Hessian Ideals of a Homogeneous Polynomial and Generalized Tjurina Algebras

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Abstract. Using the minors in Hessian matrices, we introduce new graded algebras associated to a homogeneous polynomial. When the associated projective hypersurface has isolated singularities, these algebras are related to some new local algebras associated to isolated hypersurface singularities, which generalize their Tjurina algebras. One consequence of our results is a new very rapid way to determine the number of weighted homogeneous singularities of such a hypersurface.

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1. Introduction and statement of results

Let $S = \oplus_k S_k = \mathbb{C}[x_0, ..., x_n]$ be the graded polynomial ring in $n+1$ indeterminates with complex coefficients, where S_k denotes the vector space of degree k homogeneous polynomials. Consider for a polynomial $f \in S_d$, the corresponding Jacobian ideal J_f generated by the partial derivatives f_j of f with respect to x_j for $j = 0, ..., n$, the graded Milnor algebra $M(f) = \oplus_k M(f)_k = S/J_f$ and its Hilbert series

$$HP(M(f);t) = \sum_k \dim M(f)_k t^k. \tag{1.1}$$

Assume in this note that the projective hypersurface $V = V(f) : f = 0$ in $\mathbb{P}^n$ is reduced with (at most) isolated singularities at the points $a_1, ..., a_p$. Let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of

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f and $h(f)$ be the Hessian of f, i.e. the determinant of this matrix $Hess(f)$. This Hessian $h(f)$ is a homogeneous polynomial in S of degree

$$T = (n+1)(d-2).$$

More generally, for each k satisfying $1 \leq k \leq n+1$ we denote by $h_k(f)$ the ideal in S generated by all $k \times k$-minors in the matrix $Hess(f)$. In particular, the ideal $h_{n+1}(f) = (h(f))$ is a principal ideal. For each k as above, consider the *graded k-th Hessian algebra of the polynomial f* defined by

$$H_k(f) = S/(J_f + h_k(f)), \tag{1.2}$$

whose isomorphism class is clearly a $GL(n+1, \mathbb{C})$-invariant of f. Clearly we have a sequence of epimorphisms of graded $\mathbb{C}$-algebras

$$M(f) \to H_{n+1}(f) \to H_n(f) \to \cdots \to H_1(f).$$

On the other hand, we introduce for any isolated hypersurface singularity $(V, 0)$ at the origin of $\mathbb{C}^n$, given by an analytic germ $g = 0$, the k-th Hessian ideal $h_k(g)$ to be the ideal generated by all the k-th minors in the Hessian $(n \times n)$-matrix $Hess(g) = (g_{ij})$. By convention, we set $h_{n+1}(g) = 0$. It is easy to check that the isomorphism class of the *local k-th Hessian algebra of the germ g* defined by

$$H_k(g) = \mathcal{O}_n/((g) + J_g + h_k(g)), \tag{1.3}$$

is a $\mathcal{K}$-invariant of g, i.e. depends only on the isomorphism class of the germ $(V, 0)$, see Lemma 2.1. Clearly we have now a sequence of epimorphisms of local Artinian $\mathbb{C}$-algebras

$$T(g) = H_{n+1}(g) \to H_n(g) \to \cdots \to H_1(g),$$

starting with the Tjurina algebra $T(g)$ of g. Hence one can consider the Hessian algebras $H_k(g)$ as a generalization of the Tjurina algebra $T(g)$. For $k = 1, ..., n+1$ we introduce the k-th *Hessian number* of g to be

$$\chi_k(g) = \dim H_k(g). \tag{1.4}$$

In view of the above remark, we can also write $\chi_k(g) = \chi_k(V, 0)$ and use any normal form of g to compute these new invariants. Note that $\chi_{n+1}(V, 0) = \tau(V, 0)$, the usual Tjurina number of $(V, 0)$ and one has the inequalities

$$0 \leq \chi_1(V, 0) \leq ... \leq \chi_n(V, 0) \leq \chi_{n+1}(V, 0) = \tau(V, 0).$$

Our first main result is the following.

Theorem 1.1. *Assume that $V(f)$ is a hypersurface in $\mathbb{P}^n$ with isolated singularities at the points $a_1, ..., a_p$. Then for $k = 1, 2, ..., n+1$ and any m large enough one has*

$$\dim H_k(f)_m = \sum_{i=1,p} \chi_k(V(f), a_i).$$

The following result estimates how large m should be in order to have such stabilization results. First we recall some definitions, see [12].

Definition 1.2. For a hypersurface $V : f = 0$ with isolated singularities we introduce three integers, as follows.
(i) the *coincidence threshold*

$$ct(V) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree d such that $V_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$.
(ii) the *stability threshold* $st(V) = \min\{q : \dim M(f)_k = \tau(V) \text{ for all } k \geq q\}$, where $\tau(V)$ is the total Tjurina number of V, that is $\tau(V) = \sum_{i=1,p} \tau(V(f), a_i)$.
(iii) the *minimal degree of a syzygy* $mdr(V) = \min\{q : H^n(K^*(f))_{q+n} \neq 0\}$, where $K^*(f)$ is the Koszul complex of $f_0, ..., f_n$ with the natural grading.

It is known that one has

$$ct(V) = mdr(V) + d - 2, \tag{1.5}$$

and precise estimates on $mdr(V)$ are given in [8], [9], [10], [13]. Our second main result is the following.

