Regularity of Projection Operators Attached to Worm Domains

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Abstract. We construct a projection operator on an unbounded worm domain which maps subspaces of W^s to themselves. The subspaces are determined by a Fourier decomposition of W^s according to a rotational invariance of the worm domain.

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Introduction

Our work is on the non-smooth unbounded worm domains

$$D_\beta=\{(z_1,z_2)\in\mathbb{C}^2:\mathrm{Re}\left(z_1e^{-i\log z_2\overline{z}_2}\right)>0,|\log z_2\overline{z}_2|<\beta-\pi/2\}\qquad \beta>\pi/2.$$

On a bounded version of the domains D_β, given by

$$\Omega_c=\left\{(z_1,z_2):\left|z_1+e^{i\log z_2\overline{z}_2}\right|^2<1,|\log z_2\overline{z}_2|<\beta-\pi/2\right\},$$

C. Kiselman showed the failure of the Bergman projection to preserve $C^\infty(\overline{\Omega}_c)$ [7]. The model domains, D_β, were important in [1], where the first author used them to show the Diedrich-Fornæss worm domains (constructed in [5]) provide a counterexample to regularity of the Bergman projection on a smoothly bounded pseudoconvex domain. In a detailed analysis of the Bergman kernel, Krantz and the third author, in [8], studied the L^p mapping properties of the Bergman projection on D_β, obtaining the exact range of values of p for which the mapping is bounded.

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In this article we look at regularity in terms of Sobolev spaces. We denote by $W^s(D_\beta)$ the space of functions whose derivatives of order $\leq s$ are in $L^2(D_\beta)$, and by $W^s_{\mathscr{D}}(D_\beta)$ the closure of $\mathscr{D} := C_0^\infty(D_\beta)$ in $W^s(D_\beta)$. The first author's results on smooth domains relied on the fact (proved in the same paper) that the Bergman projection on the model domain, D_β, fails to map $W^s_{\mathscr{D}}(D_\beta)$ to $W^s(D_\beta)$ for large enough s [1]. More precisely, the failure to preserve Sobolev spaces was proved on subspaces (defined as $W^s_j(D_\beta)$ below). This was instrumental in proving that Condition R, in which for each $s \geq 0$ there exists an $M \geq 0$ such that the Bergman projection is bounded as a map from $W^{s+M}_{\mathscr{D}}(\Omega)$ to $W^s(\Omega)$ ([3]), fails when Ω is the Diederich-Fornæss worm domain [4]. We point out that for a smoothly bounded pseudoconvex domain Condition R is equivalent to the apparently stronger condition in which the larger domain $W^{s+M}(\Omega)$ replaces $W^{s+M}_{\mathscr{D}}(\Omega)$. This equivalence also holds on the domains D_β: the first author constructed a composition of first order operators which allow us to consider the Bergman projection acting on functions which vanish to desired order at the boundary, without changing the resulting image of the projection (see Theorem 2.2 and specifically Theorem 3.1 in [2]).

The question remained whether there exists another (oblique) projection operator which preserves the level of the Sobolev spaces. We construct such an operator in the present article.

We now state our main result. From the rotational invariance of D_β with respect to the rotations, $\rho_\theta(z) = (z_1, e^{i\theta} z_2)$, we can decompose the Bergman space $B(D_\beta) = L^2(D_\beta) \cap \mathcal{O}(D_\beta)$ by

$$B(D_\beta) = \bigoplus_{j \in \mathbb{Z}} B_j(D_\beta),$$

where $B_j(D_\beta)$ consists of functions $f \in B(D_\beta)$ satisfying $f \circ \rho_\theta \equiv e^{ij\theta} f$. The space $L^2(D_\beta)$ admits a similar decomposition into subspaces $L^2_j(D_\beta)$, and we can define $W^s_j(D_\beta) = L^2_j(D_\beta) \cap W^s(D_\beta)$.

Our main theorem is grounded on adjustments to factors which imply the obstruction to regularity of the Bergman projection on worm domains. The Bergman kernel for each space, $B_j(D_\beta)$ is explicitly calculated and expressed as an integral in the form:

$$K_j(z,w) = \frac{1}{2\pi^2} z_2^j \overline{w}_2^j \int_{\mathbb{R}} \frac{(\xi - \frac{j+1}{2})\xi}{\sinh\left[(2\beta - \pi)(\xi - \frac{j+1}{2})\right] \sinh \pi\xi} z_1^{i\xi - 1} \overline{w}_1^{-i\xi - 1} d\xi,$$

where, with an abuse of notation, we write

$$z_1^\alpha = (z_1 e^{-i \log z_2 \overline{z}_2})^\alpha e^{i\alpha \log z_2 \overline{z}_2}.$$

Such a power of z_1 is holomorphic on D_β as is easy to see, and it is locally constant in $|z_2|$, but not constant if α is not an integer and $\beta > \pi$. In fact, in this case, the fiber over z_1 is a union of disjoint annuli in z_2 and the function is constant on each such annulus, but not globally constant.

Using the residue calculus, one can compute an asymptotic expansion of the kernel (see [1]). The poles corresponding to non-integer multiples of i of the

kernel lead to non-integer powers of z_1 and w_1 which ultimately lead to the obstruction of regularity of the operator.
We construct a kernel which, when added to the Bergman kernel, eliminates all such poles, and in this way we successfully remove the obstruction to regularity of the Bergman projection on the model domains, D_β, and construct new projections which preserve the level of Sobolev spaces:

Main Theorem. *Let $\beta > \pi/2$, and D_β be defined as above. For all $j \in \mathbb{Z}$ there exists a bounded linear projection*

$$\mathbf{T}_j : L^2(D_\beta) \to B_j(D_\beta)$$

which satisfies

$$\mathbf{T}_j : W^s_{\mathscr{D}}(D_\beta) \to W^s_j(D_\beta)$$

for every $s \geq 0$.

