On a Canonical Class of Green Currents for the Unit Sections of Abelian Schemes

For Christophe Soulé, on the occasion of his sixtieth birthday

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Abstract. We show that on any abelian scheme over a complex quasi-projective smooth variety, there is a Green current for the zero-section, which is axiomatically determined up to ∂ and $\bar{\partial}$-exact differential forms. On an elliptic curve, this current specialises to a Siegel function. We prove generalisations of classical properties of Siegel functions, like distribution relations and reciprocity laws. Furthermore, as an application of a refined version of the arithmetic Riemann-Roch theorem, we show that the above current, when restricted to a torsion section, is the realisation in analytic Deligne cohomology of an element of the (Quillen) K_1 group of the base, the corresponding denominator being given by the denominator of a Bernoulli number. This generalises the second Kronecker limit formula and the denominator 12 computed by Kubert, Lang and Robert in the case of Siegel units. Finally, we prove an analog in Arakelov theory of a Chern class formula of Bloch and Beauville, where the canonical current plays a key role.

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1 Introduction

In this article, we show that on any abelian scheme over a complex quasi-projective smooth variety, there is a Green current $\mathfrak{g}$ for the the cycle given by the zero-section of the abelian scheme, which is uniquely determined, up to ∂ and $\bar{\partial}$-exact forms, by three axioms — see Theorem 1.1.
We proceed to show that this Green current is naturally compatible with isogenies (ie it satisfies distribution relations), up to ∂ and $\bar{\partial}$-exact forms (see Theorem 1.2.1), and that it intervenes in an Arakelov-theoretic generalization of a formula of Bloch and Beauville (see [2, p. 249] for the latter), which is proven here without resorting to the Fourier-Mukai transform. See Theorem 1.2.2. Furthermore, we show that if the basis of the abelian scheme is a point (ie if the abelian scheme is an abelian variety), then $\mathfrak{g}$ is a harmonic Green current (see Theorem 1.2.3). The current $\mathfrak{g}$ is also shown to be compatible with products (see Theorem 1.2.5).
Finally, we show that the restriction of $\mathfrak{g}$ to the complement of the zero-section has a spectral interpretation. Up to a sign, it is given there by the degree $(g-1)$ part of the analytic torsion form of the Poincaré bundle of the abelian scheme. See Theorem 1.3.1 for this. In point 2 of the same theorem, we show that the restriction of the higher analytic torsion form to torsion sections, which never meet the zero-section, lies in the rational image of the Beilinson regulator from K_1 to analytic Deligne cohomology and we give a multiplicative upper bound for the denominators involved. To prove Theorem 1.3.1, we make heavy use of the arithmetic Riemann-Roch theorem in higher degrees proven in [22] and to compute the denominators described in Theorem 1.3.2, we apply the Adams-Riemann-Roch theorem in Arakelov geometry proven in [42].
If one specializes to elliptic schemes (i.e. abelian schemes of relative dimension 1) the results proven in Theorems 1.1, 1.2 and 1.3 one recovers many results contained in the classical theory of elliptic units. In particular, on elliptic schemes the current $\mathfrak{g}$ is described by a Siegel function and the spectral interpretation of $\mathfrak{g}$ specializes to the second Kronecker limit formula. The reciprocity law for elliptic units (ie the analytic description of the action of the Galois group on the elliptic units) is also easily obtained and (variants) of the results of Kubert-Lang and Robert on the fields of definition of elliptic units are recovered as a special case of the above denominator computations. Details about elliptic schemes are given in section 5 where references to the classical literature are also given. The reader will notice that even in the case of elliptic schemes, our methods of proof are quite different from the classical ones.
The current $\mathfrak{g}$ can also be used to describe the realisation in analytic Deligne cohomology of the degree 0 part of the polylogarithm on abelian schemes introduced by J. Wildeshaus in [45] (see also [25]). The fact that this should be the case was a conjecture of G. Kings. His conjecture is proven in [26].
Here is a detailed description of the results.
Let (R, Σ) be an arithmetic ring. By definition, this means that R is an excellent regular ring, which comes with a finite conjugation-invariant set Σ of

embeddings into $\mathbb{C}$ (see [19, 3.1.2]). For example R might be $\mathbb{Z}$ with its unique embedding into $\mathbb{C}$, or $\mathbb{C}$ with the identity and complex conjugation as embeddings.

Recall that an arithmetic variety over R is a scheme, which is flat and of finite type over R. In this text, all arithmetic varieties over R will also be assumed to be regular, as well as quasi-projective over R. For any arithmetic variety X over R, we write as usual

$$X(\mathbb{C}) := \coprod_{\sigma\in\Sigma}(X\times_{R,\sigma}\mathbb{C})(\mathbb{C}) = \coprod_{\sigma\in\Sigma}X(\mathbb{C})_\sigma.$$

Let $D^{p,p}(X_\mathbb{R})$ (resp. $A^{p,p}(X_\mathbb{R})$) be the $\mathbb{R}$-vector space of currents (resp. differential forms) γ on $X(\mathbb{C})$ such that

- γ is a real current (resp. differential form) of type (p,p);
- $F_\infty^*\gamma = (-1)^p\gamma$,

where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the real analytic involution given by complex conjugation. We then define

$$\widetilde{D}^{p,p}(X_\mathbb{R}) := D^{p,p}(X_\mathbb{R})/(\mathrm{im}\ \partial + \mathrm{im}\ \bar\partial)$$

(resp.

$$\widetilde{A}^{p,p}(X_\mathbb{R}) := A^{p,p}(X_\mathbb{R})/(\mathrm{im}\ \partial + \mathrm{im}\ \bar\partial) \qquad).$$

All these notations are standard in Arakelov geometry. See [44] for a compendium. It is shown in [19, Th. 1.2.2 (ii)] that the natural map $\widetilde{A}^{p,p}(X_\mathbb{R}) \to \widetilde{D}^{p,p}(X_\mathbb{R})$ is an injection.

If Z a closed complex submanifold of $X(\mathbb{C})$, we shall write more generally $D^{p,p}_Z(X_\mathbb{R})$ for the $\mathbb{R}$-vector space of currents γ on $X(\mathbb{C})$ such that

- γ is a real current of type (p,p);
- $F_\infty^*\gamma = (-1)^p\gamma$;
- the wave-front set of γ is included in the real conormal bundle of Z in $X(\mathbb{C})$.

Similarly, we then define the $\mathbb{R}$-vector spaces

$$\widetilde{D}^{p,p}_Z(X_\mathbb{R}) := D^{p,p}_Z(X_\mathbb{R})/D^{p,p}_{Z,0}(X_\mathbb{R})$$

where $D^{p,p}_{Z,0}(X_\mathbb{R})$ is the set of currents $\gamma \in D^{p,p}_Z(X_\mathbb{R})$ such that: there exists a complex current α of type $(p-1,p)$ and a complex current β of type $(p,p-1)$ such that $\gamma := \partial\alpha + \bar\partial\beta$ and such that the wave-front sets of α and β are included in the real conormal bundle of Z in $X(\mathbb{C})$.

See [24] for the definition (and theory) of the wave-front set.

It is a consequence of [11, Cor. 4.7] that the natural morphism $\widetilde{D}^{p,p}_Z(X_{\mathbb{R}}) \to \widetilde{D}^{p,p}(X_{\mathbb{R}})$ is an injection.[1] Thus the real vector space $\widetilde{D}^{p,p}_Z(X_{\mathbb{R}})$ can be identified with a subspace of the real vector space $\widetilde{D}^{p,p}(X_{\mathbb{R}})$.
Furthermore, it is a consequence of [11, Th. 4.3] that for any R-morphism $f : Y \to X$ of arithmetic varieties, there is a natural morphism of $\mathbb{R}$-vector spaces

$$f^* : \widetilde{D}^{p,p}_Z(X_{\mathbb{R}}) \to \widetilde{D}^{p,p}_{f(\mathbb{C})^*(Z)}(Y_{\mathbb{R}}),$$

provided $f(\mathbb{C})$ is transverse to Z. This morphism extends the morphism $\widetilde{A}^{p,p}(X_{\mathbb{R}}) \to \widetilde{A}^{p,p}(Y_{\mathbb{R}})$, which is obtained by pulling back differential forms.
Fix now S an arithmetic variety over R. Let $\pi : \mathcal{A} \to S$ be an abelian scheme over S of relative dimension g. We shall write as usual $\mathcal{A}^\vee \to S$ for the dual abelian scheme. Write ϵ (resp. $\epsilon^\vee$) for the zero-section of $\mathcal{A} \to S$ (resp. $\mathcal{A}^\vee \to S$) and also S_0 (resp. $S_0^\vee$) for the image of ϵ (resp. $\epsilon^\vee$). We denote by the symbol $\mathcal{P}$ the Poincaré bundle on $\mathcal{A} \times_S \mathcal{A}^\vee$. We equip the Poincaré bundle $\mathcal{P}$ with the unique metric $h_{\mathcal{P}}$ such that the canonical rigidification of $\mathcal{P}$ along the zero-section $\mathcal{A}^\vee \to \mathcal{A} \times_S \mathcal{A}^\vee$ is an isometry and such that the curvature form of $h_{\mathcal{P}}$ is translation invariant along the fibres of the map $\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} \mathcal{A}^\vee(\mathbb{C}) \to \mathcal{A}^\vee(\mathbb{C})$.
We write $\overline{\mathcal{P}} := (\mathcal{P}, h_{\mathcal{P}})$ for the resulting hermitian line bundle. Write $\overline{\mathcal{P}}^0$ be the restriction of $\mathcal{P}$ to $\mathcal{A} \times_S (\mathcal{A}^\vee \backslash S_0^\vee)$.
The aim of this text is now to prove the following three theorems.

Theorem 1.1. *There is a unique class of currents $\mathfrak{g}_{\mathcal{A}} \in \widetilde{D}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})$ with the following three properties:*

(a) *Any element of $\mathfrak{g}_{\mathcal{A}}$ is a Green current for $S_0^\vee(\mathbb{C})$.*

(b) *The identity $(S_0^\vee, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$ holds in $\widehat{\mathrm{CH}}^g(\mathcal{A}^\vee)_{\mathbb{Q}}$.*

(c) *The identity $\mathfrak{g}_{\mathcal{A}} = [n]_* \mathfrak{g}_{\mathcal{A}}$ holds for all $n \geqslant 2$.*

Here the morphism p_2 is the second projection $\mathcal{A} \times_S \mathcal{A}^\vee \to \mathcal{A}^\vee$ and $[n] : \mathcal{A}^\vee \to \mathcal{A}^\vee$ is the multiplication-by-n morphism. The symbol $\widehat{\mathrm{ch}}(\cdot)$ refers to the arithmetic Chern character and $\widehat{\mathrm{CH}}^\bullet(\cdot)$ is the arithmetic Chow group. See [19, 1.2] for the notion of Green current.
Supplement. The proof of Theorem 1.1 given below shows that if S is assumed proper over $\operatorname{Spec} R$, then the condition (b) can be replaced by the following weaker condition :
(b)' The identity of currents $\mathrm{dd}^c\mathfrak{g} + \delta_{S_0^\vee(\mathbb{C})} = (-1)^g p_{2,*}(\mathrm{ch}(\overline{\mathcal{P}}))^{(g)}$ holds.
Here $\mathrm{dd}^c := \frac{i}{2\pi}\partial\bar{\partial}$ and $\delta_{S_0^\vee(\mathbb{C})}$ is the Dirac current associated to $S_0^\vee(\mathbb{C})$ in $\mathcal{A}^\vee(\mathbb{C})$. Furthermore, $\mathrm{ch}(\overline{\mathcal{P}})$ is the Chern character form of the hermitian bundle $\overline{\mathcal{P}}$. See [20, Intro.] for this.
Remarks. (1) The condition (b) apparently makes the current $\mathfrak{g}_{\mathcal{A}}$ dependent on the arithmetic structure of $\mathcal{A}$. We shall show in 1.2.4 below that this is not

[1] many thanks to J.-I. Burgos for bringing this to our attention

the case. In particular, in defining $\mathfrak{g}_{\mathcal{A}}$, we could have assumed that $R = \mathbb{C}$. The settting of arithmetic varieties is used in Theorem 1.1 because it is the natural one for property (b). Formula (15) gives a purely analytic expression for $\mathfrak{g}_{\mathcal{A}}$. (2) It is tempting to try to refine Theorem 1.1 by using in property (b) the arithmetic Chow groups defined by J.-I. Burgos (in [10]) rather than the arithmetic Chow groups of Gillet-Soulé. One would then obtain a class of forms with certain logarithmic singularities, rather than a class of currents. Such a refinement does not seem to be easily attainable though, because of the lack of covariant functoriality of the spaces of forms mentioned in the last sentence.

The next theorem gives some properties of the class of currents $\mathfrak{g}_{\mathcal{A}}$.

Let $\mathcal{L}$ be a rigidified line bundle on $\mathcal{A}$. Endow $\mathcal{L}$ with the unique hermitian metric $h_{\mathcal{L}}$, which is compatible with the rigidification and whose curvature form is translation-invariant on the fibres of $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$. Let $\overline{\mathcal{L}} := (\mathcal{L}, h_{\mathcal{L}})$ be the resulting hermitian line bundle. Let $\phi_{\mathcal{L}} : \mathcal{A} \to \mathcal{A}^\vee$ be the polarisation morphism induced by $\mathcal{L}$.

Theorem 1.2. *1. Let $\iota : \mathcal{A} \to \mathcal{B}$ be an isogeny of abelian schemes over S. Then the identity $\iota_*^\vee(\mathfrak{g}_{\mathcal{B}}) = \mathfrak{g}_{\mathcal{A}}$ holds.*

2. Suppose that $\mathcal{L}$ is ample relatively to S and symmetric. Then the equalities

$$(S_0^\vee, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) = \frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^g)$$

are verified in $\widehat{\mathrm{CH}}^*(\mathcal{A}^\vee)_{\mathbb{Q}}$.

3. If $S \to \operatorname{Spec} R$ is the identity on $\operatorname{Spec} R$ then any element of $\mathfrak{g}_{\mathcal{A}}$ is a harmonic Green current for $S_0^\vee(\mathbb{C})$, where $\mathcal{A}^\vee(\mathbb{C})$ is endowed with a conjugation invariant Kähler metric, whose Kähler form is translation invariant.

4. The class $\mathfrak{g}_{\mathcal{A}}$ is invariant under any change of arithmetic rings $(R, \Sigma) \to (R', \Sigma')$.

5. Let $\mathcal{B} \to S$ be another abelian scheme and let $\pi_{\mathcal{A}^\vee} : \mathcal{A}^\vee \times_S \mathcal{B}^\vee \to \mathcal{A}^\vee$ (resp. $\pi_{\mathcal{B}^\vee} : \mathcal{A}^\vee \times_S \mathcal{B}^\vee \to \mathcal{B}^\vee$) be the natural projections. Then

$$\mathfrak{g}_{\mathcal{A}\times_S\mathcal{B}} = \pi_{\mathcal{A}^\vee}^*(\mathfrak{g}_{\mathcal{A}}) * \pi_{\mathcal{B}^\vee}^*(\mathfrak{g}_{\mathcal{B}}) \tag{1}$$

6. The class of currents $\mathfrak{g}_{\mathcal{A}}$ lies in $\widetilde{D}^{g-1,g-1}_{S_0^\vee(\mathbb{C})}(\mathcal{A}^\vee_{\mathbb{R}})$.

7. Let T be a an arithmetic variety over R and let $T \to S$ be a morphism of schemes over R. Let $\mathcal{A}_T$ be the abelian scheme obtained by base-change and let $\mathrm{BC} : \mathcal{A}_T \to \mathcal{A}$ be the corresponding morphism. Then $\mathrm{BC}(\mathbb{C})$ is tranverse to $S_0^\vee(\mathbb{C})$ and $\mathrm{BC}^*\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}_T}$.

Here $\iota^\vee : \mathcal{B}^\vee \to \mathcal{A}^\vee$ is the isogeny, which is dual to ι. For the notion of harmonic Green current, see [8] and [29]. The pairing $*$ appearing in the equation (1) is the $*$-product of Green currents. See [19, par. 2.2.11, p. 122] for the definition. Recall that an S-isogeny between the abelian schemes $\mathcal{A}$ and $\mathcal{B}$ is a flat and finite S-morphism $\mathcal{A} \to \mathcal{B}$, which is compatible with the group-scheme structures. The symbol $\widehat{\mathrm{c}}_1(\cdot)$ refers the first arithmetic Chern class; see [20, Intro.] for this notion.

Theorem 1.2.1 generalizes to higher degrees the distribution relations of Siegel units. See section 5 below for details. If the morphism $S \to \operatorname{Spec} R$ is the identity on $\operatorname{Spec} R$ and R is the ring of integers of a number field, then it is shown in [29, Prop. 11.1 (ii)] that Theorem 1.2.3 implies Theorem 1.2.2. Still in the situation where $S \to \operatorname{Spec} R$ is the identity on $\operatorname{Spec} R$, another construction of a Green current for $S_0^\vee(\mathbb{C})$ is described in A. Berthomieu's thesis [3]. The current constructed by Berthomieu is likely to be harmonic (it is not proven in [3], but according to the author [private communication] it can easily be shown). The current constructed in [3] satisfies the identity in Theorem 1.3.1 by construction.

