Syzygies of 5-Gonal Canonical Curves

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Abstract. We show that for 5-gonal curves of odd genus $g \geq 13$ and even genus $g \geq 28$ the $\lceil \frac{g-1}{2} \rceil$-th syzygy module of the curve is not determined by the syzygies of the scroll swept out by the special pencil of degree 5.

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1. Introduction

In this article we study minimal free resolutions of the coordinate ring S_C of 5-gonal canonically embedded curves $C \subset \mathbb{P}^{g-1}$. The *gonality* of a curve C is defined as the minimal degree of a nonconstant map $C \longrightarrow \mathbb{P}^1$.

A pencil of degree k on a canonically embedded curve C of genus g, defining a degree k map $C \to \mathbb{P}^1$ sweeps out a rational normal scroll X of dimension $d = k-1$ and degree $f = g-k+1$. It follows that the linear strand of X is a subcomplex of the linear strand of the curve C. To be more precise the scroll contributes with an Eagon-Northcott complex of length $g-k$ to the linear strand of the curve. This means in particular, that the Betti numbers of X give a lower bound for the Betti numbers of C.

The main focus of this article lies on the relation between the Betti numbers of the canonical curve $C \subset \mathbb{P}^{g-1}$ and the Betti numbers of the scroll X defined by a pencil of minimal degree on C.

>From the values of the Hilbert function H_{S_C} and the relation with Betti numbers (see [Eis05, Corollary. 9.4 and Corollary. 1.10]) one obtains the following relation for the Betti numbers of a canonical curve $C \subset \mathbb{P}^{g-1}$:

$$\beta_{i,i+1}(C) = i \cdot \binom{g-2}{i+1} - (g-i-1) \cdot \binom{g-2}{i-2} + \beta_{i-1,i+1}(C).$$

Since the minimal free resolution of a canonical curve is self-dual, we have

$$\beta_{i-1,i+1}(C) = \beta_{g-i-1,g-i}(C) \geq \beta_{g-i-1,g-i}(X)$$

and a direct computation for the case $i = \lceil \frac{g-3}{2} \rceil$ shows that if $k > 3$ and $g \geq 5$, then

$$\beta_{i,i+1}(C) \geq i \cdot \binom{g-2}{i+1} - (g-i-1) \cdot \binom{g-2}{i-2} + \beta_{g-i-1,g-i}(X) > \beta_{i,i+1}(X)$$

for all $i = 1, \ldots, \lceil \frac{g-3}{2} \rceil$. We are interested in the Betti numbers $\beta_{i,i+1}(C)$ for $i \geq \lceil \frac{g-1}{2} \rceil$.

The gonality of a general canonical curve of genus g is precisely $\lceil \frac{g+2}{2} \rceil$ and therefore $\beta_{n,n+1}(X) = 0$, where $n = \lceil \frac{g-1}{2} \rceil$. For odd genus g and ground field of characteristic 0, Voisin and Hirschowitz-Ramanan (see [Voi05] and [HR98]) showed that the locus

$$\mathcal{K}_g := \{C \in \mathcal{M}_g \mid \beta_{n,n+1}(C) \neq 0\}$$

defines an effective divisor in the moduli space of curves, the so-called *Koszul divisor*.

On the Hurwitz-scheme $\mathcal{H}_{g,k}$ a natural analogue of the Koszul divisor could be the following

$$\mathcal{K}_{g,k} := \{C \in \mathcal{H}_{g,k} \mid \beta_{n,n+1}(C) > \beta_{n,n+1}(X)\}.$$

If the genus is odd, then $\mathcal{K}_{g,k}$ is a divisor on the Hurwitz-scheme if $\beta_{n,n+1}(C) = \beta_{n,n+1}(X)$ holds for a general curve $C \in \mathcal{H}_{g,k}$ (see [HR98, §3]). For gonality $k = 3, 4$ it is known from [Sch86, §6] that the so-called iterated mapping cone construction, which we recall in Section 2, always gives a minimal free resolution of $C \subset \mathbb{P}^{g-1}$. In particular $\beta_{n,n+1}(C) = \beta_{n,n+1}(X)$ holds for general 3-gonal and 4-gonal canonical curves. For 5-gonal curves we used *Macaulay2* (see [GS]) to verify computationally, that $\beta_{n,n+1}(C) = \beta_{n,n+1}(X)$ holds for general 5-gonal canonical curves of genus $g < 13$.

We will show that $\mathcal{K}_{g,5}$ is no longer a divisor for odd $g \geq 13$ by proving the following theorem.

Theorem. *Let C be a 5-gonal canonical curve of genus g and $n = \lceil \frac{g-1}{2} \rceil$. Then*

$$\beta_{n,n+1}(C) > \beta_{n,n+1}(X)$$

for odd genus $g \geq 13$ and even genus $g \geq 28$.

The proof is based on the techniques introduced in [Sch86]. First we resolve the curve C as an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module, where $\mathbb{P}(\mathcal{E})$ is the bundle associated to the rational normal scroll swept out by the g^1_5. In the next step we resolve the $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules occurring in this resolution by Eagon-Northcott type complexes. An iterated mapping cone construction then gives a non-minimal resolution of $C \subset \mathbb{P}^{g-1}$. By determining the ranks of the maps which give rise to non-minimal parts in the iterated mapping cone we can decide whether the curve has extra syzygies. In the last section we discuss the genus 13 case in detail.

