The Eisenstein Ideal and Jacquet-Langlands Isogeny over Function Fields

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Received: May 4, 2014
Revised: March 13, 2015

Communicated by Peter Schneider

Abstract. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two distinct prime ideals of $\mathbb{F}_q[T]$. We use the Eisenstein ideal of the Hecke algebra of the Drinfeld modular curve $X_0(\mathfrak{pq})$ to compare the rational torsion subgroup of the Jacobian $J_0(\mathfrak{pq})$ with its subgroup generated by the cuspidal divisors, and to produce explicit examples of Jacquet-Langlands isogenies. Our results are stronger than what is currently known about the analogues of these problems over $\mathbb{Q}$.

2010 Mathematics Subject Classification: 11G09, 11G18, 11F12
Keywords and Phrases: Drinfeld modular curves; Cuspidal divisor group; Shimura subgroup; Eisenstein ideal; Jacquet-Langlands isogeny

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Contents

[1]The first author was supported in part by the Simons Foundation. The second author was partially supported by National Science Council and Max Planck Institute for Mathematics.

1. Introduction

1.1. Motivation. Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime number p. Let $A = \mathbb{F}_q[T]$ be the ring of polynomials in indeterminate T with coefficients in $\mathbb{F}_q$, and $F = \mathbb{F}_q(T)$ the field of fractions of A. The degree map $\deg : F \to \mathbb{Z} \cup \{-\infty\}$, which associates to a non-zero polynomial its degree in T and $\deg(0) = -\infty$, defines a norm on F by $|a| := q^{\deg(a)}$. The corresponding place of F is usually called the *place at infinity*, and is denoted by ∞. We also define a norm and degree on the ideals of A by $|\mathfrak{n}| := \#(A/\mathfrak{n})$ and $\deg(\mathfrak{n}) := \log_q |\mathfrak{n}|$. Let F_∞ denote the completion of F at ∞, and $\mathbb{C}_\infty$ denote the completion of an algebraic closure of F_∞. Let $\Omega := \mathbb{C}_\infty - F_\infty$ be the *Drinfeld half-plane*.

Let $\mathfrak{n} \lhd A$ be a non-zero ideal. The level-$\mathfrak{n}$ *Hecke congruence subgroup* of $\mathrm{GL}_2(A)$

$$\Gamma_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n} \right\}$$

plays a central role in this paper. This group acts on Ω via linear fractional transformations. Drinfeld proved in [6] that the quotient $\Gamma_0(\mathfrak{n}) \setminus \Omega$ is the space of $\mathbb{C}_\infty$-points of an affine curve $Y_0(\mathfrak{n})$ defined over F, which is a moduli space of rank-2 Drinfeld modules (we give a more formal discussion of Drinfeld modules and their moduli schemes in Section 4). The unique smooth projective curve over F containing $Y_0(\mathfrak{n})$ as an open subvariety is denoted by $X_0(\mathfrak{n})$. The *cusps* of $X_0(\mathfrak{n})$ are the finitely many points of the complement of $Y_0(\mathfrak{n})$ in $X_0(\mathfrak{n})$; the cusps generate a finite subgroup $\mathcal{C}(\mathfrak{n})$ of the Jacobian variety $J_0(\mathfrak{n})$ of $X_0(\mathfrak{n})$, called the *cuspidal divisor group*. By the Lang-Néron theorem, the group of F-rational points of $J_0(\mathfrak{n})$ is finitely generated, in particular, its torsion subgroup $\mathcal{T}(\mathfrak{n}) := J_0(\mathfrak{n})(F)_{\mathrm{tor}}$ is finite. It is known that when $\mathfrak{n}$ is square-free $\mathcal{C}(\mathfrak{n}) \subseteq \mathcal{T}(\mathfrak{n})$.

For a square-free ideal $\mathfrak{n} \lhd A$ divisible by an even number of primes, let D be the division quaternion algebra over F with discriminant $\mathfrak{n}$. The group of units $\Gamma^{\mathfrak{n}}$ of a maximal A-order in D acts on Ω, and the quotient $\Gamma^{\mathfrak{n}} \setminus \Omega$ is the space of $\mathbb{C}_\infty$-points of a smooth projective curve $X^{\mathfrak{n}}$ defined over F; this curve is a moduli space of $\mathscr{D}$-elliptic sheaves introduced in [28]. Let $J^{\mathfrak{n}}$ be the Jacobian variety of $X^{\mathfrak{n}}$.

The analogy between $X_0(\mathfrak{n})$ and the classical modular curves $X_0(N)$ over $\mathbb{Q}$ classifying elliptic curves with $\Gamma_0(N)$-structures is well-known and has been extensively studied over the last 35 years. Similarly, the modular curves $X^{\mathfrak{n}}$ are the function field analogues of Shimura curves X^N parametrizing abelian surfaces equipped with an action of the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant N.

Let $\mathbb{T}(\mathfrak{n})$ be the $\mathbb{Z}$-algebra generated by the Hecke operators $T_{\mathfrak{m}}$, $\mathfrak{m} \lhd A$, acting on the group $\mathcal{H}_0(\mathscr{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ of $\mathbb{Z}$-valued $\Gamma_0(\mathfrak{n})$-invariant cuspidal harmonic cochains on the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(F_\infty)$. The *Eisentein ideal* $\mathfrak{E}(\mathfrak{n})$ of $\mathbb{T}(\mathfrak{n})$ is the ideal generated by the elements $T_{\mathfrak{p}} - |\mathfrak{p}| - 1$, where $\mathfrak{p} \nmid \mathfrak{n}$ is prime. In this paper we study the Eisenstein ideal in the case when $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$ is a product of two distinct primes, with the goal of applying this theory to two important arithmetic problem: 1) comparing $\mathcal{T}(\mathfrak{n})$ with $\mathcal{C}(\mathfrak{n})$, and 2) constructing explicit homomorphisms $J_0(\mathfrak{n}) \to J^{\mathfrak{n}}$. Our proofs use the rigid-analytic uniformizations of $J_0(\mathfrak{n})$ and $J^{\mathfrak{n}}$ over F_∞. It seems that the existence of actual geometric fibres at ∞ allows one to prove stronger results than what is currently known about either of these problems in the classical setting; this is specific to function fields since the analogue of ∞ for $\mathbb{Q}$ is the archimedean place.

Our initial motivation for studying $\mathfrak{E}(\mathfrak{p}\mathfrak{q})$ came from an attempt to prove a function field analogue of Ogg's conjecture [37] about the so-called Jacquet-Langlands isogenies. We briefly recall what this is about. A geometric consequence of the Jacquet-Langlands correspondence [25] is the existence of Hecke-equivariant $\mathbb{Q}$-rational isogenies between the new quotient $J_0(N)^{\mathrm{new}}$ of $J_0(N)$ and the Jacobian J^N of X^N; see [45]. (Here N is a square-free integer with an even number of prime factors.) The proof of the existence of aforementioned isogenies relies on Faltings' isogeny theorem, so provides no information about them beyond the existence. It is a major open problem in this area to make the isogenies more canonical (cf. [24]). In [37], Ogg made several predictions about the kernel of an isogeny $J_0(N)^{\mathrm{new}} \to J^N$ when $N = pp'$ is a product of two distinct primes and $p = 2, 3, 5, 7, 13$. As far as the authors are aware, Ogg's conjecture remains open except for the special cases when J^N has dimension 1 ($N = 14, 15, 21, 33, 34$) or dimension 2 ($N = 26, 38, 58$). In these cases, J^N and $J_0(N)^{\mathrm{new}}$ are either elliptic curves or, up to isogeny, decompose into a product of two elliptic curves given by explicit Weierstrass equations. One can then find an isogeny $J_0(N)^{\mathrm{new}} \to J^N$ by studying the isogenies between these elliptic curves; see the proof of Theorem 3.1 in [21]. This argument does not generalize to J^N of dimension ≥ 3 because they contain absolutely simple abelian varieties of dimension ≥ 2, and one's hold on such abelian varieties is decidedly more fleeting.

Now returning to the setting of function fields, let $\mathfrak{n} \lhd A$ be a square-free ideal with an even number of prime factors. The global Jacquet-Langlands correspondence over F, combined with the main results in [6] and [28], and Zarhin's isogeny theorem, implies the existence of a Hecke-equivariant F-rational isogeny $J_0(\mathfrak{n})^{\mathrm{new}} \to J^{\mathfrak{n}}$. In Section 9, by studying the groups of connected components of the Néron models of $J_0(\mathfrak{n})$ and $J^{\mathfrak{n}}$, we propose a function field analogue of Ogg's conjecture (see Conjecture 9.3). This conjecture predicts that, when $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$ is a product of two distinct primes with $\deg(\mathfrak{p}) \leq 2$, there is a Jacquet-Langlands isogeny whose kernel comes from cuspidal divisors and is isomorphic to a specific abelian group. Our approach to proving this conjecture starts with the observation that $\mathcal{C}(\mathfrak{n})$ is annihilated by the Eisenstein ideal $\mathfrak{E}(\mathfrak{n})$ acting on $J_0(\mathfrak{n})$, so we first try to show that there is a Jacquet-Langlands isogeny whose kernel is annihilated by $\mathfrak{E}(\mathfrak{n})$, and then try to describe the kernel of the Eisenstein ideal $J[\mathfrak{E}(\mathfrak{n})]$ in $J_0(\mathfrak{n})$ explicitly enough to pin down the kernel of the isogeny. This naturally leads to the study of $J[\mathfrak{E}(\mathfrak{n})]$ for composite $\mathfrak{n}$. On the other hand, $J[\mathfrak{E}(\mathfrak{n})]$ also plays an important role in the analysis of $\mathcal{T}(\mathfrak{n})$, as was first demonstrated by Mazur in his seminal paper [33] in the case of classical modular Jacobian $J_0(p)$ of prime level. These two applications of the theory of the Eisenstein ideal constitute the main theme of this paper.

1.2. MAIN RESULTS. The *Shimura subgroup* $\mathcal{S}(\mathfrak{n})$ of $J_0(\mathfrak{n})$ is the kernel of the homomorphism $J_0(\mathfrak{n}) \to J_1(\mathfrak{n})$ induced by the natural morphism $X_1(\mathfrak{n}) \to X_0(\mathfrak{n})$ of modular curves (see Section 8.1).

Assume $\mathfrak{p} \lhd A$ is prime. Define $N(\mathfrak{p}) = \frac{|\mathfrak{p}|-1}{q-1}$ if $\deg(\mathfrak{p})$ is odd, and define $N(\mathfrak{p}) = \frac{|\mathfrak{p}|-1}{q^2-1}$, otherwise. In [38], Pál developed a theory of the Eisenstein ideal $\mathfrak{E}(\mathfrak{p})$ in parallel with Mazur's paper [33]. In particular, he showed that $J[\mathfrak{E}(\mathfrak{p})]$ is everywhere unramified of order $N(\mathfrak{p})^2$, and is essentially generated by $\mathcal{C}(\mathfrak{p})$ and $\mathcal{S}(\mathfrak{p})$, both of which are cyclic of order $N(\mathfrak{p})$. Moreover, $\mathcal{C}(\mathfrak{p}) = \mathcal{T}(\mathfrak{p})$ and $\mathcal{S}(\mathfrak{p})$ is the largest μ-type subgroup scheme of $J_0(\mathfrak{p})$. These results are the analogues of some of the deepest results from [33], whose proof first establishes that the completion of the Hecke algebra $\mathbb{T}(\mathfrak{p})$ at any maximal ideal in the support of $\mathfrak{E}(\mathfrak{p})$ is Gorenstein.

As we will see in Section 8, even in the simplest composite level case the kernel of the Eisenstein ideal $J[\mathfrak{E}(\mathfrak{n})]$ has properties quite different from its prime level counterpart. For example, $J[\mathfrak{E}(\mathfrak{n})]$ can be ramified, generally $\mathcal{S}(\mathfrak{n})$ has smaller order than $\mathcal{C}(\mathfrak{n})$, neither of these groups is cyclic, and $\mathcal{S}(\mathfrak{n})$ is not the largest μ-type subgroup scheme of $J_0(\mathfrak{n})$.

First, we discuss our results about $\mathcal{C}(\mathfrak{n})$, $\mathcal{S}(\mathfrak{n})$, and $\mathcal{T}(\mathfrak{n})$:

Theorem 1.1.

(1) *We give a complete description of* $\mathcal{C}(\mathfrak{pq})$ *as an abelian group; see Theorem 6.11.*
(2) *For an arbitrary square-free* $\mathfrak{n}$ *we show that the group scheme* $\mathcal{S}(\mathfrak{n})$ *is* μ*-type, and therefore annihilated by* $\mathfrak{E}(\mathfrak{n})$*, and we give a complete description of* $\mathcal{S}(\mathfrak{n})$ *as an abelian group; see Proposition 8.5 and Theorem 8.6.*
(3) *If* $\ell \neq p$ *is a prime number which does not divide*

$$(q-1)\cdot \gcd(|\mathfrak{p}|+1, |\mathfrak{q}|+1),$$

then the ℓ*-primary subgroups of* $\mathcal{C}(\mathfrak{pq})$ *and* $\mathcal{T}(\mathfrak{pq})$ *are equal; see Theorem 7.3.*

Usually, many of the primes dividing the order of $\mathcal{C}(\mathfrak{pq})$ satisfy the condition in (3), so, aside from a relatively small explicit set of primes, we can determine the ℓ-primary subgroup $\mathcal{T}(\mathfrak{pq})_\ell$ of $\mathcal{T}(\mathfrak{pq})$. For example, (1) and (3) imply that if ℓ does not divide $(|\mathfrak{p}|^2-1)(|\mathfrak{q}|^2-1)$, then $\mathcal{T}(\mathfrak{pq})_\ell = 0$. The most advantageous case for applying (3) is when $\deg(\mathfrak{q}) = \deg(\mathfrak{p})+1$, since then $\gcd(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$ divides $q-1$. In particular, if $q=2$ and $\deg(\mathfrak{q}) = \deg(\mathfrak{p})+1$, then we conclude that the odd part of $\mathcal{T}(\mathfrak{pq})$ coincides with $\mathcal{C}(\mathfrak{pq})$. These results are qualitatively stronger than what is currently known about the rational torsion subgroup $J_0(N)(\mathbb{Q})_{\mathrm{tor}}$ of classical modular Jacobians of composite square-free levels (cf. [4]).