The last theorem relates the current $\mathfrak{g}_{\mathcal{A}}$ to the Bismut-Köhler analytic torsion form of the Poincaré bundle (see [6, Def. 3.8, p. 668] for the definition).

Let λ be a $(1,1)$-form on $\mathcal{A}(\mathbb{C})$ defining a Kähler fibration structure on the fibration $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ (see [6, par. 1] for this notion). With the form λ, one can canonically associate a hermitian metric on the relative cotangent bundle $\Omega_{\mathcal{A}/S}$ and we shall write $\overline{\Omega}_{\mathcal{A}/S}$ for the resulting hermitian vector bundle. We suppose that λ is translation invariant on the fibres of the map $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ as well as conjugation invariant. We shall write

$$T(\lambda, \overline{\mathcal{P}}^0) \in \widetilde{A}((\mathcal{A}^\vee \backslash S_0^\vee)_{\mathbb{R}}) := \bigoplus_{p \geqslant 0} \widetilde{A}^{p,p}((\mathcal{A}^\vee \backslash S_0^\vee)_{\mathbb{R}})$$

for the Bismut-Köhler higher analytic torsion form of $\overline{\mathcal{P}}^0$ along the fibration

$$\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} (\mathcal{A}^\vee(\mathbb{C}) \backslash S_0^\vee(\mathbb{C})) \longrightarrow \mathcal{A}^\vee(\mathbb{C}) \backslash S_0^\vee(\mathbb{C}).$$

For any regular arithmetic variety X over R, the (Beilinson) regulator map gives rise to a morphism of groups

$$\mathrm{reg}_{\mathrm{an}} : K_1(X) \longrightarrow \bigoplus_{p \geqslant 0} H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p)).$$

To define the space $H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p))$ and the map $\mathrm{reg}_{\mathrm{an}}$, let us first write $H^*_{D,\mathrm{an}}(X, \mathbb{R}(\cdot))$ for the analytic real Deligne cohomology of $X(\mathbb{C})$. By definition,

$$H^q_{D,\mathrm{an}}(X, \mathbb{R}(p)) := \mathbb{H}^q(X(\mathbb{C}), \mathbb{R}(p)_{D,\mathrm{an}})$$

where $\mathbb{R}(p)_{D,\mathrm{an}}$ is the complex of sheaves of $\mathbb{R}$-vector spaces

$$0 \to \mathbb{R}(p) \to \mathcal{O}_{X(\mathbb{C})} \xrightarrow{d} \Omega^1_{X(\mathbb{C})} \to \cdots \to \Omega^{p-1}_{X(\mathbb{C})} \to 0$$

on $X(\mathbb{C})$ (for the ordinary topology). Here $\mathbb{R}(p) := (2i\pi)^p\,\mathbb{R} \subseteq \mathbb{C}$. We now define

$$H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p)) := \{\gamma \in H^{2p-1}_{D,\mathrm{an}}(X, \mathbb{R}(p)) \mid F_\infty^*\gamma = (-1)^p\gamma\}.$$

By construction, the regulator map $K_1(X) \to \oplus_{p\geqslant 0} H^{2p-1}_{D,\mathrm{an}}(X, \mathbb{R}(p))$ (see [12] for a direct construction of the regulator and further references) factors through $\oplus_{p\geqslant 0} H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p))$ and thus gives rise to a map $K_1(X) \to \oplus_{p\geqslant 0} H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p))$. This is the definition of the map $\mathrm{reg}_{\mathrm{an}}$.
It is shown in [12, par 6.1] that there is a natural inclusion $H^{2p-1}_{D,\mathrm{an}}(X_{\mathbb{R}}, \mathbb{R}(p)) \hookrightarrow \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$.
For the next theorem, define

$$N_{2g} := 2 \cdot \text{denominator}\,[(-1)^{g+1}B_{2g}/(2g)],$$

where B_{2g} is the $2g$-th Bernoulli number. Recall that the Bernoulli numbers are defined by the identity of power series:

$$\sum_{t\geqslant 1} B_t \frac{u^t}{t!} = \frac{u}{\exp(u) - 1}.$$

Theorem 1.3. *1. The class of differential forms* $\mathrm{Td}(\epsilon^*\overline{\Omega}_{\mathcal{A}/S}) \cdot T(\lambda, \overline{\mathcal{P}}^0)$ *lies in* $\widetilde{A}^{g-1,g-1}((\mathcal{A}^\vee \backslash S_0^\vee)_{\mathbb{R}})$ *and the equality*

$$\mathfrak{g}_{\mathcal{A}}|_{\mathcal{A}^\vee(\mathbb{C})\backslash S_0^\vee(\mathbb{C})} = (-1)^{g+1}\,\mathrm{Td}(\epsilon^*\overline{\Omega}_{\mathcal{A}/S}) \cdot T(\lambda, \overline{\mathcal{P}}^0)$$

holds. In particular $T(\lambda, \overline{\mathcal{P}}^0)^{(g-1)}$ *does not depend on* λ.

2. Suppose that λ *is the first Chern form of a relatively ample rigidified line bundle, endowed with its canonical metric. Let* $\sigma \in \mathcal{A}^\vee(S)$ *be an element of finite order* n*, such that* $\sigma^* S_0^\vee = \emptyset$*. Then*

$$g \cdot n \cdot N_{2g} \cdot \sigma^* T(\lambda, \overline{\mathcal{P}}^0) \in \mathrm{image}(\mathrm{reg}_{\mathrm{an}}(K_1(S))).$$

A the end of section 4.2 (see the end of the proof of Lemma 4.5), we give a statement, which is slightly stronger than Theorem 1.3.2 (but more difficult to formulate).
Theorem 1.3.1 can be viewed as a generalization to higher degrees of the second Kronecker limit formula (see [31, chap. 20, par. 5, p. 276] for the latter). Theorem 1.3.2 generalizes to higher degrees part of a classical statement on elliptic units and their fields of definition. See section 5 below.
Remark. It would be interesting to have an analogue of Theorem 1.3.2, where $\mathrm{reg}_{\mathrm{an}}$ is replaced by the analytic cycle class $\mathrm{cyc}_{\mathrm{an}}$ (see (3) below). If $S \simeq \mathrm{Spec}\,R$ and R is the ring of integers in a number field, then $\mathrm{reg}_{\mathrm{an}}$ and $\mathrm{cyc}_{\mathrm{an}}$ can be identified but this is not true in general. In particular, this suggests

that the Bernoulli number $12 = N_2/2$, which appears in the denominators of elliptic units (see the last section) should be understood as coming from the natural integral structure of the group $K_1(\cdot)_{\mathbb{Q}}$ and not from the natural integral structure of the corresponding motivic cohomology group $\bigoplus_p H^{2p-1}_{\mathcal{M}}(\cdot, \mathbb{Z}(p))_{\mathbb{Q}}$. Some of the results of this article were announced in [37].

We shall provide many bibliographical references to ease the reading but the reader of this article is nevertheless assumed to have some familiarity with the language of Arakelov theory, as expounded for instance in [44].

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Notations. Here are the main notational conventions. Some of them have already been introduced above. Recall that we wrote $\pi : \mathcal{A} \to S$ for the structure morphism of the abelian scheme $\mathcal{A}$ over S. We also write $\pi^\vee : \mathcal{A}^\vee \to S$ for the structure morphism of the abelian scheme $\mathcal{A}^\vee$ over S. Write $\mu = \mu_{\mathcal{A}} : \mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$ for the addition morphism and $p_1 : \mathcal{A} \times \mathcal{A}^\vee \to \mathcal{A}$, $p_2 : \mathcal{A} \times \mathcal{A}^\vee \to \mathcal{A}^\vee$ for the obvious projections. We shall also also write $\underline{\mathrm{p}}_1, \underline{\mathrm{p}}_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\underline{\mathrm{p}}_1^\vee, \underline{\mathrm{p}}_2^\vee : \mathcal{A}^\vee \times \mathcal{A}^\vee \to \mathcal{A}^\vee$ for more obvious projections. Recall that we wrote ϵ (resp. $\epsilon^\vee$) for the zero-section of $\mathcal{A} \to S$ (resp. $\mathcal{A}^\vee \to S$) and also that we wrote S_0 (resp. $S_0^\vee$) for the image of ϵ (resp. $\epsilon^\vee$). Write $\omega_{\mathcal{A}} := \det(\Omega_{\mathcal{A}/S})$ for the determinant of the sheaf of differentials of $\mathcal{A}$ over S. We let $\mathrm{dd}^c := \frac{i}{2\pi}\partial\bar{\partial}$.

2 Proof of Theorem 1.1

If M is a smooth complex quasi-projective variety, we shall write $H^*_D(M, \mathbb{R}(\cdot))$ for the Deligne-Beilinson cohomology of M. We recall its definition. Let $\bar{M}$ be a smooth complex projective variety, which contains M as an open subscheme. We call $\bar{M}$ a compactfication of M. Suppose furthermore that $\bar{M} \backslash M$ is the underlying set of a reduced divisor with normal crossings D. From now on, we view M and $\bar{M}$ as complex analytic spaces and we work in the category of complex analytic spaces. Let $j : M \hookrightarrow \bar{M}$ be the given open embedding. There is a natural subcomplex $\Omega^\bullet_{\bar{M}}(\log D)$ of $j_*\Omega^\bullet_M$, called the *complex of holomorphic differential forms on M with logarithmic singularities along D*. The objects of $\Omega^\bullet_{\bar{M}}(\log D)$ are locally free sheaves. We redirect the reader to [9, chap. 10] for the definition and further bibliographical references. Write $F^p\Omega^\bullet_{\bar{M}}(\log D)$ for the subcomplex

$$\Omega^p_{\bar{M}}(\log D) \longrightarrow \Omega^{p+1}_{\bar{M}}(\log D) \longrightarrow \cdots$$

of $\Omega^\bullet_{\bar{M}}(\log D)$. Write $f^0_p : F^p\Omega^\bullet_{\bar{M}}(\log D) \to j_*\Omega^\bullet_M$ for the inclusion morphism. Abusing notation, we shall identify $\mathrm{R}j_*\mathbb{R}(p)$ with the complex, which is the image by j_* of the canonical flasque resolution of $\mathbb{R}(p)$. Similarly, we shall write $\mathrm{R}j_*\Omega^\bullet_M$ for the simple complex associated to the image by j_* of the canonical flasque resolution of $\Omega^\bullet_M$ (the latter being a double complex). Write $f_p : F^p\Omega^\bullet_{\bar{M}}(\log D) \to \mathrm{R}j_*\Omega^\bullet_M$ for the morphism obtained by composing f^0_p with the canonical morphism $j_*\Omega^\bullet_M \to \mathrm{R}j_*\Omega^\bullet_M$. There is a natural morphism of complexes $\mathbb{R}(p) \to \Omega^\bullet_M$ (where $\mathbb{R}(p)$ is viewed as a complex with one object sitting in degree 0) and by the functoriality of the flasque resolution, we obtain a morphism $r_p : \mathrm{R}j_*\mathbb{R}(p) \to \mathrm{R}j_*\Omega^\bullet_M$. We now define the complex

$$\mathbb{R}(p)_D := \mathrm{simple}\big(\mathrm{R}j_*\mathbb{R}(p) \oplus F^p\Omega^\bullet_{\bar{M}}(\log D) \xrightarrow{u_p} \mathrm{R}j_*\Omega^\bullet_M\big)$$

where $u_p := r_p - f_p$. By definition, Deligne-Beilinson cohomology is now defined by the formula

$$H^q_D(M, \mathbb{R}(p)) := \mathbb{H}^q(M, \mathbb{R}(p)_D).$$

Notice that by construction, $H^q_D(M, \mathbb{R}(p)) = H^q_{D,\mathrm{an}}(M, \mathbb{R}(p))$ if M is compact (so that D is empty). More generally, there is a natural "forgetful" morphism of $\mathbb{R}$-vector spaces $H^q_D(M, \mathbb{R}(p)) \to H^q_{D,\mathrm{an}}(M, \mathbb{R}(p))$ (what is forgotten is the logarithmic structure) ; see [10, before Prop. 1.3] for this. It can be proven that Deligne-Beilinson cohomology does not depend on the choice of the compactification $\bar{M}$. By its very definition, we have a canonical long exact sequence of $\mathbb{R}$-vector spaces

$$\cdots \to H^{q-1}(M, \mathbb{C}) \to H^q_D(M, \mathbb{R}(p)) \to H^q(M, \mathbb{R}(p)) \oplus F^pH^q(M, \mathbb{C}) \to \cdots \quad (2)$$

where $F^pH^q(M, \mathbb{C})$ is the p-th term of the Hodge filtration of the mixed Hodge structure on $H^q(M, \mathbb{C})$. Furthermore the $\mathbb{R}$-vector space $H^q_D(M, \mathbb{R}(p))$ has a natural structure of contravariant functor from the category of smooth quasi-projective varieties over $\mathbb{C}$ to the category of $\mathbb{R}$-vector spaces (see [10, Prop. 1.3] for this). If we equip the singular cohomology spaces $H^q(M, \mathbb{C})$ and $H^q_D(M, \mathbb{R}(p))$ with their natural contravariant structure, then the sequence (2) becomes an exact sequence of functors.
If X is an arithmetic variety, then we define

$$H^q_D(X_\mathbb{R}, \mathbb{R}(p)) := \{\gamma \in H^q_D(X(\mathbb{C}), \mathbb{R}(p)) \mid F^*_\infty\gamma = (-1)^p\gamma\}.$$

Before beginning with the proof of Theorem 1.1, we shall prove the following key lemma.

Lemma 2.1. *Let $n \geqslant 2$. The eigenvalues of the $\mathbb{R}$-endomorphism $[n]^*$ of the Deligne-Beilinson cohomology $\mathbb{R}$-vector space $H^{2p-1}_D(\mathcal{A}^\vee(\mathbb{C}), \mathbb{R}(p))$ lie in the set $\{1, n, n^2, \ldots, n^{2p-1}\}$.*

Proof. The existence of the exact sequence of functors (2) shows that we have the following exact sequence of $\mathbb{R}$-vector spaces

$H^{2p-2}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{C}) \to H^{2p-1}_D(\mathcal{A}^\vee(\mathbb{C}), \mathbb{R}(p)) \to$

$$\to H^{2p-1}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{R}(p)) \oplus F^p H^{2p-1}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{C})$$

a and that the differentials in this sequence are compatible with the natural contravariant action of $[n]$. Hence it is sufficient to prove the conclusion of the lemma for the $\mathbb{R}$-vector spaces $H^{2p-2}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{C})$, $H^{2p-1}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{R}(p))$ and $H^{2p-1}(\mathcal{A}^\vee(\mathbb{C}), \mathbb{C})$. These spaces are more easily tractable and can be approximated by the Leray spectral sequence

$$E_2^{rs} = H^r(S(\mathbb{C}), \mathrm{R}^s\pi^\vee(\mathbb{C})_*(K)) \Rightarrow H^{r+s}(\mathcal{A}^\vee(\mathbb{C}), K)$$

where K is $\mathbb{R}$ or $\mathbb{C}$. Now notice that since $[n]$ is an S-morphism, this spectral sequence carries a natural contravariant action of $[n]$, which is compatible with the aforementionned action of $[n]$ on its abutment. Consider the index $2p-1$. We know that $[n]^*$ acts on $\mathrm{R}^s\pi^\vee(\mathbb{C})_*(K)$ by multiplication by n^s. This may be deduced from known results on abelian varieties using the proper base-change theorem. Hence $[n]^*$ acts on $H^r(S(\mathbb{C}), \mathrm{R}^s\pi^\vee(\mathbb{C})_*(K))$ by multiplication by n^s as well. Now the existence of the spectral sequence shows that $H^{2p-1}(\mathcal{A}^\vee(\mathbb{C}), K)$ has a natural filtration, which consists of subquotients of the spaces $H^r(S(\mathbb{C}), \mathrm{R}^s\pi^\vee(\mathbb{C})_*(K))$, where $r+s = 2p-1$. Since $s \leqslant 2p-1$, this proves the assertion for the index $2p-1$. The index $2p-2$ can be treated in an analogous fashion. $\square$

Proof of uniqueness. Let $\mathfrak{g}_{\mathcal{A}}$ and $\mathfrak{g}^0_{\mathcal{A}}$ be elements of $\widetilde{D}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})$ satisfying (a), (b) and (c). Let $\kappa_{\mathcal{A}} := \mathfrak{g}^0_{\mathcal{A}} - \mathfrak{g}_{\mathcal{A}} \in \widetilde{D}^{g-1,g-1}(\mathcal{A}^\vee)$ be the error term.
Recall the fundamental exact sequence

$$\mathrm{CH}^{g,g-1}(\mathcal{A}^\vee) \xrightarrow{\mathrm{cyc}_{\mathrm{an}}} \widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}^g(\mathcal{A}^\vee) \to \mathrm{CH}^g(\mathcal{A}^\vee) \to 0 \tag{3}$$

(see [19, th. 3.3.5] for this). Here $\mathrm{CH}^{g,g-1}(\cdot)$ is Gillet-Soulé's version of one of Bloch's higher Chow groups. The group $\widehat{\mathrm{CH}}^g(\mathcal{A}^\vee)$ is the g-th arithmetic Chow group and $\mathrm{CH}^g(\mathcal{A}^\vee)$ is the g-th ordinary Chow group. By construction, there are maps

$$\mathrm{CH}^{g,g-1}(\mathcal{A}^\vee) \xrightarrow{\mathrm{cyc}} H_D^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} H_{D,\mathrm{an}}^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g)) \to \widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})$$

whose composition is $\mathrm{cyc}_{\mathrm{an}}$. Here cyc is the cycle class map into Deligne-Beilinson cohomology; the second map from the left is the forgetful map and the third one is the natural inclusion mentioned before Theorem 1.3.
Now let $n \geqslant 2$. Let

$$V := \mathrm{image}\big(H_D^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} H_{D,\mathrm{an}}^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g))\big) \tag{4}$$

be the image of the forgetful map from $H_D^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g))$ to $H_{D,\mathrm{an}}^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g))$. By (b), we know that $\kappa_{\mathcal{A}} \in V$. Furthermore V is invariant under $[n]^*$. In fact, by Lemma 2.1, $[n]^*$ restrict to an injective morphism $V \to V$, which is thus an isomorphism, since V is finite dimensional. Now the projection formula shows that the equation $[n]_*[n]^* = n^{2g}$ is valid in $H_{D,\mathrm{an}}^{2g-1}(\mathcal{A}^\vee_{\mathbb{R}}, \mathbb{R}(g))$ and we conclude that that V is also invariant under $[n]_*$. The same equation $[n]_*[n]^* = n^{2g}$ now shows that the eigenvalues of $[n]_*$ on V lie in the set $\{n^{2g}, n^{2g-1}, \dots, n\}$. In particular, $[n]_*$ has no non-vanishing fixed point in V. Since $[n]_*\kappa_{\mathcal{A}} = \kappa_{\mathcal{A}}$ by (c), this proves that $\kappa_{\mathcal{A}} = 0$.

