Enumerating Exceptional Collections of Line Bundles on Some Surfaces of General Type

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Abstract. We use constructions of surfaces as abelian covers to write down exceptional collections of line bundles of maximal length for every surface X in certain families of surfaces of general type with $p_g = 0$ and $K_X^2 = 3, 4, 5, 6, 8$. We also compute the algebra of derived endomorphisms for an appropriately chosen exceptional collection, and the Hochschild cohomology of the corresponding quasiphantom category. As a consequence, we see that the subcategory generated by the exceptional collection does not vary in the family of surfaces. Finally, we describe the semigroup of effective divisors on each surface, answering a question of Alexeev.

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Contents

1 Introduction

Exceptional collections of maximal length on surfaces of general type with $p_g = 0$ have been constructed for Godeaux surfaces [13, 15], primary Burniat surfaces [2], and Beauville surfaces [24, 39]. Recently, progress has also been made for some fake projective planes [25, 23]. In this article, we present a method which can be applied uniformly to produce exceptional collections of line bundles on several surfaces with $p_g = 0$, including Burniat surfaces with $K^2 = 6$ (cf. [2]), $5, 4, 3$, Kulikov surfaces with $K^2 = 6$ and some Beauville surfaces with $K^2 = 8$ [24, 39]. In fact we do more: we enumerate all exceptional collections of line bundles corresponding to any choice of numerical exceptional collection. We can use this enumeration process to find those exceptional collections that are particularly well-suited to studying the surface itself, and possibly its moduli space.

Both [2] and [24] hinted that it should be possible to produce exceptional collections of line bundles on a wide range of surfaces of general type with $p_g = 0$. This inspired us to build the approaches of [2, 24] into the larger framework of abelian covers (see especially Section 2), an important part of which is a new formula for the pushforward of certain line bundles on any abelian cover, generalising formulas of Pardini [43]. We believe that this work is a step in the right direction, even though there remain many families of surfaces which require further study (see Section 3.1 for more details).

Let X be a surface of general type with $p_g = 0$, and let Y be a del Pezzo surface with $K_Y^2 = K_X^2$. The lattices $\operatorname{Pic} X/\operatorname{Tors} X$ and $\operatorname{Pic} Y$ are both isomorphic to $\mathbb{Z}^{1,N}$, where $N = 9 - K_X^2$, and moreover, the cohomology groups $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are completely algebraic. By exploiting this relationship between X and Y, we can study exceptional collections of line bundles on X. Indeed, exceptional collections on del Pezzo surfaces are well understood after [42], [33], and we sometimes refer to X as a fake del Pezzo surface, to emphasise this analogy.

Suppose now that X is a fake del Pezzo surface that is constructed as a branched Galois abelian cover $\varphi\colon X \to Y$, where Y is a (weak) del Pezzo surface with $K_Y^2 = K_X^2$. Many fake del Pezzo surfaces can be constructed in this way [10], but we require certain additional assumptions on the branch locus and Galois group (see Section 3.1). These assumptions ensure that there is an appropriate choice of lattice isometry $\operatorname{Pic} Y \to \operatorname{Pic} X/\operatorname{Tors} X$. This isometry is combined with our pushforward formula to calculate the coherent cohomology of any line bundle on X.

Theorem 1.1 (Theorem 2.1) *Let X be a fake del Pezzo surface satisfying our assumptions, and let L be any line bundle on X. We have an explicit formula for the line bundles M_χ appearing in the pushforward $\varphi_* L = \bigoplus_{\chi \in G^*} M_\chi$, where G is the Galois group of the cover $\varphi\colon X \to Y$.*

Working modulo torsion, we can use the above lattice isometry to lift any exceptional collection of line bundles on Y to a numerical exceptional collection on X. We then incorporate Theorem 1.1 into a systematic computer search, to find those combinations of torsion twists which correspond to an exceptional collection on X.
The search for exceptional collections on fake del Pezzo surfaces, leads naturally to the following question, which was asked by Alexeev [1]:

Can we characterise effective divisors on X in terms of those on Y?

For example, in [1], Alexeev gives an explicit description of the semigroup of effective divisors on the Burniat surface with $K^2 = 6$, and proposes similar descriptions for the other Burniat surfaces. We use our pushforward formula to prove these characterisations for the Burniat surfaces and other fake del Pezzo surfaces, cf. Theorems 3.2, 5.1.

Theorem 1.2 *Let X be a fake del Pezzo surface satisfying our assumptions. Then the semigroup of effective divisors on X is generated by the reduced pullback of irreducible components of the branch divisor, together with pullbacks of certain (-1)-curves on Y.*

Let $\mathbb{E}$ be an exceptional collection on X, and suppose $H_1(X, \mathbb{Z})$ is nontrivial. Then $\mathbb{E}$ can not be full, for K-theoretic reasons (see Section 4). Hence we have a semiorthogonal decomposition of the bounded derived category of coherent sheaves on X:

$$D^b(X) = \langle \mathbb{E}, \mathcal{A} \rangle.$$

If $\mathbb{E}$ is of maximal length, then $\mathcal{A}$ is called a quasiphantom category; that is, $K_0(\mathcal{A})$ is torsion and the Hochschild homology $HH_*(\mathcal{A})$ is trivial. Even when $H_1(X, \mathbb{Z})$ vanishes, an exceptional collection of maximal length need not be full (see [15]), and in this case $\mathcal{A}$ is called a phantom category, because $K_0(\mathcal{A})$ is trivial.
On the other hand, the Hochschild cohomology does detect the quasiphantom category $\mathcal{A}$; in fact, $HH^*(\mathcal{A})$ measures the formal deformations of $\mathcal{A}$. We calculate $HH^*(\mathcal{A})$ by considering the A_∞-algebra of endomorphisms of $\mathbb{E}$, together with the spectral sequence developed in [36]. Indeed, one of the advantages of our systematic search, is that we can find exceptional collections for which the higher multiplications in the A_∞-algebra of $\mathbb{E}$ are as simple as possible. Theorem 1.3 below serves as a prototype statement of our results for a good exceptional collection on a fake del Pezzo surface. More precise statements can be found for the Kulikov surface in Section 4.7.

Theorem 1.3 *Let $\mathcal{X} \to T$ be a family of fake del Pezzo surfaces satisfying our assumptions. Then for any t in T, there is an exceptional collection $\mathbb{E}$ of line bundles on $X = \mathcal{X}_t$ which has maximal length $12 - K_X^2$. Moreover, the subcategory of $D^b(X)$ generated by $\mathbb{E}$ does not vary with t, and the Hochschild cohomology of X agrees with that of the quasi-phantom category $\mathcal{A}$ in degrees less than or equal to two.*

The significance of Theorem 1.3 is amplified by the reconstruction theorem of [17]: if X and X' are smooth, $\pm K_X$ is ample, and $D^b(X)$ and $D^b(X')$ are equivalent bounded derived categories, then $X \cong X'$. In conjunction with Theorem 1.3, we see that if K_X is ample, then X can be reconstructed from the quasi-phantom category $\mathcal{A}$. The gluing between $\mathcal{A}$ and $\mathbb{E}$ does not vary with X, because the statement about Hochschild cohomology implies that the formal deformation spaces of X are isomorphic to the formal deformation spaces of $\mathcal{A}$. Currently, it is not clear whether there is any practical way to extract information about X from $\mathcal{A}$, although some interesting ideas are discussed in [2]. It would be interesting to know whether this "rigidity" of $\mathbb{E}$ is a general phenomenon, or just a coincidence for good choices of exceptional collection.

In Section 2 we review abelian covers, and prove our result on pushforwards of line bundles, which is valid for any abelian cover, and is used throughout. In Section 3.1, we explain our assumptions on the fake del Pezzo surface X and its Galois covering structure $\varphi\colon X \to Y$, and describe our approach to enumerating exceptional collections on the surface of general type. Section 3.2 is an extended treatment of the Kulikov surface with $K^2 = 6$, which is an example of a fake del Pezzo surface. We give a cursory review of dg-categories and A_∞-algebras in Section 4, as background to our discussion of quasi-phantom categories and the theory of heights from [36]. We then show how to compute the A_∞-algebra and height of an exceptional collection on the Kulikov surface. In Section 5 we prove Theorem 1.2 for the secondary nodal Burniat surface with $K^2 = 4$. Appendix A lists certain data relevant to the Kulikov surface example of Section 3.2, and Appendix B applies similarly to the secondary nodal Burniat surface of Section 5.

With appropriate amendments, Theorems 1.2 and 1.3 hold for the Burniat surfaces with $K^2 = 6, 5, 4, 3$ and some Beauville surfaces with $K^2 = 8$. The arguments used are similar to those appearing in Sections 3.2 and 5.1, and we refer to [20] for details. We have exceptional collections of maximal length on the tertiary Burniat surface with $K^2 = 3$. In this case it is necessary to use the Weyl group action on the Picard group to find exceptional collections. We can show that the A_∞-category is formal, but we do not yet know how to compute the Hochschild cohomology of the quasiphantom category.

In order to use results on deformations of each fake del Pezzo surface, we work over $\mathbb{C}$.

Remark 1.1 The calculation of $\varphi_* L$ according to Theorem 1.1 is elementary but repetitive; we include a few sample calculations to illustrate how to do it by hand, but when the torsion group becomes large, it is more practical to use computer algebra. Our enumerations of exceptional collections are obtained by simple exhaustive computer searches. We use Magma [12], and the annotated scripts are available from [20].

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2 Preliminaries

We collect together the relevant material on abelian covers. See especially [43], [7] or [34] for details. Unless stated otherwise, X and Y are normal projective varieties, with Y nonsingular. Let G be a finite abelian group acting faithfully on X with quotient $\varphi\colon X \to Y$. Write $\Delta = \sum \Delta_i$ for the branch locus of φ, where each Δ_i is a reduced, irreducible effective divisor on Y. The cover φ is determined by the group homomorphism

$$\Phi\colon \pi_1(Y - \Delta) \to H_1(Y - \Delta, \mathbb{Z}) \to G,$$

which assigns an element of G to the class of a loop around each irreducible component Δ_i of Δ. If Φ is surjective, then X is irreducible. The factorisation through $H_1(Y - \Delta, \mathbb{Z})$ arises because G is assumed to be abelian, so we only need to consider the map $\Phi\colon H_1(Y - \Delta, \mathbb{Z}) \to G$. For brevity, we refer to the loop around Δ_i by the same symbol, Δ_i.
Let $\widetilde{Y}$ be the blow up of Y at a point P where several branch components $\Delta_{i_1}, \ldots, \Delta_{i_k}$ intersect. Then there is an induced cover of $\widetilde{Y}$, and the image of the exceptional curve E under Φ is given by

$$\Phi(E) = \sum_{j=1}^{k} \Phi(\Delta_{i_j}). \tag{1}$$

Fix an irreducible reduced component Γ of Δ and denote $\Phi(\Gamma)$ by γ. Then the inertia group of Γ is the cyclic group $H \subset G$ generated by γ. Choosing the generator of $H^* = \mathrm{Hom}(H, \mathbb{C}^*)$ to be the dual character γ^*, we may identify H^* with $\mathbb{Z}/n$, where n is the order of γ. Composing the restriction map $\mathrm{res}\colon G^* \to H^*$ with this identification gives

$$G^* \to \mathbb{Z}/n,\ \chi \mapsto k,$$

where $\chi|_H = (\gamma^*)^k$ for some $0 \leq k \leq n-1$. On the other hand, given χ in G^* of order d, the evaluation map $\chi\colon G \to \mathbb{Z}/d$ satisfies

$$\chi(\gamma) = \tfrac{d}{n}\chi|_H(\gamma) = \tfrac{dk}{n}$$

as a residue class in $\mathbb{Z}/d$ (or as an integer between 0 and $d-1$).
The pushforward of $\varphi_*\mathcal{O}_X$ breaks into a direct sum of eigensheaves

$$\varphi_*\mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}. \tag{2}$$

Moreover, the $\mathcal{L}_\chi$ are line bundles on Y and by Pardini [43], their associated (integral) divisors L_χ are given by the formula

$$dL_\chi = \sum_i \chi \circ \Phi(\Delta_i)\Delta_i. \tag{3}$$

The line bundles $\mathcal{L}_\chi$ play a pivotal role in the sequel, and we refer to them as the *character sheaves* of the cover $\varphi\colon X \to Y$.