Outline of the Proof of Theorem 1.1. Although it was known that $\mathcal{C}(\mathfrak{n})$ is finite for any $\mathfrak{n}$ (see Theorem 6.1), there were no general results about its structure, besides the prime level case $\mathfrak{n} = \mathfrak{p}$. The curve $X_0(\mathfrak{p})$ has two cusps, so $\mathcal{C}(\mathfrak{p})$ is cyclic; its order was computed by Gekeler in [10]. The first obvious difference between the prime level and the composite level $\mathfrak{n} = \mathfrak{pq}$ is that $X_0(\mathfrak{pq})$ has 4 cusps, so $\mathcal{C}(\mathfrak{pq})$ is usually not cyclic and is generated by 3 elements. To prove the result mentioned in (1), i.e., to compute the group structure of $\mathcal{C}(\mathfrak{pq})$, we follow the strategy in [10], but the calculations become much more complicated. The idea is to use Drinfeld discriminant function to obtain upper bounds on the orders of cuspidal divisors, and then use canonical specializations of $\mathcal{C}(\mathfrak{pq})$ into the component groups of $J_0(\mathfrak{pq})$ at $\mathfrak{p}$ and $\mathfrak{q}$ to obtain lower bounds on these orders.

To deduce the group structure of $\mathcal{S}(\mathfrak{n})$ mentioned in (2) we use the rigid-analytic uniformizations of $J_0(\mathfrak{n})$ and $J_1(\mathfrak{n})$ over F_∞, and the "changing levels" result from [18], to reduce the problem to a calculation with finite groups.

The proof of (3) is similar to the proof of Theorem 7.19 in [38], although there are some important differences, too. Suppose ℓ is a prime that does not divide $q(q-1)$. Since $J_0(\mathfrak{pq})$ has split toric reduction at ∞, the ℓ-primary subgroup $\mathcal{T}(\mathfrak{pq})_\ell$ maps injectively into the component group Φ_∞ of $J_0(\mathfrak{pq})$ at ∞. Using the Eichler-Shimura relations, one shows that the image of $\mathcal{T}(\mathfrak{pq})_\ell$ in Φ_∞ can be identified with a subspace of $\mathcal{H}_0(\mathscr{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{pq})} \otimes \mathbb{Z}/\ell^n\mathbb{Z}$ annihilated by the Eisenstein ideal $\mathfrak{E}(\mathfrak{pq})$ for any sufficiently large $n \in \mathbb{N}$. Denote by

$\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z})$ the subspace of $\mathcal{H}_0(\mathscr{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{pq})} \otimes \mathbb{Z}/\ell^n\mathbb{Z}$ annihilated by $\mathfrak{E}(\mathfrak{pq})$. Then we have the inclusions

$$\mathcal{C}(\mathfrak{pq})_\ell \hookrightarrow \mathcal{T}(\mathfrak{pq})_\ell \hookrightarrow \mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

The space $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z})$ contains the reductions modulo ℓ^n of certain Eisenstein series. We prove that if ℓ does not divide $q(q-1)\mathrm{gcd}(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$, then the whole $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z})$ is generated by the reductions of these Eisenstein series (see Theorem 3.9 and Lemma 3.10). This allows us to compute $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z})$. It turns out that $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathcal{C}(\mathfrak{pq})_\ell$, and consequently $\mathcal{C}(\mathfrak{pq})_\ell = \mathcal{T}(\mathfrak{pq})_\ell$. To prove Theorem 3.9, we first prove a version of the key Theorem 1 in the famous paper by Atkin and Lehner [1] for $\mathbb{Z}/\ell^n\mathbb{Z}$-valued harmonic cochains (see Theorem 2.26). The fact that we need to work with $\mathbb{Z}/\ell^n\mathbb{Z}$ rather than $\mathbb{C}$ leads to technical difficulties, which results in the restriction $\ell \nmid q(q-1)\mathrm{gcd}(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$. Note that in our definition the Hecke algebra $\mathbb{T}(\mathfrak{pq})$ includes the operators $U_\mathfrak{p}$ and $U_\mathfrak{q}$. This is important since we need to deal systematically with "old" forms of level $\mathfrak{p}$ and $\mathfrak{q}$. The smaller algebra $\mathbb{T}(\mathfrak{pq})^0$ generated by the Hecke operators $T_\mathfrak{m}$ with $\mathfrak{m}$ coprime to $\mathfrak{pq}$ used by Pál in [38] and [39] is not sufficient for getting a handle on $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^n\mathbb{Z})$. □

Now we concentrate on the case where we investigate the Jacquet-Langlands isogenies. We fix two primes x and y of A of degree 1 and 2, respectively. This differs from our usual $\mathfrak{Fraktur}$ notation for ideals of A. This is done primarily to make it easy for the reader to distinguish the theorems which assume that the level is xy. Several sections in the paper are titled "Special case" and deal exclusively with the case $\mathfrak{pq} = xy$. Note that $X_0(\mathfrak{pq})$ has genus 0 if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct primes with $\deg(\mathfrak{pq}) \leq 2$. The genus of $X_0(xy)$ is q, so this curve is the simplest example of a Drinfeld modular curve of composite level and positive genus. Also, by a theorem of Schweizer [49], $X_0(\mathfrak{pq})$ is hyperelliptic if and only if $\mathfrak{p} = x$ and $\mathfrak{q} = y$, so one can think of this case as the hyperelliptic case.
The cusps of $X_0(xy)$ can be naturally labelled $[x], [y], [1], [\infty]$; see Lemma 2.14. Let c_x and c_y denote the classes of divisors $[x] - [\infty]$ and $[y] - [\infty]$ in $J_0(xy)$. First, we show that (see Theorem 7.13)

$$\mathcal{T}(xy) = \mathcal{C}(xy) = \langle c_x \rangle \oplus \langle c_y \rangle \cong \mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/(q^2+1)\mathbb{Z}.$$

The reason we can prove this stronger result compared to (3) of Theorem 1.1 is that we can compute $\mathcal{E}_{00}(xy, \mathbb{Z}/\ell^n\mathbb{Z})$ without any restrictions on ℓ, and we can deal with the 2-primary torsion $\mathcal{T}(xy)_2$ using the fact that $X_0(xy)$ is hyperelliptic.
To simplify the notation, for the rest of this section denote $\mathbb{T} = \mathbb{T}(xy)$, $\mathfrak{E} = \mathfrak{E}(xy)$, $\mathcal{H} := \mathcal{H}_0(\mathscr{T}, \mathbb{Z})^{\Gamma_0(xy)}$, $\mathcal{H}' := \mathcal{H}(\mathscr{T}, \mathbb{Z})^{\Gamma^{xy}}$, where this last group is the group of $\mathbb{Z}$-valued Γ^{xy}-invariant harmonic cochains on $\mathscr{T}$. We show that (see Corollary 3.18)

$$\mathbb{T}/\mathfrak{E} \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z},$$

so the residue characteristic of any maximal ideal of $\mathbb{T}$ containing $\mathfrak{E}$ divides $(q^2+1)(q+1)$. The Jacquet-Langlands correspondence over F implies that there is an isomorphism $\mathcal{H} \otimes \mathbb{Q} \cong \mathcal{H}' \otimes \mathbb{Q}$ which is compatible with the action of $\mathbb{T}$.

Theorem 1.2 (See Theorems 9.5 and 9.6).

(1) *If $\mathcal{H} \cong \mathcal{H}'$ as $\mathbb{T}$-modules, then there is an isogeny $J_0(xy) \to J^{xy}$ defined over F whose kernel is cyclic of order q^2+1 and is annihilated by $\mathfrak{E}$.*

(2) *If $\mathcal{H} \cong \mathcal{H}'$ as $\mathbb{T}$-modules and for every prime $\ell|(q^2+1)$ the completion of $\mathbb{T} \otimes \mathbb{Z}_\ell$ at $\mathfrak{M} = (\mathfrak{E}, \ell)$ is Gorenstein, then there is an isogeny $J_0(xy) \to J^{xy}$ whose kernel is $\langle c_y \rangle \cong \mathbb{Z}/(q^2+1)\mathbb{Z}$.*

Remark 1.3. An isogeny $J_0(xy) \to J^{xy}$ with kernel $\langle c_y \rangle$ does not respect the canonical principal polarizations on the Jacobians since $\langle c_y \rangle$ is not a maximal isotropic subgroup of $J_0(xy)$ with respect to the Weil pairing.

Outline of the Proof of Theorem 1.2. Both $J_0(xy)$ and J^{xy} have rigid-analytic uniformization over F_∞. The assumption that $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic $\mathbb{T}$-modules allows us to identify the uniformizing tori of both Jacobians with $\mathbb{T} \otimes \mathbb{C}_\infty^\times$. Next, we show that the groups of connected components of the Néron models of $J_0(xy)$ and J^{xy} at ∞ are annihilated by $\mathfrak{E}$. This allows us to identify the uniformizing lattices of the Jacobians with ideals in $\mathbb{T}$. These two observations, combined with a theorem of Gerritzen, imply (1). If in addition we assume that $\mathbb{T}_\mathfrak{M}$ is Gorenstein, then we get an explicit description of the kernel of the Eisenstein ideal from which (2) follows. $\square$

Proving that the assumptions in Theorem 1.2 hold seems difficult. First, even though $\mathcal{H} \otimes \mathbb{Q}$ and $\mathcal{H}' \otimes \mathbb{Q}$ are isomorphic $\mathbb{T}$-modules, the integral isomorphism is much more subtle. It is related to a classical problem about the conjugacy classes of matrices in $\mathrm{Mat}_n(\mathbb{Z})$; cf. [27]. Second, when $\ell|(q^2+1)$ the kernel of $\mathfrak{M}$ in $J_0(xy)$ is ramified, and Mazur's Eisenstein descent arguments for proving $\mathbb{T}_\mathfrak{M}$ is Gorenstein do not work in this ramified situation. (Both versions of Mazur's descent discussed in [38, §§10,11] rely on subtle arithmetic properties of $J_0(\mathfrak{p})$ which are valid only for prime level.)
Nevertheless, both assumptions in Theorem 1.2 can be verified computationally; Section 10 is devoted to these calculations. We were able to check the assumptions for several cases for each prime $q \leq 7$. In particular, we were able to go beyond dimension 2, which is currently the only dimension where the Ogg's conjecture is known to be true over $\mathbb{Q}$. Section 10 is also of independent interest since it provides an algorithm for computing the action of Hecke operators on $\mathcal{H}'$; this should be useful in other arithmetic problems dealing with X^{xy}. (An algorithm for computing the Hecke action on $\mathcal{H}$ was already known from the work of Gekeler; see Remark 10.2.)

1.3. Notation. Aside from ∞, the places of F are in bijection with non-zero prime ideals of A. Given a place v of F, we denote by F_v the completion of F at v, by $\mathcal{O}_v$ the ring of integers of F_v, and by $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$. The valuation $\mathrm{ord}_v : F_v \to \mathbb{Z}$ is assumed to be normalized by $\mathrm{ord}_v(\pi_v) = 1$, where π_v is a uniformizer of $\mathcal{O}_v$. The normalized absolute value on F_∞ is denoted by $|\cdot|$.

Given a field K, we denote by $\bar{K}$ an algebraic closure of K and K^{sep} a separable closure in $\bar{K}$. The absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is denoted by G_K. Moreover, F_v^{nr} and $\mathcal{O}_v^{\mathrm{nr}}$ will denote the maximal unramified extension of F_v and its ring of integers, respectively.

Let R be a commutative ring with identity. We denote by $R^\times$ the group of multiplicative units of R. Let $\mathrm{Mat}_n(R)$ be the ring of $n \times n$ matrices over R, $\mathrm{GL}_n(R)$ the group of matrices whose determinant is in $R^\times$, and $Z(R) \cong R^\times$ the subgroup of $\mathrm{GL}_n(R)$ consisting of scalar matrices.

If X is a scheme over a base S and $S' \to S$ any base change, $X_{S'}$ denotes the pullback of X to S'. If $S' = \mathrm{Spec}(R)$ is affine, we may also denote this scheme by X_R. By $X(S')$ we mean the S'-rational points of the S-scheme X, and again, if $S' = \mathrm{Spec}(R)$, we may also denote this set by $X(R)$.

Given a commutative finite flat group scheme G over a base S (or just an abelian group G, or a ring G) and an integer n, $G[n]$ is the kernel of multiplication by n in G, and G_ℓ is the maximal ℓ-primary subgroup of G. The Cartier dual of G is denoted by G^*.

Given an ideal $\mathfrak{n} \lhd A$, by abuse of notation, we denote by the same symbol the unique monic polynomial in A generating $\mathfrak{n}$. It will always be clear from the context in which capacity $\mathfrak{n}$ is used; for example, if $\mathfrak{n}$ appears in a matrix, column vector, or a polynomial equation, then the monic polynomial is implied. The prime ideals $\mathfrak{p} \lhd A$ are always assumed to be non-zero.

2. Harmonic cochains and Hecke operators

2.1. Harmonic cochains. Let G be an oriented connected graph in the sense of Definition 1 of §2.1 in [50]. We denote by $V(G)$ and $E(G)$ its set of vertices and edges, respectively. For an edge $e \in E(G)$, let $o(e)$, $t(e) \in V(G)$ and $\bar{e} \in E(G)$ be its origin, terminus and inversely oriented edge, respectively. In particular, $t(\bar{e}) = o(e)$ and $o(\bar{e}) = t(e)$. We will assume that for any $v \in V(G)$ the number of edges with $t(e) = v$ is finite, and $t(e) \neq o(e)$ for any $e \in E(G)$ (i.e., G has no loops). A *path* in G is a sequence of edges $\{e_i\}_{i\in I}$ indexed by the set I where $I = \mathbb{Z}$, $I = \mathbb{N}$ or $I = \{1, \dots, m\}$ for some $m \in \mathbb{N}$ such that $t(e_i) = o(e_{i+1})$ for every $i, i+1 \in I$. We say that the path is *without backtracking* if $e_i \neq \bar{e}_{i+1}$ for every $i, i+1 \in I$. We say that the path without backtracking $\{e_i\}_{i\in\mathbb{N}}$ is a *half-line* if for every vertex v of G there is at most one index $n \in \mathbb{N}$ such that $v = o(e_n)$.

Let Γ be a group acting on a graph G, i.e., Γ acts via automorphisms. We say that Γ acts with *inversion* if there is $\gamma \in \Gamma$ and $e \in E(G)$ such that $\gamma e = \bar{e}$. If Γ acts without inversion, then we have a natural quotient graph $\Gamma \setminus G$ such that $V(\Gamma \setminus G) = \Gamma \setminus V(G)$ and $E(\Gamma \setminus G) = \Gamma \setminus E(G)$, cf. [50, p. 25].

