Deriving Auslander's formula

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Abstract. Auslander's formula shows that any abelian category C is equivalent to the category of coherent functors on C modulo the Serre subcategory of all effaceable functors. We establish a derived version of this equivalence. This amounts to showing that the homotopy category of injective objects of some appropriate Grothendieck abelian category (the category of ind-objects of C) is compactly generated and that the full subcategory of compact objects is equivalent to the bounded derived category of C. The same approach shows for an arbitrary Grothendieck abelian category that its derived category and the homotopy category of injective objects are well-generated triangulated categories. For sufficiently large cardinals α we identify their α-compact objects and compare them.

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1. Introduction

Let A be a Grothendieck abelian category and let $\operatorname{Inj}\mathsf{A}$ denote the full subcategory of injective objects. Then it is known from Neeman's work [24, 25] that the derived category $\mathbf{D}(\mathsf{A})$ and the homotopy category $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ are well-generated triangulated categories. The present work describes for sufficiently large cardinals α their subcategories of α-compact objects.

Recall that any well-generated triangulated category T admits a filtration $\mathsf{T} = \bigcup_\alpha \mathsf{T}^\alpha$ where α runs through all regular cardinals and T^α denotes the full subcategory of α-compact objects [23]. This is an analogue of the filtration $\mathsf{A} = \bigcup_\alpha \mathsf{A}^\alpha$ where A^α denotes the full subcategory of α-presentable objects [9]. Note that there exists a regular cardinal α_0 such that A^α is abelian for all $\alpha \geq \alpha_0$ (Corollary 5.2). In fact, when A^α is abelian and generates A, then A^β is abelian for all $\beta \geq \alpha$ (Corollary 5.5).

We distinguish two cases, keeping in mind the notation $\mathsf{A}^{\aleph_0} = \operatorname{fp}\mathsf{A}$ and $\mathsf{T}^{\aleph_0} = \mathsf{T}^c$. The first case is a generalisation of the locally noetherian case studied in [16].

THEOREM ($\alpha = \aleph_0$). *Let* A *be a Grothendieck abelian category. Suppose that the subcategory* $\operatorname{fp}\mathsf{A}$ *of finitely presented objects is abelian and generates* A*. Then the homotopy category* $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ *is a compactly generated triangulated category and the canonical functor* $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{D}(\mathsf{A})$ *induces an equivalence* $\mathbf{K}(\operatorname{Inj}\mathsf{A})^c \xrightarrow{\sim} \mathbf{D}^b(\operatorname{fp}\mathsf{A})$.

THEOREM ($\alpha \neq \aleph_0$). *Let* A *be a Grothendieck abelian category. Suppose that* $\alpha > \aleph_0$ *is a regular cardinal such that the subcategory* A^α *is abelian and generates* A*. Then the following holds:*

(1) *The derived category* $\mathbf{D}(\mathsf{A})$ *is an* α*-compactly generated triangulated category and the inclusion* $\mathsf{A}^\alpha \to \mathsf{A}$ *induces an equivalence* $\mathbf{D}(\mathsf{A}^\alpha) \xrightarrow{\sim} \mathbf{D}(\mathsf{A})^\alpha$.
(2) *The homotopy category* $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ *is an* α*-compactly generated triangulated category and the left adjoint of the inclusion* $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{K}(\mathsf{A})$ *induces a quotient functor* $\mathbf{K}(\mathsf{A}^\alpha) \twoheadrightarrow \mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha$.

The case $\alpha = \aleph_0$ is Theorem 4.9 and for $\alpha \neq \aleph_0$ see Theorems 5.10 and 5.12. Note that in case $\alpha = \aleph_0$ the derived category $\mathbf{D}(\mathsf{A})$ need not be compactly generated; an explicit example is given by Neeman [25].

The above results are obtained by 'resolving' the abelian category A. More precisely, we use a variation of Auslander's formula (Theorem 2.2) to write A as the quotient of a functor category modulo an appropriate subcategory of effaceable functors (Corollary 5.5). Then we 'derive' this presentation of A by passing to the derived category $\mathbf{D}(\mathsf{A})$ and to the homotopy category $\mathbf{K}(\operatorname{Inj}\mathsf{A})$. This passage from a Grothendieck abelian category to a well-generated triangulated category demonstrates the amazing parallel between both concepts [15]. Also, we see the relevance of the filtration $\mathsf{A} = \bigcup_\alpha \mathsf{A}^\alpha$, which seems to be somewhat neglected in the literature.

There are at least two aspects that motivate our work. The homotopy category $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ played an import role in work with Benson and Iyengar on modular representations of finite groups [5]. For instance, a classification of localising subcategories of $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ amounts to a classification of α-localising subcategories of $\mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha$ for a sufficiently large cardinal α. On the other hand, $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ has been used to reformulate Grothendieck duality for noetherian schemes, and it seems reasonable to wonder about the non-noetherian case; see [25] for details.

This paper has two parts. The first sections form the 'finite' part, dealing with finitely presented and compact objects. Cardinals greater than $\aleph_0$ only appear in the last section, which includes a gentle introduction to locally presentable abelian and well-generated triangulated categories.

2. Functor categories and Auslander's formula

In this section we recall definitions and some basic facts about functor categories. In particular, we recall Auslander's formula.

Localisation sequences. We consider pairs of adjoint functors (F, G)

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$$

satisfying the following equivalent conditions [10, I.1.3]:

(1) The functor F induces an equivalence

$$\mathsf{C}[\Sigma^{-1}] \xrightarrow{\sim} \mathsf{D}$$

where $\Sigma := \{\sigma \in \operatorname{Mor}\mathsf{C} \mid F\sigma \text{ is invertible }\}$.

(2) The functor G is fully faithful.

(3) The morphism of functors $FG \to \mathrm{Id}_{\mathsf{D}}$ is invertible.

Definition 2.1. A *localisation sequence* of abelian (triangulated) categories is a diagram of functors

$$\mathsf{B} \underset{E'}{\overset{E}{\rightleftarrows}} \mathsf{C} \underset{F'}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{2.1}$$

satisfying the following conditions:

(1) E and F are exact functors of abelian (triangulated) categories.

(2) The pairs (E, E') and (F, F') are adjoint pairs.

(3) The functors E and F' are fully faithful.

(4) An object in C is annihilated by F iff it is in the essential image of E.

We refer to [8, 30] for basic properties, in particular for the construction of the abelian (triangulated) quotient C/B' such that F induces an equivalence $\mathsf{C}/\mathsf{B}' \xrightarrow{\sim} \mathsf{D}$, where B' denotes the full subcategory of objects in C that are annihilated by F. Thus any of the functors E, E', F, F' determines the diagram (2.1) up to equivalence.

An exact functor $F\colon \mathsf{C} \to \mathsf{D}$ of abelian (triangulated) categories is by definition a *quotient functor* if F induces an equivalence $\mathsf{C}/\mathsf{B} \xrightarrow{\sim} \mathsf{D}$, where B denotes the full subcategory of objects in C that are annihilated by F.

Finitely presented functors. Let C be an additive category. We denote by $\operatorname{mod}\mathsf{C}$ the category of finitely presented functors $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. Recall that F is *finitely presented* (or *coherent*) if it fits into an exact sequence

$$\operatorname{Hom}_{\mathsf{C}}(-, X) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(-, Y) \longrightarrow F \longrightarrow 0. \tag{2.2}$$

The *Yoneda functor* is the fully faithful functor

$$\mathsf{C} \longrightarrow \operatorname{mod}\mathsf{C}, \quad X \mapsto \operatorname{Hom}_{\mathsf{C}}(-, X).$$

Additive functors. Let C be an (essentially) small additive category. A C-*module* is an additive functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. For the category of C-modules we write

$$\operatorname{Mod}\mathsf{C} := \operatorname{Add}(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$$

and consider the following full subcategories:

$$\begin{aligned}\operatorname{Proj}\mathsf{C} &:= \text{projective } \mathsf{C}\text{-modules}\\ \operatorname{Inj}\mathsf{C} &:= \text{injective } \mathsf{C}\text{-modules}\\ \operatorname{Flat}\mathsf{C} &:= \text{flat } \mathsf{C}\text{-modules}\end{aligned}$$

The flat C-modules are precisely the filtered colimits of representable functors. Thus we may identify

$$\operatorname{Flat}\mathsf{C} = \operatorname{Ind}\mathsf{C}$$

where $\operatorname{Ind}\mathsf{C}$ denotes the category of ind-objects in the sense of [12, §8]. When C admits cokernels then

$$\operatorname{Flat}\mathsf{C} = \operatorname{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$$

where $\operatorname{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$ denotes the category of left exact functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. Note that the inclusion $\operatorname{mod}\mathsf{C} \to \operatorname{Mod}\mathsf{C}$ induces an equivalence

$$\operatorname{Ind}\operatorname{mod}\mathsf{C} \xrightarrow{\sim} \operatorname{Mod}\mathsf{C}.$$

Effaceable functors. Let C be an abelian category. Then we write $\operatorname{eff}\mathsf{C}$ for the full subcategory of functors F in $\operatorname{mod}\mathsf{C}$ that admit a presentation (2.2) with $X \to Y$ an epimorphism in C. If C is small then the inclusion $\operatorname{eff}\mathsf{C} \to \operatorname{Mod}\mathsf{C}$ induces a fully faithful functor

$$\operatorname{Eff}\mathsf{C} := \operatorname{Ind}\operatorname{eff}\mathsf{C} \longrightarrow \operatorname{Mod}\mathsf{C}.$$

This identifies $\operatorname{Eff}\mathsf{C}$ with the functors in $\operatorname{Mod}\mathsf{C}$ that are effaceable in the sense of [11, p. 141].