Remark 1.1. The proof of the main theorem does not depend on the characteristic of the ground field $\Bbbk$. However for $\mathrm{char}(\Bbbk) > 0$ it is possible that

$\beta_{n,n+1}(C) > \beta_{n,n+1}(X)$ for general 5-gonal curves of genus $g < 13$. This happens, for example, for 5-gonal curves of genus 7 over a field of characteristic 2 [Sch86, §7].

Some of the statements in this article rely on computations done with *Macaulay2* [GS]. The *Macaulay2* code which verifies these statements can be found here:
http://www.math.uni-sb.de/ag-schreyer/images/data/computeralgebra/fiveGonalFile.m2

2. Scrolls, Pencils and Canonical Curves

In this section we briefly summarize the connections between pencils on canonical curves and rational normal scrolls. Most of this section follows Schreyer's article [Sch86].

Definition 2.1. *Let $e_1 \geq e_2 \geq \cdots \geq e_d \geq 0$ be integers, $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the corresponding $\mathbb{P}^{d-1}$-bundle.*
A rational normal scroll *$X = S(e_1, \ldots, e_d)$ of type $(e_1, \ldots, e_d)$ is the image of*

$$j : \mathbb{P}(\mathscr{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r$$

where $r = f + d - 1$ with $f = e_1 + \cdots + e_d \geq 2$.

In [Har81, §3] it is shown that the variety X defined above is a non-degenerate d-dimensional variety of minimal degree $f = r - d + 1 = \operatorname{codim} X + 1$. If $e_1, \ldots, e_d > 0$, then $j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r$ is an isomorphism. Otherwise it is a resolution of singularities and since the singularities of X are rational, we can consider $\mathbb{P}(\mathscr{E})$ instead of X for most cohomological considerations.
It is furthermore shown that the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the ruling $R = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $H = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$ with intersection products

$$H^d = f, \quad H^{d-1} \cdot R = 1, \quad R^2 = 0.$$

Remark 2.2 ([Sch86, (1.3)]). For $a \geq 0$ we have an isomorphism $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, S_a(\mathscr{E})(b))$, where $S_a(\mathscr{E})$ denotes the a^{th} symmetric power of $\mathscr{E}$. Thus we can compute the cohomology of the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ explicitly.
If we denote by $\Bbbk[s,t]$ the coordinate ring of $\mathbb{P}^1$ and by $\varphi_i \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R))$ the basic sections, then we can identify sections $\psi \in H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$ with homogeneous polynomials

$$\psi = \sum_{\alpha} P_\alpha(s,t) \varphi_1^{\alpha_1} \ldots \varphi_d^{\alpha_d}$$

of degree $a = \alpha_1 + \cdots + \alpha_d$ in φ_i's and with polynomial coefficients $P_\alpha \in \Bbbk[s,t]$ of degree $\deg P_\alpha = \alpha_1 e_1 + \cdots + \alpha_d e_d + b$. Thus for $a \cdot \min\{e_i\} + b \geq -1$ we get

$$h^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) = f \binom{a+d-1}{d} + (b+1) \binom{a+d-1}{d-1}.$$

In particular $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH+bR)$ is globally generated if and only if $a \geq 0$ and $a \cdot \min\{e_i\} + b \geq 0$.

Next we want to describe how to resolve line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH+bR)$ by $\mathcal{O}_{\mathbb{P}^r}$-modules. If we denote by Φ the $2 \times f$ matrix with entries in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H))$ obtained from the multiplication map

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \otimes H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \longrightarrow H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H)),$$

then the equations of X are given by the 2×2 minors of Φ. We define

$$F := H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \otimes \mathcal{O}_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}^f$$

and

$$G := H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \otimes \mathcal{O}_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}^2$$

and regard Φ as a map $\Phi : F(-1) \to G$. For $b \geq -1$, we consider the *Eagon-Northcott type* complex $\mathcal{C}$ (see [Eis95, Appendix A2.6]), whose j^{th} term is defined by

$$\mathcal{C}_j^b = \begin{cases} \bigwedge^j F \otimes S_{b-j} G \otimes \mathcal{O}_{\mathbb{P}^r}(-j), & \text{for } 0 \leq j \leq b \\ \bigwedge^{j+1} F \otimes D_{j-b-1} G^* \otimes \mathcal{O}_{\mathbb{P}^r}(-j-1), & \text{for } j \geq b+1 \end{cases}$$

where $S_j G$ denotes the j^{th} symmetric power and $D_j G^*$ denotes the j^{th} divided power of G. The differentials $\delta_j : \mathcal{C}_j^b \to \mathcal{C}_{j-1}^b$ are given by the multiplication with Φ for $j \neq b+1$ and by $\bigwedge^2 \Phi$ for $j = b+1$.

Theorem 2.3. *The Eagon-Northcott type complex $\mathcal{C}^b(a) := \mathcal{C}^b \otimes \mathcal{O}_{\mathbb{P}^r}(a)$, defined above, gives a minimal free resolution of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH+bR)$ as an $\mathcal{O}_{\mathbb{P}^r}$-module.*

Proof. See [Sch86, §1]. □

Now let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g and let further

$$g_k^1 = \{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |D|$$

be a pencil of divisors of degree k. If we denote by $\overline{D_\lambda} \subset \mathbb{P}^{g-1}$ the linear span of the divisor, then

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(k-1)$-dimensional rational normal scroll of degree $f = g-k+1$ (see e.g. [EH87, Theorem 2]). Conversely if X is a rational normal scroll of degree f containing a canonical curve, then the ruling on X cuts out a pencil of divisors $\{D_\lambda\} \subset |D|$ such that $h^0(C, \omega_C \otimes \mathcal{O}_C(D)^{-1}) = f$.

