{-1}\Bigl(\bigcup_{i=1}^{n} r_{p_i,q_i}^{-1}(U_i)\Bigr).$$

Thus $\mathcal{B}$ is closed under finite intersections and finite unions. We have

$$\tilde{s}_t^{-1}(\pi_p^{-1}(U)) = \pi_{tp}^{-1}(s_{t,p}^{-1}(U)) \tag{6.8}$$

because $\tilde{s}_t((m_p)) = (s_{t,p}(m_{tp}))_{p\in P}$. Thus $\mathcal{B}$ is closed under $\tilde{s}_t^{-1}$ for all $t \in P$. Lemma 4.10 shows that $(\pi_t, \tilde{s}_t)\colon H^0 \to M_t \times_{s_t,X,\pi_1} H^0$ is a homeomorphism. Given $U \subseteq M_t$, consider the set of all $\omega \in H^0$ for which there is $m \in U$ with $(m, \omega) \in M_t \times_{s_t,X,\pi_1} H^0$. By definition of the fibre product, this is $\pi_1^{-1}(s_t(U))$. The homeomorphism $(\pi_t, \tilde{s}_t)$ shows, however, that it is also $\tilde{s}_t(\pi_t^{-1}(U))$. This gives (6.6). $\square$

Definition 6.9. The *indicator* of an invariant subset $A \subseteq H^0$ is the following subset of X:

$$A^\# := \{x \in X \mid \pi_1^{-1}(x) \subseteq A\}.$$

Let $X' \subseteq X$ be the subset of possible situations. By definition, any saturated subset contains $X \setminus X'$. So we lose no information if we restrict attention to $A^\# \cap X'$.

Definition and Lemma 6.10. *The indicator B of an open invariant subset is open in X and has the following two properties:*

hereditary: *if $p \in P$, $m \in M_p$ satisfy $r_p(m) \in B$, then $s_p(m) \in B$;*

saturated: *if $p \in P$, $x \in X$ satisfy $s_p(m) \in B$ for all $m \in r_p^{-1}(x)$, then $x \in B$.*

If $S \subseteq P$ is a subset that generates P, then a subset $B \subseteq X$ is hereditary and saturated if and only if it satisfies the above two conditions for all $p \in S$.

Proof. Let $B = A^\#$. First we show that $A^\#$ is open. By definition, $X \setminus A^\# = \pi_1(H^0 \setminus A)$. The map $\pi_1\colon H^0 \to X$ is proper by Lemma 4.9. So it maps the closed subset $H^0 \setminus A$ to a closed subset of X. Hence $X \setminus A^\#$ is closed and $A^\#$ is open.

Next we check that $A^\#$ is hereditary. If $s_p(m) \notin A^\#$, then there is $\eta \in H^0 \setminus A$ with $\pi_1(\eta) = s_p(m)$. The concatenation $m \cdot \eta \in H^0$ exists because $s_p(m) = \pi_1(\eta)$, and has $\pi_1(m \cdot \eta) = r_p(m)$. Since A is invariant, so is $H^0 \setminus A$. Hence $m \cdot \eta \notin A$ and $r_p(m) \notin A^\#$.

We check that $A^\#$ is saturated. Assume $s_p(m) \in A^\#$ for all $m \in r_p^{-1}(x)$, and let $\eta \in H^0$ satisfy $\pi_1(\eta) = x$. Decompose η as $\eta = m \cdot \eta'$ with $\eta' \in H^0$ and $m := \pi_p(\eta)$ by Lemma 4.10. Since $r_p(m) = \pi_1(\eta) = x$, the assumption gives $\pi_1(\eta') = s_p(m) \in A^\#$, so $\eta' \in A$. Since A is invariant, we get $\eta = m \cdot \eta' \in A$ and $x \in A^\#$.

If a subset $B \subseteq X$ satisfies the conditions of being hereditary and saturated for given $p, q \in P$, then it also satisfies them for $p \cdot q$ because $M_{pq} \cong M_p \times_{s_p,X,r_q} M_q$. Hence it suffices to verify these conditions for a set of generators for P. $\square$

THEOREM 6.11. *The complete lattice of open H-invariant subsets of H^0 is isomorphic to the complete lattice of open, hereditary, saturated subsets of X. In one direction, the isomorphism maps an open H-invariant subset $A \subseteq H^0$ to its indicator; in the other direction, it maps an open, hereditary, saturated subset $B \subseteq X$ to*

$$A := \bigcup_{p \in P} (s_p \circ \pi_p)^{-1}(B) = \{(m_p)_{p \in P} \in H^0 \mid \exists p\colon s_p(m_p) \in B\}.$$

Proof. Lemma 6.10 shows that the indicator $A^\#$ of an open invariant subset $A \subseteq H^0$ is open, hereditary and saturated. Conversely, let $B \subseteq X$ be open, hereditary and saturated, and define $A \subseteq H^0$ as above. This is clearly open. We first check that A is invariant; then we check that its indicator is B.

Let $\eta \in H^0$. Then $s_p \circ \pi_p(\eta) \in X$ is the situation at time $p \in P$ in the complete history η. Thus A consists of all complete histories that, at some time, visit $B \subseteq X$. However, if $s_p \circ \pi_p(\eta) \in B$, then $s_{pq}\pi_{pq}(\eta) \in B$ for all $q \in P$ because B is hereditary; indeed, $\pi_{pq}(\eta) \in M_{pq} \cong M_p \times_{s_p,X,r_q} M_q$ corresponds to a pair $(\pi_p(\eta), m_q)$ with $r_q(m_q) = s_p\pi_p(\eta) \in B$, so $s_{pq}(\pi_{pq}(\eta)) = s_q(m_q) \in B$. The subset $pP \subseteq P$ is cofinal. Hence $\eta \in A$ if and only if the set of $q \in P$ with $s_q \circ \pi_q(\eta) \in B$ is cofinal in P.

A subset A of H^0 is H-invariant if and only if $\tilde{s}_t^{-1}(A) = A$ for all $t \in P$. Let $\eta \in H^0$ and write it as $\eta = m \cdot \eta'$ for $m \in M_t$, $\eta' \in H^0$ with $s_t(m) = r_t(\eta')$. Thus $\tilde{s}_t(\eta) = \eta'$. Then $s_{tp}\pi_{tp}(\eta) = s_p\pi_p(\eta')$ for all $p \in P$. So if there is $p \in P$ with $s_p\pi_p(\eta') \in B$, then there is $q \in P$ with $s_q\pi_q(\eta) \in B$, namely, $q = tp$; conversely, if there is $q \in P$ with $s_q\pi_q(\eta) \in B$, then the set of such $q \in P$ is cofinal and hence contains some element of the form tp with $p \in P$ by (O1). Then $s_p\pi_p(\eta') \in B$. This shows that $\eta \in A$ if and only if $\eta' \in A$. So A is invariant as desired.

Now we check that the indicator of A is B. By construction, if $x \in B$, then $\pi_1^{-1}(x) \subseteq A$. Conversely, let $x \in X \setminus B$. We must construct $\eta \in H^0 \setminus A$ with $\pi_1(\eta) = x$. For each $p \in P$, there is $m_p \in M_p$ with $r_p(m_p) = x$ and $s_p(m_p) \notin B$ because otherwise B would not be saturated. Since $X \setminus X' \subseteq B$, and $s_p(m_p) \notin B$, there is $\eta_p \in H^0$ with $\pi_p(\eta_p) = m_p$ and hence $\pi_1(\eta_p) = x$ and $s_p \circ \pi_p(\eta_p) \notin B$.

Since B is open and the map π_p is proper by Lemma 4.9, the set K_p of all such η_p is a compact subset of H^0. We have seen above that $s_p\pi_p(\eta) \in B$ implies $s_{pq}\pi_{pq}(\eta) \in B$ for all $p, q \in P$. Hence $K_p \supseteq K_{pq}$ for all $p, q \in P$. Since P is Ore, this tells us that $\{K_p\}_{p \in P}$ is a directed set of compact, non-empty subsets in H^0. The intersection of such a family of subsets is non-empty. A point η in the intersection satisfies $\pi_1(\eta) = x$ and $s_p\pi_p(\eta) \notin B$ for all $p \in P$, so that $\eta \notin A$. Thus $A^\# = B$.