Auslander's formula. The following result is somewhat hidden in Auslander's account on coherent functors.

Theorem 2.2 ([2, p. 205]). *Let C be an abelian category. Then the Yoneda functor $\mathsf{C} \to \operatorname{mod}\mathsf{C}$ induces a localisation sequence of abelian categories.*

$$\begin{tikzcd} \operatorname{eff}\mathsf{C} \arrow[r, shift left, tail] & \operatorname{mod}\mathsf{C} \arrow[l, shift left, two heads] \arrow[r, shift left, two heads] & \mathsf{C} \arrow[l, shift left, tail] \end{tikzcd} \tag{2.3}$$

Moreover, the functor $\operatorname{mod}\mathsf{C} \to \mathsf{C}$ induces an equivalence

$$\frac{\operatorname{mod}\mathsf{C}}{\operatorname{eff}\mathsf{C}} \xrightarrow{\sim} \mathsf{C}.$$

Proof. The left adjoint of the Yoneda functor is the unique functor $\operatorname{mod}\mathsf{C} \to \mathsf{C}$ that preserves finite colimits and sends each representable functor $\operatorname{Hom}_{\mathsf{C}}(-, X)$ to X. Thus a functor F with presentation (2.2) is sent to the cokernel of the representing morphism $X \to Y$. The exactness of the left adjoint follows from a simple application of the horseshoe lemma. □

Following Lenzing [19] we call this presentation of an abelian category (via the left adjoint of the Yoneda functor) *Auslander's formula.*
A predecessor of this result for Grothendieck abelian categories is due to Gabriel.

Theorem 2.3 ([8, II.2]). *Let C be a small abelian category. Then the inclusion* $\operatorname{Ind}\mathsf{C} \to \operatorname{Mod}\mathsf{C}$ *induces a localisation sequence of abelian categories extending* (2.3).

$$\operatorname{Eff}\mathsf{C} \rightleftarrows \operatorname{Mod}\mathsf{C} \rightleftarrows \operatorname{Ind}\mathsf{C} \tag{2.4}$$

Moreover, the functor $\operatorname{Mod}\mathsf{C} \to \operatorname{Ind}\mathsf{C}$ *induces an equivalence*

$$\frac{\operatorname{Mod}\mathsf{C}}{\operatorname{Eff}\mathsf{C}} \xrightarrow{\sim} \operatorname{Ind}\mathsf{C}.$$

Proof. The left adjoint of the inclusion functor is the unique functor $\operatorname{Mod}\mathsf{C} = \operatorname{Ind}\operatorname{mod}\mathsf{C} \to \operatorname{Ind}\mathsf{C}$ that preserves filtered colimits and extends the functor $\operatorname{mod}\mathsf{C} \to \mathsf{C}$ from Theorem 2.2. Any exact sequence in $\operatorname{Mod}\mathsf{C}$ can be written as a filtered colimit of exact sequences in $\operatorname{mod}\mathsf{C}$.[1] Thus the exactness of the functor $\operatorname{mod}\mathsf{C} \to \mathsf{C}$ yields the exactness of $\operatorname{Mod}\mathsf{C} \to \operatorname{Ind}\mathsf{C}$. □

Note that for any small abelian category C the category $\operatorname{Ind}\mathsf{C}$ is a Grothendieck abelian category [8, II.3]. Thus all categories occuring in diagram (2.4) are Grothendieck abelian.

A Grothendieck abelian category A has injective envelopes and we denote by $\operatorname{Inj}\mathsf{A}$ the full subcategory of injective objects. The diagram (2.4) induces a sequence of functors[2] $\operatorname{Inj}\operatorname{Ind}\mathsf{C} \to \operatorname{Inj}\mathsf{C} \to \operatorname{Inj}\operatorname{Eff}\mathsf{C}$ since a right adjoint of an exact functor preserves injectivity.

3. Derived categories

In this section we describe a derived version of Auslander's formula for the bounded derived category of an abelian category.

Let us recall some notation. For an additive category C let $\mathbf{K}(\mathsf{C})$ denote the homotopy category of cochain complexes in C. The objects of $\mathbf{K}(\mathsf{C})$ are cochain complexes and the morphisms are homotopy classes of chain maps. When C is abelian, then $\mathbf{Ac}(\mathsf{C})$ denotes the full subcategory of acyclic complexes and the derived category $\mathbf{D}(\mathsf{C})$ is by definition the triangulated quotient $\mathbf{K}(\mathsf{C})/\mathbf{Ac}(\mathsf{C})$. The superscript ^{b} refers to the full subcategory of cochain complexes X satisfying $X^n = 0$ for $|n| \gg 0$.

For a triangulated category T and a class of objects S in T let $\operatorname{Thick}(\mathsf{S})$ denote the smallest thick subcategory of T containing S.

Lemma 3.1. *Let C be an abelian category. Then the Yoneda functor $\mathsf{C} \to \operatorname{mod}\mathsf{C}$ induces a triangle equivalence $\mathbf{K}^b(\mathsf{C}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\mathsf{C})$ that makes the following*

[1]Let $\eta\colon 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be exact. Write α as filtered colimit of morphisms $\alpha_i\colon X_i \to Y_i$ in $\operatorname{mod}\mathsf{C}$. Each α_i induces an epimorphism $\beta_i\colon Y_i \to \operatorname{Coker}\alpha_i$. Then η is the filtered colimit of the exact sequences $0 \to \operatorname{Ker}\beta_i \to Y_i \to \operatorname{Coker}\alpha_i \to 0$.

[2]The notation $\operatorname{Inj}\mathsf{C}$ is ambiguous: We mean the category of injective C-modules, and not the category of injective objects in C.

diagram commutative.

$$\begin{tikzcd} \mathbf{Ac}^b(\mathsf{C}) \arrow[r, tail] \arrow[d, "\wr"] & \mathbf{K}^b(\mathsf{C}) \arrow[r, two heads] \arrow[d, "\wr"] & \mathbf{D}^b(\mathsf{C}) \arrow[d, equal] \\ \mathrm{Thick}(\mathrm{eff}\,\mathsf{C}) \arrow[r, tail] & \mathbf{D}^b(\mathrm{mod}\,\mathsf{C}) \arrow[r, two heads] & \mathbf{D}^b(\mathsf{C}) \end{tikzcd}$$

In particular, the triangulated category $\mathbf{Ac}^b(\mathsf{C})$ *is generated by the acyclic complexes of the form*

$$\cdots \to 0 \to X^{n-1} \to X^n \to X^{n+1} \to 0 \to \cdots .$$

Proof. Each object F in $\mathrm{mod}\,\mathsf{C}$ admits a finite projective resolution

$$0 \to \mathrm{Hom}_{\mathsf{C}}(-,X) \to \mathrm{Hom}_{\mathsf{C}}(-,Y) \to \mathrm{Hom}_{\mathsf{C}}(-,Z) \to F \to 0. \tag{3.1}$$

Thus we have a triangle equivalence $\mathbf{K}^b(\mathsf{C}) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\,\mathsf{C})$ because the Yoneda functor identifies C with the full subcategory of projective objects in $\mathrm{mod}\,\mathsf{C}$. For X in $\mathbf{K}^b(\mathsf{C})$ let Y denote the corresponding complex in $\mathbf{D}^b(\mathrm{mod}\,\mathsf{C})$ and observe that Y belongs to $\mathrm{Thick}(\{H^i(Y) \mid i \in \mathbb{Z}\})$. The kernel of $\mathrm{mod}\,\mathsf{C} \to \mathsf{C}$ equals $\mathrm{eff}\,\mathsf{C}$ by Theorem 2.2. Thus Y is in $\mathrm{Thick}(\mathrm{eff}\,\mathsf{C})$ iff Y is annihilated by $\mathbf{D}^b(\mathrm{mod}\,\mathsf{C}) \to \mathbf{D}^b(\mathsf{C})$ iff X is in $\mathbf{Ac}^b(\mathsf{C})$. This yields the triangle equivalence $\mathbf{Ac}^b(\mathsf{C}) \xrightarrow{\sim} \mathrm{Thick}(\mathrm{eff}\,\mathsf{C})$.
For the final assertion of the lemma, observe that the equivalence $\mathbf{K}^b(\mathsf{C}) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\,\mathsf{C})$ identifies the complexes of the form

$$\cdots \to 0 \to X^{n-1} \to X^n \to X^{n+1} \to 0 \to \cdots$$

with the objects in $\mathrm{eff}\,\mathsf{C}$ viewed as complexes concentrated in degree $n+1$. □

COROLLARY 3.2. *The canonical functor* $\mathbf{D}^b(\mathrm{mod}\,\mathsf{C}) \to \mathbf{D}^b(\mathsf{C})$ *induces an equivalence*

$$\frac{\mathbf{D}^b(\mathrm{mod}\,\mathsf{C})}{\mathrm{Thick}(\mathrm{eff}\,\mathsf{C})} \xrightarrow{\sim} \mathbf{D}^b(\mathsf{C}). \qquad \square$$

Observe that $\mathrm{Thick}(\mathrm{eff}\,\mathsf{C})$ identifies with the full subcategory of complexes X in $\mathbf{D}^b(\mathrm{mod}\,\mathsf{C})$ such that $H^n(X)$ belongs to $\mathrm{eff}\,\mathsf{C}$ for all $n \in \mathbb{Z}$; see also Lemma 5.9. I am grateful to Xiao-Wu Chen for pointing out the following.