Definition 2.1. Fix a commutative ring R with identity. An R-valued *harmonic cochain* on G is a function $f : E(G) \to R$ that satisfies

(i)
$$f(e) + f(\bar{e}) = 0 \quad \text{for all } e \in E(G),$$

(ii)

$$\sum_{\substack{e\in E(G)\\ t(e)=v}} f(e)=0 \quad \text{for all } v\in V(G).$$

Denote by $\mathcal{H}(G,R)$ the group of R-valued harmonic cochains on G.

The most important graphs in this paper are the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(F_\infty)$, and the quotients of $\mathscr{T}$. We recall the definition and introduce some notation for later use. Fix a uniformizer π_∞ of F_∞. The sets of vertices $V(\mathscr{T})$ and edges $E(\mathscr{T})$ are the cosets $\mathrm{GL}_2(F_\infty)/Z(F_\infty)\mathrm{GL}_2(\mathcal{O}_\infty)$ and $\mathrm{GL}_2(F_\infty)/Z(F_\infty)\mathcal{I}_\infty$, respectively, where $\mathcal{I}_\infty$ is the Iwahori group:

$$\mathcal{I}_\infty=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathcal{O}_\infty)\ \middle|\ c\in\pi_\infty\mathcal{O}_\infty\right\}.$$

The matrix $\begin{pmatrix} 0 & 1\\ \pi_\infty & 0\end{pmatrix}$ normalizes $\mathcal{I}_\infty$, so the multiplication from the right by this matrix on $\mathrm{GL}_2(F_\infty)$ induces an involution on $E(\mathscr{T})$; this involution is $e\mapsto\bar{e}$. The matrices

$$E(\mathscr{T})^+=\left\{\begin{pmatrix} \pi_\infty^k & u\\ 0 & 1\end{pmatrix}\ \middle|\ \begin{matrix} k\in\mathbb{Z}\\ u\in F_\infty,\ u \bmod \pi_\infty^k\mathcal{O}_\infty\end{matrix}\right\} \tag{2.1}$$

are in distinct left cosets of $\mathcal{I}_\infty Z(F_\infty)$, and there is a disjoint decomposition (cf. [12, (1.6)])

$$E(\mathscr{T})=E(\mathscr{T})^+\bigsqcup E(\mathscr{T})^+\begin{pmatrix} 0 & 1\\ \pi_\infty & 0\end{pmatrix}.$$

We call the edges in $E(\mathscr{T})^+$ *positively oriented*.

The group $\mathrm{GL}_2(F_\infty)$ naturally acts on $E(\mathscr{T})$ by left multiplication. This induces an action on the group of R-valued functions on $E(\mathscr{T})$: for a function f on $E(\mathscr{T})$ and $\gamma\in\mathrm{GL}_2(F_\infty)$ we define the function $f|\gamma$ on $E(\mathscr{T})$ by $(f|\gamma)(e)=f(\gamma e)$. It is clear from the definition that $f|\gamma$ is harmonic if f is harmonic, and for any $\gamma,\sigma\in\mathrm{GL}_2(F_\infty)$ we have $(f|\gamma)|\sigma=f|(\gamma\sigma)$.

Let Γ be a subgroup of $\mathrm{GL}_2(F_\infty)$ which acts on $\mathscr{T}$ without inversions. Denote by $\mathcal{H}(\mathscr{T},R)^\Gamma$ the subgroup of Γ-invariant harmonic cochains, i.e., $f|\gamma=f$ for all $\gamma\in\Gamma$. It is clear that $f\in\mathcal{H}(\mathscr{T},R)^\Gamma$ defines a function f' on the quotient graph $\Gamma\setminus\mathscr{T}$, and f itself can be uniquely recovered from this function: If $e\in E(\mathscr{T})$ maps to $\tilde{e}\in E(\Gamma\setminus\mathscr{T})$ under the quotient map, then $f(e)=f'(\tilde{e})$. The conditions of harmonicity (i) and (ii) can be formulated in terms of f' as follows. Since Γ acts without inversion, (i) is equivalent to

(i$'$)

$$f'(\tilde{e})+f'(\bar{\tilde{e}})=0 \quad \text{for all } \tilde{e}\in E(\Gamma\setminus\mathscr{T}).$$

Let $v\in V(\mathscr{T})$ and $\tilde{v}\in V(\Gamma\setminus\mathscr{T})$ be its image. The stabilizer group

$$\Gamma_v=\{\gamma\in\Gamma\mid \gamma v=v\}$$

acts on the set $\{e\in E(\mathscr{T})\mid t(e)=v\}$, and the orbits correspond to

$$\{\tilde{e}\in E(\Gamma\setminus\mathscr{T})\mid t(\tilde{e})=\tilde{v}\}.$$

Let $\Gamma_e := \{\gamma \in \Gamma \mid \gamma e = e\}$; clearly Γ_e is a subgroup of $\Gamma_{t(e)}$. The *weight* of e

$$w(e) := [\Gamma_{t(e)} : \Gamma_e]$$

is the length of the orbit corresponding to e. Since $w(e)$ depends only on its image $\tilde{e}$ in $\Gamma \setminus \mathscr{T}$, we can define $w(\tilde{e}) := w(e)$. Note that $\sum_{t(\tilde{e})=\tilde{v}} w(\tilde{e}) = q+1$. We stress that, in general, $w(e)$ depends on the orientation, i.e., $w(e) \neq w(\bar{e})$. With this notation, condition (ii) is equivalent to

(ii′)

$$\sum_{\substack{\tilde{e}\in E(\Gamma\setminus\mathscr{T})\\ t(\tilde{e})=\tilde{v}}} w(\tilde{e}) f'(\tilde{e}) = 0 \quad \text{for all } \tilde{v} \in V(\Gamma \setminus \mathscr{T}),$$

cf. [18, (3.1)].

Definition 2.2. The group of R-valued *cuspidal harmonic cochains* for Γ, denoted $\mathcal{H}_0(\mathscr{T}, R)^\Gamma$, is the subgroup of $\mathcal{H}(\mathscr{T}, R)^\Gamma$ consisting of functions which have compact support as functions on $\Gamma \setminus \mathscr{T}$, i.e., functions which have value 0 on all but finitely many edges of $\Gamma \setminus \mathscr{T}$. Let $\mathcal{H}_{00}(\mathscr{T}, R)^\Gamma$ denote the image of $\mathcal{H}_0(\mathscr{T}, \mathbb{Z})^\Gamma \otimes R$ in $\mathcal{H}_0(\mathscr{T}, R)^\Gamma$.

Definition 2.3. It is known that the quotient graph $\Gamma_0(\mathfrak{n}) \setminus \mathscr{T}$ is the edge disjoint union

$$\Gamma_0(\mathfrak{n}) \setminus \mathscr{T} = (\Gamma_0(\mathfrak{n}) \setminus \mathscr{T})^0 \cup \bigcup_{s\in\Gamma_0(\mathfrak{n})\setminus\mathbb{P}^1(F)} h_s$$

of a finite graph $(\Gamma_0(\mathfrak{n})\setminus\mathscr{T})^0$ with a finite number of half-lines h_s, called *cusps*; cf. Theorem 2 on page 106 of [50]. The cusps are in bijection with the orbits of the natural action of $\Gamma_0(\mathfrak{n})$ on $\mathbb{P}^1(F)$; cf. Remark 2 on page 110 of [50].

To simplify the notation, we put

$$\mathcal{H}(\mathfrak{n}, R) := \mathcal{H}(\mathscr{T}, R)^{\Gamma_0(\mathfrak{n})}$$
$$\mathcal{H}_0(\mathfrak{n}, R) := \mathcal{H}_0(\mathscr{T}, R)^{\Gamma_0(\mathfrak{n})}$$
$$\mathcal{H}_{00}(\mathfrak{n}, R) \text{ the image of } \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \otimes R \text{ in } \mathcal{H}_0(\mathfrak{n}, R).$$

One can show that $\mathcal{H}_0(\mathfrak{n}, \mathbb{Z})$ and $\mathcal{H}(\mathfrak{n}, \mathbb{Z})$ are finitely generated free $\mathbb{Z}$-modules of rank $g(\mathfrak{n})$ and $g(\mathfrak{n}) + c(\mathfrak{n}) - 1$, respectively, where $g(\mathfrak{n})$ is the genus of $X_0(\mathfrak{n})$ and $c(\mathfrak{n})$ is the number of cusps.

From the above description it is clear that f is in $\mathcal{H}_0(\mathfrak{n}, R)$ if and only if it eventually vanishes on each h_s. It is also clear that if R is flat over $\mathbb{Z}$, then $\mathcal{H}_0(\mathfrak{n}, R) = \mathcal{H}_{00}(\mathfrak{n}, R)$. On the other hand, it is easy to construct examples where this equality does not hold.

Example 2.4. The quotient graph $\mathrm{GL}_2(A) \setminus \mathscr{T}$ is a half-line; see Figure 1. Denote the edge with origin v_i and terminus v_{i+1} by e_i. The stabilizers of vertices and edges of $\mathrm{GL}_2(A) \setminus \mathscr{T}$ are well-known, cf. [17, p. 691]. From this one computes $w(e_i) = q$ for all i, $w(\bar{e}_0) = q+1$, and $w(\bar{e}_i) = 1$ for $i \geq 1$. Therefore, if $\varphi \in \mathcal{H}(1, R)$, then $\varphi(e_i) = q^i\alpha$ $(i \geq 0)$ for some fixed $\alpha \in R[q+1]$. Now it is clear that $\mathcal{H}(1, R) = R[q+1]$ and $\mathcal{H}_0(1, R) = \mathcal{H}_{00}(1, R) = 0$.

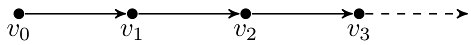

FIGURE 1. $\mathrm{GL}_2(A) \setminus \mathscr{T}$

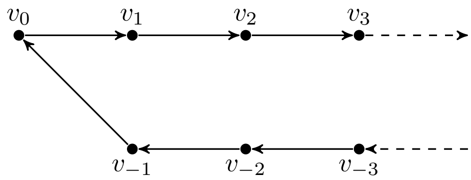

FIGURE 2. $\Gamma_0(x) \setminus \mathscr{T}$

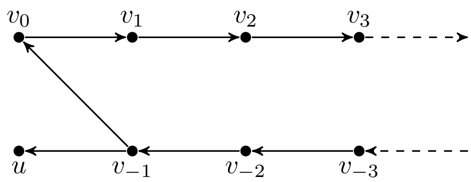

FIGURE 3. $\Gamma_0(y) \setminus \mathscr{T}$

EXAMPLE 2.5. The graph of $\Gamma_0(x) \setminus \mathcal{T}$ is given in Figure 2, where the vertex v_i ($i \in \mathbb{Z}$) is the image of $\begin{pmatrix} T^i & 0 \\ 0 & 1 \end{pmatrix} \in V(\mathscr{T})$; the positive orientation is induced from $E(\mathscr{T})^+$. Denote by e_i the edge with origin v_{i-1} and terminus v_i. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v_{-i} = v_i$ and the stabilizers of v_i ($i \geq 0$) in $\mathrm{GL}_2(A)$ are well-known (cf. [17, p. 691]), one easily computes

$$w(e_i) = \begin{cases} q & \text{if } i \geq 0 \\ 1 & \text{if } i \leq -1 \end{cases} \qquad w(\bar{e}_i) = \begin{cases} 1 & \text{if } i \geq -1 \\ q & \text{if } i \leq -2 \end{cases}$$

Suppose $\varphi \in \mathcal{H}(x, R)$ and denote $\alpha = \varphi(e_{-1})$. Since $w(e_i)\varphi(e_i) = w(\bar{e}_{i+1})\varphi(e_{i+1})$, we get

$$\varphi(e_i) = \begin{cases} \alpha q^{i+1} & \text{if } i \geq -1 \\ \alpha & \text{if } i = -2 \\ \alpha q^{-i-3} & \text{if } i \leq -3. \end{cases}$$

We conclude that $\mathcal{H}(x, R) = R$, $\mathcal{H}_0(x, R) = R_p$, and $\mathcal{H}_{00}(x, R) = 0$. (Recall that R_p denotes the p-primary subgroup of R.)

EXAMPLE 2.6. The graph $\Gamma_0(y) \setminus \mathscr{T}$ is given in Figure 3, where v_i is the image of $\begin{pmatrix} T^i & 0 \\ 0 & 1 \end{pmatrix} \in V(\mathscr{T})$ and u is the image of $\begin{pmatrix} T^{-2} & T^{-1} \\ 0 & 1 \end{pmatrix}$. We denote the edge with origin v_{i-1} and terminus v_i by e_i, and the edge with terminus u by e_u. One computes

$$w(e_i) = \begin{cases} q & \text{if } i \geq 0 \\ 1 & \text{if } i \leq -1 \end{cases} \qquad w(\bar{e}_i) = \begin{cases} 1 & \text{if } i \geq 0 \\ q & \text{if } i \leq -1 \end{cases}$$

$$w(e_u) = q+1, \qquad w(\bar{e}_u) = q-1.$$

Let $\varphi \in \mathcal{H}(y,R)$. Denote $\varphi(e_0) = \alpha$ and $\varphi(e_u) = \beta$. Then $(q+1)\beta = 0$ and

$$\varphi(e_i) = \begin{cases} \alpha q^i & \text{if } i \geq 0 \\ q^{-i-1}(\alpha + (q-1)\beta) & \text{if } i \leq -1. \end{cases}$$

This implies that $\mathcal{H}(y,R) \cong R \oplus R[q+1]$. For φ to be cuspidal we must have $q^n\alpha = 0$ and $q^n(q-1)\beta = 0$ for some $n \geq 1$. Thus, $\alpha \in R_p$ and $\beta \in R[2]$ (resp. $\beta = 0$) if p is odd (resp. 2). We get an isomorphism $\mathcal{H}_0(y,R) \cong R_p \oplus R[2]$ if p is odd and $\mathcal{H}_0(y,R) \cong R_2$ if $p = 2$. Note that $\mathcal{H}_{00}(y,R) = 0$.