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1. The Bergman projection on D_β

Following [1], we introduce the domains

$$D'_\beta = \Big\{(z_1,z_2) \in \mathbb{C}^2 : |\text{Im } z_1 - \log z_2\overline{z}_2| < \pi/2, |\log z_2\overline{z}_2| < \beta - \pi/2\Big\}$$

to aid in our study of the Bergman kernels on D_β. D'_β is related to D_β via the biholomorphic mapping

$$\begin{aligned} \Psi : D'_\beta &\to D_\beta \\ (z_1,z_2) &\mapsto (e^{z_1}, z_2). \end{aligned} \tag{1.1}$$

Let $K_{D_\beta}(z,w)$ be the Bergman kernel for D_β, and $K_j(z,w)$ the reproducing kernel for $B_j(D_\beta)$; we have the relation

$$K_{D_\beta}(z,w) = \sum_j K_j(z,w).$$

We calculate $K_j(z,w)$ using Fourier transforms as in [1].
Let S_β denote the strip

$$S_\beta := \{z = x + iy \in \mathbb{C} : |y| < \beta\},$$

and let $\omega_j(y)$ be the continuous bounded function on the interval $I_\beta := \{y : |y| < \beta\}$, given by $\omega_j = \pi\left(e^{(j+1)(\cdot)}\chi_{\beta-\pi/2}\right) * \chi_{\pi/2}$, where for $a > 0$, $\chi_a :=$

$\chi_{(-a,a)}$, the characteristic function of the interval $(-a,a)$. We denote by $\|\cdot\|_{\omega_j}$ the $L^2(S_\beta)$-norm weighted with the function ω_j:

$$\|f\|_{\omega_j} := \left(\int_{S_\beta} |f(x,y)|^2 \omega_j(y) dxdy \right)^{1/2}.$$

We further define the weighted Bergman spaces on the strip S_β by

$$B_{\omega_j} = \{f \text{ holomorphic on } S_\beta : \|f\|^2_{\omega_j} < \infty\}.$$

For $f \in B_{\omega_j}$,

$$\hat{f}(\xi,y) = \int_{\mathbb{R}} f(x+iy) e^{-ix\xi} dx$$

satisfies

$$\hat{f}(\xi,y) = e^{-y\xi} \hat{f}_{\mathbb{R}}(\xi), \tag{1.2}$$

where $\hat{f}_{\mathbb{R}}(\xi) := \hat{f}(\xi,0)$.

Here and throughout we use the notation for complex variables

$$z_1 = x + iy$$
$$w_1 = x' + iy'.$$

Define

$$k'_j(\xi,y,w_1) = \frac{1}{\hat{\omega}_j(-2i\xi)} e^{i\xi(y-\overline{w}_1)},$$

where $\hat{\omega}_j$ refers to the Fourier-Laplace transform of ω_j, and satisfies

$$\hat{\omega}_j(-2i\xi) = \pi \frac{\sinh\left[(2\beta-\pi)(\xi - \frac{j+1}{2})\right] \sinh \pi\xi}{(\xi - \frac{j+1}{2})\xi}. \tag{1.3}$$

We note that $\hat{\omega}_j$ extends to an entire function. We claim that k'_j corresponds to the kernel for the orthogonal projection on D'_β according to the following lemma:

Lemma 1.1. *Let $K'_j(z,w)$ denote the reproducing kernel of the space $B_j(D'_\beta)$. Then*

$$K'_j(z,w) = \frac{1}{2\pi^2} z_2^j \overline{w}_2^j \int_{\mathbb{R}} \frac{(\xi - \frac{j+1}{2})\xi}{\sinh\left[(2\beta-\pi)(\xi - \frac{j+1}{2})\right] \sinh \pi\xi} e^{i(z_1-\overline{w}_1)\xi} d\xi.$$

Proof. Let $\Gamma : \mathcal{B}_1 \to \mathcal{B}_2$ be a surjective isometry of two Bergman spaces. Let $K_1(z,w)$ be the reproducing kernel of the space $\mathcal{B}_1$ and $K_2(z,w)$ the kernel for $\mathcal{B}_2$. Then

$$K_2(z,w) = \overline{\Gamma_w \overline{\Gamma_z K_1(z,w)}}. \tag{1.4}$$

We now apply (1.4) to the spaces $\mathcal{B}_1 = B_{\omega_j}$ and $\mathcal{B}_2 = B_j(D'_\beta)$. From [1]

$$K_j(z_1,w_1) = \frac{1}{2\pi} \int_{\mathbb{R}} k'_j(\xi,y,w_1) e^{ix\xi} d\xi$$

is the reproducing kernel for B_{ω_j}, and

$$\begin{aligned}\Gamma : B_{\omega_j} &\to B_j(D'_\beta)\\ f(z_1) &\mapsto z_2^j f(z_1)\end{aligned}$$

is the isometry between Bergman spaces. Thus, by (1.4)

$$K'_j(z,w) = \frac{1}{2\pi} z_2^j \overline{w}_2^j \int_{\mathbb{R}} k'_j(\xi,y,w_1) e^{ix\xi} d\xi$$

from which the lemma follows. □

2. Improving the Bergman projection

Crucial to the proof in [1] of the failure of the Bergman projection to preserve $W^s(D_\beta)$ is the existence of poles of $k'_j(\xi,y,w_1)$ in the ξ variable whose imaginary part is a non-integer multiple of i. We see from (1.3) that such poles of $k'_j(\xi,y,w_1)$ are due to the zeros of $\hat{\omega}_j(-2i\xi)$ at $(j+1)/2 + ik\pi/(2\beta-\pi)$ for k a non-zero integer. In this section we deal with this obstruction by adding a correction term which eliminates such poles.
We assume initially that $j=-1$. To keep the notation that integral operators are defined by integrating functions against conjugates of functions of two variables (the kernel), we will work with terms in the kernel coming from $\hat{\omega}_j(2i\xi)$, observing that $\overline{\hat{\omega}_j(-2i\xi)} = \hat{\omega}_j(2i\xi)$. The goal in this section then is to find a function, denoted by $\hat{h}(\xi,y)$, defined in $\mathbb{C}\times I_\beta$ such that $\hat{h}(\xi,y)$, cancels the poles of the function

$$\frac{1}{\hat{\omega}_{-1}(2i\xi)} e^{-\xi y} \tag{2.1}$$

at $\xi = ik\nu_\beta$, for k a non-zero integer, and $\nu_\beta = \pi/(2\beta-\pi)$. The function $\overline{\hat{h}(\xi,y)}$ will have an inverse transform which is orthogonal to $B_{\omega_{-1}}$ and satisfy certain L^2 estimates which will be used in Section 3 to construct an integral operator. To ease notation we set

$$\tau_k(\xi) = (-1)^k \frac{e^{-k^2\nu_\beta^2}}{(2\beta-\pi)\pi} \frac{\xi^2}{\sinh(\pi\xi)} e^{-\xi^2} \qquad k\in\mathbb{Z}.$$

We define

$$\hat{h}_k(\xi,y) = \frac{\tau_k(\xi) e^{(\xi-2ik\nu_\beta)y}}{\xi - ik\nu_\beta}. \tag{2.2}$$

We note that the pole of (2.1) at $\xi = ik\nu_\beta$, for k a non-zero integer is the same as the pole of $\hat{h}_k$. Our aim is to sum $\hat{h}_k$ over k in order to produce a function which will be used to eliminate all such poles of (2.1). The following proposition shows that we can sum over k.
To keep track of the poles, we introduce the set P of all poles:

$$P := \{ik\nu_\beta : k\neq 0\} \cup \{ik : k \neq 0\}.$$

Proposition 2.1. *Let $\hat{h}_k(\xi, y)$ be defined as above. The infinite sum*

$$\sum_{k\neq 0} \hat{h}_k(\xi, \cdot)$$

converges in $L^\infty(I_\beta)$ to a function $\hat{h}(\xi, \cdot)$ uniformly in ξ on compact subsets of $\mathbb{C} \setminus P$.
Let $B_r = \cup B(ik\nu_\beta; r)$ denote the union of balls centered at elements of P for some fixed radius $r > 0$. Let U be any neighborhood of P containing B_r. Then on $\mathbb{C} \setminus U \times I_\beta$

$$|\hat{h}(\xi, y)| \lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta-\pi)|Re\xi|}, \tag{2.3}$$

with the constant of inequality depending only on U.

