# $\mathcal{L}$-Invariant for Siegel–Hilbert Forms

Giovanni Rosso

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Abstract. We prove a formula for the Greenberg–Benois $\mathcal{L}$-invariant of the spin, standard and adjoint Galois representations associated with Siegel–Hilbert modular forms. In order to simplify the calculation, we give a new definition of the $\mathcal{L}$-invariant for a Galois representation $V$ of a number field $F \neq \mathbb{Q}$; we also check that it is compatible with Benois' definition for $\mathrm{Ind}_F^{\mathbb{Q}}(V)$.

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## 1 Introduction

Since the historical results of Kummer and Kubota–Leopold on congruences for Bernoulli numbers, people have been interested in studying the $p$-adic variation of special values of $L$-functions.

More precisely, fix a motive $M$ over $\mathbb{Q}$. We suppose that $M$ is Deligne critical at $s = 0$ and that there exists a Deligne's period $\Omega(M)$ such that $\frac{L(M,0)}{\Omega(M)}$ is algebraic. Fix a prime $p$ and two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Let $V$ be the $p$-adic realization of $M$ and suppose that $V$ is semistable (à la Fontaine). Thanks to work of Coates and Perrin-Riou, we have precise conjectures on how the special values should behave $p$-adically; we fix a regular sub-module of $V$. This corresponds to the choice of a sub-$(\varphi, N)$-module of $\mathcal{D}_{\mathrm{st}}(V)$ which is a section of the exponential map

$$\mathcal{D}_{\mathrm{st}}(V) \to t(V) \cong \frac{\mathcal{D}_{\mathrm{st}}(V)}{\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(V)}.$$

Let $h$ be the valuation of the determinant of $\varphi$ on $D$. We can state the following conjecture;

CONJECTURE 1.1. *There exists a formal series $L_p^D(V,T) \in \mathbb{C}_p[[T]]$ which grows as $\log_p^h$ such that for all non-trivial, finite-order characters $\varepsilon : 1+p\mathbb{Z}_p \to \mu_{p^\infty}$ we have*

$$L_p^D(V, \varepsilon(1+p)-1) = C_\varepsilon(D)\frac{L(M\otimes\varepsilon,0)}{\Omega(M)}.$$

*Moreover, for $\varepsilon = \mathbf{1}$ we have*

$$L_p^D(V,0) = E(D)\frac{L(M,0)}{\Omega(M)},$$

*where $E(D)$ is an explicit product of Euler-type factors depending on $D$ and $(\mathcal{D}_{\mathrm{st}}(V)/D)^{N=0}$.*

It may happen that one of the factors of $E(D)$ vanishes and then we say that trivial zeros appear. Since the seminal work of [MTT86], people have been interested in describing the $p$-adic derivative of $L_p^D(V,(1+p)^s-1)$ when trivial zeros appear.
We suppose for simplicity that $L(M,0)$ is not vanishing. We have the following conjecture by Greenberg and Benois;

CONJECTURE 1.2. *Let $t$ be the number of vanishing factors of $E(D)$. Then*

- $\mathrm{ord}_{s=0}L_p^D(V,(1+p)^s-1) = t,$
- $L_p^D(V,0)^* = \mathcal{L}(V^*(1),D^*)E^*(D)\frac{L(M,0)}{\Omega(M)}.$

*Here $E^*(D)$ is the product of non-vanishing factors of $E(D)$ and $\mathcal{L}(V^*(1),D^*)$ is a number, defined in purely Galois theoretical terms (see Section 3.1), for the dual Galois representation $V^*(1)$.*

The error factor $\mathcal{L}(V,D)$ is quite mysterious. It has been calculated in only few cases for the symmetric square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms by Hida and Harron–Jorza. Unless $V$ is an elliptic curve over $\mathbb{Q}$ with multiplicative reduction at $p$ we can not prove the non-vanishing of $\mathcal{L}(V,D)$.
The aim of this paper is to calculate it in some new cases; let $F$ be a totally real field (we make no assumptions on the ramification at $p$) and $\pi$ be an automorphic representation of $\mathrm{GSp}_{2g/F}$ of weight $\underline{k} = (k_\tau)_\tau$, where $\tau$ runs through the real embeddings of $F$ and $(k_\tau) = (k_{1,\tau},\ldots,k_{g,\tau};k_0)$ (note that $k_0$ does not depend on $\tau$). We say that $\pi$ is parallel of weight $k$, $k \in \mathbb{Z}_{\geq 0}$ if $k_{i,\tau} = k$ for all $\tau$ and $i = 1,\ldots,g$ and $k_0 = gk$.
We suppose that it has Iwahoric level at all $\mathfrak{p} \mid p$. We suppose moreover that $\pi_\mathfrak{p}$ is either Steinberg (see Definition 4.8) or spherical. We partition consequently the prime ideals of $F$ above $p$ in $S^{\mathrm{Stb}} \cup S^{\mathrm{Sph}}$.
We have conjecturally two Galois representations associated with $\pi$, namely

the spinorial one $V_{\mathrm{spin}}$ and the standard one $V_{\mathrm{sta}}$. Let $V$ be one of these two representations. We choose for each prime $\mathfrak{p}$ of $F$ dividing $p$ a regular sub module $D_{\mathfrak{p}}$ of $\mathcal{D}_{\mathrm{st}}(V_{|G_{F_{\mathfrak{p}}}})$.

Consider a family of Siegel–Hilbert modular forms as in [Urb11] passing through $\pi$. Let us denote by $\beta_{\mathfrak{p}}(\kappa)$ the eigenvalue of the normalized Hecke operators $U_{1,\mathfrak{p}}$ (see Definition 4.9) on this family. Let $S^{\mathrm{Sph},1} = S^{\mathrm{Sph},1}(V,D)$ be the subset of $S^{\mathrm{Sph}}$ for which $(\mathcal{D}_{\mathrm{st}}(V_{\mathfrak{p}})/D_{\mathfrak{p}})^{N=0}$ does contain the eigenvalue 1. Conjecturally, it is empty for the spin representation. The eigenvalues 1 always appears in $\mathcal{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ for $V$ the standard representation but it may appear in $D_{\mathfrak{p}}$ (this is already the case for the symmetric square of a modular form).

Let $t_{\mathrm{Stb}}$ be the cardinality of $S^{\mathrm{Stb}}$ and $t_{\mathrm{Sph}}$ be the cardinality of $S^{\mathrm{Sph},1}$. We define $f_{\mathfrak{p}} = [F_{\mathfrak{p}}^{\mathrm{ur}} : \mathbb{Q}_p]$.

THEOREM 1.3. *Let $\pi$ be as above, of parallel weight $k$. Let $V = V_{\mathrm{spin}}$ and suppose hypothesis* LGP *of Section 4.2, then the expected number of trivial zeros for $L_p^D(V(k-1),T)$ is $t_{\mathrm{Stb}}$ and*

$$\mathcal{L}(V(k-1),D) = \prod_{\mathfrak{p}\in S^{\mathrm{Stb}}} -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p}}(k)}{\mathrm{d}k}_{|k=\underline{k}}.$$

*Let $V = V_{\mathrm{std}}$, then the conjectural number of trivial zero for $L_p^D(V,T)$ is $t_{\mathrm{Stb}} + t_{\mathrm{Sph}}$ and*

$$\mathcal{L}(V,D) = \mathcal{L}(V,D)^{\mathrm{Sph}} \prod_{\mathfrak{p}\in S^{\mathrm{Stb}}} -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p}}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

*where $\mathcal{L}(V,D)^{\mathrm{Sph}}$ is a priori global factor. It is 1 if $t_{\mathrm{Sph}} = 0$.*

In Section 4.2 we shall provide also a formula for the $\mathcal{L}$-invariant of $V_{\mathrm{std}}(s)$ $(\min(k-g-1, g-1) \geq s \geq 1)$.

The proof of the theorem is not different from the one of [Ben10, Theorem 2] which in turn is similar to the original one of [GS93].

Let now $g = 2$. Let $t$ be the number of primes above $p$ in $F$. We consider the $2t$-dimensional eigenvariety for $\mathrm{GSp}_{4/F}$ with variables $k = \{k_{\mathfrak{p},1}, k_{\mathfrak{p},2}\}_{\mathfrak{p}}$(see Section 5) and let us denote by $F_{\mathfrak{p},i}(k)$ $(i = 1,2)$ the first two graded pieces of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{spin}})$. The 10-dimensional Galois representation $\mathrm{Ad}(V_{\mathrm{spin}})$ has a natural regular sub-$(\varphi,N)$-module induced by the $p$-refinement of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{spin}})$ and which we shall denote by $D_{\mathrm{Ad}}$. With this choice of regular sub module, $\mathrm{Ad}(V_{\mathrm{spin}})$ presents $2t$ conjectural trivial zeros. In Section 5 we prove the following theorem;

THEOREM 1.4. *Let $\pi$ be an automorphic form of weight $\underline{k}$. Suppose that hypothesis* LGP *of Section 4.2 is verified for $V_{\mathrm{spin}}$ and the point corresponding* *to $\pi$ in the eigenvariety $\mathcal{X}'$ (as defined in Section 5) is étale over the weight space. We have then*

$$\mathcal{L}(\mathrm{Ad}(V_{\mathrm{spin}}(\pi)), D_{\mathrm{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \begin{pmatrix} \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,1}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,1}} \\ \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,2}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,2}} \end{pmatrix}_{1 \leq i,j \leq t|_{k=\underline{k}}}.$$

We remark that this theorem is the first to really go beyond $\mathrm{GL}_2$ and its representations $\mathrm{Sym}^n$.

The motivation for Theorem 1.3 lies in a generalization of [Ros15] to Siegel forms. In *loc. cit.* we use Greenberg–Stevens method to prove a formula for the derivative of the symmetric square $p$-adic $L$-function and calculate the analytic $\mathcal{L}$-invariant and the same method of proof could possibly be generalized to finite slope Siegel forms thanks to the overconvergent Maß-Shimura operators and overconvergent projectors of Z. Liu's thesis.
With some work, it could also be generalized to totally real field where $p$ is inert, as already done for the symmetric square [Ros13].

We hope to calculate the $\mathcal{L}$-invariant for $V_{\mathrm{std}}$ and $\mathrm{Ad}(V_{\mathrm{spin}})$ for more general forms in a future work.

In Section 2 we recall the theory of $(\varphi, \Gamma)$-module over a finite extension of $\mathbb{Q}_p$. It will be used in Section 3 to generalize the definition of the $\mathcal{L}$-invariant à la Greenberg–Benois to Galois representations $V$ over general number field $F$ (note that we do not suppose $p$ split or unramified). This definition does not require one to pass through $\mathrm{Ind}_F^{\mathbb{Q}}(V)$ to calculate the $\mathcal{L}$-invariant which in turn simplifies explicit calculation. We shall check that this definition coincides with Benois' definition for $\mathrm{Ind}_F^{\mathbb{Q}}(V)$.
We prove the above-mentioned theorems in Section 4 and 5, inspired mainly by the methods of [Hid07].