Theorem 1.3. *Assume that the hypersurface $V(f)$ in $\mathbb{P}^n$ has only isolated singularities. Then one has the following stabilization property:*

$$\dim H_n(f)_m = \tau(V(f)) - |Sing_{wh}(V(f))|$$

for any m large, where $|Sing_{wh}(V(f))|$ is the number of weighted homogeneous singularities the hypersurface $V(f)$ has. Moreover, for any integer k, $1 \leq k \leq n$, the dimension of $H_k(f)_m$ is constant for any

$$m \geq T_k := \max(T - ct(V) + k(d-2), st(V)).$$

We also have the following result, expressing $HP(H_{n+1}(f); t)$ in term of the series $HP(M(f); t)$, which was already studied in [9], [10], [12].

Proposition 1.4. *(i) Assume the hypersurface $V(f)$ is smooth. Then*

$$HP(H_{n+1}(f); t) = HP(M(f); t) - 1 = \frac{(1 - t^{d-1})^{n+1}}{(1-t)^{n+1}} - 1.$$

(ii) Assume the hypersurface $V(f)$ has isolated singularities (and it is not smooth). Then

$$HP(H_{n+1}(f); t) = HP(M(f); t).$$

A consequence of these results is the following (possibly the fastest) way to compute the total number of singularities of a projective hypersurface in cases when it is known a priori that these singularities are all weighted homogeneous.

Corollary 1.5. *Assume that the hypersurface $V(f)$ in $\mathbb{P}^n$ has only isolated singularities (and it is not smooth). Then the total number of weighted homogeneous singularities of $V(f)$ is given by*

$$|Sing_{wh}(V(f))| = \dim M(f)_m - \dim H_n(f)_m$$

for any $m \geq \tilde{T}_n = (2n+1)(d-2)$.

In the second section we prove these results, give some Corollaries and a number of Examples. In the final section we offer a discussion on the constructible partitions induced on the space of homogeneous polynomials of a fixed degree d by the values of the Hilbert-Poincaré series of the Milnor algebra $M(f)$ and of the Hessian algebras $H_k(f)$. The only results here are of an *experimental computational nature*. The computations of various invariants given in this paper were made using two computer algebra systems, namely CoCoA [3] and Singular [5]. The corresponding codes are available on request.

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2. Hessian matrix and Hessian ideals

Lemma 2.1. *The isomorphism class of the local k-th Hessian algebra*

$$H_k(g) = \mathcal{O}_n/((g) + J_g + h_k(g)),$$

of the isolated hypersurface germ $(V,0) : g = 0$ is a $\mathcal{K}$-invariant of g for any integer $k = 1, 2, ..., n$.

Proof. We have to show that if $u \in \mathcal{O}_n$ is a unit and $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ an isomorphism germ, and if we set $g'(y) = u(y) \cdot g(\phi(y))$, then the algebra $H_k(g')$ is isomorphic to the algebra $H_k(g)$ for any k. Note first that the quotients $\mathcal{O}_V = \mathcal{O}_n/(g)$ and $\mathcal{O}_{V'} = \mathcal{O}_n/(g')$ are isomorphic $\mathbb{C}$-algebras, since the isomorphism $\phi^* : \mathcal{O}_n \to \mathcal{O}_n$ given by $a(y) \mapsto a(\phi(y))$ sends the ideal (g) injectively onto the ideal (g'). Computing the partial derivatives of g' with respect to $y_1, ..., y_n$, we see that the isomorphism ϕ^* sends also the ideal $(g)+J_g$ into the ideal $(g')+J_{g'}$. The inclusion $\phi^*((g) + J_g) \subset (g') + J_{g'}$ implies $\tau(g) \geq \tau(g')$. Since the relation between g and g' is symmetrical, we get in the same way $\tau(g') \geq \tau(g)$, and hence in fact we have equalities everywhere. Now compute the second order partial derivatives of g' and note that, modulo the ideal $(g') + J_{g'}$, the Hessian matrix $Hess(g')$ is a product of 3 matrices $A(y)^t Hess(g)(\phi(y))A(y)$, where $A(y) = d\phi(y)$ is the Jacobian matrix of ϕ and $A(y)^t$ denotes its transpose. From this relation it follows that

$$\phi^*((g) + J_g + h_k(g)) \subset (g') + J_{g'} + m_k(g')$$

for any k. The equality is established exactly as above, using the symmetry between g and g'.

□

2.2. Proof of Theorem 1.1. For any k, it is clear that the ideal $J_f + h_k(f)$ defines a 0-dimensional subscheme in $\mathbb{P}^n$ whose support is contained in the singular locus $V(f)_{sing} = \{a_1, ..., a_p\}$. Hence one has for large m the following equality

$$\dim H_k(f)_m = \sum_{i=1,p} \dim H_k(f)_{a_i}, \tag{2.1}$$

where $H_k(f)_{a_i}$ denotes the analytic localization of the graded S-module $H_k(f)$ at the point a_i, exactly as in the proof of Corollary 9 in [2].

To study such a localization, assume that the coordinates have been chosen such that the singular point $a = a_1$ is located at $(1 : 0 : ... : 0)$. Then the isolated hypersurface singularity $(V(f), a)$ has a local equation

$$g(y_1, ..., y_n) = f(1, y_1, ..., y_n) = 0.$$

It follows that $g_j = f_j(1, y_1, ..., y_n)$ for any $j = 1, ..., n$, where g_j denotes the partial derivative of g with respect to y_j. Using the Euler relation for f exactly as in the proof of Corollary 9 in [2], we also get

$$d \cdot g = f_0(1, y_1, ..., y_n) + y_1 f_1(1, y_1, ..., y_n) + ... + y_n f_n(1, y_1, ..., y_n). \tag{2.2}$$