Proof.

$$\sum_{k\neq 0} \hat{h}_k(\xi, y) = \sum_{k\neq 0} \frac{\tau_k(\xi) e^{(\xi - 2ik\nu_\beta)y}}{\xi - ik\nu_\beta}$$

is a sum of terms of the form

$$e^{\xi y} \sum_{k\neq 0} a_k(\xi)$$

where

$$|a_k(\xi)| \lesssim \frac{1}{k} e^{-k^2\nu_\beta^2} |\xi|^2 e^{-Re\xi^2} e^{-\pi|Re\xi|} \qquad k \neq 0.$$

Inequality (2.3) is then a consequence of

$$\begin{aligned} |\hat{h}(\xi, y)| &= \left| e^{\xi y} \sum_k a_k(\xi) e^{-2ik\nu_\beta y} \right| \\ &\lesssim e^{\beta|Re\xi|} \sum_k |a_k(\xi)| \\ &\lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta-\pi)|Re\xi|}. \end{aligned}$$

□

We note for $f \in B_{\omega_{-1}}$:

$$\begin{aligned} \int_{\mathbb{R}} \int_{I_\beta} \overline{\hat{h}_k(\xi, y)} \hat{f}(\xi, y) \omega_{-1}(y) dy d\xi &= \int_{\mathbb{R}} \int_{I_\beta} \overline{\hat{h}_k(\xi, y)} e^{-y\xi} \hat{f}_{\mathbb{R}}(\xi) \omega_{-1}(y) dy d\xi \\ &= \int_{\mathbb{R}} \frac{\tau_k(\xi)}{\xi + ik\nu_\beta} \hat{f}_{\mathbb{R}}(\xi) \left[\int_{I_\beta} e^{2ik\nu_\beta y} \omega_{-1}(y) dy \right] d\xi \\ &= 0, \end{aligned}$$

where we use the representation of f in (1.2) in the first step, and the fact that $\int_{I_\beta} e^{2ik\nu_\beta y} \omega_{-1}(y) dy = \hat{\omega}_{-1}(-2k\nu_\beta) = 0$ in the last.
We collect the essential properties, which follow directly from the above, of the kernel function $h(x, y)$ in the following theorem:

Theorem 2.2. *There exists $h(x,y) \in L^2_{\omega_{-1}}(S_\beta)$ with the following properties:*
(i) For each $y \in I_\beta$, the poles of

$$\hat{h}(\xi, y) + \frac{1}{\hat{\omega}_{-1}(2i\xi)} e^{-\xi y}$$

with respect to ξ lie at only integer multiples of i.
(ii) The kernel given by

$$\mathcal{H}'(z,w) = \frac{1}{2\pi} \frac{1}{z_2 \overline{w}_2} \int_{\mathbb{R}} \overline{\hat{h}(\xi, y)} e^{i(x-\overline{w}_1)\xi} d\xi$$

is orthogonal to the space $B_{-1}(D'_\beta)$ in the sense that $\mathcal{H}'(\cdot, w) \perp B_{-1}(D'_\beta)$.
(iii) Let U be any neighborhood of P containing B_r for some $r > 0$. Then on $\mathbb{C} \setminus U \times I_\beta$

$$|\hat{h}(\xi, y)| \lesssim |\xi|^2 e^{-Re\xi^2} e^{(\beta-\pi)|Re\xi|},$$

with the constant of inequality depending only on U.

We also denote the horizontal lines

$$S_t = \{\mathbb{R} + it\}$$

for $t \in \mathbb{R}$. From the Theorem 2.2 iii), we have in particular, on any given S_t such that $S_t \cap P = \emptyset$, $\hat{h}(\xi, y)$ satisfies the following estimates uniformly, i.e. with constant of inequality independent of ξ:

$$\int_{I_\beta} \left| \hat{h}(\xi, y) \right|^2 dy \lesssim |\xi|^4 e^{-2Re\xi^2} e^{2(\beta-\pi)|Re\xi|}. \tag{2.4}$$

3. Mapping properties

We begin this section with some integral estimates for our constructed correction term. We let $\mathcal{H}'(z,w)$ be as in Theorem 2.2. Due to the $\overline{z}_2^{-1}$ factor in $\overline{\mathcal{H}'(z,w)}$, the operator determined by the kernel, $\mathcal{H}'(z,w)$, will have its action restricted to the $L^2_{-1}(D'_\beta)$ component of a given function in $L^2(D'_\beta)$
We use the equivalence between Bergman spaces given in Lemma 1.1 in the proof of the next proposition: for $G \in B_{-1}(D'_\beta)$, G is of the form $G = g(z_1) z_2^{-1}$, where $g \in B_{\omega_{-1}}$, and $\|G\|_{B_{-1}(D'_\beta)} = \|g\|_{B_{\omega_{-1}}}$.