Remark 3.3. The inclusion $\mathrm{eff}\,\mathsf{C} \to \mathrm{mod}\,\mathsf{C}$ induces a functor $\mathbf{D}^b(\mathrm{eff}\,\mathsf{C}) \to \mathbf{D}^b(\mathrm{mod}\,\mathsf{C})$ which is not fully faithful in general. Examples arise by taking for C the category $\mathrm{mod}\,A$ of finite dimensional modules over a finite dimensional k-algebra A, where k is any field. Then $\mathrm{eff}\,\mathsf{C}$ identifies with $\mathrm{mod}(\underline{\mathrm{mod}}\,A)$, where $\underline{\mathrm{mod}}\,A$ denotes the stable category modulo projectives [4, §6].

4. HOMOTOPY CATEGORIES OF INJECTIVES

In this section we discuss a derived version of Auslander's formula for complexes of injective objects. More precisely, we extend the presentation of a small abelian category C via Auslander's formula to the homotopy category of injective objects of $\mathrm{Ind}\,\mathsf{C}$.

Homotopically minimal complexes. Let A be a Grothendieck abelian category. Recall from [16, Proposition B.2] that every complex I in A with injective components admits a decomposition $I = I' \amalg I''$ such that I' is homotopically minimal and I'' is null homotopic. Here, a complex J is *homotopically minimal* if for all n the inclusion $\operatorname{Ker} d_J^n \to J^n$ is an injective envelope.
Recall that a full subcategory $\mathsf{B} \subseteq \mathsf{A}$ is *localising* if B is closed under forming subobjects, quotients, extensions, and coproducts.

Lemma 4.1. *Let A be a Grothendieck abelian category and B a localising subcateory. Write $t\colon \mathsf{A} \to \mathsf{B}$ for the right adjoint of the inclusion. If a complex I with injective components in A is homotopically minimal, then tI is homotopically minimal in B.*

Proof. Observe that $\operatorname{Ker} d_{tI}^n = t(\operatorname{Ker} d_I^n)$ since t is left exact. Now use that t takes injective envelopes in A to injective envelopes in B. □

Pure acyclic complexes. Let A be a Grothendieck abelian category and fix a class C of objects that generates A and is closed under finite colimits. Throughout we identify objects in A with complexes concentrated in degeee zero.
The following is a slight generalisation of a result due to Šťovíček [29].

Proposition 4.2. *Let I be a complex of injective objects in A and suppose that $\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(X, I[n]) = 0$ for all $X \in \mathsf{C}$ and $n \in \mathbb{Z}$. Then I is null homotopic.*

The proof is based on the following lemma, where $\mathbf{C}(\mathsf{A})$ denotes the abelian category of cochain complexes in A.

Lemma 4.3 ([29]). *Let $\mathsf{Y} \subseteq \mathbf{K}(\operatorname{Inj} \mathsf{A})$ be a class of objects satisfying $\mathsf{Y} = \mathsf{Y}[1]$. Then*

$$^{\perp}\mathsf{Y} := \{X \in \mathbf{K}(\mathsf{A}) \mid \operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(X, Y) = 0 \text{ for all } Y \in \mathsf{Y}\}$$

has the following properties.

(1) *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathbf{C}(\mathsf{A})$. If two terms are in $^{\perp}\mathsf{Y}$, then the third term belongs to $^{\perp}\mathsf{Y}$.*
(2) *Let $X = \bigcup X_i$ be a directed union of subobjects $X_i \subseteq X$ in $\mathbf{C}(\mathsf{A})$. If all X_i belong to $^{\perp}\mathsf{Y}$, then X belongs to $^{\perp}\mathsf{Y}$.*

Proof. (1) Let $Y \in \mathsf{Y}$. Then the induced sequence

$$0 \to \mathcal{H}om_{\mathsf{A}}(X'', Y) \to \mathcal{H}om_{\mathsf{A}}(X, Y) \to \mathcal{H}om_{\mathsf{A}}(X', Y) \to 0$$

is exact, where $\mathcal{H}om_{\mathsf{A}}(-,-)$ denotes the usual Hom complex. Now observe that

$$\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(-, Y[n]) = H^n \mathcal{H}om_{\mathsf{A}}(-, Y).$$

(2) Let $Y \in \mathsf{Y}$ and observe that $\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(-, Y[1]) = \operatorname{Ext}^1_{\mathbf{C}(\mathsf{A})}(-, Y)$. Thus the assertion follows from Eklof's lemma [7], using that it suffices to treat well-ordered chains; see [29] for details. □

Proof of Proposition 4.2. Let Y denote the class of complexes I in $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ such that $\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(X, I[n]) = 0$ for all $X \in \mathsf{C}$ and $n \in \mathbb{Z}$. We consider ${}^{\perp}\mathsf{Y} \subseteq \mathbf{K}(\mathsf{A})$ and have $\mathsf{C} \subseteq {}^{\perp}\mathsf{Y}$ by definition. Now fix an object X in C and a subobject $U \subseteq X$ in A. Write $\bigcup U_i = U$ as a directed union of subobjects which are quotients of objects in C. Thus each object X/U_i belongs to ${}^{\perp}\mathsf{Y}$ since C is closed under cokernels. Now apply Lemma 4.3. Thus each U_i is in ${}^{\perp}\mathsf{Y}$, therefore U, and finally X/U belongs to ${}^{\perp}\mathsf{Y}$. Each object in A is a directed union of subobjects of the form X/U. Thus all objects of A belong to ${}^{\perp}\mathsf{Y}$. Clearly, this implies that all complexes in Y are null homotopic. □

Let us call a class $\mathsf{C} \subseteq \mathsf{A}$ *saturated* if it is closed under finite colimits and if there is a localising subcategory $\mathsf{B} \subseteq \mathsf{A}$ such that B is generated by C.

EXAMPLE 4.4 ([13, Theorem 2.8]). Let A be a Grothendieck abelian category. Suppose that the full subcategory $\operatorname{fp}\mathsf{A}$ of finitely presented objects is abelian and that it generates A. Then any Serre subcategory of $\operatorname{fp}\mathsf{A}$ is saturated.

The following is an immediate consequence of Proposition 4.2.

PROPOSITION 4.5. *Let A be a Grothendieck abelian category. For a saturated class C of objects and a homotopically minimal complex of injective objects I, the following are equivalent:*

(1) $\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(X, I[n]) = 0$ *for all* $X \in \mathsf{C}$ *and* $n \in \mathbb{Z}$.
(2) $\operatorname{Hom}_{\mathsf{A}}(X, I^n) = 0$ *for all* $X \in \mathsf{C}$ *and* $n \in \mathbb{Z}$.

Proof. (1) $\Rightarrow$ (2): Let B denote the localising subcategory of A generated by C and write $t\colon \mathsf{A} \to \mathsf{B}$ for the right adjoint of the inclusion. The assumption implies that $\operatorname{Hom}_{\mathbf{K}(\mathsf{B})}(X, tI[n]) = 0$ for all $X \in \mathsf{C}$ and $n \in \mathbb{Z}$. Thus tI is null homotopic by Proposition 4.2, and therefore $tI = 0$ by Lemma 4.1.
(2) $\Rightarrow$ (1): Clear since $\operatorname{Hom}_{\mathbf{K}(\mathsf{A})}(X, I[n]) = H^n\, \mathcal{H}om_{\mathsf{A}}(X, I)$. □

COMPACTLY GENERATED TRIANGULATED CATEGORIES. Let T be a triangulated category and suppose that T admits small coproducts. An object X in T is *compact* if each morphism $X \to \coprod_{i\in I} Y_i$ in T factors through $\coprod_{i\in J} Y_i$ for some finite subset $J \subseteq I$. Let T^c denote the full subcategory of compact objects and observe that T^c is a thick subcategory. Following [22], the triangulated category T is *compactly generated* if T^c is essentially small (that is, the isomorphism classes of objects form a set) and T admits no proper localising subcategory containing T^c.

EXAMPLE 4.6. For a small additive category C, the derived category $\mathbf{D}(\operatorname{Mod}\mathsf{C})$ is compactly generated with subcategory of compact objects given by $\mathbf{K}^b(\mathsf{C}) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod}\mathsf{C})^c$.