Proof of existence. As very often, the proof of existence is inspired by the proof of uniqueness. Let $\mathfrak{g} \in \widetilde{D}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})$ be a class of Green currents for $S_0^\vee$ satisfying (b). To see that there is such a $\mathfrak{g}$, pick any Green current $\mathfrak{g}'$ for $S_0^\vee(\mathbb{C})$, such that $F_\infty^*\mathfrak{g}' = (-1)^{g-1}\mathfrak{g}'$. This exists by [19, th.1.3.5]. Now a basic property of the Fourier-Mukai transformation for abelian schemes (see [32, Lemme 1.2.5]) implies that $(-1)^g p_{2,*}(\mathrm{ch}(\mathcal{P}))^{(g)} = S_0^\vee$ in $\mathrm{CH}^g(\mathcal{A}^\vee)_{\mathbb{Q}}$. Hence, looking at the sequence (3), we see that there exists $\alpha \in \widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})$ such that $(a\otimes\mathbb{Q})(\alpha) = (S_0^\vee, \mathfrak{g}') - (-1)^g p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$. If we define $\mathfrak{g} := \mathfrak{g}' - \alpha$, we obtain the required class of Green currents. Now fix $n \geqslant 2$ and let $c := \mathfrak{g} - [n]_*\mathfrak{g}$. We shall prove below (see (6)) that

$$[n]_* p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} = p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}.$$

This implies that c lies in the space V defined in (4) above. Now recall that we proved that $[n]_*$ sends V on V and that 1 is not a root of the characteristic polynomial of $[n]_*$ as an endomorphism of V. Hence the linear equation in x

$$x - [n]_* x = c$$

has a unique solution in V. Call this solution c_0. By construction the current $\mathfrak{g}_0 := \mathfrak{g} + c_0$ satisfies the equation $\mathfrak{g}_0 - [n]_*\mathfrak{g}_0 = 0$. Now let $m \geqslant 2$ be another natural number. We have seen above that $\mathfrak{g}_0 - [m]_*\mathfrak{g}_0$ lies in V. On the other hand

$$[n]_*(\mathfrak{g}_0 - [m]_*\mathfrak{g}_0) = [n]_*\mathfrak{g}_0 - [m]_*[n]_*\mathfrak{g}_0 = \mathfrak{g}_0 - [m]_*\mathfrak{g}_0$$

hence $\mathfrak{g}_0 - [m]_*\mathfrak{g}_0$ is a fixed point of $[n]_*$ in V. This implies that $\mathfrak{g}_0 - [m]_*\mathfrak{g}_0 = 0$. This proves that $\mathfrak{g}_0$ satisfies (a), (b) and (c).

3 Proof of Theorem 1.2

3.1 Proof of 1.2.1

By the definition of the dual isogeny, there is a diagram

$$\begin{tikzcd} \mathcal{A} \times_S \mathcal{B}^\vee \arrow[r, "\iota \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \iota^\vee"'] & \mathcal{B} \times_S \mathcal{B}^\vee \arrow[r] & \mathcal{B}^\vee \arrow[d, "\iota^\vee"] \\ \mathcal{A} \times_S \mathcal{A}^\vee \arrow[rr] & & \mathcal{A}^\vee \end{tikzcd}$$

such that

$$(\mathrm{Id} \times \iota^\vee)^*\mathcal{P}_{\mathcal{A}} \simeq (\iota \times \mathrm{Id})^*\mathcal{P}_{\mathcal{B}}$$

and such that the outer square is cartesian. Here $\mathcal{P}_{\mathcal{A}} := \mathcal{P}$ and $\mathcal{P}_{\mathcal{B}}$ is the Poincaré bundle of $\mathcal{B}$ over S.
First notice that the arithmetic Riemann-Roch theorem [22] implies that

$$\widehat{\mathrm{ch}}((\iota \times \mathrm{Id})_*(\overline{\mathcal{O}}_{\mathcal{A}\times_S \mathcal{B}^\vee})) = \deg\ \iota$$

in $\widehat{\mathrm{CH}}^\bullet(\mathcal{B} \times_S \mathcal{B}^\vee)_{\mathbb{Q}}$.
Now we compute

$$\begin{aligned} \iota^{\vee,*} p_{2*}^{\mathcal{A}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}})) &= p_{2*}^{\mathcal{B}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{B}})\widehat{\mathrm{ch}}((\iota \times \mathrm{Id})_*(\overline{\mathcal{O}}_{\mathcal{A}\times_S \mathcal{B}^\vee}))) \\ &= (\deg\ \iota)\cdot p_{2*}^{\mathcal{B}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{B}})). \end{aligned} \tag{5}$$

Here we used the projection formula for arithmetic Chow theory (see [19, Th. 4.4.3, 7.]) and the fact that the push-forward map in arithmetic Chow theory commutes with base-change. We may now compute

$$\iota_*^\vee \iota^{\vee,*} p_{2*}^{\mathcal{A}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}})) = (\deg\ \iota)\cdot p_{2*}^{\mathcal{A}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}})) = (\deg\ \iota)\cdot \iota_*^\vee p_{2*}^{\mathcal{B}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{B}}))).$$

In other words, we have

$$p_{2*}^{\mathcal{A}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}})) = \iota_*^\vee p_{2*}^{\mathcal{B}}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{B}}))). \tag{6}$$

Furthermore, since $\iota^\vee$ restricts to an isomorphism between the zero-sections, the class of currents $\iota_*^\vee(\mathfrak{g}_{\mathcal{B}})$ consists of Green currents for $S_{0,\mathcal{A}}^\vee$. All this shows that $\mathfrak{g}_{\mathcal{A}} - \iota_*^\vee(\mathfrak{g}_{\mathcal{B}})$ lies inside the space V defined in 4. Recall that V is the image of the forgetful map $H_D^{2g-1}(\mathcal{A}_{\mathbb{R}}^\vee, \mathbb{R}(g)) \xrightarrow{\text{forgetful}} H_{D,\mathrm{an}}^{2g-1}(\mathcal{A}_{\mathbb{R}}^\vee, \mathbb{R}(g))$. To conclude, notice that for any $n \geqslant 2$, we have

$$[n]_*(\mathfrak{g}_{\mathcal{A}} - \iota_*^\vee(\mathfrak{g}_{\mathcal{B}})) = \mathfrak{g}_{\mathcal{A}} - \iota_*^\vee(\mathfrak{g}_{\mathcal{B}})$$

since $[n]$ commutes with $\iota^\vee$. It was shown just before the proof of existence in the proof of Theorem 1.1 that $[n]_*$ leaves V invariant and has no non-vanishing fixed points in V. Thus $\mathfrak{g}_{\mathcal{A}} - \iota_*^\vee(\mathfrak{g}_{\mathcal{B}}) = 0$.

3.2 Proof of 1.2.2

We shall prove the equivalent identities

$$\frac{(-1)^g}{g!\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^g) = p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} \tag{7}$$

and

$$p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(k)} = 0 \tag{8}$$

if $k \neq g$.
For the equality (8), notice that in view of (5) and the fact that $[n]^\vee = [n]$, we have

$$[n]^*(p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))) = n^{2g}\cdot p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) \tag{9}$$

for any $n \geqslant 2$. On the other hand, since $(\mathrm{Id} \times [n])^*\bar{\mathcal{P}} = \bar{\mathcal{P}}^{\otimes n}$, we have also

$$[n]^*(p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))) = \sum_{k\geqslant 0} n^{k+g} \cdot p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(k)} \tag{10}$$

and comparing equations (9) and (10) as polynomials in n proves equation (8).
We now proceed to the proof of equation (7). Notice that the line bundle $\mu^*\mathcal{L}\otimes \underline{\mathrm{p}}_1^*\mathcal{L}^\vee \otimes \underline{\mathrm{p}}_2^*\mathcal{L}^\vee$ on $\mathcal{A}\times_S\mathcal{A}$ carries a natural rigidification on the zero section $\mathcal{A} \xrightarrow{(\mathrm{Id},\epsilon)} \mathcal{A} \times_S \mathcal{A}$ and that the same line bundle is algebraically equivalent to 0 on each geometric fibre of the morphism $\underline{\mathrm{p}}_2 : \mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$. Hence there is a unique morphism $\phi_{\mathcal{L}} : \mathcal{A} \to \mathcal{A}^\vee$, the polarisation morphism induced by $\mathcal{L}$, such that there is an isomorphism of rigidified line bundles

$$(\mathrm{Id} \times \phi_{\mathcal{L}})^*\mathcal{P} \simeq \mu^*\mathcal{L} \otimes \underline{\mathrm{p}}_1^*\mathcal{L}^\vee \otimes \underline{\mathrm{p}}_2^*\mathcal{L}^\vee. \tag{11}$$

Furthermore, if we endow the line bundles on both sides of (11) with their natural metrics, this isomorphism becomes an isometry, because both line bundles carry metrics that are compatible with the rigidification and the curvature forms of both sides are translation invariant (in fact 0) on the fibres of the map $\underline{\mathrm{p}}_2(\mathbb{C})$.
We shall now give a more concrete expression for

$$\underline{\mathrm{p}}_{2*}(\widehat{\mathrm{ch}}(\mu^*\overline{\mathcal{L}})\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_1^*\overline{\mathcal{L}}^\vee)\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_2^*\overline{\mathcal{L}}^\vee)).$$

We first make the calculation

$$\begin{aligned}&\underline{\mathrm{p}}_{2*}(\widehat{\mathrm{ch}}(\mu^*\overline{\mathcal{L}})\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_1^*\overline{\mathcal{L}}^\vee)\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_1^*\overline{\mathcal{L}})) = \underline{\mathrm{p}}_{2*}(\widehat{\mathrm{ch}}(\mu^*\overline{\mathcal{L}}))\\ &= \underline{\mathrm{p}}_{2*}(\alpha^*\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}})) = \underline{\mathrm{p}}_{2*}(\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}}))\end{aligned}$$

where $\alpha : \mathcal{A} \times_S \mathcal{A} \to \mathcal{A} \times_S \mathcal{A}$ is the $\underline{\mathrm{p}}_2$-automorphism $\alpha := (\mu, \underline{\mathrm{p}}_2)$. Now notice that for any $n \geqslant 2$,

$$\begin{aligned}&\underline{\mathrm{p}}_{2*}(([n]\times \mathrm{Id})_*([n]\times \mathrm{Id})^*\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}})) = n^{2g}\underline{\mathrm{p}}_{2*}(\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}}))\\ = \quad &\underline{\mathrm{p}}_{2*}(([n]\times \mathrm{Id})^*\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}})) = \underline{\mathrm{p}}_{2*}(\sum_{l\geqslant 1} n^{2l}(\underline{\mathrm{p}}_1\widehat{\mathrm{ch}}(\overline{\mathcal{L}}))^{(l)}).\end{aligned}$$

Here we used the isometric isomorphism $[n]^*\overline{\mathcal{L}} \simeq \overline{\mathcal{L}}^{\otimes n^2}$ (recall that $\mathcal{L}$ is symmetric). We deduce that

$$\underline{\mathrm{p}}_{2*}(\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}})) = \underline{\mathrm{p}}_{2*}(\underline{\mathrm{p}}_1^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}})^{(g)}) = \sqrt{\deg(\phi_{\mathcal{L}})}$$

Thus, using the projection formula, we see that

$$\underline{\mathrm{p}}_{2*}(\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_1^*\overline{\mathcal{L}})\widehat{\mathrm{ch}}(\mu^*\overline{\mathcal{L}})\widehat{\mathrm{ch}}(\underline{\mathrm{p}}_1^*\overline{\mathcal{L}}^\vee)\underline{\mathrm{p}}_2^*\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^\vee)) = \sqrt{\deg(\phi_{\mathcal{L}})}\,\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^\vee)$$

which implies that

$$p_{2*}(\widehat{\mathrm{ch}}(p_1^*\overline{\mathcal{L}})\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) = \frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^\vee).$$

Now notice that in the proof of equation (7), we may assume without restriction of generality that $\mathcal{L}$ is relatively generated by its sections, which is to say that the natural morphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective. Indeed, for any $n \geqslant 2$, we have

$$\frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}^{\otimes n}})}}\phi_{\mathcal{L}^{\otimes n},*}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}^{\otimes n})^g) = \frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}([n]_*\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^g)$$

and for any $k \geqslant 0$,

$$[n]_*\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^k = n^{-2k}[n]_*[n]^*\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^k = n^{2g-2k}\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^k. \tag{12}$$

Hence

$$\frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}^{\otimes n}})}}\phi_{\mathcal{L}^{\otimes n},*}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}^{\otimes n})^g) = \frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^g).$$

We may thus harmlessly replace $\overline{\mathcal{L}}$ by $\overline{\mathcal{L}}^{\otimes n}$, where n is some large positive integer. In particular, we may assume (and we do) that the morphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective, since $\mathcal{L}$ is relatively ample. Now let $\mathcal{E} := \pi^*\pi_*\mathcal{L} \otimes \mathcal{L}^\vee$ and let

$$P_0^\bullet : \cdots \to \Lambda^r(\mathcal{E}) \to \Lambda^{r-1}(\mathcal{E}) \to \cdots \to \mathcal{E} \to \mathcal{O} \to 0$$

be the associated Koszul resolution. Let

$$P_1^\bullet : 0 \to \mathcal{P} \to p_1^*\mathcal{E}^\vee \otimes \mathcal{P} \to \cdots \to p_1^*\Lambda^{r-1}(\mathcal{E})^\vee \otimes \mathcal{P} \to p_1^*\Lambda^r(\mathcal{E})^\vee \otimes \mathcal{P} \to \cdots$$

be the complex $\mathcal{P} \otimes p_1^*(P_0^\bullet)^\vee$. All the bundles appearing in the complex $P_1^\bullet$ have natural hermitian metrics and we let $\eta_{\bar{P}_1}$ be the corresponding Bott-Chern class. Notice the equalities

$$\eta_{\bar{P}_1} = \widehat{\mathrm{ch}}(\Lambda_{-1}(\overline{\mathcal{E}}^\vee))\widehat{\mathrm{ch}}(\overline{\mathcal{P}}) = \widehat{c}^{\mathrm{top}}(\overline{\mathcal{E}})\widehat{\mathrm{Td}}^{-1}(\overline{\mathcal{E}})\widehat{\mathrm{ch}}(\overline{\mathcal{P}})$$

in $\widehat{\mathrm{CH}}^\bullet(\mathcal{A}\times_S\mathcal{A}^\vee)$ (see [5, last paragraph]). Here $\Lambda_{-1}(\overline{\mathcal{E}}^\vee)$ is the formal $\mathbb{Z}$-linear combination $\sum_{r\geqslant 0}(-1)^r\Lambda^r(\overline{\mathcal{E}}^\vee)$. Since $\mathrm{rk}(\mathcal{E})$ may be assumed arbitrarily large (since we may replace $\mathcal{L}$ by some of its tensor powers), we see that we may assume that $\eta_{\bar{P}_1} = 0$ in $\widehat{\mathrm{CH}}^{\bullet}(\mathcal{A} \times_S \mathcal{A}^\vee)_{\mathbb{Q}}$. Thus we may compute

$$\begin{aligned}
& p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) = p_{2*}(\widehat{\mathrm{ch}}[-p_1^*\Lambda_{-1}(\overline{\mathcal{E}}^\vee) + \overline{\mathcal{O}}]\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) \\
&= -\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r p_{2*}[\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}})^\vee))\widehat{\mathrm{ch}}(p_1^*\overline{\mathcal{L}}^{\otimes r})\widehat{\mathrm{ch}}(\overline{\mathcal{P}})] \\
&= -\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r p_{2*}[\widehat{\mathrm{ch}}(p_1^*\overline{\mathcal{L}}^{\otimes r})\widehat{\mathrm{ch}}(\overline{\mathcal{P}})]\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee) \\
&= -\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r \frac{1}{\sqrt{\deg(\phi_{\mathcal{L}^{\otimes r}})}} \phi_{\mathcal{L}^{\otimes r},*}(\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee,\otimes r}))\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee) \\
&= -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r r^{-g}\big([r]_*(\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee,\otimes r}))\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)\big)\Big] \\
&= -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r r^{-g}\big([\sum_{s\geqslant 0} r^{2g-2s}\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee,\otimes r})^{(s)}]\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)\big)\Big] \\
&= -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} \sum_{s\geqslant 0} (-1)^r r^{g-s}\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee})^{(s)}\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)\Big].
\end{aligned}$$