Proposition 3.1. *Let $\beta > \pi/2$, and $\mathbf{H}'$ be the integral operator*

$$\mathbf{H}' f(w) = \int_{D'_\beta} f(z) \overline{\mathcal{H}'(z,w)} dV(z),$$

where

$$\mathcal{H}'(z,w) = \frac{1}{2\pi} \frac{1}{z_2 \overline{w}_2} \int_{\mathbb{R}} \overline{\hat{h}(\xi, y)} e^{i(x-\overline{w}_1)\xi} d\xi.$$

Then

$$\mathbf{H}' : L^2(D'_\beta) \to B_{-1}(D'_\beta),$$

and

$$\|\mathbf{H}' f\|_{B_{-1}(D'_\beta)} \lesssim \|f\|_{L^2_{-1}(D'_\beta)}.$$

Proof. We write $D'_\beta = \mathbb{R}\times d'_\beta$, where

$$d'_\beta = \{(y,z_2)\in\mathbb{R}\times\mathbb{C} : |y-\log z_2\overline{z}_2| < \pi/2, |\log z_2\overline{z}_2| < \beta-\pi/2\}.$$

Then,

$$\begin{aligned}\mathbf{H}'f(w) &= \frac{1}{2\pi}\frac{1}{w_2}\int_{D'_\beta}\frac{1}{\overline{z}_2}\int_{\mathbb{R}}\hat{h}(\xi,y)e^{-i(x-w_1)\xi}d\xi f(z)dV(z)\\ &= \frac{1}{2\pi}\frac{1}{w_2}\int_{d'_\beta}\frac{1}{\overline{z}_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{h}(\xi,y)e^{-i(x-w_1)\xi}f(x,y,z_2)dxd\xi dydV(z_2)\\ &= \frac{1}{2\pi}\frac{1}{w_2}\int_{d'_\beta}\frac{1}{\overline{z}_2}\int_{\mathbb{R}}\hat{h}(\xi,y)e^{iw_1\xi}\hat{f}(\xi,y,z_2)d\xi dydV(z_2).\end{aligned}$$

We use a decomposition of f according to

$$f(z) = \sum_j f_j(z), \qquad f_j(z)\in L^2_j(D'_\beta).$$

Using the orthogonality of powers of z_2 (over circular regions) we can isolate any f_j by integrating through $\overline{z}_2^j$. This is used in the third step below where after integrating over z_2 only $f_{-1}(z)$ terms remain:

$$\|\mathbf{H}'f\|^2_{B_{-1}(D'_\beta)} = \frac{1}{4\pi^2}\left\|\int_{d'_\beta}\frac{1}{\overline{z}_2}\int_{\mathbb{R}}\hat{h}(\xi,y)e^{i(\cdot)\xi}\hat{f}(\xi,y,z_2)d\xi dydV(z_2)\right\|^2_{B_{\omega_{-1}}}$$

$$= \frac{1}{4\pi^2}\int_{I_\beta}\int_{\mathbb{R}}\left|\int_{d'_\beta}\frac{1}{\overline{z}_2}\int_{\mathbb{R}}\hat{h}(\xi,y)e^{-y'\xi}e^{ix'\xi}\hat{f}(\xi,y,z_2)d\xi dydV(z_2)\right|^2 dx'\omega_{-1}(y')dy'$$

$$= \frac{1}{4\pi^2}\int_{I_\beta}\int_{\mathbb{R}}\left|\int_{d'_\beta}\frac{1}{\overline{z}_2}\hat{h}(\zeta,y)e^{-y'\zeta}\hat{f}_{-1}(\zeta,y,z_2)dydV(z_2)\right|^2 d\zeta\omega_{-1}(y')dy'$$

$$\lesssim \int_{I_\beta}\int_{\mathbb{R}}\left[\left(\int_{I_\beta}\left|\hat{h}(\zeta,y)\right|^2\omega_{-1}(y)dy\right)\times\right.$$
$$\left.\left(\int_{d'_\beta}\left|\hat{f}_{-1}(\zeta,y,z_2)\right|^2 dydV(z_2)\right)e^{-2y'\zeta}\right]d\zeta\omega_{-1}(y')dy'.$$

From Theorem 2.2 (*iii*) and (2.4) we have that

$$\int_{I_\beta}\left|\hat{h}(\zeta,y)\right|^2\omega_{-1}(y)dy \lesssim |\zeta|^4 e^{-2Re\zeta^2}e^{2(\beta-\pi)|Re\zeta|}.$$

We continue with our estimate of $\|\mathbf{H}'f\|_{B_{-1}(D'_\beta)}$:

$$\begin{aligned}\|\mathbf{H}'f\|^2_{B_{-1}(D'_\beta)} &\lesssim \int_{\mathbb{R}}\int_{d'_\beta}\left|\hat{f}_{-1}(\zeta,y,z_2)\right|^2 dydV(z_2)|\zeta|^4e^{-2\zeta^2}e^{2(\beta-\pi)|\zeta|}\hat{\omega}_{-1}(-2i\zeta)d\zeta\\ &\lesssim \|f_{-1}\|^2_{L^2(D'_\beta)},\end{aligned}$$

where the last estimate follows by the fact that the term $|\zeta|^4e^{-2\zeta^2}e^{2(\beta-\pi)|\zeta|}\hat{\omega}_{-1}(-2i\zeta)$ is bounded with respect to ζ. □

We recall the biholomorphic mapping $\Psi : D'_\beta \to D_\beta$ from (1.1). Through a change of variables Ψ^{-1}, $\mathbf{H}'$ induces an integral operator on $L^2(D_\beta)$: $g \mapsto (g \circ \Psi) det(\Psi^{-1})'$, $(\Psi^{-1})'$ being the complex Jacobian of $(\Psi^{-1})'$, is an isometry between $L^2(D_\beta)$ and $L^2(D'_\beta)$, and in fact since Ψ is biholomorphic, between Bergman spaces (see also (1.4)). In this regard, we define the kernel

$$\mathcal{H}(z,w) = \frac{1}{z_1 \overline{w}_1} \mathcal{H}'(\Psi^{-1}z, \Psi^{-1}w), \tag{3.1}$$

using the fact that $det(\Psi^{-1}(z))' = \frac{1}{z_1}$.
Let $\mathbf{H}$ be the integral operator

$$\mathbf{H}f(w) = \int_{D_\beta} f(z)\overline{\mathcal{H}(z,w)}dV(z),$$

where $\mathcal{H}(z,w)$ is given by (3.1).
Then as a result of Proposition 3.1, we have the following

COROLLARY 3.2. *We have that*

$$\mathbf{H} : L^2(D_\beta) \to B_{-1}(D_\beta),$$

and

$$\|\mathbf{H}f\|_{B_{-1}(D_\beta)} \lesssim \|f\|_{L^2_{-1}(D_\beta)}.$$

We now define the projection operator $\mathbf{T}_{-1}$ as

$$\mathbf{T}_{-1} = \mathbf{P}_{-1} + \mathbf{H},$$

where $\mathbf{P}_{-1} : L^2(D_\beta) \to B_{-1}(D_\beta)$ is the orthogonal projection operator.