We have constructed two maps $A \mapsto B$ and $B \mapsto A$ from open invariant subsets of H^0 to open, hereditary, saturated subsets of X and back, and we have seen that the composite map $B \mapsto A \mapsto B$ is the identity, that is, the indicator of the subset A defined in the theorem is the given subset B. Conversely, let $A' \subseteq H^0$ be an invariant open subset. Let B be its indicator and define $A \subseteq B$ as in the statement of the theorem. We must show that $A' = A$. Since B is the indicator of A', we have $\pi_1^{-1}(B) \subseteq A'$. Since A' is invariant, this implies $(s_p\pi_p)^{-1}(B) \subseteq A'$ for all $p \in P$, that is, $A \subseteq A'$. It remains to prove $A' \subseteq A$. So we take $\eta \in A'$. Since A' is open, Lemma 6.5 gives $p \in P$ and an open subset $U \subseteq M_p$ such that $\pi_p^{-1}(U) \subseteq A'$. If $\eta' \in H^0$ satisfies $\pi_1(\eta') \in s_p(U)$, then there is $m \in U \subseteq M_p$ with $s_p(m) = \pi_1(\eta')$, so $m \cdot \eta' \in H^0$ is well-defined; it belongs to $\pi_p^{-1}(U) \subseteq A'$ by construction. Since $\tilde{s}_p(m \cdot \eta') = \eta'$ and A' is invariant, we get $\eta' \in A'$ for all $\eta' \in H^0$ with $\pi_1(\eta') \in s_p(U)$. Thus $s_p(U)$ is contained in the indicator B of A'. Then $\pi_p^{-1}(U) \subseteq A$, so $\eta \in A$ as desired. □

Corollary 6.12. *The groupoid H is minimal if and only if X has no non-trivial open, hereditary, saturated subsets.* □

Example 6.13. Let $P = (\mathbb{N}, +)$ and let r_1 be surjective, so that we are dealing with a regular topological graph (see the beginning of Section 5). Then our notion of a hereditary and saturated subset of the space of vertices X is the same one used by Katsura [45] to describe the gauge-invariant ideals of a topological graph C*-algebra in the regular case.

Example 6.14. Let $P = (\mathbb{N}^k, +)$ and assume that X is discrete and $r_p\colon M_p \to X$ is surjective for all $p \in \mathbb{N}^k$. Our data is equivalent to that of a row-finite k-graph without sources, and our Cuntz–Pimsner algebra is the higher-rank graph C*-algebra in this case. Since the standard basis $e_1, \dots, e_k$ generates $\mathbb{N}^k$, a subset is hereditary and saturated if and only if it satisfies the conditions in Definition 6.10 for $p = e_1, \dots, e_k$. Our notion of being hereditary and saturated is equivalent to the one used by Raeburn, Sims and Yeend in [68] to describe the gauge-invariant ideals in a higher-rank graph C*-algebra. Theorem 6.23 below will show that, in general, the open invariant subsets of H^0 correspond to those ideals in $C^*(H)$ that are "gauge-invariant" in a suitable sense.

6.2. Effectivity.

Definition 6.15. An étale topological groupoid H is *essentially free* if the subset of objects with trivial isotropy is dense in H^0. It is *effective* if every open subset of $H^1 \setminus H^0$ contains an arrow x with $s(x) \neq r(x)$. Equivalently, the interior of the set $\{h \in H^1 \setminus H^0 \mid r(h) = s(h)\}$ is empty.

For a second countable, locally compact, étale groupoid, being effective or essentially free are equivalent properties by [72, Proposition 3.6] or [9, Lemma 3.1] (these articles use the name "topologically principal" for "essentially free"). Being essentially free is a variant of the aperiodicity condition that is used to characterise when topological higher-rank graph C*-algebras are simple (see,

for instance, [79, Definition 5.2]). We cannot, however check whether H is essentially free without looking at points in H^0, that is, infinite paths.
As we shall see, the following definition characterises when the groupoid H is effective in terms of the original data of an action by topological correspondences:

Definition 6.16. An action $(M_p, \sigma_{p,q})$ of an Ore monoid P on a locally compact Hausdorff space X by proper topological correspondences is *effective* if for all $p, q \in P$ with $pq^{-1} \neq 1$ in G and for all non-empty, open subsets $U \subseteq X'$, there are $a, f, g \in P$ with $paf = qag$ and $y \in M_{paf} = M_{qag}$ with $r_{paf}(y) \in U$ and $\mathrm{mid}_{p,a,f}(y) \neq \mathrm{mid}_{q,a,g}(y)$ in M_a. Here $X' \subseteq X$ denotes the closed subset of possible situations; $\mathrm{mid}_{p,a,f}(y)$ denotes the component in the middle factor M_a after identifying $M_{paf} \cong M_p \times_X M_a \times_X M_f$, and similarly for $\mathrm{mid}_{q,a,g}(y)$.

Similar criteria for boundary path groupoids of higher-rank topological graphs being effective have been found by Wright [77]; for higher-rank graphs without topology, such criteria are also given in [40, 53, 74]. Before we explain the relationship between our criterion and others, we prove the theorem suggested by our notation:

Theorem 6.17. *The groupoid H is effective if and only if the action $(M_p, \sigma_{p,q})$ is effective.*

Proof. We may assume without loss of generality that $X = X'$.
First we assume that the action $(M_p, \sigma_{p,q})$ is not effective. This means that there are $p, q \in P$ with $pq^{-1} \neq 1$ in G and a non-empty open subset $U \subset X'$ such that $\mathrm{mid}_{p,a,f}(y) = \mathrm{mid}_{q,a,g}(y)$ in M_a for all $a, f, g \in P$ with $paf = qag$ and all $y \in M_{paf}$ with $r_{paf}(y) \in U$. This means that $(\tilde{s}_p x)_a = (\tilde{s}_q x)_a$ for all $x \in \pi_1^{-1}(U)$ and all $a \in P$. Hence $\tilde{s}_p x = \tilde{s}_q x$ in H^0 for all $x \in \pi_1^{-1}(U)$. Thus the elements of the form (x, pq^{-1}, x) for $x \in \pi_1^{-1}(U)$ form a bisection B in $H^1 \setminus H^0$ with $r|_B = s|_B$, which means that H is not effective.
Now assume that the action $(M_p, \sigma_{p,q})$ is effective. Let $U \subseteq H^1 \setminus H^0$ be a non-empty open subset. We need to find $(x, g, y) \in U$ with $x \neq y$.
The intersection $U \cap H^1_{p,q}$ is non-empty for some $p, q \in P$. Replacing U by $U \cap H^1_{p,q}$, we may arrange that $U \subseteq H^1_{p,q}$. The subgroupoid H_1 is an increasing union of equivalence relations, so it is certainly effective. Hence we are done if $pq^{-1} = 1$ in G. Thus we may assume from now on that $pq^{-1} \neq 1$ in G.
We may shrink U to a bisection because H is étale. We may then shrink further so that $r(U) = \pi_t^{-1}(U_t)$ for some $t \in P$ and some non-empty open subset $U_t \subseteq M_t$ because subsets of the form $\pi_t^{-1}(U_t)$ form a base for the topology on H^0 by Lemma 6.5. Since the map $s_t \colon M_t \to X$ is a local homeomorphism, we may shrink U_t even further, so that s_t becomes injective on U_t. Hence s_t restricts to a homeomorphism from $U_t \subseteq M_t$ onto an open subset $s_t(U_t) \subseteq X$. We are going to show that there is $x \in \pi_t^{-1}(U_t)$ with $\tilde{s}_p x \neq \tilde{s}_q x$. Thus (x, g, x) is not an arrow in H; since $r(U) = \pi_t^{-1}(U_t)$ and $U \subseteq H^1_{p,q}$, there must be $y \in H^0$ with $(x, g, y) \in U \subseteq H^1_{p,q}$. Since $\tilde{s}_p x = \tilde{s}_q y \neq \tilde{s}_q x$, we have $x \neq y$, as desired.

Since P is Ore, we may find $h, i \in P$ with $ph = ti$. Then we may find $h', i' \in P$ with $qhh' = ti'$. Thus $phh' = tih'$ and $qhh' = ti'$. Since $H^1_{p,q} \subseteq H^1_{phh',qhh'}$, we may replace (p, q) by (phh', qhh'). Thus we may assume without loss of generality that there are $p', q' \in P$ with $p = tp'$ and $q = tq'$.