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## 2 Some results on rank one $(\varphi, \Gamma)$-module

Let $L$ be a finite extension of $\mathbb{Q}_p$. The aim of this section is to recall certain results concerning $(\varphi, \Gamma)$-modules over the Robba ring $\mathcal{R}_L$. Let $L_0$ be the maximal unramified extension contained in $L$. Let $L'_0$ be the maximal unramified extension contained in $L_\infty := L(\mu_{p^\infty})$ and $L' = L \cdot L'_0$. Let $e_L := [L(\mu_{p^\infty}) : L_0(\mu_{p^\infty})] = [\Gamma_{\mathbb{Q}_p} : \Gamma_L]$, where $\Gamma_L := \mathrm{Gal}(L_\infty/L)$. We define

$$\mathbf{B}^{\dagger,r}_{L,\mathrm{rig}} = \Bigg\{ f = \sum_{n\in\mathbb{Z}} a_n \pi_L^n | a_n \in L'_0, \text{ such that } f(X) = \sum_{n\in\mathbb{Z}} a_n X^n \text{ is holomorphic on } p^{-\frac{1}{e_L r}} \leq |X|_p < 1 \Bigg\},$$

$$\mathbf{B}^{\dagger}_{L,\mathrm{rig}} := \bigcup_r \mathbf{B}^{\dagger,r}_{L,\mathrm{rig}},$$

where $\pi_L$ is a certain uniformizer coming from the theory of field of norms. Note that $\mathbf{B}^{\dagger}_{L,\mathrm{rig}}$ is classically called the Robba ring of $L'_0$. For sake of notation, we shall denote write $\mathcal{R}_L := \mathbf{B}^{\dagger}_{L,\mathrm{rig}}$. We hope that this will cause no confusion in what follows.

We have an action of $\varphi$ on $\mathcal{R}_L$. If $L = L_0$, there is no ambiguity and we have:

$$\varphi(\pi_L) = (1+\pi_L)^p - 1, \quad \varphi(a_n) = \varphi_{L'_0}(a_n).$$

Otherwise the action on $\pi_L$ is more complicated.

Similarly, we have a $\Gamma_L$-action. If $L = L_0$ we have

$$\gamma(\pi_L) = (1+\pi_L)^{\chi_{\mathrm{cycl}}(\gamma)} - 1,$$

where $\chi_{\mathrm{cycl}}$ is the cyclotomic character. If $L$ is ramified we also have an action of $\Gamma_L$ on the coefficients given by

$$\gamma(a_n) = \sigma_\gamma(a_n)$$

where $\sigma_\gamma$ is the image of $\gamma$ via

$$\Gamma_L \to \Gamma_L/\Gamma_{L'} \stackrel{\cong}{\to} \mathrm{Gal}(L'_0/L_0).$$

If $a_n$ is fixed by $\varphi$ and $\Gamma_L$, then is it in $\mathbb{Q}_p$. We have $\mathrm{rk}_{\mathcal{R}_{\mathbb{Q}_p}} \mathcal{R}_L = [L_\infty : \mathbb{Q}_{p,\infty}]$.

Let $\delta : L^\times \to E^\times$ be a continuous character. Let $\mathcal{R}_L(\delta)$ be the rank one $(\varphi, \Gamma_L)$-module defined as follows; fix a uniformizer $\varpi_L$ of $L$ and write $\delta = \delta_0\delta_1$ with $\delta_0|_{\mathcal{O}_L^\times} := \delta|_{\mathcal{O}_L^\times}$, $\delta_0(\varpi_L) := 1$ and $\delta_1$ is trivial on $\mathcal{O}_L^\times$ and $\delta_1(\varpi_L) := \delta(\varpi_L)$. As $\delta_0$ is a unitary character, it defines by class field theory a unique one dimensional Galois representation $\tilde{\delta}_0$. Fontaine's theorem on the equivalence of category between $(\varphi, \Gamma_L)$-modules and Galois representations [Fon90] gives

us a one dimensional $(\varphi, \Gamma_L)$-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(\tilde{\delta}_0)$.
We define $\mathcal{R}_L(\delta_1) := \mathcal{R}_L \otimes_{\mathbb{Q}_p} E e_{\delta_1}$ so that $\varphi^{f_L}(e_{\delta_1}) = \delta_1(\varpi_L)e_{\delta_1}$ (here $f_L$ is the degree of $L_0$ over $\mathbb{Q}_p$), $\gamma(e_{\delta_1}) = e_\delta$ and $\varphi$ does not act on the $E$-coefficient. Finally, we define $\mathcal{R}_L(\delta) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(\tilde{\delta}_0) \otimes_{\mathcal{R}_L} \mathcal{R}_L(\delta_1)$.

We now classify the cohomology of such a $(\varphi, \Gamma_L)$-modules. It will be useful to calculate it explicitly in terms of $C_{\varphi,\gamma}$-complexes [Ben11, §1.1.5]. We fix then a generator $\gamma_L$ of $\Gamma_L$; if clear from the context, we shall drop the subscript $_L$ and write simply $\gamma$.

Proposition 2.1. *We have $H^0(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ for all $\tau$; in this case we have $H^0(\mathcal{R}_L(\delta)) \cong E$. We shall denote its basis by $t^{-\underline{m}} \otimes e_\delta$, where*

$$t^{-\underline{m}} = (t^{-m_\tau}) \in \prod_\tau B^+_{\mathrm{dR}} \otimes_{L,\tau} E.$$

*If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then*

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

*If $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then*

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

*Otherwise*

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p].$$

*We have $H^2(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$; in this case we have $H^2(\mathcal{R}_L(\delta)) \cong E$.*

Note that when we choose $t^{-\underline{m}}$ as a basis we are implicitly using the fact that we can embed certain sub-rings of $\mathcal{R}_L$ into $B^+_{\mathrm{dR}}$ (see [Ben11, §1.2.1]).

*Proof.* The same results is stated in [Nak09, Proposition 2.14, 2.15, Lemma 2.16] for $E-B$-pairs, but the proof for $(\varphi, \Gamma)$-modules is the same.
Recall that have a canonical duality [Liu08] given by cup product

$$H^i(D) \times H^{2-i}(D^*(\chi_{\mathrm{cycl}})) \to H^2(\chi_{\mathrm{cycl}}).$$

The last fact is then a direct consequence. □

This allows us to define a canonical basis of $H^2(\mathcal{R}_L(|\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}))$.
We define $H^1_{\mathrm{f}}(D)$ as the $H^1$ of the complex

$$\mathcal{D}_{\mathrm{cris}}(D) \to t_D \oplus \mathcal{D}_{\mathrm{cris}}(D)$$

and we have immediately [Nak09, Proposition 2.7]

$$\dim_E H^1_{\mathrm{f}}(D) = \dim_E(H^0(D)) + \dim_E t_D. \tag{2.2}$$

Hence

Lemma 2.3. *If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then*

$$\dim_E H^1_{\mathrm{f}}(\mathcal{R}_L(\delta)) = 1.$$

*If $\delta(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then*

$$\dim_E H^1_{\mathrm{f}}(\mathcal{R}_L(\delta)) = d.$$

Proposition 2.4. *Let $D$ be a semi-stable $(\varphi, \Gamma)$-module over $\mathcal{R}_L$ with non-negative Hodge–Tate weight. Suppose that $\mathcal{D}_{\mathrm{st}}(D) = \mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$. Then $D$ is crystalline,*

$$D \cong \oplus \mathcal{R}_L(\delta_i)$$

*with $\delta_i(z) = \prod_\tau \tau(z)^{m_{i,\tau}}$, $m_{i,\tau} \leq 0$ and $\mathcal{D}_{\mathrm{st}}(D) = \mathcal{D}_{\mathrm{cris}}(D) = H^0(D)$.*

*Proof.* We follow closely the proof [Ben11, Proposition 1.5.8]. As $N\varphi = p\varphi N$ we obtain immediately that $N = 0$, hence $D$ is crystalline.
Let $r$ be the rank of $D$ over $\mathcal{R}_L$. We write the Hodge–Tate weight as $(\underline{m_i})_{i=1}^r$ where $\underline{m_i} = (m_{i,\tau})_\tau$.
We prove the proposition by induction; the case $r = 1$ is easy.
If $D$ is not split, for $r = 2$, we can suppose, as $D$ is de Rham, that for each $\tau$ we have $-m_{1,\tau} \leq -m_{2,\tau}$, hence $\underline{m_1} = 0$ by twisting. Let $\delta$ be defined by $\prod_\tau \tau(z)^{m_\tau}$. So we have an extension of $\mathcal{R}_L(\delta)$ by $\mathcal{R}_L$. Let $d_2$ be a lift to $D$ of a basis of $\mathcal{R}_L$. As $\varphi = 1$ we have $\varphi d_2 = d_2$. As the extension is crystalline we know that $\gamma$ acts trivially too, hence the extension splits.
Suppose now $r > 2$. Take $v$ in the $\mathrm{Fil}^{-\underline{m_0}}\mathcal{D}_{\mathrm{st}}(D)$, the smallest filtered piece of $\mathcal{D}_{\mathrm{st}}(D)$. We can associate to it $\mathcal{R}_L(\delta)$, where $\delta(z) = \prod_\tau \tau(z)^{m_{0,\tau}}$. We have

$$0 \to \mathcal{R}_L(\delta) \to D \to D' \to 0.$$

By inductive hypothesis $D' \cong \oplus_{i=1}^{d-1} \mathcal{R}_L(\delta_i)$. We can write

$$\mathrm{Ext}(D', \mathcal{R}_L(\delta)) = \oplus_{i=1}^{d-1} \mathrm{Ext}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta))$$

and we are reduced to the case $r = 2$ which has already been dealt. □

We now want to calculate $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$ for $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$. We recall the following lemma [Ben11, Lemma 1.4.3]

Lemma 2.5. *The extension $\mathrm{cl}(a, b)$ in $H^1(\mathcal{R}_L(\delta))$ corresponding to the couple $(a, b)$ is crystalline if and only if the equation $(1 - \gamma)x = b$ has a solution in $\mathcal{R}_L(\delta)\left[\frac{1}{t}\right]$*

The following proposition in an immediate consequence of the above lemma [Ben11, Theorem 1.5.7 (i)] (see also the construction of [Nak09] at page 900)

Proposition 2.6. *Let $e_\delta$ be a basis for $\mathcal{R}_L(\delta)$. Then $x_{\underline{m}} = \mathrm{cl}(t^{-\underline{m}}, 0)e_\delta$ is a basis of $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$.*

REMARK 2.7. *If $\delta$ is the trivial character then $x_0$ corresponds (via class field theory) to the unramified $\mathbb{Z}_p$-extension of* $\mathrm{Hom}(G_L, E^\times) \cong H^1(G_L, E)$.