2.1 Line bundles on X

We develop tools for calculating with torsion line bundles on X. Let $\pi'\colon A' \to X$ be the maximal abelian cover of X; that is, the étale cover of X associated to the subgroup $\pi_1(X)^{\mathrm{ab}} = H_1(X, \mathbb{Z})$ of $\pi_1(X)$. Now let ψ' be the composite map $\varphi \circ \pi'\colon A' \to Y$. It is not always true that ψ' is Galois and ramified over the same branch divisor Δ as $\varphi\colon X \to Y$ (see for example [45], [9]). So choose a maximal subgroup T of the torsion subgroup $\operatorname{Tors} X$ in $\operatorname{Pic} X$ whose associated cover $\psi\colon A \to Y$ is Galois and ramified over Δ. We have the following commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, "\varphi"] & & Y \end{tikzcd}$$

Let the Galois group of ψ be $\widetilde{G}$. Then the original group G is the quotient $\widetilde{G}/T$, so we get short exact sequences

$$0 \to T \to \widetilde{G} \to G \to 0 \tag{4}$$

and

$$0 \leftarrow T^* \leftarrow \widetilde{G}^* \leftarrow G^* \leftarrow 0 \tag{5}$$

where $G^* = \operatorname{Hom}(G, \mathbb{C}^*)$, etc. In fact, for each surface that we consider, these exact sequences are split, so that

$$\widetilde{G} = G \oplus T,\ \widetilde{G}^* = G^* \oplus T^*. \tag{6}$$

Let $\overline{\Gamma}$ be a reduced irreducible component of the branch locus Δ of an abelian cover $\varphi\colon X \to Y$ and suppose the inertia group of $\overline{\Gamma}$ is cyclic of order n. Then

Definition 2.1 (cf. [2]) *The* reduced pullback Γ *of* $\overline{\Gamma}$ *is the (integral) divisor* $\Gamma = \frac{1}{n}\varphi^*(\overline{\Gamma})$ *on* X.

Remark 2.1 The reduced pullback extends to arbitrary linear combinations $\sum_i k_i\Delta_i$ in the obvious way. We use a bar to denote divisors on Y and remove the bar when taking the reduced pullback. In other situations, it is convenient to use D_i to denote the reduced pullback of a branch divisor Δ_i.

The remainder of this section is dedicated to calculating the pushforward $\varphi_*(L\otimes\tau)$, where $L = \mathcal{O}_X(\sum_i k_i D_i)$ is the line bundle associated to the reduced pullback of $\sum_i k_i\Delta_i$, and τ is any torsion line bundle contained in $T \subset \operatorname{Tors} X$. We do this by exploiting the association of the free part L with $\varphi\colon X \to Y$, and the torsion part τ with $\pi\colon A \to X$. The formulae that we obtain are a natural extension of results in [43]. It may be helpful to skip ahead to Examples 2.2.1 and 2.4.1 before reading this section in detail.

2.2 Free case

Until further notice, we write $\overline{\Gamma} \subset Y$ for an irreducible component of the branch divisor Δ of $\varphi\colon X \to Y$. By Pardini [43], the inertia group $H \subset G$ of $\overline{\Gamma}$ is cyclic, and H is generated by $\Phi(\overline{\Gamma})$ of order n. Let $\Gamma \subset X$ be the reduced pullback of $\overline{\Gamma}$, so that $n\Gamma = \varphi^*(\overline{\Gamma})$. We start with cyclic covers.

Lemma 2.1 *Let $\alpha\colon X \to Y$ be a cyclic cover with group $H \cong \mathbb{Z}/n$, and suppose that $\overline{\Gamma}$ is an irreducible reduced component of the branch divisor. Let Γ be the reduced pullback of $\overline{\Gamma}$, and suppose $0 \le k \le n-1$. Then*

$$\alpha_*\mathcal{O}_X(k\Gamma) = \bigoplus_{i\in H^*-S} \mathcal{M}_i^{-1} \oplus \bigoplus_{i\in S} \mathcal{M}_i^{-1}(\overline{\Gamma}),$$

where $\mathcal{M}_i$ is the character sheaf associated to α with character $i \in H^*$, and

$$S = \{n-k, \dots, n-1\} \subset H^* \cong \mathbb{Z}/n.$$

Remark 2.2 If k is a multiple of n, say $k = pn$, the projection formula gives

$$\alpha_*\mathcal{O}_X(k\Gamma) = \alpha_*(\alpha^*\mathcal{O}_Y(p\overline{\Gamma})) = \alpha_*\mathcal{O}_X \otimes \mathcal{O}_Y(p\overline{\Gamma}) = \bigoplus_{i\in H^*} \mathcal{M}_i^{-1}(p\overline{\Gamma}).$$

Thus the lemma extends to any integer multiple of Γ.

Proof After removing a finite number of points from $\overline{\Gamma}$, we may choose a neighbourhood U of $\overline{\Gamma}$ such that U does not intersect any other irreducible components of Δ. Then since X and Y are normal we may calculate $\alpha_*\mathcal{O}_X(k\Gamma)$ locally on $\alpha^{-1}(U)$ and U. In what follows, we do not distinguish U (respectively $\alpha^{-1}(U)$) from Y (resp. X).
Let $g = \Phi(\overline{\Gamma})$ so that $H = \langle g\rangle \cong \mathbb{Z}/n$, and identify H^* with $\mathbb{Z}/n$ via $g^* = 1$. Locally, write $\alpha\colon \alpha^{-1}(U) \to U$ as $z^n = b$ where $b = 0$ defines $\overline{\Gamma}$ in U. Then

$$\alpha_*\mathcal{O}_X = \bigoplus_{i=0}^{n-1} \mathcal{O}_Y z^i = \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma}) = \bigoplus_{i=0}^{n-1} \mathcal{M}_i^{-1},$$

where the last equality is given by (3). Thus $\alpha_*\mathcal{O}_X$ is generated by $1, z, \dots, z^{n-1}$ as an $\mathcal{O}_Y$-module, and the $\mathcal{O}_Y$-algebra structure on $\alpha_*\mathcal{O}_X$ is induced by the equation $z^n = b$.

The calculation for $\mathcal{O}_X(k\Gamma)$ is similar,

$$\alpha_*\mathcal{O}_X(k\Gamma) = \alpha_*\mathcal{O}_X\frac{1}{z^k} = \bigoplus_{i=-k}^{n-k-1}\mathcal{O}_Y z^i = \bigoplus_{i=0}^{n-k-1}\mathcal{O}_Y z^i \oplus \bigoplus_{i=-k}^{-1}\mathcal{O}_Y\frac{z^{n+i}}{b}$$

where we use $z^n = b$ to remove negative powers of z. Thus

$$\begin{aligned}\alpha_*\mathcal{O}_X(k\Gamma) &= \bigoplus_{i=0}^{n-k-1}\mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma}) \oplus \bigoplus_{i=n-k}^{n-1}\mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma})(\overline{\Gamma})\\ &= \bigoplus_{i\in H^*-S}\mathcal{M}_i^{-1} \oplus \bigoplus_{i\in S}\mathcal{M}_i^{-1}(\overline{\Gamma}),\end{aligned}$$

where $S = \{n-k,\dots,n-1\}$. □

The lemma can be extended to any abelian group using arguments inspired by Pardini [43] Sections 2 and 4.

Proposition 2.1 *Let $\varphi\colon X \to Y$ be an abelian cover with group G, and let $k = np + \overline{k}$, where $0 \le \overline{k} \le n-1$. Then*

$$\varphi_*\mathcal{O}_X(k\Gamma) = \bigoplus_{\chi\in G^*-S_{k\overline{\Gamma}}}\mathcal{L}_\chi^{-1}(p\overline{\Gamma}) \oplus \bigoplus_{\chi\in S_{k\overline{\Gamma}}}\mathcal{L}_\chi^{-1}((p+1)\overline{\Gamma}),$$

where

$$S_{k\overline{\Gamma}} = \{\chi \in G^* : n-\overline{k} \le \chi|_H \le n-1\}.$$

Proof By the projection formula, we only need to consider the case $k = \overline{k}$ (cf. Remark 2.2). As in the proof of Lemma 2.1, after removing a finite number of points, we may take a neighbourhood U of $\overline{\Gamma}$ which does not intersect any other components of Δ. We work on U and its preimages $\varphi^{-1}(U)$, $\beta^{-1}(U)$. Factor $\varphi\colon X \to Y$ as

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y,$$

where α is a cyclic cover ramified over Γ with group $H = \langle g\rangle \cong \mathbb{Z}/n$, and β is unramified by our assumptions. As in Lemma 2.1 we denote the character sheaves of α by $\mathcal{M}_i$, and those of the composite map $\varphi = \beta\circ\alpha$ by $\mathcal{L}_\chi$. Now

$$\beta_*\mathcal{M}_i = \bigoplus_{\chi\in[i]}\mathcal{L}_\chi \tag{7}$$

where the notation $[i]$ means the preimage of i in H^* under the restriction map res: $G^* \to H^*$. That is,

$$[i] = \{\chi \in G^* : \chi|_H = i\},$$

where d is the order of χ. Since β is not ramified we combine Lemma 2.1 and (7) to get

$$\varphi_*\mathcal{O}_X(k\Gamma) = \bigoplus_{\chi\in G^*-S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1} \oplus \bigoplus_{\chi\in S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1}(\overline{\Gamma})$$

where

$$S_{k\overline{\Gamma}} = \{\chi \in G^* : n-k \leq \chi|_H \leq n-1\}$$

is the preimage of $S = \{n-k,\dots,n-1\} \subset H^*$ under res: $G^* \to H^*$. □

2.2.1 Example (Campedelli surface)

Let $\varphi\colon X \to \mathbb{P}^2$ be a $G = (\mathbb{Z}/2)^3$-cover branched over seven lines in general position. We label the lines $\Delta_1,\dots,\Delta_7$, and define Φ to induce a set-theoretic bijection between $\{\Delta_i\}$ and $(\mathbb{Z}/2)^3 - \{0\}$. We make the definition of Φ more precise later (see Example 2.4.1). It is well known ([34, §4]) that X is a surface of general type with $p_g = 0$, $K^2 = 2$ and $\pi_1 = (\mathbb{Z}/2)^3$.
Choose generators g_1, g_2, g_3 for $(\mathbb{Z}/2)^3$ so that $\Phi(\Delta_1) = g_1$. There are eight character sheaves for the cover, which we calculate using formula (3),

$$\mathcal{L}_{(0,0,0)} = \mathcal{O}_{\mathbb{P}^2},\ \mathcal{L}_\chi = \mathcal{O}_{\mathbb{P}^2}(2) \text{ for } \chi \neq (0,0,0).$$

Write D_1 for the reduced pullback of Δ_1, so that $\varphi^*(\Delta_1) = 2D_1$. Then

$$S_{\Delta_1} = \{\chi : \chi|_{\langle g_1\rangle} = 1\} = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\},$$

so that by Proposition 2.1, we have

$$\varphi_*\mathcal{O}_X(D_1) = \mathcal{O}_{\mathbb{P}^2} \oplus 4\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^2}(-2).$$

2.3 Torsion case

In this section we use the maximal abelian cover A to calculate the pushforward of a torsion line bundle on X. To simplify notation, we assume that the composite cover $A \to X \to Y$ is Galois with group $\widetilde{G}$, so that $T = \operatorname{Tors} X$.

Proposition 2.2 *Let τ be a torsion line bundle on X. Then*

$$\varphi_*\mathcal{O}_X(-\tau) = \bigoplus_{\chi\in G^*} \mathcal{L}_{\chi+\tau}^{-1}.$$

where addition $\chi + \tau$ takes place in $\widetilde{G}^* = G^* \oplus T^*$.

Remark 2.3 Note that $\mathcal{L}_{\chi+\tau}$ is a character sheaf for the $\widetilde{G}$-cover $\varphi\colon A \to Y$, and the proposition allows us to interpret $\mathcal{L}_{\chi+\tau}$ as a character sheaf for the G-cover $\varphi\colon X \to Y$. Unfortunately, there is still some ambiguity, because we do not determine which character in G^* is associated to each $\mathcal{L}_{\chi+\tau}$ under the splitting of exact sequence (5). On the other hand, the special case $\tau = 0$ gives

$$\varphi_*\mathcal{O}_X = \bigoplus_{\chi\in G^*} \mathcal{L}_\chi^{-1}.$$

Proof The structure sheaf $\mathcal{O}_A$ decomposes into a direct sum of the torsion line bundles when pushed forward to X

$$\pi_*\mathcal{O}_A = \bigoplus_{\tau \in \operatorname{Tors} X} \mathcal{O}_X(-\tau).$$

Thus $\mathcal{O}_X(\tau)$ is the character sheaf with character τ under the identification $T^* \cong \operatorname{Tors} X$. The composite $\varphi_*\pi_*\mathcal{O}_A$ breaks into character sheaves according to (2), and the image of $\mathcal{O}_X(-\tau)$ is the direct sum of those character sheaves with character contained in the coset $G^* + \tau$ of τ in $\widetilde{G}^*$ under (6). □

2.4 General case

Now we combine Propositions 2.1 and 2.2 to give our formula for pushforward of line bundles $\mathcal{O}_X(\sum_i D_i) \otimes \tau$. The formula looks complicated, but most of the difficulty is in the notation.