Lemma 2.7. *The following holds:*

(1) *If $\mathfrak{n} \lhd A$ has a prime divisor of odd degree, assume $q(q-1) \in R^\times$. Otherwise, assume $q(q^2-1) \in R^\times$. Then $\mathcal{H}_0(\mathfrak{n},R) = \mathcal{H}_{00}(\mathfrak{n},R)$.*
(2) *If $\mathfrak{n} = \mathfrak{p}$ is prime and $q(q-1) \in R^\times$, then $\mathcal{H}_0(\mathfrak{n},R) = \mathcal{H}_{00}(\mathfrak{n},R)$.*

Proof. Our proof relies on the results in [17], and is partly motivated by the proof of Theorem 3.3 in [17]. Let $\Gamma := \Gamma_0(\mathfrak{n})$. By 1.11 and 2.10 in [17], the stabilizer Γ_v for any $v \in V(\mathscr{T})$ is finite, contains the scalar matrices $Z(\mathbb{F}_q)$, and $n(v) := \#\Gamma_v/\mathbb{F}_q^\times$ either divides $(q-1)q^m$ for some $m \geq 0$, or is equal to $q+1$. Moreover, $n(v) = q+1$ is possible only if all prime divisors of $\mathfrak{n}$ have even degrees. Overall, we see that our assumptions in (1) imply that $n(v)$ is invertible in R for any $v \in V(\mathscr{T})$. Since the stabilizer Γ_e of any $e \in V(\mathscr{T})$ is a subgroup of $\Gamma_{t(e)}$ containing $Z(\mathbb{F}_q)$, we also have $n(e) := \#\Gamma_e/\mathbb{F}_q^\times \in R^\times$. Note that $n(e)$ does not depend on the orientation of e and depends only on its image $\tilde{e}$ in $\Gamma \setminus \mathscr{T}$, so we can define $n(\tilde{e}) = n(e)$.

Let $\mathcal{H}_0(\Gamma \setminus \mathscr{T}, R)$ be the subgroup of $\mathcal{H}(\Gamma \setminus \mathscr{T}, R)$ consisting of compactly supported harmonic cochains on $\Gamma \setminus \mathscr{T}$. There is an injective homomorphism

$$\begin{aligned} \mathcal{H}_0(\Gamma \setminus \mathscr{T}, R) &\to \mathcal{H}_0(\mathfrak{n}, R) \\ \varphi &\mapsto \varphi^\dagger \end{aligned} \tag{2.2}$$

defined by $\varphi^\dagger(\tilde{e}) = n(\tilde{e})\varphi(\tilde{e})$. Indeed, since $n(\tilde{e})$ does not depend on the orientation of e, $\varphi^\dagger$ clearly satisfies (i′). As for (ii′), we have

$$\sum_{\substack{\tilde{e} \in E(\Gamma\setminus\mathscr{T}) \\ t(\tilde{e}) = \tilde{v}}} w(\tilde{e})\varphi^\dagger(\tilde{e}) = \sum_{\substack{\tilde{e} \in E(\Gamma\setminus\mathscr{T}) \\ t(\tilde{e}) = \tilde{v}}} \frac{n(\tilde{v})}{n(\tilde{e})} n(\tilde{e})\varphi(\tilde{e}) = n(\tilde{v}) \sum_{\substack{\tilde{e} \in E(\Gamma\setminus\mathscr{T}) \\ t(\tilde{e}) = \tilde{v}}} \varphi(\tilde{e}) = 0.$$

The map (2.2) is also defined over $\mathbb{Z}$, and by [17, Thm. 3.3] gives an isomorphism $\mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z}) \xrightarrow{\sim} \mathcal{H}_0(\mathfrak{n}, \mathbb{Z})$. Next, there is an isomorphism

$$\mathcal{H}_0(\Gamma \setminus \mathscr{T}, R) \cong \mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z}) \otimes_\mathbb{Z} R,$$

which follows, for example, by observing that $H_1(\Gamma \setminus \mathscr{T}, R) \cong \mathcal{H}_0(\Gamma \setminus \mathscr{T}, R)$ and applying the universal coefficient theorem for simplicial homology. Hence

$$\mathcal{H}_0(\Gamma \setminus \mathscr{T}, R) \cong \mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z}) \otimes_{\mathbb{Z}} R \cong \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \otimes_{\mathbb{Z}} R.$$

Let $g = \mathrm{rank}_{\mathbb{Z}} \mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z})$. Thinking of the elements of $\mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z})$ as 1-cycles, it is easy to show by induction on g that one can choose $e_1, \dots, e_g \in E(\Gamma \setminus \mathscr{T})$ and a $\mathbb{Z}$-basis $\varphi_1, \dots, \varphi_g$ of $\mathcal{H}_0(\Gamma \setminus \mathscr{T}, \mathbb{Z})$ such that $\Gamma \setminus \mathscr{T} - \{e_1, \dots, e_g\}$ is a tree, and $\varphi_i(e_j) = \delta_{ij}$ =(Kronecker's delta), $1 \leq i, j \leq g$. By slight abuse of notation, denote the image of $\varphi_i^\dagger$ in $\mathcal{H}_{00}(\mathfrak{n}, R)$ by the same symbol. Let $\psi \in \mathcal{H}_0(\mathfrak{n}, R)$. Then

$$\psi' := \psi - \sum_{i=1}^{g} \frac{\psi(e_i)}{n(e_i)} \varphi_i^\dagger$$

is supported on a finite subtree S of $\Gamma \setminus \mathscr{T}$. Let $v \in V(S)$ be a vertex such that there is a unique $e \in E(S)$ with $t(e) = v$. Note that $w(e) \in R^\times$. Condition (ii$'$) gives $w(e)\psi'(e) = 0$, so $\psi'(e) = 0$. This process can be iterated to show that $\psi' = 0$. This implies that the natural map $\mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \otimes_{\mathbb{Z}} R \to \mathcal{H}_0(\mathfrak{n}, R)$ is surjective, which is part (1).

To prove part (2), we can assume that $\deg(\mathfrak{p})$ is even. A consequence of 2.7 and 2.8 in [17] is that there is a unique $v_0 \in V(\Gamma \setminus \mathscr{T})$ with $n(v_0) = q+1$ and a unique $e_0 \in E(\Gamma \setminus \mathscr{T})$ with $o(e_0) = v_0$. For any other $v \in V(\Gamma \setminus \mathscr{T})$, $n(v)$ divides $(q-1)q^m$. Since the stabilizer of any edge $e \in E(\Gamma \setminus \mathscr{T})$ is a subgroup of the stabilizers of both $t(e)$ and $o(e)$, we have $n(e) \in R^\times$. After this observation, we can repeat the argument used to prove (1) to reduces the problem to showing that $\psi \in \mathcal{H}_0(\mathfrak{p}, R)$ supported on a finite tree S is identically 0. We can always choose $v \in V(S)$ to be a vertex different from v_0 but such that there is a unique $e \in E(S)$ with $t(e) = v$. Since $w(e)$ is a unit in R, we can also finish as in part (1). $\square$

The conclusion in Example 2.6 that $\mathcal{H}_0(y, R) \neq \mathcal{H}_{00}(y, R)$ if $R[2] \neq 0$ is a special case of a general fact:

Lemma 2.8. *Assume p is odd and invertible in R. Let $\mathfrak{p} \lhd A$ be prime of even degree. If $R[2] \neq 0$, then $\mathcal{H}_0(\mathfrak{p}, R) \neq \mathcal{H}_{00}(\mathfrak{p}, R)$.*

Proof. Let $\Gamma := \Gamma_0(\mathfrak{p})$. As in Lemma 2.7, let v_0 be the unique vertex of $\Gamma \setminus \mathscr{T}$ with $n(v_0) = q+1$, and let $e_0 \in E(\Gamma \setminus \mathscr{T})$ be the unique edge with $o(e_0) = v_0$. Note that $w(\bar{e}_0) = q+1$. As we already mentioned in the proof of Lemma 2.7, for any other vertex v in $\Gamma \setminus \mathscr{T}$, $n(v)$ divides $(q-1)q^m$. Moreover, it is easy to see, for example by case (a) of Lemma 2.7 in [17], that there is at least one vertex v such that $n(v)$ is divisible by $q-1$. Consider all the paths without backtracking connecting v_0 to such a vertex, and fix a path of shortest length $\{e_0, e_1, \dots, e_m\}$. Then $w(\bar{e}_i)$ $(1 \leq i \leq m)$ is invertible in R, but $w(e_m)$ is divisible by $q-1$. For a fixed non-zero $\alpha \in R[2]$, define f on $E(\Gamma \setminus \mathscr{T})$ by $f(e_0) = \alpha$, $f(e_i) = \frac{w(e_{i-1})}{w(\bar{e}_i)} f(e_{i-1})$ $(1 \leq i \leq m)$, $f(\bar{e}_j) = f(e_j)$ $(0 \leq j \leq m)$, and $f(e) = 0$ for all other edges. It is easy to see that $f \in \mathcal{H}_0(\mathfrak{p}, R)$. On the other hand, any function $\varphi \in \mathcal{H}_0(\mathfrak{p}, \mathbb{Z})$ must be zero on e_0, since condition (ii′) for v_0 gives $(q+1)\varphi(\bar{e}_0) = 0$. Therefore, $f \notin \mathcal{H}_{00}(\mathfrak{p}, R)$. □

Remark 2.9. The fact stated in Lemma 2.8 is deduced in [38] by different (algebro-geometric) methods. Our combinatorial proof seems to answer the question in Remark 11.9 in [38].

2.2. HECKE OPERATORS AND ATKIN-LEHNER INVOLUTIONS. Assume $\mathfrak{n} \lhd A$ is fixed. Given a non-zero ideal $\mathfrak{m} \lhd A$, define an R-linear transformation of the space of R-valued functions on $E(\mathscr{T})$ by

$$f|T_{\mathfrak{m}} = \sum f| \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $f|\gamma$ for $\gamma \in \mathrm{GL}_2(F_\infty)$ is defined in Section 2.1, and the above sum is over $a, b, d \in A$ such that a, d are monic, $(ad) = \mathfrak{m}$, $(a) + \mathfrak{n} = A$, and $\deg(b) < \deg(d)$. This transformation is the $\mathfrak{m}$*-th Hecke operator*. Following a common convention, for a prime divisor $\mathfrak{p}$ of $\mathfrak{n}$ we often write $U_{\mathfrak{p}}$ instead of $T_{\mathfrak{p}}$.

PROPOSITION 2.10. *The Hecke operators preserve the spaces $\mathcal{H}(\mathfrak{n}, R)$ and $\mathcal{H}_0(\mathfrak{n}, R)$, and satisfy the recursive formulas:*

$$\begin{aligned} T_{\mathfrak{m}\mathfrak{m}'} &= T_{\mathfrak{m}} T_{\mathfrak{m}'} \quad \textit{if} \quad \mathfrak{m} + \mathfrak{m}' = A, \\ T_{\mathfrak{p}^i} &= T_{\mathfrak{p}^{i-1}} T_{\mathfrak{p}} - |\mathfrak{p}| T_{\mathfrak{p}^{i-2}} \quad \textit{if} \quad \mathfrak{p} \nmid \mathfrak{n}, \\ T_{\mathfrak{p}^i} &= T_{\mathfrak{p}}^i \quad \textit{if} \quad \mathfrak{p} | \mathfrak{n}. \end{aligned}$$

Proof. The group-theoretic proofs of the analogous statement for the Hecke operators acting on classical modular forms work also in this setting; cf. [34, §4.5]. □

DEFINITION 2.11. Let $\mathbb{T}(\mathfrak{n})$ be the commutative subalgebra of $\mathrm{End}_{\mathbb{Z}}(\mathcal{H}_0(\mathfrak{n}, \mathbb{Z}))$ with the same unity element generated by all Hecke operators. Let $\mathbb{T}(\mathfrak{n})^0$ to be the subalgebra of $\mathbb{T}(\mathfrak{n})$ generated by the Hecke operators $T_{\mathfrak{m}}$ with $\mathfrak{m}$ coprime to $\mathfrak{n}$.

For every ideal $\mathfrak{m}$ dividing $\mathfrak{n}$ with $\gcd(\mathfrak{m}, \mathfrak{n}/\mathfrak{m}) = 1$, let $W_{\mathfrak{m}}$ be any matrix in $\mathrm{Mat}_2(A)$ of the form

$$\begin{pmatrix} a\mathfrak{m} & b \\ c\mathfrak{n} & d\mathfrak{m} \end{pmatrix} \tag{2.3}$$

such that $a, b, c, d, \in A$ and the ideal generated by $\det(W_{\mathfrak{m}})$ in A is $\mathfrak{m}$. It is not hard to check that for $f \in \mathcal{H}(\mathfrak{n}, R)$, $f|W_{\mathfrak{m}}$ does not depend on the choice of the matrix for $W_{\mathfrak{m}}$ and $f|W_{\mathfrak{m}} \in \mathcal{H}(\mathfrak{n}, R)$. Moreover, as R-linear endomorphisms of $\mathcal{H}(\mathfrak{n}, R)$, $W_{\mathfrak{m}}$'s satisfy

$$W_{\mathfrak{m}_1} W_{\mathfrak{m}_2} = W_{\mathfrak{m}_3}, \quad \text{where} \quad \mathfrak{m}_3 = \frac{\mathfrak{m}_1 \mathfrak{m}_2}{\gcd(\mathfrak{m}_1, \mathfrak{m}_2)^2}. \tag{2.4}$$

Therefore, the matrices $W_{\mathfrak{m}}$ acting on the R-module $\mathcal{H}(\mathfrak{n}, R)$ generate an abelian group $\mathbb{W} \cong (\mathbb{Z}/2\mathbb{Z})^s$, called the group of *Atkin-Lehner involutions*, where s is the number of prime divisors of $\mathfrak{n}$. The following proposition, whose proof we omit, follows from calculations similar to those in [1, §2].

Proposition 2.12. *Let*

$$B_{\mathfrak{m}} = \begin{pmatrix} \mathfrak{m} & 0 \\ 0 & 1 \end{pmatrix}.$$

(1) *If* $\mathfrak{n}$ *is coprime to* $\mathfrak{m}$ *and* $f \in \mathcal{H}(\mathfrak{n}, R)$*, then*

$$(f|B_{\mathfrak{m}})|W_{\mathfrak{m}} = f,$$

where $W_{\mathfrak{m}}$ *is the Atkin-Lehner involution acting on* $\mathcal{H}(\mathfrak{n}\mathfrak{m}, R)$. (*Note that by Lemma 2.25,* $f|B_{\mathfrak{m}} \in \mathcal{H}(\mathfrak{n}\mathfrak{m}, R)$.)

