On Two Higher Chow Groups of Schemes over a Finite Field

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Abstract. Given a separated scheme X of finite type over a finite field, its higher Chow groups $CH_{-1}(X,1)$ and $CH_{-2}(X,3)$ are computed explicitly.

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1. Introduction

Let $\mathbb{F}_q$ be the field of q elements of characteristic p. For a separated scheme X which is essentially of finite type over $\operatorname{Spec}\mathbb{F}_q$, we define the Borel-Moore motivic homology group $H_i^{\mathrm{BM}}(X,\mathbb{Z}(j))$ as the homology group $H_{i-2j}(z_j(X,\bullet)) = \mathrm{CH}_j(X,i-2j)$ of Bloch's cycle complex $z_j(X,\bullet)$ ([Bl, Introduction, p. 267] see also [Ge-Le2, 2.5, p. 60] to remove the condition that X is quasi-projective; see [Le1] for the labeling using dimension and not codimension). If $j > i$ or $j > \dim X$, then $H_i^{\mathrm{BM}}(X,\mathbb{Z}(j)) = 0$ for trivial reason. When X is essentially smooth over $\operatorname{Spec}\mathbb{F}_q$, it coincides with the motivic cohomology group defined in [Le1, Part I, Chapter I, 2.2.7, p. 21] or [Vo-Su-Fr] (cf. [Le2, Theorem 1.2, p. 300], [Vo, Corollary 2, p. 351]). For an abelian group M, we set $H_i^{\mathrm{BM}}(X,M(j)) = H_{i-2j}(z_j(X,\bullet)\otimes_{\mathbb{Z}} M)$.
For a scheme X, we let $\mathcal{O}(X) = H^0(X,\mathcal{O}_X)$. The aim of this paper is to prove the following theorem.

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Theorem 1.1. *Let X be a connected scheme which is separated and of finite type over* $\operatorname{Spec}\mathbb{F}_q$*. Then for $j = -1, -2$, the pushforward map*

$$\alpha_X : H^{\mathrm{BM}}_{-1}(X, \mathbb{Z}(j)) \to H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X), \mathbb{Z}(j))$$

is an isomorphism if X is proper, and the group $H^{\mathrm{BM}}_{-1}(X, \mathbb{Z}(j))$ is zero if X is not proper.

Theorem 1.1 is a generalization of a theorem of Akhtar [Ak, Theorem 3.1, p. 285] where the claim is proved for $j = -1$ and X smooth projective over $\operatorname{Spec}\mathbb{F}_q$. Our proof of Theorem 1.1 is independent of [Ak], and we do not require a Bertini-type theorem.

If we assume Parshin's conjecture, then the statement in the theorem holds for any $j \leq -1$. Moreover we also obtain $H^{\mathrm{BM}}_i(X, \mathbb{Z}(j)) = 0$ for any $i \leq -2$ and $j \leq -1$. The method is explained in Section 4

We define the étale Borel-Moore (not motivic) homology with $\mathbb{Z}_\ell$-coefficients, where ℓ is a prime different from p, in Remark 4.3. Then we compute it explicitly, and deduce that $H^{\mathrm{BM}}_i(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong H^{\mathrm{BM,et}}_i(X, \mathbb{Z}_\ell(j))$ in the range $i \leq -1$ and $j \leq -1$ (using Parshin's conjecture where it is needed for the computation of the Borel-Moore motivic homology groups).

The original version of this paper was written without using the Bloch-Kato-Milnor conjecture. We use it as a theorem of Rost and Voevodsky. It is used via theorems of Geisser and Levine (e.g., [Ge-Le2, Corollary 1.2, p.56]).

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2. Higher Chow groups of smooth curves over a finite field

A curve will mean a scheme of pure dimension one, separated and of finite type over a field. The aim of this section is to compute the higher Chow groups $\mathrm{CH}^i(X, j)$ for a smooth curve X over a finite field in the range $i, j \geq 0$.

Lemma 2.1. *Let X be a connected smooth curve over a finite field. Then*

$$\mathrm{CH}^i(X, 0) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathrm{Pic}(X), & i = 1, \\ 0, & i \geq 2. \end{cases}$$

Proof. These are the classical Chow groups and the computation is known. For $i \geq 2$, it vanishes by dimension reason. See also [Bl, THEOREM (6.1), p.287]. □

Lemma 2.2. *Let $j \geq 2$. Let X be a smooth curve over a finite field $\mathbb{F}_q$ of characteristic p. Then we have $\mathrm{CH}^i(X,j) = 0$ for $i > j$, and for $i \leq j$, the cycle map in* [Ge-Le2, (1.2), p.56] *gives an isomorphism*

$$\mathrm{CH}^i(X,j) \cong \bigoplus_{\ell \neq p} H^{2i-j-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)).$$

The right hand side is zero unless $2i - j = 1, 2, 3$. If moreover X is affine, the right hand side is zero for $2i - j = 3$.

Proof. We first note that $\mathrm{CH}^i(X,j) = 0$ if $i > j + 1$ by dimension reason. Henceforth we consider the case $i \leq j + 1$.
Recall Bloch's formula ([Bl, THEOREM(9.1), p.296]):

$$K_j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i \geq 0} \mathrm{CH}^i(X,j) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{2.1}$$

where K_j is the j-th algebraic K-group. Recall also Harder's result (the result [Hard, 3.2.5 Korollar, p.175] is not correctly stated; we refer to [Gr, THEOREM 0.5, p.70] and the remark there for the explanation and the corrected statement) which implies that $K_j(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $j \geq 2$. Hence $\mathrm{CH}^i(X,j)$ is torsion for $j \geq 2$.
We recall the definition of motivic cohomology given in [Ge-Le2, Section 2.5, p.60]. For a smooth scheme X over a field, define the cohomological cycle complex by $Z^j(X,i) = z^j(X, 2j-i)$ where $z^*(-,*)$ is Bloch's cycle complex ([Bl, INTRODUCTION, p.267], see also [Ge-Le2, 2.2, p.58]). Then, for an abelian group A, define $H^j_{\mathcal{M}}(X, A(i)) = H^i(Z^j(X,*) \otimes_{\mathbb{Z}} A)$. We have $H^{2i-j}_{\mathcal{M}}(X, \mathbb{Z}(i)) = \mathrm{CH}^i(X,j)$.
The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives an exact sequence

$$\begin{aligned} &H^{2i-j-1}_{\mathcal{M}}(X, \mathbb{Q}(i)) \to H^{2i-j-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(i)) \xrightarrow{(1)} H^{2i-j}_{\mathcal{M}}(X, \mathbb{Z}(i)) \\ &\to H^{2i-j}_{\mathcal{M}}(X, \mathbb{Q}(i)). \end{aligned}$$

The first and the last terms are zero as was remarked above. The map (1) is hence an isomorphism.
Since we are in the range $i \leq j+1$ (equivalently, $2i-j-1 \leq i$), we apply [Ge-Le2, Corollary 1.2, p.56] and the computation ([Ge-Le1, Theorem 8.4, p.491]) by Geisser and Levine of p-torsion motivic cohomology to obtain

$$H^{2i-j-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(i)) \cong \bigoplus_{\ell \neq p} H^{2i-j-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \oplus \varinjlim_r H^{i-j-1}(X_{\mathrm{Zar}}, \nu^i_r).$$

(We refer to [Ge-Le1] for the definition of $H^*(X_{\mathrm{Zar}}, \nu^i_r)$.) One can compute the right hand side explicitly. The p-part is zero since we are in the range $i \leq j+1$ and $j \geq 2$.