We now have to cut out another "canonical" one-dimensional subspace in $H^1(\mathcal{R}_L(\delta))$ which trivially intersects $H^1_{\mathrm{f}}(\mathcal{R}_L(\delta))$ (and reduces to the cyclotomic $\mathbb{Z}_p$-extension in the sense of the previous remark).
We recall that for $L = \mathbb{Q}_p$ Benois has defined in [Ben11, Proposition 1.5.9] a canonical complement $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$ of $H^1_{\mathrm{f}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$ inside $H^1(\mathcal{R}_{\mathbb{Q}_p}(z^m))$. He has also defined a canonical basis $y_m$ of $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^m))$.
We hence define the extension

$$y_{\underline{m}} := \frac{1}{e_L} \log_p(\chi_{\mathrm{cycl}}(\gamma_L))\mathrm{cl}(0, t^{-\underline{m}})e_\delta.$$

When $L = \mathbb{Q}_p$, this is the same element $y_m$ as defined by Benois.
We can calculate cohomology of induced $(\varphi, \Gamma_{\mathbb{Q}_p})$-module. Indeed, we now consider two $p$-adic fields $K$ and $L$, $L$ a finite extension of $K$. The main reference for this part is [Liu08, §2.2]. Let $D$ be a $(\varphi, \Gamma_L)$-module, we define

$$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D) = \left\{ f : \Gamma_K \to D | f(hg) = hf(g) \ \forall h \in \Gamma_L \right\}.$$

It has rank $[L : K]\mathrm{rk}_{\mathcal{R}_L}(D)$ over $\mathcal{R}_K$; indeed $\mathcal{R}_L$ is a $\mathcal{R}_K$-module of rank $[L : K]/|\Gamma_K/\Gamma_L| = [L'_0 : K'_0]$. (The unramified part of $L/K$ plus the ramified part which is disjoint by $K_\infty$. See after [Liu08, Theorem 2.2].) If $D$ comes from a $G_L$-representation $V$ we have

$$\mathbf{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{G_L}^{G_K}(V)) = \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)).$$

We have then the equivalent of Shapiro's lemma

$$H^i(D) \cong H^i(\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)).$$

Moreover, the aforementioned duality for $(\varphi, \Gamma)$-modules is compatible with induction [Liu08, Theorem 2.2].
If $D \cong \mathcal{R}_L(\delta)$ is free of rank one, then we have an explicit description of $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$. Let $e_\infty = |\Gamma_K/\Gamma_L|$, we write $\left\{\omega^i\right\}_{i=0}^{e_\infty - 1}$ for $(\Gamma_K/\Gamma_L)^\wedge$. The $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$ is the $\mathcal{R}_L$-span of $f_i$, where $f_i(g) = \omega^i(g)\delta(\chi_{\mathrm{cycl}}(g))e_\delta$.

We go back to the previous setting where $K = \mathbb{Q}_p$ (hence $e_\infty = e_L$). Suppose $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ and let $D = \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\mathbb{Q}_p}}(\mathcal{R}_L(\delta))$. Note that in this case $\mathcal{D}_{\mathrm{st}}(D) \cong E^{f_L}$ is a filtered $\varphi$-module where $\varphi$ acts as a permutation of length $f_L$. To $\mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$ corresponds (by Proposition 2.4 over $\mathbb{Q}_p$) a rank-one $(\varphi, \Gamma)$-module $\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})$, for $m_0$ the minimum of the $m_\tau$'s (hence $-m_0$ is the greatest Hodge–Tate weight of $D$).
The identifications

$$H^0(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})) = \mathcal{D}_{\mathrm{st}}(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))^{\varphi=1} = \mathcal{D}_{\mathrm{st}}(D)^{\varphi=1} = H^0(D) = H^0(\mathcal{R}_L(\delta))$$

induces (via the maps cl(0, ) and cl( ,0)) an injection

$$H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})) \hookrightarrow H^1(\mathrm{Ind}_L^{\mathbb{Q}_p}(\mathcal{R}_L(\delta))). \tag{2.8}$$

which sends $x_{m_0}$ to $x_{\underline{m}}$ and $y_{m_0}$ to $y_{\underline{m}}$.

We consider a $(\varphi,\Gamma)$-module $M$ which sits in the non-split exact sequence

$$0 \to M_0 := \oplus_{i=1}^r \mathcal{R}_L(\delta_i) \to M \to M_1 := \oplus_{i=1}^r \mathcal{R}_L(\delta_i') \to 0, \tag{2.9}$$

where $\delta_i(z) = |\mathrm{N}_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{m_{i,\tau}}$ with $m_{i,\tau} \geq 1$ for all $\tau$ and $\delta_i'(z) = \prod_\tau \tau(z)^{k_{i,\tau}}$ with $k_{i,\tau} \leq 0$ for all $\tau$. We say that $M$ is of type $U_{m,k}$ if the image of $M$ in $H^1(M_1)$ is crystalline.

Proposition 2.10. *Suppose that $M$ as above is not of type $U_{m,k}$. Then we have $\dim_E(H^1(M)) = 2[L:\mathbb{Q}_p]r$ and $H^2(M) = H^0(M) = 0$. Moreover, if we write*

$$0 \to H^0(M_1) \xrightarrow{\Delta_0} H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \xrightarrow{\Delta_1} H^2(M_0) \to 0$$

*we have $H^1(M_0) = \mathrm{Im}(\Delta_1) \oplus H^1_{\mathrm{f}}(M_0)$, $\mathrm{Im}(f_1) = H^1_{\mathrm{f}}(M)$ and $H^1(M_1) = \mathrm{Im}(g_1) \oplus H^1_{\mathrm{f}}(M_1)$.*

*Proof.* We have $H^0(M) = 0$ by definition of $M$. Note that $M^*(\chi_{\mathrm{cycl}})$ is a module of the same type, hence $H^2(M) = H^0(M^*(\chi_{\mathrm{cycl}})) = 0$. We can write

$$0 \to H^0(M_1) \to H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \to H^2(M_0) \to 0$$

and conclude by Proposition 2.1.
Note that $\dim_E H^1_{\mathrm{f}}(M) = rd$ by (2.2).
By hypothesis, we have that $\mathrm{Im}(\Delta_1) \cap H^1_{\mathrm{f}}(M_0) = 0$ and the first statement follows from dimension counting.
The third statement follows from duality.
For the second statement $H^1_{\mathrm{f}}(M_0)$ injects into $H^1_{\mathrm{f}}(M)$. As both have the same dimension, we conclude. ☐

We give the following key lemma for the definition of the $\mathcal{L}$-invariant

Lemma 2.11. *The intersection of $T := \mathrm{Im}(H^1(M))$ and $\mathrm{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})))$ in $\mathrm{Im}(H^1(M_1))$ is one dimensional.*

*Proof.* The intersection is non-empty as the sum of their dimension is $d+2$ and $\mathrm{Im}(H^1(M_1))$ has dimension $d+1$. We have that $H^1_{\mathrm{f}}(M_1)$ is contained in the image of $H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$ via (2.8) and by the previous proposition the former is not in the image of $g_1$ and we are done. ☐

In particular, we deduce that $T$ surjects into the image of $H^1_{\mathrm{c}}(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$.

## 3 $\mathcal{L}$-invariant over number fields

Let $F$ be a number field. We consider a global Galois representation

$$V : G_F \to \mathrm{GL}_n(E)$$

where $E$ is $p$-adic field. We suppose that it is unramified outside a finite number of places $S$ containing all the $p$-adic places. We suppose moreover that it is semistable at all places above $p$ (*i.e.* $\mathcal{D}_{\mathrm{st}}(V_{|F_{\mathfrak{p}}})$ is of rank $n$ over $F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$, being $F_{\mathfrak{p}}^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Q}_p$ contained in $F_{\mathfrak{p}}^{\mathrm{ur}}$).
In this section we generalize Greenberg–Benois definition of the $\mathcal{L}$-invariant for $V$ whenever it presents trivial zeros. Note that we do not require $p$ split or unramified in $F$.
Let $t$ be the number of trivial zeros. The classical definition by Greenberg [Gre94] describes the $\mathcal{L}$-invariant as the "slope" of a certain $t$-dimension subspace of $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p^t)$ which is a $2t$-dimensional space with a canonical basis given by $\mathrm{ord}_p$ and $\log_p$.
In our setting, the main obstacle is that the cohomology of the $(\varphi, \Gamma)$-module $\mathcal{R}_{F_{\mathfrak{p}}}$ is no longer two-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida's work for symmetric powers of Hilbert forms [Hid07], we consider the image of $H^1(\mathcal{R}_{\mathbb{Q}_p})$ inside $H^1(\mathcal{R}_{F_{\mathfrak{p}}})$.
If $t$ denotes the number of expected trivial zeros, we show that we can define, similarly to [Ben11], a $t$-dimensional subspace of $H^1(G_{F,S}, V)$ whose image in $H^1(\mathcal{R}_{\mathbb{Q}_p})$ has trivial intersection with the crystalline cocycle. This is enough to define the $\mathcal{L}$-invariant; we further check that our definition is compatible with Benois'.

### 3.1 Definition of the $\mathcal{L}$-invariant

We define local cohomological conditions $L_v$ in order to define a Selmer group; we denote by $G_v$ a fixed decomposition group at $v$ in $G_{F,S}$ and by $I_v$ the inertia. For $v \nmid p$ we define

$$L_v := \mathrm{Ker}\left(H^1(G_v, V) \to H^1(I_v, V)\right).$$

If $v \mid p$ we define

$$L_v := H^1_{\mathrm{f}}(F_v, V) = \mathrm{Ker}(H^1(G_v, V) \to H^1(G_v, V \otimes_E \mathbf{B}_{\mathrm{cris}})).$$

If $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ denotes the $(\varphi, \Gamma)$-module associated with $V$ we also have $L_p = H^1_{\mathrm{f}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. We define then the Bloch-Kato Selmer group

$$H^1_{\mathrm{f}}(V) := \mathrm{Ker}\left(H^1(G_{F,S}, V) \to \prod_{v \in S} \frac{H^1(G_v, V)}{L_v}\right).$$

We make the following additional hypotheses:

C1) $H^1_{\mathrm{f}}(V) = H^1_{\mathrm{f}}(V^*(1)) = 0$,

C2) $H^0(G_{F,S}, V) = H^0(G_{F,S}, V^*(1)) = 0$,

C3) $\varphi$ on $\mathbf{D}_{\mathrm{st}}(V_{|F_{\mathfrak{p}}})$ is semisimple at $1 \in F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$ and $p^{-1} \in F_{\mathfrak{p}}^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} E$ for all $\mathfrak{p} \mid p$,

C4) $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{|F_{\mathfrak{p}}})$ has no saturated sub-quotient of type $U_{m,k}$ for all $\mathfrak{p} \mid p$.