Now notice that the expression

$$\begin{aligned}
& [n]_* p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) = p_{2*}((\mathrm{Id} \times [n])_*\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) \\
&= p_{2*}((\mathrm{Id} \times [n])_*(\mathrm{Id} \times [n])^* \sum_{k\geqslant 0} n^{-k}\widehat{\mathrm{ch}}(\overline{\mathcal{P}})^{(k)}) = p_{2*}(\sum_{k\geqslant 0} n^{2g-k}\widehat{\mathrm{ch}}(\overline{\mathcal{P}})^{(k)})
\end{aligned}$$

is a Laurent polynomial in $n \geqslant 2$. The equation (12) shows that the expression

$$[n]_*\Big(-\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} \sum_{s\geqslant 0} (-1)^r r^{g-s}\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee})^{(s)}\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)\Big]\Big)$$

is also a Laurent polynomial in $n \geqslant 2$. We may thus identify the coefficients of these polynomials. We obtain the following : if $g + k$ is even, then

$$\begin{aligned}
& p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(k)} = \\
& \quad -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee})^{((g+k)/2)}[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r r^{g-(g+k)/2}\widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)]\Big]
\end{aligned}$$

and

$$p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(k)} = 0$$

if $g + k$ is odd. Note that we have already proven the stronger fact that $p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(k)} = 0$ if $k \neq g$. Thus

$$p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} = -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}\Big[\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^{\vee})^{g}[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})} (-1)^r \widehat{\mathrm{ch}}(\Lambda^r(\pi^*\pi_*(\overline{\mathcal{L}}))^\vee)]\Big].$$

Using furthermore that the left-hand side expression in the last equality is of pure degree g in $\widehat{\mathrm{CH}}^{\bullet}(\mathcal{A}^\vee)_{\mathbb{Q}}$, we deduce that

$$p_{2*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} = -\frac{1}{\sqrt{\deg(\phi_{\mathcal{L}})}}\phi_{\mathcal{L},*}\Big[\widehat{\mathrm{ch}}(\overline{\mathcal{L}}^\vee)^g[\sum_{r=1}^{\mathrm{rk}(\mathcal{E})}(-1)^r\binom{\mathrm{rk}(\mathcal{E})}{r}]\Big].$$

Now notice that by the binomial formula $\sum_{r=1}^{\mathrm{rk}(\mathcal{E})}(-1)^r\binom{\mathrm{rk}(\mathcal{E})}{r} = (1-1)^{\mathrm{rk}(\mathcal{E})}-1 = -1$. This proves equation (7).

3.3 Proof of 1.2.3

Let Z be an analytic cycle of pure codimension c on $\mathcal{A}(\mathbb{C})$. In view of the assumption on S, $\mathcal{A}(\mathbb{C})$ is a finite disjoint union of abelian varieties and so we may (and do) choose a translation invariant Kähler form on $\mathcal{A}(\mathbb{C})$. A current $\mathfrak{g}$ on $\mathcal{A}(\mathbb{C})$ of type $(c-1,c-1)$ is said to be a harmonic Green current for Z (with respect to the Kähler form), if it satisfies the following properties : $\mathfrak{g}$ is a Green current for Z, the differential form $\mathrm{dd}^c\mathfrak{g}+\delta_Z$ is harmonic and $\int_{\mathcal{A}(\mathbb{C})}\mathfrak{g}\wedge\kappa = 0$ for any harmonic form κ on $\mathcal{A}(\mathbb{C})$. Notice now that a differential form on $\mathcal{A}(\mathbb{C})$ is harmonic if and only if it is translation invariant (see for instance [15, p. 648]). Hence the concept of harmonic Green current is independent of the choice of the translation invariant Kähler form.

The property (a) in Theorem 1.1 shows that $\mathfrak{g}_{\mathcal{A}}$ is a Green current for $S_0^\vee(\mathbb{C})$ and the property 2 in Theorem 1.2 shows that $\mathrm{dd}^c\mathfrak{g}+\delta_{S_0^\vee(\mathbb{C})}$ is harmonic. Let now κ be a harmonic form of type $(1,1)$ on $\mathcal{A}(\mathbb{C})$. We know that κ is d-closed and that $[n]^*\kappa = n^2\cdot\kappa$ for any $n \geqslant 2$. We may thus compute

$$\int_{\mathcal{A}(\mathbb{C})}\mathfrak{g}_{\mathcal{A}}\wedge\kappa = \int_{\mathcal{A}(\mathbb{C})}[n]_*(\mathfrak{g}_{\mathcal{A}}\wedge\kappa) = n^{-2}\int_{\mathcal{A}(\mathbb{C})}[n]_*(\mathfrak{g}_{\mathcal{A}}\wedge[n]^*\kappa) = n^{-2}\int_{\mathcal{A}(\mathbb{C})}\mathfrak{g}_{\mathcal{A}}\wedge\kappa$$

and hence $\int_{\mathcal{A}(\mathbb{C})}\mathfrak{g}_{\mathcal{A}}\wedge\kappa = 0$. Thus $\mathfrak{g}_{\mathcal{A}}$ is harmonic.

3.4 Proof of 1.2.4

In the next section, we shall give an expression for $\mathfrak{g}_{\mathcal{A}}$, which depends only on $\mathcal{A}_{\mathbb{C}}$ (see the formula (15)). This implies the assertion.

3.5 Proof of 1.2.5

The proof of 1.2.5 is postponed to the end of the proof of Theorem 1.3.1. See the paragraph before subsection 4.2.

3.6 Proof of 1.2.6

This is a direct consequence of Theorem 1.1.1 and [11, Cor. 4.7 (i)] (thanks to J.-I. Burgos for providing this proof).

3.7 Proof of 1.2.7

We leave the proof that $\mathrm{BC}(\mathbb{C})$ is transverse to $S_0^\vee(\mathbb{C})$ to the reader. The equation $\mathrm{BC}^*\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}_T}$ follows from formula (15) for $\mathfrak{g}_{\mathcal{A}}$, which will be proved in the next section, together with [11, Th. 9.11 (ii)] and the fact that the higher analytic torsion forms of Bismut-Köhler are compatible with base-change.

4 Proof of Theorem 1.3

4.1 Proof of 1.3.1

This is the most difficult point to prove. We shall construct a class of currents $\mathfrak{g}^0_{\mathcal{A}}$ which naturally restricts to the degree $(g-1, g-1)$ part of the analytic torsion and we shall prove that $\mathfrak{g}^0_{\mathcal{A}}$ satisfies the axioms defining $\mathfrak{g}_{\mathcal{A}}$. The arithmetic Riemann-Roch theorem in higher degrees plays a crucial role here.

4.1.1 Definition of $\mathfrak{g}^0_{\mathcal{A}}$

Let

$$V : 0 \to \mathcal{O}_{\mathcal{A}} \to V_0 \to \cdots \to V_r \to V_{r+1} \to \cdots$$

be a resolution of $\mathcal{O}_{\mathcal{A}}$ by π_*-acyclic vector bundles. Dualising, we get a resolution

$$V^\vee : \cdots \to V^\vee_{r+1} \to V^\vee_r \to \cdots \to V^\vee_0 \to \mathcal{O}_{\mathcal{A}} \to 0$$

of $\mathcal{O}_{\mathcal{A}}$ on the left. The first hypercohomology spectral sequence of the complex $V^\vee \otimes \mathcal{P}$ for the functor $p_{2,*}$ provides an exact sequence

$$\begin{aligned}\mathcal{H} : \cdots \to \mathrm{R}^g p_{2*}(V^\vee_r \otimes \mathcal{P}) \to \mathrm{R}^g p_{2*}(V^\vee_{r-1} \otimes \mathcal{P}) \to \dots \\ \to \mathrm{R}^g p_{2*}(V^\vee_0 \otimes \mathcal{P}) \to {\epsilon^\vee}_*(\omega^\vee_{A/S}) \to 0.\end{aligned}$$

Now endow the vector bundles V_r with conjugation-invariant hermitian metrics. The line bundle $\overline{\omega}_{\mathcal{A}/S}$ is endowed with its L^2-metric. This metric does not depend on the choice of λ. This follows from the explicit formula for the L^2-metric on Hodge cohomology given in [38, Lemma 2.7]. The arithmetic Riemann-Roch [22] in higher degrees applied to $\overline{\mathcal{P}}$ and p_2 is the identity

$$\begin{aligned}(-1)^g\widehat{\mathrm{ch}}(\sum_{r\geqslant 0}(-1)^r\mathrm{R}^g p_{2*}(\overline{V}^\vee_r \otimes \overline{\mathcal{P}})) - \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}^\vee_r \otimes \overline{\mathcal{P}}) + \\ + \int_{p_2} \mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee} \\ = p_{2,*}(\widehat{\mathrm{Td}}(\overline{\mathrm{T}}p_2)\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) - \int_{p_2}\mathrm{ch}(\mathcal{P})R(\mathrm{T}p_2)\,\mathrm{Td}(\mathrm{T}p_2)\end{aligned}$$

in $\widehat{\mathrm{CH}}^\bullet(\mathcal{A}^\vee)_{\mathbb{Q}}$. Here $\eta_{\bar{V}^\vee}$ is the Bott-Chern secondary class of $\overline{V}^\vee$, where $\mathcal{O}_{\mathcal{A}}$ has index 0. We have identified λ with $p_1^*\lambda$.

Notice first that

$$
\begin{aligned}
&[\int_{p_2} \mathrm{ch}(\mathcal{P}) R(\mathrm{T}p_2)\,\mathrm{Td}(\mathrm{T}p_2)]^{(g-1)}\\
&\qquad = [\int_{p_2} \mathrm{ch}(\mathcal{P}) R(\mathrm{T}\pi)\,\mathrm{Td}(\mathrm{T}\pi)]^{(g-1)}\\
&\qquad = [\pi^{\vee,*}(\epsilon^*(R(\mathrm{T}\pi)\,\mathrm{Td}(\mathrm{T}\pi)))\int_{p_2} \mathrm{ch}(\mathcal{P})]^{(g-1)} = 0
\end{aligned}
$$

where we used (7).
Write $\mathcal{T}(\overline{\mathcal{H}})$ for the homogenous secondary class in the sense of Bismut-Burgos-Litcanu (see [11, sec. 6]) of the resolution $\mathcal{H}$. By its very definition, $-\mathcal{T}(\overline{\mathcal{H}})^{(g-1)}$ is a class of Green currents for $S_0^\vee(\mathbb{C})$ and it is shown in [11, Th. 10.28] that

$$
[\widehat{\mathrm{ch}}(\sum_{r\geqslant 0}(-1)^r \mathrm{R}^g p_{2*}(\overline{V}_r^\vee \otimes \overline{\mathcal{P}}))]^{(g)} = (S_0^\vee, -\mathcal{T}(\overline{\mathcal{H}})^{(g-1)}) \tag{13}
$$

in $\widehat{\mathrm{CH}}^g(\mathcal{A}^\vee)_{\mathbb{Q}}$ and equation (7) shows that

$$
\begin{aligned}
[p_{2,*}(\widehat{\mathrm{Td}}(\overline{\mathrm{T}}p_2)\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))]^{(g)} &= [p_{2,*}(\widehat{\mathrm{Td}}(\overline{\mathrm{T}}\pi)\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))]^{(g)}\\
&= [p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))\pi^{\vee,*}(\epsilon^*(\widehat{\mathrm{Td}}(\overline{\mathrm{T}}\pi)))]^{(g)} = p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}
\end{aligned}
$$

hence we are led to the equality

$$
\begin{aligned}
p_{2,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)} = (-1)^g(S_0^\vee, -\mathcal{T}(\overline{\mathcal{H}})^{(g-1)}) - \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}_r^\vee \otimes \overline{\mathcal{P}})^{(g-1)}\\
+ [\int_{p_2} \mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}]^{(g-1)}. \quad (14)
\end{aligned}
$$

This motivates the definition:

$$
\boxed{
\begin{aligned}
\mathfrak{g}^0_{\mathcal{A}} \quad := \quad & -\mathcal{T}(\overline{\mathcal{H}})^{(g-1)} + (-1)^{g+1}\sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}_r^\vee \otimes \overline{\mathcal{P}})^{(g-1)}\\
& +(-1)^g[\int_{p_2} \mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}]^{(g-1)} \qquad (15)
\end{aligned}
}
$$

Lemma 4.1. *The class of currents $\mathfrak{g}^0_{\mathcal{A}}$ does not depend on the resolution V, nor on the metrics on the bundles V_r, nor on the translation invariant Kähler form λ.*

Proof. We first prove that the class of currents $\mathfrak{g}^0_{\mathcal{A}}$ does not depend on V and that it does not depend on the hermitian metrics or on the bundles V_r.

Suppose that there is a second resolution V' dominating V :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
V' : 0 \arrow[r] & \mathcal{O}_{\mathcal{A}} \arrow[r] \arrow[d, "\mathrm{Id}"'] & V'_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & V'_r \arrow[r] \arrow[d] & V'_{r+1} \arrow[r] \arrow[d] & \cdots \\
V : 0 \arrow[r] & \mathcal{O}_{\mathcal{A}} \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & V_r \arrow[r] \arrow[d] & V_{r+1} \arrow[r] \arrow[d] & \cdots \\
Q : & 0 \arrow[r] & Q_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_r \arrow[r] \arrow[d] & Q_{r+1} \arrow[r] \arrow[d] & \cdots \\
 & & 0 & & 0 & 0 &
\end{tikzcd}$$

By assumption the complex Q is exact and we assume that its objects are π_*-acyclic. We endow everything with hermitian metrics. We shall write $\mathcal{H}'$ for the exact sequence

$$\mathcal{H}' : \cdots \to \mathrm{R}^g p_{2*}(V_r^{',\vee} \otimes \mathcal{P}) \to \mathrm{R}^g p_{2*}(V_{r-1}^{',\vee} \otimes \mathcal{P}) \to \dots$$
$$hfill \to \mathrm{R}^g p_{2*}(V_0^{',\vee} \otimes \mathcal{P}) \to \epsilon^\vee{}_*(\omega^\vee_{A/S}) \to 0.$$

In order to emphasize the dependence of $\mathfrak{g}^0_{\mathcal{A}}$ on the resolution V together with the collection of hermitian metrics on the V_r, we shall write $\mathfrak{g}^0_{\overline{V}} := \mathfrak{g}^0_{\mathcal{A},\overline{V}}$ instead of $\mathfrak{g}^0_{\mathcal{A}}$. Recall that $\eta_{\overline{V}^\vee}$ is the Bott-Chern secondary class of the sequence $\overline{V}^\vee$, with $\overline{\mathcal{O}}_{\mathcal{A}}$ sitting at the index 0. We shall accordingly write $\eta_{\overline{V}',\vee}$ for the Bott-Chern secondary class of the sequence $\overline{V}^{',\vee}$, with $\overline{\mathcal{O}}_{\mathcal{A}}$ sitting at the index 0. By definition, we have

$$\begin{aligned}
(-1)^g(\mathfrak{g}^0_{\overline{V}} - \mathfrak{g}^0_{\overline{V}'}) = {} & (-1)^{g+1}\mathcal{T}(\overline{\mathcal{H}})^{(g-1)} \\
& - \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}_r^\vee \otimes \overline{\mathcal{P}})^{(g-1)} + [\int_{p_2} \mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}]^{(g-1)} \\
& - (-1)^{g+1}\mathcal{T}(\overline{\mathcal{H}}')^{(g-1)} + \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}_r^{',\vee} \otimes \overline{\mathcal{P}})^{(g-1)} \\
& - [\int_{p_2} \mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}',\vee}]^{(g-1)}.
\end{aligned}$$