Set $a = 2i - j - 1$. Let us show that $H^a_{\rm et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ if $a \geq 3$. If $a \geq 4$, it follows from the fact that the cohomological dimension of a curve over a finite field is 3 ([SGA4-3, Exposé X, Corollaire 4.3, p.15]). Suppose $a = 3$. We have an exact sequence

$$H^3_{\rm et}(X, \mathbb{Q}_\ell(i)) \to H^3_{\rm et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^4_{\rm et}(X, \mathbb{Z}_\ell(i)).$$

The third term is zero because of the cohomological dimension reason. The Hochschild-Serre spectral sequence reads

$$E_2^{p,q} = H^p_{\rm et}(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), H^q_{\rm et}(\overline{X}, \mathbb{Q}_\ell(i)) \Rightarrow H^{p+q}_{\rm et}(X, \mathbb{Q}_\ell(i))$$

where $\overline{X} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\overline{\mathbb{F}}_q$. We have $E_2^{0,3} = 0$ since $H^3_{\rm et}(\overline{X}, \mathbb{Q}_\ell(i)) = 0$. To show $E_2^{1,2} = 0$, note that the weight of $H^2_{\rm et}(\overline{X}, \mathbb{Q}_\ell(i))$ is $2 - 2i$. Since $j \geq 2$ and $a = 3$, the weight $2 - 2i$ is nonzero, hence $E_2^{1,2} = 0$. We have $E_2^{2,1} = 0$ because the cohomological dimension of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is one. This proves the claim in this case.

Suppose $a = 2$ and X is affine. We have an exact sequence

$$H^2_{\rm et}(X, \mathbb{Q}_\ell(i)) \to H^2_{\rm et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \to H^3_{\rm et}(X, \mathbb{Z}_\ell(i)).$$

The third term is zero since the cohomological dimension of an affine curve over a finite field is 2 ([SGA4-3, Exposé XIV, Théorème 3.1, p.15]). We use the Hochshild-Serre spectral sequence as above. We have $E_2^{0,2} = 0$ using the same cohomological dimension reason. Note that the (possible) weights of $H^1_{\rm et}(\overline{X}, \mathbb{Q}_\ell(i))$ are $1 - 2i$ and $2 - 2i$. Since neither of them is zero, we have $E_2^{1,1} = 0$. These imply that $H^2_{\rm et}(X, \mathbb{Q}_\ell(i)) = 0$ hence the claim in this case. □

Lemma 2.3. *Let X be a smooth curve over a finite field. We have*

$$\mathrm{CH}^i(X, 1) = \begin{cases} 0, & i = 0, \\ \mathcal{O}(X)^\times, & i = 1, \\ 0, & i \geq 3. \end{cases}$$

Proof. The case $i = 0$ is trivial. The case $i = 1$ is found in Bloch's paper ([Bl, THEOREM (6.1), p.287]). For $i \geq 3$, the claim follows by dimension reason. □

Lemma 2.4. *Let U be an affine smooth curve over a finite field. Then $\mathrm{CH}^2(U, 1) = 0$.*

Proof. The group $SK_1(U)$ sits in the following exact sequence:

$$0 \to SK_1(U) \to K_1(U) \to \mathcal{O}(U)^\times \to 0.$$

We use the result [Gr, THEOREM 0.5, p.70], which says that $SK_1(U) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for an affine smooth curve U. Using Lemma 2.3, it follows from counting the dimension of both sides of Bloch's formula (2.1) that $\dim_{\mathbb{Q}} H^3_{\mathcal{M}}(U, \mathbb{Q}(2)) = 0$. For the rest of the proof, one proceeds as in Lemma 2.2. □

Lemma 2.5. *Let Z be a scheme which is finite over* $\operatorname{Spec}\mathbb{F}_q$. *Then we have isomorphisms*

$$\begin{aligned}&H^{\mathrm{BM}}_{-1}(Z,\mathbb{Z}(j)) \overset{(1)}{\cong} H^{\mathrm{BM}}_{-1}(Z_{\mathrm{red}},\mathbb{Z}(j))\\ &\overset{(2)}{\cong} H^{\mathrm{BM}}_{0}(Z_{\mathrm{red}},\mathbb{Q}/\mathbb{Z}(j)) \overset{(3)}{\cong} \bigoplus_{\ell\neq p} H^0_{\mathrm{et}}(Z_{\mathrm{red}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))\end{aligned}$$

for $j \le -1$, which are functorial with respect to pushforwards. Here Z_{red} denotes the reduced scheme associated to Z.

Proof. For any scheme W of finite type over $\mathbb{F}_q$ and an abelian group A, we have $H^{\mathrm{BM}}_i(W,A(j)) \cong H^{\mathrm{BM}}_i(W_{\mathrm{red}},A(j))$ for any i,j, since the cycle complexes are canonically isomorphic by definition. This gives the isomorphism (1).
For (2), we use the long exact sequence of the universal coefficient theorem for higher Chow groups:

$$\begin{aligned}&H^{\mathrm{BM}}_0(Z_{\mathrm{red}},\mathbb{Q}(j)) \to H^{\mathrm{BM}}_0(Z_{\mathrm{red}},\mathbb{Q}/\mathbb{Z}(j))\\ &\to H^{\mathrm{BM}}_{-1}(Z_{\mathrm{red}},\mathbb{Z}(j)) \to H^{\mathrm{BM}}_{-1}(Z_{\mathrm{red}},\mathbb{Q}(j)).\end{aligned}$$