We shall need the following well-known result about Bousfield localisation for compactly generated triangulated categories.

PROPOSITION 4.7. *Let T be a compactly generated triangulated category and $\mathsf{S} \subseteq \mathsf{T}^c$ a triangulated subcategory. Then the triangulated category*

$$\mathsf{S}^{\perp} := \{Y \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(X, Y) = 0 \textit{ for all } X \in \mathsf{S}\}$$

has small coproducts and is compactly generated. Moreover, the left adjoint of the inclusion $\mathsf{S}^\perp \to \mathsf{T}$ *induces (up to direct summands) an equivalence* $\mathsf{T}^c/\mathsf{S} \xrightarrow{\sim} (\mathsf{S}^\perp)^c$.

Proof. Combine [21, Theorem 2.1] and [23, Theorem 9.1.16]. ☐

Homotopy categories of injectives. We describe the homotopy category of injective objects for Grothendieck abelian categories of the form $\operatorname{Ind}\mathsf{C}$ given by a small abelian category C. The following lemma provides the basis; it is the special case where $\operatorname{Ind}\mathsf{C}$ is replaced by $\operatorname{Mod}\mathsf{C}$.

Lemma 4.8. *Let* C *be a small abelian category. Then the canonical functor*

$$Q\colon \mathbf{K}(\operatorname{Inj}\mathsf{C}) \longrightarrow \mathbf{D}(\operatorname{Mod}\mathsf{C})$$

is a triangle equivalence which restricts to an equivalence

$$\mathbf{K}(\operatorname{Inj}\mathsf{C})^c \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\mathsf{C}). \tag{4.1}$$

Proof. Recall from [28] that the restriction of Q to the full subcategory of K-injective complexes is a triangle equivalence. Moreover, each X in $\mathbf{K}(\operatorname{Mod}\mathsf{C})$ fits into an exact triangle $aX \to X \to iX \to$ with aX acyclic and iX K-injective.
Each $F \in \operatorname{mod}\mathsf{C}$ admits a finite projective resolution (3.1) since C is abelian. Then a standard argument yields $\operatorname{Hom}_{\mathbf{K}(\operatorname{Mod}\mathsf{C})}(F, I[n]) = 0$ for all $n \in \mathbb{Z}$ and each acyclic complex I of injectives, since this holds when F is projective. Thus I is null homotopic by Proposition 4.2, and we conclude that Q is an equivalence. It remains to observe that $\mathbf{K}^b(\mathsf{C}) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\mathsf{C})$ by Lemma 3.1. ☐

Theorem 4.9. *Let* C *be a small abelian category. Then the triangulated category* $\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})$ *has small coproducts and is compactly generated. Moreover, the canonical functor* $\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \to \mathbf{D}(\operatorname{Ind}\mathsf{C})$ *induces a triangle equivalence*

$$\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})^c \xrightarrow{\sim} \mathbf{D}^b(\mathsf{C}). \tag{4.2}$$

Proof. The inclusion $\operatorname{Ind}\mathsf{C} \to \operatorname{Mod}\mathsf{C}$ identifies

$$\operatorname{Inj}\operatorname{Ind}\mathsf{C} = \{I \in \operatorname{Inj}\mathsf{C} \mid \operatorname{Hom}(X, I) = 0 \text{ for all } X \in \operatorname{eff}\mathsf{C}\}.$$

This follows from the localisation sequence in Theorem 2.3, since

$$\operatorname{Ind}\mathsf{C} = \{Y \in \operatorname{Mod}\mathsf{C} \mid \operatorname{Hom}(X, Y) = 0 = \operatorname{Ext}^1(X, Y) \text{ for all } X \in \operatorname{Eff}\mathsf{C}\}$$

by [8, III.3]. Then Proposition 4.5 implies that the inclusion $\mathbf{K}(\operatorname{Ind}\mathsf{C}) \to \mathbf{K}(\operatorname{Mod}\mathsf{C})$ identifies

$$\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) = \{I \in \mathbf{K}(\operatorname{Inj}\mathsf{C}) \mid \operatorname{Hom}(X, I[n]) = 0 \text{ for all } X \in \operatorname{eff}\mathsf{C},\ n \in \mathbb{Z}\},$$

where $\operatorname{eff}\mathsf{C} \subseteq \mathbf{D}^b(\operatorname{mod}\mathsf{C})$ is viewed as subcategory of $\mathbf{K}(\operatorname{Inj}\mathsf{C})$ via the equivalence (4.1). Now apply Proposition 4.7 and use that $\mathbf{K}(\operatorname{Inj}\mathsf{C})$ is compactly generated by Lemma 4.8. Thus $\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})$ is compactly generated, and $\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})^c$ identifies with $\mathbf{D}^b(\mathsf{C})$ thanks to Corollary 3.2. ☐

Remark 4.10. The first theorem from the introduction (labelled $\alpha = \aleph_0$) is a consequence of Theorem 4.9, since a Grothendieck abelian category A that is generated by the full subcategory $\operatorname{fp}\mathsf{A}$ of finitely presented objects is equivalent to $\operatorname{Ind}\operatorname{fp}\mathsf{A}$ via the functor $\operatorname{Ind}\operatorname{fp}\mathsf{A} \to \mathsf{A}$ induced by the inclusion $\operatorname{fp}\mathsf{A} \to \mathsf{A}$.

COROLLARY 4.11. *The inclusion* $\operatorname{Inj}\operatorname{Ind}\mathsf{C} \to \operatorname{Inj}\mathsf{C}$ *induces a functor*

$$\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \longrightarrow \mathbf{K}(\operatorname{Inj}\mathsf{C})$$

that admits a left and a right adjoint. The left adjoint makes the following diagram commutative.

$$\begin{tikzcd}
\mathbf{D}^b(\operatorname{mod}\mathsf{C}) \arrow[rr, two heads] \arrow[d, tail] & & \mathbf{D}^b(\mathsf{C}) \arrow[d, tail] \\
\mathbf{K}(\operatorname{Inj}\mathsf{C}) \arrow[rr, two heads, shift left=2] \arrow[rr, two heads, shift right=2] & & \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \arrow[ll, tail]
\end{tikzcd}$$

Proof. The left adjoint of the inclusion $F\colon \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \to \mathbf{K}(\operatorname{Inj}\mathsf{C})$ exists by construction and restricts to $\mathbf{D}^b(\operatorname{mod}\mathsf{C}) \to \mathbf{D}^b(\mathsf{C})$; see Proposition 4.7. Next observe that F preserves coproducts since its essential image are the objects perpendicular to a set of compact objects. Thus F admits a right adjoint by Brown representability. $\square$

The following consequence of Theorem 4.9 is due to Šťovíček; his proof is different and based on an analysis of fp-injective modules.

COROLLARY 4.12 ([29]). *Let A be a coherent ring. Then* $\mathbf{K}(\operatorname{Inj}A)$ *is a compactly generated triangulated category and the canonical functor* $\mathbf{K}(\operatorname{Inj}A) \to \mathbf{D}(\operatorname{Mod}A)$ *induces a triangle equivalence*

$$\mathbf{K}(\operatorname{Inj}A)^c \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}A). \qquad \square$$

FUNCTORIALITY. We consider the assignment $\mathsf{C} \mapsto \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})$ and discuss its functoriality.

Fix an additive functor $f\colon \mathsf{C} \to \mathsf{D}$ between small additive categories. Then

$$f^*\colon \operatorname{Mod}\mathsf{D} \longrightarrow \operatorname{Mod}\mathsf{C}, \quad X \mapsto X \circ f$$

admits a right adjoint f_* and a left adjoint $f_!$ [12, §5]. Note that $f_!$ extends f, that is, $f_!$ sends each representable functor $\operatorname{Hom}_{\mathsf{C}}(-, X)$ to $\operatorname{Hom}_{\mathsf{D}}(-, f(X))$. Thus $f_!$ restricts to a functor $\operatorname{Ind}\mathsf{C} \to \operatorname{Ind}\mathsf{D}$.