This implies that the localization of the Jacobian ideal J_f at the point a coincides with the ideal spanned by $g, g_1, ..., g_n$, i.e. the Tjurina ideal of the isolated hypersurface singularity $(V(f), a)$. Let's look now at the second order derivatives. One clearly has $g_{ij} = f_{ij}(1, y_1, ..., y_n)$ for all $1 \leq i, j \leq n$. If we take the partial derivative with respect to y_j for some $j > 0$ of the polynomials entering into the equality (2.2), we get

$$d \cdot g_j = f_{0j}(1, y_1, ..., y_n) + y_1 f_{1j}(1, y_1, ..., y_n) + ...+ \tag{2.3}$$

$$+(f_j(1, y_1, ..., y_n) + y_j f_{jj}(1, y_1, ..., y_n)) + ... + y_n f_{nj}(1, y_1, ..., y_n).$$

Similarly, starting with the Euler relation for f_0, taking the partial derivative with respect to x_0 and then setting $x_0 = 1$ and $x_j = y_j$ for $j > 0$ we get

$$(d-1) \cdot f_0(1, y_1, ..., y_n) = \tag{2.4}$$

$$= (f_0(1, y_1, ..., y_n) + f_{00}(1, y_1, ..., y_n)) + \sum_{j>0} y_j f_{0j}(1, y_1, ..., y_n).$$

The formulas (2.3) and (2.4) give a proof of the following result.

Proposition 2.3. *The localization of the ideal $J_f + h_k(f)$ at the singular point $a = (1 : 0 : ... : 0)$ coincides with the ideal in the corresponding local ring $\mathcal{O}_n$ spanned by the local equation $g(y) = f(1, y_1, ..., y_n)$, its partial derivatives g_j with respect to y_j for $j = 1, ..., n$ and all $k \times k$ minors in the local $n \times n$ Hessian matrix $Hess(g) = (g_{ij})$.*

Proof. Indeed, modulo the Tjurina ideal, the matrix $Hess(f)(1, y_1, ..., y_n)$ can be transformed using the above formulas by obvious line and column operations into a matrix obtained from the local Hessian matrix $Hess(g)$ by adding a first line and a first column formed by zeros only.

□

Combining this definition with Proposition 2.3 and the equality (2.1) clearly completes the proof of Theorem 1.1.

Example 2.4. Assume that $(V, 0)$ is an A_1-singularity, hence a node. Then we can choose an equation of the form $g = y_1^2 + ... + y_n^2$ and it is clear that $\chi_k(V, 0) = 0$ for $k = 1, 2, ..., , n$ and $\chi_{n+1}(V, 0) = 1$. In particular, if $V(f)$ is a nodal hypersurface then $H_k(f)_m = 0$ for $1 \leq k \leq n$ and m large. Hence the corresponding Hilbert-Poincaré series

$$HP(H_k(f);t) = \sum_m \dim H_k(f)_m t^m$$

is a polynomial for $1 \leq k \leq n$.

Example 2.5. Assume that $n = 2$ and $(V, 0)$ is an A_q-singularity. Then we can choose an equation of the form $g = y_1^2 + y_2^{q+1}$ and it is clear that $\chi_1(V, 0) = 0$, $\chi_2(V, 0) = q - 1$ and $\chi_3(V, 0) = q$. In particular, if $V(f)$ is a cuspidal curve then $\dim H_k(f)_m = (k-1)\kappa$ for m large, where κ is the number of cusps A_2. To have a numerical example, let $f = x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z)$. Then $f = 0$ has $\kappa = 3$ cusps and no other singularities, see also [6], p.129 for more on this quartic curve. Numerical computations show that

$$HP(H_1(f);t) = 1 + 3t = 1 + 3t + 0(t^2 + t^3 + ...)$$
$$HP(H_2(f);t) = 1 + 3t + 6t^2 + 7t^3 + 3(t^4 + t^5 + ...)$$
$$HP(H_3(f);t) = 1 + 3t + 6t^2 + 7t^3 + 6(t^4 + t^5 + ...).$$

In particular

$$\dim H_k(f)_m = 3(k-1)$$

for $k = 1, 2, 3$ and $m \geq 4$.

Example 2.6. Assume that $n = 2$ and $(V, 0)$ is an D_q-singularity, $q \geq 4$. Then we can choose an equation of the form $g = y_1^2y_2 + y_2^{q-1}$ and it is clear that $\chi_1(V, 0) = 1$, $\chi_2(V, 0) = q - 1$ and $\chi_3(V, 0) = q$. Moreover, it clear that for a plane curve singularity $(V, 0)$, one has $\chi_1(V, 0) = 1$, if and only if $(V, 0)$ has type D_q for some $q \geq 4$.

Using Theorem 1.1 and Example 2.5 we get the following.

Example 2.7. The curve $V(f)$ in $\mathbb{P}^2$ has only A_q singularities for various q's (i.e. $V(f)$ has only corank 1 singularities) if and only if $H_1(f)_m = 0$ for m large. If this holds, then the total number of singularities of $V(f)$ is given by

$$|Sing(V(f))| = \dim H_3(f)_m - \dim H_2(f)_m$$

for m large.

2.8. Proof of Theorem 1.3. Consider the local case of an isolated weighted homogeneous singularity $(V, 0)$ given by $g = 0$. Then $g \in J_g$ and in the local Gorenstein ring $M(g) = \mathcal{O}_n/J_g$, the Hessian $h(g)$ of g spans the socle which is one dimensional, see [15]. Clearly we have $H_n(g) = M(g)/(h(g))$, and hence $\chi_n(g) = \tau(g) - 1 = \chi_{n+1}(g) - 1$. Next, if g is not weighted homogeneous, then the ideal (g) in $M(g)$ is nonzero, and hence contains the minimal ideal $(h(g))$. It follows that in such a case $\chi_n(g) = \tau(g)$. The result then follows for large m using Theorem 1.1 .

