A Combinatorial Interpretation for Schreyer's Tetragonal Invariants

Wouter Castryck and Filip Cools

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Abstract. Schreyer has proved that the graded Betti numbers of a canonical tetragonal curve are determined by two integers b_1 and b_2, associated to the curve through a certain geometric construction. In this article we prove that in the case of a smooth projective tetragonal curve on a toric surface, these integers have easy interpretations in terms of the Newton polygon of its defining Laurent polynomial. We can use this to prove an intrinsicness result on Newton polygons of small lattice width.
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1 Introduction

Let k be an algebraically closed field of characteristic 0 and let $\mathbb{T}^2 = (k^*)^2$ be the two-dimensional torus over k. Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon and consider the associated toric surface $\text{Tor}(\Delta)$ over k, i.e. the Zariski closure of the image of

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{\sharp(\Delta\cap\mathbb{Z}^2)-1} : (\alpha, \beta) \mapsto (\alpha^i\beta^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

Let

$$f = \sum_{(i,j)\in\mathbb{Z}^2} c_{i,j}x^iy^j \in k[x^{\pm1}, y^{\pm1}]$$

be an irreducible Laurent polynomial and consider its Newton polygon

$$\Delta(f) = \text{conv}\left\{ (i,j) \in \mathbb{Z}^2 \;\middle|\; c_{i,j} \neq 0 \right\}.$$

Let $U_f \subset \mathbb{T}^2$ be the curve cut out by f. We say that f is Δ-non-degenerate if $\Delta(f) \subset \Delta$ and for every face $\tau \subset \Delta$ (vertex, edge, or Δ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. Here

$$f_\tau = \sum_{(i,j) \in \tau \cap \mathbb{Z}^2} c_{i,j} x^i y^j .$$

For a fixed instance of Δ and given that $\Delta(f) \subset \Delta$, the condition of Δ-non-degeneracy is generically satisfied. It implies that the Zariski closure C_f of $\varphi_\Delta(U_f)$ inside $\mathrm{Tor}(\Delta)$ is non-singular. A curve that is isomorphic to C_f for some Δ-non-degenerate Laurent polynomial is in turn called Δ-non-degenerate.

Non-degenerate curves form an attractive class of objects from the point of view of explicit algebraic geometry. On the one hand they vastly generalize well-known families such as elliptic curves, hyperelliptic curves, trigonal curves[1], smooth plane curves, $C_{a,b}$ curves, ... covering a much broader range of geometric situations. On the other hand they remain very tangible, because many important geometric invariants can be told by simply looking at the combinatorics of Δ. Two notable instances are:

- the (geometric) *genus* g, which equals $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$, where $\Delta^{(1)}$ is the convex hull of the interior lattice points of Δ; see [10];

- the *gonality* γ, which equals $\mathrm{lw}(\Delta)$, except if $\Delta \cong 2\Upsilon$ or $\Delta \cong d\Sigma$ for some $d \geq 2$, where

  $$\Upsilon = \mathrm{conv}\{(-1,-1),(1,0),(0,1)\} \quad \text{and} \quad \Sigma = \mathrm{conv}\{(0,0),(1,0),(0,1)\},$$

  in which case it equals $\mathrm{lw}(\Delta) - 1$; here lw denotes the lattice width, and $\cong$ indicates unimodular equivalence; see [4, Lem. 6.2]. (Shorter characterization: $\gamma = \mathrm{lw}(\Delta^{(1)}) + 2$ except if $\Delta \cong 2\Upsilon$ in which case $\gamma = 3$.)

Similar interpretations exist for the *Clifford index* and the *Clifford dimension* [4, §8], and in some cases for the *minimal degree of a plane model* [6]. The current paper extends the list of combinatorial features of non-degenerate curves, by focusing on tetragonal curves. Namely, we give the following interpretation for the invariants b_1 and b_2, as introduced by Schreyer in [14, (6.2)]. The definition of these invariants will be recalled in Section 2 below.

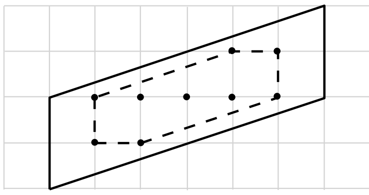
[1] Strictly spoken, there do exist trigonal curves that are not non-degenerate; for example see [4, Lem. 4.4]. But all trigonal curves are 'morally' non-degenerate, in the sense that they can always be embedded in a toric surface, which is sufficient for most applications. See also the remark at the end of this section.

Theorem 1. *Let C be a tetragonal Δ-non-degenerate curve. Then Schreyer's corresponding set of invariants $\{b_1, b_2\}$ is given by*

$$\left\{ \ \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) - 4\,,\ \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - 1 \ \right\}.$$

Here ∂ denotes the boundary and $\Delta^{(2)} = \Delta^{(1)(1)}$ is the convex hull of the interior lattice points of $\Delta^{(1)}$.

Example 2. The Laurent polynomial $f = 1 + y^2 - x^6y^2 + x^6y^4 \in \mathbb{C}[x, y]$ is Δ-non-degenerate, where Δ is as follows.



The dashed lines indicate $\Delta^{(1)}$. One verifies, purely by looking at the Newton polygon, that C_f is a tetragonal curve of genus 9 with $b_1 = b_2 = 2$. (In view of [4, Cor. 6.3, Thm. 9.1], one can even say that it carries a unique g^1_4, whose scrollar invariants read $1, 1, 4$; see Remark 2 below for more background on this terminology.)

Schreyer's invariants are known to determine the *Betti diagram of the canonical ideal*, and vice versa [14, (6.2)]. In particular, Theorem 1 implies that in the tetragonal case, the Betti diagram is combinatorially determined. We believe that this holds in much greater generality (work in progress).