Recall that s_t restricts to a homeomorphism from U_t onto $V := s_t(U_t)$. Hence $x \mapsto \tilde{s}_t x$ is a homeomorphism from $\pi_t^{-1}(U_t)$ onto $\pi_1^{-1}(V)$ (compare Lemma 4.10). By assumption, there are $a, f, g \in P$ with $p'af = q'ag$ and $y \in M_{p'af}$ with $r_{p'af}(y) \in V$ and $\mathrm{mid}_{p',a,f}(y) \neq \mathrm{mid}_{q',a,g}(y)$ in M_a. By construction, there is $z \in M_t$ with $s_t(z) = r_{p'af}(y)$. Then $(z, y) \in M_t \times_X M_{p'af} \cong M_{tp'af}$. We have $\tilde{s}_p(z, y) = \tilde{s}_{p'}(y)$ and $\tilde{s}_q(z, y) = \tilde{s}_{q'}(y)$ because $p = tp'$ and $q = tq'$. The M_a-component of $\tilde{s}_{p'}(y)$ is $\mathrm{mid}_{p',a,f}(y)$, and that of $\tilde{s}_{q'}(y)$ is $\mathrm{mid}_{q',a,g}(y)$. Since these are different, $\tilde{s}_p(x) \neq \tilde{s}_q(x)$ for any $x \in H^0$ with $tp'af$-component (z, y). Such x exist because we have restricted to possible histories throughout, making the maps $\pi_p \colon H^0 \to M_p$ surjective for all $p \in P$. □

Theorem 6.18. *Assume that P is countable and X is second countable or, more generally, that H is second countable. The Cuntz–Pimsner algebra $\mathcal{O}$ or, equivalently, the* C*-*algebra* $\mathrm{C}^*(H)$, *is simple if and only if the following three conditions are satisfied:*

(1) $\mathrm{C}^*(H) = \mathrm{C}^*_{\mathrm{r}}(H)$;
(2) *the action $(M_p, \sigma_{p,q})$ is effective;*
(3) *any non-empty, closed, hereditary, saturated of X contains X'.*

The first condition above follows if H is amenable and, in particular, if G is amenable.

Proof. We use [9, Theorem 5.1], which characterises when the groupoid C*-algebra of a second countable, locally compact, Hausdorff, étale groupoid is simple. The groupoid H is always locally compact, Hausdorff, and étale. The third condition is equivalent to the minimality of H by Corollary 6.12. Since H is second countable, it is essentially free if and only if it is effective by [9, Lemma 3.1]. This is equivalent to the effectivity of the action $(M_p, \sigma_{p,q})$ by Theorem 6.17. Thus our conditions are equivalent to the three conditions in [9, Theorem 5.1]. □

Kwaśniewski and Szymański [49] also provide an aperiodicity criterion and use it to prove simplicity results and a uniqueness theorem for certain Cuntz–Pimsner algebras over Ore monoids. On the one hand, their criterion still works if the unit fibre A of a product system is only a liminal C*-algebra. On the other hand, it does not imply the simplicity of the Cuntz algebra $\mathcal{O}_n$ because it only uses the (partial, multivalued) action of P on the spectrum of A induced by the product system. This contains no information if $A = \mathbb{C}$ as in the standard construction of $\mathcal{O}_n$.

Now we simplify the definition of being effective for the monoids $P = (\mathbb{N}^k, +)$. Some of the steps also work for other monoids.

Lemma 6.19. *The definition of an effective action of* $\mathbb{N}^k$ *does not change if we assume, in addition, that the pair* $(p,q) \in \mathbb{N}^k$ *is* reduced*, that is, there is no* $t \in \mathbb{N}^k$ *with* $p,q \in \mathbb{N}^k t$.

Proof. Take $p = p't$, $q = q't$ for $p',q',t \in \mathbb{N}^k$. Since $U \subseteq X'$ is non-empty and consists of possible situations, there is $m \in M_t$ with $r_t(m) \in U$. Since r_t is continuous, there is an open neighbourhood V of m with $r_t(V) \subseteq U$, and then $U' := s_t(V)$ is open as well. Let a, f, g and y' verify the effectivity criterion for p', q', U'. Then the same a, f, g and $y = m \cdot y'$ verify it for p, q, U. $\square$

The role of f, g in Definition 6.16 is merely as padding to make $paf = qag$. In a commutative monoid such as $\mathbb{N}^k$, we may simply take $f = q$ and $g = p$; or we may take $f = (p \vee q) - p$ and $g = (p \vee q) - p$, where $p \vee q$ denotes the maximum of p and q in $\mathbb{N}^k$. The latter choice is minimal and may therefore be optimal.

Lemma 6.20. *Let* $P = \mathbb{N}^k$. *Then the action on* X *is effective if and only if, for all reduced* $p, q \in \mathbb{N}^k$ *with* $p - q \neq 0$ *in* $\mathbb{Z}^k$ *and for all non-empty, open subsets* $U \subseteq X'$, *there are* $a \in \mathbb{N}^k$ *and* $y \in M_{p+a+f} = M_{q+a+g}$ *with* $r_{p+a+f}(y) \in U$ *and* $\mathrm{mid}_{p,a,f}(y) \neq \mathrm{mid}_{q,a,g}(y)$ *in* M_a, *with* $f = (p \vee q) - q$ *and* $g = (p \vee q) - p$.

Proof. The criterion in the lemma differs from Definition 6.16 in two ways. First, we assume (p,q) reduced, which makes no difference by Lemma 6.19. Secondly, we choose particular f, g. To see that this makes no difference, choose f', g', y' as in Definition 6.16. Then $f' - f = g' - g = h \in \mathbb{N}^k$. The truncation $y := r_{p+a+f}(y') \in M_{p+a+f}$ still has the same mid-part in M_a and hence also verifies the criterion in Definition 6.16. $\square$

The condition in Lemma 6.20 is exactly Condition (ii) in [77, Theorem 3.1]. As shown in [77], this condition is equivalent to Yeend's aperiodicity condition (A), which characterises when the groupoid model of the topological higher-rank graph is essentially free; Wright also gives an example where her finite-path version of aperiodicity is much easier to check than the original criterion.

Proposition 6.21. *Let* $P = (\mathbb{N}, +)$ *and assume* $X = X'$. *Then the action of* P *on* X *is effective if and only if the set of base points of loops without entrances has empty interior in* X.

We will explain during the proof what loops without entrances are. The condition we arrive at characterises when the groupoid model for a regular topological graph is effective, compare [45, Definition 6.6].

Proof. Any reduced pair in $\mathbb{N}$ is of the form $(p,0)$ or $(0,p)$. Since our notion is symmetric, we may as well take $q = 0$. Thus the condition in Lemma 6.20 says that for any non-empty, open subset $U \subseteq X'$ and any $p \in \mathbb{N}$ there is $a \in \mathbb{N}$ and $y \in M_{pa}$ with $r_{pa}(y) \in U$ and $\mathrm{mid}_{p,a,0}(y) \neq \mathrm{mid}_{0,a,p}(y)$ in M_a. If there is $y \in M_p$ with $r_p(y) \in U$ and $s_p(y) \neq r_p(y)$, then $a = 0$ and y will do because the relevant mid-parts are $s_p(y)$ and $r_p(y)$, respectively. Thus we may assume as well that $s_p(m) = r_p(m)$ for all $m \in M_p$ with $r_p(m) \in U$; such a path is called a *loop*, and $r_p(m) \in X$ is its *base point.*

Take $m \in M_p$ with $r_p(m) \in U$. Identify $M_p \cong M_1^{\times\times p}$ and write m as a path

$$x_1 \xleftarrow{m_1} x_2 \xleftarrow{m_2} x_3 \xleftarrow{m_3} x_4 \to \cdots \xleftarrow{m_p} x_{p+1} = x_1$$

with $x_1, \dots, x_p \in X$, $m_1, \dots, m_p \in M_1$. An *entrance* for this loop is $1 \le i \le p$ and an arrow $m'\colon x' \to x_i$ with $m' \neq m_i$. If t is such an entrance, then take $a = i$ and let $y \in M_{pa}$ be the concatenation of the loop m and the path

$$x_1 \xleftarrow{m_1} x_2 \xleftarrow{m_2} x_3 \leftarrow \cdots \leftarrow x_i \xleftarrow{m'} x'.$$

The relevant length-a mid-parts of this concatenation are the paths from x' to x_1 and x_p to x_1 involving m' and m_i, respectively. Hence an entrance to some loop gives the data (a, y) required in Definition 6.16. If the set of base points of loops without entrances has empty interior, then we may find some loop with an entrance with base point in U, so the action is effective.
Conversely, assume that U is an open subset so that all points in U are base points of some loop without an entrance. Then there is only one path with range x for any $x \in U$: we must follow the loop based at that point because it has no entrances (note that our paths go backwards). A continuity argument shows that the period of the loop is locally constant. Shrinking U, we may arrange that it is constant equal to p for all $x \in U$. Then $\mathrm{mid}_{p,a,0}(m) = \mathrm{mid}_{0,a,p}(m)$ for all $m \in M_{p+a}$ with $r_{p+a}(m) \in U$. □

If X is discrete, so that we are dealing with an ordinary graph C*-algebra, then Proposition 6.21 simplifies further to the condition that there are no loops without entrances, which is a standard condition in the theory of graph C*-algebras (see [66]).