Let I be the saturation of the Jacobian ideal J_f. Then Prop. 2. in [7] tells us that

$$\dim S_q/I_q = \tau(V(f)) \tag{2.5}$$

for any $q \geq T - ct(D)$. In other words, for such q's, the evaluation map

$$ev_q : S_q/I_q \to \oplus \mathcal{O}_n/((g) + J_g) \tag{2.6}$$

is a bijection, where the sum is taken over all singularities $(V, a) : g = 0$ of the hypersurface $V = V(f)$. In particular, for any fixed singularity $a \in V_{sing}$ there is a polynomial $h_a \in S_q$ such that the class of h_a in the summand $T(g) = \mathcal{O}_n/((g) + J_g)$ corresponding to a is 1 and its classes in all the other summands are 0. The k-minors in the matrix $Hess(f)$ have degree $k(d-2)$ and the computation done in Proposition 2.3 shows that for any singular point $a \in V_{sing}$ there are linear combinations m_s of such minors (arising from a change of coordinates on $\mathbb{P}^n$) such that the classes of m_s in the corresponding local ring $T(g)$ generate the ideal $h_k(g)$. Corollary 2 in [7] tells us that

$$I_q = J_{f,q} \tag{2.7}$$

for all $q \geq q_1 = \max(T - ct(V(f)), st(V(f)))$.
Consider the bijection ev_q for $q \geq T_k := \max(T - ct(V) + k(d-2), st(V)) \geq q_1$. Since all the products $h_a m_s \in h_k(f)_q$, it follows that

$$ev_q((J_{f,q} + h_k(f)_q)/J_{f,q}) = \oplus((g) + J_g + h_k(g)/((g) + J_g).$$

This gives the following bijection for $q \geq T_k$

$$ev_q : H_k(f)_q \to \oplus \mathcal{O}_n/((g) + J_g + h_k(g))$$

which completes the proof of Theorem 1.3.

Remark 2.9. Note that $T_k \leq \tilde{T}_k = T + k(d-2) = (n+k+1)(d-2)$ for any k. So one can use this larger bound when there is no information on $ct(V)$ and $st(V)$, i.e. we always have stabilization for $\dim H_k(f)_m$ when $m \geq \tilde{T}_k$. When we have information on $ct(V)$ but not on $st(V)$, as it is often the case, one can use the better general bound

$$\widehat{T}_k = T + \max(k(d-1) - ct(V), 0).$$

Corollary 2.10. *(i) The hypersurface $V(f)$ in $\mathbb{P}^n$ is nodal if and only if $H_n(f)_m = 0$ for $m \geq \widehat{T}_n = 3n(d-2)/2$. In particular, for $n = 2$, we have $H_2(f)_m = 0$ for $m \geq T = \widehat{T}_2 = 3d - 6$.*
(ii) Assume the hypersurface $V(f)$ in $\mathbb{P}^n$ is nodal and $n \geq 4$. Then $H_{n-1}(f)_m = 0$ for $m \geq \widehat{T}_{n-1} = (3n-2)(d-2)/2$.

Proof. For a nodal hypersurface, it is known that $ct(V(f)) \geq (n+2)(d-2)/2$, see [9]. It follows that

$$T - ct(V(f)) + n(d-2) \leq T + (n-2)(d-2)/2 = 3n(d-2)/2 \geq T,$$

which proves the result (i). The second claim (ii) follows along the same lines. The condition $n \geq 4$ is necessary to insure that $\widehat{T}_{n-1} \geq T$.

□

Example 2.11. For the nodal curves $f = xyz^{d-2} + x^d + y^d = 0$, a direct computation shows that the stabilization $H_2(f)_m = 0$ occurs for $m \geq 3d - 7$ for low values of d, so only one unit better than our prediction. For the Chebyshev curves considered in [11] or in [22], a direct computation shows that the stabilization $H_2(f)_m = 0$ occurs for $m \geq 3d - 8$ for low values of d, so just 2 units better than our general prediction in Corollary 2.10.

Example 2.12. Let $V : f = 0$ be the Chebyshev 4-fold of degree d in $\mathbb{P}^5$ considered in [10] or in [22]. A direct computation shows that the stabilization $H_4(f)_k = 0$ occurs for $k \geq 13m - 10$ for low values of $d = 2m + 1$, and for $k \geq 13m - 17$ for low values of $d = 2m$. The corresponding (integral) values given by T_4' in these cases are $13m - 7$ for low values of $d = 2m+1$ and $13m - 13$ for low values of $d = 2m$, so the difference between the theoretical bound and the real value of k is less than 4 for any degree d. Similarly, the stabilization $H_5(f)_k = 0$ occurs for $k \geq 15m - 10$ for low values of $d = 2m + 1$, and for $k \geq 15m - 17$ for low values of $d = 2m$. The corresponding (integral) values given by T_5' in these cases are $15m - 8$ for low values of $d = 2m+1$ and $15m - 15$ for low values of $d = 2m$, so the difference between the theoretical bound and the real value of k is less than 2 for any degree d.

Corollary 2.13. *Assume the curve $V(f)$ in $\mathbb{P}^2$ has ν nodes and κ cusps as singularities. Then $\dim H_2(f)_m = \kappa$ and $\dim M(f)_m = \dim H_3(f)_m = \nu + 2\kappa$ for $m \geq T + d/6$. In other words, the asymptotic behavior of $\dim H_k(f)_m$ for $k = 2, 3$ determines the number of nodes and cusps.*

Proof. For a plane curve with nodes and cusps, it is known that $ct(V(f)) \geq 11d/6 - 4$, see [13]. It follows that $T_1 \leq T + d/6$, which proves the result. □

The stabilization value of $H_{n-1}(f)_m$ is not easy to describe gemetrically beyond Theorem 1.1, as the following example shows.