Definition 2.2 Let n_i be the order of $\Psi(\Delta_i)$ in $\widetilde{G}$, and write $k_i = n_i p_i + \overline{k}_i$, where $0 \le \overline{k}_i \le n_i - 1$. Then given any subset $I \subset \{1, \dots, m\}$, we define

$$S_I[\tau] = \bigcap_{i \in I} S_{k_i\Delta_i}[\tau] \cap \bigcap_{j \in I^c} S_{k_j\Delta_j}[\tau]^c,$$

where

$$S_{k\overline{\Gamma}}[\tau] = \{\chi \in G^* : n - \overline{k} \le \tfrac{n}{d}(\chi + \tau)(\Psi(\overline{\Gamma})) \le n - 1\}$$

for any reduced irreducible component $\overline{\Gamma}$ of the branch locus Δ. Note that for fixed τ in T^*, the collection of all $S_I[\tau]$ partitions G^*.

Theorem 2.1 *Let $D = \sum_{i=1}^m k_i D_i$ be the reduced pullback of the linear combination of branch divisors $\sum_{i=1}^m k_i \Delta_i$ on Y. Then*

$$\varphi_*\mathcal{O}_X(D - \tau) = \bigoplus_I \bigoplus_{\chi \in S_I[\tau]} \mathcal{L}^{-1}_{\chi+\tau}(\Delta_I),$$

where I is any subset of $\{1, \dots, m\}$ and $\Delta_I = \sum_{i \in I} \Delta_i$.

Remark 2.4 For simplicity, we have assumed that $k_i = \overline{k}_i$ for all i in the statement and proof of the theorem. When this is not the case, by the projection formula (cf. Remark 2.2) we twist by $\mathcal{O}_Y(\sum_{i=1}^m p_i\Delta_i)$.

Proof Fix i and let D_i be the reduced pullback of an irreducible component Δ_i of the branch divisor. Choose a neighbourhood of Δ_i which does not intersect any other Δ_j. This may also require us to remove a finite number of points from D_i. We work locally in this neighbourhood and its preimages under φ, π. Now by the projection formula,

$$\pi_*\pi^*\mathcal{O}_X(k_i D_i) = \pi_*\mathcal{O}_A \otimes \mathcal{O}_X(k_i D_i),$$

and thus

$$\psi_*\pi^*\mathcal{O}_X(k_iD_i) = \bigoplus_{\tau\in T}\varphi_*\mathcal{O}_X(k_iD_i - \tau).$$

Then we combine Propositions 2.1 and 2.2 to obtain

$$\varphi_*\mathcal{O}_X(k_iD_i - \tau) = \bigoplus_{\chi\in G^* - S_{k_i\Delta_i}[\tau]} \mathcal{L}^{-1}_{\chi+\tau} \oplus \bigoplus_{\chi\in S_{k_i\Delta_i}[\tau]} \mathcal{L}^{-1}_{\chi+\tau}(\Delta_i),$$

where the indexing is explained in Definition 2.2.
To extend to the global setting and linear combinations $\sum k_iD_i$, we just need to keep track of which components of Δ should appear as a twist of each $\mathcal{L}^{-1}_{\chi+\tau}$ in the direct sum. This book-keeping is precisely the purpose of Definition 2.2. □

Using the formula

$$K_X = \varphi^*\big(K_Y + \sum_i \tfrac{n_i-1}{n_i}\Delta_i\big) \tag{8}$$

and the Theorem, we give an alternative proof of the decomposition of $\varphi_*\mathcal{O}_X(K_X)$.

Corollary 2.1 *[43, Proposition 4.1] We have*

$$\varphi_*\mathcal{O}_X(K_X) = \bigoplus_{\chi\in G^*}\mathcal{L}_{\chi^{-1}}(K_Y).$$

Proof Let D_i be the reduced pullback of Δ_i. Then by (8) and the projection formula, we have

$$\begin{aligned}\varphi_*(\mathcal{O}_X(K_X)) &= \varphi_*\Big(\varphi^*\mathcal{O}_Y(K_Y)\otimes\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big)\Big)\\ &= \mathcal{O}_Y(K_Y)\otimes\varphi_*\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big).\end{aligned}$$

Now by definition,

$$S_{(n_i-1)\Delta_i} = \{\chi\in G^* : 1 \le \tfrac{n_i}{d}\chi(\Phi(\Delta_i)) \le n_i - 1\} = \{\chi\in G^* : \chi(\Phi(\Delta_i)) \ne 0\}.$$

Thus in the decomposition of $\varphi_*\mathcal{O}_X\big(\sum_i(n_i-1)D_i\big)$ given by Theorem 2.1, the summand $\mathcal{L}_\chi^{-1}$ is twisted by $\sum_{j\in J}\Delta_j$, where J is the set of indices j with $\chi(\Phi(\Delta_j)) \ne 0$. Then by (3),

$$\mathcal{L}_\chi^{-1}\Big(\sum_{i\in J}\Delta_i\Big) = \sum_i(1-\tfrac{1}{d})\chi(\Phi(\Delta_i))\Delta_i = \mathcal{L}_{\chi^{-1}},$$

where the last equality is because $\chi^{-1}(g) = -\chi(g) = d - \chi(g)$ for any g in G. Thus we obtain

$$\varphi_*\Big(\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big)\Big) = \bigoplus_{\chi\in G^*}\mathcal{L}_{\chi^{-1}},$$

and the Corollary follows. □

2.4.1 Example 2.2.1 continued

We resume our discussion of the Campedelli surface. The fundamental group of X is $(\mathbb{Z}/2)^3$, and so the maximal abelian cover $\pi\colon A \to X$ is a $(\mathbb{Z}/2)^6$-cover $\psi\colon A \to \mathbb{P}^2$ branched over Δ. Choose generators $g_1, \ldots, g_6$ of $(\mathbb{Z}/2)^6$. As promised in Example 2.2.1, we now fix Φ and Ψ:

Δ_i	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
$\Phi(\Delta_i)$	g_1	g_2	g_3	g_1+g_2	g_1+g_3	g_2+g_3	$g_1+g_2+g_3$
$\Psi(\Delta_i)-\Phi(\Delta_i)$	0	0	0	g_4	g_5	g_6	$g_4+g_5+g_6$

For clarity, the table displays the difference between $\Psi(\Delta_i)$ and $\Phi(\Delta_i)$. In order that A be the maximal abelian cover, Ψ is defined so that each $\Psi(\Delta_i)$ generates a distinct summand of $(\mathbb{Z}/2)^6$, excepting $\Psi(\Delta_7)$, which is chosen so that $\sum_i \Psi(\Delta_i) = 0$. This last equality is induced by the relation $\sum_i \Delta_i = 0$ in $H_1(\mathbb{P}^2 - \Delta, \mathbb{Z})$.

The torsion group $\operatorname{Tors} X$ is generated by g_4^*, g_5^*, g_6^*. As an illustration of Theorem 2.1, we calculate $\varphi_*\mathcal{O}_X(D_1)\otimes\tau$, where τ is the torsion line bundle on X associated to g_4^*. Suppose $\varphi_*\mathcal{O}_X(D_1)\otimes\tau = \bigoplus_{\chi\in G^*}\mathcal{M}_\chi$, where $\mathcal{M}_\chi$ are the line bundles to be calculated. In the table below, we collect the data relevant to Theorem 2.1.

χ	$\mathcal{L}^{-1}_{\chi+\tau}$	$(\chi+\tau)\circ\Psi(D_1)$	Twist by Δ_1?	$\mathcal{M}_\chi$
$(0,0,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(1,0,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}$
$(0,1,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(0,0,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,1,0)$	$\mathcal{O}_{\mathbb{P}^2}(-3)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,0,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(0,1,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,1,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-1)$

Summing the last column of the table, we get

$$\varphi_*\mathcal{O}_X(D_1)\otimes\tau = \mathcal{O}_{\mathbb{P}^2} \oplus 4\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^2}(-2).$$

In particular, we see that the linear system on X associated to the line bundle $\mathcal{O}_X(D_1)\otimes\tau$ contains a single effective divisor.

3 Exceptional collections of line bundles on surfaces

3.1 Overview and definitions

We outline our method for producing exceptional collections, starting with some definitions and fundamental observations. A good reference for semi-orthogonal decompositions is [37], and Proposition 3.1 is proved in [26].

Definition 3.1 *An object E in $D^b(X)$ is called* exceptional *if*

$$\mathrm{Ext}^k(E,E) = \begin{cases} \mathbb{C} & \text{if } k=0, \\ 0 & \text{otherwise.} \end{cases}$$

An exceptional collection $\mathbb{E} \subset D^b(X)$ *is a sequence of exceptional objects* $\mathbb{E} = (E_0, \dots, E_n)$ *such that if* $0 \le i < j \le n$ *then*

$$\mathrm{Ext}^k(E_j, E_i) = 0 \text{ for all } k.$$

Remark 3.1 *Some authors prefer the term exceptional sequence rather than exceptional collection.*

It follows from Definition 3.1 that a line bundle on a surface is exceptional if and only if $p_g = q = 0$. Moreover, if $\mathbb{E}$ is an exceptional collection of line bundles, and L is any line bundle, then $\mathbb{E} \otimes L = (E_0 \otimes L, \dots, E_n \otimes L)$ is again an exceptional collection, so we always normalise $\mathbb{E}$ so that $E_0 = \mathcal{O}_X$.
Let $\mathcal{E} = \langle \mathbb{E} \rangle$ denote the smallest full triangulated subcategory of $D^b(X)$ containing all objects in $\mathbb{E}$. Then $\mathcal{E}$ is an admissible subcategory of $D^b(X)$, and so we have a *semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle,$$

where $\mathcal{A}$ is the left orthogonal to $\mathcal{E}$. That is, $\mathcal{A}$ consists of all objects F in $D^b(X)$ such that $\mathrm{Ext}^k(F,E) = 0$ for all k and for all E in $\mathcal{E}$. We say that the exceptional collection $\mathbb{E}$ is *full* if $D^b(X) = \mathcal{E}$. The K-theory is additive across semiorthogonal decompositions:

Proposition 3.1 *If $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then*

$$K_0(X) = K_0(\mathcal{A}) \oplus K_0(\mathcal{B}).$$

Moreover, if $\mathbb{E}$ is an exceptional collection of length n, then $K_0(\mathcal{E}) = \mathbb{Z}^n$. Thus if $K_0(X)$ is not free, then X can never have a full exceptional collection. The maximal length of an exceptional collection on X is less than or equal to the rank of $K(X)$.

3.1.1 Exceptional collections on del Pezzo surfaces

Let Y be the blow up of $\mathbb{P}^2$ in n points, and write H for the pullback of the hyperplane section, $\overline{E}_i$ for the ith exceptional curve. Then by work of Kuleshov and Orlov [42], [33] there is an exceptional collection of sheaves on Y

$$\mathcal{O}_{\overline{E}_1}(-1), \dots, \mathcal{O}_{\overline{E}_n}(-1), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H).$$

Note that the blown up points do not need to be in general position, and can even be infinitely near. We prefer an exceptional collection of line bundles on Y, so we mutate past $\mathcal{O}_Y$ to get

$$\mathcal{O}_Y,\ \mathcal{O}_Y(\overline{E}_1), \dots, \mathcal{O}_Y(\overline{E}_n),\ \mathcal{O}_Y(H),\ \mathcal{O}_Y(2H). \tag{9}$$

In fact, we only use the numerical properties of a given exceptional collection of line bundles on Y. Choose a basis $e_0, \dots, e_n$ for the lattice $\operatorname{Pic} Y \cong \mathbb{Z}^{1,n}$ with intersection form $\operatorname{diag}(1, -1^n)$. Then we write equation (9) numerically as

$$0,\ e_1, \dots, e_n,\ e_0,\ 2e_0.$$

3.1.2 From del Pezzo to general type

Let X be a surface of general type with $p_g = 0$ which admits an abelian cover $\varphi\colon X \to Y$ of a del Pezzo surface Y with $K_Y^2 = K_X^2$. In addition, we suppose that the maximal abelian cover $A \to X \to Y$ is also Galois. Otherwise choose a maximal subgroup $T \subset \operatorname{Tors} X$ for which the associated cover is Galois, and replace A, as in Section 2. The branch divisor is $\Delta = \sum_i \Delta_i$ and we assume that Δ is sufficiently reducible so that

(A1) $\operatorname{Pic} Y$ is generated by integral linear combinations of Δ_i.

Now the Picard lattices of X and Y are isomorphic. Thus if G is not too complicated, e.g. of the form $\mathbb{Z}/p \times \mathbb{Z}/q$, we might hope to have:

(A2) The reduced pullbacks D_i of Δ_i (see Definition 2.1) generate $\operatorname{Pic} X / \operatorname{Tors} X$.