A second aim of this paper is to initiate a discussion on the *intrinsicness* of Δ. Namely, given the many geometric invariants that are encoded in the Newton polygon, one might wonder to what extent it is possible to reconstruct Δ from the abstract geometry of a given Δ-non-degenerate curve C_f. The best one can hope for is to find back Δ up to unimodular equivalence, because unimodular transformations correspond to automorphisms of $\mathbb{T}^2$. Another relaxation is that (usually) one can only expect to recover $\Delta^{(1)}$, rather than all of Δ. For example, let $f \in k[x, y]$ be $d\Sigma$-non-degenerate for some integer $d \geq 2$ and let $(x_0, y_0) \in U_f$ be sufficiently generic. Then $f' = f(x + x_0, y + y_0)$ is Δ-non-degenerate, where Δ is obtained from $d\Sigma$ by clipping off the point $(0, 0)$. In this case $\Delta \not\cong d\Sigma$, while clearly $C_f \cong C_{f'}$. More generally, pruning a vertex off a lattice polygon Δ without affecting its interior boils down to forcing the curve through a certain non-singular point of $\mathrm{Tor}(\Delta)$, which is usually not intrinsic. One is naturally led to the following question.

Question 3 (intrinsicness). *Let Δ, Δ' be two-dimensional lattice polygons for which there exists a curve that is both Δ-non-degenerate and Δ'-non-degenerate. Does it follow that $\Delta^{(1)} \cong \Delta'^{(1)}$?*

Our conjecture is that for 'most' pairs of polygons the answer is yes. E.g., this is known to be true as soon as

(a) $\Delta^{(1)}$ is one-dimensional, because a Δ-non-degenerate curve is hyperelliptic of genus $g \geq 2$ if and only if $\Delta^{(1)} \cap \mathbb{Z}^2$ consists of g collinear points [11, Lem. 3.2.9],

(b) $\Delta^{(1)} = \emptyset$ or $\Delta^{(1)} \cong (d-3)\Sigma$ for some integer $d \geq 3$, because a Δ-non-degenerate curve is abstractly isomorphic to a smooth plane curve if and only if $\Delta^{(1)}$ is a multiple of the standard simplex (up to equivalence) [4, Cor. 8.2].

(c) $\Delta^{(1)} \cong [0,a] \times [0,b]$ for some integers $a \geq b \geq -1$ with $(a+1)(b+1) \neq 4$, because a Δ-non-degenerate curve of genus $g \neq 4$ can be embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta^{(1)}$ is a standard rectangle (up to equivalence); see [5]. The assumption $g \neq 4$ is necessary: see the discussion following (d) below.

Let us indicate why we expect Question 3 to have an affirmative answer for many more instances of Δ, while gathering some material that will be needed in Section 2. Our starting point is a theorem by Khovanskii [10], stating that there exists a canonical divisor K_Δ on C_f such that a basis for the Riemann-Roch space $H^0(C_f, K_\Delta)$ is given by

$$\left\{x^i y^j\right\}_{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2}. \tag{1}$$

Here x, y are to be viewed as functions on C_f through φ_Δ. Note that one recovers the statements that $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$ and that C_f is hyperelliptic if and only if $\Delta^{(1)}$ is one-dimensional; see [7, Lem. 5.1] for more details. If $\Delta^{(1)}$ is two-dimensional, then Khovanskii's theorem implies that the canonical model C_f^{can} of C_f satisfies

$$C_f^{\text{can}} \subset \text{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

But surfaces of the form $\text{Tor}(\Delta^{(1)})$ are very special. Most notably, they are of low degree, and they are generated by binomials. The idea is that they are so special that there is room for at most one such surface containing C_f^{can}. This idea is not always true, but the exceptions seem rare. If it *is* true, then the following general and seemingly new statement allows one to recover $\Delta^{(1)}$. A proof will be given in Section 3.

Theorem 4. *Let Δ, Δ' be two-dimensional lattice polygons with*

$$\sharp(\Delta \cap \mathbb{Z}^2) - 1 = \sharp(\Delta' \cap \mathbb{Z}^2) - 1 = N,$$

and suppose that $\text{Tor}(\Delta), \text{Tor}(\Delta') \subset \mathbb{P}^N$ can be obtained from one another using a projective transformation. Then $\Delta \cong \Delta'$.

Using this, we can immediately extend the above list to the case where

(d) $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 5$ and $\Delta^{(2)} = \emptyset$, which holds if and only if C_f is trigonal of genus $g \geq 5$, or isomorphic to a smooth plane quintic [4, §8]. In this case $\mathrm{Tor}(\Delta^{(1)})$ can be characterized as the unique irreducible surface containing C_f^{can} that is generated by quadrics. Indeed, the fact that it is generated by quadrics follows from [12], while uniqueness follows from Petri's theorem [13].

The above argument breaks down in the genus 4 case where $\Delta \cong 2\Upsilon$, because $\mathrm{Tor}((2\Upsilon)^{(1)}) = \mathrm{Tor}(\Upsilon)$ is not generated by quadrics. And indeed, using this, it is not hard to cook up examples of (2Υ)-non-degenerate curves that are non-degenerate with respect to $[0,3] \times [0,3]$, and also of (2Υ)-non-degenerate curves that are non-degenerate with respect to $\mathrm{conv}\{(0,0),(4,0),(0,2)\}$. (See §5.6 of our unpublished `arXiv` paper `1304.4997` for an extended discussion; see also Example 13 below.)

In Section 2 we will give a similar but more complicated recipe for recovering $\mathrm{Tor}(\Delta^{(1)})$ in most tetragonal cases. More precisely, we extend the list with the situation where

(e) $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 5$, which holds if and only if C_f is tetragonal and $b_1 \geq b_2 + 2$. In this case $\mathrm{Tor}(\Delta^{(1)})$ can be characterized as the unique surface containing C_f^{can} that is linearly equivalent to $2H - b_1R$, when viewed as a divisor inside the scroll spanned by a g^1_4.

More explanation will be given in Section 4. Of course, in establishing this, we will make extensive use of Theorem 1 and its proof.

Remark 5. Even though we formulate our results in terms of non-degenerate curves, they remain valid for the slightly more general class of *arbitrary* smooth curves in toric surfaces. Indeed, to a smooth (non-torus-invariant) curve C in a toric surface $\varphi : \mathbb{T}^2 \hookrightarrow X$ one can always associate a 'defining Laurent polynomial' $f \in k[x^{\pm 1}, y^{\pm 1}]$, by which we mean a generator of the ideal of $\varphi^{-1}C$. It is well-defined up to multiplication by cx^iy^j for some $c \in k^*$ and $(i,j) \in \mathbb{Z}^2$. One then just proceeds with f and $\Delta = \Delta(f)$, as if f were Δ-non-degenerate. We refer to [4, §4] for a more extended discussion.