6.3. Gauge-invariant ideals. We now describe which ideals in $\mathrm{C}^*(H)$ come from open H-invariant subsets of H^0 if H is not effective. Recall that the groupoid H is graded by the group completion G of P, $H = \bigsqcup_{g \in G} H_g$, and that this corresponds to the Fell bundle structure on the Cuntz–Pimsner algebra $\mathcal{O}$, that is, $\mathcal{O}_1 = \mathrm{C}^*(H_1)$ is the groupoid C*-algebra of the subgroupoid H_1. The canonical projection to $\mathcal{O}_1$ is a conditional expectation $E\colon \mathcal{O} \to \mathcal{O}_1$.

Definition 6.22. We call an ideal $I \subseteq \mathcal{O}$ *gauge-invariant* if I is equal to the ideal in $\mathcal{O}$ generated by $E(I) \subseteq \mathcal{O}_1$.

If G is an Abelian group, so that there is a dual action of $\hat{G}$ on $\mathcal{O}$, then standard arguments show that an ideal is "gauge-invariant" in the sense of Definition 6.22 if and only if it is invariant under the dual action of the compact group $\hat{G}$ on $\mathcal{O}$ in the usual sense because $E(x) = \int_{\hat{G}} \alpha_\gamma(x)\,\mathrm{d}\gamma$.

Theorem 6.23. *The gauge-invariant ideals in $\mathcal{O}$ are in bijection with open invariant subsets of H^0 or, equivalently, with open, hereditary and saturated subsets of X.*

Proof. The bijection between invariant subsets of H^0 and open, hereditary and saturated subsets of X is Theorem 6.11. An open and invariant subset $U \subseteq H^0$ corresponds to the ideal $\mathrm{C}^*(H|_U) \subseteq \mathrm{C}^*(H)$, where $H|_U$ denotes the subgroupoid of H with object space $U \subseteq H^0$ and arrow space $r^{-1}(U) = s^{-1}(U) \subseteq H^1$. This ideal is gauge-invariant with $E(\mathrm{C}^*(H|_U)) = \mathrm{C}^*(H_1|_U)$.

Conversely, let $I \subseteq \mathrm{C}^*(H)$ be a gauge-invariant ideal. Then $E(I)$ is an ideal in $\mathrm{C}^*(H_1)$. The groupoid H_1 is an approximately proper equivalence relation or a hyperfinite relation. This implies that it is amenable and that any restriction of H_1 to an invariant closed subset remains effective. Now [9, Corollary 5.9] implies that ideals in $\mathrm{C}^*(H_1)$ are in bijection with H_1-invariant open subsets of H^0, where $U \subseteq H^0$ corresponds to the ideal $\mathrm{C}^*(H_1|_U)$. This can only be of the form $E(I)$ for an ideal $I \subseteq \mathrm{C}^*(H)$ if U is invariant under the whole groupoid H, and then the ideal I generated by $\mathrm{C}^*(H_1|_U)$ is $\mathrm{C}^*(H|_U)$. Thus any gauge-invariant ideal I is of the form $\mathrm{C}^*(H|_U)$ for an open H-invariant subset U of H^0. □

6.4. Invariant measures. In the following, a "measure" on a locally compact space X means a Radon measure or, equivalently, a positive linear functional on $\mathrm{C_c}(X)$. We assume X to be second countable and P to be countable, so that H is second countable. Let $c\colon P \to (0,\infty)$ be a homomorphism to the multiplicative group of positive real numbers. This extends to the group completion G and then to H, by letting $c|_{H^1_{p,q}} = c(p)/c(q)$. This cocycle on H generates a 1-parameter group of automorphisms of $\mathrm{C}^*(H)$, see [69, Section 5]. The one-parameter automorphism groups of $\mathrm{C}^*(H)$ described above are not as special as it may seem:

Proposition 6.24. *Let H be effective. An automorphism of $\mathrm{C}^*(H)$ that acts trivially on the C^*-subalgebra $\mathrm{C}^*(H_1)$ is given by pointwise multiplication with a homomorphism $G \to \mathbb{T}$. A one-parameter group of such automorphisms comes from a homomorphism $c\colon P \to (0,\infty)$.*

Proof. Renault's construction of a groupoid model for C*-algebras with a Cartan subalgebra in [72] is natural. More precisely, any automorphism of a twisted groupoid C*-algebra $\mathrm{C}^*(H,L)$ for an étale, effective, Hausdorff, locally compact groupoid H and a Fell line bundle L over H that maps the subalgebra $\mathrm{C}_0(H^0)$ into itself must come from an automorphism of the pair (H,L).

The automorphism fixes $\mathrm{C}_0(H^0) \subseteq \mathrm{C}^*(H_1)$ if and only if the induced automorphism of H acts trivially on objects. For an effective groupoid, this implies that the automorphism acts identically on the inverse semigroup of bisections. Hence it is the identity automorphism. Thus the only source of such automorphisms of $\mathrm{C}^*(H,L)$ are automorphisms of the Fell line bundle L; this is the trivial Fell line bundle in our case.

Any automorphism of a Fell line bundle over a groupoid H acts by pointwise multiplication with a continuous groupoid homomorphism $H \to \mathbb{T}$. Since we want the automorphism to act identically on $\mathrm{C}^*(H_1)$, this homomorphism must be constant on $H_1 \subseteq H$. Hence it is constant on the subspaces H_g and thus comes from a group homomorphism on G. Since G is the group completion of P, homomorphisms $G \to \mathbb{T}$ are in bijection with homomorphisms $P \to \mathbb{T}$.

A one-parameter automorphism group is equivalent to a continuous homomorphism $P \times \mathbb{R} \to \mathbb{T}$; this is equivalent to a continuous homomorphism

$P \to \mathrm{Hom}(\mathbb{R},\mathbb{T}) \cong \mathbb{R} \cong (\mathbb{R}_{>0},\cdot)$. Hence all one-parameter automorphism groups of $\mathrm{C}^*(H)$ that fix $\mathrm{C}^*(H_1)$ come from a homomorphism $P \to (0,\infty)$. □

Definition 6.25. A measure μ on H^0 is *c-invariant* if $\mu(r(B)) = c(g)\mu(s(B))$ for any bisection $B \subseteq H^1_g$. If $c \equiv 1$, we speak simply of invariant measures.

A c-invariant measure on H^0 gives a KMS-weight on $\mathrm{C}^*(H)$ for the corresponding automorphism group (with temperature 1); conversely, if H has trivial isotropy groups, then any KMS-weight on $\mathrm{C}^*(H)$ for this 1-parameter group of automorphisms is of this form for a unique c-invariant measure on H^0 (see [69, Proposition 5.4]); this result is generalised by Neshveyev [63] to KMS states on groupoids with non-trivial isotropy, and by Thomsen [75] to KMS weights. In particular, invariant measures on H^0 give tracial weights on $\mathrm{C}^*(H)$; if the set of objects with non-trivial isotropy has measure zero for all invariant measures on H^0, then all KMS weights are of this form.
We are going to describe invariant measures on H^0 in terms of measures on X. This requires two operations on measures: push-forwards along continuous maps and pull-backs along local homeomorphisms. The first is standard: if $f\colon X \to Y$ is a continuous map and μ is a measure on X, then $f_*\mu$ is the measure on Y defined by $f_*\mu(B) = \mu(f^{-1}(B))$ for Borel subsets $B \subseteq Y$. If $f\colon X \to Y$ is a local homeomorphism and λ is a measure on Y, then $f^*\lambda$ is the measure on X defined by

$$f^*\lambda(B) = \int_Y |f^{-1}(y) \cap B| \,\mathrm{d}\lambda(y).$$

If $h\colon X \to \mathbb{C}$ is Borel measurable with compact support, then

$$\int_X h(x)\,\mathrm{d}(f^*\lambda)(x) = \int_Y \sum_{\{x\in X \mid f(x)=y\}} h(x)\,\mathrm{d}\lambda(y)$$

because this holds for characteristic functions of Borel subsets.

Definition 6.26. A measure λ on X is *c-invariant* if $\lambda = c(p)(r_p)_*s_p^*(\lambda)$ for all $p \in P$.