Note that if $V$ satisfies the previous four conditions, so does $V^*(1)$.
The first two conditions tell us that the Poitou–Tate sequence reduces to

$$H^1(G_{F,S}, V) \cong \bigoplus_{v \in S} \frac{H^1(G_v, V)}{H^1_{\mathrm{f}}(F_v, V)}. \tag{3.1}$$

For each $\mathfrak{p} \mid p$ we denote by $V_{\mathfrak{p}}$ the restriction to $G_{F_{\mathfrak{p}}}$ of $V$. We choose a regular sub-module $D_{\mathfrak{p}} \subset \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ and define a filtration $(D_{\mathfrak{p},i})$ of $\mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$.

$$D_{\mathfrak{p},i} = \begin{cases} 0 & i = -2, \\ (1 - p^{-1}\varphi)D_{\mathfrak{p}} + N(D_{\mathfrak{p}}^{\varphi=1}) & i = -1, \\ D_{\mathfrak{p}} & i = 0, \\ D_{\mathfrak{p}} + \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})^{\varphi=1} \cap N^{-1}(D_{\mathfrak{p}}^{\varphi=p^{-1}}) & i = 1, \\ \mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}}) & i = 2. \end{cases} \tag{3.2}$$

We have that $D_{\mathfrak{p},1}/D_{\mathfrak{p},-1}$ coincides with the eigenvectors of $\varphi$ on $\mathbf{D}_{\mathrm{st}}(V_{\mathfrak{p}})$ of eigenvalue 1 (resp. $p^{-1}$) and which are in the kernel (resp. in the image) of $N$.
This filtration induces a filtration on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})$. Namely, we pose

$$F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}) \cap (D_{\mathfrak{p},i} \otimes \mathcal{R}_{F_{\mathfrak{p}},\log}[t^{-1}]).$$

We define

$$W_{\mathfrak{p}} := F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}}).$$

The same proof as [Ben11, Proposition 2.1.7] tells us that we can find a unique decomposition

$$W_{\mathfrak{p}} = W_{\mathfrak{p},0} \bigoplus W_{\mathfrak{p},1} \bigoplus M_{\mathfrak{p}}$$

such that $t_{\mathfrak{p},0} = \dim_E H^0(W_{\mathfrak{p}}^*(1)) = \mathrm{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_{\mathfrak{p},0}$, $t_{\mathfrak{p},1} = \dim_E H^0(W_{\mathfrak{p}}) = \mathrm{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_{\mathfrak{p},1}$ and $M_{\mathfrak{p}}$ sits in a sequence

$$0 \to M_{\mathfrak{p},0} \xrightarrow{f} M_{\mathfrak{p}} \xrightarrow{g} M_{\mathfrak{p},1} \to 0$$

such that $\mathrm{gr}^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})) = W_{\mathfrak{p},0} \oplus M_{\mathfrak{p},0}$ and $\mathrm{gr}^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{p}})) = W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}$. Moreover $M_{\mathfrak{p}}$ is non-split; by construction we have $H^0(M_{\mathfrak{p}}) = H^2(M_{\mathfrak{p}}) = 0$ and if the exact sequence were split we would have $H^0(M_{\mathfrak{p}}) \neq 0$ and $H^2(M_{\mathfrak{p}}) \neq 0$.

We can prove exactly in the same way as [Ben11, Proposition 2.1.7 (i)] that C4 implies $\mathrm{rank}_{\mathcal{R}_{F_\mathfrak{p}}} M_{\mathfrak{p},1} = \mathrm{rank}_{\mathcal{R}_{F_\mathfrak{p}}} M_{\mathfrak{p},0}$.
In order to define the $\mathcal{L}$-invariant we shall follow verbatim Benois' construction. For sake of notation, we write $\mathbf{D}^\dagger_\mathfrak{p}$ for $\mathbf{D}^\dagger_{\mathrm{rig}}(V_\mathfrak{p})$. We obtain from [Ben11, Proposition 1.4.4 (i)]

$$H^1_{\mathrm{f}}(\mathrm{gr}^2(\mathbf{D}^\dagger_\mathfrak{p})) = H^0(\mathrm{gr}^2(\mathbf{D}^\dagger_\mathfrak{p})) = 0.$$

We deduce the following isomorphism

$$H^1_{\mathrm{f}}(F_1\mathbf{D}^\dagger_\mathfrak{p}) = H^1_{\mathrm{f}}(\mathbf{D}^\dagger_\mathfrak{p}) = H^1_{\mathrm{f}}(F_\mathfrak{p}, V). \tag{3.3}$$

As the Hodge–Tate weights of $F_{-1}\mathbf{D}^\dagger_\mathfrak{p}$ are $< 0$, we obtain from [Ben11, Proposition 1.5.3 (i)] and Poiteau–Tate duality $H^2(F_{-1}\mathbf{D}^\dagger_\mathfrak{p}) = 0$. Using the long exact sequence associated with

$$0 \to F_{-1}\mathbf{D}^\dagger_\mathfrak{p} \to F_1\mathbf{D}^\dagger_\mathfrak{p} \to W_\mathfrak{p} \to 0$$

we see that

$$\frac{H^1(W_\mathfrak{p})}{H^1_{\mathrm{f}}(W_\mathfrak{p})} = \frac{H^1(F_{-1}\mathbf{D}^\dagger_\mathfrak{p})}{H^1_{\mathrm{f}}(F_\mathfrak{p}, V)}.$$

As Greenberg and Benois do, we make the extra assumption that

C5) $W_{\mathfrak{p},0} = 0$ for all $\mathfrak{p} \mid p$.

Using Proposition 2.4 we can write $\mathrm{gr}^1(\mathbf{D}^\dagger_\mathfrak{p}) = \oplus_{i=1}^{t_{\mathfrak{p},1}+r_\mathfrak{p}} \mathcal{R}_{F_\mathfrak{p}}(\prod_{\tau_\mathfrak{p}} \tau_\mathfrak{p}(z)^{m_{i,\tau_\mathfrak{p}}})$. We define the $2(t_{\mathfrak{p},1} + r_\mathfrak{p})$-dimensional subspace obtained as the image of

$$\mathrm{Ind}_\mathfrak{p} := \left\{ \sum_{i=0}^{t_{\mathfrak{p},1}+r_\mathfrak{p}} Ex_{\underline{m_i}} + Ey_{\underline{m_i}} \right\} \subset H^1(\mathrm{gr}^1(\mathbf{D}^\dagger_\mathfrak{p})). \tag{3.4}$$

We define

$$T_\mathfrak{p} = (H^1(F_1\mathbf{D}^\dagger_\mathfrak{p}) \cap \mathrm{Ind}_\mathfrak{p})/H^1_{\mathrm{f}}(F_\mathfrak{p}, V).$$

It has dimension $t_{\mathfrak{p},1} + r_\mathfrak{p}$.
Write $t = \sum_\mathfrak{p} t_{\mathfrak{p},1} + r_\mathfrak{p}$. We have a unique $t$-dimensional subspace $H^1(D, V)$ of $H^1(G_{F,S}, V)$ projecting via (3.1) to $\oplus_\mathfrak{p} T_\mathfrak{p}$. We have an isomorphism (cfr. [Ben11, Proposition 1.5.9])

$$\mathrm{Ind}_\mathfrak{p} \cong \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}) \oplus \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}) \cong E^{t_{\mathfrak{p},1}+r_\mathfrak{p}} \oplus E^{t_{\mathfrak{p},1}+r_\mathfrak{p}},$$

where the first (resp. the second) factor is identified with $E^{t_{\mathfrak{p},1}+r_\mathfrak{p}}$ via the basis $\left\{x_{\underline{m_i}}\right\}$ (resp. $\left\{y_{\underline{m_i}}\right\}$). We shall denote the two projections by $\iota_{\mathrm{f},\mathfrak{p}}$ and $\iota_{\mathrm{c},\mathfrak{p}}$.
We denote by $\iota_{\mathrm{f}}$ (resp. $\iota_{\mathrm{c}}$) the projection of $H^1(D, V)$ to $E^t$ via $\oplus\iota_{\mathrm{f},\mathfrak{p}}$ (resp. $\oplus\iota_{\mathrm{c},\mathfrak{p}}$). By the remark after Lemma 2.11 and the definition of $T_\mathfrak{p}$, we have that $\iota_{\mathrm{c}}$ is surjective.
Summing up, we can give the following definition;

Definition 3.5. *The $\mathcal{L}$-invariant of the pair $(V,D)$ is*

$$\mathcal{L}(V,D) := \det(\iota_{\mathrm{f}} \circ \iota_{\mathrm{c}}^{-1}),$$

*where the determinant is calculated w.r.t. the basis $(x_{\underline{m_i}}, y_{\underline{m_j}})_{1 \leq i,j \leq t}$.*

Remark 3.6. *There is no a priori reason for which $\mathcal{L}(V,D)$ should be non-zero.*

In the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ we see from the description of $H^1(F_1\mathbf{D}_{\mathfrak{p}}^{\dagger})$ that the space $T_{\mathfrak{p}}$ depends only on $V_{|F_{\mathfrak{p}}}$ exactly as in the classical case.

## 3.2 Comparison with Benois' definition

Fix a global field $F$ and let $\{\mathfrak{p}\}$ be the set of primes above $p$.
Let $G_p$ denote a fixed decomposition group at $p$ in $G_{\mathbb{Q}}$ and let $\mathfrak{p}_0$ be the corresponding place of $F$. Let $G_{\mathfrak{p}_0,F}$ be the decomposition group at $\mathfrak{p}_0$ in $G_F$. For each other place $\mathfrak{p}$ above $p$ in $F$, we have $G_{\mathfrak{p}} = \tau_{\mathfrak{p}} G_p \tau_{\mathfrak{p}}^{-1}$. We shall denote by $G_{\mathfrak{p},F}$ the corresponding decomposition group in $G_F$. Consider a $p$-adic Galois representation

$$V : G_F \to \mathrm{GL}_n(E).$$

We shall suppose $E$ big enough to contain the Galois closure of $F_{\mathfrak{p}}$, for all $\mathfrak{p}$. As before, we suppose $V$ semistable at all primes above $p$. We have then

$$\mathrm{Ind}_F^{\mathbb{Q}}(V)_{|G_p} \cong \bigoplus_{\mathfrak{p}} \tau_{\mathfrak{p}}^{-1} \mathrm{Ind}_{G_{\mathfrak{p},F}}^{G_{\mathfrak{p}}} V_{|G_{\mathfrak{p},F}}$$

where $\tau_{\mathfrak{p}} \in G_p \setminus \mathrm{Hom}(F, \overline{\mathbb{Q}})$.
Consider the $(\varphi,\Gamma)$-module

$$\mathbf{D}^{\dagger} := \mathbf{D}_{\mathrm{rig}}^{\dagger}\left(\mathrm{Ind}_F^{\mathbb{Q}} V\right).$$

We let $D$ be the regular $(\varphi,N)$-module of $\mathcal{D}_{\mathrm{st}}(\mathbf{D}^{\dagger})$ induced by $\{D_{\mathfrak{p}}\}_{\mathfrak{p}}$. As before we have a filtration $(F_i\mathbf{D}^{\dagger})$ on $\mathbf{D}^{\dagger}$ induced by the filtration on $D$. We denote by $W$ the quotient $F_1\mathbf{D}^{\dagger}/F_{-1}\mathbf{D}^{\dagger}$. Note that it is semistable. We write $W = W_0 \oplus M \oplus W_1$. We suppose that $V$ satisfies the hypotheses C1-C5 of the previous section.