Notation 2.4. *During the rest of this article $C \subset \mathbb{P}^{g-1}$ will denote a canonical curve with a basepoint free g_k^1. The variety $X = S(e_1, \dots, e_d)$ is the scroll of degree $f = g-k+1$ and dimension $d = k-1$ defined by this pencil and $\mathbb{P}(\mathcal{E})$ will denote the $\mathbb{P}^{d-1}$-bundle corresponding to X.*

The next important theorem due to Schreyer explains how to obtain a free resolutions of k-gonal canonical curves $C \subset \mathbb{P}^{g-1}$ by an iterated mapping cone construction.

Theorem 2.5. *i) $C \subset \mathbb{P}(\mathscr{E})$ has a resolution $F_\bullet$ of type*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-(k-2)H + a_j^{(k-3)}R) \longrightarrow$$

$$\dots \longrightarrow \sum_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H + a_j^{(1)}R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})} \longrightarrow \mathscr{O}_C \longrightarrow 0$$

with $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

ii) The complex $F_\bullet$ is self dual, i.e.,

$$\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R)) \cong F_\bullet$$

iii) If all $a_k^{(j)} \geq -1$, then an iterated mapping cone

$$\left[\left[\dots\left[\mathscr{C}^{(f-2)}(-k) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{C}^{(a_j^{(k-3)})}(-k+2)\right] \longrightarrow \dots\right] \longrightarrow \mathscr{C}^0\right]$$

gives a, not necessarily minimal, resolution of C as an $\mathscr{O}_{\mathbb{P}^{g-1}}$-module.

Proof. See [Sch86, Corollary 4.4] and [Sch86, Lemma 4.2]. □

Remark 2.6. The $a_i^{(k)}$'s in part i) of the theorem above satisfy certain linear equations obtained from the Euler characteristic of the complex $F_\bullet$ and part (ii) of the Theorem above. In particular in [Sch86, §6] it is shown, that for 5-gonal curves C

$$\textstyle\sum_{i=1}^5 a_i = 2g-12 \quad \text{and} \quad a_i + b_i = f-2,$$

where $a_i := a_i^{(1)}$, $b_i := a_i^{(2)}$ and $f = g-4$ is the degree of the rational normal scroll swept out by the g_5^1 on C.

Definition 2.7. *We call a partition $(e_1, \dots, e_d)$* balanced *if* $\max_{i,j} |e_i - e_j| \leq 1$. *A rational normal scroll $X = S(e_1, \dots, e_d)$ of dimension d is said to be of* balanced type *if $(e_1, \dots, e_d)$ is a balanced partition.*
With the same notation as in the remark above we say that a 5-gonal canonical curve satisfies the balancing conditions *if*

- *the scroll X of dimension 4 swept out by the g_5^1 on C is of balanced type and*
- *the partition $(a_1, \dots, a_5)$ of $2g-12$ is balanced.*

Remark 2.8. One can show (see e.g. [GV06, Corollary 3.3]) that a generic k-gonal curve sweeps out a scroll of balanced type. The proof uses Ballico's Theorem [Bal89, Proposition 2.1.1] and the fact that the type of the scroll is uniquely determined by the g_k^1 (see [Sch86, (2.4)]).

For a section $\Psi : \mathscr{O}_X(-H+bR) \longrightarrow \mathscr{O}_X(aR)$ in $H^0(X, \mathscr{O}_X(H-(b-a)R))$ the induced comparison maps $\psi_\bullet : \mathscr{C}^b_\bullet(-1) \longrightarrow \mathscr{C}^a_\bullet$ between the corresponding Eagon-Northcott type complexes are determined by Ψ up to homotopy. For example

$$\operatorname{Hom}(\mathscr{C}^b_{a+1}(-1), \mathscr{C}^a_{a+2}) = \operatorname{Hom}(\mathscr{C}^b_a(-1), \mathscr{C}^a_{a+1}) = 0$$

by degree reasons, and therefore the $(a+1)^{st}$−comparison map ψ_{a+1} is uniquely determined by Ψ (not only up to homotopy). The following lemma is due to Martens and Schreyer.

LEMMA 2.9. *If* $\operatorname{Hom}(\mathscr{C}^b_j(-1), \mathscr{C}^a_{j+1}) = \operatorname{Hom}(\mathscr{C}^b_{j-1}(-1), \mathscr{C}^a_j) = 0$*, then the* j^{th}*-comparison map* $\psi_j : \mathscr{C}^b_j \to \mathscr{C}^a_j$ *is given (up to a scalar factor) by the composition*

$$\psi_j : \bigwedge^j F \otimes S_{b-j}G \longrightarrow \bigwedge^j F \otimes S_{b-j}G \otimes S_{j-a-1}G \otimes D_{j-a-1}G^*$$

$$\xrightarrow{id\otimes mult\otimes id} \bigwedge^j F \otimes S_{b-a-1}G \otimes D_{j-a-1}G^* \xrightarrow{id\otimes\Psi\otimes id} \bigwedge^j F \otimes F \otimes D_{j-a-1}G^*$$

$$\xrightarrow{\wedge\otimes id} \bigwedge^{j+1} F \otimes D_{j-a-1}G^*.$$

Proof. In [MS86, Lemma of the Appendix] this is shown for the $(a+1)^{st}$ comparison map but the proof immediately generalizes to our situation as long as $\operatorname{Hom}(\mathscr{C}^b_j(-1), \mathscr{C}^a_{j+1}) = \operatorname{Hom}(\mathscr{C}^b_{j-1}(-1), \mathscr{C}^a_j) = 0$. □

3. PROOF OF THE MAIN THEOREM

The aim of this section is to prove Theorem 3.1 and then to show that a general 5-gonal curve satisfies the balancing conditions.