(2) *Let* $\mathfrak{m}|\mathfrak{n}$ *with* $\gcd(\mathfrak{m}, \mathfrak{n}/\mathfrak{m}) = 1$*, and* $\mathfrak{b}$ *be coprime to* $\mathfrak{m}$*. If* $f \in \mathcal{H}(\mathfrak{n}, R)$*, then*

$$(f|B_{\mathfrak{b}})|W_{\mathfrak{m}} = (f|W_{\mathfrak{m}})|B_{\mathfrak{b}},$$

where on the left hand-side $W_{\mathfrak{m}}$ *denotes the Atkin-Lehner involution acting on* $\mathcal{H}(\mathfrak{n}\mathfrak{b}, R)$ *and on the right hand-side* $W_{\mathfrak{m}}$ *denotes the involution acting on* $\mathcal{H}(\mathfrak{n}, R)$.

(3) *Let* $f \in \mathcal{H}(\mathfrak{n}, R)$*. If* $\mathfrak{q}$ *is a prime ideal which divides* $\mathfrak{n}$ *but does not divide* $\mathfrak{n}/\mathfrak{q}$*, then* $f|(U_{\mathfrak{q}} + W_{\mathfrak{q}}) \in \mathcal{H}(\mathfrak{n}/\mathfrak{q}, R)$.

The vector space $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$ is equipped with a natural (Petersson) inner product

$$\langle f, g \rangle = \sum_{e \in E(\Gamma_0(\mathfrak{n}) \backslash \mathscr{T})} n(e)^{-1} f(e) g(e),$$

where $n(e)$ is defined in the proof of Lemma 2.7. The Hecke operator $T_{\mathfrak{m}}$ is self-adjoint with respect to this inner product if $\mathfrak{m}$ is coprime to $\mathfrak{n}$; one can prove this by an argument similar to the proof of Lemma 13 in [1].

Definition 2.13. Let $\mathfrak{m}$ be a divisor of $\mathfrak{n}$ and $\mathfrak{d}$ be a divisor of $\mathfrak{n}/\mathfrak{m}$. By Lemma 2.25, the map $\varphi \mapsto \varphi|B_{\mathfrak{d}}$ gives an injective homomorphism

$$i_{\mathfrak{d},\mathfrak{m}} : \mathcal{H}_0(\mathfrak{m}, \mathbb{Q}) \to \mathcal{H}_0(\mathfrak{n}, \mathbb{Q}).$$

We denote the subspace generated by the images of all $i_{\mathfrak{d},\mathfrak{m}}$ ($\mathfrak{m} \neq \mathfrak{n}$) by $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})^{\mathrm{old}}$. The orthogonal complement of $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})^{\mathrm{old}}$ with respect to the Petersson product is the *new* subspace of $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$, and will be denoted by $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})^{\mathrm{new}}$. The new subspace of $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$ is invariant under the action $\mathbb{T}(\mathfrak{n})$ (this again can be proven as in [1]). We denote by $\mathbb{T}(\mathfrak{n})^{\mathrm{new}}$ the quotient of $\mathbb{T}(\mathfrak{n})$ through which $\mathbb{T}(\mathfrak{n})$ acts on $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})^{\mathrm{new}}$.

As we mentioned, the cusps of $\Gamma_0(\mathfrak{n})$ are in bijection with the orbits of the action of $\Gamma_0(\mathfrak{n})$ on

$$\mathbb{P}^1(F) = \mathbb{P}^1(A) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in A, \gcd(a, b) = 1, a \text{ is monic} \right\},$$

where $\Gamma_0(\mathfrak{n})$ acts on $\mathbb{P}^1(F)$ from the left as on column vectors. We leave the proof of the following lemma to the reader.

Lemma 2.14. *Assume* $\mathfrak{n}$ *is square-free.*

(1) *For* $\mathfrak{m}|\mathfrak{n}$ *let* $[\mathfrak{m}]$ *be the orbit of* $\begin{pmatrix}1\\ \mathfrak{m}\end{pmatrix}$ *under the action of* $\Gamma_0(\mathfrak{n})$. *Then* $[\mathfrak{m}] \neq [\mathfrak{m}']$ *if* $\mathfrak{m} \neq \mathfrak{m}'$, *and the set* $\{[\mathfrak{m}] \mid \mathfrak{m}|\mathfrak{n}\}$ *is the set of cusps of* $\Gamma_0(\mathfrak{n})$. *In particular, there are* 2^s *cusps, where* s *is the number of prime divisors of* $\mathfrak{n}$.

(2) *Since* $W_\mathfrak{m}$ *normalizes* $\Gamma_0(\mathfrak{n})$, *it acts on the set of cusps of* $\Gamma_0(\mathfrak{n})$. *There is the formula*

$$W_\mathfrak{m}[\mathfrak{n}] = [\mathfrak{n}/\mathfrak{m}].$$

The cusp $[\mathfrak{n}]$ is usually called the *cusp at infinity.* We will denote it by $[\infty]$.

2.3. Fourier expansion. An important observation in [38] is that the theory of Fourier expansions of automorphic forms over function fields developed in [57] works over more general rings than $\mathbb{C}$. Here we follow Gekeler's reinterpretation [12] of Weil's adelic approach as analysis on the Bruhat-Tits tree, but we will extend [12] to the setting of these more general rings.

Definition 2.15. Following [38] we say that R is a *coefficient ring* if $p \in R^\times$ and R is a quotient of a discrete valuation ring $\tilde{R}$ which contains p-th roots of unity. Note that the image of the p-th roots of unity of $\tilde{R}$ in R is exactly the set of p-th roots of unity of R. For example, any algebraically closed field of characteristic different from p is a coefficient ring.

Let

$$\begin{aligned} \eta : F_\infty &\to R^\times \\ \sum a_i \pi_\infty^i &\mapsto \eta_0\Big(\mathrm{Trace}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)\Big) \end{aligned}$$

where $\eta_0 : \mathbb{F}_p \to R^\times$ is a non-trivial additive character fixed once and for all. Let f be an R-valued function on $E(\mathscr{T})$, which is invariant under the action of

$$\Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(A) \right\},$$

and is alternating (i.e., satisfies $f(e) = -f(\bar{e})$ for all $e \in E(\mathscr{T})$). The *constant Fourier coefficient* of f is the R-valued function f^0 on $\pi_\infty^\mathbb{Z}$ defined by

$$f^0(\pi_\infty^k) = \begin{cases} q^{1-k} \sum_{u \in (\pi_\infty)/(\pi_\infty^k)} f\left(\begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix}\right) & \text{if } k \geq 1 \\ f\left(\begin{pmatrix} \pi_\infty^k & 0 \\ 0 & 1 \end{pmatrix}\right) & \text{if } k \leq 1. \end{cases}$$

For a divisor $\mathfrak{m}$ on F, the $\mathfrak{m}$-*th Fourier coefficient* $f^*(\mathfrak{m})$ of f is

$$f^*(\mathfrak{m}) = q^{-1-\deg(\mathfrak{m})} \sum_{u \in (\pi_\infty)/(\pi_\infty^{2+\deg(\mathfrak{m})})} f\left(\begin{pmatrix} \pi_\infty^{2+\deg(\mathfrak{m})} & u \\ 0 & 1 \end{pmatrix}\right) \eta(-mu),$$

if $\mathfrak{m}$ is non-negative, and $f^*(\mathfrak{m}) = 0$, otherwise; here $m \in A$ is the monic polynomial such that $\mathfrak{m} = \mathrm{div}(m) \cdot \infty^{\deg(\mathfrak{m})}$.

Theorem 2.16. *Let f be an R-valued function on $E(\mathscr{T})$, which is Γ_∞-invariant and alternating. Then*

$$f\left(\begin{pmatrix}\pi_\infty^k & y\\ 0 & 1\end{pmatrix}\right) = f^0(\pi_\infty^k) + \sum_{\substack{0\neq m\in A\\ \deg(m)\leq k-2}} f^*(\mathrm{div}(m)\cdot\infty^{k-2})\cdot\eta(my).$$

In particular, f is uniquely determined by the functions f^0 and f^*.

Proof. This follows from [38, §2] and [12, §2]. □

Lemma 2.17. *Assume f is alternating and Γ_∞-invariant. Then f is a harmonic cochain if and only if*

(i) $f^0(\pi_\infty^k) = f^0(1)q^{-k}$ *for any* $k\in\mathbb{Z}$;
(ii) $f^*(\mathfrak{m}\infty^k) = f^*(\mathfrak{m})q^{-k}$ *for any non-negative divisor* $\mathfrak{m}$ *and* $k\in\mathbb{Z}_{\geq 0}$.

Proof. See Lemma 2.13 in [12]. □

Lemma 2.18. *For an ideal $\mathfrak{m}\lhd A$ and $f\in\mathcal{H}(\mathfrak{n},\mathbb{Z})$ we have*

$$(f|T_\mathfrak{m})^*(\mathfrak{r}) = \sum_{\substack{a\ \text{monic}\\ a|\gcd(\mathfrak{m},\mathfrak{r})\\ (a)+\mathfrak{n}=A}} \frac{|\mathfrak{m}|}{|a|} f^*\left(\frac{\mathfrak{r}\mathfrak{m}}{a^2}\right).$$

In particular,

$$(f|T_\mathfrak{m})^*(1) = |\mathfrak{m}|f^*(\mathfrak{m}).$$

Proof. See Lemma 3.2 in [38]. □

Lemma 2.19. *Assume $\mathfrak{n}$ is square-free. A harmonic cochain $f\in\mathcal{H}(\mathfrak{n},R)$ is cuspidal if and only if $(f|W)^0(1)=0$ for all $W\in\mathbb{W}$.*

Proof. By definition, f is cuspidal if and only if it vanishes on all but finitely many edges of each cusp $[\mathfrak{m}]$. The positively oriented edges of the cusp $[\infty]$ are given by the matrices $\begin{pmatrix}\pi_\infty^k & 0\\ 0 & 1\end{pmatrix}$, $k\leq 1$. By definition of f^0 and Lemma 2.17,

$$f\left(\begin{pmatrix}\pi_\infty^k & 0\\ 0 & 1\end{pmatrix}\right) = f^0(\pi_\infty^k) = q^{-k}f^0(1).$$

Since q is invertible in R, we see that f eventually vanishes on $[\infty]$ if and only if $f^0(1)=0$. Next, by Lemma 2.14, f vanishes on $[\mathfrak{n}/\mathfrak{m}]$ if and only if $f|W_\mathfrak{m}$ vanishes on $[\infty]$, which is equivalent to $(f|W_\mathfrak{m})^0(1)=0$. □

Theorem 2.20. *If R is a coefficient ring, then the bilinear $\mathbb{T}(\mathfrak{n})\otimes R$-equivariant pairing*

$$\begin{aligned}(\mathbb{T}(\mathfrak{n})\otimes R)\times\mathcal{H}_{00}(\mathfrak{n},R) &\to R\\ T, f &\mapsto (f|T)^*(1)\end{aligned}$$

is perfect.

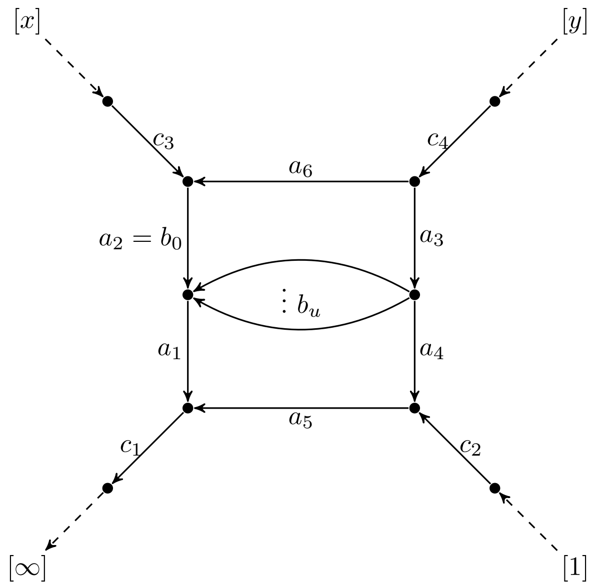

FIGURE 4. $\Gamma_0(xy) \setminus \mathscr{T}$

Proof. Theorem 3.17 in [11] says that the pairing

$$\begin{aligned} \mathbb{T}(\mathfrak{n}) \times \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) &\to \mathbb{Z} \\ T, f &\mapsto (f|T)^*(1) \end{aligned} \tag{2.5}$$

is non-degenerate and becomes a perfect pairing after tensoring with $\mathbb{Z}[p^{-1}]$. Since p is invertible in R by assumption, the claim follows. $\square$

It is not known if in general the pairing (2.5) is perfect. This is in contrast to the situation over $\mathbb{Q}$ where the analogous pairing between the Hecke algebra and the space of weight-2 cusp forms on $\Gamma_0(N)$ with integral Fourier expansions is perfect (cf. [46, Thm. 2.2]). This dichotomy comes from the formula $(f|T_\mathfrak{m})^*(1) = |\mathfrak{m}| f^*(\mathfrak{m})$; in the classical situation the first Fourier coefficient of $f|T_m$ is just the mth Fourier coefficient of f.