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2 Schreyer's tetragonal invariants

Let C/k be a tetragonal curve of genus $g \geq 5$ and assume it to be canonically embedded in $\mathbb{P}^{g-1}$. Fix a gonality pencil g^1_4 on C and consider

$$S = \bigcup_{D \in g^1_4} \langle D \rangle \subset \mathbb{P}^{g-1},$$

where $\langle D \rangle \subset \mathbb{P}^{g-1}$ denotes the linear span of D. One can show that S is a rational normal threefold scroll whose type we denote by (e_1, e_2, e_3), where we assume $0 \leq e_1 \leq e_2 \leq e_3$. One has $\deg S = e_1 + e_2 + e_3 = g - 3$, and S is non-singular if and only if $e_1 > 0$. If $e_1 = 0$ then the singularities are resolved by the natural map $\mu : \mathbb{P}(\mathcal{E}) \to S$, where $\mathcal{E}$ is the locally free sheaf $\mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3)$ on $\mathbb{P}^1$; if $e_1 > 0$ then μ is an isomorphism. The Picard group of $\mathbb{P}(\mathcal{E})$ is freely generated by the hyperplane class $H = [\mu^*(\mathcal{O}(1))]$ and the ruling class R consisting of the fibers of the projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. The following intersection-theoretic identities hold: $H^3 = g - 3$, $H^2 \cdot R = 1$ and $R^2 = 0$. For more general background and references, see [4, §9] and [14, §2-4].

Remark 6. The numbers e_1, e_2, e_3 are called the *scrollar invariants* of C with respect to our g^1_4.

Now let C' be the strict transform under μ of our canonical curve $C \subset S$. Schreyer proved that C' is the complete intersection of surfaces Y and Z in $\mathbb{P}(\mathcal{E})$, with $Y \sim 2H - b_1 R$, $Z \sim 2H - b_2 R$, $b_1 + b_2 = g - 5$ and $-1 \leq b_2 \leq b_1 \leq g - 4$. He moreover showed that b_1, b_2 are invariants of the curve: they depend neither on the canonical embedding, nor on the choice of the g^1_4, nor on the choice of Y and Z. If $b_1 > b_2$, which is automatic if g is even, then Y is in fact unique, and $\mu(Y) \subset \mathbb{P}^{g-1}$ is independent of the chosen g^1_4. For these particular statements we refer to [14, (6.2)].

The goal of this section is to prove the combinatorial interpretation for Schreyer's invariants b_1, b_2 stated in Theorem 1. Using the abbreviations

$$B = \sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) - 4, \qquad B^{(1)} = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) - 1,$$

we will in fact show:

Theorem 7. *Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, and suppose that C_f is tetragonal. Then its invariants b_1, b_2 satisfy $\{b_1, b_2\} = \{B, B^{(1)}\}$. If moreover $B > B^{(1)}$ then the surface $\mu(Y)$ associated to the canonical model C_f^{can} from Section 1 equals* $\mathrm{Tor}(\Delta^{(1)})$.

Proof. The assumption that C_f is tetragonal is equivalent to $\mathrm{lw}(\Delta^{(1)}) = 2$ and $\Delta \not\cong 2\Upsilon$. We can also suppose that $\Delta \not\cong 5\Sigma$, because this case can be reduced to

$$\Delta \cong \mathrm{conv}\{(1,0), (5,0), (0,5), (0,1)\}$$

by means of a coordinate transformation, as explained in the discussion preceding Question 3. By [4, Lem. 5.2] we can therefore suppose that

$$\Delta^{(1)} \subset \left\{ (X,Y) \in \mathbb{R}^2 \,|\, 0 \leq Y \leq 2 \right\} \quad \text{and} \quad \Delta \subset \left\{ (X,Y) \in \mathbb{R}^2 \,|\, -1 \leq Y \leq 3 \right\}.$$

Then the projection map $U_f \to \mathbb{T}^1 : (x,y) \mapsto x$ has degree 4, i.e. it gives rise to a g^1_4 on C_f. As remarked in Section 1, the canonical model C_f^{can} obtained using the basis (1) of $H^0(C_f, K_\Delta)$ satisfies

$$C_f^{\mathrm{can}} \subset \mathrm{Tor}(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

The scroll S corresponding to our g^1_4 is easily seen to be the Zariski closure of the image of the map

$$\begin{aligned} &\mathbb{T}^3 \hookrightarrow \mathbb{P}^{g-1} : \\ &\quad (\alpha,\beta,\gamma) \mapsto \left((\alpha^i)_{(i,0)\in\Delta^{(1)}\cap\mathbb{Z}^2} : (\beta\alpha^i)_{(i,1)\in\Delta^{(1)}\cap\mathbb{Z}^2} : (\gamma\alpha^i)_{(i,2)\in\Delta^{(1)}\cap\mathbb{Z}^2} \right). \end{aligned}$$

(Note that the scrollar invariants e_1, e_2, e_3 are precisely the numbers

$$\sharp\{(i',j') \in \Delta^{(1)} \cap \mathbb{Z}^2 \,|\, j' = j\} - 1$$

for $j = 0,1,2$, up to order; for a generalization of this observation, see [4, §9].) Moreover, one verifies that S contains $\mathrm{Tor}(\Delta^{(1)})$, i.e. the above chain of inclusions extends to

$$C_f^{\mathrm{can}} \subset \mathrm{Tor}(\Delta^{(1)}) \subset S \subset \mathbb{P}^{g-1}.$$