Example 2.14. Let $V : f = 0$ be a plane curve having only simple singularities, hence singularities of type A_k, D_k, E_6, E_7 and E_8. A direct computation shows that $\chi_1(g) = 2$ for g of type E_6 and E_7, and $\chi_1(g) = 3$ for g of type E_8. Using now Example 2.6, it follows that, for $m \geq \tilde{T}_1 = 4d - 8$, one has

$$\dim H_1(f)_m = \sum_k \sharp D_k + 2(\sharp E_6 + \sharp E_7) + 3\sharp E_8.$$

Example 2.15. Consider the line arrangement in $\mathbb{P}^2$ given by the following equation

$$f = (x^2 - y^2)(y^2 - z^2)(x^2 - z^2).$$

Then the curve $V(f) : f = 0$ is a union of 6 lines and it has 3 nodes A_1 and 4 triple points D_4. In this case $ct(V) = 6$ as shown by $HP(H_3(f); t) = HP(M(f); t)$, and hence $T_2 = \max(12 - 6 + 8, st(V)) = 14$ and $T_1 = T_2 - (d - 2) = 10$. Numerical computations give the following.

$$HP(H_1(f); t) = 1 + 3t + 6t^2 + 10t^3 + 10t^4 + 7t^5 + 5t^6 + 4(t^7 + t^8 + ...)$$

(hence again a stabilization 3 units before T_1)

$HP(H_2(f);t) = 1{+}3t{+}6t^2{+}10t^3{+}15t^4{+}18t^5{+}19t^6{+}19t^7{+}13t^8{+}12(t^9{+}t^{10}{+}...)$

(hence again a stabilization 2 units before T_2) and

$HP(H_3(f);t) = HP(M(f);t) = 1{+}3t{+}6t^2{+}10t^3{+}15t^4{+}18t^5{+}19(t^6{+}t^7{+}...)$,

as predicted by Proposition 1.4.

The following consequence of the proof of Theorem 1.3 might be useful sometimes.

Corollary 2.16. *An isolated hypersurface singularity $g = 0$ is weighted homogeneous if and only if $\chi_n(g) = \tau(g) - 1$. Otherwise $\chi_n(g) = \tau(g)$.*

2.17. Proof of Proposition 1.4. In the case (i), it is enough to note that $M(f)$ is an Artinian Gorenstein ring whose 1-dimensional socle is spanned by the Hessian polynomial $h(f)$ which has degree T, see for instance [15].

In the case (ii), the class of the Hessian $h(f)$ will now vanish in the local Tjurina ring $T(g)$ attached to any singularity of $V(f)$ by Proposition 2.3. Indeed, recall that the first row and first column are zero in the Hessian matrix $Hess(f)$ (in suitable coordinates) modulo the Tjurina ideal $(g) + J_g$. Moreover, as in the proof of Theorem 1.3, the evaluation map

$$ev_T : S_T/J_{f,T} \to \oplus\mathcal{O}_n/((g) + J_g) \tag{2.8}$$

is an isomorphism. It follows that $h(f) \in J_f$ and hence $HP(H_{n+1}(f);t) = HP(M(f);t)$ in this case.

The Hilbert-Poincaré series of the Hessian rings $H_k(f)$ for $1 \leq k \leq n$ are quite mysterious, as the following example shows.

Example 2.18. Consider the Fermat curve $V(f) : f = x^6 + y^6 + z^6 = 0$ and its "deformation" $V(f') : f' = f + 2xyz^4 + 3xzy^4 + 5yzx^4 = 0$. One obtains the following results by direct computation.

$$HP(M(f);t) = HP(M(f');t) = \tag{2.9}$$
$$= 1{+}3t{+}6t^2{+}10t^3{+}15t^4{+}18t^5{+}19t^6{+}18t^7{+}15t^8{+}10t^9{+}6t^{10}{+}3t^{11}{+}t^{12}.$$

In particular, $V(f')$ is smooth as well and $HP(H_3(f);t) = HP(H_3(f');t)$ can be computed using Proposition 1.4. For $k = 2$ we get the following results.

$HP(H_2(f);t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19t^6 + 18t^7 + 12t^8 + 7t^9 + 3t^{10}$

and

$HP(H_2(f');t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19t^6 + 18t^7 + 9t^8.$

Note that the 2-minors have degree 8, which explains the fact that the monomials containing t^m for $m < 8$ are unchanged. For f there are only 3 such minors, which explain the drop from 15 to 12 in the coefficient of t^8. For f', there are 6 distinct minors (due to the symmetry of the Hessian matrix) and these 2-minors span a 6-dimensional vector space modulo $J_{f',8}$ in this case. Finally, for $k = 1$ we get

$$HP(H_1(f); t) = 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + 10t^6 + 6t^7 + 3t^8 + t^9$$

and

$$HP(H_1(f'); t) = 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 3t^5.$$

Note that the 1-minors have degree 4, which explains the fact that the monomials containing t^m for $m < 3$ are unchanged. For f there are only 3 such minors, which explain the drop from 15 to 12 in the coefficient of t^4. For f', there are 6 distinct 1-minors (due to the symmetry of the Hessian matrix) and these 1-minors are linearly independent in this case as seen by looking at the coefficient of t^4. Moreover, if we look at the coefficient of t^5 in $HP(H_1(f'); t)$, it says that *the only relations among the 18 polynomials $x_m f'_{ij}$ with $i \leq j$ and the 3 partial derivatives f'_0, f'_1, f'_3 are the obvious 3 relations given by the Euler's formulas*

$$(d-1)f'_q = x_0 f'_{q0} + x_1 f'_{q1} + x_2 f'_{q2},$$

for $q = 0, 1, 2$. This accounts for the drop $18 - 15 = 3$ when we pass from $M(f')$ to $H_1(f')$.