In very good cases, reduced pullback actually induces an isometry of lattices

(A3) $f\colon \operatorname{Pic} Y \to \operatorname{Pic} X / \operatorname{Tors} X$, such that $f(K_Y) = -K_X$ modulo $\operatorname{Tors} X$.

We say that a surface satisfies assumption (A) if (A1), (A2) and (A3) hold. These conditions are quite strong, and are not strictly necessary for our methods. For example, we could replace (A3) with an isometry of lattices from the abstract lattice $\mathbb{Z}^{1,n}$ to $\operatorname{Pic} X / \operatorname{Tors} X$.

Definition 3.2 *A sequence $\mathbb{E} = (E_0, \dots, E_n)$ of line bundles on X is called* numerically exceptional *if $\chi(E_j, E_i) = 0$ whenever $0 \le i < j \le n$.*

Assume X satisfies (A), and let $(\Lambda_i) = (\Lambda_0, \dots, \Lambda_n)$ be an exceptional collection on Y. Now define $(L_i) = (L_0, \dots, L_n)$ by $L_i = f(\Lambda_i)^{-1}$. A calculation with the Riemann–Roch formula shows that (L_i) is a numerically exceptional collection on X. This is explained in [2].

Given a numerically exceptional collection (L_i) of line bundles on X, the remaining obstacle is to determine whether (L_i) is genuinely exceptional rather than just numerically so. Indeed, most numerically exceptional collections on X are not exceptional. The standard trick (see [13]) is to choose torsion line bundles τ_i in such a way that the twisted sequence $(L_i \otimes \tau_i)$ is an exceptional collection. We examine these choices of τ_i more carefully in what follows.

3.1.3 Acyclic line bundles

We discuss acyclic line bundles following [24].

Definition 3.3 *Let L be a line bundle on X. If $H^i(X,L)=0$ for all i, then we call L an* acyclic *line bundle. We define the acyclic set associated to L to be*

$$\mathcal{A}(L)=\{\tau\in\operatorname{Tors}X : L\otimes\tau \text{ is acyclic}\}.$$

We call L numerically acyclic *if $\chi(X,L)=0$. Clearly, an acyclic line bundle must be numerically acyclic.*

Remark 3.2 In the notation of [24], $\tau=-\chi$.

Lemma 3.1 ([24], Lemma 3.4) *A numerically exceptional collection $L_0=\mathcal{O}_X, L_1\otimes\tau_1,\dots,L_n\otimes\tau_n$ on X is exceptional if and only if*

$$\begin{aligned}-\tau_i&\in\mathcal{A}(L_i^{-1}) \text{ for all } i, \text{ and}\\ \tau_i-\tau_j&\in\mathcal{A}(L_j^{-1}\otimes L_i) \text{ for all } j>i.\end{aligned}\tag{10}$$

Thus to enumerate all exceptional collections on X of a particular numerical type, it suffices to calculate the relevant acyclic sets, and systematically test the above conditions (10) on all possible combinations of τ_i.

3.1.4 Calculating cohomology of line bundles

Given a torsion twist $L\otimes\tau$, Theorem 2.1 gives a decomposition

$$\varphi_*(L\otimes\tau)=\bigoplus_{\chi\in G^*}\mathcal{M}_\chi,$$

for some line bundles $\mathcal{M}_\chi$ on Y, which may be computed explicitly. Since φ is finite, we have

$$h^p(L\otimes\tau)=\sum_{\chi\in G^*}h^p(\mathcal{M}_\chi)$$

for all p.

Thus $L\otimes\tau$ is acyclic if and only if each summand $\mathcal{M}_\chi$ is acyclic on Y. Now if $\chi(Y,\mathcal{M}_\chi)=0$ and $h^0(\mathcal{M}_\chi)=h^2(\mathcal{M}_\chi)=0$, we see that $h^1(\mathcal{M}_\chi)=0$. Thus by Serre duality and the Riemann–Roch theorem, we are reduced to calculating Euler characteristics and determining effectivity for (lots of) divisor classes on the del Pezzo surface Y.

3.1.5 Coordinates on $\operatorname{Pic}X/\operatorname{Tors}X$

Under assumption (A), we make the following definition.

DEFINITION 3.4 *Choose a basis $B_1, \dots, B_n$ for* $\operatorname{Pic} X/\operatorname{Tors} X$ *consisting of linear combinations of reduced pullbacks. Then any line bundle L on X may be written uniquely as*

$$L = \mathcal{O}_X(d_1, \dots, d_n) \otimes \tau$$

so that $L = \mathcal{O}_X\big(\sum_{i=1}^n d_i B_i\big) \otimes \tau$. We call d (respectively τ) the multidegree (resp. torsion twist) of L with respect to the chosen basis.

The torsion twist associated to any line bundle on X may be calculated using Theorem 2.1 and the following immediate lemma. See Lemma 3.4 for an example.

LEMMA 3.2 *If τ is a torsion line bundle, then $h^0(\tau) \neq 0$ implies $\tau = 0$.*

REMARK 3.3 Definition 3.4 fixes a basis for $\operatorname{Pic} Y = \mathbb{Z}^{1,9-K^2}$ via the isometry with $\operatorname{Pic} X/\operatorname{Tors} X$. This basis corresponds to a geometric marking on the del Pezzo surface Y, and the multidegree d of L is just the image of L in $\operatorname{Pic} Y$ under the isometry. In fixing our basis, we break some of the symmetry of the coordinates. This is necessary in order to use the computer to search for exceptional collections. We can recover the symmetry later using the Weyl group action (see Section 3.1.7).

3.1.6 Determining effectivity of divisor classes

For each fake del Pezzo surface, we have the following theorem.

THEOREM 3.1 *Suppose X is a fake del Pezzo surface satisfying assumption (A) and with $T = \operatorname{Tors} X$. Let $\mathfrak{E}$ denote the semigroup generated by the reduced pullbacks D_i of the irreducible branch components Δ_i, and pullbacks of the other (-1)- and (-2)-curves on Y. Then $\mathfrak{E}$ is the semigroup of all effective divisors on X.*

We prove this theorem for the secondary nodal Burniat surface with $K^2 = 4$ in Section 5 (cf. [1] for the Burniat surface with $K^2 = 6$). The other fake del Pezzo surfaces work in the same way, see [20].
Moreover, $\mathfrak{E}$ is graded by multidegree, and we define a homomorphism

$$t \colon \mathfrak{E} \to \operatorname{Tors} X$$

sending D_i to its torsion twist under Definition 3.4. The image under t of the graded summand $\mathfrak{E}_d$ of multidegree d is the set of torsion twists τ for which $\mathcal{O}_X(\sum d_i B_i) \otimes \tau$ is effective.

3.1.7 Group actions on the set of exceptional collections

We consider a dihedral group action and the Weyl group action on the set of exceptional collections on X. Mutations are not considered systematically in this article, since a mutation of a line bundle need not be a line bundle.

Let $\mathbb{E} = (E_1, \dots, E_n)$ be an exceptional collection of line bundles on X. If we normalise the first line bundle of any exceptional collection to be $\mathcal{O}_X$, then there is an obvious dihedral group action on the set of exceptional collections of length n on X, generated by $\mathbb{E} \mapsto (E_2, \dots, E_n, E_1(-K_X))$ and $\mathbb{E} \mapsto \mathbb{E}^{-1} = (E_n^{-1}, \dots, E_1^{-1})$.
The Weyl group of $\operatorname{Pic} Y$ is generated by reflections in (-2)-classes. That is, suppose α is a class in $\operatorname{Pic} Y$ with $K_Y \cdot \alpha = 0$ and $\alpha^2 = -2$. Then

$$r_\alpha \colon L \mapsto L + (L \cdot \alpha)\alpha$$

is a reflection on $\operatorname{Pic} Y$ which fixes K_Y. Any reflection sends an exceptional collection on Y to another exceptional collection. Thus the Weyl group action on numerical exceptional collections on Y induces an action on numerical exceptional collections on X under assumption (A). This action accounts for the choices made in giving Y a geometric marking (see Definition 3.4).

3.2 The Kulikov surface with $K^2 = 6$

For details on the Kulikov surface (first described in [34]), its torsion group and moduli space, see [19]. The Kulikov surface X is a $(\mathbb{Z}/3)^2$-cover of the del Pezzo surface Y of degree 6. Figure 1 shows the associated cover of $\mathbb{P}^2$ branched over six lines in special position. The configuration has just one free parameter, and in fact, the Kulikov surfaces form a 1-dimensional, irreducible, connected component of the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 6$.

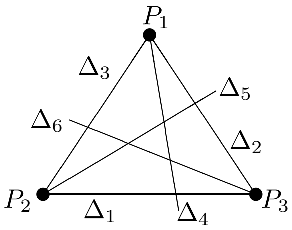

Figure 1: The Kulikov configuration

To obtain a nonsingular cover, we blow up the plane at three points P_1, P_2, P_3, giving a $(\mathbb{Z}/3)^2$-cover of a del Pezzo surface of degree 6. The exceptional curves are denoted $\overline{E}_i$. By results of [19], the torsion group $\operatorname{Tors} X$ is isomorphic to $(\mathbb{Z}/3)^3$, so the maximal abelian cover $\psi \colon A \to Y$ has group $\widetilde{G} \cong (\mathbb{Z}/3)^5$. Let g_i generate $\widetilde{G}$, and write g_i^* for the dual generators of $\widetilde{G}^*$. As explained in Section 2, the covers are determined by $\Phi \colon H_1(\mathbb{P}^2 - \Delta, \mathbb{Z}) \to G$ and $\Psi \colon H_1(\mathbb{P}^2 - \Delta, \mathbb{Z}) \to \widetilde{G}$, which are defined in the table below.

D	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
$\Phi(D)$	g_1	g_1	g_1	g_2	$g_1 + g_2$	$2g_1 + g_2$
$\Psi(D) - \Phi(D)$	0	g_3	$2g_3 + g_4$	$2g_4$	g_5	$2g_5$

The images of the exceptional curves $\overline{E}_i$ under Φ and Ψ are computed using formula (1):

$$\Phi(\overline{E}_1) = 2g_1 + g_2,\ \Phi(\overline{E}_2) = g_2,\ \Phi(\overline{E}_3) = g_1 + g_2,\ \text{etc.}$$

LEMMA 3.3 *The Kulikov surface satisfies assumptions (A1) and (A2). That is, the free part of* $\operatorname{Pic} X$ *is generated by the reduced pullbacks of* $\Delta_1 + \overline{E}_2 + \overline{E}_3$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, *and the intersection pairing* $\operatorname{diag}(1, -1, -1, -1)$ *is inherited from* Y.

PROOF Define $e_0 = D_1 + E_2 + E_3$, $e_1 = E_1$, $e_2 = E_2$, $e_3 = E_3$ in $\operatorname{Pic} X$. These are integral divisors, since they are reduced pullbacks, and the intersection pairing is $\operatorname{diag}(1, -1, -1, -1)$, which is unimodular. For example, by definition of reduced pullback, $3e_0 = \varphi^*(\Delta_1 + \overline{E}_2 + \overline{E}_3)$, and so

$$(3e_0)^2 = \varphi^*(\Delta_1 + \overline{E}_2 + \overline{E}_3)^2 = 9 \cdot 1,$$

or $e_0^2 = 1$. Hence we have an isomorphism of lattices. □

Using the basis chosen in this lemma, we compute the coordinates (Definition 3.4) of the reduced pullback D_i of each irreducible branch component Δ_i.

LEMMA 3.4 *We have*

$$\begin{aligned} \mathcal{O}_X(D_1) &= \mathcal{O}_X(1, 0, -1, -1), & \mathcal{O}_X(D_4) &= \mathcal{O}_X(1, -1, 0, 0)[2, 1, 2], \\ \mathcal{O}_X(D_2) &= \mathcal{O}_X(1, -1, 0, -1)[1, 0, 2], & \mathcal{O}_X(D_5) &= \mathcal{O}_X(1, 0, -1, 0)[2, 1, 0], \\ \mathcal{O}_X(D_3) &= \mathcal{O}_X(1, -1, -1, 0)[2, 0, 2], & \mathcal{O}_X(D_6) &= \mathcal{O}_X(1, 0, 0, -1)[2, 1, 1], \end{aligned}$$

where $[a, b, c]$ *in* $(\mathbb{Z}/3)^3$ *denotes a torsion line bundle on* X.

PROOF We prove that $\mathcal{O}_X(D_2) = \mathcal{O}_X(1, -1, 0, -1)[1, 0, 2]$. The other cases are similar. It is clear that $\Delta_2 \sim \Delta_1 - \overline{E}_1 + \overline{E}_2$ on Y, so the multidegree is correct. It remains to check the torsion twist, by showing that $\mathcal{F} = \mathcal{O}_X(D_2 - D_1 + E_1 - E_2 - \tau)$ has a global section when $\tau = [1, 0, 2]$. Then by Lemma 3.2, we have the desired equality.