Now let $\mu : \mathbb{P}(\mathcal{E}) \to S$ be as above and denote by C' the strict transform of C_f^{can} under μ. Similarly, denote by T' the strict transform of $\mathrm{Tor}(\Delta^{(1)})$. Write the divisor class of T' as $aH + bR$ with $a, b \in \mathbb{Z}$. Let F be the fiber of π above $\alpha \in \mathbb{T}^1 \subset \mathbb{P}^1$. Then $\mu(F)$ is a $\mathbb{P}^2$ whose intersection with $\mathrm{Tor}(\Delta^{(1)})$ has $\beta = y$ and $\gamma = y^2$ as parameter equations on $\mathbb{T}^2 \subset \mathbb{P}^2$. In particular this intersection is a conic, so we have that

$$a = (aH + bR) \cdot H \cdot R = T' \cdot H \cdot R = 2.$$

Next, we compute the intersection product $T' \cdot H^2$ in two ways. On the one hand we find the degree of $\mathrm{Tor}(\Delta^{(1)})$, which equals $2\mathrm{Vol}(\Delta^{(1)})$ because the Hilbert polynomial of $\mathrm{Tor}(\Delta^{(1)})$ equals the Ehrhart polynomial of $\Delta^{(1)}$, see [8, Prop. 9.4.3]. On the other hand one has

$$T' \cdot H^2 = (2H + bR) \cdot H^2 = 2(g-3) + b.$$

We obtain that $b = 2\mathrm{Vol}(\Delta^{(1)}) - 2(g-3) = -B$, where the latter equality follows from Pick's theorem. In conclusion, $T' \sim 2H - BR$. Now

- if $Y = T'$ then it is immediate that $b_1 = B$ and, consequently, $b_2 = B^{(1)}$,

- if $Y \neq T'$ then if we intersect $Y \sim 2H - b_1R$ and $T' \sim 2H - BR$ on $\mathbb{P}(\mathcal{E})$, we obtain a (possibly reducible) curve whose image under μ has degree

  $$H\cdot(2H-BR)\cdot(2H-b_1R) = 4(g-3)-2b_1-2B \leq 4(g-3)-2(g-5) = 2g-2.$$

  This follows from $2b_1 \geq b_1 + b_2 = g-5$ and $2B \geq B + B^{(1)} = g-5$ if $B \geq B^{(1)}$, and from $2b_1 \geq b_1+b_2+1 = g-4$ and $2B = g-6$ if $B < B^{(1)}$; see Lemma 9 below. In both cases, if either one of the inequalities would be strict, then we would run into a contradiction because C' is contained in this intersection (and $\mu(C') = C_f^{\mathrm{can}}$, being a canonical curve, has degree $2g-2$). We conclude that $b_1 = b_2 = B = B^{(1)} = \frac{g-5}{2}$ or $b_1 = B^{(1)} = \frac{g-4}{2}$ and $b_2 = B = \frac{g-6}{2}$.

All conclusions follow. □

Remark 8. Assume that C_f is not isomorphic to a smooth plane quintic, i.e. $\Delta^{(1)} \not\cong 2\Sigma$. Then by Petri's theorem [13] the ideal of C_f^{can} is generated by quadrics. In this case we can construct (instances of) Schreyer's surfaces $Y, Z \subset \mathbb{P}(\mathcal{E})$ in a concrete way, by explicitly giving the defining equations of $\mu(Y), \mu(Z) \subset S$. Indeed, by [3, Thm. 4] the ideal of C_f^{can} is minimally generated by quadrics

$$b_1, \ldots, b_r, b'_1 \ldots, b'_s, \mathcal{F}_{2,w_1}, \ldots, \mathcal{F}_{2,w_t},$$

where

- the $r = \binom{g-3}{2}$ binomials b_i generate $\mathcal{I}(S)$,
- the $s = (4g-6) - \sharp(2\Delta^{(1)} \cap \mathbb{Z}^2)$ binomials b'_i cut $\mathrm{Tor}(\Delta^{(1)})$ out in S,
- $t = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) = B^{(1)} + 1$ and the quadrics $\mathcal{F}_{2,w_i}$ are constructed in the explicit manner described in [3]. Note that there is some freedom in the way these quadrics arise.

Then if $\mathcal{F}_f \subset \mathbb{P}(\mathcal{E})$ denotes the strict transform under μ of the joint zero locus of the quadrics $\mathcal{F}_{2,w_i}$, one can verify that $\mathcal{F}_f \sim 2H - B^{(1)}R$, so that one can take $Y = T'$ and $Z = \mathcal{F}_f$ if $B \geq B^{(1)}$, and $Y = \mathcal{F}_f$ and $Z = T'$ if $B < B^{(1)}$.

We end this section by explicitly listing the lattice polygons for which $B \leq B^{(1)}$. We will need the following property of two-dimensional lattice polygons of the form $\Delta^{(1)}$. An edge τ of a two-dimensional lattice polygon Γ is always supported on a line $a_\tau X + b_\tau Y = c_\tau$ with $a_\tau, b_\tau, c_\tau \in \mathbb{Z}$ and a_τ, b_τ coprime. When signs are chosen appropriately, we can assume that Γ is contained in the half-plane $a_\tau X + b_\tau Y \leq c_\tau$. Then the line $a_\tau X + b_\tau Y = c_\tau + 1$ is called the *outward shift* of τ. It is denoted by $\tau^{(-1)}$, and the polygon (which may take vertices outside $\mathbb{Z}^2$) that arises as the intersection of the half-planes $a_\tau X + b_\tau Y \leq c_\tau + 1$ is denoted by $\Gamma^{(-1)}$. If $\Gamma = \Delta^{(1)}$ for some lattice polygon Δ, then the outward shifts of two adjacent edges of Γ always intersect in a lattice point, and in fact $\Gamma^{(-1)} = \Delta^{(1)(-1)}$ is a lattice polygon. Moreover,

$\Delta \subset \Delta^{(1)(-1)}$, i.e. $\Delta^{(1)(-1)}$ is the maximal lattice polygon with respect to inclusion for which the convex hull of the interior lattice points equals $\Delta^{(1)}$. See [9, §4] or [11, §2.2] for proofs.

Even though the following statement is purely combinatorial, given its geometric interpretation, it is natural to abbreviate $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$. Similarly, we will write $g^{(1)} = \sharp(\Delta^{(2)} \cap \mathbb{Z}^2)$.

Lemma 9. *Let Δ be a lattice polygon with $\operatorname{lw}(\Delta^{(1)}) = 2$. Then we have:*

- *$B < B^{(1)}$ if and only if*

$$\Delta^{(1)} \cong \Gamma_{4k+4} := \operatorname{conv}\{(0,0),(k,0),(2k+2,1),(k+1,2),(1,2)\}$$

for some integer $k \geq 0$. In this case $g = 4k+4$, $B = 2k-1$ and $B^{(1)} = 2k$.