4. PROPERTIES OF THE PROJECTION $\mathbf{T}_{-1}$

THEOREM 4.1. *Let $\beta > \pi/2$ and $\mathbf{T}_{-1} = \mathbf{P}_{-1} + \mathbf{H}$. Then*

$$\mathbf{T}_{-1} : L^2(D_\beta) \to B_{-1}(D_\beta).$$

Furthermore, $\mathbf{T}_{-1}$ is a projection, and has the regularity property

$$\mathbf{T}_{-1} : W^k_{\mathscr{D}}(D_\beta) \to W^k_{-1}(D_\beta) \qquad \forall k, \tag{4.1}$$

and

$$\|\mathbf{T}_{-1}f\|_{W^k_{-1}(D_\beta)} \lesssim \|f\|_{W^k(D_\beta)}$$

for $f \in W^k_{\mathscr{D}}(D_\beta)$.

Proof. The mapping from $L^2(D_\beta)$ to $B_{-1}(D_\beta)$ follows from the corresponding properties of $\mathbf{P}_{-1}$ and $\mathbf{H}$ (see Corollary 3.2).
That $\mathbf{T}_{-1}$ is a projection follows from $\mathbf{P}_{-1}$ being a projection and from the restriction of $\mathbf{H}$ to $B_{-1}(D_\beta)$ being equivalently 0 (from Theorem 2.2 *ii.*):

$$\begin{aligned}\mathbf{T}^2_{-1} &= \mathbf{P}^2_{-1} + \mathbf{P}_{-1}\mathbf{H} + \mathbf{H}\mathbf{P}_{-1} + \mathbf{H}^2 \\ &= \mathbf{P}_{-1} + \mathbf{H} \\ &= \mathbf{T}_{-1}.\end{aligned}$$

Since $\mathbf{T}_{-1}f$ is holomorphic, to prove (4.1) we estimate the L^2 norm of holomorphic derivatives of $\mathbf{T}_{-1}f$. Also, $\mathbf{T}_{-1}f$ is of the form $g(w_1,|w_2|)w_2^{-1}$, where the function $g(w_1,|w_2|)$ is holomorphic and locally constant in w_2, so its derivatives in w_2 are zero and we only need to estimate the derivatives with respect to the first variable. To prove the theorem we thus show

$$\left\| \frac{\partial^k}{\partial w_1^k}\mathbf{T}_{-1}f \right\|_{L^2(D_\beta)} \lesssim \|f_{-1}\|_{W^k(D_\beta)}, \tag{4.2}$$

for $f \in W^k_{\mathscr{D}}(D_\beta)$.

The domain D'_β is related to D_β via the biholomorphic mapping Ψ. We can then read off the kernels attached to the domain D_β from the transformation formula applied to the corresponding kernels on D'_β, as in (3.1). We have the relations

$$\begin{aligned} K_{-1}(z,w) &= \frac{1}{z_1\overline{w}_1}K'_{-1}(\Psi^{-1}z,\Psi^{-1}w)\\ \mathcal{H}(z,w) &= \frac{1}{z_1\overline{w}_1}\mathcal{H}'(\Psi^{-1}z,\Psi^{-1}w)\\ \mathcal{T}_{-1}(z,w) &= \frac{1}{z_1\overline{w}_1}\mathcal{T}'_{-1}(\Psi^{-1}z,\Psi^{-1}w), \end{aligned}$$

where K_{-1}, $\mathcal{H}$, $\mathcal{T}_{-1}$ (resp. K'_{-1}, $\mathcal{H}'$, $\mathcal{T}'_{-1}$) are the kernels for, respectively, $\mathbf{P}_{-1}$, $\mathbf{H}$, $\mathbf{T}_{-1}$ (resp. $\mathbf{P}'_{-1}$, $\mathbf{H}'$, $\mathbf{T}'_{-1}$).

Using integration by parts, we relate $\frac{\partial^k}{\partial w_1^k}\mathbf{T}_{-1}f$ to k^{th} order derivatives falling on f.

From above, we have

$$\mathbf{T}_{-1}f(w) = \int_{D_\beta} \overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z),$$

where

$$\begin{aligned} \overline{\mathcal{T}_{-1}(z,w)} = \frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}\Bigg(&\frac{1}{\hat{\omega}_{-1}(2i\xi)}\overline{z}_1^{-i\xi-1}w_1^{i\xi-1}\\ &+\hat{h}(\xi,(\log z_1-\log\overline{z}_1)/2i)z_1^{-i\xi/2-1}\overline{z}_1^{-i\xi/2}w_1^{i\xi-1}\Bigg)d\xi. \end{aligned}$$

By virtue of the factor $\overline{z}_2^{-1}$ in $\overline{\mathcal{T}_{-1}(z,w)}$, all action is isolated on $f_{-1}(z)$. Thus,

$$\begin{aligned} \mathbf{T}_{-1}f(w) &= \int_{D_\beta}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)\\ &= \int_{D_\beta}\overline{\mathcal{T}_{-1}(z,w)}f_{-1}(z)dV(z). \end{aligned}$$

Furthermore,

$$\frac{\partial^k}{\partial w_1^k}\mathbf{T}_{-1}f = \int_{D_\beta}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f_{-1}(z)dV(z), \tag{4.3}$$

and

$$\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)} =$$

$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}(i\xi-1)(i\xi-2)\cdots(i\xi-k)\Bigg(\frac{1}{\hat{\omega}_{-1}(2i\xi)}\overline{z}_1^{-i\xi-1}w_1^{i\xi-k-1}$$

$$+\hat{h}(\xi,(\log z_1-\log\overline{z}_1)/2i)z_1^{-i\xi/2-1}\overline{z}_1^{-i\xi/2}w_1^{i\xi-k-1}\Bigg)d\xi. \tag{4.4}$$

Our strategy is roughly as follows: we use shifts of contours of integration to write the integrands of (4.4) using derivatives with respect to z_1; we make sure Fubini's theorem applies with respect to the z and ξ integrals and then we take the z_1 derivatives outside the ξ integrals; finally we can then perform an integration by parts in the z_1 variable in (4.3).

When shifting the contour of integration, in order to verify that Fubini's theorem applies, we work with the two cases, each of which determines a different direction of shift:

i) $|w_1|<|z_1|$

ii) $|z_1|<|w_1|$.