The symmetry of the polynomial $HP(H_1(f); t)$ is easy to explain in general. Indeed, if we denote by $F(d)$ the Fermat polynomial in $n+1$ variables of degree d, then it is clear that

$$H_1(F(d)) = M(F(d-1)).$$

3. Constructible partitions for the space of homogeneous polynomials

Consider the projective space $\mathbb{P}(S_d)$ of non-zero homogeneous polynomials of degree d and let $G = PGL_n(\mathbb{C})$ act on $\mathbb{P}(S_d)$ in the usual way, i.e. by linear changes of coordinates. Then there is a Zariski open and dense, G-invariant subset $U_s(n, d)$ in $\mathbb{P}(S_d)$, parametrizing the smooth hypersurfaces of degree d in $\mathbb{P}^n$. There is a larger G-invariant Zariski open subset $U_{iso}(n, d)$ such that $f \in U_{iso}(n, d)$ if and only if the hypersurface $V(f) : f = 0$ in $\mathbb{P}^n$ has only isolated singularities. Moreover, for each list of (distinct or not) simple isolated hypersurface singularities $(Y_1, Y_2, ..., Y_p)$ we can consider the constructible subset $S(Y_1, Y_2, ..., Y_p) \subset U_{iso}(n, d)$ defined by the condition that $f \in S(Y_1, Y_2, ..., Y_p)$ if and only if the hypersurface $V(f)$ has exactly p singularities of types $Y_1, ..., Y_p$. It is known that if this stratum is non-empty, then

$$\operatorname{codim} S(Y_1, Y_2, ..., Y_p) \leq \sum_{i=1,p} \tau(Y_i)$$

with equality if the degree d is large, e.g. if

$$d \geq \sum_{i=1,p} s_i + p - 1, \tag{3.1}$$

where s_i is the $\mathcal{K}$-determinacy order of Y_i, see Prop. 1.3.9 in [6], p.18. Moreover in this case the stratum $S(Y_1, Y_2, ..., Y_p)$ is smooth and connected. For more precise and more general results, see Theorem 1 and Theorem 5 in [16].
When all of the singularities Y_i are nodes and $n = 2$, then the corresponding stratum $S_0(Y_1, Y_2, ..., Y_p)$ consisting of all reduced *irreducible* curves with p nodes is connected. This is the famous Severi problem solved by J. Harris [18]. However, in general, the strata $S(Y_1, Y_2, ..., Y_p)$ are not connected even in this case, see Example 3.1 below. For more general and precise results in this direction, as well as for an extensive bibliography on the subject, we refer to the excellent survey [16].
Since $\dim M(f)_k = \tau(V(f))$, for $k > T = (n+1)(d-2)$, see [2], it follows that there is a finite constructible *Jacobian partition* $\mathcal{S}_Y(M(f))$ of each stratum $S(Y) = S(Y_1, Y_2, ..., Y_p)$, such that $f, g \in S(Y)$ are in the same stratum of $\mathcal{S}_Y$ if and only if $HP(M(f); t) = HP(M(g); t)$. It is known that $f_s \in U_s(n, d)$ if and only if

$$(3.2) \qquad HP(M(f_s); t) = \frac{(1 - t^{d-1})^{n+1}}{(1 - t)^{n+1}},$$

and hence the Jacobian partition on $U_s(n, d)$ consists of just one stratum. However, for a nodal hypersurface $V(f)$ with 3 nodes, the value of $HP(M(f); t)$ depends on whether the nodes are collinear or not, see Example 4.3 (iii) in [12]. It follows that the Jacobian stratification on the stratum $S(3A_1)$ is not trivial in general.

Example 3.1. Consider the case $n = 2$, $d = 4$ and the stratum $S(3A_1)$. It is known that this stratum $S(A_3)$ has two irreducible components, $X = S_0(3A_1)$ containing the irreducible quartics with 3 nodes (necessarily non-collinear) and Y, containing the reduced quartics with 3 collinear nodes obtained from a smooth cubic and a trisecant line, see Remark 4.7.21 in [21]. Then one has $\operatorname{codim} X = \operatorname{codim} Y = 3$, as follows from Theorem 4.7.18 in [21].
Example 4.3 (iii) in [12] tells us that there are two possibilities for $H(M(f); t)$ on the stratum $S(3A_1)$, namely the *minimal value* (coefficient by coefficient) is

$$HP(M(f); t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + 3t^6 + ...$$

and this is attained exactly on X and the other possible value is

$$HP(M(g); t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3t^6 + ...$$

and which is attained precisely on Y. In other words, in this case the Jacobian partition separates the irreducible components of the stratum $S(3A_1)$.
Consider now the case $n = 2$, but $d \geq 8$. Since the $\mathcal{K}$-determinacy order of a node is $s_i = 2$, it follows from (3.1) that the stratum $S(3A_1)$ is smooth and connected in this case. The minimal value of $HP(M(f); t)$, which is

$$(3.3) \qquad \frac{(1 - t^{d-1})^3}{(1 - t)^3} + 2t^T + 3t^{T+1}(1 + t + t^2 + ...),$$

by Example 4.3 (iii) in [12], is attained on a Zariski open dense subset S' of the stratum $S(3A_1)$, while the larger value

$$\frac{(1-t^{d-1})^3}{(1-t)^3}+t^{T-1}+2t^T+3t^{T+1}(1+t+t^2+...), \tag{3.4}$$

is attained on the closed subset $S''=S(3A_1)\setminus S'$ in $S(3A_1)$.