Now suppose that f is an exact functor between abelian categories. Then f^* restricts to a functor $\operatorname{Ind}\mathsf{D} \to \operatorname{Ind}\mathsf{C}$, since $\operatorname{Ind}\mathsf{C} = \operatorname{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$ and $\operatorname{Ind}\mathsf{D} = \operatorname{Lex}(\mathsf{D}^{\mathrm{op}}, \mathsf{Ab})$. Thus $(f_!, f^*)$ yields an adjoint pair of functors $\operatorname{Ind}\mathsf{C} \rightleftarrows \operatorname{Ind}\mathsf{D}$ and $f_!$ is exact. Therefore f^* restricts to a functor

$$\operatorname{Inj}\operatorname{Ind}\mathsf{D} \longrightarrow \operatorname{Inj}\operatorname{Ind}\mathsf{C}.$$

PROPOSITION 4.13. *An exact functor* $f\colon \mathsf{C} \to \mathsf{D}$ *induces a functor*

$$F^*\colon \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{D}) \longrightarrow \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})$$

that admits a left and a right adjoint. The left adjoint makes the following diagram commutative.

$$\begin{tikzcd}
\mathbf{D}^b(\mathsf{C}) \arrow[rr, two heads, "\mathbf{D}^b(f)"] \arrow[d, hook] & & \mathbf{D}^b(\mathsf{D}) \arrow[d, hook] \\
\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \arrow[rr, shift left=2, "F_!"] \arrow[rr, shift right=2, "F_*"'] & & \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{D}) \arrow[ll, "F^*" description]
\end{tikzcd}$$

Proof. In Corollary 4.11 the assertion has been established for the canonical functors

$$p\colon \operatorname{mod}\mathsf{C} \longrightarrow \mathsf{C} \quad \text{and} \quad q\colon \operatorname{mod}\mathsf{D} \longrightarrow \mathsf{D}.$$

Now consider the sequence $(f_!, f^*, f_*)$ of functors making the following diagram commutative.

$$\begin{tikzcd}
\operatorname{mod}\mathsf{D} \arrow[r, tail] & \operatorname{Mod}\mathsf{D} \arrow[d, "f^*" description] & \operatorname{Inj}\operatorname{Ind}\mathsf{D} \arrow[l, tail] \arrow[d] \\
\operatorname{mod}\mathsf{C} \arrow[r, tail] \arrow[u] & \operatorname{Mod}\mathsf{C} \arrow[u, shift left=3, "f_!"] \arrow[u, shift right=3, "f_*"'] & \operatorname{Inj}\operatorname{Ind}\mathsf{C} \arrow[l, tail]
\end{tikzcd}$$

We extend this diagram to complexes and obtain the following diagram.

$$\begin{tikzcd}
\mathbf{D}^b(\operatorname{mod}\mathsf{D}) \arrow[r, tail] & \mathbf{D}(\operatorname{Mod}\mathsf{D}) \arrow[r, two heads, shift left=2, "Q_!"] \arrow[r, two heads, shift right=2, "Q_*"'] \arrow[d, "f^*" description] & \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{D}) \arrow[l, tail, "Q^*" description] \arrow[d, "F^*" description] \\
\mathbf{D}^b(\operatorname{mod}\mathsf{C}) \arrow[r, tail] \arrow[u] & \mathbf{D}(\operatorname{Mod}\mathsf{C}) \arrow[u, shift left=3, "f_!"] \arrow[u, shift right=3, "\mathbf{R}f_*"'] \arrow[r, two heads, shift left=2, "P_!"] \arrow[r, two heads, shift right=2, "P_*"'] & \mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C}) \arrow[l, tail, "P^*" description] \arrow[u, shift left=3, "F_!"] \arrow[u, shift right=3, "F_*"']
\end{tikzcd}$$

Here, $\mathbf{R}f_*$ denotes the right derived functor. Thus it remains to describe the vertical functors on the right. In fact, F^* is determined by the identity $P^*F^* = f^*Q^*$. Now set $F_! := Q_! f_! P^*$ and $F_* := Q_* \mathbf{R}f_* P^*$. Then we have

$$\begin{aligned}
\operatorname{Hom}(F_!, -) &= \operatorname{Hom}(Q_! f_! P^*, -) \\
&= \operatorname{Hom}(-, P_* f^* Q^*) \\
&= \operatorname{Hom}(-, P_* P^* F^*) \\
&= \operatorname{Hom}(-, F^*)
\end{aligned}$$

and similarly

$$\operatorname{Hom}(-, F_*) = \operatorname{Hom}(F^*, -).$$

It remains to show that $F_!$ restricts on compacts to $\mathbf{D}^b(f)$, after identifying the full subcategory of compacts in $\mathbf{K}(\operatorname{Inj}\operatorname{Ind}\mathsf{C})$ with $\mathbf{D}^b(\mathsf{C})$ via (4.2). This assertion holds for $P_!$, $Q_!$, and $f_!$. Then the identity $F_! P_! = Q_! f_!$ yields the assertion for $F_!$, using that the diagram

$$\begin{tikzcd} \mathbf{K}^b(\mathsf{C}) \arrow[r, "\sim"] \arrow[d, "\mathbf{K}^b(f)"'] & \mathbf{D}^b(\operatorname{mod}\mathsf{C}) \arrow[r, two heads, "\mathbf{D}^b(p)"] \arrow[d] & \mathbf{D}^b(\mathsf{C}) \arrow[d, "\mathbf{D}^b(f)"] \\ \mathbf{K}^b(\mathsf{D}) \arrow[r, "\sim"] & \mathbf{D}^b(\operatorname{mod}\mathsf{D}) \arrow[r, two heads, "\mathbf{D}^b(q)"] & \mathbf{D}^b(\mathsf{D}) \end{tikzcd}$$

is commutative. □

5. GROTHENDIECK ABELIAN CATEGORIES

In this section we generalise the results from the previous sections to arbitrary Grothendieck abelian categories. This involves the concepts of locally presentable abelian and well-generated triangulated categories.

LOCALLY PRESENTABLE ABELIAN CATEGORIES. Grothendieck abelian categories are well-known to be locally presentable in the sense of Gabriel and Ulmer [9]. We recall this concept, refering to [1, 9] for details and unexplained terminology.

Let A be a cocomplete category and fix a regular cardinal α. An object X in A is *α-presentable* if the representable functor $\operatorname{Hom}_{\mathsf{A}}(X,-)$ preserves α-filtered colimits. We denote by A^α the full subcategory which is formed by all α-presentable objects. Observe that A^α is closed under α-small colimits in A. The category A is called *locally α-presentable* if A^α is essentially small and each object is an α-filtered colimit of α-presentable objects. Moreover, A is *locally presentable* if it is locally β-presentable for some cardinal β. Note that we have for each locally presentable category A a filtration $\mathsf{A} = \bigcup_\beta \mathsf{A}^\beta$ where β runs through all regular cardinals.

Let C be a small additive category and fix a regular cardinal α. When C has α-small colimits we write

$$\operatorname{Ind}_\alpha \mathsf{C} := \operatorname{Lex}_\alpha(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$$

for the category of left exact functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ preserving α-small products. This category is locally α-presentable. Conversely, for any locally α-presentable additive category A the assignment $X \mapsto \operatorname{Hom}_{\mathsf{A}}(-,X)|_{\mathsf{A}^\alpha}$ induces an equivalence

$$\mathsf{A} \xrightarrow{\sim} \operatorname{Ind}_\alpha \mathsf{A}^\alpha.$$

GROTHENDIECK ABELIAN CATEGORIES. We begin with a discussion of the localisation theory for Grothendieck abelian categories.

PROPOSITION 5.1. *Let A be a Grothendieck abelian category and α a regular cardinal. Suppose that A is locally α-presentable and that A^α is abelian. For a localising subcategory $\mathsf{B} \subseteq \mathsf{A}$ such that $\mathsf{B} \cap \mathsf{A}^\alpha$ generates B, the following holds:*

(1) *B and A/B are locally α-presentable Grothendieck abelian categories.*
(2) *$\mathsf{B}^\alpha = \mathsf{B} \cap \mathsf{A}^\alpha$ and the quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{B}$ induces an equivalence*

$$\mathsf{A}^\alpha/\mathsf{B}^\alpha \xrightarrow{\sim} (\mathsf{A}/\mathsf{B})^\alpha.$$

(3) *The inclusion* $\mathsf{B} \to \mathsf{A}$ *induces a localisation sequence.*

$$\begin{tikzcd} \mathsf{B}^\alpha \arrow[r, tail] \arrow[d, tail] & \mathsf{A}^\alpha \arrow[r, two heads] \arrow[d, tail] & \mathsf{A}^\alpha/\mathsf{B}^\alpha \arrow[d, tail] \\ \mathsf{B} \arrow[r, tail, shift left] & \mathsf{A} \arrow[l, two heads, shift left] \arrow[r, two heads, shift left] & \mathsf{A}/\mathsf{B} \arrow[l, tail, shift left] \end{tikzcd}$$

Proof. For the case $\alpha = \aleph_0$, see Theorems 2.6 and 2.8 of [13]. The general case is analogous; it amounts to identifying the sequence $\mathsf{B} \rightarrowtail \mathsf{A} \twoheadrightarrow \mathsf{A}/\mathsf{B}$ with the sequence $\operatorname{Ind}_\alpha \mathsf{B}^\alpha \to \operatorname{Ind}_\alpha \mathsf{A}^\alpha \to \operatorname{Ind}_\alpha(\mathsf{A}^\alpha/\mathsf{B}^\alpha)$ which is induced by $\mathsf{B}^\alpha \rightarrowtail \mathsf{A}^\alpha \twoheadrightarrow \mathsf{A}^\alpha/\mathsf{B}^\alpha$. $\square$

The Popesco–Gabriel theorem yields the following well-known[3] consequence.