We know that the higher K-groups of a finite field are torsion from [Qu, THEOREM 8, p.583]. Then using a formula of Bloch (2.1), we see that the groups in the sequence above with $\mathbb{Q}$-coefficient are zero.
Since $\mathbb{F}_q$ is perfect, $\operatorname{Spec} Z_{\mathrm{red}}$ is smooth over $\mathbb{F}_q$. The map (3) is the cycle map in [Ge-Le2] (which is defined for smooth schemes over a field). The fact that the cycle map is an isomorphism follows from [Ge-Le2, Corollary 1.2, p. 56] and [Me-Su, (11.5), THEOREM, p. 328], and [Ge-Le1, Theorem 1.1, p406].
It is clear that the isomorphisms (1) and (2) are functorial with respect to pushforwards. Let Z' be another scheme which is finite over $\operatorname{Spec}\mathbb{F}_q$ and let $f : Z' \to Z$ be a morphism over $\operatorname{Spec}\mathbb{F}_q$. We prove that the isomorphisms (3) for Z and Z' are compatible with the pushforward maps with respect to f. We are easily reduced to the case when both Z and Z' are spectra of finite extensions of $\mathbb{F}_q$. Let $Z'' = Z' \times_Z Z'$ and let $\mathrm{pr}_1, \mathrm{pr}_2 : Z'' \to Z'$ denote the projections to the first and the second factor, respectively. Then the diagram

$$\begin{tikzcd} H^{\mathrm{BM}}_0(Z',\mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "f_*"] \arrow[d, "\mathrm{pr}_2^*"'] & H^{\mathrm{BM}}_0(Z,\mathbb{Q}/\mathbb{Z}(j)) \arrow[d, "f^*"] \\ H^{\mathrm{BM}}_0(Z'',\mathbb{Q}/\mathbb{Z}(j)) \arrow[r, "(\mathrm{pr}_1)_*"] & H^{\mathrm{BM}}_0(Z',\mathbb{Q}/\mathbb{Z}(j)) \end{tikzcd}$$

is commutative, and a similar commutativity holds for the corresponding étale cohomology groups. Since the pullback map $f^* : H^0_{\mathrm{et}}(Z,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \to H^0_{\mathrm{et}}(Z',\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))$ is injective, it suffices to prove that the isomorphisms (3) for Z'' and Z' are compatible with the pushforward maps with respect to pr_1. Since Z'' is isomorphic to the disjoin union of a finite number of copies of Z', the last claim can be checked easily. The lemma is proved. □

The statement is better understood using étale Borel-Moore homology groups. See Remark 4.3.

REMARK 2.6. *Suppose X is a connected scheme which is proper over* $\operatorname{Spec}\mathbb{F}_q$. *Then Theorem 1.1 says that the group $H_{-1}^{\mathrm{BM}}(X,\mathbb{Z}(j))$ is isomorphic to $H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\mathcal{O}(X),\mathbb{Z}(j))$. We can then use Lemma 2.5 and compute this group explicitly. The computation of étale cohomology group shows that this group is a cyclic group whose order is*

$$|\mathcal{O}(X_{\mathrm{red}})|^{-j}-1$$

for each $j=-1,-2$.

LEMMA 2.7. *Let $\mathbb{F}'_1$, $\mathbb{F}'_2$ be two finite extensions of $\mathbb{F}_q$ with $\mathbb{F}'_1\subset\mathbb{F}'_2$. Then for $j\le -1$, the pushforward map $H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\mathbb{F}'_2,\mathbb{Z}(j))\to H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\mathbb{F}'_1,\mathbb{Z}(j))$ is surjective.*

Proof. By Lemma 2.5, the cycle class map gives an isomorphism α : $H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\mathbb{F}'_k,\mathbb{Z}(j))\cong\bigoplus_{\ell\ne p}H^0_{\mathrm{et}}(\operatorname{Spec}\mathbb{F}'_k,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))$ for $k=1,2$. The cycle class map is compatible with the pushforward by a finite morphism ([Ge-Le2, Lemma 3.5(2), p.69]). Thus the claim follows from the corresponding statement for the étale cohomology groups. (See [So, Lemme 6 iii), p.269] and [So, IV.1.7, p.283].) □

3. PROOF OF THEOREM 1.1

LEMMA 3.1. *Let X be an integral scheme which is of finite type over* $\operatorname{Spec}\mathbb{F}_q$. *Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ in $\mathcal{O}(X)$. Then the degree $[\mathbb{F}:\mathbb{F}_q]$ divides the degree $[\kappa(x):\mathbb{F}_q]$ for all closed points $x\in X_0$. If moreover X is normal, we have the equality $[\mathbb{F}:\mathbb{F}_q]=\gcd_{x\in X_0}[\kappa(x):\mathbb{F}_q]$.*

Proof. For each $x\in X_0$, the composite $\mathbb{F}\hookrightarrow\mathcal{O}(X)\to\kappa(x)$, where the second map is induced from the pullback map by the closed immersion, is injective since $\mathbb{F}$ is a field. Hence $[\mathbb{F}:\mathbb{F}_q]$ divides $[\kappa(x):\mathbb{F}_q]$.
Suppose that X is normal of dimension d. Then X is geometrically irreducible as a scheme over $\operatorname{Spec}\mathbb{F}$. Let $d_0=(\gcd_{x\in X_0}[\kappa(x):\mathbb{F}_q])/[\mathbb{F}:\mathbb{F}_q]$. Let $\mathbb{F}\subset\mathbb{F}_0$ denote an extension of degree d_0. The canonical morphism $f:X\times_{\operatorname{Spec}\mathbb{F}}\operatorname{Spec}\mathbb{F}_0\to X$ is a finite étale cover in which every closed point of X splits completely. It follows from a Chebotarev density type theorem ([La]; we refer to [Ra, Lemma 1.7, p.98] for the statement which is ready for our use) that f is an isomorphism. Hence $d_0=1$. This completes the proof. □

LEMMA 3.2. *Let $d\ge 0$ be an integer. Suppose that Theorem 1.1 holds for all connected normal schemes over $\operatorname{Spec}\mathbb{F}_q$ of dimension d which are not proper over $\operatorname{Spec}\mathbb{F}_q$. Then Theorem 1.1 holds for all connected normal schemes of dimension d which are proper over $\operatorname{Spec}\mathbb{F}_q$.*

Proof. Let X be a connected normal scheme of dimension d which is proper over $\operatorname{Spec}\mathbb{F}_q$. Let $j\in\{-1,-2\}$. Let $x\in X_0$ be a closed point. The pushforward map α_X in the statement of Theorem 1.1 is surjective since its composite with the pushforward map $\iota_{x*}:H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\kappa(x),\mathbb{Z}(j))\to H_{-1}^{\mathrm{BM}}(X,\mathbb{Z}(j))$ is surjective by Lemma 2.7. By assumption, the group $H_{-1}^{\mathrm{BM}}(X\setminus\{x\},\mathbb{Z}(j))$ is zero. Hence the localization sequence shows that the pushforward map ι_{x*} is surjective. This implies that $H^{\mathrm{BM}}_{-1}(X,\mathbb{Z}(j))$ is of order dividing $\gcd_{x\in X_0}(|\kappa(x)|^{-j}-1) = q^{(-j)\cdot\gcd_{x\in X_0}[\kappa(x):\mathbb{F}_q]}-1$. This equals $q^{[\mathbb{F}:\mathbb{F}_q]}$ by Lemma 3.1 where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$ in $\mathcal{O}(X)$. We know the order of the target of α_X is also equal to this value from Lemma 2.5. Hence the bijectivity of α_X follows. $\square$