Example 3.2. Consider the case $n=2$, $d=6$ and the stratum $S(6A_2)$. It is known by the classical work of Zariski that this stratum has two irreducible components, X containing the irreducible sextics with 6 cusps on a conic and Y, containing the irreducible sextics with 6 cusps not on a conic. As an example of a sextic in X one can take

$$V(f): f=(x^2+y^2)^3+(y^3+z^3)^2=0$$

and then a direct computation yields

$$HP(M(f);t)= \tag{3.5}$$
$$=1+3t+6t^2+10t^3+15t^4+18t^5+19t^6+18t^7+16t^8+13t^9+12(t^{10}+\cdots.$$

As an example of a sextic in Y one can take

$$V(g): g=27(x+y)^3(x+y-z)^2(x+y+2z)-27(x+y)^2(x+y-z)^2((x-y)^2-z^2)+$$
$$+9((x+y)^2-z^2)((x-y)^2-z^2)^2-((x-y)^2-z^2)^3=0,$$

see [20], formula (5.7), and again a direct computation yields

$$HP(M(g);t)=1+3t+6t^2+10t^3+15t^4+18t^5+19t^6+18t^7+15t^8+12(t^9+....$$

We do not know whether the Hilbert-Poincaré series is constant on X and/or Y, but it seems that again it can separate distinct irreducible components in a stratum.

We have seen above, especially in Example 2.18, that though one can also define *the Hessian partition* by looking at the strata inside a given $S(Y_1,...,Y_p)$ where the corresponding Hilbert-Poincaré series $HP(H_k(f);t)$ are constant for $k=1,2,...,n+1$, the behavior of these partitions is rather mysterious. Indeed, Example 2.18 shows that already the Hessian partition on the stratum $U_s(n,d)$ can be quite subtle.

To investigate the possibilities for the Hilbert series for $HP(M(f);t)$ as well as for $HP(H_k(f);t)$ for various k's, in the neighborhood of a given point $g\in\mathbb{P}(S_d)$, one can proceed as follows. It is known that the tangent space to the G-orbit of g (where all the invariants are clearly preserved) is given by $(J_g)_d$. Let $h_1,...,h_q$ be a (monomial) basis of $M(g)_d$ and consider the family of polynomials

$$f=g+a_1h_1+...+a_qh_q, \tag{3.6}$$

with $a_j\in\mathbb{C}$, which is exactly a transversal to the G-orbit of g. If we choose the a_j small enough, the values obtained for $HP(M(f);t)$ as well as for $HP(H_k(f);t)$ for various k's, will cover all the possibilities that occur in a neighborhood of g (in the strong topology). If we allow any values for a_j, they will give sometimes interesting values, even though not related to the fixed polynomial g (but covering this time a Zariski open neighborhood of g).

Remark 3.3. Consider the case of smooth plane curves of increasing degree d.
(i) When $d = 2$, then $U_s(2,2)$ is a single G-orbit, hence all the invariants $HP(H_k(f))$ are constant on $U_s(2,2)$, i.e. the Hessian partition is trivial. This fact clearly holds for any $n \geq 2$ as well.
(ii) When $d = 3$, it is known that any polynomial in $U_s(2,3)$ is G-equivalent to a Hesse normal form

$$f = x^3 + y^3 + z^3 + 3axyz,$$

with $a^3 \neq -1$. This comes essentially from the fact that if $g = x^3 + y^3 + z^3$ is the corresponding Fermat polynomial, then xyz is a basis for the 1-dimensional vector space $M(g)_3$.
(iii) When $d = 4$, if we take $g = x^4 + y^4 + z^4$, then a basis for $M(g)_4$ is given by the following 6 monomials

$$x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2.$$

But doing computation with 6 parameter families seems too complicated for the moment. That is why in the following examples we take $h_1, ..., h_q$ just *a family of independent elements* of $M(g)_d$, i.e. we explore what happens only in *a fixed direction in the normal space* to the orbit of g.

Example 3.4. Consider the case $n = 2$, $d = 6$, $g = x^6 + y^6 + z^6$ and the 3-parameter family

$$E : f = g + ax^4yz + bxy^4z + cxyz^4,$$

which is a partial transversal as explained above. We will describe all the possibilities for the series $HP(H_1(f); t)$ when f has this form and $f \in U_s(2,6)$. By direct computation one gets the following Hessian strata in E, where for each strata we list the corresponding series $HP(H_1(f); t)$ and the conditions on a, b, c to get such a series, up to the action of the permutation group on x, y, z.

$S_1 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 3t^5$ with a, b, c generic .

This is the largest stratum, a Zariski open dense subset in $E = \mathbb{C}^3$.

$S_2 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 4t^5$ with $a = b \neq 0$ and c generic .

$S_3 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 6t^5$ with $a = b = c$ generic .

$S_4 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 4t^5 + t^6$ with $a = b \neq 0$ and $c = 0$.

$S_5 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 6t^5 + 2t^6$ with a, b, c a collection of finite nonzero values, for instance $a = b = c = 2$.

$S_6 : 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 7t^5 + 3t^6 + t^7$ with $a = b = 0$ and c generic .

$S_7 : 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + 10t^6 + 6t^7 + 3t^8 + t^9$ with $a = b = c = 0$.