Theorem 6.27. *The map $(\pi_1)_*$ induced by the projection $\pi_1\colon H^0 \to X$ is a bijection between c-invariant measures on H^0 and c-invariant measures on X.*

Proof. A measure μ on H^0 gives measures $\mu_p := (\pi_p)_*(\mu)$ on M_p for all $p \in P$. These are linked by $(r_{p,q})_*\mu_{pq} = \mu_p$ for all $p,q \in P$ because $r_{p,q} \circ \pi_{pq} = \pi_p$. Conversely, we claim that any family of measures $(\mu_p)_{p\in P}$ with $(r_{p,q})_*\mu_{pq} = \mu_p$ for all $p,q \in P$ comes from a unique measure μ on H^0. This is because $\bigcup_{p\in P} \pi_p^*(\mathrm{C_c}(M_p))$ is a dense subspace in $\mathrm{C_c}(H^0)$. The consistency condition $r_{p,q} \circ \pi_{pq} = \pi_p$ implies that the positive linear maps $\mathrm{C_c}(M_p) \to \mathbb{C}$, $f \mapsto \int_{M_p} f(x)\,\mathrm{d}\mu_p(x)$, well-define a positive linear map on $\bigcup_{p\in P} \pi_p^*(\mathrm{C_c}(M_p))$. This extends uniquely to a positive linear map on $\mathrm{C_c}(H^0)$. Thus we may replace a measure μ on H^0 by a family of measures $(\mu_p)_{p\in P}$ on the spaces M_p whenever this is convenient.
Next we claim that μ is c-invariant if and only if $\mu_p = c(p)s_p^*\mu_1$ for all $p \in P$.

Assume first that μ is c-invariant. Let $p \in P$ and let $U \subseteq M_p$ be an open, relatively compact subset such that $s_p|_U : U \to X$ is injective. Then

$$\{(x, p, \tilde{s}_p x) \mid x \in \pi_p^{-1}(U)\} \subseteq H^1_{p,1}$$

is a bisection with range $\pi_p^{-1}(U)$ and source $\pi_1^{-1}(s_p(U))$ by Lemma 6.5. Since μ is c-invariant,

$$\mu_p(U) = \mu(\pi_p^{-1}(U)) = c(p)\mu(\pi_1^{-1}(s_p(U))) = c(p)\mu_1(s_p(U)).$$

This equality also holds for all open subsets of U because s_p is still injective on them. This implies $\mu_p(B) = c(p)\mu_1(s_p(B))$ for all Borel subsets B of U.
If $B \subseteq M_p$ is an arbitrary Borel subset, then we may cover B by open, relatively compact subsets on which s_p is injective because s_p is a local homeomorphism. Then we may decompose B as a countable disjoint union $B = \bigsqcup_{i\in\mathbb{N}} B_i$ of Borel subsets with $B_i \subseteq U_i$ for open, relatively compact subsets U_i such that $s_p|_{U_i}$ is injective for all i. Applying the formula above for each i gives

$$\mu_p(B) = c(p) \int_X |s_p^{-1}(x) \cap B| \,\mathrm{d}\mu_1(x) = c(p) s_p^* \mu_1(B).$$

Thus $\mu_p = c(p)s_p^*\mu_1$ if μ is c-invariant.
Conversely, assume $\mu_p = c(p)s_p^*\mu_1$. Let $g = pq^{-1} \in G$ and let $V \subseteq H^1_g$ be a bisection. We may assume without loss of generality that $V \subseteq H^1_{p,q}$, replacing (p, q) by (ph, qh) for some $h \in P$ if necessary. Decomposing V into disjoint Borel subsets, we may further reduce to the case where $s(V) \subseteq H^0$ is one of the base neighbourhoods in Lemma 6.5; say, $s(V) = \pi_t^{-1}(U)$ for an open subset $U \subseteq M_t$. Replacing (p, q, t) by (pa, qa, tb) for suitable $a, b \in P$, we may arrange $t = p$ by the Ore condition. We assume this from now on. Decomposing V even further, we may arrange that $s_p|_U$ is injective. Then V is the product of two bisections, one of the form $\{(x, p, \tilde{s}_p(x)) \mid x \in \pi_p^{-1}(U)\}$, the other of the form $\{(y, q, \tilde{s}_q(y))^{-1} \mid y \in r(V)\}$; here $\tilde{s}_q r(V) = \tilde{s}_p \pi_p^{-1}(U) = \pi_1^{-1}(s_p(U))$. So $r(V)$ must be of the form $\pi_q^{-1}(W)$ for some $W \subseteq M_q$ for which $s_q|_W$ is a homeomorphism onto $s_p(U)$.
The upshot of these reductions is that μ is c-invariant once the c-invariance condition holds for bisections of the form $\{(x, p, \tilde{s}_p(x)) \mid x \in \pi_p^{-1}(U)\}$ with $p \in P$ and an open subset $U \subseteq M_p$. But this is exactly the condition $\mu_p = c(p)s_p^*\mu_1$. This finishes the proof of the claim.
The claim shows that the family of measures $(\mu_p)_{p\in P}$ and hence the measure μ is determined uniquely by the measure μ_1 on $M_1 = X$ provided μ is c-invariant. If we are given a measure λ on X, then $\mu_p := c(p)s_p^*(\lambda)$ is the only possibility for a c-invariant measure on H^0 with $\mu_1 = \lambda$. This family of measures gives a measure on H^0 if and only if $(r_{p,q})_*\mu_{pq} = \mu_p$ for all $p, q \in P$. In particular, for $p = 1$ and $q \in P$, this gives the condition $c(q)(r_q)_* s_q^*\lambda = (r_q)_*\mu_q = \lambda$, that is, λ has to be c-invariant.
The proof of the theorem will be finished by checking that

$$c(p)s_p^*\lambda = c(pq)(r_{p,q})_* s_{pq}^*\lambda$$

holds for all $p, q \in P$ if λ is a c-invariant measure on X. Since $c(pq) = c(p)c(q)$, we have to prove $c(q)(r_{p,q})_* s_{pq}^* \lambda = s_p^* \lambda$. Let $U \subseteq M_p$ be an open, relatively compact subset. On the one hand, $(s_p^*\lambda)(U) := \int_X |s_p^{-1}(x) \cap U| \,\mathrm{d}\lambda(x)$. Substituting $c(q)(r_q)_* s_q^* \lambda$ for λ, this becomes

$$\begin{aligned} (6.28) \quad c(q) \int_X |s_p^{-1}(x) \cap U| \,\mathrm{d}(r_q)_* s_q^* \lambda(x) &= c(q) \int_{M_q} |s_p^{-1}(r_q(z)) \cap U| \,\mathrm{d}s_q^* \lambda(z) \\ &= c(q) \int_X \sum_{\{z \in M_q \mid s_q(z) = x\}} |s_p^{-1}(r_q(z)) \cap U| \,\mathrm{d}\lambda(x). \end{aligned}$$

On the other hand,

$$(6.29) \quad c(q)((r_{p,q})_* s_{pq}^* \lambda)(U) = c(q) s_{pq}^* \lambda(r_{p,q}^{-1}(U)) = c(q) \int_X |s_{pq}^{-1}(x) \cap r_{p,q}^{-1}(U)| \,\mathrm{d}\lambda(x).$$

We may identify $M_{pq} \cong M_p \times_{s_p, X, r_q} M_q$ and $r_{p,q}$ with the projection to the first factor. Hence $s_{pq}^{-1}(x) \cap r_{p,q}^{-1}(U)$ is the set of pairs (y, z) with $y \in U \subseteq M_p$, $z \in M_q$, $s_p(y) = r_q(z)$ and $s_q(z) = x$. The cardinality of this set is the sum over all $z \in s_q^{-1}(x)$ of the cardinalities of $s_p^{-1}(r_q(z)) \cap U$. Hence the right hand sides in (6.28) and (6.29) are equal, as desired. $\square$

Similar arguments as in the end of the proof of Theorem 6.27 show the following:

Remark 6.30. The map $\alpha_p := (r_p)_*(s_p)^*$ on the space of measures on X is an action of P, that is, $\alpha_{pq} = \alpha_p \circ \alpha_q$ for all $p, q \in P$. Therefore, if $S \subseteq P$ generates P, then it suffices to check whether a measure is invariant for $p \in S$.

Example 6.31. Assume that X is discrete. Then a (positive) measure on X is simply a function $\lambda \colon X \to \mathbb{R}_+$. We have $(r_p)_*(s_p)^*(\lambda)(x) = \sum_{r_p(m)=x} \lambda(s_p(m))$; a c-invariant measure is a non-negative joint eigenvector of these maps with eigenvalue $p \mapsto c(p)$. If $P = \mathbb{N}$, then it suffices to look at the generator $1 \in \mathbb{N}$, and the resulting map on the measures of X is matrix-vector multiplication with the adjacency matrix of the graph described by X_1. For $P = (\mathbb{N}^k, +)$, it suffices to look at the k generators of $\mathbb{N}^k$. Thus the c-invariant weights on H^0 are exactly those joint eigenvalues of the adjacency matrices that have a non-negative eigenvector.