THEOREM 3.1. *Let* $C \subset \mathbb{P}^{g-1}$ *be a general* 5*-gonal canonical curve of genus* g *satisfying the balancing conditions. If* X *is the scroll swept out by the* g^1_5 *and* $n = \lceil \frac{g-1}{2} \rceil$*, then*

$$\beta_{n,n+1}(C) > \beta_{n,n+1}(X) \text{ for odd } g \geq 13 \text{ and even } g \geq 28.$$

Throughout this section, $C \subset \mathbb{P}^{g-1}$ will be a 5-gonal canonical curve of genus g. In this case X is a $d = 4$ dimensional rational normal scroll of degree $f = g-4$. Recall from Theorem 2.5 and Remark 2.6 that $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+(f-2)R) \to \sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR)$$

$$\xrightarrow{\Psi} \sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C$$

where $\sum_{i=1}^5 a_i = 2g-12$, $a_i + b_i = f-2$.
The matrix Ψ is skew symmetric by the structure theorem for Gorenstein ideals in codimension 3 and the 5 Pfaffians of Ψ generate the ideal of C (see [BE77, Theorem 2.1]).

As in Theorem 2.3, we denote by $F = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} \cong \mathscr{O}^f_{\mathbb{P}^{g-1}}$ and by $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}} \cong \mathscr{O}^2_{\mathbb{P}^{g-1}}$. By abuse of notation, we will also refer to the vector spaces $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ and $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ by F and G, respectively.

Resolving the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules occurring in the minimal resolution of $C \subset \mathbb{P}(\mathscr{E})$, we get

$$\begin{tikzcd}
\sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR) \arrow[r, "\Psi"] & \sum_{i=1}^5 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+b_iR) \\
\vdots \arrow[u] & \vdots \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-2}(-3) \arrow[r, "\psi_{n-2}"] \arrow[u] & \sum_{i=1}^5 \mathscr{C}^{a_i}_{n-2}(-2) \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-1}(-3) \arrow[r, "\psi_{n-1}"] \arrow[u] & \sum_{i=1}^5 \mathscr{C}^{a_i}_{n-1}(-2) \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

where $\operatorname{rank}(\sum_{i=1}^5 \mathscr{C}^{b_i}_{n-2}) \leq \operatorname{rank}(\sum_{i=1}^5 \mathscr{C}^{a_i}_{n-2})$ (with equality for odd genus). Next note that if C is a canonical curve of odd genus $g = 2n+1$ satisfying the balancing conditions, then we obtain the following inequality

$$\min\{b_i\} = \left\lfloor \frac{\sum_{i=1}^5 b_i}{5} \right\rfloor = \left\lfloor \frac{6n-15}{5} \right\rfloor \geq n-2 \geq \left\lceil \frac{4n-10}{5} \right\rceil = \max\{a_i\} \tag{3.1}$$

if $n \geq 5$. For even genus $g = 2n$ and $n \geq 8$ we similarly get

$$\min\{b_i\} = \left\lfloor \frac{6n-18}{5} \right\rfloor \geq n-2 \geq \left\lceil \frac{4n-12}{5} \right\rceil = \max\{a_i\}.$$

It follows that in these cases

$$\mathscr{C}^{b_i}_{n-2}(-3) = \bigwedge^{n-2} F \otimes S_{b_i-n+2}G(-n-1)$$

and

$$\mathscr{C}^{a_i}_{n-2}(-2) = \bigwedge^{n-1} F \otimes D_{n-a_i-3}G^*(-n-1).$$

Thus, if C is a canonical curve of odd genus $g \geq 11$ or even genus $g \geq 16$, then the $(n-2)^{\text{th}}$ comparison map in the diagram above

$$\psi := \psi_{n-2} : \sum_{i=1}^5 \mathscr{C}^{b_i}_{n-2}(-3) \longrightarrow \sum_{i=1}^5 \mathscr{C}^{a_i}_{n-2}(-2)$$

has entries in the field $\Bbbk$ and its rank determines the Betti number $\beta_{n,n+1}(C)$. To be more precise, we have

$$\beta_{n,n+1}(C) = \beta_{n,n+1}(X) + \dim\ker(\psi),$$

where the Betti number $\beta_{n,n+1}(X)$ is given by $\operatorname{rank}(\mathscr{C}^0_n) = n \cdot \binom{f}{n+1}$.

For the proof of Theorem 3.1 we restrict ourselves to curves of odd genus since the theorem is proved in exactly the same way for even genus. For curves of odd genus $g = 2n+1$, we distinguish 5 types of curves satisfying the balancing conditions. These types depend on the congruence class of n modulo 5, i.e., on the block structure of the skew symmetric matrix Ψ. Setting $r := \lfloor \frac{n}{5} \rfloor$ we have the following five possibilities.

Type I: $(a_1,\dots,a_5) = (a,a,a,a,a), (b_1,\dots,b_5) = (b,b,b,b,b) \Leftrightarrow a = 4r+2$, $b = 6r+3$ and $n = 5r+5$.

Type II: $(a_1,\dots,a_5) = (a-1,a,a,a,a), (b_1,\dots,b_5) = (b+1,b,b,b,b) \Leftrightarrow a = 4r-1$, $b = 6r-2$ and $n = 5r+1$.

Type III: $(a_1,\dots,a_5) = (a-1,a-1,a,a,a), (b_1,\dots,b_5) = (b+1,b+1,b,b,b) \Leftrightarrow a = 4r$, $b = 6r-1$ and $n = 5r+2$.