PROPOSITION 2.21. *In the special case $\mathfrak{n} = xy$, the pairing (2.5)*

$$\mathbb{T}(xy) \times \mathcal{H}_0(xy, \mathbb{Z}) \to \mathbb{Z}$$

is perfect. Moreover, as $\mathbb{Z}$-modules,

$$\mathbb{T}(xy)^0 = \mathbb{T}(xy) \cong \mathbb{Z} \oplus \bigoplus_{\substack{\deg(\mathfrak{p})=1 \\ \mathfrak{p} \neq x}} \mathbb{Z} T_\mathfrak{p}.$$

Proof. Take $\alpha_x,\ \beta_x \in \mathbb{F}_q$ such that $y = x^2 + \alpha_x x + \beta_x$. Let $\varpi_x := x^{-1}$, which is also a uniformizer at ∞. The quotient graph $\Gamma_0(xy) \backslash \mathscr{T}$ is depicted in Figure

4 with positively oriented edges

$$c_1 = \begin{pmatrix} \varpi_x & 0 \\ 0 & 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} \varpi_x^3 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_3 = \begin{pmatrix} \varpi_x^4 & \varpi_x \\ 0 & 1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} \varpi_x^5 & y^{-1} \\ 0 & 1 \end{pmatrix};$$

$$a_1 = \begin{pmatrix} \varpi_x^2 & \varpi_x \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} \varpi_x^3 & \varpi_x \\ 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} \varpi_x^4 & y^{-1} \\ 0 & 1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} \varpi_x^3 & \varpi_x^2 \\ 0 & 1 \end{pmatrix};$$

$$a_5 = \begin{pmatrix} \varpi_x^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} \varpi_x^4 & \varpi_x - \beta_x \varpi_x^3 \\ 0 & 1 \end{pmatrix};$$

$$b_u = \begin{pmatrix} \varpi_x^3 & \varpi_x + u\varpi_x^2 \\ 0 & 1 \end{pmatrix}, \quad u \in \mathbb{F}_q.$$

Note that in this notation $a_2 = b_0$. A small calculation shows that

$$w(a_1) = w(\bar{a}_2) = w(\bar{a}_3) = w(a_4) = q - 1,$$

and the weights of all other edges in $(\Gamma_0(xy) \setminus \mathscr{T})^0$ are 1.
It is easy to see that the map

$$\begin{aligned} \mathcal{H}_0(xy, \mathbb{Z}) &\to \bigoplus_{u \in \mathbb{F}_q} \mathbb{Z} \\ f &\mapsto (f(b_u))_{u \in \mathbb{F}_q} \end{aligned}$$

is an isomorphism, so the harmonic cochains $f_v \in \mathcal{H}_0(xy, \mathbb{Z})$, $v \in \mathbb{F}_q$, defined by $f_v(b_u) = \delta_{v,u}$=(Kronecker's delta) form a $\mathbb{Z}$-basis. Let $f \in \mathcal{H}_0(xy, \mathbb{Z})$ and $\kappa \in \mathbb{F}_q$. By Lemma 2.18

$$q(f|T_{x-\kappa})^*(1) = q^2 f^*(x-\kappa) = \sum_{w \in \varpi_x \mathcal{O}_\infty / \varpi_x^3 \mathcal{O}_\infty} f\left(\begin{pmatrix} \varpi_x^3 & w \\ 0 & 1 \end{pmatrix}\right) \eta\left(-(x-\kappa)w\right)$$

$$\begin{aligned} =& f\left(\begin{pmatrix} \varpi_x^3 & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{\beta \in \mathbb{F}_q^\times} f\left(\begin{pmatrix} \varpi_x^3 & \beta\varpi_x^2 \\ 0 & 1 \end{pmatrix}\right) \eta\left(-(\varpi_x^{-1} - \kappa)\beta\varpi_x^2\right) \\ &+ \sum_{u \in \mathbb{F}_q} \sum_{\beta \in \mathbb{F}_q^\times} f\left(\begin{pmatrix} \varpi_x^3 & \beta(\varpi_x + u\varpi_x^2) \\ 0 & 1 \end{pmatrix}\right) \eta\left(-(\varpi_x^{-1} - \kappa)\beta(\varpi_x + u\varpi_x^2)\right). \end{aligned}$$

Since the double class of $\begin{pmatrix} \varpi_x^3 & w \\ 0 & 1 \end{pmatrix}$ does not change if w is replaced by βw ($\beta \in \mathbb{F}_q^\times$), $f\left(\begin{pmatrix} \varpi_x^3 & 0 \\ 0 & 1 \end{pmatrix}\right) = f(c_2) = 0$, and $\sum_{\beta \in \mathbb{F}_q^\times} \eta(\beta\varpi_x) = -1$, the above sum reduces to

$$-f(a_4) + \sum_{u \in \mathbb{F}_q} f(b_u)(q\delta_{u,\kappa} - 1).$$

Using (ii′),

$$(q-1)f(a_1) + f(a_5) = 0, \quad (q-1)f(a_4) + f(\bar{a}_5) = 0, \quad f(a_1) = \sum_{u \in \mathbb{F}_q} f(b_u).$$

Therefore, $f(a_4) = -\sum_{u\in\mathbb{F}_q} f(b_u)$ and we get

$$(f|T_{x-\kappa})^*(1) = f(b_\kappa).$$

In particular, $(f_v|T_{x-\kappa})^*(1) = \delta_{\kappa,v}$. This implies that the homomorphism

$$\mathbb{T}(xy) \to \mathrm{Hom}(\mathcal{H}_0(xy,\mathbb{Z}),\mathbb{Z}) \tag{2.6}$$

induced by the pairing (2.5) is surjective. Comparing the ranks of both sides, we conclude that this map is in fact an isomorphism, which is equivalent to the pairing being perfect. Let M be the $\mathbb{Z}$-submodule of $\mathbb{T}(xy)$ generated by $\{T_{x-\kappa} \mid \kappa \in \mathbb{F}_q\}$. The composition of $M \hookrightarrow \mathbb{T}(xy)$ with (2.6) gives a surjection $M \to \mathrm{Hom}(\mathcal{H}_0(xy,\mathbb{Z}),\mathbb{Z})$. This implies that $M = \mathbb{T}(xy)$ and $M \cong \bigoplus_{\kappa\in\mathbb{F}_q} \mathbb{Z}T_{x-\kappa}$.

An easy consequence of the definitions is that $f^*(1) = -f(a_1)$, cf. [11, (3.16)]. If we denote $S = \sum_{\kappa\in\mathbb{F}_q} T_{x-\kappa}$, then

$$(f|S)^*(1) = \sum_{\kappa\in\mathbb{F}_q} f(b_\kappa) = f(a_1) = -f^*(1). \tag{2.7}$$

The non-degeneracy of the pairing implies that $S = -1$. Therefore

$$\mathbb{T}(xy) = \mathbb{Z} \oplus \bigoplus_{\kappa\in\mathbb{F}_q^\times} \mathbb{Z}T_{x-\kappa} \subseteq \mathbb{T}(xy)^0,$$

which implies $\mathbb{T}(xy) = \mathbb{T}(xy)^0$. $\square$

Remark 2.22. In [44], we have extended the statement of Proposition 2.21 to arbitrary $\mathfrak{n} \lhd A$ of degree 3. More precisely, we proved that the pairing (2.5) is perfect if $\deg(\mathfrak{n}) = 3$. Moreover, if $\mathfrak{n}$ has degree 3 but is not a product of three distinct primes of degree 1, then $\mathbb{T}(\mathfrak{n}) = \mathbb{T}(\mathfrak{n})^0$. Finally, if $\mathfrak{n}$ is a product of three distinct primes of degree 1, then $\mathbb{T}(\mathfrak{n})/\mathbb{T}(\mathfrak{n})^0$ is finite but non-zero.

2.4. ATKIN-LEHNER METHOD. For $b \in A$, let $S_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Define a linear operator $U_\mathfrak{p}$ on the space of R-valued functions on $E(\mathscr{T})$ by

$$f|U_\mathfrak{p} = \sum_{\substack{b\in A \\ \deg(b)<\deg(\mathfrak{p})}} f|B_\mathfrak{p}^{-1}S_b.$$

Note that the action of $B_\mathfrak{m}^{-1}$ on functions on $E(\mathscr{T})$ is the same as the action of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{m} \end{pmatrix}$ (since the diagonal matrices act trivially), so this operator agrees with the Hecke operator $U_\mathfrak{p}$ when restricted to $\mathcal{H}(\mathfrak{n},R)$ for any $\mathfrak{n}$ divisible by $\mathfrak{p}$.

LEMMA 2.23. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two distinct prime ideals of A. If $f \in \mathcal{H}(\mathscr{T},R)^{\Gamma_\infty}$, then*

$$(f|B_\mathfrak{p})|U_\mathfrak{p} = |\mathfrak{p}| \cdot f,$$
$$(f|B_\mathfrak{p})|U_\mathfrak{q} = (f|U_\mathfrak{q})|B_\mathfrak{p}.$$

Proof. We have

$$(f|B_\mathfrak{p})|U_\mathfrak{p} = \sum_{\substack{b\in A\\ \deg(b)<\deg(\mathfrak{p})}} (f|B_\mathfrak{p})|B_\mathfrak{p}^{-1}S_b = \sum_{\substack{b\in A\\ \deg(b)<\deg(\mathfrak{p})}} f|S_b.$$

Since $S_b \in \Gamma_\infty$, we have $f|S_b = f$ for all b, so the last sum is equal to $|\mathfrak{p}|f$. Next, for $b \in A$ representing a residue modulo $\mathfrak{q}$ we have

$$B_\mathfrak{p} B_\mathfrak{q}^{-1} S_b = \begin{pmatrix} \mathfrak{p} & b\mathfrak{p} \\ 0 & \mathfrak{q} \end{pmatrix}.$$

By the division algorithm there is $a \in A$ and $b' \in A$ with $\deg(b') < \deg(\mathfrak{q})$ such that $b\mathfrak{p} = a\mathfrak{q} + b'$. Now

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathfrak{p} & b\mathfrak{p} \\ 0 & \mathfrak{q} \end{pmatrix} = \begin{pmatrix} \mathfrak{p} & b' \\ 0 & \mathfrak{q} \end{pmatrix} = B_\mathfrak{q}^{-1} S_{b'} B_\mathfrak{p}.$$

As b runs over the residues modulo $\mathfrak{q}$, b' runs over the same set since $\mathfrak{p} \neq \mathfrak{q}$. Thus, using Γ_∞-invariance of f, we get $(f|B_\mathfrak{p})|U_\mathfrak{q} = (f|U_\mathfrak{q})|B_\mathfrak{p}$. □

Lemma 2.24. *For any non-zero ideal $\mathfrak{m} \lhd A$ and $f \in \mathcal{H}(\mathscr{T}, R)^{\Gamma_\infty}$*

$$(f|B_\mathfrak{m})^0(\pi_\infty^k) = f^0(\pi_\infty^{k-\deg(\mathfrak{m})}), \quad (f|B_\mathfrak{m})^*(\mathfrak{n}) = f^*(\mathfrak{n}/\mathfrak{m}).$$

Proof. See Proposition 2.10 in [12]. □

Given ideals $\mathfrak{n}, \mathfrak{m} \lhd A$, denote

$$\Gamma_0(\mathfrak{n}, \mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \mid c \in \mathfrak{n}, b \in \mathfrak{m} \right\}.$$

Lemma 2.25. *If $f \in \mathcal{H}(\mathfrak{n}, R)$, then $f|B_\mathfrak{m}$ is $\Gamma_0(\mathfrak{n}\mathfrak{m})$-invariant and $f|B_\mathfrak{m}^{-1}$ is $\Gamma_0(\mathfrak{n}/\gcd(\mathfrak{n},\mathfrak{m}), \mathfrak{m})$-invariant.*

Proof. This follows from a straightforward manipulation with matrices. □

Theorem 2.26. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two distinct primes such that $\mathfrak{p}\mathfrak{q}$ divides $\mathfrak{n}$, and $\mathfrak{p}\mathfrak{q}$ is coprime to $\mathfrak{n}/\mathfrak{p}\mathfrak{q}$. Let $\varphi \in \mathcal{H}(\mathfrak{n}, R)$. Assume $\varphi^*(\mathfrak{m}) = 0$ unless $\mathfrak{p}$ or $\mathfrak{q}$ divides $\mathfrak{m}$. Then there exist $\psi_1 \in \mathcal{H}(\mathfrak{n}/\mathfrak{p}, R)$ and $\psi_2 \in \mathcal{H}(\mathfrak{n}/\mathfrak{q}, R)$ such that*

$$s_{\mathfrak{p},\mathfrak{q}} \cdot \varphi = \psi_1|B_\mathfrak{p} + \psi_2|B_\mathfrak{q},$$

where $s_{\mathfrak{p},\mathfrak{q}} = \gcd(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$.

Proof. Take $\phi_2 := |q|^{-1} \cdot \varphi|U_\mathfrak{q} \in \mathcal{H}(\mathfrak{n}, R)$. We have

$$(\phi_2)^0(\pi_\infty^k) = \varphi^0(\pi_\infty^{k+\deg(\mathfrak{q})}), \quad \phi_2^*(\mathfrak{m}) = \varphi^*(\mathfrak{m}\mathfrak{q}).$$

Let $\varphi_1 := \varphi - \phi_2|B_\mathfrak{q} \in \mathcal{H}(\mathfrak{n}\mathfrak{q}, R)$. Then by Lemma 2.24,

$$(\varphi_1)^0(\pi_\infty^k) = 0, \quad \varphi_1^*(\mathfrak{m}) = \varphi^*(\mathfrak{m}) \text{ if } \mathfrak{q} \nmid \mathfrak{m}, \quad \varphi_1^*(\mathfrak{m}) = 0 \text{ if } \mathfrak{q}|\mathfrak{m}.$$

Let $\phi_1 := \varphi_1|B_\mathfrak{p}^{-1}$, which is $\Gamma_0(\mathfrak{n}\mathfrak{q}/\mathfrak{p}, \mathfrak{p})$-invariant by Lemma 2.25. In particular, $\varphi_1^*(\mathfrak{m}) = 0$ unless $\mathfrak{p}|\mathfrak{m}$, which implies that ϕ_1 is Γ_∞-invariant. Since Γ_∞ and $\Gamma_0(\mathfrak{n}\mathfrak{q}/\mathfrak{p}, \mathfrak{p})$ generates $\Gamma_0(\mathfrak{n}\mathfrak{q}/\mathfrak{p})$, we get $\phi_1 \in \mathcal{H}(\mathfrak{n}\mathfrak{q}/\mathfrak{p}, R)$ with

$$(\phi_1)^0(\pi_\infty^k) = 0, \quad \phi_1^*(\mathfrak{m}) = \varphi^*(\mathfrak{m}\mathfrak{p}) \text{ if } \mathfrak{q} \nmid \mathfrak{m}, \quad \phi_1^*(\mathfrak{m}) = 0 \text{ if } \mathfrak{q}|\mathfrak{m},$$

and

$$\varphi = \phi_1|B_\mathfrak{p} + \phi_2|B_\mathfrak{q}.$$

By Proposition 2.12, $\psi_1 := \varphi|(U_\mathfrak{p} + W_\mathfrak{p}) \in \mathcal{H}(\mathfrak{n}/\mathfrak{p}, R)$. Using Proposition 2.12 and Lemma 2.23,

$$(\phi_1|B_\mathfrak{p})|(U_\mathfrak{p} + W_\mathfrak{p}) = \phi_1|B_\mathfrak{p}|U_\mathfrak{p} + \phi_1|B_\mathfrak{p}|W_\mathfrak{p} = |\mathfrak{p}|\phi_1 + \phi_1 = (|\mathfrak{p}| + 1)\phi_1.$$

On the other hand, using the fact that $\phi_2 \in \mathcal{H}(\mathfrak{n}, R)$, we have

$$(\phi_2|B_\mathfrak{q})|(U_\mathfrak{p} + W_\mathfrak{p}) = \phi_2|(U_\mathfrak{p} + W_\mathfrak{p})|B_\mathfrak{q}.$$

If we denote $\psi := \phi_2|(U_\mathfrak{p} + W_\mathfrak{p})$, then we proved that

$$\psi_1 = (|\mathfrak{p}| + 1)\phi_1 + \psi|B_\mathfrak{q} \in \mathcal{H}(\mathfrak{n}/\mathfrak{p}, R).$$

Therefore,

$$\begin{aligned}(|\mathfrak{p}| + 1)\varphi &= (|\mathfrak{p}| + 1)\phi_1|B_\mathfrak{p} + (|\mathfrak{p}| + 1)\phi_2|B_\mathfrak{q} \\ &= ((|\mathfrak{p}| + 1)\phi_1 + \psi|B_\mathfrak{q})|B_\mathfrak{p} + ((|\mathfrak{p}| + 1)\phi_2 - \psi|B_\mathfrak{p})|B_\mathfrak{q} = \psi_1|B_\mathfrak{p} + \psi_2|B_\mathfrak{q},\end{aligned}$$

where $\psi_2 := (|\mathfrak{p}| + 1)\phi_2 - \psi|B_\mathfrak{p}$. We already proved that $\psi_1 \in \mathcal{H}(\mathfrak{n}/\mathfrak{p}, R)$. Obviously $\psi_2|B_\mathfrak{q} \in \mathcal{H}(\mathfrak{n}, R)$. By Lemma 2.25, ψ_2 is $\Gamma_0(\mathfrak{n}/\mathfrak{q}, \mathfrak{q})$-invariant. Since it is also Γ_∞-invariant, we conclude $\psi_2 \in \mathcal{H}(\mathfrak{n}/\mathfrak{q}, R)$.