Any c-invariant weight on H^0 gives a KMS state on $\mathrm{C}^*(H)$ by first applying the conditional expectation $\mathrm{C}^*(H) \to \mathrm{C}_0(H^0)$ that restricts functions to H^0 and then integrating against the measure on H^0. If *all* isotropy groups are trivial, then these are all KMS states. Neshveyev [63] shows how to describe KMS states if there is non-trivial isotropy. A particularly simple case is if the set of points in H^0 with non-trivial isotropy is a null set for all c-invariant measures on H^0. In that case, all KMS states for the $\mathbb{R}$-action generated by the cocycle c factor through the conditional expectation, so we get the same answer as if there were no isotropy groups.

If the set of non-trivial isotropy has positive measure, then Neshveyev's description of KMS states uses essentially invariant measurable families of states on the C*-algebras of the isotropy groups. If there is lots of isotropy, these may be hard to classify. The situation is tractable, however, if the set of points in H^0 with non-trivial isotropy is countable.

This always happens for (separable) graph algebras. The set of orbits of points in H^0 with non-trivial isotropy is in bijection with simple loops in the graph. The orbit of a point in H^0 and the set of simple loops are both countable. Thus for a graph C*-algebra, we may get all KMS states in two steps. First we find the non-negative eigenvectors of the adjacency matrix. Then we examine which loops in the graph give atoms for the induced measure on H^0. For each such loop, we take a state on $C^*(\mathbb{Z})$, that is, a probability measure on the circle. The KMS states for a non-negative eigenvector of the adjacency matrix are in bijection with the set of all such families of probability measures on the circle. In the case of (higher-rank) graphs, the measures on H^0 that we get are essentially product measures coming from a measure on the discrete set of vertices V of the graph. If the higher-rank graph is effective, then the set of points with non-trivial isotropy in H^0 is often also a null set for such product type measures on H^0. This explains why the descriptions of the KMS states or weights on (higher-rank) graph C*-algebras in, for example, [37,38,75], are often equivalent to what we found above, namely, those joint eigenvalues of the k adjacency matrices that have a non-negative eigenvector. As usual, our analysis only applies to regular (higher-rank) graphs.

Remark 6.32. It is shown in [35] that any stably finite, exact, unital C*-algebra has a tracial state. Conversely, a C*-algebra with a tracial state is stably finite. Thus a unital, exact C*-algebra has a tracial state if and only if it is stably finite. We have already remarked that $C^*(H)$ is unital if and only if X' is compact, and exactness follows if H is amenable, compare Theorem 6.3. Hence Theorem 6.27 gives a necessary and sufficient criterion for $C^*(H)$ to be stably finite in the case where X' is compact metrisable, P is countable, and H is amenable.

6.5. Local contractivity.

Definition 6.33 ([3]). A locally compact, Hausdorff, étale groupoid $\mathcal{G}$ is *locally contracting* if, for every non-empty open subset $U \subset \mathcal{G}^0$, there are an open subset $V \subseteq U$ and a bisection B of $\mathcal{G}$ such that $\overline{V} \subseteq s(B)$ and $r(B\overline{V}) \subsetneq V$.

Since H is an étale, locally compact groupoid, [3, Proposition 2.4] shows that $C^*_r(H)$ is purely infinite (that is, every hereditary C*-subalgebra contains an infinite projection) if H is essentially free and locally contracting. Actually, we only need H to be effective here by [9, Lemma 3.1.(4)]: this is exactly the condition in [3, Lemma 2.3] that is used in the proof of [3, Proposition 2.4]. We would, however, not expect local contractivity to be necessary for $C^*_r(H)$ to be purely infinite.

The following definition characterises when the groupoid H is locally contracting in terms of the original action by topological correspondences:

Definition 6.34. An action $(M_p, \sigma_{p,q})$ of an Ore monoid P on a locally compact Hausdorff space X by proper topological correspondences is *locally contracting* if for any relatively compact, open subset $S \subseteq X'$, there are $n \in \mathbb{N}$ and $p_i, q_i, a_i, b_i \in P$ and $W_i \subseteq M'_{p_i} \times_{s_{p_i}, X', s_{q_i}} M'_{q_i}$ for $1 \le i \le n$ such that

(LC1) $p_1 a_1 = p_2 a_2 = \dots = p_n a_n = q_1 b_1 = \dots = q_n b_n$;
(LC2) the restricted coordinate projections $\mathrm{pr}_1 \colon W_i \to M'_{p_i}$ and $\mathrm{pr}_2 \colon W_i \to M'_{q_i}$ are injective and open;
(LC3) the subsets

$$\mathrm{pr}_1(W_i) M'_{a_i} := \{\sigma^{-1}_{p_i,a_i}(x_1, x_2) \mid x_1 \in \mathrm{pr}_1(W_i),\ x_2 \in M'_{a_i}, s_{p_i}(x_1) = r_{a_i}(x_2)\}$$

of $M'_{p_i a_i}$ are disjoint, and so are the subsets $\mathrm{pr}_2(W_i) M'_{b_i}$;
(LC4) $\overline{\bigsqcup_{i=1}^n \mathrm{pr}_1(W_i) M'_{a_i}} \subsetneq \bigsqcup_{i=1}^n \mathrm{pr}_2(W_i) M'_{b_i}$;
(LC5) $r_{q_i} \, \mathrm{pr}_2(W_i) \subseteq S$;
(LC6) $p_i q_i^{-1} \neq p_j q_j^{-1}$ in G for $i \neq j$.

Here $X' \subseteq X$ is the closed invariant subspace of possible situations, which is different from X if some of the range maps are not surjective.

The choice of a_i, b_i does not really matter: if the conditions hold for one choice satisfying (LC1), then also for all others. This follows from Lemma 4.16 and the surjectivity of the maps $r_{p,q}$ on the M'_{pq}.
Giving up some symmetry, we may use the Ore conditions to simplify the data above slightly: we may assume either $p_1 = p_2 = \dots = p_n$ and $a_1 = a_2 = \dots = a_n$ or $q_1 = q_2 = \dots = q_n$ and $b_1 = b_2 = \dots = b_n$.
Condition (LC6) says, roughly speaking, that we cannot make n smaller by combining the data for any $i \neq j$. This is its only role, and it could be left out.

Theorem 6.35. *The groupoid H is locally contracting if and only if the action $(M_p, \sigma_{p,q})$ is locally contracting.*

Proof. To simplify notation, we replace X by X' throughout, so that the maps r_p are all surjective.
Call a subset U of H^0 good if there are V, B as in Definition 6.33. If $U_1 \subseteq U_2$ and U_1 is good, then so is U_2. If $T \subseteq H^1$ is a bisection, then $r(T)$ is good if and only if $s(T)$ is good: given V, B for $s(T)$, then TV and TBT^{-1} will work for $r(T)$. If $U \subseteq H^0$ is an arbitrary open subset, then there are $p \in P$ and $U_p \subseteq M_p$ such that $\pi_p^{-1}(U_p) \subseteq U$, $s_p|_{U_p}$ is injective, and U_p is relatively compact and open; here we use that subsets of the form $\pi_p^{-1}(U_p)$ for $U_p \subseteq M_p$ open form a base (Lemma 6.5), that M_p is locally compact, and that s_p is a local homeomorphism for each p (Lemma 4.10). Hence there is a bisection with range $\pi_p^{-1}(U_p)$ and source $\pi_1^{-1}(s_p(U_p))$. Thus U is good when $\pi_1^{-1}(s_p(U_p))$ is good. Summing up, all non-empty open subsets of H^0 are good once all non-empty subsets of the form $\pi_1^{-1}(S)$ with $S \subseteq X$ relatively compact and open are good.
Assume now that the action $(M_p, \sigma_{p,q})$ is locally contracting. Let $S \subset X$ be a non-empty, open, relatively compact set. Pick $n \in \mathbb{N}$ and $p_i, q_i, a_i, b_i \in P$ and

subsets $W_i \subseteq M_{p_i} \times_{s_{p_i},X,s_{q_i}} M_{q_i}$ as in Definition 6.34. Let

$$B_i := \{(x_1 y, p_i q_i^{-1}, x_2 y) \mid (x_1, x_2) \in W_i,\ y \in H^0,\ s_{p_i}(x_1) = \pi_1(y)\};$$

here $x_1 y$ and $x_2 y$ are well-defined because $s_{p_i}(x_1) = s_{q_i}(x_2) = \pi_1(y)$, which uses that $s_{p_i}(x_1) = s_{q_i}(x_2)$ for $(x_1, x_2) \in W_i$. Since $\tilde{s}_{p_i}(x_1 y) = y = \tilde{s}_{q_i}(x_2 y)$, we have $B_i \subseteq H^1_{p_i,q_i}$. Condition (LC2) ensures that the range and source maps are open and injective on each B_i, so these are bisections. For different i, they have disjoint sources and ranges by (LC3). Hence $B := B_1 \sqcup B_2 \sqcup \dots \sqcup B_n$ is a bisection as well. Condition (LC4) gives $\overline{r(B)} \subsetneq s(B)$, and (LC5) gives $s(B) \subseteq \pi_1^{-1}(S)$. Since we assume S to be relatively compact, the closed subset $\overline{r(B)}$ is compact. So there is an open subset V with $\overline{r(B)} \subsetneq V \subseteq \overline{V} \subseteq s(B)$. Hence $\pi_1^{-1}(S)$ is good. This implies that H is locally contracting.