Lemma 3.3. *Let $S_1 \xleftarrow{\alpha_1} S_3 \xrightarrow{\alpha_2} S_2$ be a diagram of sets and let R be an integral domain. For $i=1,2,3$, let* $\mathrm{Map}(S_i,R)$ *denote the R-module of R-valued functions on S_i. Then the cokernel of the homomorphism*

$$\beta : \mathrm{Map}(S_1,R)\oplus\mathrm{Map}(S_2,R)\to\mathrm{Map}(S_3,R)$$

which sends (f_1,f_2) to $f_1\circ\alpha_1 - f_2\circ\alpha_2$ is R-torsion free.

Proof. Let $e : \mathrm{Map}(S_3,R)\to\operatorname{Coker}\beta$ denote the quotient map. Let $f\in\mathrm{Map}(S_3,R)$ and suppose that $e(f)$ is an R-torsion element in $\operatorname{Coker}\beta$. We prove that $e(f)=0$. Since $e(f)$ is an R-torsion element, there exist a non-zero element $a\in R$ and an element $(f_1,f_2)\in\mathrm{Map}(S_1,\mathbb{Z})\oplus\mathrm{Map}(S_2,\mathbb{Z})$ satisfying $af = f_1\circ\alpha_1 - f_2\circ\alpha_2$. Let us take a complete set $T\subset R$ of representatives of R/aR. For $i=1,2$, let $\overline{f}_i$ denote the unique T-valued function on S_i satisfying $\overline{f}_i(x)\equiv f_i(x) \mod aR$ for every $x\in S_i$. We then have

$$\overline{f}_1\circ\alpha_1 \equiv f_1\circ\alpha_1 \equiv f_2\circ\alpha_2 \equiv \overline{f}_2\circ\alpha_2$$

modulo $a\mathrm{Map}(S_3,R)$. Since both $\overline{f}_1\circ\alpha_1$ and $\overline{f}_2\circ\alpha_2$ are T-valued functions, we have $\overline{f}_1\circ\alpha_1 = \overline{f}_2\circ\alpha_2$. For $i=1,2$, let g_i denote the unique R-valued function on S_i satisfying $f_i = \overline{f}_i + ag_i$. Then

$$af = (\overline{f}_1 + ag_1)\circ\alpha_1 - (\overline{f}_2 + ag_2)\circ\alpha_2 = a(g_1\circ\alpha_1 - g_2\circ\alpha_2).$$

Since $\mathrm{Map}(S_3,R)$ is R-torsion free, we have $f = g_1\circ\alpha_1 - g_2\circ\alpha_2$. This shows that $e(f)=0$, which proves the claim. $\square$

Lemma 3.4. *Let $d\geq 0$ be an integer. Suppose that Theorem 1.1 holds for all connected schemes of dimension smaller than d which are proper over* $\operatorname{Spec}\mathbb{F}_q$ *and for all connected normal schemes over* $\operatorname{Spec}\mathbb{F}_q$ *of dimension d. Then Theorem 1.1 holds for all connected schemes of dimension d which are proper over* $\operatorname{Spec}\mathbb{F}_q$.

Proof. Let X be a connected scheme of dimension d which is proper over $\operatorname{Spec}\mathbb{F}_q$. Without loss of generality we may assume that X is reduced. Suppose that X is not normal. Let $\pi : X'\to X$ denote the normalization of X. The scheme X' is proper over $\operatorname{Spec}\mathbb{F}_q$ since π is finite by [EGAII, Remarque 6.3.10, p. 120]. Take a reduced closed subscheme $Y\subset X$ of dimension less than that of X such that $X\setminus Y$ is normal and set $Y' = (Y\times_X X')_{\mathrm{red}}$. By assumption, Theorem 1.1 holds for each connected component of $X\setminus Y$, X', Y and Y'.

Let $j \in \{-1, -2\}$. Let us consider the commutative diagram

$$\begin{tikzcd} H^{\mathrm{BM}}_{-1}(Y, \mathbb{Z}(j)) \arrow[r, "\beta"] \arrow[d, "\alpha_Y"', "\cong"] & H^{\mathrm{BM}}_{-1}(X, \mathbb{Z}(j)) \arrow[d, "\alpha_X"'] \\ H^{\mathrm{BM}}_{-1}(\operatorname{Spec} \mathcal{O}(Y), \mathbb{Z}(j)) \arrow[r, "\gamma"] & H^{\mathrm{BM}}_{-1}(\operatorname{Spec} \mathcal{O}(X), \mathbb{Z}(j)) \end{tikzcd}$$

where all the homomorphisms are pushforwards. Since α_Y is an isomorphism and we know that γ is surjective using Lemma 2.5 and Lemma 2.7, the homomorphism α_X is surjective.
Since $H^{\mathrm{BM}}_{-1}(X \setminus Y, \mathbb{Z}(j))$ is zero, the localization sequence shows that the map β is surjective.
We use the following notation for short: for a scheme Z, we denote $\operatorname{Spec} \mathcal{O}(Z)_{\mathrm{red}}$ by $a(Z)$.

Lemma 3.5. *The diagram*

$$\begin{tikzcd} H^0_{\mathrm{et}}(a(Y'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[r] \arrow[d] & H^0_{\mathrm{et}}(a(X'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[d] \\ H^0_{\mathrm{et}}(a(Y), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[r] & H^0_{\mathrm{et}}(a(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)), \end{tikzcd}$$

where the arrows are the pushforward homomorphisms, is cocartesian for every prime number $\ell \neq p$.