Type IV: $(a_1,\dots,a_5) = (a,a,a,a+1,a+1), (b_1,\dots,b_5) = (b,b,b,b-1,b-1) \Leftrightarrow a = 4r$, $b = 6r+1$ and $n = 5r+3$.

Type V: $(a_1,\dots,a_5) = (a,a,a,a,a+1), (b_1,\dots,b_5) = (b,b,b,b,b-1) \Leftrightarrow a = 4r+1$, $b = 6r+2$ and $n = 5r+4$.

Proof of Theorem 3.1.
Since the proof of the theorem is similar for all different types above, we will only carry out the proof for curves of type II, leaving the other cases to the reader. We show that the map

$$\psi : \left(\mathscr{C}_{n-2}^{(b+1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{b}\right)(-3) \longrightarrow \left(\mathscr{C}_{n-2}^{(a-1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{a}\right)(-2)$$

induced by the skew-symmetric matrix

$$\Psi : \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^{\oplus 4} \end{matrix} \longrightarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^{\oplus 4} \end{matrix}$$

has a non-trivial decomposable element in the kernel. Note that the map

$$\Psi_{(11)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(a-1)R)$$

is zero by the skew-symmetry of Ψ. Thus it is sufficient to find an element in the kernel of the map $\psi_{(41)} : \mathscr{C}_{n-2}^{(b+1)}(-3) \longrightarrow \sum_{i=1}^{4} \mathscr{C}_{n-2}^{(a)}(-2)$, induced by the submatrix

$$\Psi_{(41)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^{\oplus 4}$$

of the matrix Ψ. By Lemma 2.9, the map $\psi_{(41)}$ is uniquely determined and is given as the composition

$$\mathscr{C}_{n-2}^{b+1}(-3) = \bigwedge^{n-2} F \otimes S_{b-n+3}G = \bigwedge^{n-2} F \otimes S_{n-a-2}G$$

$$\hookrightarrow \bigwedge^{n-2} F \otimes S_{n-a-2}G \otimes S_{n-a-3}G \otimes D_{n-a-3}G^*$$

$$\longrightarrow \bigwedge^{n-2} F \otimes S_{2n-2a-5}G \otimes D_{n-a-3}G^* \xrightarrow{id\otimes\Psi_{(41)}\otimes id} \bigwedge^{n-2} F \otimes F^{\oplus 4} \otimes D_{n-a-3}G^*$$

$$\xrightarrow{\wedge\otimes id}\left(\bigwedge^{n-1}F\right)^{\oplus 4}\otimes D_{n-a-3}G^{*}=\sum_{i=1}^{4}\mathscr{C}^{a}_{n-2}(-2)\ .$$

Since the multiplication map $S_{n-a-2}G\otimes S_{n-a-3}G\longrightarrow S_{2n-2a-5}G$ is not injective, we show that the existence of an $f\in\bigwedge^{n-2}F$ and a $g\in S_{n-a-2}G$ such that $f\wedge\Psi_{(41)}(g\cdot g')=0$ for all $g'\in S_{n-a-3}G$.
To this end, let $g\in S_{n-a-2}G$ be an arbitrary element and let $\{g'_1,\dots,g'_{n-a-2}\}$ be a basis of $S_{n-a-3}G$. For $i=1,\dots,(n-a-2)$, we define

$$(f_1^{(i)},f_2^{(i)},f_3^{(i)},f_4^{(i)})^t:=\Psi_{(41)}(g\cdot g'_i)\in F^4$$

and choose a maximal linearly independent subset $\{f_k\}_{k=1,\dots,p}$ of $\{f_j^{(i)}\}\subset F$. Since

$$\begin{aligned}n-2=5r-1\geq\#\{f_j^{(i)}\}&=4\cdot\dim_{\Bbbk}(S_{n-a-3}G)=4(n-a-2)=4r\\ &\geq\#\{f_k\}=p\end{aligned}$$

holds for all $r\geq 1$ (i.e. $g\geq 13$), we find a nonzero element f of the form

$$f=f_1\wedge f_2\wedge\cdots\wedge f_p\wedge\tilde f\in\bigwedge^{n-2}F$$

for some $\tilde f\in\bigwedge^{n-p-2}F$. By construction, $f\otimes g$ is in the kernel of $\psi_{(41)}$, and hence $(f\otimes g,0,0,0,0)^t$ lies in the kernel of ψ. $\square$

We can immediately generalize Theorem 3.1.

Theorem 3.2. *Let $C\subset\mathbb{P}^{g-1}$ be a general 5-gonal canonical curve satisfying the balancing conditions. Then*

$$\begin{aligned}\beta_{n+c,n+c+1}(C)&>\beta_{n+c,n+c+1}(X)\quad\textit{for odd genus } g=2n+1\geq 30c+13\\ \beta_{n+c,n+c+1}(C)&>\beta_{n+c,n+c+1}(X)\quad\textit{for even genus } g=2n\geq 30c+28.\end{aligned}$$

Proof. With the above notation we always have

$$\operatorname{rank}(\sum_{i=1}^{5}\mathscr{C}^{b_i}_{n-2+c})\leq\operatorname{rank}(\sum_{i=1}^{5}\mathscr{C}^{a_i}_{n-2+c}).$$