Lemma 3.7. *Let $M$ be as in (2.9). We have*

$$0 \to \mathrm{Ind}(M_0) \to \mathrm{Ind}(M) \to \mathrm{Ind}(M_1) \to 0.$$

We can now compare our definition of $\mathcal{L}$-invariant with Benois'.

PROPOSITION 3.8. *We have a commutative diagram*

$$\begin{tikzcd} H^1(G_{\mathbb{Q},S},\mathrm{Ind}(V)) \arrow[d] & H^1(\mathrm{Ind}(V),\mathrm{Ind}(D)) \arrow[l] \arrow[r, "\mathrm{Res}_p"] \arrow[d] & \frac{H^1(F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)))}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}(V))} = \frac{H^1(F_{-1}\mathbf{D}^\dagger)}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}(V))}. \arrow[d, "\iota_p"] \\ H^1(G_{F,S},V) & H^1(V,D) \arrow[l] \arrow[r, "\oplus_{\mathfrak{p}}\mathrm{Res}_{\mathfrak{p}}"] & \prod_{\mathfrak{p}} T_{\mathfrak{p}} \end{tikzcd}$$

*whose vertical arrows are isomorphism.*

*Proof.* We follow [Hid06, §3.4.4]. Recall that we wrote $\mathbf{D}^\dagger_{\mathfrak{p}}$ for $\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}})$. Shapiro's lemma tells us that

$$\frac{H^1(G_p,\mathrm{Ind}_F^{\mathbb{Q}}V)}{H^1_{\mathrm{f}}(G_p,\mathrm{Ind}_F^{\mathbb{Q}}V)} \stackrel{\iota_p}{\cong} \bigoplus_{\mathfrak{p}} \frac{H^1(\mathbf{D}^\dagger_{\mathfrak{p}})}{H^1_{\mathrm{f}}(\mathbf{D}^\dagger_{\mathfrak{p}})}.$$

We are left to show that $H^1(F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)))$ is sent by $\iota_p$ into $(H^1(F_1\mathbf{D}^\dagger_{\mathfrak{p}})\cap\mathrm{Inv}_{\mathfrak{p}})$ and we shall conclude by dimension counting.
We have then an injection

$$F_1\mathbf{D}^\dagger(\mathrm{Ind}(V)) \hookrightarrow \oplus_{\mathfrak{p}}\mathrm{Ind}(F_1(\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}}))).$$

Then clearly the image of $\iota_p$ lands in $H^1(F_1\mathbf{D}^\dagger_{\mathfrak{p}})$. But we have also the injection

$$\mathrm{gr}^1(\mathbf{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}V)) \hookrightarrow \oplus_{\mathfrak{p}}\mathrm{Ind}(\mathrm{gr}^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{p}})))$$

which by (2.8) tells us that the image of $\iota_p$ lands in $\mathrm{Inv}_{\mathfrak{p}}$ and we are done. □

COROLLARY 3.9. *We have* $\mathcal{L}(V,D) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(V),\mathrm{Ind}_F^{\mathbb{Q}}(D))$.

## 4 SIEGEL–HILBERT MODULAR FORMS, THE LOCAL CASE

The calculation of the $\mathcal{L}$-invariant requires to produce explicit cocycles in $H^1(D,V)$; when $V$ appears in $\mathrm{Ad}(V')$ for a certain representation $V'$ we can sometimes use the method of Mazur and Tilouine [MT90] to produce these cocycles. This has been done in many case for the symmetric square [Hid04, Mok12] and generalized to symmetric powers of the Galois representation associated with Hilbert modular forms in [Hid07, HJ13]. The main limit of this approach is that for most representations $V$ it is computationally heavy to obtain $V$ as the quotient of an adjoint representation.
In the case $\mathbf{D}^\dagger_{\mathrm{rig}}(V) = W = M$ the situation is way simpler; if $t=1$ it has been proved in [Ben10] that to produce the cocycle in $H^1(V,D)$ it is enough to find deformations of $V|_{\mathbb{Q}_p}$.
We shall generalized the method of Benois to our situation in the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. This will allow us to give a complete formula for the $\mathcal{L}$-invariant of the Galois representations associated with a Siegel–Hilbert modular form which is Steinberg at all primes above $p$.

### 4.1 The case $t_{\mathfrak{p}} = r_{\mathfrak{p}} = 1$

We now suppose that $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. For sake of notation, in this section we shall drop the index $_{\mathfrak{p}}$. In particular, in this subsection $F = F_{\mathfrak{p}}$.
All that we have to do is to check that the calculation of [Ben11, Theorem 2] works in our setting.
We write as before

$$0 \to M_0 \to M \to M_1 \to 0$$

and, only in this subsection, we shall write $\delta$ for the character defining $M_0$ and $\psi$ for the character defining $M_1$. We suppose $\delta = \delta' \circ \mathrm{N}_{F/\mathbb{Q}_p}$ for $\delta'(z) = |z|_p z^k$ with $k \geq 1$ and $\psi = \psi' \circ \mathrm{N}_{F/\mathbb{Q}_p}$ with $\psi'(z) = z^m$ with $m \leq 0$. We consider an infinitesimal deformation

$$0 \to M_{0,A} \to M_A \to M_{1,A} \to 0,$$

over $A = E[T]/(T^2)$. We suppose that $M_{0,A}$ (resp. $M_{1,A}$) is an infinitesimal deformation of $M_0$ (resp. $M_1$) which still factors through $\mathrm{N}_{F/\mathbb{Q}_p}$.
We shall write $\delta_A$, $\delta'_A$, $\psi_A$ and $\psi'_A$ for the corresponding one-dimensional character.

Theorem 4.1. *Suppose that* $\mathrm{d}\log_p(\delta'_A {\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})) \neq 0$*; then*

$$\mathcal{L}(M, M_0) = -\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})) \frac{f^{-1}\mathrm{d}\log_p(\delta_A \psi_A^{-1})(\varpi)}{\mathrm{d}\log_p(\delta'_A {\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))}.$$

*Proof.* Recall the definition of Ind in (3.4). We have a vector $v = ax_m + by_m$ in $H^1(F_1\mathbf{D}^\dagger) \cap \mathrm{Ind}$. By definition $\mathcal{L}(M) = ab^{-1}$. The extension $M_{j,A}$ provides us with connecting morphisms $B_j^i : H^i(M_j) \to H^{i+1}(M_j)$. We have by definition

$$\begin{aligned} B_1^0(t^{-m}e_m) =& \mathrm{cl}(\mathrm{dlog}(\psi'_A)(p)t^{-m}e_m, \mathrm{dlog}(\psi'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))t^{-m}e_m) \\ =& \mathrm{dlog}(\delta'_A)(p)x_m + \mathrm{dlog}(\delta'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))y_m. \end{aligned} \tag{4.2}$$

As in [Ben10, §3.2] we consider the dual extension

$$0 \to M_1^*(\chi_{\mathrm{cycl}}) \to M^*(\chi_{\mathrm{cycl}}) \to M_0^*(\chi_{\mathrm{cycl}}) \to 0,$$

and we shall denote with a $^*$ the corresponding map in the long exact sequence of cohomology.
We have hence $\ker(\Delta_1) \perp \mathrm{Im}(\Delta_0^*)$ under duality, and a map

$$H^1(M_1^*) \to H^1(\mathcal{R}_{\mathbb{Q}_p}(|z|z^{1-m})).$$

By duality again, we deduce that the image of $\Delta_0^*$ inside the target of the above arrow is

$$a\alpha_{1-m} + b\beta_{1-m},$$

where $\alpha_{1-m}$ (resp. $\beta_{1-m}$) is the dual of $x_m$ (resp. $y_m$) as in [Ben10, Proposition 1.1.5].

We now consider the map

$$B_1^{1^*} : H^1(M_1^*(\chi_{\mathrm{cycl}})) \to H^2(M_1^*(\chi_{\mathrm{cycl}})) = H^2(\mathcal{R}_{\mathbb{Q}_p}(|z|z^m)) \cong E.$$

We can use [Ben10, Proposition 2.4] to see that after the above identification of $H^2$ with $E$ we have

$$B_1^{1^*}(\alpha_{1-m}) = c\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))^{-1}\mathrm{d}\log_p({\psi'_A}^{-1}(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})), \tag{4.3}$$

$$B_1^{1^*}(\beta_{1-m}) = c\mathrm{d}\log_p({\psi'_A}^{-1}(p)), \tag{4.4}$$

where $c \in E^\times$. We consider the following anti-commutative diagram

$$\begin{array}{ccc}
H^0(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_0^*} & H^1(M_1^*(\chi_{\mathrm{cycl}})) \\
\downarrow{\scriptstyle B_0^{1^*}} & & \downarrow{\scriptstyle B_1^{1^*}} \\
H^1(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_1^*} & H^2(M_1^*(\chi_{\mathrm{cycl}}))
\end{array}$$

which means

$$B_1^{1^*}\Delta_0^* = -\Delta_1^* B_0^{1^*}.$$

We calculate this identity on $t^{1-k}e_{1-k}$. Applying (4.3) and (4.4) to ${\psi'_A}^{-1}\chi_{\mathrm{cycl}}$, (4.2) to ${\delta'_A}^{-1}\chi_{\mathrm{cycl}}$ and using [Ben10, (3.6)] which says

$$\Delta_1^* B_0^{1^*}(t^{1-k}) = c\left(b\mathrm{d}\log_p(\delta'_A)(p) + a\mathrm{d}\log_p(\delta'_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))\right)$$

we get

$$b^{-1}a = -\log_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))\frac{\mathrm{d}\log_p(\delta'_A{\psi'_A}^{-1})(p)}{\mathrm{d}\log_p(\delta'_A{\psi'_A}^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))}.$$

We conclude as $\delta'_A(p)^f = \delta_A(\varpi)$. $\square$

REMARK 4.5. *In particular, this theorem proves that this definition of $\mathcal{L}$-invariant is compatible with the Fontaine-Mazur one [Pot14, Zha14].*