Corollary 5.2. *Any Grothendieck abelian category is locally presentable. Moreover, there exists a regular cardinal α such that A^α is abelian.*

Proof. Let A be a Grothendieck abelian category with generator U and set $\Gamma := \operatorname{End}_{\mathsf{A}}(U)$. Then the functor $\operatorname{Hom}_{\mathsf{A}}(U,-)\colon \mathsf{A} \to \operatorname{Mod}\Gamma$ is fully faithful and admits an exact left adjoint $-\otimes_\Gamma U$; it is the unique colimit preserving functor sending Γ to U. This induces an equivalence $\operatorname{Mod}\Gamma/\mathsf{C} \xrightarrow{\sim} \mathsf{A}$, where $\mathsf{C} \subseteq \operatorname{Mod}\Gamma$ denotes the localising subcategory of objects annihilated by $-\otimes_\Gamma U$; see [27]. Now choose α so that $\operatorname{mod}_\alpha \Gamma$ is abelian (see Lemma 5.3 below) and contains a generator of C. Then apply Proposition 5.1. $\square$

Let C be a small additive category and fix a regular cardinal α. We write

$$\operatorname{mod}_\alpha \mathsf{C} := (\operatorname{Mod}\mathsf{C})^\alpha \qquad \text{and} \qquad \operatorname{proj}_\alpha \mathsf{C} := \operatorname{Proj}\mathsf{C} \cap \operatorname{mod}_\alpha \mathsf{C}.$$

The next lemma shows that $\operatorname{mod}_\alpha \mathsf{C}$ is abelian when α is sufficiently large.

Lemma 5.3. *The following conditions are equivalent:*

(1) *The kernel of each morphism in* $\operatorname{mod}\mathsf{C}$ *belongs to* $\operatorname{mod}_\alpha \mathsf{C}$.
(2) *The category* $\operatorname{proj}_\alpha \mathsf{C}$ *has* pseudo-kernels*, that is, for each morphism $Y \to Z$ there exists a morphism $X \to Y$ making the sequence $X \to Y \to Z$ exact.*
(3) *The category* $\operatorname{mod}_\alpha \mathsf{C}$ *is abelian.*

Proof. (1) $\Rightarrow$ (2): The objects in $\operatorname{proj}_\alpha \mathsf{C}$ are precisely the direct summands of coproducts $Y = \coprod_{i\in I} \operatorname{Hom}_{\mathsf{C}}(-,Y_i)$ with $\operatorname{card} I < \alpha$. Clearly, Y is the filtered colimit of subobjects $\coprod_{i\in J} \operatorname{Hom}_{\mathsf{C}}(-,Y_i)$ with $\operatorname{card} J < \aleph_0$. This colimit is α-small, and it follows that any morphism $Y \to Z$ in $\operatorname{proj}_\alpha \mathsf{C}$ is an α-small filtered colimit of morphisms $Y_\lambda \to Z_\lambda$ in $\operatorname{proj}_{\aleph_0} \mathsf{C} \subseteq \operatorname{mod}\mathsf{C}$. Thus

$$\operatorname{Ker}(Y \to Z) = \operatorname*{colim}_\lambda \operatorname{Ker}(Y_\lambda \to Z_\lambda)$$

belongs to $\operatorname{mod}_\alpha \mathsf{C}$. It remains to observe that each object in $\operatorname{mod}_\alpha \mathsf{C}$ is the quotient of an object in $\operatorname{proj}_\alpha \mathsf{C}$.
(2) $\Rightarrow$ (3): This follows from a standard argument [3, III.2] since each object in $\operatorname{mod}_\alpha \mathsf{C}$ is the cokernel of a morphism in $\operatorname{proj}_\alpha \mathsf{C}$.

[3]References are [9, p. 4] or [12, 9.11.3].

(3) ⇒ (1): Clear. □

When C has α-small colimits, then the Yoneda functor $\mathsf{C} \to \operatorname{mod}_\alpha \mathsf{C}$ admits a left adjoint; it is the α-small colimit preserving functor $\operatorname{mod}_\alpha \mathsf{C} \to \mathsf{C}$ taking each representable functor $\operatorname{Hom}_{\mathsf{A}}(-, X)$ to X. Let $\operatorname{eff}_\alpha \mathsf{C}$ denote the full subcategory of $\operatorname{mod}_\alpha \mathsf{C}$ consisting of the objects annihilated by this left adjoint, and set $\operatorname{Eff}_\alpha \mathsf{C} := \operatorname{Ind}_\alpha \operatorname{eff}_\alpha \mathsf{C}$.

PROPOSITION 5.4. *Let C be a small abelian category with α-small coproducts and suppose that $\operatorname{Ind}_\alpha \mathsf{C}$ is Grothendieck abelian. Then the inclusion $\operatorname{Ind}_\alpha \mathsf{C} \to \operatorname{Mod} \mathsf{C}$ induces a localisation sequence of abelian categories*

$$\operatorname{Eff}_\alpha \mathsf{C} \rightleftarrows \operatorname{Mod} \mathsf{C} \rightleftarrows \operatorname{Ind}_\alpha \mathsf{C} \tag{5.1}$$

which restricts to the localisation sequence

$$\operatorname{eff}_\alpha \mathsf{C} \rightleftarrows \operatorname{mod}_\alpha \mathsf{C} \rightleftarrows \mathsf{C}.$$

Proof. The inclusion $\operatorname{Ind}_\alpha \mathsf{C} \to \operatorname{Mod} \mathsf{C}$ has a left adjoint; it is the colimit preserving functor which is the identity on the representable functors [8, V.1]. This left adjoint is exact by the Popesco–Gabriel theorem [27], and it sends α-presentable objects to α-presentable objects, since the right adjoint preserves α-filtered colimits. This yields the left adjoint of the Yoneda functor $\mathsf{C} \to \operatorname{mod}_\alpha \mathsf{C}$. The rest follows from Proposition 5.1. □

There is an interesting consequence which seems worth mentioning.

COROLLARY 5.5. *Let A be a locally α-presentable Grothendieck abelian category such that A^α is abelian. Then*

$$\frac{\operatorname{Mod} \mathsf{A}^\alpha}{\operatorname{Eff}_\alpha \mathsf{A}^\alpha} \xrightarrow{\sim} \mathsf{A}$$

and A^β is abelian for every regular cardinal $\beta \geq \alpha$.

Proof. We have a quotient functor $Q\colon \operatorname{Mod} \mathsf{A}^\alpha \to \mathsf{A}$ by Proposition 5.4, and this yields the presentation of A. Now observe that $\operatorname{mod}_\beta \mathsf{A}^\alpha$ is abelian for all $\beta \geq \alpha$ by Lemma 5.3. Thus Q restricts to an exact quotient functor of abelian categories $\operatorname{mod}_\beta \mathsf{A}^\alpha \to \mathsf{A}^\beta$ by Proposition 5.1. □

WELL-GENERATED TRIANGULATED CATEGORIES. The triangulated analogue of a Grothendieck abelian category is a well-generated triangulated category in the sense of Neeman [23]. Such triangulated categories admit small coproducts and are α-compactly generated for some regular cardinal α. Here, we collect their essential properties and refer to [14, 23] for further details.

Fix a triangulated category T and suppose that T has small coproducts. Recall that a full triangulated subcategory $\mathsf{S} \subseteq \mathsf{T}$ is *localising* if S is closed under all coproducts. For a regular cardinal α, a full triangulated subcategory $\mathsf{S} \subseteq \mathsf{T}$ is *α-localising* if it is closed under α-small coproducts. An object X in T is called *α-small* if every morphism $X \to \coprod_{i \in I} Y_i$ in T factors through $\coprod_{i \in J} Y_i$ for some subset $J \subseteq I$ with $\operatorname{card} J < \alpha$.

A triangulated category T with small coproducts is *α-compactly generated* if there is a full subcategory T^α satisfying the following:

(1) $\mathsf{T}^\alpha \subseteq \mathsf{T}$ is an essentially small α-localising subcategory consisting of α-small objects.
(2) T admits no proper localising subcategory containing T^α.
(3) Given a family $(X_i \to Y_i)_{i\in I}$ of morphisms in T such that the induced map $\mathrm{Hom}_{\mathsf{T}}(C, X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, Y_i)$ is surjective for all $C \in \mathsf{T}^\alpha$ and $i \in I$, the induced map $\mathrm{Hom}_{\mathsf{T}}(C, \coprod_i X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \coprod_i Y_i)$ is surjective.

Then T^α is uniquely determined by (1)–(3) and the objects in T^α are called *α-compact*. Also, T is β-compactly generated for every regular cardinal $\beta \geq \alpha$, and T^β is the smallest β-localising subcategory of T containing T^α. In particular, $\mathsf{T} = \bigcup_\beta \mathsf{T}^\beta$ where β runs through all regular cardinals.
The $\aleph_0$-compactly generated triangulated categories are precisely the usual compactly generated triangulated categories.
The most important aspect of the theory is that well-generated categories behave well under localisation; this is in complete analogy to Grothendieck abelian categories.

LOCALISATION THEORY FOR WELL-GENERATED TRIANGULATED CATEGORIES. We recall the basic facts from the localisation theory for well-generated triangulated categories. For further details, see [17, 23].
The following is the analogue of Proposition 5.1 for abelian categories.