- *$B = B^{(1)}$ if and only if either*

$$\Delta^{(1)} \cong \Gamma^m_{4k+5} := \operatorname{conv}\{(0,0),(k,0),(2k+2,1),(k+m,2),(m,2),(0,1))\}$$

for some integers $k \geq 0$ and $0 \leq m \leq k+2$ (in these cases, $g = 4k+5$ and $B = B^{(1)} = 2k$), or

$$\Delta^{(1)} \cong \Gamma_{4k+3} := \operatorname{conv}\{(0,0),(k,0),(2k+1,1),(k+1,2),(1,2)\}$$

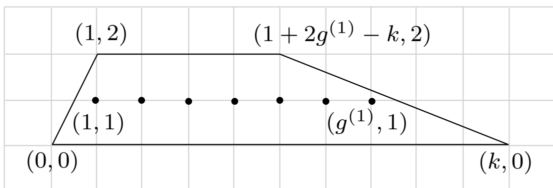
for some integer $k \geq 1$ (in this case, $g = 4k+3$ and $B = B^{(1)} = 2k-1$), or

$$\Delta^{(1)} \cong \Gamma_{4k+1} := \operatorname{conv}\{(0,0),(k,0),(2k,1),(k,2),(1,2)\}$$

for some integer $k \geq 2$ (in this case, $g = 4k+1$ and $B = B^{(1)} = 2k-2$).

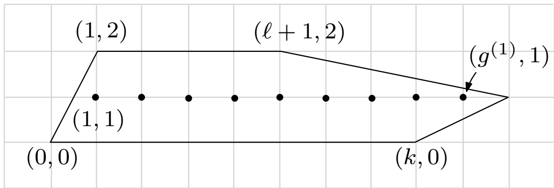
Proof. First we consider the polygons with $g^{(1)}$ equal to 0 and 1 separately. If $g^{(1)} = 0$ then $\Delta^{(1)} \cong 2\Sigma$, hence $B = 2 > B^{(1)} = -1$. If $g^{(1)} = 1$ then $B^{(1)} = 0$, hence $B \leq B^{(1)}$ if and only if $g \leq 5$. It is easy to check that there is one such polygon in genus 4 (namely $\Delta \cong 2\Upsilon$, so $\Delta^{(1)} \cong \Upsilon = \Gamma_4$) and three such polygons in genus 5 (corresponding to $\Delta^{(1)} \cong \Gamma^0_5, \Gamma^1_5, \Gamma^2_5$). Each of these appear in the classification.
If $g^{(1)} \geq 2$, we can use Koelman's classification [11, Section 4.3] of lattice polygons Γ with lattice width 2. One can assume that $\Gamma = \Delta^{(1)}$ is contained in the strip $\{(X,Y) \in \mathbb{R}^2 \,|\, 0 \leq Y \leq 2\}$. Koelman subdivided these polygons into three types:

- *Type 0: there is no boundary lattice point of Γ with $Y = 1$.*
Then up to equivalence $\Gamma = \Delta^{(1)}$ is of the form



with $g^{(1)} \leq k \leq 2g^{(1)}$. One sees that $B = 2g^{(1)} - 2$ and $B^{(1)} = g^{(1)} - 1$, so $B \leq B^{(1)}$ implies that $g^{(1)} \leq 1$: a contradiction.

- *Type* 1*: there is one boundary lattice point of* Γ *with* $Y = 1$.
  Up to equivalence $\Gamma = \Delta^{(1)}$ is of the form

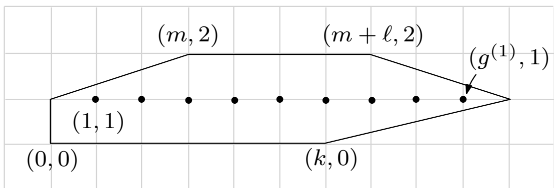


with $0 \leq k \leq 2g^{(1)} + 1$ and

$$\begin{cases} 0 \leq \ell \leq k & \text{if} \quad 0 \leq k \leq g^{(1)}, \\ 0 \leq \ell \leq 2g^{(1)} - k + 1 & \text{if} \quad g^{(1)} < k \leq 2g^{(1)} + 1. \end{cases}$$

Since moreover Γ is an interior lattice polygon we have that $\Gamma^{(-1)}$ takes its vertices inside $\mathbb{Z}^2$, leading to the inequalities $k \geq \frac{g^{(1)}-1}{2}$ and $\ell \geq \frac{g^{(1)}-1}{2}$. For this type, $B = k+\ell-1 \geq g^{(1)}-2$ and $B^{(1)} = g^{(1)}-1$. So if $B \leq B^{(1)}$ then either $k = \ell = \frac{g^{(1)}-1}{2}$ (and $g = 4k + 4 \equiv 0 \bmod 4$), or $k = \ell = \frac{g^{(1)}}{2}$ (and $g = 4k + 3 \equiv 3 \bmod 4$), or $k = \frac{g^{(1)}+1}{2}$ and $\ell = \frac{g^{(1)}-1}{2}$ (and $g = 4k + 1 \equiv 0 \bmod 4$). We find back the polygons $\Gamma_{4k+1}, \Gamma_{4k+3}, \Gamma_{4k+4}$ from the statement of the lemma.

- *Type* 2*: there are two boundary lattice points of* Γ *with* $Y = 1$.
  Up to equivalence $\Gamma = \Delta^{(1)}$ is of the form



with $0 \leq m \leq g^{(1)} + 1$, $0 \leq k \leq 2g^{(1)} + 2 - 2m$ and

$$\begin{cases} 0 \leq \ell \leq k & \text{if} \ \ 0 \leq k \leq g^{(1)} + 1 - m, \\ 0 \leq \ell \leq 2g^{(1)} - k - 2m + 2 & \text{if} \ \ g^{(1)} + 1 - m < k \leq 2g^{(1)} + 2 - 2m. \end{cases}$$

Since moreover Γ is an interior lattice polygon, we also get the inequalities $k \geq \frac{g^{(1)}-1}{2}$ and $\ell \geq \frac{g^{(1)}-1}{2}$. If $B \leq B^{(1)}$ then since $B = k + \ell \geq g^{(1)} - 1 = B^{(1)}$, we have that $k = \ell = \frac{g^{(1)}-1}{2}$, $B = B^{(1)} = 2k$ and $g = 4k + 5$. So we get the polygons Γ^m_{4k+5} from the statement.