## 4.2 CALCULATION OF THE $\mathcal{L}$-INVARIANT FOR STEINBERG FORMS

We fix a totally real field $F$. Let $I$ be the set of real embeddings. Fix two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

as before. We partition $I = \sqcup_{\mathfrak{p}} I_{\mathfrak{p}}$ according to the $p$-adic place which each embedding induces. We shall denote by $q_{\mathfrak{p}} = p^{f_{\mathfrak{p}}}$ the residual cardinality for each prime ideal $\mathfrak{p}$. We consider an irreducible representation $\pi$ of $\mathrm{GSp}_{2g/F}$, algebraic of weight $k = (k_\tau)_\tau$, where $(k_\tau) = (k_{\tau,1}, \dots, k_{\tau,g}; k_0)$ ($k_0$ is a parallel weight for $\mathrm{Res}_F^{\mathbb{Q}}(\mathbb{G}_m)$) with $k_{\tau,1} \leq k_{\tau,2} \dots \leq k_{\tau,g}$. If $k_{\tau,1} \geq g+1$ for all $\tau$, then the weight is cohomological. The cohomological weight of $\pi$ is then

$$(\mu_\tau)_\tau = (k_\tau)_\tau - (g+1, \dots, g+1; 0)_\tau.$$

For parallel weights $k$, we shall choose $k_0 = gk$.
We now describe the conjectural Galois representation associated with $\pi$. We have a spin Galois representation $V_{\mathrm{spin}}$ (whose image is contained in $\mathrm{GL}_{2^g}$) and a standard Galois representation $V_{\mathrm{sta}}$ (whose image is contained in $\mathrm{GL}_{2g+1}$) given respectively by the spinorial and the standard representation of $\mathrm{GSpin}_{2g+1} = {}^L\mathrm{GSp}_{2g}$.
Thanks to the work of Scholze [Sch15] we now dispose of the standard Galois representation (see for example [HJ13, Theorem 18]). We also know the existence of the spin representation in many cases [KS14].
We now recall some expected properties of these Galois representations. Our main reference is [HJ13, §3.3]. We will make the following assumption on $\pi$ at $p$;

for each $\mathfrak{p} \mid p$ either $\pi_\mathfrak{p}$ is spherical or Steinberg.

We explain what we mean by Steinberg. Consider the Satake parameters at $\mathfrak{p}$, normalized as in [BS00, Corollary 3.2], $(\alpha_{\mathfrak{p},1}, \dots, \alpha_{\mathfrak{p},g})$. We have the following theorem on Iwahori-spherical representation of $\mathrm{GSp}_{2g}(F_\mathfrak{p})$ [Tad94, Theorem 7.9].

THEOREM 4.6. *Let $\alpha_1, \dots, \alpha_g, \alpha$ be $g+1$ character of $F_\mathfrak{p}^\times$. Let $B_{\mathrm{GSp}_{2g}}$ be the Borel subgroup of $\mathrm{Sp}_{2g}(F_\mathfrak{p})$. Then $\mathrm{Ind}_{B_{\mathrm{GSp}_{2g}}}^{\mathrm{GSp}_{2g}(F_\mathfrak{p})}(\alpha_1 \times \dots \times \alpha_g \rtimes \alpha)$ is not irreducible if and only if one of the following conditions is satisfied:*

*i) There exist at least three indexes $i$ such that $\alpha_i$ has exact order two and the $\alpha_i$'s are mutually distinct;*

*ii) There exists $i$ such that $\alpha_i = |\mathrm{N}(\ )|_\mathfrak{p}{}^{\pm 1}$;*

*iii) There exist $i$ and $j$ such that $\alpha_i = |\mathrm{N}(\ )|_\mathfrak{p}{}^{\pm 1}\alpha_j{}^{\pm 1}$.*

REMARK 4.7. *As shown in [HJ13, Lemma 19], such a points are contained in a proper subset of the Hecke eigenvariety for $\mathrm{GSp}_{2g}$.*

DEFINITION 4.8. *We say that $\pi_\mathfrak{p}$ is Steinberg if $\alpha_i = |\mathrm{N}(\ )|_\mathfrak{p}^{i-1}\alpha_1$.*

If $\pi_\mathfrak{p}$ is Steinberg at $p$, then $\alpha_{\mathfrak{p},i}(\varpi_\mathfrak{p}) = q_\mathfrak{p}^i \alpha_{\mathfrak{p},1}(\varpi_\mathfrak{p})$.
Trivial zeros appear also for automorphic forms which are only partially Steinberg at $\mathfrak{p}$ and can be dealt exactly at the same way as the parallel one but for the sake of notation we prefer not to deal with them.

To each $g+1$ non-zero elements $(t_1,\dots,t_g;t_0) \in (A^\times)^{g+1}$ we associate the diagonal matrix

$$u(t_1,\dots,t_g;t_0) := (t_1,\dots,t_g,t_0t_g^{-1},\dots,t_0t_1^{-1})$$

of $\mathrm{GSp}_{2g}(A)$.
For $1 \leq i \leq g-1$ we denote by $u_{\mathfrak{p},i}$ the diagonal matrix associated with $(1,\dots,1,\varpi_{\mathfrak{p}}^{-1},\dots,\varpi_{\mathfrak{p}}^{-1};\varpi_{\mathfrak{p}}^{-2})$, where $\varpi_{\mathfrak{p}}$ appears $i$ times; we also denote by $u_{\mathfrak{p},0}$ the diagonal matrix corresponding to $(1,\dots,1;\varpi_{\mathfrak{p}}^{-1})$.

DEFINITION 4.9. *The Hecke operators $U_{\mathfrak{p},i}$, for $1 \leq i \leq g$ are defined as the double coset operator* $[\mathrm{Iw}u_{\mathfrak{p},g-i}\mathrm{Iw}]$.

We have that $U_{\mathfrak{p},g}$ is the "classical" $U_p$ operator [BS00, §0]. We shall say then that $\pi$ is of finite slope for $U_{\mathfrak{p},g}$ if $U_{\mathfrak{p},g}$ has eigenvalue $\alpha_{\mathfrak{p},0} \neq 0$ on $\pi_{\mathfrak{p}}$.

We are interested to study the possible $p$-stabilization of $\pi$ (*i.e.* Iwahori fixed vectors). If $\pi_{\mathfrak{p}}$ is unramified at $\mathfrak{p}$, we have then $2^g g!$ choices (see [HJ13, Lemma 16] or [BS00, Proposition 9.1]). If $\pi_{\mathfrak{p}}$ is Steinberg, we have instead only one possible choice, as the monodromy $N$ has maximal rank.

Suppose that we can lift $\pi$ to an automorphic representation $\pi^{(2^g)}$ of $\mathrm{GL}_{2^g}$. We suppose also that we can lift $\pi$ to an automorphic representation $\pi^{(2g+1)}$ of $\mathrm{GL}_{2g+1}$.
Let $V = V_{\mathrm{spin}}$ (resp. $V_{\mathrm{sta}}$) be the Galois representation associated with $\pi^{(2^g)}$ (resp. $\pi^{(2g+1)}$). We make the following assumption

LGP) $V$ is semistable at all $\mathfrak{p} \mid p$ and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for $f$ as above. Arthur's transfer from $\mathrm{GSp}_{2g}$ to $\mathrm{GL}_{2g+1}$ has been proven in [Xu] (note that it is now unconditional [MW]) and for $V = V_{\mathrm{sta}}$ this hypothesis is then verified thanks to [Car14, Theorem 1.1]. These hypotheses are also satisfied in many cases for $V = V_{\mathrm{spin}}$ in genus 2 (see [AS06, PSS14]).
Roughly speaking, we require that

$$\mathrm{WD}(V_{|F_{\mathfrak{p}}})^{\mathrm{ss}} \cong \iota_n^{-1}\pi_{\mathfrak{p}}^{(n)},$$

where $\mathrm{WD}(V_{|F_{\mathfrak{p}}})$ is the Weil-Deligne representation associated with $V_{|F_{\mathfrak{p}}}$ à la Berger, $\pi_{\mathfrak{p}}^{(n)}$ is the component at $\mathfrak{p}$ of $\pi^{(n)}$, and $\iota_n$ is the local Langlands correspondence for $\mathrm{GL}_n(F_{\mathfrak{p}})$ geometrically normalized ($n = 2g+1$ when $V$ is the standard representation and $n = 2^g$ when $V$ is the spinorial representation).
When $\pi_{\mathfrak{p}}$ is an irreducible quotient of $\mathrm{Ind}_B^{\mathrm{GSp}_{2g}}(\alpha_{\mathfrak{p},1} \otimes \cdots \otimes \alpha_{\mathfrak{p},g})$ we have that

the Frobenius eigenvalues on $\mathrm{WD}(V_{\mathrm{spin}_{|F_\mathfrak{p}}})^{\mathrm{ss}}$ are the $2^g$ numbers

$$\left(\alpha_{\mathfrak{p},0} \prod_{\substack{0 \leq r \leq g \\ 1 \leq i_1 < \ldots < i_r \leq g}} \alpha_{\mathfrak{p},i_1}(\varpi_\mathfrak{p}) \cdots \alpha_{\mathfrak{p},i_r}(\varpi_\mathfrak{p})\right).$$

The ones on $\mathrm{WD}(V_{\mathrm{sta}|F_\mathfrak{p}})^{\mathrm{ss}}$ are

$$\left(\alpha_{\mathfrak{p},g}^{-1}(\varpi_\mathfrak{p}), \ldots, \alpha_{\mathfrak{p},1}^{-1}(\varpi_\mathfrak{p}), 1, \alpha_{\mathfrak{p},1}(\varpi_\mathfrak{p}), \ldots, \alpha_{\mathfrak{p},g}(\varpi_\mathfrak{p})\right).$$

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel) if we are Steinberg or be trivial otherwise. (This is also a consequence of the weight-monodromy conjecture for $V$.)
Let $\mathfrak{p}$ be a $p$-adic place of $F$ and let $\tau$ be a complex place in $I_\mathfrak{p}$. The Hodge–Tate weights of $V_{\mathrm{spin}_{|F_\mathfrak{p}}}$ at $\tau$ are then

$$\left(\frac{k_0}{2} + \frac{1}{2}\sum_{i=1}^{g} \varepsilon(i)(k_{i,\tau} - i)\right)_\varepsilon,$$

where $\varepsilon$ ranges among the $2^g$ maps from $\{1, \ldots, g\}$ to $\{\pm 1\}$.
The one of $V_{\mathrm{sta}_{|F_\mathfrak{p}}}$ are $(1 - k_{\tau,g}, \ldots, g - k_{\tau,1}, 0, k_{\tau,1} - g, \ldots, k_{\tau,g} - 1)$.
Thanks to work of Tilouine-Urban [TU99], Urban [Urb11], Andreatta-Iovita-Pilloni [AIP15] we have families of Siegel modular forms;

THEOREM 4.10. *Let* $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times \times ((\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times)^g, \mathbb{C}_p^\times\right)$ *be the weight space. There exist an affinoid neighborhood* $\mathcal{U}$ *of* $\kappa_0 = \left((z, (z_i)_{i=1}^g) \mapsto z^{k_0} \prod_{\tau \in I} \prod_i \tau(z_i)^{k_{i,\tau}}\right)$ *in* $\mathcal{W}$*, an equidimensional rigid variety* $\mathcal{X} = \mathcal{X}_\pi$ *of dimension* $dg + 1$*, a finite surjective map* $w : \mathcal{X} \to U$*, a character* $\Theta : \mathcal{H}^{Np} \to \mathcal{O}(\mathcal{X})$*, and a point* $x$ *in* $\mathcal{X}$ *above* $\underline{k}$ *such that* $x \circ \Theta$ *corresponds to the Hecke eigensystem of* $\pi$*.*
*Moreover, there exists a dense set of points* $x$ *of* $\mathcal{X}$ *coming from classical cuspidal Siegel–Hilbert automorphic forms of weight* $(k_{i,\tau}; k_0)$ *which are regular and spherical at* $p$*.*

REMARK 4.11. *Assuming Leopoldt's conjecture, the multiplicative group appearing in the definition of* $\mathcal{W}$ *is, up to a finite subgroup,* $((\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times)^{g+1} / \overline{\mathcal{O}_F^\times}$ *(i.e. the* $\mathbb{Z}_p$*-points of the torus of* $\mathrm{Res}_F^\mathbb{Q}(\mathrm{GSp}_{2g})$ *modulo the* $\mathbb{Z}_p$*-points of the center).*

This allows us to define two pseudo-representations $R_? : G_\mathbb{Q} \to \mathcal{O}(\mathcal{X})$, for $? =$ spin, sta, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Suppose now that $V_?$ is absolutely irreducible (this is conjectured to hold when $\pi$ is Steinberg at least at one prime); we have then, shrinking $\mathcal{U}$ around $\underline{k}$ if necessary, a *big* Galois representation $\rho_?$ with value in $\mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ such that $\mathrm{Tr}(\rho_?) = R_?$ [BC09, page 214].