PROPOSITION 5.6. *Let T be a triangulated category and α a regular cardinal. Suppose that T is α-compactly generated. For a localising subcategory $\mathsf{S} \subseteq \mathsf{T}$ such that $\mathsf{S} \cap \mathsf{T}^\alpha$ generates S, the following holds:*

(1) *S and T/S are α-compactly generated triangulated categories.*
(2) *$\mathsf{S}^\alpha = \mathsf{S} \cap \mathsf{T}^\alpha$ and the quotient functor $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ induces (up to direct summands) a triangle equivalence $\mathsf{T}^\alpha/\mathsf{S}^\alpha \xrightarrow{\sim} (\mathsf{T}/\mathsf{S})^\alpha$.*
(3) *The inclusion $\mathsf{S} \to \mathsf{T}$ induces a localisation sequence.*

$$\begin{tikzcd} \mathsf{S}^\alpha \arrow[r, tail] \arrow[d, tail] & \mathsf{T}^\alpha \arrow[r, two heads] \arrow[d, tail] & \mathsf{T}^\alpha/\mathsf{S}^\alpha \arrow[d, tail] \\ \mathsf{S} \arrow[r, tail, shift left] & \mathsf{T} \arrow[l, two heads, shift left] \arrow[r, two heads, shift left] & \mathsf{T}/\mathsf{S} \arrow[l, tail, shift left] \end{tikzcd}$$

Proof. See Theorem 4.4.9 in [23]. □

The following generalises Proposition 4.7, which treats the case $\alpha = \aleph_0$.

PROPOSITION 5.7. *Let T be an α-compactly generated triangulated category and $\mathsf{S} \subseteq \mathsf{T}^\alpha$ a triangulated subcategory that is closed under α-small coproducts. Then the triangulated category*

$$\mathsf{S}^\perp := \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(X, Y) = 0 \textit{ for all } X \in \mathsf{S}\}$$

has small coproducts and is α-compactly generated. Moreover, the left adjoint of the inclusion $\mathsf{S}^\perp \to \mathsf{T}$ induces (up to direct summands) an equivalence $\mathsf{T}^\alpha/\mathsf{S} \xrightarrow{\sim} (\mathsf{S}^\perp)^\alpha$.

Proof. Let $\bar{\mathsf{S}} \subseteq \mathsf{T}$ denote the smallest localising subcategory containing S. Then the assertion follows from Proposition 5.6, since $\bar{\mathsf{S}} \cap \mathsf{T}^\alpha = \mathsf{S}$ and the composite $\mathsf{S}^\perp = \bar{\mathsf{S}}^\perp \rightarrowtail \mathsf{T} \twoheadrightarrow \mathsf{T}/\bar{\mathsf{S}}$ is an equivalence by [23, Theorem 9.1.16]. □

The derived category of a Grothendieck abelian category. The derived category of a Grothendieck abelian category is known to be a well-generated triangulated category [24]. For the derived categories of rings and schemes, one finds a discussion of α-compact objects in [20]. Here, we give a description of the full subcategory of α-compacts for any Grothendieck abelian category, provided the cardinal α is sufficiently large. This is based on the following special case.

Proposition 5.8. *Let C be a small additive category and $\alpha > \aleph_0$ a regular cardinal such that $\operatorname{mod}_\alpha \mathsf{C}$ is abelian. Then the derived category $\mathbf{D}(\operatorname{Mod} \mathsf{C})$ is α-compactly generated and the inclusion $\operatorname{mod}_\alpha \mathsf{C} \to \operatorname{Mod} \mathsf{C}$ induces an equivalence*

$$\mathbf{D}(\operatorname{mod}_\alpha \mathsf{C}) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} \mathsf{C})^\alpha.$$

Proof. First observe that $\operatorname{mod}_\alpha \mathsf{C}$ is an abelian category with enough projective objects. Indeed, any α-small coproduct of representable functors belongs to $\operatorname{mod}_\alpha \mathsf{C}$ and any object in $\operatorname{mod}_\alpha \mathsf{C}$ is a quotient of such a projective object. Now identify $\mathbf{D}(\operatorname{Mod} \mathsf{C})$ with the full subcategory of $\mathbf{K}(\operatorname{Proj} \mathsf{C})$ consisting of the K-projective complexes [28]. Because $\mathbf{D}(\operatorname{Mod} \mathsf{C})$ is compactly generated, the subcategory $\mathbf{D}(\operatorname{Mod} \mathsf{C})^\alpha$ identifies with the smallest α-localising subcategory containing all perfect complexes. The latter equals the full subcategory of K-projectives in $\mathbf{K}(\operatorname{proj}_\alpha \mathsf{C})$, and this in turn identifies with $\mathbf{D}(\operatorname{mod}_\alpha \mathsf{C})$. □

Let A be an abelian category and $\mathsf{B} \subseteq \mathsf{A}$ a Serre subcategory. Define the full subcategory

$$\mathbf{D}_{\mathsf{B}}(\mathsf{A}) := \{X \in \mathbf{D}(\mathsf{A}) \mid H^n(X) \in \mathsf{B} \text{ for all } n \in \mathbb{Z}\} \subseteq \mathbf{D}(\mathsf{A})$$

and note that the quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{B}$ induces a functor

$$\mathbf{D}(\mathsf{A})/\mathbf{D}_{\mathsf{B}}(\mathsf{A}) \longrightarrow \mathbf{D}(\mathsf{A}/\mathsf{B}).$$

We will need the following fact.

Lemma 5.9. *The functor $\mathbf{D}(\mathsf{A})/\mathbf{D}_{\mathsf{B}}(\mathsf{A}) \to \mathbf{D}(\mathsf{A}/\mathsf{B})$ is a triangle equivalence when the quotient functor $\mathsf{A} \to \mathsf{A}/\mathsf{B}$ admits a right adjoint.*

Proof. Let s denote right adjoint of $q\colon \mathsf{A} \to \mathsf{A}/\mathsf{B}$. The composite

$$\mathbf{K}(\mathsf{A}/\mathsf{B}) \xrightarrow{s} \mathbf{K}(\mathsf{A}) \twoheadrightarrow \mathbf{D}(\mathsf{A}) \twoheadrightarrow \mathbf{D}(\mathsf{A})/\mathbf{D}_{\mathsf{B}}(\mathsf{A})$$

annihilates each acyclic complex since $qs \cong \operatorname{Id}_{\mathsf{A}/\mathsf{B}}$. Thus s induces an exact functor $\mathbf{D}(\mathsf{A}/\mathsf{B}) \to \mathbf{D}(\mathsf{A})/\mathbf{D}_{\mathsf{B}}(\mathsf{A})$ such that $qsX \cong X$ for X in $\mathbf{D}(\mathsf{A}/\mathsf{B})$. On the other hand, for any complex Y in $\mathbf{D}(\mathsf{A})$ the cone of the adjunction morphism $Y \to sqY$ belongs to $\mathbf{D}_{\mathsf{B}}(\mathsf{A})$; thus $Y \cong sqY$ in $\mathbf{D}(\mathsf{A})/\mathbf{D}_{\mathsf{B}}(\mathsf{A})$. □

The following result describes for any Grothendieck abelian category the subcategory of α-compact objects, provided the cardinal α is sufficiently large.

THEOREM 5.10. *Let* A *be a Grothendieck abelian category and* $\alpha > \aleph_0$ *a regular cardinal. Suppose that* A^α *is abelian and generates* A. *Then the derived category* $\mathbf{D}(\mathsf{A})$ *is* α*-compactly generated and the inclusion* $\mathsf{A}^\alpha \to \mathsf{A}$ *induces a triangle equivalence* $\mathbf{D}(\mathsf{A}^\alpha) \xrightarrow{\sim} \mathbf{D}(\mathsf{A})^\alpha$.

Proof. We set $\mathsf{C} := \mathsf{A}^\alpha$ and identify $\mathsf{A} \xrightarrow{\sim} \operatorname{Ind}_\alpha \mathsf{C}$. Consider the following commutative diagram which is obtained from the pair of localisation sequences in Proposition 5.4 by forming derived categories and using Lemma 5.9.