This concludes the proof. □

Remark 10. For each lattice polygon $\Gamma = \Gamma_g, \Gamma_g^m$ appearing in the statement of the lemma, there is only one polygon Δ for which $\Delta^{(1)} = \Gamma$, namely $\Delta = \Gamma^{(-1)}$. Note that $(\Gamma_4)^{(-1)} \cong 2\Upsilon$ and recall that a (2Υ)-non-degenerate curve is trigonal, rather than tetragonal.

3 From toric surfaces to polygons

This section is devoted to proving Theorem 4. As an a priori remark, note that it is important to impose that $\operatorname{Tor}(\Delta)$ and $\operatorname{Tor}(\Delta')$ are obtained from one another using a transformation of $\mathbb{P}^N$, rather than just isomorphic. For instance, let

$$\Delta = \operatorname{conv}\{(0,0),(3,0),(3,2),(0,2)\} \text{ and } \Delta' = \operatorname{conv}\{(0,0),(5,0),(5,1),(0,1)\},$$

then $\operatorname{Tor}(\Delta), \operatorname{Tor}(\Delta') \subset \mathbb{P}^{11}$ are isomorphic (because their normal fans are the same), but not projectively equivalent, as they have different degrees (6 resp. 5). Here clearly $\Delta \not\cong \Delta'$.

Proof. We assume familiarity with the theory of divisors on toric surfaces, along the lines of [4, §3]. Notation-wise, we will write

- Σ_Δ for the (inner) normal fan associated to a given two-dimensional lattice polygon Δ, and
- Δ_D for the polygon (well-defined up to translation) corresponding to a Weil divisor (or a Cartier divisor, or an invertible sheaf) D on a given toric surface.

The proof then works as follows. Let Δ and Δ' be as in the statement of Theorem 4. The projective transformation induces an automorphism $\operatorname{Tor}(\Delta) \to \operatorname{Tor}(\Delta)$ that sends $\mathcal{O}_{\operatorname{Tor}(\Delta)}(1)$ to $\mathcal{O}_{\operatorname{Tor}(\Delta')}(1)$. Because

$$\Delta \cong \Delta_{\mathcal{O}_{\operatorname{Tor}(\Delta)}(1)} \qquad \text{and} \qquad \Delta' \cong \Delta_{\mathcal{O}_{\operatorname{Tor}(\Delta')}(1)}$$

it suffices to prove the following general statement: if

$$\iota : \operatorname{Tor}(\Delta) \xrightarrow{\cong} \operatorname{Tor}(\Delta')$$

is an isomorphism between two toric surfaces, and if D is a Weil divisor on $\operatorname{Tor}(\Delta)$, then

$$\Delta_D \cong \Delta_{\iota(D)}.$$

Now it is known that two isomorphic toric varieties always admit a toric isomorphism between them [1, Thm. 4.1], i.e. an isomorphism that is induced by a $\mathrm{GL}_2(\mathbb{Z})$-transformation taking Σ_Δ to $\Sigma_{\Delta'}$. It is clear that such an isomorphism preserves polygons (up to equivalence). Therefore we may assume that $\Sigma_\Delta = \Sigma_{\Delta'}$ and that ι is an automorphism of $\operatorname{Tor}(\Delta)$. Every such automorphism can be written as the composition of

- a toric automorphism,
- the automorphism induced by the action of an element of $\mathbb{T}^2$,
- a number of automorphisms of the form e_v^λ, where $\lambda \in k$ and $v \in \mathbb{Z}^2$ is a *column vector* of Δ, i.e. a primitive vector v for which there exists an edge $\tau \subset \Delta$ such that $u + v \in \Delta$ for all $u \in (\Delta \setminus \tau) \cap \mathbb{Z}^2$. To describe e_v^λ explicitly, assume that $v = (0, -1)$ and that τ lies horizontally (the general case can be reduced to this case by using an appropriate unimodular transformation). Then $\mathrm{Tor}(\Delta)$ can be viewed as a compactification of $\mathbb{T}^2 \cup (x\text{-axis})$ rather than just $\mathbb{T}^2$. On $\mathbb{T}^2 \cup (x\text{-axis})$, e_v^λ acts as $(x, y) \mapsto (x, y+\lambda)$. The column vector property ensures that this extends nicely to all of $\mathrm{Tor}(\Delta)$.

  Example. Let $\Delta = [0,1] \times [0,1]$ and consider the map

  $$\varphi : \mathbb{T}^2 \cup (x\text{-axis}) \hookrightarrow \mathrm{Tor}(\Delta) : (x, y) \mapsto (1, x, y, xy).$$

  The point $(x, y + \lambda)$ is mapped to $(1 : x : y + \lambda : xy + \lambda x)$. So here

$$e_{(0,-1)}^\lambda : (X_{0,0} : X_{1,0} : X_{0,1} : X_{1,1}) \mapsto (X_{0,0} : X_{1,0} : X_{0,1} + \lambda X_{0,0} : X_{1,1} + \lambda X_{1,0}).$$

See [2, Thm. 3.2] for a proof of this statement, along with a more elaborate discussion. Now the first type of automorphisms preserves polygons up to equivalence, as before. The second type also preserves polygons because it preserves torus-invariant Weil divisors. As for the third type, let D_τ be the torus-invariant prime divisor corresponding to the base edge τ of v. Then by adding a divisor of the form $\mathrm{div}(x^i y^j)$ if needed, one can always find a torus-invariant Weil divisor that is equivalent to D and whose support does not contain D_τ; see [4, §4] for more details. But such a divisor is preserved by e_v^λ, hence the theorem follows. □