For $1 \leq j < g$ we define $\lambda_{\mathfrak{p}}(u_{\mathfrak{p},g-j}) = \varpi_{\mathfrak{p}}^{\sum_{\tau \in I_{\mathfrak{p}}} k_{\tau,1}+\cdots+k_{\tau,j}-k_0}$ and $\lambda_{\mathfrak{p}}(u_{\mathfrak{p},0}) = \varpi_{\mathfrak{p}}^{\sum_{\tau \in I_{\mathfrak{p}}}(k_{\tau,1}+\cdots+k_{\tau,g}-k_0)/2}$. We have analytic functions $\beta_{\mathfrak{p},j} := \Theta(U_{\mathfrak{p},j}|\lambda_{\mathfrak{p}}(u_{\mathfrak{p},g-j})|_p) \in \mathcal{O}(\mathcal{X})$. We proceed now as in [HJ13]. We recall the following theorem [Liu13, Theorem 0.3.4];

THEOREM 4.12. *Let $\rho : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ be a continuous representation. Suppose that there exist $\kappa_1(x), \ldots, \kappa_n(x)$ in $F_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} \mathcal{O}(\mathcal{X})$, $F_1(x), \ldots, F_d(x)$ in $\mathcal{O}(\mathcal{X})$, and a Zariski dense set of points $Z \subset \mathcal{X}$ such that*

- *for any $x$ in $\mathcal{X}$, the Hodge–Tate weights of $\rho_x$ are $\kappa_1(x), \ldots, \kappa_n(x)$;*
- *for any $z$ in $Z$, $\rho_z$ is crystalline;*
- *for any $z$ in $Z$, $\kappa_{\tau,1}(z) < \ldots < \kappa_{\tau,n}(z)$, for all $\tau \in I_{\mathfrak{p}}$;*
- *for any $z$ in $Z$, the eigenvalues of $\varphi^{f_{\mathfrak{p}}}$ on $\mathcal{D}_{\mathrm{cris}}(V_z)$ are $\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{\kappa_{\tau,1}(z)} F_1(z), \ldots, \prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{\kappa_{\tau,n}(z)} F_n(z)$;*
- *for any $C$ in $\mathbb{R}$, defines $Z_C \subset Z$ as the set of points $z$ such that for all $I, J \subset \{1, \ldots, n\}$ such that $|\sum_{i \in I} \kappa_{i,\tau}(z) - \sum_{j \in J} \kappa_{\tau,j}(z)| > C$ for all $\tau \in I_{\mathfrak{p}}$. We require that for all $z \in Z$ and $C \in \mathbb{R}$, $Z_C$ accumulates at $z$.*
- *for $1 \leq i \leq n$ there exist character $\chi_i : \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \to \mathcal{O}(\mathcal{X})^{\times}$ such that its derivative at 1 is $\kappa_i$ and at each $z \in Z$ we have $\chi_i(u) = \prod_{\tau} \tau(u)^{\kappa_{\tau,i}(z)}$.*

*Then, for all $x$ in $\mathcal{X}$ non-critical and regular ($\kappa_1(x) < \ldots < \kappa_n(x)$ and the eigenvalues of $\varphi$ on $\bigwedge^i \mathcal{D}_{\mathrm{cris}}(V_x)$ are distinct for all $i$) there exists a Zariski neighbourhood $U$ of $x$ such that $\rho_U$ is trianguline and its graded pieces are $\mathcal{R}_U(\chi_i)$.*

Here the rank one $(\varphi, \Gamma)$-module $\mathcal{R}_U(\chi_i)$ over $U$ is defined similarly as in Section 2 following [Liu13, §0.2].
We can apply this theorem and show that the $(\varphi, \Gamma)$-module associated with $\rho_{?|G_{\mathbb{Q}_p}}$ is trianguline. We now explicit the triangulation, given in [HJ13, §3.3].

As seen before, a $p$-stabilization of $\pi_{\mathfrak{p}}$ corresponds to a permutation $\nu$ and a map $\varepsilon$.
The eigenvalues of $\varphi^{f_{\mathfrak{p}}}$ are given by

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_1+\mu_{1,\tau}} \beta_{\mathfrak{p},1},$$

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_i+\mu_{i,\tau}} \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}},$$

$$\prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_g+\mu_{g,\tau}} \frac{\beta_{\mathfrak{p},g-1}}{\beta_{\mathfrak{p},g}^2}$$

where $c_i$'s are a positive integer independent of the weight.
We define the following characters of $F_{\mathfrak{p}}$ with value in $\mathcal{O}(\mathcal{X})$:

$$\chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p},1},$$
$$\chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}) = \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}},$$
$$\chi_{\mathfrak{p},g}(\varpi_{\mathfrak{p}}) = \frac{\beta_{\mathfrak{p},g-1}}{\beta^2_{\mathfrak{p},g}},$$

and $\chi_{\mathfrak{p},i}(u) = \prod_{\tau\in I_{\mathfrak{p}}} \tau(u)^{c_i+\mu_{i,\tau}}$.

From [HJ13, Lemma 19] we have that the graded pieces of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathrm{sta}|_{\mathfrak{p}}})$ are then given by the characters $\chi_{\mathfrak{p},g}, \ldots, 1, \ldots, \chi^{-1}_{\mathfrak{p},g}$.

Concerning $V_{\mathrm{spin}}$, we number the subsets of $\{1, \ldots, g\}$ as $I_1, I_2, \ldots, I_{2^g}$. Each $I_j$ corresponds to a map $\varepsilon_j : \{1, \ldots, g\} \to \pm 1$.
We have then the graded pieces $\delta_{\mathfrak{p},j}$ are given by the characters

$$\delta_{\mathfrak{p},\varepsilon_j}(u) = \prod_{\tau\in I_{\mathfrak{p}}} \tau(u)^{d_j + \frac{k_0+\sum_i \varepsilon_j(i)k_{i,\tau}}{2}},$$
$$\delta_{\mathfrak{p},\varepsilon_j}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p},g} \prod_{i\in I_j} \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}).$$

Let $V$ be either $V_{\mathrm{Sta}}$ or $V_{\mathrm{spin}}$. If $\pi_{\mathfrak{p}}$ is Steinberg, there is only one choice of a regular $(\varphi, N)$-sub-module $D_{\mathfrak{p}}$ of $\mathbf{D}_{\mathrm{st}}(V_{G_{F_{\mathfrak{p}}}})$, where $V$ is one of the two representations associated with $\pi$ described above. If the form is not Steinberg at $\mathfrak{p}$ many different regular sub-module can be chosen.
In any case, we expect (and we shall assume in the follow) that there is at most one trivial zero for each $\mathfrak{p}$. Consider now the representation $\pi$ of parallel weight $\underline{k}$ (*i.e.* associated with $\mathrm{N}_{F/\mathbb{Q}}(\det^{\underline{k}})$, $\underline{k} \in \mathbb{Z}$) as in the introduction.
We give a preliminary proposition on the factorization of the $\mathcal{L}$-invariant. Recall the set $S^{\mathrm{Sph},1}$ and $S^{\mathrm{Stb}}$ defined in the introduction, we have the following;

Proposition 4.13. *We have the following factorization*

$$\mathcal{L}(V, D) = \mathcal{L}(V, D)^{\mathrm{Sph}} \prod_{\mathfrak{p}\in S^{\mathrm{Stb}}} \mathcal{L}(V, D)_{\mathfrak{p}},$$

*where $\mathcal{L}(V, D)^{\mathrm{Sph}}$ comes from the prime in $S^{\mathrm{sph}}$ and the factors $\mathcal{L}(V, D)_{\mathfrak{p}}$ are local.*

*Proof.* We follow [Hid07, §1.3]. In the notation of Section 3, we write $W_1 = \oplus_{\mathfrak{p}\in S^{\mathrm{Stb}}} W_{\mathfrak{p},1}$ and $M_1 = \oplus_{\mathfrak{p}\in S^{\mathrm{Sph},1}} M_{\mathfrak{p},1}$. We are left to show that the endomorphism $\iota_{\mathrm{f}} \circ \iota_{\mathrm{c}}^{-1}$ of $\mathcal{D}_{\mathrm{cris}}(W_1 \oplus M_1) \cong E^t$ keeps stable $\mathcal{D}_{\mathrm{cris}}(M_1)$ and on the quotient it respects the direct sum decomposition $\oplus_{\mathfrak{p}\in S^{\mathrm{Stb}}} \mathcal{D}_{\mathrm{cris}}(W_{\mathfrak{p},1})$.