To illustrate the cases, we consider integrals of the form

$$\phi_t(w_1)=\int_U\frac{1}{\overline{z}_2}\int_{Im(\xi)=t}\sigma_{w_1}(\xi,z_1,\overline{z}_1)f_{-1}(z)d\xi dV(z),$$

where σ_{w_1} will be either

$$(i\xi-1)(i\xi-2)\cdots(i\xi-k)\frac{1}{\hat{\omega}_{-1}(2i\xi)}\frac{1}{z_1w_1^{k+1}}\left(\frac{w_1}{\overline{z}_1}\right)^{i\xi}$$

or

$$(i\xi-1)(i\xi-2)\cdots(i\xi-k)\hat{h}(\xi,(\log z_1-\log\overline{z}_1)/2i)\frac{1}{z_1w_1^{k+1}}\left(\frac{w_1}{|z_1|}\right)^{i\xi},$$

and the domain of integration U will be either $D_\beta\bigcap\{|w_1|<|z_1|\}$ or $D_\beta\bigcap\{|z_1|<|w_1|\}$. Using the estimates for $\hat{\omega}_{-1}(2i\xi)$ and the estimate in (2.3) for $\hat{h}$, we have

$$|\phi_t(w_1)|\lesssim\int_U\left|\frac{1}{\overline{z}_2}\frac{1}{z_1w_1^{k+1}}\right|\left(\frac{|z_1|}{|w_1|}\right)^t|f_{-1}(z)|dV(z). \tag{4.5}$$

We see Fubini's theorem applies in case *i)* when $t<0$ and in case *ii)* when $t>0$. The signs of t correspond to shifts in the lower- and upper half planes, respectively.

We now proceed to the write an expression for the kernel $\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}$ in terms of derivatives with respect to the z variable, corresponding to the two cases. It will be shown in both cases we are lead to the same expression.

Case *i)*. By construction of the term h in Section 2 the integrand exhibits poles only at integer multiples of i, of which those at $-i,-2i,\ldots,-ik$ are cancelled. We therefore deform the contour of integration in (4.4) to $\mathbb{R} - ik$. The contribution of the sides of the contour are null due to the exponential decay in ξ of the integrand.

We now work with the contour of integration in (4.4) deformed to $\mathbb{R} - ik$. We first consider

$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}-ik}(i\xi-1)(i\xi-2)\cdots(i\xi-k)\frac{1}{\hat{\omega}_{-1}(2i\xi)}\overline{z}_1^{-i\zeta-1}w_1^{i\xi-k-1}d\xi =$$
$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}(i\zeta+k-1)(i\zeta+k-2)\cdots(i\zeta)\frac{1}{\hat{\omega}_{-1}(2i(\zeta-ik))}\overline{z}_1^{-i\zeta-k-1}w_1^{i\zeta-1}d\zeta.$$

We use

$$\frac{1}{\hat{\omega}_{-1}(2i(\zeta-ik))} = (-1)^k\frac{1}{\pi}\frac{(\zeta-ik)^2}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)},$$

hence

$$\begin{aligned}
&\frac{(i\zeta+k-1)(i\zeta+k-2)\cdots(i\zeta)}{\hat{\omega}_j(2i(\zeta-ik))}\overline{z}_1^{-i\zeta-k-1}w_1^{i\zeta-1}\\
&\qquad = (-1)^k\frac{1}{\pi}\frac{(\zeta-ik)(\zeta-ik)}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\times\\
&\qquad\qquad (i\zeta+k-1)(i\zeta+k-2)\cdots(i\zeta)\overline{z}_1^{-i\zeta-k-1}w_1^{i\zeta-1}\\
&\qquad = (-1)^k\frac{1}{\pi}\frac{(\zeta-ik)(i\zeta)}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\times\\
&\qquad\qquad (\zeta-ik)(i\zeta+k-1)\cdots(i\zeta+1)\overline{z}_1^{-i\zeta-k-1}w_1^{i\zeta-1}\\
&\qquad = \frac{1}{\pi}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z}_1{}^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}.
\end{aligned}$$

For the integral above we thus have

$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}-ik}(i\xi-1)(i\xi-2)\cdots(i\xi-k)\frac{1}{\hat{\omega}_{-1}(2i\xi)}\overline{z}_1^{-i\zeta-1}w_1^{i\xi-k-1}d\xi =$$
$$\frac{1}{2\pi^2}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z}_1{}^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta.$$

Similarly, we work with

$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}-ik}(i\xi-1)(i\xi-2)\cdots(i\xi-k)\hat{h}(\xi,(\log z_1-\log\overline{z}_1)/2i)\times z_1^{-i\xi/2-1}\overline{z}_1^{-i\xi/2}w_1^{i\xi-k-1}d\xi. \tag{4.6}$$

Let us write

$$\hat{h}(\xi,(\log z_1-\log\overline{z}_1)/2i) = \frac{\xi}{\sinh(\pi\xi)}g(\xi,z_1),$$

and note that $g(\xi,z_1)$ has the property

$$\Lambda_t g(\xi,z_1) = 0,$$

where

$$\Lambda_t := \left(\left(\frac{z_1}{\overline{z}_1}\right)^{1/2}\frac{\partial}{\partial z_1} + \left(\frac{\overline{z}_1}{z_1}\right)^{1/2}\frac{\partial}{\partial \overline{z}_1}\right).$$

The integrand in (4.6) can thus be written according to

$$\begin{aligned}
&(i\zeta+k-1)(i\zeta+k-2)\cdots(i\zeta)\hat{h}(\zeta-ik,(\log z_1-\log\overline{z}_1)/2i)\times\\
&\qquad\qquad z_1^{(-i\zeta-k)/2-1}\overline{z}_1^{(-i\zeta-k)/2}w_1^{i\zeta-1}\\
&=(-1)^k\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)\times\\
&\qquad(i\zeta+k)(i\zeta+k-1)\cdots(i\zeta+1)z_1^{(-i\zeta-k)/2-1}\overline{z}_1^{(-i\zeta-k)/2}w_1^{i\zeta-1}\\
&=\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)\times\\
&\qquad\left(\left(\frac{z_1}{\overline{z}_1}\right)^{1/2}\frac{\partial}{\partial z_1}+\left(\frac{\overline{z}_1}{z_1}\right)^{1/2}\frac{\partial}{\partial\overline{z}_1}\right)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}\\
&=\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)(\Lambda_t)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{|w_1|<|z_1|}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)=-\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\times\\
&\quad\int_{Rew_1<Rez_1}\left[\frac{1}{\pi}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z}_1{}^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta+\right.\\
&\quad\left.\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)(\Lambda_t)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta\right]f_{-1}(z)dV(z).
\end{aligned}$$