4 INTRINSICNESS FOR TETRAGONAL CURVES

We are ready to explain why intrinsicness holds for lattice polygons Δ satisfying

$$\mathrm{lw}(\Delta^{(1)}) = 2 \quad \text{and} \quad B \geq B^{(1)} + 2,$$

that is, for the polygons of type (e) from the introduction. Let C be a Δ-non-degenerate curve. Then it is a tetragonal curve (indeed, $B \geq B^{(1)} + 2$ implies $\Delta \not\cong 2\Upsilon$) whose Schreyer invariants b_1, b_2 satisfy $b_1 \geq b_2 + 2$. By Theorem 7 we find that Schreyer's surface $\mu(Y) \subset \mathbb{P}^{g-1}$ equals $\mathrm{Tor}(\Delta^{(1)})$. Now suppose that C is also Δ'-non-degenerate for some two-dimensional lattice polygon Δ'. By the tetragonality of C we have $\mathrm{lw}(\Delta'^{(1)}) = 2$. In analogy with the previous notation, write

$$B' = \sharp(\partial\Delta'^{(1)} \cap \mathbb{Z}^2) - 4, \qquad B'^{(1)} = \sharp(\Delta'^{(2)} \cap \mathbb{Z}^2) - 1,$$

so that $\{B', B'^{(1)}\} = \{b_1, b_2\}$ by Theorem 7. It follows that either

$$B' \geq B'^{(1)} + 2 \qquad \text{or} \qquad B'^{(1)} \geq B' + 2.$$

But the latter is impossible by Lemma 9, which states that $B'^{(1)}$ is at most $B' + 1$. Therefore $B' > B'^{(1)}$ and, again by Theorem 7, we find that $\mu(Y)$ is given by $\mathrm{Tor}(\Delta'^{(1)})$. We conclude that $\mathrm{Tor}(\Delta^{(1)})$ and $\mathrm{Tor}(\Delta'^{(1)})$ are equal, possibly modulo a projective transformation. Intrinsicness now follows from Theorem 4.

This argument can be refined. For instance, in genus $g \not\equiv 0 \bmod 4$ it suffices that $B \geq B^{(1)} + 1$, because in this case Lemma 9 yields the sharper bound $B'^{(1)} \leq B'$. In genus $g \equiv 2 \bmod 4$ one sees that this is automatically satisfied.

By pushing this type of reasoning, we obtain the following statement.

Theorem 11. *Let Δ, Δ' be two-dimensional lattice polygons and let there be a curve that is both Δ-non-degenerate and Δ'-non-degenerate. Suppose that $\mathrm{lw}(\Delta^{(1)}) = 2$ and define $g = \sharp(\Delta^{(1)} \cap \mathbb{Z}^2) = \sharp(\Delta'^{(1)} \cap \mathbb{Z}^2)$.*

- *Case $g \equiv 0 \bmod 4$. If $\Delta^{(1)}, \Delta'^{(1)} \not\cong \Gamma_g$ then $\Delta^{(1)} \cong \Delta'^{(1)}$. This is automatic if $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 5$.*
- *Case $g \equiv 1 \bmod 4$. If $\Delta^{(1)}, \Delta'^{(1)} \not\cong \Gamma_g^m$ for all $1 \leq m \leq (g+3)/4$ then $\Delta^{(1)} \cong \Delta'^{(1)}$. This is automatic if $\sharp(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq \sharp(\Delta^{(2)} \cap \mathbb{Z}^2) + 4$.*
- *Cases $g \equiv 2, 3 \bmod 4$. Here one always has $\Delta^{(1)} \cong \Delta'^{(1)}$.*

Proof. The cases $g \equiv 0, 2 \bmod 4$ follow along the above lines of thought. For the case $g \equiv 3 \bmod 4$ one remarks that Schreyer's invariants coincide if and only if $B = B^{(1)}$, which by Lemma 9 happens if and only if $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_g$. If not then $B \geq B^{(1)} + 1$, and one proceeds as before.
The most subtle case is when $g \equiv 1 \bmod 4$. If $g = 5$ then Schreyer's invariants coincide if and only if $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_5^0$ (indeed, the polygons Γ_5^1 and Γ_5^2 appearing in Lemma 9 were excluded in the statement), so this is analogous to the $g \equiv 3 \bmod 4$ case. If $g > 5$ then one draws the weaker conclusion that Schreyer's invariants coincide if and only if $\Delta^{(1)}$ and $\Delta'^{(1)}$ are among Γ_g and Γ_g^0. To distinguish between both cases, one notes that the scrollar invariants e_1, e_2, e_3 are

$$\frac{g-5}{4}, \frac{g-1}{4}, \frac{g-3}{2} \qquad \text{and} \qquad \frac{g-5}{4}, \frac{g-5}{4}, \frac{g-1}{2},$$

respectively. Here we implicitly used that our curve carries a unique g_4^1 by [4, Cor. 6.3], so it does make sense to talk about *the* scrollar invariants. We conclude that $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_g^0$ if the curve has two coinciding scrollar invariants, and that $\Delta^{(1)} \cong \Delta'^{(1)} \cong \Gamma_g$ if not. ∎

Remark 12. Note that the theorem remains valid if we replace 'for all $1 \leq m \leq (g+3)/4$' by 'for all $m \in \{0, \ldots, (g+3)/4\} \setminus \{i\}$', for whatever i.

Example 13. Let $g \geq 4$ satisfy $g \equiv 0 \bmod 4$, and denote by Δ_g the (unique) lattice polygon for which $\Delta_g^{(1)} = \Gamma_g$. Then it is possible that a Δ_g-non-degenerate curve is also non-degenerate with respect to a lattice polygon Δ' for which $\Delta'^{(1)} \not\cong \Gamma_g$. For instance, consider $f = 1 - x^2y^4 - x^{\frac{g}{2}+2}y^2$ and $f' = (y^4-1)x^{\frac{g}{2}+1} + 4y^2$. Both polynomials are non-degenerate with respect to their respective Newton polygons. Note that $\Delta(f) \cong \Delta_g$ and that $\Delta(f')^{(1)} \not\cong \Gamma_g$. Now the rational maps

$$U_f \to U_{f'} : (x,y) \mapsto \left(x, \frac{1-xy^2}{x^{\frac{g}{4}+1}y}\right)$$

$$U_{f'} \to U_f : (x,y) \mapsto \left(x, \frac{2y}{x^{\frac{g}{4}+1}(1+y^2)}\right)$$

are inverses of each other, so C_f and $C_{f'}$ are isomorphic. We conclude that C_f is both Δ_g-non-degenerate and $\Delta(f')$-non-degenerate.