Consider a prime $\mathfrak{p}_0 \in S^{\mathrm{Stb}}$ and a cocycle $c \in H^1(D,V)$ such that $\mathrm{res}_{\mathfrak{p}}(c) = 0$ for all $\mathfrak{p} \neq \mathfrak{p}_0$. This means that $\mathrm{res}_{\mathfrak{p}}(c) \in H^1_{\mathrm{f}}(F_{\mathfrak{p}}, V) = H^1_{\mathrm{f}}(F_{\mathfrak{p}}, M_{\mathfrak{p}})$ (by (3.3)). We have hence $\iota_{\mathrm{c},\mathfrak{p}}(c) = 0$ for all primes $\mathfrak{p} \neq \mathfrak{p}_0$ as $H^1_{\mathrm{c}}$ is the direct sum complement of $H^1_{\mathrm{f}}$ (see [Ben11, Proposition 1.5.9]).
If $\mathfrak{p}$ in $S^{\mathrm{Stb}}$ by Proposition 2.10 we also have $\iota_{\mathrm{f},\mathfrak{p}}(c) = 0$.
The proposition then follows from standard linear algebra as in [Hid07, Corollary 1.9]. ☐

Remark 4.14. *A key ingredient in the proof of the factorization at Steinberg places is that each prime ideal brings a single trivial zero.*

We now consider the case $V = V_{\mathrm{sta}}$. We have a contribution to trivial zeros from the $\pi_{\mathfrak{p}}$'s which are Steinberg and possibly from the $\pi_{\mathfrak{p}}$ which are spherical. In particular, if we choose the regular sub-module coming from an ordinary filtration, we always have a trivial zero coming from each place.
For all $1 \leq s \leq \min(k-g-1, g-1)$ we have also $e_{\mathrm{Stb}}$ trivial zeros for $V(s)$.

Theorem 4.15. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V,D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p},1}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

*where $k$ is the parallel weight variable.*
*For $1 \leq s \leq \min(k-g-1, g-2)$ we also have*

$$\mathcal{L}(V(s),D(s))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p (\beta_{\mathfrak{p},s-1}\beta^{-1}_{\mathfrak{p},s}(k))}{\mathrm{d}k}_{|k=\underline{k}}$$

*and if $g-1 \leq k-g-1$ we have*

$$\mathcal{L}(V(g-1),D(g-1))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p (\beta_{\mathfrak{p},g-1}\beta^{-2}_{\mathfrak{p},g}(k))}{\mathrm{d}k}_{|k=\underline{k}}.$$

*Proof.* We note that we can specialize to a parallel family, so that no contribution from the denominator appears. We can apply Theorem 4.1 for $\delta_A \psi_A^{-1}(\varpi_{\mathfrak{p}}) = \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}})$. The factor $\log_p(u)$ disappears because of the change of variable $T \mapsto u^k - 1$ ($u$ any topological generator of $\mathbb{Z}_p^\times$). ☐

Remark 4.16. *The presence of $f_{\mathfrak{p}}$ in the denominator can be explained in terms of the $p$-adic $L$-function for the induced representation, its missing Euler factors at $p$ and Conjecture 1.2. See [Hid09, pag. 1348].*

From now on, $V = V_{\mathrm{spin}}(k-1)$ ($s = k-1$ is the only critical integer); if $\pi_{\mathfrak{p}}$ is spherical it should not give any trivial zeros (as the corresponding $p$-adic representation is conjectured to be crystalline and consequently the $\beta_i$'s are Weil numbers of non-zero weight).
So we are left to see what happen at the primes Steinberg at $\mathfrak{p}$. Twisting by $\beta_{\mathfrak{p},g}$ the triangulated $(\varphi,\Gamma)$-module of $\rho_{\mathrm{spin}}$ we are in the hypothesis of Theorem 4.1 and we have

THEOREM 4.17. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V,D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}}\frac{\mathrm{d}\log_p \beta_{\mathfrak{p},1}(k)}{\mathrm{d}k}_{|k=\underline{k}},$$

*where $k$ is a parallel weight variable.*

## 5 THE CASE OF THE ADJOINT REPRESENTATION

We prove Theorem 1.4 of the introduction. We consider only the case $g = 2$. Fix an automorphic representation $\pi$ of weight $\underline{k} = (\underline{k}_{\tau,1}, \ldots, \underline{k}_{\tau,g}; \underline{k}_0)_\tau$ and let $V = V_{\mathrm{spin}}$ be the spin representation associated with $\pi$. Let $\rho = \rho_{\mathrm{spin}}$ be the corresponding big Galois representation.

We specializes the eigenvariety $\mathcal{X}$ of Theorem 4.10 to the subspace of the weight space given by the equations $k_{i,\tau} = k_{i,\tau'}$ if $\tau$ and $\tau'$ induce the same $p$-adic place $\mathfrak{p}$ and $k_0 = \underline{k}_0$. We shall denote the new variable by $k_{\mathfrak{p},i}$ and this eigenvariety by $\mathcal{X}'$. For simplicity, we rewrite the graded pieces of $V$ as

$$\begin{aligned}
\delta_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},1}^{-1}(k),\ \ \delta_{\mathfrak{p},1}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},1}+k_{\mathfrak{p},2}-3}{2}},\\
\delta_{\mathfrak{p},2}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},2}^{-1}(k),\ \ \delta_{\mathfrak{p},2}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},2}-k_{\mathfrak{p},1}+1}{2}},\\
\delta_{\mathfrak{p},3}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},2}(k),\ \ \delta_{\mathfrak{p},3}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},2}+k_{\mathfrak{p},1}-1}{2}},\\
\delta_{\mathfrak{p},4}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},1}(k),\ \ \delta_{\mathfrak{p},4}(u) = \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},1}-k_{\mathfrak{p},2}+3}{2}}
\end{aligned}$$

where $k = (k_{\mathfrak{p},1}, k_{\mathfrak{p}_2}; k_0)_{\mathfrak{p}}$.
The representation space of $\mathrm{Ad}(V)$ is given by the matrices

$$\mathfrak{Sp}_4 = \left\{X \in \mathfrak{SL}_4 | X^t J + JX = 0\right\}.$$

The $p$-stabilization on $V$ induces a natural $p$-stabilization and consequently a regular sub-module $D_{\mathrm{Ad}}$ on $\mathrm{Ad}(V_{\mathrm{spin}})$. We have

$$\begin{aligned}
D_{\mathrm{Ad}-1} &= \{\text{nilpotent } X\},\\
D_{\mathrm{Ad}0} &= \{\text{unipotent } X\}.
\end{aligned}$$

The basis for the space $D_{\mathrm{Ad}0}/D_{\mathrm{Ad}-1}$ is given by the two diagonal matrices $d_1 = [-1, 0, 0, 1]$ and $d_2 = [0, -1, 1, 0]$. We shall denote by $d_{\mathfrak{p},i}$ these matrices when seen as a vector for $\mathrm{Ad}(V_{\mathfrak{p}})$.

PROPOSITION 5.1. *Suppose that* C1-C4 *holds for $V$. Suppose that the classical $E$-point $x$ in the eigenvariety $\mathcal{X}'$ corresponding to $\pi$ is étale above the weight space. Then, the space $\mathcal{L}(D_{\mathrm{Ad}}, V)$ is generated by the image of* $\left(\frac{\mathrm{d}\log_p \delta_{\mathfrak{p},i}}{\mathrm{d}k_{\mathfrak{p}',j}} d_{\mathfrak{p},i}\right)_{\mathfrak{p}',j=1,2}$.

*Proof.* The proof is standard and goes back to [MT90], so we shall only sketch it. Let $A = E[T]/(T^2)$. Consider an infinitesimal deformation of $\rho$ given by

$$\rho_A = V \oplus \rho';$$

note that $\rho'$ can be written as the first order truncation of $\frac{\partial \rho}{\partial v}$, where $v$ is any direction in the weight space.
From $\rho_A$ we can construct a cocyle $c_{x,A}$ defined by

$$G_F \ni \sigma \mapsto \rho'(\sigma) V^{-1}(\sigma).$$

It is easy to check that this defines a cocycle with values in $V \otimes V^*$. Moreover its image lands in $\mathrm{Ad}(V) \subset V \otimes V^*$ as the determinant is fixed (by our choice of the Hodge–Tate weight on $\mathcal{X}'$). Writing explicitly the matrix for the $(\varphi, \Gamma)$-module associated with $\rho_A$ we obtain

$$\begin{pmatrix} \frac{\partial \delta_{\mathfrak{p},1}}{\partial v} & * & * & * \\ & \frac{\partial \delta_{\mathfrak{p},2}}{\partial v} & * & * \\ & & \frac{\partial \delta_{\mathfrak{p},3}}{\partial v} & * \\ & & & \frac{\partial \delta_{\mathfrak{p},4}}{\partial v} \end{pmatrix}_{|k=\underline{k}} \begin{pmatrix} \delta_{\mathfrak{p},1}^{-1} & * & * & * \\ & \delta_{\mathfrak{p},2}^{-2} & * & * \\ & & \delta_{\mathfrak{p},3}^{-1} & * \\ & & & \delta_{\mathfrak{p},4}^{-1} \end{pmatrix}_{|k=\underline{k}}$$

In particular, they are upper triangular and their projection via $\iota_{\mathrm{f}}$ onto the vector $d_{\mathfrak{p},1}$ is $\frac{\mathrm{d} \log_p F_{\mathfrak{p},1}(k)}{\mathrm{d} v}_{|k=\underline{k}}$. Similarly for $d_{\mathfrak{p},2}$.
We also have that the projection via $\iota_{\mathrm{c}}$ onto $d_{\mathfrak{p},1}$ is $-\frac{\partial (k_{\mathfrak{p},1}+k_{\mathfrak{p},2})/2}{\partial v}_{|k=\underline{k}}$.
By hypothesis, the projection to the weight space is étale at $x$ and hence $\left\{\frac{\partial}{\partial k_{\mathfrak{p},i}}\right\}_{\mathfrak{p},i=1,2}$ is a base of the tangent space at $x$ in $\mathcal{X}'$ and we are done. □

We can now prove Theorem 1.4 which we recall;

THEOREM 5.2. *Let $\pi$ be an automorphic form of weight $k$. Suppose that hypothesis* LGP *is verified for $V_{\mathrm{spin}}$ and the point corresponding to $\pi$ in the eigenvariety $\mathcal{X}'$ is étale over the weight space. We have then*

$$\mathcal{L}(\mathrm{Ad}(V_{\mathrm{spin}}), D_{\mathrm{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \begin{pmatrix} \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,1}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,1}} \\ \frac{\partial \log_p F_{\mathfrak{p}_i,1}(k)}{\partial k_{\mathfrak{p}_j,2}} & \frac{\partial \log_p F_{\mathfrak{p}_i,2}(k)}{\partial k_{\mathfrak{p}_j,2}} \end{pmatrix}_{1 \leq i,j \leq t_{|k=\underline{k}}}.$$

*Proof.* By hypothesis we can use Proposition 5.1, so we just have to follow the proof of [Hid06, Theorem 3.73]. The matrix of $\iota_{\mathrm{f}}$ is exactly what appears in the Theorem, while the matrix of $\iota_{\mathrm{c}}$ can be directly calculated using the formula $\frac{\mathrm{d} \log_p (u^{\pm k_{\mathfrak{p},i}})}{\mathrm{d} k_{\mathfrak{p}',j}} = \pm \delta_{\mathfrak{p},\mathfrak{p}'} \delta_{i,j}$ (where $\delta_{a,b}$ here is Kronecker delta) and gives a contribution of $2^{-1}$ for each prime ideal $\mathfrak{p}$. □

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Giovanni Rosso
DPMMS, Centre for
Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WB
United Kingdom
gr385@cam.ac.uk