The pushforward $\varphi_*\mathcal{F}$ splits into a direct sum of line bundles $\bigoplus \mathcal{M}_\chi$, one for each character $\chi = (a, b)$ in G^*. The following table collects the data required to calculate each $\mathcal{M}_\chi$ via Theorem 2.1. The second column is calculated using equation (3), and the next four columns evaluate $\chi + \tau$ on each $\Psi(\Gamma)$, where Γ

is any one of Δ_1, Δ_2, $\overline{E}_1$ and $\overline{E}_2$. The final column is explained below.

χ	$\mathcal{L}^{-1}_{\chi+\tau}$	$(\chi+\tau)\circ\Psi(\Gamma)$ Δ_1	Δ_2	$\overline{E}_1$	$\overline{E}_2$	$\mathcal{M}_\chi$
$(0,0)$	$\mathcal{O}_Y(-2,1,1,0)$	0	1	0	1	$\mathcal{O}_Y(-3,1,2,1)$
$(1,0)$	$\mathcal{O}_Y(-1,0,0,1)$	1	2	2	1	$\mathcal{O}_Y$
$(0,1)$	$\mathcal{O}_Y(-2,1,0,1)$	0	1	1	2	$\mathcal{O}_Y(-3,1,1,2)$
$(2,0)$	$\mathcal{O}_Y(-2,0,1,1)$	2	0	1	1	$\mathcal{O}_Y(-2,0,1,1)$
$(1,1)$	$\mathcal{O}_Y(-2,1,0,1)$	1	2	0	2	$\mathcal{O}_Y(-1,0,0,0)$
$(0,2)$	$\mathcal{O}_Y(-2,1,1,0)$	0	1	2	0	$\mathcal{O}_Y(-3,2,1,1)$
$(2,1)$	$\mathcal{O}_Y(-2,0,1,0)$	2	0	2	2	$\mathcal{O}_Y(-2,1,1,0)$
$(1,2)$	$\mathcal{O}_Y(-3,1,1,1)$	1	2	1	0	$\mathcal{O}_Y(-2,0,0,0)$
$(2,2)$	$\mathcal{O}_Y(-2,1,1,1)$	2	0	0	0	$\mathcal{O}_Y(-2,1,0,1)$

Now by the projection formula (cf. Remark 2.2),

$$\varphi_*\mathcal{F} = \varphi_*\mathcal{O}_X(2D_1 + D_2 + E_1 + 2E_2 - \tau) \otimes \mathcal{O}_Y(-\Delta_1 - \overline{E}_2).$$

So according to Theorem 2.1 and the remark following it, each $\mathcal{M}_\chi$ is a twist of $\mathcal{L}^{-1}_{\chi+\tau}(-\Delta_1 - \overline{E}_2)$ by a certain combination of Δ_1, Δ_2, $\overline{E}_1$ and $\overline{E}_2$. By Definition 2.2, the rules governing the twists are:

$$\begin{aligned}
\text{twist by } \Delta_1 &\iff (\chi+\tau)\circ\Psi(\Delta_1) = 1 \text{ or } 2\\
\text{twist by } \Delta_2 &\iff (\chi+\tau)\circ\Psi(\Delta_2) = 2\\
\text{twist by } \overline{E}_1 &\iff (\chi+\tau)\circ\Psi(\overline{E}_1) = 2\\
\text{twist by } \overline{E}_2 &\iff (\chi+\tau)\circ\Psi(\overline{E}_2) = 1 \text{ or } 2.
\end{aligned}$$

Thus $\varphi_*\mathcal{F}$ is given by the direct sum of the line bundles $\mathcal{M}_\chi$ listed in the final column. Note that $\mathcal{M}_{(1,0)} = \mathcal{O}_Y$, so $h^0(\varphi_*\mathcal{F}) = 1$. Hence $D_2 - D_1 + E_1 - E_2 - \tau \sim 0$. □

Corollary 3.1 *By formula* (8)*, we have*

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1)[0,0,2].$$

Thus the Kulikov surface satisfies (A3).

Proof The multidegree is clear by (8), but the torsion twist requires some care. Since K_X is the pullback of an integral divisor on Y, it should be torsion-neutral with respect to our coordinate system on $\operatorname{Pic} X$. Thus by Lemma 3.4, we see that the required twist is $[0,0,2]$. □

Theorem 3.2 *The semigroup $\mathfrak{E}$ of effective divisors on the Kulikov surface is generated by the nine reduced pullbacks of components of the branch divisor $D_1, \dots, D_6, E_1, E_2, E_3$.* □

This Theorem is proved using an easier variant of the proof of Theorem 5.1. The situation here is easier, because all of the (-1)-curves on Y are branch divisors, and there are no (-2)-curves.
Thus we have a homomorphism of semigroups $t\colon \mathfrak{E} \to \operatorname{Tors} X$, which sends an effective divisor to its associated torsion twist (see Lemma 3.3), under the choice of basis (from Lemma 3.4).

3.2.1 Acyclic line bundles on the Kulikov surface

Let us start with the following numerical exceptional collection on Y:

$$\Lambda\colon 0,\ e_0 - e_1,\ e_0 - e_2,\ e_0 - e_3,\ 2e_0 - \textstyle\sum_{i=1}^{3} e_i,\ e_0.$$

Given assumptions (A), we see that Λ corresponds to the following numerically exceptional sequence of line bundles on X:

$$\begin{aligned} L_0 = \mathcal{O}_X,\ L_1 = \mathcal{O}_X(-1,1,0,0), L_2 = \mathcal{O}_X(-1,0,1,0),\\ L_3 = \mathcal{O}_X(-1,0,0,1),\ L_4 = \mathcal{O}_X(-2,1,1,1),\ L_5 = \mathcal{O}_X(-1,0,0,0). \end{aligned} \tag{11}$$

We find all collections of torsion twists $L_i \otimes \tau_i$ which are exceptional collections on X. The first step is to find the acyclic sets associated to the various $L_j^{-1} \otimes L_i$.

Proposition 3.2 *The acyclic sets $\mathcal{A}(L_j^{-1} \otimes L_i)$ for $j > i \geq 0$ are listed in Appendix A.*

First Proof By Theorem 3.2, it is an easy exercise to check each entry in the table. As an illustration, we calculate $\mathcal{A}(L_1^{-1})$. The effective divisors on X of multidegree $(1,-1,0,0)$ are $D_2 + E_3$, $D_3 + E_2$, D_4. Thus applying the homomorphism t to each of these effective divisors, we see that $[1,0,2]$, $[2,0,2]$, $[2,1,2]$ do not appear in $\mathcal{A}(L_1^{-1})$. Next we consider degree two cohomology via Serre duality. The effective divisors of multidegree $(2,0,-1,-1)$ are

$$\begin{gathered} 2D_1 + E_2 + E_3, D_1 + D_2 + E_1 + E_3, D_1 + D_3 + E_1 + E_2,\\ D_2 + D_3 + 2E_1, D_1 + D_4 + E_1, D_1 + D_5 + E_2, D_1 + D_6 + E_3,\\ D_2 + D_5 + E_1, D_3 + D_6 + E_1, D_5 + D_6. \end{gathered}$$

Again, applying t we find that $[0,0,2]$, $[2,0,0]$, $[1,0,0]$, $[0,0,1]$, $[1,2,0]$, $[1,2,2]$, $[1,2,1]$, $[0,2,0]$, $[2,2,2]$, $[2,1,1]$ can not appear in $\mathcal{A}(L_1^{-1})$. The acyclic set is made up of those elements of $\operatorname{Tors} X$ which do not appear in either of the two lists above. □

SECOND PROOF As a sanity check, an alternative proof is to use Theorem 2.1 repeatedly, to calculate the cohomology of all possible torsion twists of L_1. □

Both methods are implemented in our computer script [20].

3.2.2 EXCEPTIONAL COLLECTIONS ON THE KULIKOV SURFACE

We now find all exceptional collections on X which are numerically of the form (11). Lemma 3.1 reduces us to a simple search, which can be done systematically [20].

THEOREM 3.3 *The surface X has nine exceptional collections $L_0 = \mathcal{O}_X$, $L_1 \otimes \tau_1, \ldots, L_5 \otimes \tau_5$ which are numerically of the form* (11). *They are given in Table 1 below. Each row lists the required torsion twists τ_i for $i = 1, \ldots, 5$ as elements of $(\mathbb{Z}/3)^3$.*

	τ_1	τ_2	τ_3	τ_4	τ_5
1	$[0,0,0]$	$[0,2,2]$	$[2,2,1]$	$[2,2,1]$	$[0,0,1]$
2	$[2,2,0]$	$[2,1,2]$	$[0,0,1]$	$[1,1,1]$	$[2,2,1]$
3	$[2,2,1]$	$[2,1,2]$	$[0,0,1]$	$[1,1,1]$	$[2,0,2]$
4	$[2,2,0]$	$[2,0,1]$	$[0,2,0]$	$[2,2,1]$	$[2,1,2]$
5	$[1,1,0]$	$[1,0,2]$	$[2,2,0]$	$[1,1,1]$	$[2,2,1]$
6	$[1,1,0]$	$[1,0,2]$	$[0,0,1]$	$[1,1,1]$	$[2,2,1]$
7	$[1,1,0]$	$[1,0,2]$	$[2,2,1]$	$[1,1,1]$	$[0,0,1]$
8	$[2,0,2]$	$[2,2,0]$	$[0,1,2]$	$[1,1,1]$	$[2,2,1]$
9	$[2,0,2]$	$[2,2,1]$	$[0,1,2]$	$[1,1,1]$	$[1,0,2]$

Table 1: Exceptional collections on the Kulikov surface

REMARK 3.4
1. The precise number of exceptional collections is not important. Rather, the fact that we have definitively enumerated all exceptional collections of numerical type Λ, means that we can sift through the list to find one with the most desirable properties.

2. Let Λ' be any translation of Λ under the Weyl group action of $A_1 \times A_2$ on $\operatorname{Pic} Y$. Then Λ' is another numerical exceptional collection on X (see Section 3.1.7), so we may enumerate exceptional collections on X of numerical type Λ'. For the Kulikov surface, each element of the orbit corresponds to either 9, 14, 18 or 24 exceptional collections on X. Thus, the Weyl group action does not "lift" to X in a way which is compatible with the covering $X \to Y$. On occasion, this incompatibility is used to

our advantage (see [20]). We return to these exceptional collections in Section 4.

4 Heights of exceptional collections

Let X be a surface of general type with $p_g = q = 0$, $\operatorname{Tors} X \neq 0$ with an exceptional collection of line bundles $\mathbb{E} = (E_0, \dots, E_{n-1})$. Write $\mathcal{E}$ for the smallest full triangulated subcategory of $D^b(X)$ containing $\mathbb{E}$. In this section we calculate some invariants of $\mathbb{E}$. The invariants we consider are essentially determined by the derived category, but we must enhance the derived category in order to make computations. For completeness, we discuss some background first.

4.1 Motivation from del Pezzo surfaces

Let Y be a del Pezzo surface and let $\mathbb{E}$ be a strong exceptional collection of line bundles on Y. Recall that $\mathbb{E}$ is *strong* if $\operatorname{Ext}^k(E_i, E_j) = 0$ for all i, j and for all $k > 0$. We define the *partial tilting bundle* of $\mathbb{E}$ to be $T = \bigoplus_i E_i$. Then the derived endomorphism ring $\operatorname{Ext}^*(T,T) = \bigoplus_{i,j} \operatorname{Hom}(E_i, E_j)$ is an associative algebra, and we have an equivalence of categories $\mathcal{E} \cong D^b(\text{mod-} \operatorname{Ext}^*(T,T))$ (see [16]).
From now on, we assume that $\mathbb{E}$ is an exceptional collection on a fake del Pezzo surface X, so that we do not have the luxury of choosing a strong exceptional collection. Instead, we recover $\mathcal{E}$ by studying the higher multiplications coming from the A_∞-algebra structure on $\operatorname{Ext}^*(T,T)$.

4.2 Digression on dg-categories

We sketch the construction of a *differential graded* (or dg) enhancement $\mathcal{D}$ of $D^b(X)$. Objects in $\mathcal{D}$ are the same as those in $D^b(X)$, but morphisms $\operatorname{Hom}^\bullet_{\mathcal{D}}(F,G)$ form a chain complex, with differential d of degree $+1$. Composition of maps $\operatorname{Hom}^\bullet_{\mathcal{D}}(F,G) \otimes \operatorname{Hom}^\bullet_{\mathcal{D}}(G,H) \to \operatorname{Hom}^\bullet_{\mathcal{D}}(F,H)$ is a morphism of complexes (the Leibniz rule), and for any object F in $\mathcal{D}$, we require $d(\mathrm{id}_F) = 0$. For a precise definition of $\operatorname{Hom}^\bullet_{\mathcal{D}}(F,G)$, one could use the Čech complex, and we refer to [36] for details. The main point is that the cohomology of $\operatorname{Hom}^\bullet_{\mathcal{D}}(F,G)$ in degree k is $\operatorname{Ext}^k_{D^b(X)}(F,G)$, so in particular, we have $H^0(\operatorname{Hom}^\bullet_{\mathcal{D}}(F,G)) = \operatorname{Hom}_{D^b(X)}(F,G)$.