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{eff}_\alpha \mathsf{C}}(\operatorname{mod}_\alpha \mathsf{C}) \arrow[r, tail] \arrow[d, tail] & \mathbf{D}(\operatorname{mod}_\alpha \mathsf{C}) \arrow[r, two heads] \arrow[d, tail] & \mathbf{D}(\mathsf{C}) \arrow[d, tail] \\
\mathbf{D}_{\mathrm{Eff}_\alpha \mathsf{C}}(\operatorname{Mod} \mathsf{C}) \arrow[r, tail, shift left] & \mathbf{D}(\operatorname{Mod} \mathsf{C}) \arrow[l, two heads, shift left] \arrow[r, two heads, shift left] & \mathbf{D}(\operatorname{Ind}_\alpha \mathsf{C}) \arrow[l, tail, shift left]
\end{tikzcd}$$

The assertion follows from the localisation theory for α-compactly generated triangulated categories; see Proposition 5.6. More precisely, we know from Proposition 5.8 that

$$\mathbf{D}(\operatorname{Mod} \mathsf{C})^\alpha = \mathbf{D}(\operatorname{mod}_\alpha \mathsf{C}),$$

and we need to show that $\mathbf{D}_{\mathrm{eff}_\alpha \mathsf{C}}(\operatorname{mod}_\alpha \mathsf{C})$ generates the localising subcategory $\mathbf{D}_{\mathrm{Eff}_\alpha \mathsf{C}}(\operatorname{Mod} \mathsf{C})$ of $\mathbf{D}(\operatorname{Mod} \mathsf{C})$. To see this, set $\mathsf{T} := \mathbf{D}(\operatorname{Mod} \mathsf{C})$ and let $Q\colon \operatorname{Mod} \mathsf{C} \to \mathsf{A}$ denote the exact left adjoint of the functor sending $X \in \mathsf{A}$ to $\operatorname{Hom}_{\mathsf{A}}(-, X)|_{\mathsf{C}}$ from (5.1). Consider the cohomological functor

$$H\colon \mathsf{T} \xrightarrow{H^*} \operatorname{Mod} \mathsf{C} \xrightarrow{Q} \mathsf{A}$$

and observe that H restricts to $\mathsf{T}^\alpha \to \mathsf{A}^\alpha$, since Q restricts to $\operatorname{mod}_\alpha \mathsf{C} \to \mathsf{A}^\alpha$. The kernel of H equals $\mathbf{D}_{\mathrm{Eff}_\alpha \mathsf{C}}(\operatorname{Mod} \mathsf{C})$, and it is generated by the homotopy colimits of countable sequences of morphisms in T^α annihilated by H; see [17, §7.5]. It remains to note that these homotopy colimits belong to $\mathbf{D}_{\mathrm{eff}_\alpha \mathsf{C}}(\operatorname{mod}_\alpha \mathsf{C})$ since $\alpha > \aleph_0$. □

COROLLARY 5.11. *For a Grothendieck abelian category* A, *the filtration* $\mathsf{A} = \bigcup_\alpha \mathsf{A}^\alpha$ *induces a filtration*

$$\mathbf{D}(\mathsf{A}) = \bigcup_{\alpha \text{ regular}} \mathbf{D}(\mathsf{A}^\alpha). \qquad \square$$

HOMOTOPY CATEGORIES OF INJECTIVES. In recent work of Neeman [25] it is shown that for any Grothendieck abelian category the homotopy category of injective objects is well-generated. Here we are slightly more specific and provide an analogue of Theorem 4.9 for uncountable regular cardinals.

THEOREM 5.12. *Let* A *be a Grothendieck abelian category and* $\alpha > \aleph_0$ *a regular cardinal. Suppose that* A^α *is abelian and generates* A. *Then the following holds:*

(1) *The category* $\mathbf{K}(\operatorname{Inj} \mathsf{A})$ *has small coproducts and is* α*-compactly generated.*
(2) *The left adjoint of the inclusion* $\mathbf{K}(\operatorname{Inj} \mathsf{A}) \to \mathbf{K}(\mathsf{A})$ *restricts to a quotient functor* $\mathbf{K}(\mathsf{A}^\alpha) \twoheadrightarrow \mathbf{K}(\operatorname{Inj} \mathsf{A})^\alpha$.

(3) *The canonical functor* $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{D}(\mathsf{A})$ *restricts to a quotient functor* $\mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha \twoheadrightarrow \mathbf{D}(\mathsf{A}^\alpha)$.

Proof. We adapt the proof of Theorem 4.9 using the description of A via Proposition 5.4. As before, set $\mathsf{C} := \mathsf{A}^\alpha$ and identify $\mathsf{A} \xrightarrow{\sim} \operatorname{Ind}_\alpha \mathsf{C}$. Consider

$$\operatorname{eff}_\alpha \mathsf{C} \subseteq \mathbf{D}(\operatorname{mod}_\alpha \mathsf{C}) = \mathbf{D}(\operatorname{mod}\mathsf{C})^\alpha$$

and let

$$\mathsf{S} := \operatorname{Loc}(\operatorname{eff}_\alpha \mathsf{C}) \subseteq \mathbf{D}(\operatorname{Mod}\mathsf{C})$$

denote the localising subcategory generated by $\operatorname{eff}_\alpha \mathsf{C}$. Then $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ identifies with $\mathsf{S}^\perp$ in $\mathbf{D}(\operatorname{Mod}\mathsf{C})$, and it follows from Proposition 5.7 that $\mathbf{K}(\operatorname{Inj}\mathsf{A})$ is α-compactly generated. Moreover,

$$\mathbf{D}(\operatorname{Mod}\mathsf{C})/\mathsf{S} \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj}\mathsf{A}) \qquad \text{and} \qquad \mathbf{D}(\operatorname{mod}_\alpha \mathsf{C})/\mathsf{S}^\alpha \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha.$$

Our assertions about $\mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha$ follow by inspection of the following commuting diagram.

$$\begin{tikzcd}
\mathbf{K}(\operatorname{mod}_\alpha \mathsf{C}) \arrow[rr, two heads] \arrow[dd, two heads] \arrow[dr, tail] & & \mathbf{K}(\mathsf{C}) \arrow[rr, equal] \arrow[dd, two heads] \arrow[dr, tail] & & \mathbf{K}(\mathsf{C}) \arrow[dd, two heads] \arrow[dr, tail] & \\
& \mathbf{K}(\operatorname{Mod}\mathsf{C}) \arrow[rr, two heads, shift left] \arrow[dd, two heads, shift right] & & \mathbf{K}(\mathsf{A}) \arrow[ll, tail, shift left] \arrow[rr, equal] \arrow[dd, two heads, shift right] & & \mathbf{K}(\mathsf{A}) \arrow[dd, two heads, shift right] \\
\mathbf{D}(\operatorname{mod}_\alpha \mathsf{C}) \arrow[rr, two heads] \arrow[dr, tail] & & \mathbf{K}(\operatorname{Inj}\mathsf{A})^\alpha \arrow[rr, two heads] \arrow[dr, tail] & & \mathbf{D}(\mathsf{A})^\alpha \arrow[dr, tail] & \\
& \mathbf{D}(\operatorname{Mod}\mathsf{C}) \arrow[rr, two heads, shift left] \arrow[uu, tail, shift right] & & \mathbf{K}(\operatorname{Inj}\mathsf{A}) \arrow[ll, tail, shift left] \arrow[rr, two heads, shift left] \arrow[uu, tail, shift right] & & \mathbf{D}(\mathsf{A}) \arrow[ll, tail, shift left] \arrow[uu, tail, shift right]
\end{tikzcd}$$

We omit details but explain the construction of the diagram. The two bottom rows are obtained by localising

$$\mathbf{K}(\operatorname{Inj}\mathsf{C}) = \mathbf{D}(\operatorname{Mod}\mathsf{C})$$

with respect to $\operatorname{Loc}(\operatorname{eff}_\alpha \mathsf{C})$ and $\mathbf{D}_{\operatorname{Eff}_\alpha \mathsf{C}}(\operatorname{Mod}\mathsf{C})$, restricting the left adjoints to the full subcategories of α-compact objects, and keeping in mind that

$$\operatorname{Loc}(\operatorname{eff}_\alpha \mathsf{C}) \subseteq \mathbf{D}_{\operatorname{Eff}_\alpha \mathsf{C}}(\operatorname{Mod}\mathsf{C}).$$

The two top rows follow from Proposition 5.4. The left adjoint of the inclusion $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{K}(\mathsf{A})$ is obtained by taking the composite

$$\mathbf{K}(\mathsf{A}) \rightarrowtail \mathbf{K}(\operatorname{Mod}\mathsf{C}) \twoheadrightarrow \mathbf{D}(\operatorname{Mod}\mathsf{C}) \twoheadrightarrow \mathbf{K}(\operatorname{Inj}\mathsf{A}). \qquad \square$$

Corollary 5.13. *The inclusion* $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{K}(\mathsf{A})$ *admits a left adjoint.*

Proof. The proof of Theorem 5.12 yields an explicit left adjoint; for other constructions see [18, Example 5] and [25, Theorem 2.13]. $\square$

The stable derived category. Following Buchweitz [6] and Orlov [26], we define the *stable derived category* of a Grothendieck abelian category A as the full subcategory of acyclic complexes in $\mathbf{K}(\operatorname{Inj}\mathsf{A})$. This is precisely the definition given in [16] for a locally noetherian category, and we denote this category by $\mathbf{S}(\mathsf{A})$.

Corollary 5.14. *Let A be a Grothendieck abelian category and $\alpha > \aleph_0$ a regular cardinal. Suppose that A^α is abelian and generates A. Then the stable derived category $\mathbf{S}(\mathsf{A})$ is α-compactly generated and fits into the following localisation sequence:*

$$\mathbf{S}(\mathsf{A}) \rightleftarrows \mathbf{K}(\operatorname{Inj}\mathsf{A}) \rightleftarrows \mathbf{D}(\mathsf{A})$$

Moreover, the left adjoints preserve α-compactness.

Proof. The canonical functor $\mathbf{K}(\operatorname{Inj}\mathsf{A}) \to \mathbf{D}(\mathsf{A})$ is a functor between α-compactly generated triangulated categories by Theorems 5.10 and 5.12; it determines the localisation sequence. In particular, its kernel is α-compactly generated by Proposition 5.6. □

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