Since the curve C satisfies the balancing conditions, we obtain as in (3.1)

$$\min\{b_i\}\geq n-2+c\geq\max\{a_i\}$$

for odd genus $g=2n+1$ if $n\geq 5c+5$ or even genus $g=2n$ if $n\geq 5c+8$. Thus, it follows that the Betti number $\beta_{n+c,n+c+1}(C)$ is determined by the rank of the map

$$\psi_{n-2+c}:\sum_{i=1}^{5}\mathscr{C}^{b_i}_{n-2+c}(-3)\longrightarrow\sum_{i=1}^{5}\mathscr{C}^{a_i}_{n-2+c}(-2)$$

which has entries in the ground field $\Bbbk$. By the same construction as in the proof of Theorem 3.1, we find a decomposable element in the kernel of ψ_{n-2+c} if $r=\lfloor\frac{n}{5}\rfloor\geq 3c+1$ (for odd genus) or $r=\lfloor\frac{n}{5}\rfloor\geq 3c+3$ (for even genus). This gives precisely the range for the genus, as stated in the theorem. $\square$

Next we want to show that the generic 5-gonal curve satisfies the balancing conditions. The balancing conditions on a 5-gonal canonical curve are open conditions. It is therefore sufficient to find an example for each g.

PROPOSITION 3.3. *For any odd $g \geq 13$ (and even $g \geq 28$), there exists a smooth and irreducible 5-gonal canonical genus g curve satisfying the balancing conditions.*

Proof. We illustrate the proof for odd genus curves of type II:
To this end, let $X = S(e_1, \dots, e_4) \cong \mathbb{P}(\mathscr{E})$ with $e_1 \geq \dots \geq e_4$ be a fixed 4-dimensional rational normal scroll of balanced type and of degree $f = g - 4$. Let further

$$(a_1, \dots, a_5) = (a-1, a, a, a, a) \text{ and } (b_1, \dots, b_5) = (b+1, b, b, b, b)$$

be balanced partitions such that $\sum_{i=1}^{5} a_i = 2g - 12$, $g = 2n + 1$, $n = 5r + 1$, $a = 4r - 1$ and $a_i + b_i = g - 6$. We consider a general skew-symmetric morphism

$$\underbrace{\begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \\ \bigoplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^{\oplus 4} \end{matrix}}_{=:\mathscr{F}} \xrightarrow{\Psi} \underbrace{\begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R) \\ \bigoplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^{\oplus 4} \end{matrix}}_{=:\mathscr{F}^* \otimes \mathscr{L}}$$

If $\bigwedge^2 \mathscr{F}^* \otimes \mathscr{L}$ is globally generated, then it follows by a Bertini type theorem (see e.g. [Oko94, §3]) that the scheme $\mathrm{Pf}(\Psi)$ cut out by the Pfaffians of Ψ is smooth of codimension 3 or empty. Recall from Remark 2.2, that a line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H + cR)$ is globally generated if and only if $e_4 + c \geq 0$. Thus since for $r \geq 1$

$$\min\{e_i\} = \left\lfloor \frac{g-4}{4} \right\rfloor = \left\lfloor \frac{10r-1}{4} \right\rfloor = 2r + \left\lfloor \frac{2r-1}{4} \right\rfloor \geq (b - a + 1) = 2r,$$

we conclude that in this case

$$\bigwedge^2 \mathscr{F}^* \otimes \mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - (b-a+1)R)^{\oplus 4} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - (b-a)R)^{\oplus 6}$$

is globally generated. It follows that $C = \mathrm{Pf}(\Psi)$ is smooth of codimension 3 or empty. The iterated mapping cone construction gives a free resolution of $C \subset \mathbb{P}^{g-1}$ of length $g-2$ which is not null-homotopic. Therefore it follows from the Auslander-Buchsbaum formula that C is a non-empty (and therefore smooth) arithmetically Cohen-Macaulay scheme. In particular we have a surjective map

$$\underbrace{H^0(\mathbb{P}^{g-1}, \mathscr{O}_{\mathbb{P}^{g-1}})}_{\cong \Bbbk} \to H^0(C, \mathscr{O}_C) \to 0$$

and therefore C is smooth and connected and hence an irreducible curve.
Doing the same for curves of type I, III, IV and V and the even genus cases the result follows for all genera except for $g = 15$ (in this case $\bigwedge^2 \mathscr{F}^* \otimes \mathscr{L}$ is not globally generated). For the $g = 15$ case one can verify the statement by using *Macaulay2* (see [GS]). $\square$

COROLLARY 3.4. *$\mathcal{K}_{g,5} = \{C \in \mathcal{H}_{g,5} \mid \beta_{n,n+1}(C) > \beta_{n,n+1}(X)\}$ equals $\mathcal{H}_{g,5}$ for odd genus $g \geq 13$ and even genus $g \geq 28$.*

Proof. For odd genus $g \geq 13$ and even genus $g \geq 28$ it follows by Theorem 3.1 and Proposition 3.3 that $\mathcal{K}_{g,5}$ is a non empty and dense subset of $\mathcal{H}_{g,5}$. The conclusion follows by semi-continuity on the Betti numbers. □

4. A FIRST EXAMPLE

In this section, we discuss the case of a general 5-gonal canonical curve of genus 13. The rational normal scroll X swept out by the g^1_5 on C is therefore a 4-dimensional scroll of type $S(3,2,2,2)$ and degree $f = 9$. The curve $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+7R) \to \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+5R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+4R)^{\oplus 4} \end{matrix} \xrightarrow{\Psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+3R)^{\oplus 4} \end{matrix} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C$$

where Ψ is a skew-symmetric matrix with entries as indicated below

$$(\Psi) \sim \begin{pmatrix} 0 & (H-2R) & (H-2R) & (H-2R) & (H-2R) \\ & 0 & (H-R) & (H-R) & (H-R) \\ & & 0 & (H-R) & (H-R) \\ & & & 0 & (H-R) \\ & & & & 0 \end{pmatrix} \tag{4.1}$$

We resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules in the resolution above by Eagon-Northcott type complexes and determine the rank of the maps which give rise to non-minimal parts in the iterated mapping cone. As in Section 3, we denote by $F = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ and by $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$.

$$\begin{tikzcd}
\sum_{i=1}^5 \mathscr{C}_3^{b_i}(-3) = \begin{matrix} \bigwedge^3 F\otimes S_2G(-6) \\ \oplus \\ (\bigwedge^3 F\otimes S_1G(-6))^{\oplus 4} \end{matrix} \arrow[r, "\psi_3"] & \sum_{i=1}^5 \mathscr{C}_3^{a_i}(-2) = \begin{matrix} \bigwedge^4 F(-6) \\ \oplus \\ (\bigwedge^4 F(-5))^{\oplus 4} \end{matrix} \\
\sum_{i=1}^5 \mathscr{C}_4^{b_i}(-3) = \begin{matrix} \bigwedge^4 F\otimes S_1G(-7) \\ \oplus \\ (\bigwedge^4 F(-7))^{\oplus 4} \end{matrix} \arrow[u] \arrow[r, "\psi_4"] & \sum_{i=1}^5 \mathscr{C}_4^{a_i}(-2) = \begin{matrix} \bigwedge^5 F\otimes D_1G^*(-7) \\ \oplus \\ (\bigwedge^5 F(-7))^{\oplus 4} \end{matrix} \arrow[u] \\
\sum_{i=1}^5 \mathscr{C}_5^{b_i}(-3) = \begin{matrix} \bigwedge^5 F(-8) \\ \oplus \\ (\bigwedge^5 F(-9))^{\oplus 4} \end{matrix} \arrow[u] \arrow[r, "\psi_5"] & \sum_{i=1}^5 \mathscr{C}_5^{a_i}(-2) = \begin{matrix} \bigwedge^6 F\otimes D_2G^*(-8) \\ \oplus \\ (\bigwedge^6 F\otimes D_1G^*(-8))^{\oplus 4} \end{matrix} \arrow[u]
\end{tikzcd}$$

Note that by degree reasons, the maps indicated above are the only ones which give rise to possibly non-minimal parts in the iterated mapping cone.
By the Gorenstein property of canonical curves, it follows that the maps

$$\psi_3' : (\bigwedge^3 F \otimes S_1G)^{\oplus 4}(-6) \to \bigwedge^4 F(-6)$$

and

$$\psi_5' : \bigwedge^5 F(-8) \to (\bigwedge^6 F \otimes D_1 G^*)^{\oplus 4}(-8)$$

are dual to each other and one can easily check the surjectivity of ψ_3'. It remains to compute the rank of ψ_4.

Proposition 4.1. *Let Ψ be a general skew symmetric matrix with entries as indicated above. Then the induced matrix $\psi_4 : \sum_{i=1}^5 \mathscr{C}_4^{b_i}(-3) \to \sum_{i=1}^5 \mathscr{C}_4^{a_i}(-2)$ has a six dimensional kernel.*

Proof. According to Section 2, we can write down the relevant cohomology groups. Let $\{s,t\}$ be a basis of $G = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ and $\{\varphi_1\}$ be a basis of $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-3R))$ then a basis of $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$ is given by $\{s\varphi_1, t\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. We consider the submatrix $\psi_{(41)} : \bigwedge^4 F \otimes S_1 G(-7) \to \left(\bigwedge^5 F(-7)\right)^4$ of ψ_4 induced by the first column of Ψ. As in the proof of Theorem 3.1, the map $\psi_{(41)}$ is given as the composition

$$\begin{array}{c} \bigwedge^4 F \otimes S_1 G \cong \bigwedge^4 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \\ \Big\downarrow {\scriptstyle id \otimes \Psi_{(41)}} \\ \bigwedge^4 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))^{\oplus 4} \cong \bigwedge^4 F \otimes F^{\oplus 4} \xrightarrow{\wedge} (\bigwedge^5 F)^{\oplus 4}. \end{array}$$

By our generality assumption on C, we can assume that the 4 entries of $\Psi_{(41)}$ are independent and after acting with an element in $\mathrm{Aut}(X)$, we can furthermore assume that $\Psi_{(41)} = (s\varphi_1, \varphi_2, \varphi_3, \varphi_4)^t$. It follows that elements of the form

$(\lambda s + \mu t)s\varphi_1 \wedge (\lambda s + \mu t)\varphi_2 \wedge (\lambda s + \mu t)\varphi_3 \wedge (\lambda s + \mu t)\varphi_4 \otimes (\lambda s + \mu t)$, with $\lambda, \mu \in \Bbbk$

lie in the kernel of $\psi_{(41)}$. Expanding those elements we get

$$\lambda^5 s^2 \varphi_1 \wedge s\varphi_2 \wedge s\varphi_3 \wedge s\varphi_4 \otimes s + \cdots + \mu^5 st\varphi_1 \wedge t\varphi_2 \wedge t\varphi_3 \wedge t\varphi_4 \otimes t$$

and conclude that a rational normal curve of degree 5 lies in $\mathbb{P}(\mathrm{Syz})$ where $\mathrm{Syz} \subset \mathrm{Tor}_6^T(T/I_C)_7$ is the subspace of the 6^{th} syzygy module spanned by the extra syzygies and $I_C \subset T$ denotes the ideal of the canonical curve C. We get $\beta_{6,7}(C) \geq \beta_{6,7}(X) + 6 = 222$ and by computing one example using *Macaulay2*, it follows that ψ_4 has a 6-dimensional kernel in general. $\square$

Remark 4.2. A direct computation using *Macaulay2* shows that none of the entries of the skew symmetric matrix Ψ can be made zero by suitable row and column operations respecting the skew symmetric structure of Ψ. By [Sch86, §5] this implies that the 6 extra syzygies are not induced by an additional linear series on C.