4.3 Hochschild homology

We first compute some additive invariants, only making implicit use of the dg-structure. The Hochschild homology of X is given by the Hochschild–Kostant–Rosenberg isomorphism

$$HH_k(X) \cong \bigoplus_p H^{p+k}(X, \Omega^p_X),$$

so $HH_0(X) = \mathbb{C}^{12-K^2}$ and $HH_k(X) = 0$ in all other degrees. Moreover, Hochschild homology is additive over semiorthogonal decompositions.

Theorem 4.1 *[35] If $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then*

$$HH_k(X) = HH_k(\mathcal{A}) \oplus HH_k(\mathcal{B}).$$

Assuming the Bloch conjecture on algebraic zero-cycles, we have

$$K_0(X) = \mathbb{Z}^{12-K^2} \oplus \operatorname{Tors} X,$$

and we note that K-theory is also additive over semiorthogonal decompositions (see Proposition 3.1).

Now for an exceptional collection of length n, $K_0(\mathcal{E}) = \mathbb{Z}^n$ and

$$HH_k(\mathcal{E}) = \begin{cases} \mathbb{C}^n & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the maximal length of $\mathbb{E}$ is at most $12 - K_X^2$, and such an exceptional sequence of maximal length effects a semiorthogonal decomposition $D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle$ with nontrivial semiorthogonal complement $\mathcal{A}$. We say that $\mathcal{A}$ is a *quasiphantom* category; by additivity, the Hochschild homology vanishes, but $K_0(\mathcal{A}) \supseteq \operatorname{Tors} X \neq 0$, so $\mathcal{A}$ can not be trivial.

4.4 Height

The Hochschild cohomology groups of X may be computed via the other Hochschild–Kostant–Rosenberg isomorphism (cf. [35]):

$$HH^k(X) = \bigoplus_{p+q=k} H^q(X, \Lambda^p T_X).$$

Thus for a surface of general type with $p_g = 0$, we have

$$HH^0(X) \cong H^0(\mathcal{O}_X) = \mathbb{C},\ HH^1(X) = 0,\ HH^2(X) \cong H^1(T_X),$$
$$HH^3(X) \cong H^2(T_X),\ HH^4(X) \cong H^0(2K_X) = \mathbb{C}^{1+K^2}.$$

Recall that the degree two (respectively three) Hochschild cohomology is the tangent space (resp. obstruction space) to the formal deformations of a category [32].

In principle, [36] gives an algorithm for computing $HH^*(\mathcal{A})$ using a spectral sequence and the notion of height of an exceptional collection. Moreover, by [36, Prop. 6.1], for an exceptional collection to be full, its height must vanish. Thus the height may be used to prove existence of phantom categories without reference to the K-theory. We outline the algorithm of [36] below.

Given an exceptional collection $\mathbb{E}$ on X, there is a long exact sequence (induced by a distinguished triangle)

$$\ldots \to NHH^k(\mathbb{E}, X) \to HH^k(X) \to HH^k(\mathcal{A}) \to NHH^{k+1}(\mathbb{E}, X) \to \ldots$$

where $NHH(\mathbb{E}, X)$ is the *normal Hochschild cohomology* of the exceptional collection $\mathbb{E}$. The normal Hochschild cohomology can be computed using a spectral sequence with first page

$$\mathrm{E}^1_{-p,q} = \bigoplus_{\substack{0\le a_0<\cdots<a_p\le n-1\\ k_0+\cdots+k_p=q}} \mathrm{Ext}^{k_0}(E_{a_0}, E_{a_1}) \otimes \cdots$$
$$\cdots \otimes \mathrm{Ext}^{k_{p-1}}(E_{a_{p-1}}, E_{a_p}) \otimes \mathrm{Ext}^{k_p}(E_{a_p}, S^{-1}(E_{a_0})).$$

The spectral sequence relies on the dg-structure on $\mathcal{D}$; the initial differentials d' and d'' are induced by the differential on $\mathcal{D}$ and the composition map respectively, while the higher differentials are related to the A_∞-algebra structure on Ext-groups, (see Section 4.6).

The existing examples of exceptional collections on surfaces of general type with $p_g = 0$ suggest that $NHH^k(\mathbb{E}, X)$ vanishes for small k. Thus the *height* $h(\mathbb{E})$ of an exceptional collection $\mathbb{E} = (E_0, \dots, E_{n-1})$ is defined to be the smallest integer m for which $NHH^m(\mathbb{E}, X)$ is nonzero. Alternatively, m is the largest integer such that the canonical restriction morphism $HH^k(X) \to HH^k(\mathcal{A})$ is an isomorphism for all $k \le m-2$ and injective for $k = m-1$.

4.5 Pseudoheight

The height may be rather difficult to compute in practice, requiring a careful analysis of the Ext-groups of $\mathbb{E}$ and the maps in the spectral sequence. The pseudoheight is easier to compute and sometimes gives a good lower bound for the height.

Definition 4.1 *The* pseudoheight $ph(\mathbb{E})$ *of an exceptional collection* $\mathbb{E} = (E_0, \dots, E_{n-1})$ *is*

$$ph(\mathbb{E}) = \min_{0\le a_0<\cdots<a_p\le n-1} \big(e(E_{a_0}, E_{a_1}) + \cdots$$
$$+ e(E_{a_{p-1}}, E_{a_p}) + e(E_{a_p}, E_{a_0}(-K_X)) - p + 2\big),$$

where $e(F, F') = \min\{i : \mathrm{Ext}^i(F, F') \ne 0\}$.

The pseudoheight is just the total degree of the first nonzero term in the first page of the spectral sequence, where the shift by 2 takes care of the Serre functor.

Consider the length $2n$ anticanonical extension of the sequence $\mathbb{E}$ (see also Section 3.1.7):

$$E_0, \dots, E_{n-1}, E_n = E_0(-K_X), \dots, E_{2n-1} = E_{n-1}(-K_X). \tag{12}$$

If the E_i are line bundles, then we have a numerical lower bound for the pseudoheight.

Lemma 4.1 *[36, Lem. 4.10, Lem. 5.1] If K_X is ample and $E_i \cdot K_X \geq E_j \cdot K_X$ for all $i < j$ and for all E_i, E_j in the anticanonically extended sequence* (12), *then $ph(\mathbb{E}) \geq 3$.*

The numerical conditions required by the Lemma are not particularly stringent. For example, all the exceptional collections we have exhibited on the Kulikov surface in Section 3.2 have pseudoheight at least 3, even before we consider the Ext-groups more carefully.

Remark 4.1 If L is a line bundle, then $\dim \operatorname{Ext}^k(L, L(-K_X)) = h^{2-k}(2K_X)$ by Serre duality, which is the case $p = 0$ in Definition 4.1. Thus any exceptional collection of line bundles on a surface of general type with $p_g = 0$ has pseudoheight at most 4. Moreover, if $ph(\mathbb{E}) = 4$, then $h(\mathbb{E}) = 4$ by [36].

4.6 The A_∞-algebra of an exceptional collection

Let $\mathbb{E} = (E_0, \ldots, E_{n-1})$ be an exceptional collection on X, and define $T = \oplus_{i=0}^{n-1} E_i$. Then $B = \operatorname{Hom}^\bullet_{\mathcal{D}}(T, T)$ is a differential graded algebra via the dg-structure on $\mathcal{D}$ (see Section 4.2). It can be difficult to compute the dg-algebra structure on B directly, so we pass to the A_∞-algebra H^*B.

We discuss A_∞-algebras, referring to [29] for details and further references. An A_∞*-algebra* is a graded vector space $A = \bigoplus_{p\in\mathbb{Z}} A^p$, together with graded multiplication maps $m_n \colon A^{\otimes n} \to A$ of degree $2 - n$, for each $n \geq 1$. These multiplication maps satisfy an infinite sequence of relations, starting with

$$\begin{aligned} &m_1 m_1 = 0, \\ &m_1 m_2 = m_2(m_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_1). \end{aligned}$$

These first two relations ensure that m_1 is a differential on A, satisfying the Leibniz rule with respect to m_2. The third relation is

$$\begin{aligned} &m_2(\mathrm{id}_A \otimes m_2 - m_2 \otimes \mathrm{id}_A) = \\ &\qquad m_1 m_3 + m_3(m_1 \otimes \mathrm{id}_A \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \mathrm{id}_A \otimes m_1), \end{aligned}$$

which shows that m_2 is not associative in general, but if $m_n = 0$ for all $n \geq 3$, then A is an ordinary associative differential graded algebra.

In fact, by the above discussion, we can view B as an A_∞-algebra, with m_1 being the differential, m_2 the multiplication, and $m_n = 0$ for $n \geq 3$. By a theorem of Kadeishvili (cf. [29]), the homology $H^*B = H^*(B, m_1)$ has a canonical A_∞-algebra structure, for which $m_1 = 0$, m_2 is induced by the multiplication on B, and H^*B and B are quasi-isomorphic as A_∞-algebras. This canonical A_∞-structure is unique, and H^*B is called a *minimal model* for B. We say that B is *formal* if it has a minimal model H^*B for which $m_n = 0$ for all $n \geq 3$, so that H^*B is just an associative graded algebra.

The A_∞-algebra of $\mathbb{E}$ is

$$H^*B = \operatorname{Ext}^*(T, T) = \bigoplus_k \bigoplus_{0 \leq i,j \leq n-1} \operatorname{Ext}^k(E_i, E_j),$$

and m_2 coincides with the Yoneda product on Ext-groups. Clearly, if the exceptional collection $\mathbb{E}$ consists of sheaves, then H^*B has only three nontrivial graded summands, in degrees 0, 1 and 2. Since m_n has degree $2-n$, the summands of degree 0 and 1 are crucial in determining the A_∞-algebra structure.

4.6.1 Recovering $\mathcal{E}$ from H^*B

According to [16], [30], the subcategory $\mathcal{E}$ of $\mathcal{D}$ generated by the exceptional collection $\mathbb{E}$ is equivalent to the triangulated subcategory $\operatorname{Perf}(B) \subset D^b(\text{mod-}B)$ of perfect objects over the dg-algebra B. A perfect object is a differential graded B-module that is quasi-isomorphic to a bounded chain complex of projective and finitely generated modules. As mentioned above, it is preferable to consider the A_∞-algebra H^*B instead, noting that $\mathcal{E}$ is in turn equivalent to the triangulated category of perfect A_∞-modules over H^*B. If B is formal, the equivalence reduces to $\mathcal{E} \cong D^b(\text{mod-}H^*B)$, which should be compared with Section 4.1.

We search for exceptional collections whose Hom- and Ext^1-groups are mostly zero. In good cases, this implies that B is formal, and H^*B has no deformations. It then follows that $\mathcal{E}$ is rigid, i.e. constant in families.

4.7 Quasiphantoms on the Kulikov surface

We study some properties of the exceptional collections on the Kulikov surface from Section 3.2. For the purposes of the discussion, we fix the following exceptional collection

$$\mathbb{E}\colon \mathcal{O},\ L_1[2,2,0],\ L_2[2,1,2],\ L_3[0,0,1],\ L_4[1,1,1],\ L_5[2,2,1],$$

which can be found in the second row of Table 1 in Section 3.2.

Using Theorem 2.1, we may compute the Ext-groups of the extended sequence (12). We present the results in Table 2 below. The ijth entry of the table is the following formal polynomial in q

$$\sum_{k\in\mathbb{Z}} \dim \operatorname{Ext}^k(E_i, E_{i+j}) q^k,$$

where $0 \le i, j \le 5$, and the zigzag delineates those entries whose target E_{i+j} is in the anticanonically extended part of (12).

Lemma 4.2 *The only nonzero* Ext^1*-groups are* $\operatorname{Ext}^1(E_1, E_4)$ *which is* 2*-dimensional, and* $\operatorname{Ext}^1(E_1, E_5)$ *which is* 1*-dimensional.* □

Remark 4.2 The lemma shows that $\mathbb{E}$ does not have 3-block structure. A 3-block structure means the exceptional collection can be split into three mutually orthogonal blocks (cf. [28]). In fact, every exceptional collection in Table 1, and every exceptional collection in the Weyl group orbit (cf. Section 3.1.7), has some non-zero Ext^1-groups. This is in contrast with the exceptional collections on the Burniat surface exhibited in [2], which are of the same numerical type, and have 3-block structure.