Conversely, assume that H is locally contracting. Let $S \subseteq X$ be a relatively compact, non-empty, open subset and let $U := \pi_1^{-1}(S)$. Then U is relatively compact, non-empty and open because $\pi_1 \colon H^0 \to X$ is surjective, continuous, and proper. Since H is locally contracting, there is a bisection $B \subseteq H^1$ with $\overline{r(B)} \subsetneq s(B) \subseteq U$. Next we have to analyse this bisection B locally. This does not yet use the special feature of B and gives slightly more, namely, a "base" for the inverse semigroup of bisections of H. By this we mean an inverse subsemigroup closed under finite intersections that covers H^1. This can be used to study actions of H on C*-algebras as in [11].

Lemma 6.36. *Let $B \subseteq H^1$ be a bisection and $\eta \in B$. Then there is an open bisection B_η with $\eta \in B_\eta \subseteq B$ that has the following form:*

$$B_\eta := \{(x_1 y, pq^{-1}, x_2 y) \mid (x_1, x_2) \in W,\ y \in H^0,\ s_p(x_1) = \pi_1(y)\}$$

for some $p, q \in P$ and a subset $W \subseteq M_p \times_{s_p,X,s_q} M_q$ such that $\mathrm{pr}_1 \colon W \to M_p$ and $\mathrm{pr}_2 \colon W \to M_q$ are injective and open.

If $pt = p_1$, $qt = q_1$ and

$$W_1 := \{(x_1 y, x_2 y) \mid (x_1, x_2) \in W,\ y \in W_t,\ s_p(x_1) = r_t(y)\}$$

for some $t \in P$, then the data (p, q, W) and (p_1, q_1, W_1) define the same bisection B_η.

Proof. We write B_* for the homeomorphism $s(B) \to r(B)$ induced by B; this is determined uniquely by $B_*(s(h)) = r(h)$ for all $h \in B$. Let $\xi := s(\eta)$.

There are $p, q \in P$ with $\eta \in H^1_{p,q}$ because these subsets cover H^1. Since the subsets B and $H^1_{p,q}$ and the map $s \colon H^1 \to H^0$ are open, $s(B \cap H^1_{p,q})$ is an open neighbourhood of ξ. Let $h \in B \cap H^1_{p,q}$. Lemma 4.10 shows that there are unique $x_2 \in M_q$ and $y \in H^0$ with $s_q(x_2) = \pi_1(y)$ and $s(h) = x_2 y$, namely, $x_2 = \pi_q(s(h))$ and $y = \tilde{s}_q(s(h))$. Similarly, there are unique $x_1 \in M_p$ and $y' \in H^0$ with $s_p(x_1) = \pi_1(y')$ and $r(h) = x_1 y'$, namely, $x_1 = \pi_p(r(h))$ and $y' = \tilde{s}_p(r(h))$. The assumption $h \in H^1_{p,q}$ means exactly that $y = y'$. Thus $h = (x_1 y, pq^{-1}, x_2 y)$ for $x_1 \in M_p$, $x_2 \in M_q$, $y \in H^0$ with $s_p(x_1) = s_q(x_2) = \pi_1(y)$.

Since s_p is a local homeomorphism, there is an open neighbourhood V around $\pi_p(B_* \xi) \in M_p$ so that $s_p|_V \colon V \to X$ is a homeomorphism onto an open subset of X. The subset

$$V' := \{x \in s(B \cap H^1_{p,q}) \mid \pi_p(B_* x) \in V\}$$

is still an open neighbourhood of ξ. It contains a neighbourhood of ξ that belongs to the base in Lemma 6.5. This gives us $q_2 \in P$ and $V'' \subset M_{q_2}$ with $\xi \in \pi_{q_2}^{-1}(V'') \subseteq V'$. Since s_{q_2} is a local homeomorphism as well, we may further shrink V'' so that $s_{q_2}|_{V''}$ becomes injective; we assume this. The first Ore condition gives us $a, b \in P$ with $qa = q_2 b$. Let $p' := pa$ and $q' := qa = q_2 b$. Let

$$W := \{(\pi_{p'}(r(h)), \pi_{q'}(s(h))) \mid h \in B,\ \pi_{q_2}(s(h)) \in V''\}.$$

We claim that p', q', W have the asserted properties.
Let $h \in B$ satisfy $\pi_{q_2}(s(h)) \in V''$. Write $s(h) = x_2 y$, $r(h) = x_1 y$ with $x_1 \in M_p$, $x_2 \in M_q$, $y \in H^0$ and $s_p(x_1) = s_q(x_2) = \pi_1(y)$ as above. Then $x_1 \in V$, so x_1 is the unique point in V with $s_p(x_1) = s_q(x_2)$. Now write $y = y_1 y_2$ with $y_1 \in M_a$, $y_2 \in H^0$, $s_a(y_1) = \pi_1(y_2)$. Then $s(h) = x_1 y_1 y_2$ and $r(h) = x_2 y_1 y_2$. The point $x_1 y_1 \in M_{pa} = M_{p'}$ is the unique one in $r_{p,a}^{-1}(V)$ with $s_{p,a}(x_1 y_1) = y_1$. This shows that $\pi_{p'}(r(h))$ is determined by $\pi_{q'}(s(h))$ and that the map that takes $\pi_{q'}(s(h))$ to $\pi_{q'}(r(h))$ is continuous. Hence the second coordinate projection $\mathrm{pr}_2 \colon W \to M_{q'}$ is injective and open. Since we assumed s_{q_2} to be injective on V'', the same holds for $\mathrm{pr}_1 \colon W \to M_{p'}$.
If $(x_1 y, pq^{-1}, x_2 y) \in B_\eta$, then $\tilde{s}_p(x_1 y) = y = \tilde{s}_q(x_2 y)$, so $(x_1 y, pq^{-1}, x_2 y) \in H^1_{p,q}$ and $B_\eta \subseteq H^1$. Since $\mathrm{pr}_1 \colon W \to M_{p'}$ and $\mathrm{pr}_2 \colon W \to M_{q'}$ are injective and open, so are the maps $s, r \colon B_\eta \to H^0$ because y_2 is common to both source and range and $x_1 y_1$ and $x_2 y_1$ determine each other uniquely and continuously. Thus B_η is a bisection. It is clear from the construction that $\eta \in B_\eta \subseteq B$. The last statement about different data giving the same B_η is implicitly shown above. $\square$

Fix $\xi_0 \in s(B) \setminus \overline{r(B)}$. The subset $\overline{r(B)} \cup \{\xi_0\} \subseteq s(B) \subseteq U = \pi_1^{-1}(S)$ is closed and contained in the relatively compact subset $\pi_1^{-1}(S)$, so it is compact. For each $x \in \overline{r(B)} \cup \{\xi_0\} \subseteq s(B)$, there is a unique $\eta \in B$ with $s(\eta) = x$. Let $B_\eta \subseteq B$ be some bisection as in Lemma 6.36. The open subsets $s(B_\eta)$ for these chosen bisections cover $\overline{r(B)} \cup \{\xi_0\}$. By compactness, this only needs finitely many of them, say, $B_1, \dots, B_\ell$. Let $B' := B_1 \cup B_2 \dots \cup B_\ell \subseteq B$. This is still an open bisection, and it satisfies $\overline{r(B')} \subseteq \overline{r(B)} \subseteq s(B') \subseteq s(B) \subseteq U$. Since $\xi_0 \in s(B') \setminus \overline{r(B)}$, we get $\overline{r(B')} \subsetneq s(B')$. Hence we may replace B by B'.
Each B_i is constructed from certain $p_i, q_i \in P$ and $W_i \subseteq M_{p_i} \times M_{q_i}$ as in Lemma 6.36. The Ore conditions also provide $e, a_i, b_i \in P$ with $p_i a_i = e = q_i b_i$ for all i. Our construction so far already achieves the most crucial properties (LC1), (LC2), (LC4) and (LC5); the last one follows from $s(B') \subseteq s(B) \subseteq \pi_1^{-1}(S)$. So far, however, the subsets $s(B_i)$ and $r(B_i)$ may still overlap, and some among the group elements $p_i q_i^{-1}$ may well be equal. We now rectify this. Let $p_i q_i^{-1} = p_j q_j^{-1}$ for some $i \neq j$. We may replace (p_i, q_i) by $(p_i c, q_i c)$ for $c \in P$ using the last statement in Lemma 6.36. By this, we can arrange that