Let now

$$C_r : 0 \to Q_r^\vee \otimes \mathcal{P} \to V_r^\vee \otimes \mathcal{P} \to V_r^{',\vee} \otimes \mathcal{P} \to 0$$

be the natural exact sequence. All the bundles appearing on C_r are endowed with natural hermitian metrics. By the symmetry formula [4, Th. 2.7, p. 271], we may compute

$$\mathcal{T}(\overline{\mathcal{H}})-\mathcal{T}(\overline{\mathcal{H}}')=\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(Q^\vee\otimes\mathcal{P}))-\sum_{r\geqslant 0}(-1)^r\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(C_r)).$$

On the other hand, the anomaly formula [6, Th. 3.10, p. 670] tells us that

$$\begin{aligned}
&\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{V}_r^\vee\otimes\overline{\mathcal{P}})-\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{V}_r^{',\vee}\otimes\overline{\mathcal{P}})\\
=\ &\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{Q}_r^\vee\otimes\overline{\mathcal{P}})+(-1)^g\sum_{r\geqslant 0}(-1)^r\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(C_r))\\
&-\int_{p_2}\sum_{r\geqslant 0}(-1)^r\,\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{C}_r)
\end{aligned}$$

and

$$\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{Q}_r^\vee\otimes\overline{\mathcal{P}})=\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{Q}^\vee\otimes\overline{\mathcal{P}})-(-1)^g\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(Q^\vee\otimes\mathcal{P})).$$

Furthermore the symmetry formula [4, Th. 2.7, p. 271] again implies that

$$\begin{aligned}
&\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\eta_{\bar{V}',\vee}\,\mathrm{ch}(\overline{\mathcal{P}})-\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\eta_{\bar{V}^\vee}\,\mathrm{ch}(\overline{\mathcal{P}})\\
&\qquad=-\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{Q}^\vee\otimes\overline{\mathcal{P}})+\int_{p_2}\sum_{r\geqslant 0}(-1)^r\,\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{C}_r).
\end{aligned}$$

Putting everything together, we see that

$$\begin{aligned}
&(-1)^g(\mathfrak{g}^0_{\overline{V}}-\mathfrak{g}^0_{\overline{V}'})\\
&=\Big[-(-1)^g\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(Q^\vee\otimes\mathcal{P}))+(-1)^g\sum_{r\geqslant 0}(-1)^r\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(C_r))\\
&-(-1)^g\sum_{r\geqslant 0}(-1)^r\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(C_r))+\int_{p_2}\sum_{r\geqslant 0}(-1)^r\,\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{C}_r)\\
&-\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{Q}^\vee\otimes\overline{\mathcal{P}})+(-1)^g\widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(Q^\vee\otimes\mathcal{P}))\\
&+\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{Q}^\vee\otimes\overline{\mathcal{P}})-\int_{p_2}\sum_{r\geqslant 0}(-1)^r\,\mathrm{Td}(\overline{\mathrm{T}}p_2)\widetilde{\mathrm{ch}}(\overline{C}_r)\Big]^{(g-1)}=0.
\end{aligned}$$

Homological algebra tells us that there always exists a resolution dominating simultaneously two other ones. Furthermore, we might assume that this resolution satisfies the above conditions of π_*-acyclicity (see [30, chap XX, par. 3, proof of Th. 3.5, p. 773]). Hence we have proven that $\mathfrak{g}^0_{\overline{V}}$ does not depend on V and that it does not depend on the hermitian metrics on the bundles V_r.

We shall now prove that $\mathfrak{g}^0_{\mathcal{A}}$ does not depend on the choice of λ. So let λ' be another Kähler fibration, which is translation invariant on the fibres. To emphasize the dependence of $\mathfrak{g}^0_{\mathcal{A}}$ on λ, let us write $\mathfrak{g}^0_{\mathcal{A}} := \mathfrak{g}^{0,\lambda}$. Write $\overline{\mathcal{H}}^{\lambda}$ (resp. $\overline{\mathcal{H}}^{\lambda'}$) for the sequence $\mathcal{H}$ together with the hermitian metrics induced by λ (resp. λ').
We compute as before

$$\begin{aligned}
&(-1)^g(\mathfrak{g}^{0,\lambda} - \mathfrak{g}^{0,\lambda'})\\
&\quad = (-1)^{g+1}\mathcal{T}(\overline{\mathcal{H}}^{\lambda})^{(g-1)} - \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}^{\vee}_r \otimes \overline{\mathcal{P}})^{(g-1)}\\
&\quad + [\int_{p_2} \mathrm{Td}(\mathrm{T}p_2, \lambda)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^{\vee}}]^{(g-1)}\\
&\quad - (-1)^{g+1}\mathcal{T}(\overline{\mathcal{H}}^{\lambda'})^{(g-1)} + \sum_{r\geqslant 0}(-1)^r T(\lambda', \overline{V}^{\vee}_r \otimes \overline{\mathcal{P}})^{(g-1)}\\
&\quad - [\int_{p_2} \mathrm{Td}(\mathrm{T}p_2, \lambda')\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^{\vee}}]^{(g-1)}.
\end{aligned}$$

We compute

$$\mathcal{T}(\overline{\mathcal{H}}^{\lambda'}) - \mathcal{T}(\overline{\mathcal{H}}^{\lambda}) = -\sum_{r\geqslant 0}(-1)^r \widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(V^{\vee}_r \otimes \mathcal{P}), \lambda', \lambda)$$

and by the anomaly formula [6, Th. 3.10, p. 670]

$$\begin{aligned}
&\sum_{r\geqslant 0}(-1)^r T(\lambda', \overline{V}^{\vee}_r \otimes \overline{\mathcal{P}}) - \sum_{r\geqslant 0}(-1)^r T(\lambda, \overline{V}^{\vee}_r \otimes \overline{\mathcal{P}})\\
&\quad = (-1)^g \sum_{r\geqslant 0}(-1)^r \widetilde{\mathrm{ch}}(\mathrm{R}^g\pi_*(V^{\vee}_r \otimes \mathcal{P}), \lambda', \lambda)\\
&\quad\quad - \int_{p_2}\sum_{r\geqslant 0}(-1)^r \widetilde{\mathrm{Td}}(\lambda', \lambda)\,\mathrm{ch}(\overline{V}^{\vee}_r \otimes \overline{\mathcal{P}}).
\end{aligned}$$

Now notice that by the Leibniz formula (see [42, 6.2, (7)] for details)

$$\begin{aligned}
\int_{p_2}\big(\mathrm{Td}(\mathrm{T}p_2, \lambda) - \mathrm{Td}(\mathrm{T}p_2, \lambda')\big)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^{\vee}} &= -\int_{p_2} \mathrm{dd}^c(\widetilde{\mathrm{Td}}(\lambda', \lambda))\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^{\vee}}\\
&= -\int_{p_2}\widetilde{\mathrm{Td}}(\lambda', \lambda)\mathrm{dd}^c(\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^{\vee}})\\
&= \sum_{r\geqslant 0}(-1)^r\int_{p_2}\widetilde{\mathrm{Td}}(\lambda', \lambda)\,\mathrm{ch}(\overline{V}^{\vee}_r \otimes \overline{\mathcal{P}}) - \int_{p_2}\widetilde{\mathrm{Td}}(\lambda', \lambda)\,\mathrm{ch}(\overline{\mathcal{P}}).
\end{aligned}$$

Now, by the projection formula, since λ and λ' are translation invariant, we have

$$\int_{p_2}\widetilde{\mathrm{Td}}(\lambda', \lambda)\,\mathrm{ch}(\overline{\mathcal{P}}) = \epsilon^{\vee,*}(\widetilde{\mathrm{Td}}(\lambda', \lambda))\int_{p_2}\mathrm{ch}(\overline{\mathcal{P}})$$

and by the formula (7) we have $[\epsilon^{\vee,*}(\widetilde{\mathrm{Td}}(\lambda',\lambda))\int_{p_2}\mathrm{ch}(\overline{\mathcal{P}})]^{(g-1)}=0$. Assembling everything, we get that $\mathfrak{g}^{0,\lambda}-\mathfrak{g}^{0,\lambda'}=0$. □

4.1.2 End of proof of 1.3.1

We keep the notation of the last subsection. By construction, the class of currents $\mathfrak{g}^0_{\mathcal{A}}$ has the property that

$$\mathfrak{g}^0_{\mathcal{A}}|_{\mathcal{A}^\vee(\mathbb{C})\backslash S_0^\vee(\mathbb{C})}=(-1)^{g+1}T(\lambda,\overline{\mathcal{P}}^0)^{(g-1)} \tag{16}$$

in $\widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee\backslash S_0^\vee)$ and contemplating equation (13), we see that the elements of $\mathfrak{g}^0_{\mathcal{A}}$ are Green currents for $S_0^\vee$. Looking at the equation (14) we see that $\mathfrak{g}^0_{\mathcal{A}}$ satisfies (a) and (b) in Theorem 1.1.
Now we want to prove that

$$\mathfrak{g}^0_{\mathcal{A}}=[n]_*\mathfrak{g}^0_{\mathcal{A}} \tag{17}$$

for all $n\geqslant 2$.
Fix $n\geqslant 2$. We claim that to prove equation (17), we may assume that $\ker\,[n]_{\mathcal{A}}$ is a constant diagonalisable subgroup-scheme of $\mathcal{A}$. Indeed, both sides of equation (17) depend on $\mathcal{A}^\vee_{\mathbb{C}}$ only. In proving equation (17), we thus may (and do) replace R by its fraction field $\mathrm{Frac}(R)$. We may also assume without restriction of generality that S is connected, hence integral (since S is regular by assumption). Now let S' be the normalisation of S in the composite of the field extensions of the function field $\kappa(S)$ of S, which are defined by the residue fields of the n-torsion points in the generic fibre $\mathcal{A}_{\kappa(S)}$ of $\mathcal{A}$. Then $b:S'\to S$ is finite (see [23, II, 6.3.10]) and étale (see [39, Cor. 20.8, p. 147]). Since $[n]_{\mathcal{A}}$ is étale, the group scheme of n-torsion points on $\mathcal{A}_{S'}$ is then a constant group scheme. If we again replace S' by a finite étale cover, we may assume that $\Gamma(S',\mathcal{O}_{S'})$ contains the n-th roots of 1 and the group scheme of n-torsion points on $\mathcal{A}_{S'}$ then become diagonalisable (and constant).
Now, by the projection formula the pull-back morphism $b^*:\widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee_{\mathbb{R}})\to\widetilde{A}^{g-1,g-1}(\mathcal{A}^\vee_{S',\mathbb{R}})$ is injective. Furthermore, since an étale finite morphism is a local isomorphism in the category of complex manifolds, we have $b^*([n]_*\mathfrak{g}^0_{\mathcal{A}})=[n]_*(b^*\mathfrak{g}^0_{\mathcal{A}})$. For the same reason, we also have $b^*\mathfrak{g}^0_{\mathcal{A}}=\mathfrak{g}^0_{\mathcal{A}_{S'}}$.

Lemma 4.2. *There is an isometric isomorphism*

$$[n]_*\mathcal{O}_{\mathcal{A}}\simeq\bigoplus_{\mathcal{M}\in\mathcal{A}^\vee[n](S)}\mathcal{M}$$

where the left-hand side is endowed with its L^2-metric and the direct summands on the right-hand side are endowed with the metric induced by the Poincaré bundle.

Proof. Let us denote by $\mathcal{F}_{\mathcal{A}}:D^b(\mathcal{A})\to D^b(\mathcal{A}^\vee)$ the Fourier-Mukai transformation. If X is a scheme, the category $D^b(X)$ is a full subcategory of the category $D(X)$ derived from the category of sheaves in $\mathcal{O}_X$-modules. Its objects are the complexes with a finite number of non-zero homology sheaves, all of which are coherent. The functor $\mathcal{F}_{\mathcal{A}}$ is given by the formula

$$K^\bullet \mapsto \mathrm{R}p_{2,*}(p_1^* K^\bullet \otimes \mathcal{P}).$$

It can be proven that in our situation, there is a natural isomorphism of functors

$$\mathcal{F}_{\mathcal{A}^\vee} \circ \mathcal{F}_{\mathcal{A}}(\cdot) \simeq \big(([-1]^*(\cdot)) \otimes \omega_{\mathcal{A}}^\vee\big)[-g]$$

(here we have identified $\mathcal{A}$ with $(\mathcal{A}^\vee)^\vee$) and

$$\mathcal{F}_{\mathcal{A}} \circ [n]_*(\cdot) \simeq [n]^* \circ \mathcal{F}_{\mathcal{A}}(\cdot).$$

See [32] for this. Now we compute

$$\begin{aligned}\mathcal{F}_{\mathcal{A}^\vee} \circ \mathcal{F}_{\mathcal{A}}([n]_*(\mathcal{O}_{\mathcal{A}})) &\simeq [n]_*(\mathcal{O}_{\mathcal{A}}) \otimes \omega_{\mathcal{A}}^\vee[-g] \simeq \mathcal{F}_{\mathcal{A}^\vee} \circ [n]^*(\mathcal{F}_{\mathcal{A}}(\mathcal{O}_{\mathcal{A}}))\\ &\simeq \mathcal{F}_{\mathcal{A}^\vee}([n]^*\big(\epsilon_*^\vee(\epsilon^{\vee,*}(\omega_{\mathcal{A}}^\vee))[-g]\big)) \simeq \pi_{\mathcal{A},*}(\mathcal{P}_{\mathcal{A}}|_{\mathcal{A}\times_S \ker\,[n]_{\mathcal{A}^\vee}}) \otimes \omega_{\mathcal{A}}^\vee[-g]\end{aligned}$$

and we thus obtain a canonical isomorphism

$$[n]_*\mathcal{O}_{\mathcal{A}} \simeq \bigoplus_{\mathcal{M}\in\mathcal{A}^\vee[n](S)} \mathcal{M}. \tag{18}$$

Now we make a different computation. Let G be a finite group such that $G_S \simeq \ker\,[n]_{\mathcal{A}}$. Let $L := \Gamma(S, \mathcal{O}_S)$. If $\chi : G \to L^*$ is a character of G we let $([n]_*\mathcal{O}_{\mathcal{A}})_\chi$ be the locally free subbundle of $[n]_*\mathcal{O}_{\mathcal{A}}$ which is the largest subbundle $\mathcal{S}$ of $[n]_*\mathcal{O}$ such that the action of G on $\mathcal{S}$ is given by multiplication by χ. Since G_L is a diagonalisable group scheme over L, this gives a direct sum decomposition

$$[n]_*\mathcal{O}_{\mathcal{A}} = \oplus_\chi([n]_*\mathcal{O}_{\mathcal{A}})_\chi. \tag{19}$$

Now we use the equivariant form of Bismut's relative curvature formula (see [7]). Let $a \in G$ be any non-zero element. Since the fixed point scheme of a on $\mathcal{A}$ is empty, we get

$$\mathrm{ch}_a([n]_*(\overline{\mathcal{O}}_A)) = 0. \tag{20}$$

Here $\mathrm{ch}_a(\cdot)$ is the equivariant Chern character form associated to the action of a. See [7] for details. The non-equivariant relative curvature formula gives

$$\mathrm{ch}([n]_*(\overline{\mathcal{O}}_A)) = \mathrm{rk}([n]_*\mathcal{O}_A). \tag{21}$$

From the equations (20) and (21) and the fact that $\mathrm{rk}([n]_*\mathcal{O}_A) = \#G$, we deduce (by finite Fourier theory) that all the $([n]_*\mathcal{O}_{\mathcal{A}})_\chi$ are of rank 1 and that $\mathrm{c}_1(([n]_*\overline{\mathcal{O}}_{\mathcal{A}})_\chi) = 0$, where $([n]_*\overline{\mathcal{O}}_{\mathcal{A}})_\chi$ is endowed with the metric induced by $[n]_*\overline{\mathcal{O}}_{\mathcal{A}}$.
A completely similar computation using the relative geometric fixed point formula (see [1]) shows that each line bundle $([n]_*\mathcal{O}_{\mathcal{A}})_\chi$ is actually a torsion line

bundle. Notice also that since G acts by isometries on $\mathcal{O}_A$, the direct sum decomposition (19) is an orthogonal direct sum.

Now notice that since $[n]$ is finite and flat there is an isometric equivariant isomorphism

$$\epsilon^*([n]_*\overline{\mathcal{O}}_{\mathcal{A}}) \simeq \overline{\mathcal{O}}_{\mathcal{A}}^G$$

where $\mathcal{O}_{\mathcal{A}}^G$ is endowed with the G-action induced by the action of G on itself.

This shows that there exist isometric rigidifications $\epsilon^*(([n]_*\overline{\mathcal{O}}_{\mathcal{A}})_\chi) \simeq \overline{\mathcal{O}}_S$.