	0	1	2	3	4	5
0	1	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
1	1	0	0	$2q+3q^2$	$q+2q^2$	$4q^2$
2	1	0	q^2	q^2	$4q^2$	$6q^2$
3	1	q^2	q^2	$4q^2$	$6q^2$	$6q^2$
4	1	0	$3q^2$	$5q^2$	$5q^2$	$5q^2$
5	1	$3q^2$	$5q^2$	$5q^2$	$5q^2$	$6q^2$

Table 2: Ext-table of an exceptional collection on the Kulikov surface

Proposition 4.1 *The A_∞-algebra of $\mathbb{E}$ is formal, and the product m_2 of any two elements with strictly positive degree is trivial.*

Proof The A_∞-algebra H^*B of $\mathbb{E}$, is the direct sum of all Ext-groups appearing above the zigzag in the table. By [46, Lemma 2.1] or [38, Theorem 3.2.1.1], we may assume that $m_n(\dots, id_{E_i}, \dots) = 0$ for all E_i and all $n > 2$. We show that every product m_3 must be zero for degree reasons. By Lemma 4.2, there are only two nonzero arrows in degree 1, and they can not be composed with one another, since they have the same source. Thus the product m_3 of any 3 composable elements of H^*B has degree at least $\deg m_3 + 1 + 2 + 2 = 4$, and is therefore identically zero, because the graded piece H^4B is trivial. The same argument applies for all products m_n with $n \geq 3$. Thus H^*B is a formal A_∞-algebra. In fact, we see from the table that any product m_2 of two elements of nonzero degree also vanishes for degree reasons. □

Moreover, we calculate the Hochschild cohomology of $\mathcal{A}$ using heights.

Proposition 4.2 *We have $HH^0(\mathcal{A}) = \mathbb{C}$, $HH^1(\mathcal{A}) = 0$, $HH^2(\mathcal{A}) = \mathbb{C}$, and $HH^3(\mathcal{A})$ contains a copy of $\mathbb{C}^3$.*

Proof The pseudoheight of $\mathbb{E}$ may also be computed from the table, where now we also need the portion below the zigzag. The minimal contribution to the pseudoheight is achieved by incorporating one of the nonzero Ext^1-groups. For example,

$$e(E_1, E_4) + e(E_4, E_1 \otimes \omega_X) - 1 + 2 = 1 + 2 - 1 + 2 = 4,$$

so $ph(\mathbb{E}) = 4$. In this case, by [36], the height and pseudoheight are equal. Hence $HH^k(\mathcal{A}) = HH^k(X)$ for $k \leq 2$, and $HH^3(\mathcal{A}) \supset HH^3(X)$. By the Hochschild–Kostant–Rosenberg isomorphism, the dimensions of $HH^k(X)$ follow from the infinitesimal deformation theory of the Kulikov surface, which was studied in [19]: $H^1(T_X) = 1$ and $H^2(T_X) = 3$. □

In summary, we have

Theorem 4.2 *Every Kulikov surface X has a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle$$

where $\mathcal{E}$ is generated by the exceptional collection $\mathbb{E}$, and $\mathcal{E}$ is rigid, i.e. $\mathcal{E}$ does not vary with X. The semiorthogonal complement $\mathcal{A}$ is a quasiphantom category whose formal deformation space is isomorphic to that of $D^b(X)$, and therefore X may be reconstructed from $\mathcal{A}$.

5 Secondary Burniat surfaces and effective divisors

Burniat surfaces were discovered in [18], and an alternate construction is given in [27]. There are several cases X_k, with $K^2 = k$ for $2 \leq k \leq 6$. For details we refer to [44], [4]. Exceptional collections on primary Burniat surfaces X_6 with $K^2 = 6$ were first constructed and studied in [2], where two 3-block exceptional collections are exhibited. Burniat surfaces with $K^2 = 3, 4, 5, 6$ can be constructed as abelian covers satisfying assumptions (A), and so we are able to enumerate exceptional collections on all these Burniat surfaces. We do not reproduce these computations here, but see [20]. Exceptional collections of line bundles of maximal length on the Burniat–Campedelli surface X_2 with $K^2 = 2$ remain elusive, because this surface does not satisfy assumption (A1).
In computing exceptional collections on fake del Pezzo surfaces, it becomes clear that a characterisation of effective line bundles is very useful. In this section we prove the following theorem for the secondary nodal Burniat surface.

Theorem 5.1 *Let X be a nodal secondary Burniat surface with $K^2 = 4$. Then the semigroup of effective divisors on X is generated by the reduced pullbacks of irreducible components of the branch divisor, together with the pullbacks E_4, E_5 of two (-1)-curves on Y.*

With appropriate changes, the same proof works for the other surfaces satisfying assumptions (A). Indeed, Theorem 3.2 above for the Kulikov surface is an easier case of this result. The additional complexity here arises from two sources: some of the exceptional curves on Y are not branch divisors, and there is a (-2)-curve.

5.1 Burniat surfaces revisited

We first describe the nodal secondary Burniat line configuration. Take the three coordinate points P_1, P_2, P_3 in $\mathbb{P}^2$, and label the edges $\overline{A}_0 = P_1P_2$, $\overline{B}_0 = P_2P_3$, $\overline{C}_0 = P_3P_1$. Then let $\overline{A}_1$, $\overline{A}_2$ (respectively $\overline{B}_i$, $\overline{C}_i$) be two lines passing through P_1 (resp. P_2, P_3). We require that $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_2$ are concurrent in P_4 (respectively $\overline{A}_1$, $\overline{B}_2$, $\overline{C}_1$ in P_5). This gives nine lines in total, four passing through each of P_1, P_2, P_3 and three passing through each of P_4, P_5. Moreover, $\overline{A}_1$ passes through three triple points. Blow up the five points P_i to obtain a weak del Pezzo surface Y of degree 4. The strict transforms of these nine lines (for which we use the same labels) together with the three exceptional curves $\overline{E}_i$ for $i = 1, 2, 3$, are called the *nodal secondary Burniat configuration* (see Figure 2).

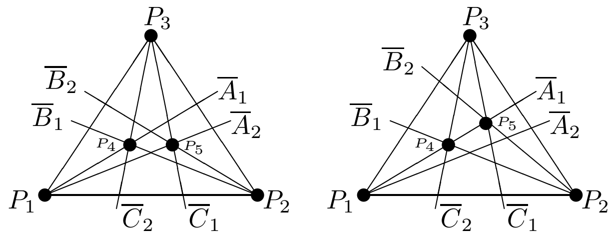

Figure 2: The secondary Burniat configurations with $K^2 = 4$ (nodal configuration is on the right)

The nodal secondary Burniat surface X_4 with $K_X^2 = 4$ is a $(\mathbb{Z}/2)^2$-cover of Y branched in the configuration of Figure 2, and X_4 is a surface of general type with $p_g = 0$, $K^2 = 4$ and $\operatorname{Tors} X = (\mathbb{Z}/2)^4$. The cover is not ramified in $\overline{E}_4$, $\overline{E}_5$. The weak del Pezzo surface Y has a (-2)-curve, $\overline{A}_1$ and the canonical model of X_4 is a $(\mathbb{Z}/2)^2$-cover of a nodal quartic. Following the description of [5, 6], the nodal secondary Burniat surfaces form an irreducible closed family, inside a 3-dimensional irreducible connected component of the moduli space. This component is given by the union of the family of nodal secondary Burniat surfaces with the *extended secondary Burniat surfaces*, which form an open subset. We do not directly consider extended Burniat surfaces here.

In Appendix B, we show that the secondary Burniat surface satisfies assumptions (A). More precisely, we exhibit an explicit basis $e_0, \dots, e_5$ for $\operatorname{Pic} X/\operatorname{Tors} X$, in terms of reduced pullbacks of irreducible branch divisors. The appendix also lists coordinates for the reduced pullback of each irreducible component of the branch divisor, according to Definition 3.4.

We define $\mathfrak{C}$ to be the semigroup generated by the reduced pullbacks $A_0, \dots, C_2$, E_1, E_2, E_3 together with ordinary pullbacks E_4, E_5. There is a multigrading on $\mathfrak{C}$ by multidegree in $\operatorname{Pic} X/\operatorname{Tors} X$, and we write $\mathfrak{C}(d)$ for graded piece of multidegree d. Using the coordinates from Appendix B, we define a homomorphism $t\colon \mathfrak{C} \to \operatorname{Tors} X$, sending each generator of $\mathfrak{C}$ to its associated torsion twist. Remember that $t(E_4) = t(E_5) = 0$ because these are pulled back from Y.

5.2 Proof of Theorem 5.1.

The stategy of proof is similar to that of Alexeev, [1], but for completeness, we outline the whole proof. The main differences are the (-2)-curve on Y and the (-1)-curves which are not branch divisors. These introduce new complications which are not present in [1]. We are able to resolve these issues because we can use the pushforward formula and Appendix B to check effectivity in a systematic manner.

Suppose D is an effective divisor and C is an effective curve class on X for which $D \cdot C < 0$. Then $C^2 < 0$ and C is in the base locus of D, so we define $D' = D - aC$ where a is the smallest positive integer for which $(D-aC)\cdot C \geq 0$. In this way, we can reduce an effective divisor on X to one which has positive intersection with all curve classes in $\mathfrak{E}$. Such divisor classes form a rational polyhedral cone $\mathcal{P}$ in $N^1(X, \mathbb{R})$.

To describe the generators of $\mathcal{P}$ in the most geometric way, we first construct certain divisor classes on X in terms of reduced pullbacks and the birational transformations the del Pezzo surface Y. Suppose we take a standard Cremona transformation of $\mathbb{P}^2$ centred on any three non-collinear triple points P_i, P_j and P_k. The numerical class of the hyperplane section of the image $\mathbb{P}^2$ is $h_{ijk} = 2e_0 - e_i - e_j - e_k$ for any $\{i, j, k\} \neq \{1, 4, 5\}$, or $h_0 = e_0$. There are also natural fibrations on Y which arise from the pencil of hyperplanes passing through a fixed P_k on some copy of $\mathbb{P}^2$. The numerical classes of these fibrations are denoted $f_i = e_0 - e_i$ or $f_{ijkl} = 2e_0 - e_i - e_j - e_k - e_l$ with $\{1, 4, 5\} \not\subset \{i, j, k, l\}$.

Lemma 5.1 *The polyhedron $\mathcal{P}$ is generated by the ten hyperplane classes h_0, h_{ijk} and eight fibrations f_i, f_{ijkl} defined above, together with the four additional classes*

$$g_1 = 3e_0 - e_1 - 2e_2 - e_3 - e_4 - e_5, \quad g_2 = 3e_0 - e_1 - e_2 - 2e_3 - e_4 - e_5,$$
$$g_3 = 3e_0 - e_1 - 2e_2 - e_4 - e_5, \quad g_4 = 3e_0 - e_1 - 2e_3 - e_4 - e_5.$$

Proof Any generator D of $\mathfrak{E}$ determines a linear function $\cdot D$, which in turn defines a collection of hyperplanes supporting the polyhedron $\mathcal{P}$. We use the computer [20] to calculate the integral generators of the cone.

We examine the additional generators. The class g_1 is the hyperplane section of the copy of $\mathbb{P}^2$ obtained by contracting $\overline{A}_0$, $\overline{A}_1$, $\overline{B}_0$, $\overline{B}_1$ and $\overline{B}_2$ on Y, and g_3 is the hyperplane section of the quadric cone given by contracting $\overline{A}_0$, $\overline{A}_1$, $\overline{E}_3$, $\overline{B}_1$ and $\overline{B}_2$. There are similar descriptions of g_2 and g_4. $\square$

Lemma 5.2 *Suppose D is an effective divisor on X with $K_X \cdot D \leq 4$. Then D is in $\mathfrak{E}$.*

Proof We may assume that D is in $\mathcal{P}$. This is a finite (and small) number of classes to check, and we do this directly using the computer implementation [20] of our pushforward formula Theorem 2.1. $\square$

Proposition 5.1 *Suppose D is an effective divisor on X with $K_X \cdot D > 4$ and $\chi(D) > 0$. Then D is in $\mathfrak{E}$.*

Proof Since $K_X \cdot D > 4$ we have that $(K_X - D) \cdot K_X < 0$ and so $K_X - D$ can not be effective. By Serre duality, $h^2(D) = h^0(K_X - D) = 0$, hence D is effective.