We remark that, as outlined above, the ζ and z integrations can be switched (just consider the integral ϕ_k in (4.5)).
Case ii). We begin by writing (4.4) in the form:

$$\begin{aligned}
&\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}=\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\times\\
&\qquad\int_{\mathbb{R}}\left[\frac{(-1)^k}{\pi}\frac{(\xi+ik)(\xi)}{\sinh\left[(2\beta-\pi)(\xi)\right]\sinh(\pi\xi)}\frac{\partial^k}{\partial\overline{z}_1^k}\overline{z}_1^{-i\xi+(k-1)}w_1^{i\xi-1-k}+\right.\\
&\qquad\qquad\left.(-1)^k\frac{\xi+ik}{\sinh(\pi\xi)}g(\xi,z_1)\left(\Lambda_t\right)^k z_1^{-i\xi/2+k/2-1}\overline{z}_1^{-i\xi/2+k/2}w_1^{i\xi-k-1}\right]d\xi,
\end{aligned}$$

which is also obtained by deforming the contour of integration to $\mathbb{R}+ik$ (using that the sides of the contour give no contributions in the same manner as that of case i)) of the following integral

$$-\frac{1}{2\pi}\int_{\mathbb{R}}\left[\frac{1}{\pi}\frac{\zeta(\zeta-ik)}{\sinh\left[(2\beta-\pi)(\zeta-ik)\right]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z}_1^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}\right.$$
$$\left.+\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)\left(\Lambda_t\right)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}\right]d\zeta,$$

noting that the contribution from the poles at integer multiples of i are cancelled due to the differential operators.
Combining the results in cases i) and ii), we have

$$\int_{D_\beta}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)=$$
$$\int_{|w_1|<|z_1|}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)+\int_{|w_1|>|z_1|}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z),$$

where

$$\int_{|w_1|<|z_1|}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)=$$
$$-\frac{1}{2\pi}\int_{|w_1|<|z_1|}\frac{1}{\overline{z}_2w_2}\times$$
$$\left[\frac{1}{\pi}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z_1}^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta+\right.$$
$$\left.\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)(\Lambda_t)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta\right]f_{-1}(z)dV(z)$$

and

$$\int_{|w_1|>|z_1|}\frac{\partial^k}{\partial w_1^k}\overline{\mathcal{T}_{-1}(z,w)}f(z)dV(z)=$$
$$-\frac{1}{2\pi}\int_{|w_1|>|z_1|}\frac{1}{\overline{z}_2w_2}\times$$
$$\left[\frac{1}{\pi}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\frac{\partial^k}{\partial\overline{z_1}^k}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta+\right.$$
$$\left.\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)(\Lambda_t)^k z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta\right]f_{-1}(z)dV(z).$$

We now use Fubini's theorem in both case i) and ii) to take the derivatives outside of the ζ integrals, and then combine the results above. Before doing so, we note

$$\begin{aligned}\Lambda_t =&\left(\left(\frac{z_1}{\overline{z}_1}\right)^{1/2} \frac{\partial}{\partial z_1} + \left(\frac{\overline{z}_1}{z_1}\right)^{1/2} \frac{\partial}{\partial \overline{z}_1} \right)\\ =&\partial_{r_1},\end{aligned}$$

where $r_1 = |z_1|$, is a tangential differential operator. To calculate the adjoint of Λ_t we note for fixed z_2, z_1 can be written with coordinates t_1 and d_1, d_1 representing the distance to the boundary $\mathrm{Re}\ z_1 e^{-i \log z_2 \overline{z}_2} = 0$, via

$$z_1 = (t_1 + id_1)e^{i\alpha} \tag{4.7}$$

where $\alpha = \log |z_2|^2 - \pi/2$. In these coordinates we calculate

$$\Lambda_t = \frac{t_1}{\sqrt{t_1^2 + d_1^2}} \frac{\partial}{\partial t_1} + \frac{d_1}{\sqrt{t_1^2 + d_1^2}} \frac{\partial}{\partial d_1}.$$

Then,

$$\begin{aligned}(\Lambda_t)^* =& - \Lambda_t - \frac{\partial}{\partial t_1}\left(\frac{t_1}{\sqrt{t_1^2 + d_1^2}} \right) - \frac{\partial}{\partial d_1}\left(\frac{d_1}{\sqrt{t_1^2 + d_1^2}} \right)\\ =& - \Lambda_t - \frac{1}{\sqrt{t_1^2 + d_1^2}}\\ =& - \Lambda_t - \frac{1}{|z_1|}. \qquad (4.8)\end{aligned}$$

Furthermore, from the relation (4.7), we can write

$$\frac{\partial}{\partial \overline{z}_1} = \alpha_1 \frac{\partial}{\partial z_1} + \alpha_2 \frac{\partial}{\partial t_1},$$

where $\alpha_1(|z_2|)$ and $\alpha_2(|z_2|)$ are bounded away from 0 and depend smoothly on $|z_2|$.

We recall that that $g(\xi, z_1)$ has the property $\Lambda_t g(\xi, z_1) = 0$, and so

$$g(\zeta - ik, z_1)(\Lambda_t)^k z_1^{-i\zeta/2-1} \overline{z}_1^{-i\zeta/2} = (\Lambda_t)^k \left[z_1^{-i\zeta/2-1} \overline{z}_1^{-i\zeta/2} g(\zeta - ik, z_1) \right].$$

We thus have, after commuting the z derivatives with the ζ integrals,

$$\begin{aligned}&\int_{D_\beta} \frac{\partial^k}{\partial w_1^k} \overline{\mathcal{T}_{-1}(z,w)} f(z) dV(z) =\\ &-\frac{1}{2\pi} \int_{D_\beta} \frac{1}{\overline{z}_2 w_2} \times\\ &\qquad \left[\frac{1}{\pi} \frac{\partial^k}{\partial \overline{z}_1{}^k} \int_{\mathbb{R}} \frac{(\zeta - ik)\zeta}{\sinh[(2\beta - \pi)(\zeta - ik)] \sinh(\pi\zeta)} \overline{z}_1^{-i\zeta-1} w_1^{i\zeta-1} d\zeta + \right.\\ &\left. (\Lambda_t)^k \int_{\mathbb{R}} \frac{\zeta}{\sinh(\pi\zeta)} g(\zeta - ik, z_1) z_1^{-i\zeta/2-1} \overline{z}_1^{-i\zeta/2} w_1^{i\zeta-1} d\zeta \right] f_{-1}(z) dV(z). \qquad (4.9)\end{aligned}$$

Integrating by parts in the first integral on the right in (4.9) gives

$$
\begin{aligned}
&-\frac{1}{2\pi^2}\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}\times \\
&\quad\left[\frac{\partial^k}{\partial\overline{z}_1{}^k}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta\right]f_{-1}(z)dV(z) \\
&=-\frac{1}{2\pi^2}\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}\times \\
&\quad\left[(\alpha_2\partial_{t_1})^k\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta\right]f_{-1}(z)dV(z) \\
&=(-1)^{k+1}\frac{1}{2\pi^2}\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}\times \\
&\quad\left[\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta\right](\alpha_2\partial_{t_1})^k f_{-1}(z)dV(z).
\end{aligned}
\tag{4.10}
$$

Similarly, we perform an integration by parts in the second integral in (4.9), using (4.8).