Summing up the discussion of the previous paragraphs, we see that there exists an isometric isomorphism of vector bundles

$$[n]_*\overline{\mathcal{O}}_{\mathcal{A}} = \oplus_\chi([n]_*\overline{\mathcal{O}}_{\mathcal{A}})_\chi \tag{22}$$

where the direct sum is orthogonal and where each $([n]_*\overline{\mathcal{O}}_{\mathcal{A}})_\chi$ is a torsion line bundle, which is isometrically rigidified and which carries a hermitian metric, whose curvature form vanishes.

Now consider the isomorphism of vector bundles

$$\bigoplus_\chi([n]_*\mathcal{O}_{\mathcal{A}})_\chi \simeq \bigoplus_{\mathcal{M}\in\mathcal{A}^\vee[n](S)} \mathcal{M} \tag{23}$$

induced by the isomorphisms (18) and (22).

We shall need the

Claim. Let $\mathcal{M}$ and $\mathcal{L}$ be torsion line bundles on $\mathcal{A}$. If there is a non-zero morphism of line bundles (without rigidification) $\mathcal{L} \to \mathcal{M}$ then $\mathcal{L}$ and $\mathcal{M}$ are isomorphic.

This follows from the fact that the Picard functor of $\mathcal{A}/S$ is representable by a scheme, which is separated over S, together with the fact that a non-trivial torsion bundle on an abelian variety has no global sections. Details are left to the reader.

The claim shows that the isomorphism (23) send each $([n]_*\mathcal{O}_{\mathcal{A}})_\chi$ into exactly one $\mathcal{M} \in \mathcal{A}^\vee[n](S)$ and that this morphism is an isomorphism of line bundles. After possibly rescaling the isomorphism $([n]_*\mathcal{O}_{\mathcal{A}})_\chi \simeq \mathcal{M}$ by an element of $\Gamma(S, \mathcal{O}_S^*)$, we obtain an isomorphism $([n]_*\mathcal{O}_{\mathcal{A}})_\chi \simeq \mathcal{M}$ of rigidified line bundles. Since both $([n]_*\mathcal{O}_{\mathcal{A}})_\chi$ and $\mathcal{M}$ are endowed with the unique metrics, whose curvature form is translation invariant on the fibres of $\mathcal{A}(\mathbb{C})/S(\mathbb{C})$ and which are compatible with the given rigidifications, this implies that this is an isometric isomorphism. This completes the proof of the lemma. □

As at the beginning of the proof of assertion 1 in Theorem 1.2, we have a diagram

$$\begin{tikzcd}
\mathcal{A}\times_S\mathcal{A}^\vee \arrow[r, "{[n]\times\mathrm{Id}}"] \arrow[d, "{\mathrm{Id}\times[n]}"'] & \mathcal{A}\times_S\mathcal{A}^\vee \arrow[r] & \mathcal{A}^\vee \arrow[d, "{[n]}"'] \\
\mathcal{A}\times_S\mathcal{A}^\vee \arrow[rr] & & \mathcal{A}^\vee
\end{tikzcd}$$

such that

$$(\mathrm{Id}\times[n])^*\mathcal{P}\simeq([n]\times\mathrm{Id})^*\mathcal{P}$$

and such that the outer square is cartesian. Write $q:=p_2\circ([n]\times\mathrm{Id})$. To clarify further computations, write $\mathrm{R}^\bullet p_{2,*}(V^\vee\otimes\mathcal{P})$ for $\mathcal{H}$. We now compute

$$\begin{aligned}
(-1)^g[n]^*\mathfrak{g}^0_{\mathcal{A}} &= [n]^*\Big[(-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet p_{2,*}(V^\vee\otimes\mathcal{P})) && (24)\\
&\quad -\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{V}^\vee_r\otimes\overline{\mathcal{P}})+\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}\Big]\\
&= (-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet q_*(V^\vee\otimes(\mathrm{Id}\times[n])^*\mathcal{P})) && (25)\\
&\quad -\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{V}^\vee_r\otimes(\mathrm{Id}\times[n])^*\overline{\mathcal{P}})\\
&\quad +\int_q\mathrm{Td}(\overline{\mathrm{T}}q)\,\mathrm{ch}((\mathrm{Id}\times[n])^*\overline{\mathcal{P}})\eta_{\bar{V}^\vee}\\
&= (-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet q_*(V^\vee\otimes([n]\times\mathrm{Id})^*\mathcal{P})) && (26)\\
&\quad -\sum_{r\geqslant 0}(-1)^r T(\lambda,\overline{V}^\vee_r\otimes([n]\times\mathrm{Id})^*\overline{\mathcal{P}})\\
&\quad +\int_q\mathrm{Td}(\overline{\mathrm{T}}q)\,\mathrm{ch}(([n]\times\mathrm{Id})^*\overline{\mathcal{P}})\eta_{\bar{V}^\vee}
\end{aligned}$$

In view of Lemma 4.1, we may replace $\overline{V}$ by $[n]^*\overline{V}_{\mathcal{A}}$ and λ by $[n]^*\lambda_{\mathcal{A}}$ in the string of equalities (27) without changing its truth-value. Thus

$$
\begin{aligned}
(-1)^g[n]^*\mathfrak{g}^0_{\mathcal{A}} &= (-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet q_*(([n]\times \mathrm{Id})^*(V^\vee\otimes\mathcal{P})))\\
&\quad -\sum_{r\geqslant 0}(-1)^r T([n]^*\lambda,([n]\times\mathrm{Id})^*(\overline{V}^\vee_r\otimes\overline{\mathcal{P}}))\\
&\quad +\int_q \mathrm{Td}(\overline{\mathrm{T}}q)([n]\times\mathrm{Id})^*(\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee})\\
&\overset{(*)}{=} (-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet p_{2,*}(([n]\times\mathrm{Id})_*(\mathcal{O})\otimes V^\vee\otimes\mathcal{P}))\\
&\quad -\sum_{r\geqslant 0}(-1)^r T(([n]\times\mathrm{Id})_*(\overline{\mathcal{O}})\otimes\overline{V}^\vee_r\otimes\overline{\mathcal{P}})\\
&\quad +\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)([n]\times\mathrm{Id})_*(1)\,\mathrm{ch}(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}\\
&\overset{(**)}{=} \sum_{\tau\in\mathcal{A}^\vee(S)}\Big[(-1)^{g+1}\mathcal{T}(\mathrm{R}^\bullet p_{2,*}((\mathrm{Id}\times(\tau\circ\pi))^*(\mathcal{P})\otimes V^\vee\otimes\mathcal{P}))\\
&\quad -\sum_{r\geqslant 0}(-1)^r T((\mathrm{Id}\times(\tau\circ\pi))^*(\overline{\mathcal{P}})\otimes\overline{V}^\vee_r\otimes\overline{\mathcal{P}})\\
&\qquad +\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)\,\mathrm{ch}((\mathrm{Id}\times(\tau\circ\pi))^*(\overline{\mathcal{P}})\otimes\overline{\mathcal{P}})\eta_{\bar{V}^\vee}\Big]\\
&= \sum_{\tau\in\mathcal{A}^\vee(S)}\Big[(-1)^{g+1}\tau_*(\mathcal{T}(\mathrm{R}^\bullet p_{2,*}(V^\vee\otimes\overline{\mathcal{P}})))-\tau_*[\sum_{r\geqslant 0}(-1)^r T(\overline{V}^\vee_r\otimes\overline{\mathcal{P}})]\\
&\quad +\tau_*[\int_{p_2}\mathrm{Td}(\overline{\mathrm{T}}p_2)(\overline{\mathcal{P}})\eta_{\bar{V}^\vee}]\Big]\\
&= (-1)^g\sum_{\tau\in\mathcal{A}^\vee(S)}\tau_*\mathfrak{g}^0_{\mathcal{A}}.
\end{aligned}
$$

The equality (*) is justified by the following

Proposition 4.3. *The equality*

$$T([n]^*\lambda,([n]\times\mathrm{Id})^*(\overline{V}^\vee_r\otimes\overline{\mathcal{P}})) = T(([n]\times\mathrm{Id})_*(\overline{\mathcal{O}})\otimes\overline{V}^\vee_r\otimes\overline{\mathcal{P}})$$

is verified for any r.

Proof. (of Proposition 4.3). This is a direct consequence of [35, Intro., Th. 0.1]. □

For the equality (**), we used Lemma 4.2. We may now compute

$$[n]_*[n]^*\mathfrak{g}^0_{\mathcal{A}} = n^{2g}\mathfrak{g}^0_{\mathcal{A}} = [n]_*(\sum_{\tau\in\mathcal{A}^\vee(S)}\tau_*\mathfrak{g}^0_{\mathcal{A}}) = n^{2g}[n]_*\mathfrak{g}^0_{\mathcal{A}}$$

i.e.

$$\mathfrak{g}^0_{\mathcal{A}} = [n]_*\mathfrak{g}^0_{\mathcal{A}}.$$

We have thus proven that $\mathfrak{g}^0_{\mathcal{A}}$ satisfies the conditions (a), (b), (c) in Theorem 1.1. Thus $\mathfrak{g}^0_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}}$ and looking at equation (16), we see that we have almost concluded the proof of Theorem 1.3.1. To finish, we quote [27], where it is shown that $T(\lambda, \overline{\mathcal{P}}^0) = \mathrm{Td}^{-1}(\epsilon^*\overline{\Omega})\gamma$ for some real differential form γ of type $(g-1, g-1)$ on $\mathcal{A}^\vee \backslash S_0^\vee$. From the above, we see that $\gamma = (-1)^{g+1}\mathfrak{g}_{\mathcal{A}}$ and we are done.

Proof of Theorem 1.2.5.

We revert to the hypotheses of the introduction and of Theorem 1.2.5 (in particular, we do not suppose anymore that for some $n \geqslant 2$, the group scheme $\ker[n]$ is the constant group scheme).

To verify the equation $\mathfrak{g}_{\mathcal{A}\times_S\mathcal{B}} = \pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})$, it is sufficient to check that the class of currents $\pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})$ satisfies the axioms (a), (b), (c) in Theorem 1.1.

The fact that the elements of $\pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})$ are Green currents for the unit section of $\mathcal{A}^\vee \times_S \mathcal{B}^\vee$ follows immediately from the definitions. This settles (a).

To verify (b), we consider the commutative diagram

$$\begin{tikzcd}
 & & \mathcal{A}\times\mathcal{B}\times\mathcal{A}^\vee\times\mathcal{B}^\vee \arrow[dl, "p_{\mathcal{A}\mathcal{A}^\vee\mathcal{B}^\vee}"'] \arrow[dd, "p_{\mathcal{A}^\vee\mathcal{B}^\vee}"] \arrow[dr, "p_{\mathcal{A}^\vee\mathcal{B}\mathcal{B}^\vee}"] & & \\
\mathcal{A}\times\mathcal{A}^\vee & \mathcal{A}\times\mathcal{A}^\vee\times\mathcal{B}^\vee \arrow[l, "q_{\mathcal{A}\mathcal{A}^\vee}"'] \arrow[dr, "q_{\mathcal{A}^\vee\mathcal{B}^\vee}"'] & & \mathcal{A}^\vee\times\mathcal{B}\times\mathcal{B}^\vee \arrow[r, "r_{\mathcal{B}\mathcal{B}^\vee}"] \arrow[dl, "r_{\mathcal{A}^\vee\mathcal{B}^\vee}"] & \mathcal{B}\times\mathcal{B}^\vee \\
 & & \mathcal{A}^\vee\times\mathcal{B}^\vee & &
\end{tikzcd}$$

where the morphisms are the obvious ones. Notice that in this diagram, the square is cartesian. We compute

$$\begin{aligned}
& (-1)^{g_{A\times B}} p_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}\times\mathcal{B}}))^{g_{A\times B}} \\
= \;& (-1)^{(g_A+g_B)} p_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}\times\mathcal{B}}))^{(g_A+g_B)} = (-1)^{(g_A+g_B)} p_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}\times\mathcal{B}})) \\
= \;& (-1)^{(g_A+g_B)} p_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(p^*_{\mathcal{A}\mathcal{A}^\vee\mathcal{B}^\vee} q^*_{\mathcal{A}\mathcal{A}^\vee}\overline{\mathcal{P}}_{\mathcal{A}}) \cdot \widehat{\mathrm{ch}}(p^*_{\mathcal{A}^\vee\mathcal{B}\mathcal{B}^\vee} r^*_{\mathcal{B}\mathcal{B}^\vee}\overline{\mathcal{P}}_{\mathcal{B}})) \\
= \;& (-1)^{(g_A+g_B)} r_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(p_{\mathcal{A}^\vee\mathcal{B}\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(p^*_{\mathcal{A}\mathcal{A}^\vee\mathcal{B}^\vee} q^*_{\mathcal{A}\mathcal{A}^\vee}\overline{\mathcal{P}}_{\mathcal{A}}) \cdot \widehat{\mathrm{ch}}(p^*_{\mathcal{A}^\vee\mathcal{B}\mathcal{B}^\vee} r^*_{\mathcal{B}\mathcal{B}^\vee}\overline{\mathcal{P}}_{\mathcal{B}}))) \\
= \;& (-1)^{(g_A+g_B)} r_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(p_{\mathcal{A}^\vee\mathcal{B}\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(p^*_{\mathcal{A}\mathcal{A}^\vee\mathcal{B}^\vee} q^*_{\mathcal{A}\mathcal{A}^\vee}\overline{\mathcal{P}}_{\mathcal{A}})) \cdot \widehat{\mathrm{ch}}(r^*_{\mathcal{B}\mathcal{B}^\vee}\overline{\mathcal{P}}_{\mathcal{B}})) \\
= \;& (-1)^{(g_A+g_B)} r_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(r^*_{\mathcal{A}^\vee\mathcal{B}^\vee}(q_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(q^*_{\mathcal{A}\mathcal{A}^\vee}\overline{\mathcal{P}}_{\mathcal{A}}))) \cdot \widehat{\mathrm{ch}}(r^*_{\mathcal{B}\mathcal{B}^\vee}\overline{\mathcal{P}}_{\mathcal{B}})) \\
= \;& (-1)^{(g_A+g_B)} r_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(r^*_{\mathcal{B}\mathcal{B}^\vee}\overline{\mathcal{P}}_{\mathcal{B}})) \cdot q_{\mathcal{A}^\vee\mathcal{B}^\vee,*}(\widehat{\mathrm{ch}}(q^*_{\mathcal{A}\mathcal{A}^\vee}\overline{\mathcal{P}}_{\mathcal{A}})) \\
= \;& (-1)^{g_A}\pi^*_{\mathcal{A}^\vee}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{A}})) \cdot (-1)^{g_B}\pi^*_{\mathcal{B}^\vee}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}_{\mathcal{B}})) = \pi^*_{\mathcal{A}^\vee}((S_0^{\vee,\mathcal{A}}, \mathfrak{g}_{\mathcal{A}})) \cdot \pi^*_{\mathcal{B}^\vee}((S_0^{\vee,\mathcal{B}}, \mathfrak{g}_{\mathcal{B}})) \\
= \;& (S_0^{\vee,\mathcal{A}\times\mathcal{B}}, \pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{B}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}}))
\end{aligned}$$

Here we have used the projection formula repeatedly, as well as the fact that direct images in arithmetic Chow theory are compatible with base-change. We also used Theorem 1.2.2. (in the second equation before last) and the definition

of the intersection product in arithmetic Chow theory (in the last equation). This settles (b).
To verify (c) we revert to the hypothesis made at the beginning of this subsection. In particular, we suppose that for some $n \geqslant 2$, the scheme $\ker[n]$ is the constant group scheme in $\mathcal{A}$. As explained at the beginning of this subsection, this does not restrict generality. First notice that by the definition of the symbols $[n]_*$ and $[n]^*$, we have

$$[n]^*[n]_*\mathfrak{g}_{\mathcal{A}} = \sum_{\tau\in\mathcal{A}^\vee[n](S)} \tau^*(\mathfrak{g}_{\mathcal{A}})$$

and thus by Theorem 1.1(c),

$$[n]^*\mathfrak{g}_{\mathcal{A}} = \sum_{\tau\in\mathcal{A}^\vee[n](S)} \tau^*(\mathfrak{g}_{\mathcal{A}}) \tag{27}$$

Of course, similar equations hold for $\mathfrak{g}_{\mathcal{B}}$. Notice also that equation (27) implies that $[n]_*\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}}$, as can be see by applying $[n]_*$ to both sides of equation (27) (see the calculation made after Proposition 4.3). Thus the equation (27) is actually equivalent to the equation $[n]_*\mathfrak{g}_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}}$.
Now we may compute

$$\begin{aligned}
& [n]^*_{\mathcal{A}^\vee\times\mathcal{B}^\vee}(\pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})) = \pi^*_{\mathcal{A}^\vee}([n]^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}})) * \pi^*_{\mathcal{B}^\vee}([n]^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})) \\
= \; & \pi^*_{\mathcal{A}^\vee}\Big(\sum_{\tau\in\mathcal{A}^\vee[n](S)} \tau^*(\mathfrak{g}_{\mathcal{A}})\Big) * \pi^*_{\mathcal{B}^\vee}\Big(\sum_{\tau\in\mathcal{B}^\vee[n](S)} \tau^*(\mathfrak{g}_{\mathcal{B}})\Big) \\
= \; & \sum_{\tau\in\mathcal{A}^\vee[n](S)\times\mathcal{B}^\vee[n](S)} \tau^*\big(\pi^*_{\mathcal{A}^\vee}(\mathfrak{g}_{\mathcal{A}}) * \pi^*_{\mathcal{B}^\vee}(\mathfrak{g}_{\mathcal{B}})\big)
\end{aligned}$$

which settles (c). Here we used in the first line the fact that the $*$-product is naturally compatible with finite étale pull-back.