Proof. Let $\overline{X} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$ and define $\overline{Y}$, $\overline{X}'$ and $\overline{Y}'$ in a similar manner. By [EGAIII-I, (1.4.16.1), p.94], we have $a(\overline{X}) = a(X) \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$ and similarly for Y, X', and Y'. Since $j \neq 0$, the weight argument shows that the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-coinvariants of any quotient $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module and of any divisible $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-submodule of $H^0_{\mathrm{et}}(a(\overline{Y}'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))$ vanish. Hence it suffices to show that the diagram

$$\begin{tikzcd} H^0_{\mathrm{et}}(a(\overline{Y}'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[r] \arrow[d] & H^0_{\mathrm{et}}(a(\overline{X}'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[d] \\ H^0_{\mathrm{et}}(a(\overline{Y}), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \arrow[r] & H^0_{\mathrm{et}}(a(\overline{X}), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \end{tikzcd}$$

is cocartesian in the category of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules and that the kernel of the homomorphism $H^0_{\mathrm{et}}(a(\overline{Y}'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \to H^0_{\mathrm{et}}(a(\overline{X}'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \oplus H^0_{\mathrm{et}}(a(\overline{Y}), \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))$ is divisible. By taking the Pontryagin dual, we prove that the diagram

$$\begin{tikzcd} H^0_{\mathrm{et}}(a(\overline{X}), \mathbb{Z}_\ell(j)) \arrow[r] \arrow[d] & H^0_{\mathrm{et}}(a(\overline{X}'), \mathbb{Z}_\ell(j)) \arrow[d] \\ H^0_{\mathrm{et}}(a(\overline{Y}), \mathbb{Z}_\ell(j)) \arrow[r] & H^0_{\mathrm{et}}(a(\overline{Y}'), \mathbb{Z}_\ell(j)), \end{tikzcd}$$

where the arrows are the pullback homomorphisms, is cartesian in the category of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules and that the cokernel of the homomorphism

$$H^0_{\mathrm{et}}(a(\overline{X}'),\mathbb{Z}_\ell(j))\oplus H^0_{\mathrm{et}}(a(\overline{Y}),\mathbb{Z}_\ell(j))\to H^0_{\mathrm{et}}(a(\overline{Y}'),\mathbb{Z}_\ell(j)) \tag{3.1}$$

is torsion free.

Let Z be a scheme which is of finite type over $\mathbb{F}_q$. Let us write $\overline{Z}=Z\times_{\mathrm{Spec}\,\mathbb{F}_q}\mathrm{Spec}\,\overline{\mathbb{F}}_q$. Then we have an isomorphism

$$H^0_{\mathrm{et}}(a(\overline{Z}),\mathbb{Z}_\ell(j))\cong \mathrm{Map}(\pi_0(\overline{Z}),\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathbb{Z}_\ell(j) \tag{3.2}$$

of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules, which is functorial with respect to pullbacks. It then follows from Lemma 3.3 that the homomorphism (3.1) has a torsion free cokernel. Hence it suffices to show that the diagram

$$\begin{tikzcd} \pi_0(\overline{X}) & \pi_0(\overline{X}') \arrow[l] \\ \pi_0(\overline{Y}) \arrow[u] & \pi_0(\overline{Y}') \arrow[l, "\psi"'] \arrow[u, "\varphi"'] \end{tikzcd} \tag{3.3}$$

is cocartesian in the category of sets.

As $X'\to X$ is a normalization, it is surjective. As surjectivity is preserved under base change, the map $\overline{Y}'\to\overline{Y}$ is surjective, hence ψ is surjective. This implies that the pushout of the diagram

$$\pi_0(\overline{X}')\xleftarrow{\varphi}\pi_0(\overline{Y}')\xrightarrow{\psi}\pi_0(\overline{Y})$$

is isomorphic to the quotient of $\pi_0(\overline{X}')$ by the following equivalence relation. We define a binary relation $\sim$ on $\pi_0(\overline{X}')$ as follows. We say $x_1'\sim x_2'$ if there exist $y_1',y_2'\in\pi_0(\overline{Y}')$ such that $x_1'=\varphi(y_1'),x_2'=\varphi(y_2')$, and $\psi(y_1')=\psi(y_2')$. We also write $\sim$ for the equivalence relation on $\pi_0(\overline{X}')$ generated by the binary relation above. Let us show that the map $\phi:\pi_0(\overline{X}')/\sim\to\pi_0(\overline{X})$ obtained from the diagram (3.3) is an isomorphism.

As the étale base change of a normalization, $\overline{X}'\to\overline{X}$ is a normalization. Hence $\pi_0(\overline{X}')$ coincides with the set of irreducible components of $\overline{X}$. As a normalization is a surjective morphism, the map ϕ is surjective.

Let C_1,C_2 be two distinct irreducible components of $\overline{X}$. We claim that if $C_1\cap C_2\neq\emptyset$ then the classes of C_1 and C_2 in $\pi_0(\overline{X}')/\sim$ coincide. Let $y\in C_1\cap C_2$. Then the local ring $\mathcal{O}_{\overline{X},y}$ is not an integral domain. Since we chose Y so that $X\setminus Y$ is normal, y belongs to $\overline{Y}$. One can take $y_1,y_2\in\overline{X}'$ lying over y such that y_i lies in the same connected component as C_i for each $i=1,2$. Note that $y_1,y_2\in\overline{Y}'$ since they both lie over $y\in\overline{Y}$. Then using the definition of the equivalence relation above for y_1 and y_2, we see that $C_1\sim C_2$.

Let C_1' and C_2' be two irreducible components of $\overline{X}$. It follows from the discussion above that if they belong to the same connected component, then $C_1'\sim C_2'$. This implies the injectivity of ϕ. This proves the claim. □

We return to the proof of Lemma 3.4. It follows from Lemma 2.5 and Lemma 3.5 that the diagram

$$\begin{array}{ccc} H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(Y'),\mathbb{Z}(j)) & \longrightarrow & H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X'),\mathbb{Z}(j)) \\ \downarrow & & \downarrow \\ H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(Y),\mathbb{Z}(j)) & \longrightarrow & H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X),\mathbb{Z}(j)) \end{array}$$

is cocartesian. We saw that γ is surjective. Taking a lift and composing with $\beta\circ(\alpha_Y)^{-1}$ we obtain a map $H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X),\mathbb{Z}(j)) \to H^{\mathrm{BM}}_{-1}(X,\mathbb{Z}(j))$. The fact that the diagram above is cocartesian and some diagram chasing imply that this map does not depend on the choice of a lift and this map is a homomorphism. We then see that the homomorphism β factors through the homomorphism

$$H^{\mathrm{BM}}_{-1}(Y,\mathbb{Z}(j)) \xrightarrow{\alpha_Y} H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(Y),\mathbb{Z}(j)) \\ \xrightarrow{\gamma} H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X),\mathbb{Z}(j)).$$

This proves that the order of $H^{\mathrm{BM}}_{-1}(X,\mathbb{Z}(j))$ divides the order of

$$H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X),\mathbb{Z}(j)).$$

Hence α_X is an isomorphism. This completes the proof. $\square$

Lemma 3.6. *Let U be a nonempty open subscheme of a separated connected scheme V over $\operatorname{Spec}\mathbb{F}_q$ such that $V\setminus U \neq \emptyset$. Then U is not proper over $\operatorname{Spec}\mathbb{F}_q$.*

Proof. As V is separated, the diagonal $\Delta \subset V\times_{\operatorname{Spec}\mathbb{F}_q} V$ is closed, hence the restriction $\Delta\cap(U\times_{\operatorname{Spec}\mathbb{F}_q} V)\subset U\times_{\operatorname{Spec}\mathbb{F}_q} V$ is closed. The image of this closed set under the second projection $U\times_{\operatorname{Spec}\mathbb{F}_q} V\to V$ is U, hence it is not closed in V since V is connected. This shows the structure map $U\to\operatorname{Spec}\mathbb{F}_q$ is not universally closed, hence it is not proper. $\square$

Proof of Theorem 1.1. First suppose $d=1$. The claim for X normal and non-proper follows from Lemmas 2.4 and 2.2. Then the claim for X proper follows from Lemmas 3.2 and 3.4.