Finally, interchanging the roles of $\mathfrak{p}$ and $\mathfrak{q}$ we obtain

$$(|\mathfrak{q}| + 1)\varphi = \psi_1'|B_\mathfrak{p} + \psi_2'|B_\mathfrak{q}$$

with $\psi_1' \in \mathcal{H}(\mathfrak{n}/\mathfrak{p}, R)$ and $\psi_2' \in \mathcal{H}(\mathfrak{n}/\mathfrak{q}, R)$. This implies the claim of the theorem. □

3. Eisenstein harmonic cochains

3.1. Eisenstein series. In this section R always denotes a coefficient ring, in particular, p is invertible in R. We say that a harmonic cochain $\varphi \in \mathcal{H}(\mathfrak{n}, R)$ is *Eisenstein* if $\varphi|T_\mathfrak{p} = (|\mathfrak{p}| + 1)\varphi$ for every prime ideal $\mathfrak{p} \lhd A$ not dividing $\mathfrak{n}$. It is clear that the Eisenstein harmonic cochains form an R-submodule of $\mathcal{H}(\mathfrak{n}, R)$ which we denote by $\mathcal{E}(\mathfrak{n}, R)$.

The Drinfeld half-plane

$$\Omega = \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(F_\infty) = \mathbb{C}_\infty - F_\infty$$

has a natural structure of a smooth connected rigid-analytic space over F_∞; see [18, §1]. The group $\Gamma_0(\mathfrak{n})$ acts on Ω via linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

This action is discrete, so the quotient

$$Y_0(\mathfrak{n})(\mathbb{C}_\infty) = \Gamma_0(\mathfrak{n}) \setminus \Omega \tag{3.1}$$

has a natural structure of a rigid-analytic curve over F_∞, which is in fact an affine algebraic curve; cf. [6, Prop. 6.6]. If we denote $\overline{\Omega} = \Omega \cup \mathbb{P}^1(F)$, then

$$X_0(\mathfrak{n})(\mathbb{C}_\infty) = \Gamma_0(\mathfrak{n}) \setminus \overline{\Omega}$$

is the projective closure of $Y_0(\mathfrak{n})$. The points $X_0(\mathfrak{n})(\mathbb{C}_\infty) - Y_0(\mathfrak{n})(\mathbb{C}_\infty)$ are called the *cusps* of $X_0(\mathfrak{n})$, and they are in natural bijection with the cusps of $\Gamma_0(\mathfrak{n}) \setminus \mathscr{T}$ in Definition 2.3.

The Hecke operator $T_\mathfrak{p}$ induces a correspondence on $X_0(\mathfrak{n})(\mathbb{C}_\infty)$

$$T_\mathfrak{p} : z \mapsto \sum_{\substack{b \in A \\ \deg(b) < \deg(\mathfrak{p})}} \frac{z+b}{\mathfrak{p}} + \mathfrak{p}z \mod \Gamma_0(\mathfrak{n}); \tag{3.2}$$

$U_\mathfrak{p}(z)$ is given by the same sum but without the last summand.

Let $\mathcal{O}(\Omega)^\times$ be the group of nowhere vanishing holomorphic functions on Ω. The group $\mathrm{GL}_2(F_\infty)$ act on $\mathcal{O}(\Omega)^\times$ via $(f|\gamma)(z) = f(\gamma z)$. To each $f \in \mathcal{O}(\Omega)^\times$ van der Put associated a harmonic cochain $r(f) \in \mathcal{H}(\mathscr{T}, \mathbb{Z})$ so that the sequence

$$0 \to \mathbb{C}_\infty^\times \to \mathcal{O}(\Omega)^\times \xrightarrow{r} \mathcal{H}(\mathcal{T}, \mathbb{Z}) \to 0 \tag{3.3}$$

is exact and $\mathrm{GL}_2(F_\infty)$-equivariant. As is explained in [18], the map r plays the role of a logarithmic derivation.

Lemma 3.1. *Assume $\mathfrak{n}$ is square-free and $f \in \mathcal{O}(\Omega)^\times$ is $\Gamma_0(\mathfrak{n})$-invariant. Then $r(f)$ is Eisenstein.*

Proof. Put

$$e_\mathfrak{n}(z) = z \prod_{0 \neq a \in \mathfrak{n}} \left(1 - \frac{z}{a}\right) \quad \text{and} \quad \Gamma_\infty^u = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in \mathfrak{n} \right\}.$$

For $z \in \Omega$, let $|z|_i = \inf\{|z - s| \mid s \in F_\infty\}$ be its "imaginary" absolute value. The subspace $\Omega_d = \{z \in \Omega \mid |z|_i \geq d\}$ of Ω is stable under Γ_∞^u, and for $d \gg 0$, the function $t(z) = e_A(z)^{-1}$ identifies $\Gamma_\infty^u \setminus \Omega_d$ with a small punctured disc $D_\varepsilon^0 = \{t \in \mathbb{C}_\infty \mid 0 < |t| \leq \varepsilon\}$. The function f is Γ_∞^u-invariant, so can be considered as a holomorphic non-vanishing function on D_ε^0. By the non-archimedean analogue of Picard's Big Theorem [55, (1.3)], f has at worst a pole at $t = 0$, or equivalently, at the cusp $[\infty]$. Now let $[c]$ be any other cusp of $\Gamma_0(\mathfrak{n})$ and $\gamma \in \mathrm{GL}_2(A)$ be such that $\gamma[\infty] = [c]$. The function $f|\gamma \in \mathcal{O}(\Omega)^\times$ is invariant under $\Gamma' = \gamma^{-1}\Gamma_0(\mathfrak{n})\gamma$. The stabilizer of $[\infty]$ in Γ' contains Γ_∞^u, so the previous argument shows that $f|\gamma$ is meromorphic at $[\infty]$. Thus, f is meromorphic at $[c]$. We conclude that f descends to a rational function on $X_0(\mathfrak{n})$ whose divisor is supported at the cusps.

Now we use an idea from the proof of Lemma 6.2 in [38]. The Hecke correspondence $T_\mathfrak{p}$ defines a map from the group of divisors on $X_0(\mathfrak{n})$ supported at the cusps to itself (cf. (3.2)):

$$T_\mathfrak{p}[\mathfrak{d}] = \begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{d} \end{pmatrix} + \sum_{\substack{b \neq 0 \\ \deg(b) < \deg(\mathfrak{p})}} \begin{pmatrix} 1 & b \\ 0 & \mathfrak{p} \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{d} \end{pmatrix} = \begin{pmatrix} \mathfrak{p} \\ \mathfrak{d} \end{pmatrix} + \sum_{\substack{b \neq 0 \\ \deg(b) < \deg(\mathfrak{p})}} \begin{pmatrix} 1 + b\mathfrak{d} \\ \mathfrak{p}\mathfrak{d} \end{pmatrix}.$$

The orbit of the cusp $[\mathfrak{d}]$ consists exactly of those $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{P}^1(A)$ such that β is divisible by $\mathfrak{d}$ and is coprime to $\mathfrak{n}/\mathfrak{d}$, so $T_\mathfrak{p}[\mathfrak{d}] = (1 + |\mathfrak{p}|)[\mathfrak{d}]$. Thus, $f|T_\mathfrak{p}$ and

$f^{|\mathfrak{p}|+1}$, as rational functions on $X_0(\mathfrak{n})$, have the same divisor. This implies that $(f|T_\mathfrak{p})/f^{|\mathfrak{p}|+1}$ is a constant function. Applying r, we get

$$r(f|T_\mathfrak{p}) = (|\mathfrak{p}|+1)r(f).$$

Since r is $\mathrm{GL}_2(F_\infty)$-equivariant, $r(f|T_\mathfrak{p}) = r(f)|T_\mathfrak{p}$, which finishes the proof. $\square$

Lemma 3.1 gives a natural source of $\mathbb{Z}$-valued Eisenstein harmonic cochains. Let $\Delta(z)$ be the Drinfeld discriminant function on Ω defined on page 183 of [14]. This is a Drinfeld modular form of weight (q^2-1) and type 0 for $\mathrm{GL}_2(A)$, which vanishes nowhere on Ω. Let $\Delta_\mathfrak{n} := \Delta|B_\mathfrak{n} = \Delta(\mathfrak{n}z)$. By page 194 of [14], $\Delta/\Delta_\mathfrak{n}$ is a $\Gamma_0(\mathfrak{n})$-invariant function in $\mathcal{O}(\Omega)^\times$. Hence $r(\Delta/\Delta_\mathfrak{n}) \in \mathcal{E}(\mathfrak{n},\mathbb{Z})$. Define

$$\nu(\mathfrak{n}) = \begin{cases} 1, & \text{if } \deg(\mathfrak{n}) \text{ is even} \\ q+1, & \text{if } \deg(\mathfrak{n}) \text{ is odd} \end{cases}$$

and

$$E_\mathfrak{n} = \frac{\nu(\mathfrak{n})}{(q-1)(q^2-1)} r(\Delta/\Delta_\mathfrak{n}). \tag{3.4}$$

By [14, (3.18)], $E_\mathfrak{n}$ is $\mathbb{Z}$-valued and primitive (i.e., $E_\mathfrak{n}$ is not a scalar multiple of another harmonic cochain in $\mathcal{H}(\mathfrak{n},\mathbb{Z})$ except for $\pm E_\mathfrak{n}$). We call $E_\mathfrak{n} \in \mathcal{E}(\mathfrak{n},\mathbb{Z})$ the *Eisenstein series*. The Fourier expansion of $E_\mathfrak{n}$ can be deduced from [14]:

$$E_\mathfrak{n}\left(\begin{pmatrix} \pi_\infty^k & y \\ 0 & 1 \end{pmatrix}\right) = \nu(\mathfrak{n}) \cdot q^{-k+1} \cdot \left[\frac{1-|\mathfrak{n}|}{1-q^2} + \sum_{\substack{0\neq m\in A,\\ \deg(m)\leq k-2}} \sigma_\mathfrak{n}(m)\eta(my)\right],$$

where $\sigma_\mathfrak{n}(m) := \sigma(m) - |\mathfrak{n}|\cdot\sigma(m/\mathfrak{n})$, and σ is the divisor function

$$\sigma(m) := \begin{cases} \sum\limits_{\substack{\text{monic } m'\in A,\\ m'|m}} |m'|, & \text{if } m \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.2. Note that for each prime $\mathfrak{p} \lhd A$ and $m \in A$,

$$\sigma(m\mathfrak{p}) = \sigma(\mathfrak{p})\sigma(m) - |\mathfrak{p}|\sigma(m/\mathfrak{p}).$$

Therefore the Fourier expansion of $E_\mathfrak{n}|T_\mathfrak{p}$ for each prime $\mathfrak{p}$ not dividing $\mathfrak{n}$ also tells us that $E_\mathfrak{n} \in \mathcal{E}(\mathfrak{n},\mathbb{Z})$.

Example 3.3. Let E_0 be the R-valued function on $E(\mathscr{T})$ defined by

$$E_0\left(\begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix}\right) = -E_0\left(\begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = q^{-k}.$$

This function is alternating and Γ_∞-invariant. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$, let $\omega := \mathrm{ord}_\infty(cu+d)$. By the calculations in [12, p. 379]

$$(E_0|\gamma)\left(\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} -q^{k-2\deg(c)-1} & \text{if } \omega \geq k-\deg(c) \\ q^{2\omega-k} & \text{if } \omega < k-\deg(c). \end{cases}$$

Now it is easy to see that if $\alpha \in R[q+1]$, then αE_0 is the function in $\mathcal{H}(1,R)$ discussed in Example 2.4. The Hecke operator $T_\mathfrak{p}$ acts on αE_0 by

$$\begin{aligned}(\alpha E_0|T_\mathfrak{p})\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right) =& \alpha E_0\left(\begin{pmatrix}\pi^k\mathfrak{p} & u\mathfrak{p}\\ 0 & 1\end{pmatrix}\right)\\ &+ \sum_{\deg(b)<\deg(\mathfrak{p})} \alpha E_0\left(\begin{pmatrix}\pi^k/\mathfrak{p} & (u+b)/\mathfrak{p}\\ 0 & 1\end{pmatrix}\right)\end{aligned}$$

$$= \alpha(-1)^{k-\deg(\mathfrak{p})} + q^{\deg(\mathfrak{p})}\alpha(-1)^{k+\deg(\mathfrak{p})} = (1+|\mathfrak{p}|)\alpha E_0\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right).$$

Therefore, αE_0 is Eisenstein and $\mathcal{H}(1,R) = \mathcal{E}(1,R) \cong R[q+1]$.

Example 3.4. Let $\varphi \in \mathcal{H}_0(x,R)$. As we saw in Example 2.5, φ is uniquely determined by $\varphi(e_0) = \alpha$. Note that $\varphi^0(1) = \varphi(e_1) = q\alpha$. On the other hand, $E_x^0(1) = q$, so $\mathcal{H}(x,R) = \mathcal{E}(x,R) \cong R$ is generated by E_x.