The question arises of how the extra syzygies of a 5-gonal canonical curve $C \subset \mathbb{P}^{12}$ differ from the syzygies induced by the scroll swept out by the g_5^1 on C. At least in the genus $g = 13$ case we can give an answer in this direction by considering *syzygy schemes*, originally introduced in [Ehb94].

Definition and Remark 4.3. *Let $C \subset \mathbb{P}^{g-1}$ be a smooth and irreducible canonical curve and let $I_C \subset S$ be the ideal of C. Let further*

$$\mathrm{L}_\bullet : S \leftarrow S(-2)^{\beta_{1,2}} \leftarrow S(-3)^{\beta_{2,3}} \leftarrow \cdots$$

be the linear strand of a minimal free resolution of $S_C = S/I_C$. For a p^{th} linear syzygy $s \in L_p$, let V_s be the smallest vector space such that the following diagram commutes

$$\begin{array}{ccc} L_{p-1} & \longleftarrow & L_p \\ \cup & & \cup \\ V_s \otimes S(-p) & \longleftarrow & S(-p-1) \cong \langle s \rangle \ . \end{array}$$

The rank of the syzygy s *is defined to be* $\mathrm{rank}(s) := \dim V_s$.
Since $\mathrm{Hom}(\mathrm{L}_\bullet, S)$ *is a free complex and the Koszul complex is exact, it follows that the maps of the dual diagram extend to a morphism of complexes. Dualizing again we get*

$$\begin{tikzcd} S & L_1 \arrow[l] & \cdots \arrow[l] & L_{p-1} \arrow[l] & L_p \arrow[l] \\ \bigwedge^p V_s \otimes S(-1) \arrow[u, "\varphi_p"] & \bigwedge^{p-1} V_s \otimes S(-2) \arrow[l] \arrow[u] & \cdots \arrow[l] & V_s \otimes S(-p) \arrow[l] \arrow[u] & S(-p-1) \arrow[l] \arrow[u] \ . \end{tikzcd}$$

By degree reasons there are only trivial homotopies and therefore all the vertical maps except φ_p are unique. The syzygy scheme $\mathrm{Syz}(s)$ *of s is the scheme defined by the ideal*

$$I_s = \mathrm{im}(S \longleftarrow \bigwedge^{p-1} V_s \otimes S(-2)).$$

*The p^{th}-*syzygy scheme $\mathrm{Syz}_p(C)$ *of a curve C is defined to be the intersection* $\bigcap_{s \in L_p} \mathrm{Syz}(s)$.

Any p^{th}-syzygy of a canonical curve has rank $\geq p+1$ and the syzygies of rank $p+1$ are called scrollar syzygies. The name is justified by a theorem due to von Bothmer:

Theorem 4.4 ([GvB07, Corollary 5.2]). *Let $s \in L_p$ be a p^{th} scrollar syzygy. Then $\mathrm{Syz}(s)$ is a rational normal scroll of degree $p+1$ and codimension p that contains the curve C.*

We can now come back to our example of a 5-gonal genus 13 curve. Recall that in this case the space of extra syzygies can be identified with the kernel of the map

$$\mathscr{C}_4^5(-3) \to (\mathscr{C}_4^3)^{\oplus 4}(-2)$$

which is induced by the first column of the skew symmetric matrix Ψ.
We denote by $\mathrm{Pf}_1, \ldots, \mathrm{Pf}_4 \in H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-3R))$ the 4 Pfaffians of the matrix Ψ that involve the first column and consider the iterated mapping cone

$$\left[\left[\mathscr{C}^5 \to (\mathscr{C}^3)^{\oplus 4}\right] \to \mathscr{C}^0\right],$$

where $\sum_{i=1}^4 \mathscr{C}^3 \to \mathscr{C}^0$ is induced by the multiplication with $(\mathrm{Pf}_1, \ldots, \mathrm{Pf}_4)$. This complex is a resolution of the ideal J generated by the 4 Pfaffians as an

$\mathscr{O}_{\mathbb{P}^{12}}$-module. In particular the minimized resolution is a subcomplex of the minimal free resolution of S_C.
Since all extra syzygies are induced by the first column of Ψ, it follows that the 6^{th} syzygy modules in the linear strand of these minimal resolutions are canonically isomorphic. Therefore $\mathrm{Syz}_6(V(J))$ and $\mathrm{Syz}_6(C)$ coincide and $V(J) \subset \mathrm{Syz}_6(C)$.

PROPOSITION 4.5. *Let* $C \subset \mathbb{P}^{12}$ *be a general* 5*-gonal canonical curve. Then* $\mathrm{Syz}_6(C)$ *is the scheme cut out by the* 4 *Pfaffians of* Ψ *involving the first column. In particular* $\mathrm{Syz}_6(C) = C \cup p$ *for some point* $p \in X$.

Proof. One inclusion follows from the discussion above. The other inclusion follows by computing one example using *Macaulay2*. □

REMARK 4.6. Since we do not have a full description of the space of extra syzygies for 5-gonal canonical curves of higher genus, a similar approach does not work in these cases.

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