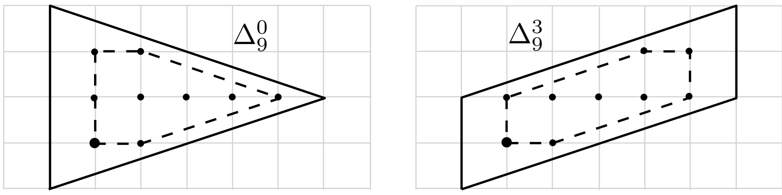
Example 14. We conjecture that for each $g \geq 5$ with $g \equiv 1 \bmod 4$ and each $0 \leq n, m \leq (g+3)/4$, there exists a curve that is both Δ_g^n- and Δ_g^m-non-degenerate. Here Δ_g^n and Δ_g^m are the unique lattice polygons having Γ_g^n and Γ_g^m as their respective interiors.
Loosely speaking, we believe that the following strategy for finding such a curve always works (although we could not prove this). From Sections 1 and 2 we know that the canonical model C_f^{can} of a Δ_g^n-non-degenerate curve C_f satisfies $C_f^{\mathrm{can}} \subset \mathrm{Tor}(\Gamma_g^n) \subset S \subset \mathbb{P}^{g-1}$, where S is a rational normal scroll of type

$$\left(\frac{g-5}{4}, \frac{g-5}{4}, \frac{g-1}{2}\right),$$

and that C_f^{can} arises as the intersection of two surfaces Y and Z inside the class

$$2H - \frac{g-5}{2}R$$

(the role of μ, which is only relevant for $g=5$, is ignored for the sake of exposition). Recall from Remark 8 that one can take $Y = \mathrm{Tor}(\Gamma_g^n)$, and $Z = \mathcal{F}_f$. The idea is to switch the role of Y and Z, in the sense that one chooses f such that $\mathcal{F}_f = \theta(\mathrm{Tor}(\Gamma_g^m))$ for some $\theta \in \mathrm{Aut}(S) \subset \mathrm{Aut}(\mathbb{P}^{g-1})$. Because non-degeneracy is generically satisfied, one expects $\theta^{-1}(Y)$ to be of the form $\mathcal{F}_{f'}$ for some Δ_g^m-non-degenerate Laurent polynomial f'.
Explicit examples in genus $g=5$ can be found in our unpublished `arXiv` paper `1304.4997`.
For $g = 9$ and $\{n, m\} = \{0, 3\}$



we used the above approach to find that the Δ_9^0-non-degenerate Laurent polynomial

$$f = 8x^5y + 36x^4y + 66x^3y - x^2y^2 + 62x^2y - x^2 + 33xy + \\ +9y - 2x^{-1}y^3 - 2x^{-1}y^2 - 4x^{-1}y - 3x^{-1} - 3x^{-1}y^{-1}$$

and the Δ_9^3-non-degenerate Laurent polynomial

$$f' = 2x^5y^3 + x^5y^2 - x^5y - 6x^4y - 15x^3y + 2x^2y^2 - 14x^2y + \\ +x^2 - 15xy - 6y - x^{-1}y + 3x^{-1} + 3x^{-1}y^{-1}$$

define birationally equivalent curves in $\mathbb{T}^2$. To describe the automorphism θ explicitly, we need to pick coordinates of $\mathbb{P}^{g-1}$. When thought of as the ambient space of $\mathrm{Tor}(\Gamma_9^0)$, we will write

$$\mathbb{P}^{g-1} = \operatorname{Proj} V \text{ with } V = k[X_{0,0}, X_{1,0}, X_{0,1}, X_{1,1}, X_{2,1}, X_{3,1}, X_{4,1}, X_{0,2}, X_{1,2}],$$

where $X_{i,j}$ is the coordinate corresponding to the lattice point $(i,j) \in \Gamma_9^0$ (the origin is understood to be the bold-marked lattice point). Similarly, when thought of as the ambient space of $\mathrm{Tor}(\Gamma_9^3)$ we write

$$\mathbb{P}^{g-1} = \operatorname{Proj} W \text{ with } W = k[X_{0,0}, X_{1,0}, X_{0,1}, X_{1,1}, X_{2,1}, X_{3,1}, X_{4,1}, X_{3,2}, X_{4,2}].$$

Then, on the level of coordinate rings, $\theta : V \to W$ can be defined by

$$\begin{pmatrix} \theta(X_{0,1}) \\ \theta(X_{1,1}) \\ \theta(X_{2,1}) \\ \theta(X_{3,1}) \\ \theta(X_{4,1}) \\ \theta(X_{0,2}) \\ \theta(X_{1,2}) \\ \theta(X_{0,0}) \\ \theta(X_{1,0}) \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 9 & 7 & 2 & 0 & 0 & 0 & 0 \\ 1 & 6 & 13 & 12 & 4 & 0 & 0 & 0 & 0 \\ 1 & 7 & 18 & 20 & 8 & 0 & 0 & 0 & 0 \\ 1 & 8 & 24 & 32 & 16 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 4 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} X_{0,1} \\ X_{1,1} \\ X_{2,1} \\ X_{3,1} \\ X_{4,1} \\ X_{3,2} \\ X_{4,2} \\ X_{0,0} \\ X_{1,0} \end{pmatrix}.$$

We leave it to the reader to verify that θ maps S to S and sends $\mathrm{Tor}(\Gamma_9^3)$ to $\mathcal{F}_f$ and $\mathcal{F}_{f'}$ to $\mathrm{Tor}(\Gamma_9^0)$ (for an appropriate choice of defining equations for $\mathcal{F}_f$ and $\mathcal{F}_{f'}$).

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Wouter Castryck
Vakgroep Wiskunde
Universiteit Gent
Krijgslaan 281
9000 Gent, Belgium
and
Departement Elektrotechniek
KU Leuven
Kasteelpark Arenberg 10/2452
3001 Leuven, Belgium
wouter.castryck@gmail.com

Filip Cools
Dept. of Mathematics
and Applied Mathematics
University of Cape Town
Private Bag X1
Rondebosch 7701
South Africa
filip.cools@uct.ac.za