Let us prove the claim for non-proper X. We use induction on the number of irreducible components n of X. Suppose $n=1$. We may without loss of generality assume X is reduced so that X is integral. Take an open immersion from X to a connected scheme X' of dimension one which is proper over $\operatorname{Spec}\mathbb{F}_q$ such that the complement $X'\setminus X$ is zero dimensional. Let $j\in\{-1,-2\}$. We have proved that the pushforward map $H^{\mathrm{BM}}_{-1}(X',\mathbb{Z}(j))\to H^{\mathrm{BM}}_{-1}(\operatorname{Spec}\mathcal{O}(X'),\mathbb{Z}(j))$ is an isomorphism. This implies, using Lemma 2.7, that the pushforward map $H^{\mathrm{BM}}_{-1}(X'\setminus X,\mathbb{Z}(j))\to H^{\mathrm{BM}}_{-1}(X',\mathbb{Z}(j))$ is surjective. Hence, by the localization sequence, we have $H^{\mathrm{BM}}_{-1}(X,\mathbb{Z}(j))=0$.

Suppose $n\geq 2$. We take a (non-empty) zero dimensional closed subscheme $Y\subset X$ such that $X\setminus Y = X_1\coprod\cdots\coprod X_r$ (disjoint union of schemes) with the following properties:

(1) X_i is a connected one dimensional open subscheme of X,

(2) the number of irreducible components of X_i is less than n,
(3) the closure $\overline{X}_i$ of X_i in X equals $X_i \cup Y$,

for each $1 \leq i \leq r$.
We can for example take as Y the following scheme. Let Y_0 be a zero dimensional subscheme of X such that the complement $X \setminus Y_0$ is not connected. We order the set of such Y_0's by inclusion, and let Y be a minimal one with respect to this ordering. Let us check the properties (1)(2)(3). Let $\{X_i\}_{1 \leq i \leq r}$ be the set of connected components of $X \setminus Y$ then (1) holds true. We have $r \geq 2$ by construction. Since the number of irreducible components of X equals the sum of the number of irreducible components of the X_i's, the property (2) holds true. The closure $\overline{X}_i$ of X_i in X is contained in $X_i \cup Y$ by construction. Suppose $\overline{X}_i \neq X_i \cup Y$ for some $1 \leq i \leq r$. Let $y \in (X_i \cup Y) \setminus \overline{X}_i$. Then the minimality condition on the construction of Y implies that $X \setminus (Y \setminus \{y\}) = (X_1 \coprod \cdots \coprod X_r) \cup \{y\}$ is connected. This implies in particular that $y \in \overline{X}_i$, which is a contradiction, so (3) holds true.
Taking $U = X_i$ and $V = \overline{X}_i$ in Lemma 3.6, we see that X_i is not proper. By the non-properness of X (and changing the indexing) we may suppose that $\overline{X}_1$ is not proper. The localization sequence gives the exact sequence

$$H_0^{\mathrm{BM}}(X_1, \mathbb{Z}(j)) \xrightarrow{\varphi} H_{-1}^{\mathrm{BM}}(Y, \mathbb{Z}(j)) \to H_{-1}^{\mathrm{BM}}(\overline{X}_1, \mathbb{Z}(j)).$$

By the inductive hypothesis, we have $H_{-1}^{\mathrm{BM}}(\overline{X}_1, \mathbb{Z}(j)) = 0$, hence φ is a surjection. Now use the following localization sequence

$$\begin{aligned}&\bigoplus_{i=1}^{r} H_0^{\mathrm{BM}}(X_i, \mathbb{Z}(j)) \xrightarrow{\psi} H_{-1}^{\mathrm{BM}}(Y, \mathbb{Z}(j)) \to H_{-1}^{\mathrm{BM}}(X, \mathbb{Z}(j)) \\ &\to \bigoplus_{i=1}^{r} H_{-1}^{\mathrm{BM}}(X_i, \mathbb{Z}(j)).\end{aligned}$$

Since φ is surjective, ψ is surjective. By the inductive hypothesis, $\bigoplus_{i=1}^{r} H_{-1}^{\mathrm{BM}}(X_i, \mathbb{Z}(j)) = 0$. It follows that $H_{-1}^{\mathrm{BM}}(X, \mathbb{Z}(j)) = 0$. The claim is proved in the case $d = 1$.
Next suppose that $d \geq 2$ and X is affine. Let $j \in \{-1, -2\}$. The localization sequence gives an exact sequence

$$\begin{aligned}&\varinjlim_Y H_{-1}^{\mathrm{BM}}(Y, \mathbb{Z}(j)) \to H_{-1}^{\mathrm{BM}}(X, \mathbb{Z}(j)) \\ &\to \varinjlim_Y H_{-1}^{\mathrm{BM}}(X \setminus Y, \mathbb{Z}(j)),\end{aligned}$$

where Y runs over the reduced closed subschemes of X of pure codimension one. For dimension reasons, we have $\varinjlim_Y H_{-1}^{\mathrm{BM}}(X \setminus Y, \mathbb{Z}(j)) = 0$. Hence by induction on d, we have $H_{-1}^{\mathrm{BM}}(X, \mathbb{Z}(j)) = 0$.
Next suppose that $d \geq 2$ and X is not proper. Using a similar argument as above, we are reduced, by induction on the number of irreducible components of X, to the case where X is integral. Take an open immersion from X to a connected scheme X' of dimension d, which is proper over $\operatorname{Spec} \mathbb{F}_q$, such that X is dense in X'. Take a non-empty affine open subscheme $U \subset X$ and set $Y = X' \setminus U$. Let us take an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. By [Hart, Chapter

II, Proposition 3.1, p. 66] and [Hart, Chapter II, Proposition, p. 67] (originally due to [Go]), for each irreducible component X'' of $X' \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$, we know that $X'' \setminus U \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$ is connected and is of pure codimension one in X'. This shows that Y is of pure codimension one in X'.