4.2 Proof of 1.3.2

We shall apply the Adams-Riemann-Roch theorem in Arakelov geometry proven in [42, Th. 3.6]. Let $\overline{\mathcal{M}}$ be the rigidified hermitian line bundle on $\mathcal{A}$ corresponding to σ. By assumption, there is an isomorphism $\overline{\mathcal{M}}^{\otimes n} \simeq \overline{\mathcal{O}}_{\mathcal{A}}$ of rigidified hermitian line bundles. Let k, l be two positive integers such that $k = l \pmod n$. Let $\overline{\Omega} := \overline{\Omega}_{\mathcal{A}/S}$. The theorem [42, Th. 3.6] implies that the identity

$$\begin{aligned}
& \psi^k(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \psi^k(T(\lambda,\overline{\mathcal{M}})) \\
& \qquad = \theta^k(\epsilon^*\overline{\Omega})^{-1}\mathrm{R}^\bullet\pi_*(\overline{M}^{\otimes k}) - \mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega})^{-1})T(\lambda,\overline{\mathcal{M}}^{\otimes k})
\end{aligned} \tag{28}$$

holds in $\widehat{K}_0(S)[\frac{1}{k}]$ and that the identity

$$\begin{aligned}
& \psi^l(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \psi^l(T(\lambda,\overline{\mathcal{M}})) \\
& \qquad = \theta^l(\epsilon^*\overline{\Omega})^{-1}\mathrm{R}^\bullet\pi_*(\overline{M}^{\otimes l}) - \mathrm{ch}(\theta^l(\epsilon^*\overline{\Omega})^{-1})T(\lambda,\overline{\mathcal{M}}^{\otimes l})
\end{aligned} \tag{29}$$

holds in $\widehat{K}_0(S)[\frac{1}{l}]$. Here the symbols $\psi^*(\cdot)$ refer to the Adams operations acting on arithmetic K_0-theory; see [42, sec. 2 and before Th. 3.6] for the exact definition. For the definition of the symbol $\theta^*(\cdot)$, see [42, sec. 2]. In the following computations, we shall need the following properties of these symbols. Define $\phi^t(\cdot)$ to be the additive operator, which sends a differential form η of type (r,r) to the differential form $t^r\cdot\eta$. It is proven in [42, Prop. 4.2] that $\theta^k(\epsilon^*\overline{\Omega})$ is a unit in $\widehat{K}_0(S)[\frac{1}{k}]$ and we have

$$\mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega})) = k^{\mathrm{rk}(\Omega)}\,\mathrm{Td}(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{\mathrm{Td}}^{-1}(\epsilon^*\overline{\Omega}^\vee)) \tag{30}$$

(and similarly for l instead of k). See [42, Lemma 6.11] for this. Secondly, if $\eta\in\widetilde{A}(S_\mathbb{R})$, then we have

$$\psi^k(\eta) = k\cdot\phi^k(\eta)$$

in $\widehat{K}_0(S)$. In view of the fact that $\theta^k(\epsilon^*\overline{\Omega})$ is a unit in $\widehat{K}_0(S)[\frac{1}{k}]$, we get the equation

$$\begin{aligned}\mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(T(\lambda,\overline{\mathcal{M}}))&\\ = \mathrm{R}^\bullet\pi_*(\overline{M}^{\otimes k}) - T(\lambda,\overline{\mathcal{M}}^{\otimes k})&\end{aligned}$$

in $\widehat{K}_0(S)[\frac{1}{k}]$ from equation (28). Similarly, we get

$$\mathrm{ch}(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \mathrm{ch}(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(T(\lambda,\overline{\mathcal{M}})) = \mathrm{R}^\bullet\pi_*(\overline{M}^{\otimes l}) - T(\lambda,\overline{\mathcal{M}}^{\otimes l})$$

in $\widehat{K}_0(S)[\frac{1}{l}]$. Since $k=l \pmod n$, we obtain the equation

$$\begin{aligned}\mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \mathrm{ch}(\theta^k(\epsilon^*\overline{\Omega}))\psi^k(T(\lambda,\overline{\mathcal{M}}))&\\ = \mathrm{ch}(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(\mathrm{R}^\bullet\pi_*\overline{\mathcal{M}}) - \mathrm{ch}(\theta^l(\epsilon^*\overline{\Omega}))\psi^l(T(\lambda,\overline{\mathcal{M}}))&\end{aligned}$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$. In view of equation (30) and of the fact that $\mathrm{R}^r\pi_*\mathcal{M}=0$ for all $r\geqslant 0$, this translates to the equation

$$\begin{aligned}k^g\,\mathrm{Td}(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{\mathrm{Td}}^{-1}(\epsilon^*\overline{\Omega}^\vee))\psi^k(T(\lambda,\overline{\mathcal{M}}))&\\ = l^g\,\mathrm{Td}(\epsilon^*\overline{\Omega}^\vee)\phi^l(\overline{\mathrm{Td}}^{-1}(\epsilon^*\overline{\Omega}^\vee))\psi^l(T(\lambda,\overline{\mathcal{M}}))&\end{aligned}$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$. Recall that in [27] it is shown that $T(\lambda,\overline{\mathcal{M}}) = \mathrm{Td}^{-1}(\epsilon^*\overline{\Omega})\gamma$, where γ is a real differential form of type $(g-1,g-1)$ on S. So we may rewrite

$$\begin{aligned}k^{g+1}\,\mathrm{Td}(\epsilon^*\overline{\Omega}^\vee)\phi^k(\overline{\mathrm{Td}}^{-1}(\epsilon^*\overline{\Omega}^\vee))\phi^k(\mathrm{Td}^{-1}(\epsilon^*\overline{\Omega}))\phi^k(\gamma)&\\ = l^{g+1}\,\mathrm{Td}(\epsilon^*\overline{\Omega}^\vee)\phi^l(\overline{\mathrm{Td}}^{-1}(\epsilon^*\overline{\Omega}^\vee))\phi^l(\mathrm{Td}^{-1}(\epsilon^*\overline{\Omega}))\phi^l(\gamma)&\end{aligned} \tag{31}$$

Furthermore

Lemma 4.4. *We have* $\mathrm{Td}^{-1}(\epsilon^*\overline{\Omega}^\vee\oplus\epsilon^*\overline{\Omega}) = 1$.

Proof. We may (and do) assume that $R = \mathbb{C}$. Consider the relative Hodge extension

$$0 \to \mathrm{R}^0\pi_*(\Omega) \to H^1_{\mathrm{dR}}(\mathcal{A}/S) \to R^1\pi_*(\mathcal{O}_{\mathcal{A}}) \to 0 \tag{32}$$

where $H^1_{\mathrm{dR}}(\mathcal{A}/S) := R^1\pi_*(\Omega^\bullet_{\mathcal{A}/S})$ is the first relative de Rham cohomology sheaf. The sequence (32) is the expression of the filtration on $R^1\pi_*(\Omega^\bullet_{\mathcal{A}/S})$, which comes from the relative Hodge to de Rham spectral sequence. This spectral sequence is the first hypercohomology spectral sequence of the relative de Rham complex $\Omega^\bullet_{\mathcal{A}/S}$ and it degenerates by [13, Prop. 5.3]. The relative form of the GAGA theorem shows that there is an isomorphism of holomorphic vector bundles $H^1_{\mathrm{dR}}(\mathcal{A}/S)(\mathbb{C}) \simeq (R^1\pi(\mathbb{C})_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S(\mathbb{C})}$ (see [14, p. 31]) and via this isomorphism we endow $H^1_{\mathrm{dR}}(\mathcal{A}/S)(\mathbb{C})$ with the fibrewise Hodge metric, whose formula is given in [38, before Lemma 2.7]. Since the metric on $H^1_{\mathrm{dR}}(\mathcal{A}/S)(\mathbb{C})$ is locally constant by construction, the curvature matrix of the hermitian vector bundle $H^1_{\mathrm{dR}}(\mathcal{A}/S)(\mathbb{C}) \simeq (R^1\pi(\mathbb{C})_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S(\mathbb{C})}$ vanishes. Now the formula in [38, Lemma 2.7] shows that in the sequence (32), the L^2-metric on the first term corresponds to the induced metric and the L^2-metric on the end term corresponds to the quotient metric. We now view the sequence (32) as a sequence of hermitian vector bundles with the metrics described above. Using [36, Th. 3.4.1], we see that the secondary class $\widetilde{\mathrm{Td}}$ of the sequence (32) is dd^c-closed. Thus

$$\mathrm{Td}(R^1\pi_*(\mathcal{O}_{\mathcal{A}}, L^2) \oplus \mathrm{R}^0\pi_*(\Omega, L^2)) = \mathrm{Td}((H^1_{\mathrm{dR}}(\mathcal{A}/S), \text{Hodge metric})) = 1.$$

Now notice that by relative Lefschetz duality for Hodge cohomology (see [13, Lemme 6.2]) and Grothendieck duality, there is an isomorphism of $\mathcal{O}_S$-modules $\phi_\lambda : R^1\pi_*(\mathcal{O}_{\mathcal{A}}) \xrightarrow{\sim} \mathrm{R}^0\pi_*(\Omega)^\vee$, which is dependent on λ. To describe it, let $S = \operatorname{Spec}\mathbb{C}$. Under the Hodge-de Rham splitting of the sequence (32), the morphism ϕ_λ is given by the formula

$$\omega \mapsto \omega \wedge \lambda^{g-1} \mapsto \int_{\mathcal{A}} (\omega \wedge \lambda^{g-1}) \wedge (\cdot)$$

(notice that this formula does actually not depend on the splitting). Now comparing the last formula with the formula for the Hodge metric in [38, before Lemma 2.7], we see that, up to a constant factor, ϕ_λ induces an isometry between $R^1\pi_*(\mathcal{O}_{\mathcal{A}}, L^2)$ and the dual of the hermitian vector bundle $\mathrm{R}^0\pi_*(\Omega, L^2)$. To complete the proof, notice that since the volume of the fibres of $\pi(\mathbb{C})$ is locally constant (by the assumption on λ), the natural isomorphism of vector bundles $\mathrm{R}^0\pi_*(\Omega, L^2) \simeq \epsilon^*\overline{\Omega}$ is an isometry up to a locally constant factor. Hence the Chern forms of $\mathrm{R}^0\pi_*(\Omega, L^2)$ and $\epsilon^*\overline{\Omega}$ are the same. This completes the proof. □

Together with Lemma 4.4, we see that (31) gives the identity

$$k^{2g}\,\mathrm{Td}(\overline{\Omega}^\vee)\gamma = l^{2g}\,\mathrm{Td}(\overline{\Omega}^\vee)\gamma$$

or in other words

$$\boxed{(k^{2g} - l^{2g})T(\lambda, \overline{\mathcal{M}}) = 0} \tag{33}$$

in $\widehat{K}_0(S)[\frac{1}{kl}]$.
We shall now show that equation (33) implies the identity

$$2g \cdot n \cdot N_{2g} \cdot T(\lambda, \overline{\mathcal{M}}) = 0 \tag{34}$$

in $\widehat{K}_0(S)$. For this consider the following combinatorial lemma.

Lemma 4.5. *Let G be an abelian group, written additively. Let $c \geqslant 1$ and let $\alpha \in G$. Suppose that for all $k, l > 0$ such that $k = l \pmod n$, we have*

$$(l^c - k^c) \cdot \alpha = 0$$

in $G[\frac{1}{kl}]$. Then

$$\mathrm{order}(\alpha) \mid 2 \cdot n \cdot c \cdot [\prod_{\substack{p \text{ prime}, \\ p \nmid n,\ (p-1)|c}} p]$$

Proof. Rephrasing the hypotheses of the lemma, we find that for any $k, l > 0$ such that $k = l \pmod n$, there exist integers $a, b \geqslant 0$ (depending on the couple (k, l)) such that:

$$k^a l^b (l^c - k^c) \cdot g = 0. \tag{35}$$

For any (k, l) as above, we will denote by γ the integer such that $l = k + \gamma n$. In what follows, p will be a prime dividing $\mathrm{order}(\alpha)$ and we will write δ_p (or in short δ if no confusion can occur) the positive valuation $v_p(\mathrm{order}(\alpha))$.
We deduce from equation (35) that:

$$p^\delta \mid k^a l^b (l^c - k^c).$$

From now on, k and l will be chosen such that $k, l \neq 0 \pmod p$, this implying in particular that the classes of k and l in $\mathbb{Z}/p^\delta\mathbb{Z}$ are invertible. We thus get:

$$p^\delta \mid (l^c - k^c)$$

or equivalently:

$$(l/k)^c = 1 \pmod{p^\delta}$$

in $(\mathbb{Z}/p^\delta\mathbb{Z})^*$. We will denote by $C \subset (\mathbb{Z}/p^\delta\mathbb{Z})^*$ the set of the classes (l/k) with l and k as above. The restriction of the map $\varphi : x \mapsto x^c$ to C is then identically equal to 1. In what follows, in order to bound δ, we determine the set C. We must distinguish between two cases.

Case 1. The prime p doesn't divide n.
Taking $k = n$, we have $(l/k) = (k + \gamma n)/k = 1 + \gamma$ for all the integers γ satisfying $(1 + \gamma) \wedge p = 1$. We deduce from this that necessarily:

$$C = (\mathbb{Z}/p^\delta\mathbb{Z})^*.$$

We now need the following well-known group isomorphisms, which we recall for the sake of the exposition (the group laws are multiplicative on the left-hand side and additive on the right-hand side):

- If $p \neq 2$, $(\mathbb{Z}/p^\delta\mathbb{Z})^* \simeq \mathbb{Z}/p^{\delta-1}(p-1)\mathbb{Z}$.
- If $p = 2$, $(\mathbb{Z}/2\mathbb{Z})^* \simeq \{0\}$, $(\mathbb{Z}/4\mathbb{Z})^* \simeq \mathbb{Z}/2\mathbb{Z}$ and for $\delta \geqslant 3$, $(\mathbb{Z}/2^\delta\mathbb{Z})^* \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{\delta-2}\mathbb{Z})$, the projection on the first factor being the obvious reduction map to $(\mathbb{Z}/4\mathbb{Z})^* \simeq \mathbb{Z}/2\mathbb{Z}$.

Let's then discuss the two subcases $p \neq 2$ and $p = 2$ separately.
If $p \neq 2$, the restriction of φ to $C = (\mathbb{Z}/p^\delta\mathbb{Z})^* \simeq \mathbb{Z}/p^{\delta-1}(p-1)\mathbb{Z}$ is equal to 1 identically. On the righten side, φ is the multiplication by $\overline{c}$ the class of c in $\mathbb{Z}/p^{\delta-1}(p-1)\mathbb{Z}$. One must then have $\overline{c} = 0$, and thus:

$$(p-1)p^{\delta-1} \mid c,$$

i.e. the two conditions $p - 1 \mid c$ and $\delta_p \leqslant 1 + v_p(c)$.
If $p = 2$, using the same argument as above, the discussion falls into three different subsubcases:

- $\delta_2 = 1$, no condition.
- $\delta_2 = 2$, we find that $2 \mid c$.
- $\delta_2 \geqslant 3$, we get that $2 \mid c$ and $2^{\delta_2-2} \mid c$, i.e. $\delta_2 \leqslant 2 + v_2(c)$.

Summing up those three subsubcases, we finally conclude that if c is odd then $\delta_2 = 1$ and if c is even then $\delta_2 \leqslant 2 + v_2(c)$.

Let's now come to the:

Case 2. The prime p divide n.
Let's define $\beta := v_p(n) \geqslant 1$ and $n' := n/p^\beta$ and let's suppose in addition that $\delta > \beta$.
We compute in $(\mathbb{Z}/p^\beta\mathbb{Z})^*$

$$l/k = (k + \gamma n)/k = 1 + \gamma n/k = 1 + (\gamma n'/k)p^\beta = 1$$

and thus the set C is contained in the kernel K of the reduction morphism:

$$(\mathbb{Z}/p^\delta\mathbb{Z})^* \longrightarrow (\mathbb{Z}/p^\beta\mathbb{Z})^*.$$

The integer n' being prime to p, its class in $(\mathbb{Z}/p^\delta\mathbb{Z})$ is invertible and one can take $k = n'$. We find that for any integer γ:

$$l/k = 1 + \gamma n/n' = 1 + \gamma p^\beta$$

in $(\mathbb{Z}/p^\delta\mathbb{Z})^*$, this implying the set equality:

$$C = K.$$

We are thus left to determine K. Again, we will distinguish between two subcases:

If $p \neq 2$ we have:

$$\#K = \#(\mathbb{Z}/p^{\delta}\mathbb{Z})^*/\#(\mathbb{Z}/p^{\beta}\mathbb{Z})^* = p^{\delta-1}(p-1)/p^{\beta-1}(p-1) = p^{\delta-\beta}$$

from which we deduce first that $K \simeq \mathbb{Z}/p^{\delta-\beta}\mathbb{Z}$ and then immediately that $p^{\delta-\beta} \mid c$, i.e.

$$\delta \leqslant \beta + v_p(c) = v_p(n) + v_p(c).$$

If $p = 2$, the discussion now falls into five different subsubcases.