In the next picture (a Hasse diagram as in the theory of hyperplane arrangements) we draw a line between S_i and S_j with S_i above S_j if the inequality $HP(H_1(f^i);t) \geq HP(H_1(f^j);t)$ (coefficient by coefficient) holds for some $f^i \in S_i$ and $f^j \in S_j$ and we call the resulting poset a Hessian poset. This inequality is *a necessary condition such that the closure of the stratum S_j intersects the stratum S_i*, by obvious semicontinuity properties. This condition is *not sufficient*, since for instance the stratum S_5 consists of finitely many points, hence its closure coincides with S_5, and has empty intersection with the strata S_6 and S_7. However, all the other strata S_j contain g in their closure. Note also that if, instead of looking at strata S_j inside $E \cap U_s(2,6)$, we consider the corresponding strata S'_j in $U_s(2,6)$ defined by the same series $HP(H_1(f);t)$, the above claim that S_5 is finite it is no longer valid for S'_5, and hence it is possible that the closure of S'_5 in $U_s(2,6)$ contains g. To decide such a question by this approach would involve computation with a family having $\dim M(g)_6 = 19$ parameters. On the positive side, the above computation shows that the strata S'_1, S'_2, S'_3, S'_4 and S'_6 contain g in their closure.

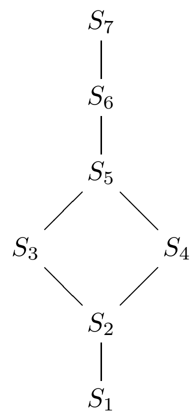

Figure 1. The Hasse diagram of a Hessian poset in the case $n = 2$, $d = 6$

Here is an example for the smooth curves of degree 6, which can be regarded as a continuation of Example 2.18 and where we consider both series $HP(H_1(f);t)$ and $HP(H_2(f);t)$.

Example 3.5. Consider the case $n = 2$, $d = 6$. Let $g = f'' = x^6 + y^6 + z^6 + 3x^2y^2z^2$. Then the curve $V(f'') : f'' = 0$ is smooth and we have

$HP(H_2(f'');t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19t^6 + 18t^7 + 9t^8$, hence the same series as for the polynomial f' in Example 2.18 and

$HP(H_1(f'');t) = 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 6t^5$, hence a strictly larger series than for the polynomial f' since the coefficients of t^5 satisfy $3 < 6$.

Consider now the 1-parameter deformation $f''_a = f'' + axyz^4$ of f''. For small values of a we get

$HP(H_2(f_a''); t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19t^6 + 18t^7 + 9t^8$, hence the same series as for the polynomials f' and f''. Moreover

$HP(H_1(f_a''); t) = 1+3t+6t^2+10t^3+9t^4+4t^5$, hence a series between that of the polynomial f' and that of polynomial f'', since the corresponding coefficients of t^5 satisfy $3 < 4 < 6$. However, for the value $a = 3$ we get different results, namely

$HP(H_2(f_3''); t) = 1+3t+6t^2+10t^3+15t^4+18t^5+19t^6+18t^7+9t^8+2t^9$, and

$HP(H_1(f_3''); t) = 1 + 3t + 6t^2 + 10t^3 + 9t^4 + 4t^5$, hence this polynomial has both series distinct from the minimal values given by the polynomial f' in Example 2.18. Note that the series $HP(H_2(f_3''); t)$ is strictly bigger than the series $HP(H_2(f''); t)$, while the series $HP(H_1(f_3''); t)$ is stricly smaller than $HP(H_1(f''); t)$. *The conclusion is that the strata of the Hessian partition to which belong f_3'' and respectively f'' are not adjacent, i.e. their closures are disjoint.*

Remark 3.6. It is not at all clear what the series $HP(H_k(f); t))$ tells us about the geometry of the smooth hypersurface $V(f) : f = 0$.

(i) In the curve case, when $n = 2$, the equations $f = h(f) = 0$ define the *inflection points* of the curve $V(f)$, and by Bezout Theorem, their number is at most $d(3d - 6)$, with equality for a generic curve, see for instance [1]. This number is lower for some curves, e.g. for the Fermat curve $x^d + y^d + z^d = 0$ we have only $3d$ inflection points (counted without multiplicities). However, the series $HP(H_k(f); t))$ do not seem to be related to such differences. Indeed, the two smooth curves $V(f')$ and $V(f_3'')$ of degree 6 in Example 3.5 have each 72 inflection points (by direct computation), but the corresponding series are different.

(ii) The subvariety of $V(f) \subset \mathbb{P}^n$ given by the vanishing of the ideal $h_{n+2-k}(f)$ corresponds to the closure of the *Thom-Boardman singularity set $S_k(\phi_f)$ of the dual mapping $\phi_f : V(f) \to \mathbb{P}^n$* associated to f. For a generic f, the stratum $S_k(\phi_f)$ is smooth, of codimension $k(k+1)/2$ in $V(f)$ (in particular empty if if $n - 1 < k(k+1)/2$) and connected if $n - 1 > k(k+1)/2$, see [1].

(iii) When $d = 3$, there is a graded Artinian Gorenstein algebra $R(f)$ with socle degree 3 associated to f via the so-called *Macaulay inverse systems*, see [4], [17], [19]. By results in classical algebraic geometry in [14], the variety given by the vanishing of the ideal $h_k(f)$ corresponds to the linear forms $\ell \in R(f)_1$ such that the multiplication by ℓ gives a linear map $R(f)_1 \to R(f)_2$ of rank $< k$, see Lemma 6.1 in [4]. In this way relations to the *Lefschetz properties of Artinian graded rings* are established. For such properties, and a definition of generalized Hessians playing a key role in this theory, see [17], in particular Definition 3.75.

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