Let us show that Y is connected. Write $f\colon X' \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q \to X'$ for the canonical projection. We note that the map f is surjective, and, as the canonical morphism $\operatorname{Spec}\overline{\mathbb{F}}_q \to \operatorname{Spec}\mathbb{F}_q$ is universally closed by [EGAII, Proposition (6.1.10)], the map f is a closed map. Let $\xi \in X$ denote the generic point of X. As X is dense in X', the closure of ξ in X' equals X'. Take $\xi' \in f^{-1}(\xi)$ and let X'' be an irreducible component of $X' \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$ that contains ξ'. Using that an irreducible component is closed, we see that X'' contains the closure in $X' \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$ of ξ'. Then as f is a closed map, the morphism $f|_{X''} : X'' \to X'$ is surjective. Using the fact above by Goodman, we have that $X'' \setminus U \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$ is connected. Then as $X'' \setminus (U \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q)$ surjects onto $X' \setminus U = Y$, we have that Y is connected as the continuous image of a connected space.

Write $X \cap Y = Z_1 \coprod \cdots \coprod Z_r$ so that each Z_i is connected. We claim that each Z_i is not proper. As $X \subset X'$ is an open subset, $X \cap Y \subset Y$ is an open subset of Y, hence each $Z_i \subset Y$ is an open subset of Y. As Y is connected, $\overline{Z}_i^Y \neq Z_i$ where $\overline{Z}_i^Y$ denotes the closure of Z_i in Y. This implies that Z_i is the complement of a non-empty closed set, namely $\overline{Z}_i^Y \setminus Z_i$, of a connected proper scheme $\overline{Z}_i^Y$. It follows from Lemma 3.6 that Z_i is not proper.

Let $j \in \{-1, -2\}$. Since U is affine, from the localization sequence

$$H^{\mathrm{BM}}_{-1}(Y \cap X, \mathbb{Z}(j)) \to H^{\mathrm{BM}}_{-1}(X, \mathbb{Z}(j)) \to H^{\mathrm{BM}}_{-1}(U, \mathbb{Z}(j))$$

it follows by induction on d that $H^{\mathrm{BM}}_{-1}(X, \mathbb{Z}(j))$ is zero (to remove the hypothesis that the schemes in the localization sequence are quasi-projective, we refer to [Le2, Theorem 1.7, p. 301] and [Ge-Le2, 2.6, p. 60]). This proves the claim for X not proper.

The claim for X proper follows from Lemmas 3.2 and 3.4. This completes the proof. □

4. Under Parshin's conjecture

We assume Parshin's conjecture in this section and draw some consequences. Parshin's conjecture states that for any projective smooth scheme Z over a finite field, $H^a_{\mathcal{M}}(Z, \mathbb{Q}(b)) = 0$ unless $a = 2b$. We note that it is a theorem of Harder for curves (we refer to [Gr, THEOREM 0.5, p.70] for the correct implication of Harder's result).

Proposition 4.1. *Assume that Parshin's conjecture holds. Then the statement in Theorem 1.1 holds true for any $j \leq -1$. We also have $H^{\mathrm{BM}}_i(X, \mathbb{Z}(j)) = 0$ for $i \leq -2$ and $j \leq -1$.*

We begin with a lemma.

Lemma 4.2. (1) *Let U be a connected scheme of pure dimension $d \geq 1$ over $\mathbb{F}_q$. Then $\varinjlim_V H_i^{\mathrm{BM}}(V, \mathbb{Z}(j)) = 0$, where V runs over the (non-empty) open subschemes of U, for $i \leq -1$ and $j \leq -1$ assuming Parshin's conjecture.*

(2) *Let V be a zero dimensional scheme over $\mathbb{F}_q$. Then $H_i^{\mathrm{BM}}(V, \mathbb{Z}(j)) = 0$ for $i \leq -2$ and $j \leq -1$ assuming Parshin's conjecture.*

Proof. Let K denote the function field of U. If $d > i-j$, then $H_i^{\mathrm{BM}}(V, \mathbb{Z}(j)) = \mathrm{CH}_j(V, i-2j) = \mathrm{CH}^{d-j}(V, i-2j)$. So the limit is $\mathrm{CH}^{d-j}(\operatorname{Spec} K, i-2j)$, which equals zero by dimension reason.
Suppose $d \leq i-j$. Let V be an open smooth subscheme of U. We proceed as in the proof of Lemma 2.2. We have

$$H_i^{\mathrm{BM}}(V, \mathbb{Z}(j)) = H_{\mathcal{M}}^{2d-i}(V, \mathbb{Z}(d-j)) = H_{\mathcal{M}}^{2d-i-1}(V, \mathbb{Q}/\mathbb{Z}(d-j)),$$

where the second equality follows from [Ge, Theorem 4.7 ii), p.312] (this uses Parshin's conjecture). Since we are in the range $d \leq i-j$, we can use [Ge-Le2, Corollary 1.2, p.56] and [Ge-Le1, Theorem 8.4, p.491] to see that the quantity above is isomorphic to $\bigoplus_{\ell \neq p} H_{\mathrm{et}}^{2d-i-1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-j)) \oplus \varinjlim_r H^{d+j-i-1}(X_{\mathrm{Zar}}, \nu_r^{d-j})$. The p-part is zero since we are in the range $d \leq i-j$.
We may assume that V is affine. Then $H_{\mathrm{et}}^{2d-i-1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-j)) = 0$ for $d-3 \geq i$ since the cohomological dimension of V is $d+1$ ([SGA4-3, Exposé XIV, Théorème 3.1, p.15]). Suppose $d \geq 2$. Then the claim follows from this immediately since $i \leq -1$. Suppose $d = 1$. The vanishing $H_{\mathrm{et}}^2(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1-j)) = 0$ can be shown using the same method as in the proof of Lemma 2.2. This proves (1). Let V be as in (2). The remaining case is $i = -2$. We proceed as in the proof of Lemma 2.2. We have an exact sequence

$$H_{\mathrm{et}}^1(V, \mathbb{Q}_\ell(-j)) \to H_{\mathrm{et}}^1(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \to H_{\mathrm{et}}^2(V, \mathbb{Z}_\ell(-j)).$$

The third term is zero since V is zero dimensional. We use the Hochschild-Serre spectral sequence as before. We have $E_2^{0,1} = E_2^{1,0} = 0$ using the weight argument. The claim then follows. This completes the proof. □