Choose $\bar{D}$ in $\operatorname{Pic} Y$ such that the numerical class of $K_Y + \bar{D}$ in $\operatorname{Pic} Y$ is the same as that of $D - K_X$ in $\operatorname{Pic} X/\operatorname{Tors} X$ by assumption (A). Then

$$\chi(K_Y + \bar{D}) = 1 + \frac{1}{2}(K_Y + \bar{D})\bar{D} = 1 + \frac{1}{2}(D - K_X)D = \chi(D) > 0.$$

Moreover, $h^2(K_Y + \bar{D}) = h^0(-\bar{D}) = 0$ because $-K_Y \cdot -\bar{D} < 0$, so by the same argument as above, we see that $K_Y + \bar{D}$ is effective on Y.
Now, any effective divisor on Y is a positive linear combination of branch divisors $\overline{A}_0, \ldots, \overline{C}_2, \overline{E}_1, \overline{E}_2, \overline{E}_3$ and exceptional curves $\overline{E}_4$ and $\overline{E}_5$. So taking the reduced pullback, we get the following expression for the numerical class of D in $\operatorname{Pic} X/\operatorname{Tors} X$:

$$D = K_X + (\text{combination of } A_0, \ldots, E_3) + \tfrac{1}{2}(\text{combination of } E_4, E_5).$$

The coefficient of $\frac{1}{2}$ appears because $\overline{E}_4$ and $\overline{E}_5$ are not branch divisors. It remains to show that $D + \tau$ is in $\mathfrak{E}$ for any τ such that $D + \tau$ is effective. This is implied by the following lemma:

Lemma 5.3 *(1) Let L be any of the following line bundles on X:*

$$\mathcal{O}_X(K_X + \gamma) \otimes \tau,\ \mathcal{O}_X(K_X + \tfrac{1}{2}E_4) \otimes \tau,\ \mathcal{O}_X(K_X + \tfrac{1}{2}E_5) \otimes \tau,\ \textit{or}\ \mathcal{O}_X(K_X + \tfrac{1}{2}(E_4 + E_5)) \otimes \tau$$

where γ is any generator of $\mathfrak{E}$ and τ is any element of $\operatorname{Tors} X$. Then L is effective and in $\mathfrak{E}$ unless $L = \mathcal{O}_X(K_X + A_1)$.

(2) The line bundles $L = \mathcal{O}_X(K_X + kA_1)$ are not effective for any $k > 0$.

Proof

(1) Suppose $L = \mathcal{O}_X(K_X + A_0) \otimes \tau$, and take the graded piece of $\mathfrak{E}$ with multidegree $d = (4, -2, -2, -1, -1, -1)$. We use the computer [20] to check that the image of $\mathfrak{E}(d)$ under t is all of $\operatorname{Tors} X$. This proves that L is effective and in $\mathfrak{E}$ for any τ. The same computation works for all multidegrees listed in the statement, except when $L = \mathcal{O}_X(K_X + A_1) \otimes \tau$, for which we refer to the proof of part (2).

(2) When $L = \mathcal{O}_X(K_X + A_1) \otimes \tau$, the same computation as above shows that the image of $\mathfrak{E}(4, -2, -1, -1, -2, -2)$ under t is $\operatorname{Tors} X - \{[1, 0, 0, 0]\}$. Thus $\mathcal{O}_X(K_X + A_1)$ is not in $\mathfrak{E}$. Indeed, the pushforward is

$$\varphi_* L = \mathcal{O}_Y(e_2 - e_1) \oplus \mathcal{O}_Y(e_3 - e_2) \oplus \mathcal{O}_Y(-2e_0 + e_2 + e_3) \oplus \mathcal{O}_Y(e_0 - e_3 - e_4 - e_5),$$

which is not effective. Moreover, by the projection formula, we have

$$\varphi_* L(2mA_2) = \varphi_* L \otimes \mathcal{O}_Y(m\overline{A}_2) = \varphi_* L \otimes \mathcal{O}_Y(m(e_0 - e_1 - e_4 - e_5)),$$

which is not effective for any m, and so $\mathcal{O}_X(K_X + kA_2)$ is not effective for any odd $k = 1 + 2m$. For even k, the proof is similar, starting from $\varphi_*\mathcal{O}_X(K_X)$. $\square$

Remark 5.1 Since A_2 is a (-2)-curve, we have $K_X \cdot (K_X + kA_1) = K_X^2 = 4$ for all k. Thus we do not need part (2) of the above lemma, because $K_X + kA_1$ does not satisfy the assumptions of Proposition 5.1.

Finally, we take care of the cases with $\chi(D) \leq 0$.

Lemma 5.4 *Suppose D is an effective divisor on X with numerical class in $\mathcal{P}$ and $\chi(D) \leq 0$. Then D is in one of the following classes:*

(1) h or $2h$ for any hyperplane generator h;

(2) g or $2g_1$ or $2g_3$, where g refers to any of the additional generators described in Lemma 5.1;

(3) nf, $nf + f'$, $nf + h$, $nf + g$ for any $n \geq 1$ where f is a fibration and f' is another fibration with intersection $f \cdot f' = 1$, h is a hyperplane generator with $f \cdot h = 1$, g is an additional generator with $f \cdot g = 1$.

Proof This is a systematic induction. We note that each generator γ of $\mathcal{P}$ has $\chi(\gamma) = 0$. Moreover, if $D = D_1 + D_2$ then $\chi(D) = \chi(D_1) + \chi(D_2) + D_1 \cdot D_2 - 1$. So for example, starting from f_1, we choose another fibration generator f'. Either $f_1 \cdot f' = 0$, in which case $f_1 = f'$ and $\chi(2f_1) = -1$, or $f_1 \cdot f' = 1$, so that $\chi(f_1 + f') = 0$. Now adding a further generator γ to $f_1 + f'$ yields $\chi(f_1 + f' + \gamma) > 0$ by simple consideration of the intersection numbers, unless γ is one of f_1 or f'. We continue in this way, to produce the list of possibilities.$\square$

Lemma 5.5 *Suppose L is an effective line bundle with numerical class one of the exceptional cases from Lemma 5.4. Then L is in $\mathfrak{E}$.*

Proof We give a proof for nf_1. The other possibilities listed in Lemma 5.4(3) work in the same way, and cases (1) and (2) can be checked by a direct computation [20]. As in the proof of Lemma 5.3, we split into even and odd cases and make use of the projection formula.

Let $L = \mathcal{O}_X(2f_1) \otimes \tau$ for some τ in the image of $t(\mathfrak{E}(2f_1))$, so that in particular, L is effective. Since $C_0 + E_3$ is a section of $\mathcal{O}_X(f_1)$, it follows that $\mathcal{O}_X(nf_1) \otimes \tau$ is effective and in $\mathfrak{E}$ for any $n \geq 2$.

Now suppose τ is any torsion element in $\operatorname{Tors} X - t(\mathfrak{E}(2f_1))$, so that L is not in $\mathfrak{E}$. For example, $\tau = [0, 0, 0, 1]$. Then we compute

$$\begin{aligned}\varphi_* L = \mathcal{O}_Y(0, -1, 0, 0, 0, 1) &\oplus \mathcal{O}_Y(-2, 1, 1, 1, 1, 1)\\ &\oplus \mathcal{O}_Y(-1, -1, 1, 1, 1, 0) \oplus \mathcal{O}_Y(-2, 0, 1, 1, 1, 1),\end{aligned}$$

which is clearly not effective. Moreover, by the projection formula, we see that $\varphi_* L \otimes \mathcal{O}_X(2mf_1) = \varphi_* L \otimes \mathcal{O}_Y(mf_1)$ is not effective for any m either, for degree reasons. This completes the proof for any even multiple of f_1. A similar computation proves the odd case, starting from $3f_1$. $\square$

A Appendix: Acyclic bundles on the Kulikov surface

For reference, here are the acyclic line bundles on the Kulikov surface used in Section 3.2.

L	$\mathcal{A}(L)$
L_1^{-1}	$[0,0,0],[0,1,0],[1,1,0],[2,1,0],[2,2,0],[1,0,1],[2,0,1],[0,1,1],$ $[1,1,1],[0,2,1],[2,2,1],[0,1,2],[1,1,2],[0,2,2]$
L_2^{-1}	$[0,1,0],[1,1,0],[2,2,0],[2,0,1],[0,1,1],[1,1,1],[2,1,1],[1,2,1],$ $[2,2,1],[0,0,2],[1,0,2],[0,1,2],[1,1,2],[1,2,2]$
L_3^{-1}	$[0,1,0],[1,1,0],[1,0,1],[0,1,1],[1,1,1],[0,2,1],[1,2,1],[0,0,2],$ $[2,0,2],[0,1,2],[1,1,2],[2,1,2],[0,2,2],[1,2,2]$
L_4^{-1}	$[0,0,0],[0,1,0],[2,1,0],[0,2,0],[2,2,0],[1,0,1],[2,0,1],[0,1,1],$ $[1,1,1],[2,1,1],[0,2,1],[2,2,1],[1,1,2],[0,2,2],[2,2,2]$
L_5^{-1}	$[0,1,0],[1,1,0],[2,2,0],[1,0,1],[2,0,1],[0,1,1],[1,1,1],[0,2,1],$ $[1,2,1],[2,2,1],[0,0,2],[0,1,2],[1,1,2],[0,2,2],[1,2,2]$
$L_2^{-1}\otimes L_1$	$[1,0,0],[2,0,0],[2,1,0],[0,1,1],[0,1,2],[2,1,2],[0,2,2]$
$L_3^{-1}\otimes L_1$	$[0,0,0],[1,0,0],[2,0,0],[1,1,0],[2,1,0],[2,2,0],[1,1,2],[2,1,2],$ $[2,2,2]$
$L_4^{-1}\otimes L_1$	$[0,1,0],[1,1,0],[0,1,1],[1,1,1],[1,2,1],[0,0,2],[1,0,2],[2,0,2],$ $[0,1,2],[1,1,2],[1,2,2]$
$L_5^{-1}\otimes L_1$	$[1,0,0],[2,0,0],[1,1,0],[2,1,0],[2,2,0],[0,1,1],[0,0,2],[0,1,2],$ $[1,1,2],[2,1,2],[0,2,2],[2,2,2]$
$L_3^{-1}\otimes L_2$	$[1,0,1],[1,1,1],[2,1,1],[2,0,2],[1,1,2],[2,1,2],[1,2,2]$
$L_4^{-1}\otimes L_2$	$[0,0,0],[0,1,0],[1,1,0],[1,0,1],[0,1,1],[1,1,1],[0,2,1],[2,0,2],$ $[0,1,2],[1,1,2],[0,2,2]$
$L_5^{-1}\otimes L_2$	$[0,1,0],[1,0,1],[0,1,1],[1,1,1],[2,1,1],[0,2,1],[0,0,2],[2,0,2],$ $[1,1,2],[2,1,2],[0,2,2],[1,2,2]$
$L_4^{-1}\otimes L_3$	$[0,0,0],[0,1,0],[1,1,0],[2,2,0],[2,0,1],[0,1,1],[1,1,1],[2,2,1],$ $[1,0,2],[0,1,2],[1,1,2]$
$L_5^{-1}\otimes L_3$	$[0,1,0],[1,1,0],[2,1,0],[2,2,0],[2,0,1],[1,1,1],[2,1,1],[1,2,1],$ $[0,0,2],[1,0,2],[0,1,2],[1,2,2]$
$L_5^{-1}\otimes L_4$	$[1,0,0],[2,0,0],[1,1,0],[2,2,0],[0,0,2],[0,1,2],[2,1,2],[2,2,2]$

B Appendix: Nodal Secondary Burniat surface with $K^2 = 4$

The maps $\Psi_4, \Psi_4^n \colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^6$ determining respectively the non-nodal and nodal Burniat surfaces, differ from one another slightly. We tabulate them below.

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Psi_4(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	0	$g_4 + g_6$	$g_3 + g_5$
$\Psi_4^n(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	$g_3 + g_4$	$g_3 + g_6$	$g_3 + g_5$

The restriction imposed by P_5 is $\Psi_4(\overline{A}_2 + \overline{B}_2 + \overline{C}_1) = 0$ in the non-nodal case, and $\Psi_4^n(\overline{A}_1 + \overline{B}_2 + \overline{C}_1) = 0$ in the nodal case. Either way, g_7 is eliminated, so the torsion group is $(\mathbb{Z}/2)^4$, generated by $g_3^*, \ldots, g_6^*$.
We extend the basis chosen for the free part of $\mathrm{Pic}(X_5)$. The basis is the same for non-nodal and nodal surfaces

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,$$
$$e_4 = C_0 - C_2 + E_1,\ e_5 = B_0 - B_2 + E_3.$$

Coordinates for non-nodal surface:

	Multidegree						Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	0	0	$[1,1,0,0]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	0	$[1,0,0,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	-1	$[0,1,1,0]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	0	0	$[0,0,1,1]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	-1	0	$[0,0,1,0]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	0	-1	$[0,0,1,1]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0	0	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	0	-1	$[0,0,1,0]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	-1	0	0

Coordinates for nodal surface are the same (with same multidegrees) except for the following:

	Multidegree						Torsion
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	-1	$[1,0,1,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	0	$[0,1,0,0]$

In both cases, $\mathcal{O}_X(K_X) = \mathcal{O}(3,-1,-1,-1,-1,-1)[0,0,1,0]$.

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