$$
\begin{aligned}
&-\frac{1}{2\pi}\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}(\Lambda_t)^k\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta f_{-1}(z)dV(z) \\
&\quad=(-1)^{k+1}\frac{1}{2\pi}\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}\left[\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta\right]\times \\
&\qquad\left(\Lambda_t+|z_1|^{-1}\right)^k f_{-1}(z)dV(z).
\end{aligned}
\tag{4.11}
$$

To finish the proof we note that the proof of Proposition 3.1 , with $\hat{h}(\xi,y)e^{-ix\xi}$ replaced with

$$\frac{(\xi-ik)\xi}{\sinh[(2\beta-\pi)(\xi-ik)]\sinh(\pi\xi)}e^{-i\overline{z}_1\xi}$$

may be followed to show that the operator from (4.10) with kernel

$$\int_{D_\beta}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}\frac{(\zeta-ik)\zeta}{\sinh[(2\beta-\pi)(\zeta-ik)]\sinh(\pi\zeta)}\overline{z}_1^{-i\zeta-1}w_1^{i\zeta-1}d\zeta$$

maps $L^2(D_\beta)$ to $L^2_{-1}(D_\beta)$. Similarly, the proof of Proposition 3.1 shows that the operator with kernel

$$\frac{1}{2\pi}\frac{1}{\overline{z}_2 w_2}\int_{\mathbb{R}}\frac{\zeta}{\sinh(\pi\zeta)}g(\zeta-ik,z_1)z_1^{-i\zeta/2-1}\overline{z}_1^{-i\zeta/2}w_1^{i\zeta-1}d\zeta$$

occurring in (4.11) maps $L^2(D_\beta)$ to $L^2_{-1}(D_\beta)$. We have estimates for the term in (4.11) when $f \in W^s_{\mathscr{D}}(D_\beta)$:

$$\begin{aligned}
\sum_{\alpha \le k} \left\| |z_1|^{-k+\alpha} \Lambda_t^\alpha f_{-1} \right\|_{L^2(D_\beta)} & \\
&\lesssim \sum_{\alpha \le k} \left\| |z_1|^{-k+\alpha} \Lambda_t^\alpha f_{-1} \right\|_{L^2((\mathbb{D}_1 \times \mathbb{C}) \cap D_\beta)} + \|f_{-1}\|_{W^k(D_\beta)} \\
&\lesssim \sum_{\alpha \le k} \left\| t_1^{-k+\alpha} \Lambda_t^\alpha f_{-1} \right\|_{L^2((\mathbb{D}_1 \times \mathbb{C}) \cap D_\beta)} + \|f_{-1}\|_{W^k(D_\beta)} \\
&\lesssim \|f_{-1}\|_{W^k(D_\beta)},
\end{aligned}$$

where $\mathbb{D}_1 := \{|z_1| \le 1\}$, the variable t_1 is as in (4.7), and the last step follows from Theorem 1.4.4.4 in [6] (with a slight variation in the argument we can also apply Theorem 11.8 in [9] which holds for smooth domains).
Then, together (4.10) and (4.11) show

$$\begin{aligned}
\left\| \frac{\partial^k}{\partial w_1^k} \mathbf{T}_{-1} f \right\|_{L^2(D_\beta)} &\lesssim \|f_{-1}\|_{W^k(D_\beta)} + \sum_{\alpha \le k} \left\| |z_1|^{-k+\alpha} \Lambda_t^\alpha f_{-1} \right\|_{L^2(D_\beta)} \\
&\lesssim \|f_{-1}\|_{W^k(D_\beta)}.
\end{aligned}$$

The estimate in (4.2) is verified, completing the proof of the theorem. □

5. The case $j \neq -1$

We construct operators

$$\mathbf{T}_j : W^k_{\mathscr{D}}(D_\beta) \to W^k_j(D_\beta) \qquad \forall k,$$

for the cases $j \neq -1$ as follows.
We let Q_j be the projection from $L^2(D_\beta)$ to $L^2_j(D_\beta)$ given by

$$Q_j f(z_1, z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_1, e^{i\theta} z_2) e^{-ij\theta} d\theta.$$

Then we take the operator $\mathbf{T}_j$ to be given by

$$\mathbf{T}_j f = w_2^{j+1} \mathbf{T}_{-1}(z_2^{-j-1} Q_j f).$$

For each $\mathbf{T}_j$, due to properties of the operator $\mathbf{T}_{-1}$, we have a theorem similar to Theorem 4.1:

Theorem 5.1. *Let $\beta > \pi/2$, and D_β be defined as above. For all $j \in \mathbb{Z}$ there exists a bounded linear projection*

$$\mathbf{T}_j : L^2(D_\beta) \to B_j(D_\beta)$$

which satisfies

$$\mathbf{T}_j : W^k_{\mathscr{D}}(D_\beta) \to W^k_j(D_\beta) \qquad \forall k,$$

and

$$\|\mathbf{T}_j f\|_{W^k_j(D_\beta)} \lesssim \|f\|_{W^k(D_\beta)}.$$

This proves the Main Theorem.

6. Remarks

We end with a few remarks. We first note that in our proof of Theorem 4.1, we worked with Sobolev spaces, W^k for integer k. The general case for all $s \geq 0$ follows by interpolation.

Secondly, there are infinitely many projection operators which have the same regularity properties as our constructed projection in the Main Theorem. Other projections can be constructed for instance by changing the factor $\tau_k(\xi)$ in Section 2 with the replacement of the term $e^{-\xi^2}$ with another $e^{-m\xi^2}$ for any positive integer m. Then the rest of the arguments could be followed verbatim. By [2], if the Bergman projection were to map $C_0^\infty(\overline{D_\beta})$ continuously into $C^\infty(\overline{D_\beta})$ (it does not) then we would automatically have continuity from the larger space $C^\infty(\overline{D_\beta})$ as well. Thus, it would be of interest to find an improvement to the projection, along the lines presented here, which preserves $W_j^s(D_\beta)$ for all $s \geq 0$.

We lastly note that, while it would be ideal to obtain an operator which would map W^s to itself, without the restriction to the space W_j^s, by summing the operators in Main Theorem over j, the dependence of the norms in Theorem 5.1 on j prohibit the convergence of such a summation. Following the calculations of the proof of Proposition 3.1 leads to the estimates for the norms of $\mathbf{T}_j$:

$$\|T_j\| \lesssim \frac{\sinh[(j+1)(\beta - \pi/2)]}{j+1}.$$

This exponential growth of the estimates thus prohibits us from using results such as the Cotlar–Stein almost orthogonality lemma to conclude any convergence of a sum over the operators $\mathbf{T}_j$.

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