Proof of Proposition 4.1. Suppose $i = -1$. For the proof of Theorem 1.1 to work for general i, we need as an input the vanishing of $\varinjlim_Y H_{-1}^{\mathrm{BM}}(X \setminus Y, \mathbb{Z}(j))$ (the notation as in the proof of Theorem 1.1), and this is all that we need. As we have seen in Lemma 4.2 above, the vanishing holds under Parshin's conjecture, hence the claim follows if $i = -1$.
Suppose $i \leq -2$. We show by induction on the dimension d that $H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) = 0$. The case $d = 0$ is Lemma 4.2(2). Consider the exact sequence

$$\varinjlim_Y H_i^{\mathrm{BM}}(Y, \mathbb{Z}(j)) \to H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) \to \varinjlim_Y H_i^{\mathrm{BM}}(X \setminus Y, \mathbb{Z}(j))$$

where Y runs over closed subschemes of X. By the inductive hypothesis, the first term is zero, and the third term is zero by Lemma 4.2(1). □

For fixed (i,j), we only need to assume Parshin's conjecture for projective smooth schemes of dimension (less than or equal to) $i-j$. Theorem 1.1 treats the cases $(i,j)=(-1,-1)$ and $(-1,-2)$. Hence we can use Harder's result and need not assume Parshin's conjecture.

Remark 4.3. For this remark, we do not use Parshin's conjecture. Let us define and compute the étale Borel-Moore (not motivic) homology groups for a scheme X separated and of finite type over $\mathbb{F}_q$ in the same range as that of Proposition 4.1. We will see that the étale Borel-Moore homology groups and the Borel-Moore motivic homology groups are isomorphic in this range. Let ℓ be a prime number prime to p. We define the étale Borel-Moore homology group to be

$$H_i^{\mathrm{BM,et}}(X,\mathbb{Z}/\ell^n(j)) = H_{\mathrm{et}}^{-i}(X, Rf^!(\mathbb{Z}/\ell^n(-j)))$$

for $i,j\in\mathbb{Z}$ and $n\geq 1$. Since this is isomorphic to

$$\mathrm{Hom}_{\mathbb{Z}/\ell^n}(H_{\mathrm{et},c}^{i+1}(X,\mathbb{Z}/\ell^n(-j)),\mathbb{Z}/\ell^n),$$

we set

$$H_i^{\mathrm{BM,et}}(X,\mathbb{Z}_\ell(j)) = \mathrm{Hom}_{\mathbb{Q}_\ell/\mathbb{Z}_\ell}(H_{\mathrm{et},c}^{i+1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)),\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Then it is easy to see that a statement similar to the one in Proposition 4.1 holds for étale Borel-Moore (not motivic) homology groups with $\mathbb{Z}_\ell$-coefficient. Namely, we have $H_i^{\mathrm{BM,et}}(X,\mathbb{Z}_\ell(j))=0$ for $i\leq -2$, and, for X connected and for $i=-1$, the pushforward by the structure morphism is an isomorphism if X is proper, and $H_{-1}^{\mathrm{BM,et}}(X,\mathbb{Z}_\ell(j))=0$ otherwise.

References

[Ak] Akhtar, R., Zero cycles on varieties over finite fields. Commun. Algebra. 32, no. 1, (2004) 279–294.

[Bl] Bloch, S., Algebraic cycles and higher K-theory. Adv. Math. 61, no. 3, (1986) 267–304.

[Ge] Geisser, T., Applications of de Jong's theorem on alterations. Hauser, H. (ed.) et al., Resolution of singularities. A research textbook in tribute to Oscar Zariski. Based on the courses given at the working week in Obergurgl, Austria, September 7-14, 1997. Basel: Birkhäuser. Prog. Math. 181, 299-314 (2000)

[Ge-Le1] Geisser, T., Levine, M., The K-theory of fields in characteristic p. Invent. Math. 139, no. 3, (2000) 459–493.

[Ge-Le2] Geisser, T., Levine, M., The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. J. Reine Angew. Math. 530, (2001) 55–103.

[Go] Goodman, J. E., Affine open subsets of algebraic varieties and ample divisors. Ann. Math. (2) 89, (1969) 160–183.

[Gr] Grayson, D. R., Finite generation of K-groups of a curve over a finite field (After Daniel Quillen). *Algebraic K-theory*, Proc. Conf. Oberwolfach 1980, Part I, Lect. Notes Math. 966, (Springer-Verlag, Berlin-Heidelberg-New York 1982) 69–90.

[Hard] Harder, G., Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern. (German) Invent. Math. 42, 135-175 (1977)

[Hart] Hartshorne, R, *Ample subvarieties of algebraic varieties* Lect. Notes Math. 156, Springer-Verlag, Berlin-Heidelberg-New York (1970)

[La] Lang, S., Sur les séries L d'une variétè algébrique. Bull. Soc. Math. Fr. 84, 385-407 (1956)

[Le1] Levine, M., *Mixed Motives* Math. Surv. Monogr. 57, (American Mathematical Society, Providence, RI, (1998)

[Le2] Levine, M., Techniques of localization in the theory of algebraic cycles. J. Algebr. Geom. 10, no. 2, (2001) 299–363.

[Me-Su] Merkurjev, A. S., Suslin, A. A., K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. Math. USSR, Izv. 21, (1983) 307–340; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 46, no. 5, (1982) 1011–1046

[Qu] Quillen, D., On the cohomology and K-theory of the general linear groups over a finite field. Ann. Math. (2) 96, 552–586 (1972)

[Ra] Raskind, W., Abelian class field theory of arithmetic schemes. Jacob, Bill (ed.) et al., K-theory and algebraic geometry: connections with quadratic forms and division algebras. Summer Research Institute on quadratic forms and division algebras, July 6-24, 1992, University of California, Santa Barbara, USA. Providence, RI: American Mathematical Society. Proc. Symp. Pure Math. 58, Part 1, 85-187 (1995)

[So] Soulé, C., K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. Invent. Math. 55, (1979) 251–295.

[Vo] Voevodsky, V., Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. Int. Math. Res. Not. no. 7, (2002) 351–355.

[Vo-Su-Fr] Voevodsky, V., Suslin, A. A., Friedlander, E. *Cycles, transfers, and motivic homology theories* Ann. Math. Stud. 143, (Princeton University Press, Princeton, NJ, 2000)

[EGAII] Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphisms. Publ. Math., Inst. Hautes Étud. Sci. 8, (1961) 1–222.

[EGAIII-I] Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Publ. Math. Inst. Hautes Études Sci. No. 11 (1961) 167 pp.

[SGA4-3] Séminaire de gèomètrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de P. Deligne, B. Saint-Donat. Tome 3. Exposés IX à XIX.) Lecture Notes in Mathematics. 305. Berlin-Heidelberg-New York: Springer-Verlag. vi, 640 pp. (1973)

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