$p_i = p_j$ and $q_i = q_j$, which we now assume. Then

$$B_i \cup B_j = \{(x_1 y, pq^{-1}, x_2 y) \mid (x_1, x_2) \in W_i \cup W_j,\ y \in H^0,\ s_p(x_1) = \pi_1(y)\}.$$

We know that $B_i \cup B_j$ is a bisection because it is contained in the bisection B. Since $X = X'$, this implies that the coordinate projections remain injective on $W_i \cup W_j$. They are open because this holds locally on W_i and W_j. Hence we may simply merge the data (p_i, q_i, W_i) and (p_j, q_j, W_j) into one piece, without changing the bisection B'. We go on merging part of our data until all group elements $g_i := p_i q_i^{-1}$ are different. This achieves (LC6). By construction, $B_i \subseteq H^1_{p_i,q_i} \subseteq H^1_{g_i}$, and these subsets are disjoint for different g_i. Hence the B_i are disjoint bisections. Since their union remains a bisection, their ranges and sources must be disjoint, which is (LC3). Hence all the conditions in Definition 6.34 hold. □

Our condition for local contractivity is rather complicated because it is necessary and sufficient. If we restricted to $n = 1$ in Definition 6.34, the condition would no longer be necessary, and it would depend on the G-grading on H, so it would not be a property of the groupoid H alone. Nevertheless, the case $n = 1$ of our criterion gives a useful sufficient condition:

Corollary 6.37. *The groupoid H is locally contracting if, for any relatively compact, open subset $S \subseteq X'$, there are $p, q, a, b \in P$ with $pa = qb$ and a subset $W \subseteq M'_p \times_{s_p,X',s_q} M'_q$ such that the projections $\mathrm{pr}_1\colon W \to M'_p$ and $\mathrm{pr}_2\colon W \to M'_q$ are injective and open, $r_q(\mathrm{pr}_2(W)) \subseteq S$, and $\overline{\mathrm{pr}_1(W)} \cdot M'_a \subsetneq \mathrm{pr}_2(W) \cdot M'_b$ as subsets of $M'_{pa} = M'_{qb}$.*

Proof. Besides restricting to $n = 1$, we also have rewritten

$$\overline{\mathrm{pr}_1(W) \cdot M'_a} = \overline{\mathrm{pr}_1(W)} \cdot M'_a,$$

using the closure of $\mathrm{pr}_1(W)$ in M'_p. This is because $\mathrm{pr}_1(W) \cdot M'_a = \mathrm{pr}_1(W) \times_{s_p,X',s_q} M'_a \subseteq M'_p \times_{s_p,X',s_q} M'_a$, and for such a subset the closure operation clearly works on the first entry only. □

Specialising further, we may assume W to have the form $W = W'_p \times_{s_p,X',s_q} W'_q$ for open subsets $W_p \subseteq M'_p$ and $W_q \subseteq M'_q$; if W has this form, then we may choose W_p and W_q minimal given W by taking $W_p = \mathrm{pr}_1(W)$ and $W_q = \mathrm{pr}_2(W)$. Then $s_p(W_p) = s_q(W_q)$ because $s_p \circ \mathrm{pr}_1 = s_q \circ \mathrm{pr}_2$ on W.

Lemma 6.38. *The maps $\mathrm{pr}_1|_W$ and $\mathrm{pr}_2|_W$ are injective and open if and only if $W_p \subseteq M'_p$ and $W_q \subseteq M'_q$ are open and the restrictions $s_p|_{W_p}$ and $s_q|_{W_q}$ are injective.*

Proof. Assume first that $\mathrm{pr}_1|_W$ and $\mathrm{pr}_2|_W$ are injective and open. Then $W_p = \mathrm{pr}_1(W)$ and $W_q = \mathrm{pr}_2(W)$ are open. If $x \in W_p$, then any $y \in W_q$ with $s_q(y) = s_p(x)$ gives a point $(x, y) \in W$ with $\mathrm{pr}_1(x, y) = x$. Hence $s_q|_{W_q}$ is injective if $\mathrm{pr}_1|_W$ is injective. Conversely, assume that $W_p \subseteq M'_p$ and $W_q \subseteq M'_q$ are open subsets and the restrictions $s_p|_{W_p}$ and $s_q|_{W_q}$ are injective. Since s_p and s_q are local homeomorphisms, so are their restrictions to the open subsets W_p and W_q.

Being also injective and continuous, they are homeomorphisms onto $s_p(W_p) = s_q(W_q)$. Hence the coordinate projections on W are homeomorphisms onto W_p and W_q, respectively. Since these subsets are open, the coordinate projections are injective and open. $\square$

Corollary 6.39. *The groupoid H is locally contracting if, for any relatively compact, open subset $S \subseteq X'$, there are $p, q, a, b \in P$ with $pa = qb$ and open subsets $W_p \subseteq M'_p$ and $W_q \subseteq M'_q$ such that $s_p(W_p) = s_q(W_q)$, the restrictions $s_p|_{M'_p}$ and $s_q|_{M'_q}$ are injective, $r_q(W_q) \subseteq S$, and $\overline{W_p} \cdot M'_a \subsetneq W_q \cdot M'_b$ as subsets of $M'_{pa} = M'_{qb}$.*

Proof. Specialise Corollary 6.37 to the case where $W = W_p \times_{s_p, X', s_q} W_q$. $\square$

The role of $a, b \in P$ is only as padding to be able to compare $\overline{W_p}$ and W_q. Since we restricted to possible histories, so that all range maps are surjective, we have

$$\overline{W_p} \cdot M'_a \subsetneq W_q \cdot M'_b \quad \iff \quad \overline{W_p} \cdot M'_{at} \subsetneq W_q \cdot M'_{bt}$$

for any $t \in P$. This shows that the choice of a, b does not matter: if the criterion holds for one choice, it holds for all choices. The same is true, of course, for Corollary 6.39, and an analogous statement holds for Theorem 6.35.
If P is a lattice-ordered Ore monoid, we therefore get equivalent criteria if we take $a = p^{-1}(p \vee q)$ and $b = q^{-1}(p \vee q)$ in Corollary 6.37 or Corollary 6.39. We write down the variant of Corollary 6.39:

Corollary 6.40. *Assume that P is Ore and lattice-ordered. The groupoid H is locally contracting if, for any relatively compact, open subset $S \subseteq X'$, there are $p, q \in P$ and open subsets $W_p \subseteq M'_p$ and $W_q \subseteq M'_q$ such that $s_p(W_p) = s_q(W_q)$, the restrictions $s_p|_{M'_p}$ and $s_q|_{M'_q}$ are injective, $r_q(W_q) \subseteq S$, and $\overline{W_p} \cdot M'_a \subsetneq W_q \cdot M'_b$ as subsets of $M'_{pa} = M'_{qb}$, where $a = p^{-1}(p \vee q)$ and $b = q^{-1}(p \vee q)$.*

The inclusion $\overline{W_p} \cdot M'_a \subseteq W_q \cdot M'_b$ for $a = p^{-1}(p \vee q)$ and $b = q^{-1}(p \vee q)$ means that for any $x_p \in \overline{W_p}$ there is $x_q \in W_q$ so that x_p and x_q have a common extension; the minimal such extension lives in $M'_{pa} = M'_{qb}$. The meaning of $\overline{W_p} \cdot M'_a \neq W_q \cdot M'_b$ is that some $y \in W_q$ has no common extension with any element of $\overline{M_p}$.
Corollary 6.40 involves the same ingredients as the sufficient condition for the boundary path groupoid of a higher-rank topological graph to be locally contracting in [73, Proposition 5.8]. The proof of [73, Lemma 5.9] shows that our sufficient criterion is more general than the one in [73]. We are, however, not aware of any applications of the criterion in [73].
Our criteria have the advantage that we understand very well which assumptions we have imposed on the contracting bisections for H. In Corollary 6.37, the only assumption is that the bisection is contained in H^1_g for some $g \in G$. In Corollaries 6.39 and 6.40, we assume this and that the bisection has a product form.

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