- $\beta = 1$ and $\delta_2 = 1$, no condition.
- $\beta = 1$ and $\delta_2 = 2$, we find that $2 \mid c$, i.e. c is even.
- $\beta = 1$ and $\delta_2 \geqslant 3$, then we get $2 \mid c$ and $2^{\delta_2-2} \mid c$, i.e. $\delta_2 \leqslant 2 + v_2(c)$ with c being necessary even.

  To summarize those three first cases let's write that when $\beta = 1$, if c odd then $\delta_2 = 1$ and if c is even then $\delta_2 \leqslant 2 + v_2(c)$.
- $\beta = 2$ and $\delta_2 = 2$, no condition.
- In all other cases, the kernel K is contained in $(\mathbb{Z}/2^{\delta_2-2}\mathbb{Z})$ and as a consequence is cyclic. We thus get:

  $$\#K = \#(\mathbb{Z}/2^{\delta_2}\mathbb{Z})^*/\#(\mathbb{Z}/2^{\beta}\mathbb{Z})^* = 2^{\delta_2-1}/2^{\beta-1} = 2^{\delta_2-\beta}$$

  and so $2^{\delta_2-\beta} \mid c$, from which we deduce:

  $$\delta_2 \leqslant \beta + v_2(c) = v_2(n) + v_2(c).$$

In conclusion of the subcase $p = 2$, we find that:

$$\delta_2 \leqslant v_2(n) + v_2(c) + w_2,$$

with $w_2 = 1$ if $v_2(n) = 1$ and c is even, and $w_2 = 0$ otherwise.

Putting everything together, we have finally proven that:

$$\mathrm{order}(\alpha) \quad \mid \quad F_2 \times \prod_{\substack{p \text{ prime, } p\neq 2, \\ p \text{ doesn't divide } n,\ (p-1)\mid c}} p^{1+v_p(c)} \times \prod_{\substack{p \text{ prime,} \\ p\neq 2,\ p\mid n}} p^{v_p(n)+v_p(c)}$$

where the F_2 factor is given by the following rules:

- if n and c are odd then $F_2 = 2$,
- if n is odd and c is even then $F_2 = 2^{2+v_2(c)}$,
- if n is even then $F_2 = 2^{v_2(n)+v_2(c)+w_2}$; with $w_2 = 1$ if $v_2(n) = 1$ and c is even, and $w_2 = 0$ otherwise.

The lemma's statement is then a direct consequence of this (more precise) assertion. □

In view of Lemma 4.5, the set of identities (33) implies that the order of $T(\lambda, \overline{\mathcal{M}})$ in $\widehat{K}_0(S)$ divides $2 \cdot n \cdot 2g \cdot [\prod_{p \text{ prime},\ p \nmid n,\ (p-1)|2g} p]$.

Now use the notations of the last lemma. It is shown in [40, Appendix B] that the equality

$$2 \cdot \text{denominator}[(-1)^{\frac{c+2}{2}} B_c/c] = 2 \cdot \prod_{\substack{p \text{ prime,} \\ (p-1)|c}} p^{\mathrm{ord}_p(c)+1} \tag{36}$$

holds if c is even. This proves the identity (34) .

To conclude the proof of Theorem 1.3.2, recall that there is an exact sequence

$$K_1(\mathcal{A}^\vee) \xrightarrow{-2\mathrm{reg}_{\mathrm{an}}} \oplus_{p \geqslant 0} \widetilde{A}^{p,p}(\mathcal{A}^\vee_{\mathbb{R}}) \xrightarrow{a} \widehat{K}_0(\mathcal{A}^\vee) \to K_0(\mathcal{A}^\vee) \to 0$$

(see [21, Th. 6.2 (i)] for this).

5 The case of elliptic schemes

In this last section, we shall consider elliptic schemes and compare the conclusions of Theorems 1.1, 1.2 and 1.3 with classical results on elliptic units.

So suppose that $\mathcal{A}$ is of relative dimension 1, i.e. that $\mathcal{A}$ is an elliptic scheme over S. Suppose also that the structural morphism $S \to \operatorname{Spec} R$ is the identity on $\operatorname{Spec} R$. Let $\sigma \in \Sigma$ be an embedding of R into $\mathbb{C}$. There exists an isomorphism of complex Lie groups

$$\mathcal{A}(\mathbb{C})_\sigma := (\mathcal{A} \times_{R,\sigma} \mathbb{C})(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot \tau_\sigma) \tag{37}$$

where $\tau_\sigma \in \mathbb{C}$ lies in the upper half plane.

Proposition 5.1. (a) *The restriction of* $\mathfrak{g}_{\mathcal{A}^\vee}$ *to* $\mathcal{A}(\mathbb{C})_\sigma = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot \tau_\sigma)\backslash\{0\}$ *is given by the function*

$$\phi(z) = \phi_{\mathcal{A},\sigma}(z) := -2\log|e^{-z\cdot\eta(z)/2}\mathrm{sigma}(z)\Delta(\tau_\sigma)^{\frac{1}{12}}|$$

(b) *Endow* $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\cdot\tau_\sigma)$ *with its Haar measure of total measure* 1*. The function* ϕ *then defines an* L^1*-function on* $\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot \tau_\sigma)$ *and the restriction of* $\mathfrak{g}_{\mathcal{A}^\vee}$ *to* $\mathcal{A}(\mathbb{C})_\sigma$ *is the current* $[\phi]$ *associated with* ϕ.

Here $\Delta(\bullet)$ is the discriminant modular form, sigma(z) is the Weierstrass sigma-function associated with the lattice $[1, \tau_\sigma]$ and η is the quasi-period map associated with the lattice $[1, \tau_\sigma]$, extended $\mathbb{R}$-linearly to all of $\mathbb{C}$ (see [43, I, Prop. 5.2] for the latter).

Proof. The formula (a) follows from Theorem 1.3.1 and the formula for the Ray-Singer analytic torsion of a flat line bundle on an elliptic curve given in [41, Th. 4.1].

For statement (b), notice that there exists a Green form η of log type along 0 (see [19, Th. 1.3.5, p.106]). In this situation η is a real-valued C^∞ function on $\mathcal{A}(\mathbb{C})_\sigma\backslash\{0\}$, which is locally and hence globally L^1 on $\mathcal{A}(\mathbb{C})_\sigma$ (because $\mathcal{A}(\mathbb{C})_\sigma$ is compact). By definition the current $[\eta]$ associated with η is a Green current for 0 and by [19, Lemma 1.2.4], there exists a C^∞ real-valued function f on $\mathcal{A}(\mathbb{C})_\sigma$, such that $\mathfrak{g}_{\mathcal{A}^\vee}|_{\mathcal{A}(\mathbb{C})_\sigma} = [\eta] + [f] = [\eta + f]$. Now by construction the restriction of $\mathfrak{g}_{\mathcal{A}^\vee}$ to $\mathcal{A}(\mathbb{C})_\sigma\backslash\{0\}$ is given by the current associated with the restriction to $\mathcal{A}(\mathbb{C})_\sigma\backslash\{0\}$ of the locally L^1-function $\eta + f$. Since $\eta + f$ and ϕ are both C^∞ on $\mathcal{A}(\mathbb{C})_\sigma\backslash\{0\}$, they must actually coincide on $\mathcal{A}(\mathbb{C})_\sigma\backslash\{0\}$. This proves (b). $\square$

The distribution relations for $\mathfrak{g}_{\mathcal{A}^\vee}$ (i.e. Theorem 1.1.3) imply that the function $\phi(z)$ has the property that

$$\sum_{\substack{w\in\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\cdot\tau_\sigma),\\ n\cdot w=z}} \phi_{\mathcal{A},\sigma}(w) = \phi_{\mathcal{A},\sigma}(z).$$

for any $n \geqslant 2$ and $z \in \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\cdot\tau_\sigma)\backslash\{0\}$. This also follows from the more precise distribution relations given in [28, Th. 4.1, p. 43].

Now suppose that R is a Dedekind ring. We suppose given $\underline{z} \in \mathcal{A}(S)$, whose image is disjoint from the unit section and such that $n\cdot\underline{z} = 0$. Let $\underline{z}_\sigma \in \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\cdot\tau_\sigma)$ be the element corresponding to $\underline{z}$. We compute from the definition that $N_2 = 24$. Theorem 1.3.2 now implies that there exists $u \in R^*$, which does not depend on $\sigma \in \Sigma$, such that

$$\log|\sigma(u)| = 24\cdot n\cdot\phi_{\mathcal{A},\sigma}(\underline{z}_\sigma) \tag{38}$$

In particular the real number

$$\exp(24\cdot n\cdot\phi_{\mathcal{A},\sigma}(\underline{z}_\sigma))$$

is an algebraic unit. If R is the ring of integers of a number field, n has at least two distinct prime factors and $\Sigma = \{\sigma, \bar\sigma\}$ then (38) is also a consequence of [28, Th. 2.2, p. 37]. Notice that (38) overlaps with part of the reciprocity law for elliptic units, if $\mathcal{A}_{\overline{\mathrm{Frac}(R)}}$ is assumed to have complex multiplication by the ring of integers of an imaginary quadratic field (see [31, chap. 19, par. 3, Th. 3]). Special instances of elliptic units were first (implicitly) constructed by Eisenstein in his analytic proof of cubic and quartic reciprocity laws (cf. [18] and [16], [17]). For a thorough historical discussion of Eisenstein's contribution and additional references, see [33, §8].

Remark. Our proof of the fact that the real number $\exp(24\cdot n\cdot\phi_{\mathcal{A},\sigma}(\underline{z}_\sigma))$ is an algebraic unit shows that 24 naturally comes from a Bernoulli number via von Staudt's theorem. Indeed, von Staudt's theorem is the main tool in the proof of (36). The proof of the fact that the number $\exp(24\cdot n\cdot\phi_{\mathcal{A},\sigma}(\underline{z}_\sigma))$ (in fact even the number $\exp(12\cdot n\cdot\phi_{\mathcal{A},\sigma}(\underline{z}_\sigma))$) is an algebraic unit, which is given in [28, Th. 2.2, p. 37], does not seem to establish such a link.

References

[1] Paul Baum, William Fulton, and George Quart, *Lefschetz-Riemann-Roch for singular varieties*, Acta Math. 143 (1979), no. 3-4, 193–211.

[2] Arnaud Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. 273 (1986), no. 4, 647–651 (French).

[3] Alain Berthomieu, *Comparaison de métriques de Quillen sur l'inverse du déterminant de la cohomologie du fibré de Poincaré*, Thèse, Orsay, 1994. Unpublished.

[4] Jean-Michel Bismut, Henri Gillet, and Christophe Soulé, *Bott-Chern currents and complex immersions*, Duke Math. J. 60 (1990), no. 1, 255–284.

[5] ______, *Complex immersions and Arakelov geometry*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 249–331.

[6] Jean-Michel Bismut and Kai Köhler, *Higher analytic torsion forms for direct images and anomaly formulas*, J. Algebraic Geom. 1 (1992), no. 4, 647–684.

[7] Jean-Michel Bismut and Xiaonan Ma, *Familles d'immersions holomorphes et formes de torsion analytique équivariantes*, C. R. Math. Acad. Sci. Paris 334 (2002), no. 10, 893–897, DOI 10.1016/S1631-073X(02)02372-5.

[8] Jean-Benoît Bost, *Green's currents and height pairing on complex tori*, Duke Math. J. 61 (1990), no. 3, 899–912.

[9] Jose Ignacio Burgos, *The regulators of Beilinson and Borel*, CRM Monograph Series, vol. 15, American Mathematical Society, Providence, RI, 2002.

[10] ______, *Arithmetic Chow rings and Deligne-Beilinson cohomology*, J. Algebraic Geom. 6 (1997), no. 2, 335–377.

[11] Jose Ignacio Burgos and Razvan Litcanu, *Singular Bott-Chern classes and the arithmetic Grothendieck Riemann Roch theorem for closed immersions*, Documenta Mathematica 15 (2010), no. 15, 73–176.

[12] Jose Ignacio Burgos and Steve Wang, *Higher Bott-Chern forms and Beilinson's regulator*, Invent. Math. 132 (1998), no. 2, 261–305.

[13] Pierre Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. 35 (1968), 259–278 (French).

[14] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982.

[15] Georges de Rham, *L'oeuvre d'Élie Cartan et la topologie*, Élie Cartan, 1869–1951 (hommage de l'Acad. République Socialiste de Roumanie à l'occasion du centenaire de sa naissance), Editura Acad. R.S.R., Bucharest, 1975, pp. 11–20 (French).

[16] Gotthold Eisenstein, *Briefe an Gauss* (25.01.1845). Math. Werke II, 825–831.

[17] ______, *Briefe an Gauss* (18.02.1845). Math. Werke II, 832–837.

[18] ______, *Application de l'algèbre à l'arithmétique transcendante*, J. Reine Angew. Math. 29 (1845), 177–184. Math. Werke I, 291–298.

[19] Henri Gillet and Christophe Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. 72 (1990), 93–174 (1991).

[20] ______, *Characteristic classes for algebraic vector bundles with Hermitian metric. I*, Ann. of Math. (2) 131 (1990), no. 1, 163–203.

[21] ______, *Characteristic classes for algebraic vector bundles with Hermitian metric. II*, Ann. of Math. (2) 131 (1990), no. 2, 205–238.

[22] Henri Gillet, Damian Rössler, and Christophe Soulé, *An arithmetic Riemann-Roch theorem in higher degrees*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 6, 2169–2189 (English, with English and French summaries).

[23] Alexander Grothendieck, *Éléments de géométrie algébrique.* Inst. Hautes Études Sci. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32 (1960-1967).

[24] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis; Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[25] Guido Kings, *K-theory elements for the polylogarithm of abelian schemes*, J. Reine Angew. Math. 517 (1999), 103–116.

[26] Guido Kings and Damian Rössler, *Currents, polylogarithms and norm compatible elements on abelian schemes*, Preprint. arXiv:1412.2925.

[27] Kai Köhler, *Complex analytic torsion forms for torus fibrations and moduli spaces*, Regulators in analysis, geometry and number theory, Progr. Math., vol. 171, Birkhäuser Boston, Boston, MA, 2000, pp. 167–195.

[28] Daniel S. Kubert and Serge Lang, *Modular units*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 244, Springer-Verlag, New York, 1981.

[29] Klaus Künnemann, *Arakelov Chow groups of abelian schemes, arithmetic Fourier transform, and analogues of the standard conjectures of Lefschetz type*, Math. Ann. 300 (1994), no. 3, 365–392, DOI 10.1007/BF01450492.

[30] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[31] ______, *Elliptic functions*, 2nd ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987. With an appendix by J. Tate.

[32] Gérard Laumon, *Transformation de Fourier generalisée*, arXiv alg-geom/9603004.

[33] Franz Lemmermeyer, *Reciprocity laws*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. From Euler to Eisenstein.

[34] Andrey Levin, *Polylogarithmic currents on abelian varieties*, Regulators in analysis, geometry and number theory, Progr. Math., vol. 171, Birkhäuser Boston, Boston, MA, 2000, pp. 207–229.

[35] Xiaonan Ma, *Formes de torsion analytique et familles de submersions. I*, Bull. Soc. Math. France 127 (1999), no. 4, 541–621 (French, with English and French summaries).

[36] Vincent Maillot, *Un calcul de Schubert arithmétique*, Duke Math. J. 80 (1995), no. 1, 195–221 (French).

[37] Vincent Maillot and Damian Rössler, *Elements of the group K_1 associated to abelian schemes.*, Abstract of a talk given during the 2008 Oberwolfach meeting on arithmetic algebraic geometry (August 3rd-9th, 2008). Oberwolfach Reports 5 (2008), no. 3, 2013-2014.

[38] Vincent Maillot and Damian Roessler, *On the periods of motives with complex multiplication and a conjecture of Gross-Deligne*, Ann. of Math. (2) 160 (2004), no. 2, 727–754.

[39] James S. Milne, *Abelian varieties*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 103–150.

[40] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[41] Daniel B. Ray and Isadore M. Singer, *Analytic torsion for complex manifolds*, Ann. of Math. (2) 98 (1973), 154–177.

[42] Damian Roessler, *An Adams-Riemann-Roch theorem in Arakelov geometry*, Duke Math. J. 96 (1999), no. 1, 61–126.

[43] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

[44] Christophe Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[45] Jörg Wildeshaus, *Realizations of polylogarithms*, Lecture Notes in Mathematics, vol. 1650, Springer-Verlag, Berlin, 1997.

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