Example 3.5. Finally, we return to the setting of Example 2.6. The Eisenstein series E_y, as a function on $\Gamma_0(y)\setminus\mathscr{T}$, can be explicitly described by $E_y(e_i) = E_y(e_{-i-1}) = q^i$ for $i \geq 0$, and $E_y(e_u) = 0$. Next, the function αE_0, for any $\alpha \in R[q+1]$, can be considered as a function on $\Gamma_0(y)\setminus\mathscr{T}$, and as such it is given by $\alpha E_0(e_u) = -\alpha$ and $\alpha E_0(e_i) = \alpha(-1)^{i+1}$ ($\forall i \in \mathbb{Z}$). Since any $f \in \mathcal{H}(y,R)$ is uniquely determined by its values on e_u and e_0, we see that

$$\mathcal{H}(y,R) = \mathcal{E}(y,R) = RE_y \oplus \{\alpha E_0 \mid \alpha \in R[q+1]\} \cong R \oplus R[q+1].$$

3.2. Cuspidal Eisenstein harmonic cochains. We set

$$\mathcal{E}_0(\mathfrak{n},R) := \mathcal{E}(\mathfrak{n},R) \cap \mathcal{H}_0(\mathfrak{n},R), \quad \mathcal{E}_{00}(\mathfrak{n},R) := \mathcal{E}(\mathfrak{n},R) \cap \mathcal{H}_{00}(\mathfrak{n},R).$$

Let $\mathfrak{p} \lhd A$ be a prime. Theorem 6.6 in [38] states that $\mathcal{E}_0(\mathfrak{p},R) \cong R\left[\frac{|\mathfrak{p}|-1}{q-1}\right]$, if $\deg(\mathfrak{p})$ is odd, and $\mathcal{E}_0(\mathfrak{p},R) \cong R\left[2\frac{|\mathfrak{p}|-1}{q^2-1}\right]$, if $\deg(\mathfrak{p})$ is even. Note that Examples 2.6 and 3.5 imply that $\mathcal{E}_0(y,R) \cong R[2]$, which is a special case of this theorem. The main result of this section is a similar description of $\mathcal{E}_0(\mathfrak{pq},R)$ and $\mathcal{E}_{00}(\mathfrak{pq},R)$ under certain assumptions. Note that as a consequence of the Ramanujan-Petersson conjecture over function fields $\mathcal{E}_0(\mathfrak{n},\mathbb{C}) = 0$ for any $\mathfrak{n}$. Therefore, $\mathcal{E}_0(\mathfrak{n},R)$ can be non-trivial only if R has non-trivial additive torsion.

Definition 3.6. The *Eisenstein ideal* $\mathfrak{E}(\mathfrak{n})$ of $\mathbb{T}(\mathfrak{n})$ is the ideal generated by the elements $\{T_\mathfrak{p} - |\mathfrak{p}| - 1 \mid \mathfrak{p} \text{ is prime}, \mathfrak{p} \nmid \mathfrak{n}\}$. We say that a maximal ideal $\mathfrak{M} \lhd \mathbb{T}(\mathfrak{n})$ is Eisenstein if $\mathfrak{E}(\mathfrak{n}) \subset \mathfrak{M}$.

Lemma 3.7. *Let $\mathfrak{p} \lhd A$ be a prime, and S be a set of prime ideals of A of density one that does not contain $\mathfrak{p}$. A cochain $f \in \mathcal{H}_{00}(\mathfrak{p},R)$ is Eisenstein if and only if*

$$f|T_\mathfrak{q} = (|\mathfrak{q}|+1)f, \quad \forall \mathfrak{q} \in S.$$

Proof. Let $\mathfrak{I}(\mathfrak{p})$ be the ideal of $\mathbb{T}(\mathfrak{p})$ generated by the elements $T_\mathfrak{q} - |\mathfrak{q}| - 1$, where $\mathfrak{q} \in S$. It is enough to show that $\mathfrak{E}(\mathfrak{p}) \otimes R = \mathfrak{I}(\mathfrak{p}) \otimes R$ in $\mathbb{T}(\mathfrak{p}) \otimes R$. The proof of the analogous statement over $\mathbb{Q}$ can be found in [4, Lem. 4]. We briefly sketch the argument over F.

Let $\mathfrak{M} \lhd \mathbb{T}(\mathfrak{p})$ be a maximal ideal such that the characteristic ℓ of $\mathbb{T}(\mathfrak{p})/\mathfrak{M}$ is different from p. There is a unique semi-simple representation

$$\rho_\mathfrak{M} : G_F \to \mathrm{GL}_2(\mathbb{T}(\mathfrak{p})/\mathfrak{M}),$$

which is unramified away from $\mathfrak{p}$ and ∞, and such that for all primes $\mathfrak{q} \lhd A$, $\mathfrak{q} \neq \mathfrak{p}$, the following relations hold:

$$\mathrm{Tr}\rho_\mathfrak{M}(\mathrm{Frob}_\mathfrak{q}) = T_\mathfrak{q} \bmod \mathfrak{M}, \quad \det \rho_\mathfrak{M}(\mathrm{Frob}_\mathfrak{q}) = |\mathfrak{q}| \bmod \mathfrak{M}.$$

The existence of such residual representations for $G_\mathbb{Q}$ is well-known. The corresponding statement over F can be proved along the same lines (cf. [40, Prop. 2.6]); this relies on Drinfeld's fundamental results in [6]. If $\mathfrak{M} \supset \mathfrak{I}(\mathfrak{p})$, then $T_\mathfrak{q} \equiv (1 + |\mathfrak{q}|) \pmod{\mathfrak{M}}$ for all $\mathfrak{q} \in S$. In view of the Chebotarev density and the Brauer-Nesbitt theorems, we conclude that $\rho_\mathfrak{M}$ is the direct sum $\mathbf{1} \oplus \chi_\ell$ of the trivial and cyclotomic characters. But this means that $T_\mathfrak{q} \equiv (1 + |\mathfrak{q}|) \pmod{\mathfrak{M}}$ for all $\mathfrak{q} \neq \mathfrak{p}$, and therefore $\mathfrak{M}$ is Eisenstein. Now it suffices to show that $\mathfrak{I}(\mathfrak{p}) \otimes R \subseteq \mathfrak{E}(\mathfrak{p}) \otimes R$ is an equality in the completion $(\mathbb{T}(\mathfrak{p}) \otimes R)_\mathfrak{M}$ at any maximal Eisenstein $\mathfrak{M}$ of residue characteristic $\neq p$. (Recall that p is invertible in R.)

A consequence of the proof of Theorem 2.20 in [11], the argument in the proof of Theorem 2.26, and the fact that $\mathcal{H}_0(1, R) = 0$ is that $\mathbb{T}(\mathfrak{p})^0 \otimes R = \mathbb{T}(\mathfrak{p}) \otimes R$. On the other hand, by Theorem 5.13 in [39], the ideal $\mathfrak{E}(\mathfrak{p})_\mathfrak{M}$ in $\mathbb{T}(\mathfrak{p})^0_\mathfrak{M}$ is principal, generated by $T_\mathfrak{q} - |\mathfrak{q}| - 1$ for any "good" prime $\mathfrak{q}$. Since the density of these good primes is positive (see [38, p. 186]), there is a good prime in S. Therefore $(\mathfrak{I}(\mathfrak{p}) \otimes R)_\mathfrak{M}$ contains a generator of $(\mathfrak{E}(\mathfrak{p}) \otimes R)_\mathfrak{M}$ and must be equal to it. $\square$

Fix two distinct primes $\mathfrak{p}$ and $\mathfrak{q}$. Set

$$\nu(\mathfrak{p}, \mathfrak{q}) = \begin{cases} 1, & \text{if } \deg(\mathfrak{p}) \text{ or } \deg(\mathfrak{q}) \text{ is even}, \\ q + 1, & \text{otherwise}. \end{cases}$$

Let $E_{(\mathfrak{p},\mathfrak{q})} \in \mathcal{E}(\mathfrak{p}\mathfrak{q}, \mathbb{Z})$ be the Eisenstein series defined by

$$E_{(\mathfrak{p},\mathfrak{q})}\left(\begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix}\right) = \nu(\mathfrak{p}, \mathfrak{q}) \cdot q^{-k+1} \cdot \left[\frac{(1 - |\mathfrak{p}|)(1 - |\mathfrak{q}|)}{1 - q^2} + \sum_{\substack{0 \neq m \in A, \\ \deg(m) \leq k-2}} \sigma'_{\mathfrak{p}\mathfrak{q}}(m) \eta(mu)\right],$$

where

$$\sigma'_\mathfrak{n}(m) := \sum_{\substack{\text{monic } m' \in A,\ (m', \mathfrak{n}) = 1, \\ \text{and } m' | m}} |m'|.$$

It is clear that $E_{(\mathfrak{p},\mathfrak{q})} = E_{(\mathfrak{q},\mathfrak{p})}$, and comparing the Fourier expansions we get

$$\frac{\nu(\mathfrak{p})}{\nu(\mathfrak{p}, \mathfrak{q})} \cdot E_{(\mathfrak{p},\mathfrak{q})} = \left(E_\mathfrak{p} - E_\mathfrak{p} | B_\mathfrak{q}\right). \tag{3.5}$$

Lemma 3.8.

(1) *Viewing* $E_{\mathfrak{p}}$ *as a harmonic cochain in* $\mathcal{H}(\mathfrak{pq},\mathbb{Z})$*, we get*

$$E_{\mathfrak{p}}|U_{\mathfrak{p}} = E_{\mathfrak{p}} = -E_{\mathfrak{p}}|W_{\mathfrak{p}} \quad \text{and} \quad E_{\mathfrak{p}}|W_{\mathfrak{q}} = E_{\mathfrak{p}}|B_{\mathfrak{q}}.$$

(2)

$$E_{(\mathfrak{p},\mathfrak{q})} = E_{(\mathfrak{p},\mathfrak{q})}|U_{\mathfrak{p}} = E_{(\mathfrak{p},\mathfrak{q})}|U_{\mathfrak{q}} = E_{(\mathfrak{p},\mathfrak{q})}|W_{\mathfrak{pq}} = -E_{(\mathfrak{p},\mathfrak{q})}|W_{\mathfrak{p}} = -E_{(\mathfrak{p},\mathfrak{q})}|W_{\mathfrak{q}}.$$

Proof. The proof is straightforward. □

Let $\mathcal{E}'(\mathfrak{pq},R)$ be the R-submodule of $\mathcal{E}(\mathfrak{pq},R)$ spanned by $E_{\mathfrak{p}}$, $E_{\mathfrak{q}}$ and $E_{(\mathfrak{p},\mathfrak{q})}$, i.e., $\mathcal{E}'(\mathfrak{pq},R) = RE_{\mathfrak{p}} + RE_{\mathfrak{q}} + RE_{(\mathfrak{p},\mathfrak{q})}$. Denote

$$\mathcal{E}'_0(\mathfrak{pq},R) = \mathcal{E}'(\mathfrak{pq},R)\cap\mathcal{H}_0(\mathfrak{pq},R) \quad \text{and} \quad \mathcal{E}'_{00}(\mathfrak{pq},R) = \mathcal{E}'(\mathfrak{pq},R)\cap\mathcal{H}_{00}(\mathfrak{pq},R).$$

Theorem 3.9. *The following holds:*

(1) *If* $\nu(\mathfrak{p},\mathfrak{q})$ *is invertible in* R*, then* $\mathcal{E}'(\mathfrak{pq},R)$ *is a free* R*-module of rank* 3.

(2) *If* $q-1$ *and* $s_{\mathfrak{p},\mathfrak{q}} = \gcd(|\mathfrak{p}|+1,|\mathfrak{q}|+1)$ *are both invertible in* R*, then*

$$\mathcal{E}_{00}(\mathfrak{pq},R) = \mathcal{E}'_{00}(\mathfrak{pq},R).$$

Proof. By (3.5), $E_{\mathfrak{p}}|B_{\mathfrak{q}}$ and $E_{\mathfrak{q}}|B_{\mathfrak{p}}$ are both in $\mathcal{E}'(\mathfrak{pq},R)$, and

$$\mathcal{E}'(\mathfrak{pq},R) = R(E_{\mathfrak{p}}|B_{\mathfrak{q}}) + R(E_{\mathfrak{q}}|B_{\mathfrak{p}}) + RE_{(\mathfrak{p},\mathfrak{q})}.$$

Suppose $f = aE_{\mathfrak{p}}|B_{\mathfrak{q}} + bE_{\mathfrak{q}}|B_{\mathfrak{p}} + cE_{(\mathfrak{p},\mathfrak{q})} = 0$. Then $f^*(1) = cE^*_{(\mathfrak{p},\mathfrak{q})}(1) = c\cdot\nu(\mathfrak{p},\mathfrak{q}) = 0$. Since $\nu(\mathfrak{p},\mathfrak{q})$ is invertible in R under our assumption, we get $c = 0$. Without loss of generality we can assume that either $\deg(\mathfrak{p})$ is even, or $\deg(\mathfrak{p})$ and $\deg(\mathfrak{q})$ are both odd. Now $E^*_{\mathfrak{p}}(1) = (E_{\mathfrak{p}}|B_{\mathfrak{q}})^*(\mathfrak{q}) = \nu(\mathfrak{p})$ is invertible in R. Therefore from $f^*(\mathfrak{q}) = a\cdot\nu(\mathfrak{p}) = 0$ we get $a = 0$. From the fact that $E_{\mathfrak{p}}$ is primitive, we have $b = 0$. The proof of Part (1) is complete.

To prove Part (2), it suffices to show that $\mathcal{E}_{00}(\mathfrak{pq},R)$ is contained in $\mathcal{E}'(\mathfrak{pq},R)$. Given an Eisenstein harmonic cochain $f\in\mathcal{E}_{00}(\mathfrak{pq},R)$, let

$$f_1 := f - f^*(1)\nu(\mathfrak{p},\mathfrak{q})^{-1}E_{(\mathfrak{p},\mathfrak{q})} \in \mathcal{E}(\mathfrak{pq},R).$$

Then for every ideal $\mathfrak{m}\lhd A$, $f_1^*(\mathfrak{m}) = 0$ unless $\mathfrak{p}$ or $\mathfrak{q}$ divides $\mathfrak{m}$. By the method of Section 2, $(|\mathfrak{p}|+1)f_1 = \psi_1|B_{\mathfrak{p}} + \psi_2|B_{\mathfrak{q}}$, where $\psi_1\in\mathcal{H}(\mathfrak{q},R)$ and $\psi_2\in\mathcal{H}(\mathfrak{p},R)$. Moreover, by the proof of Theorem 2.26, we can take

$$\psi_1 = f_1|(U_{\mathfrak{p}}+W_{\mathfrak{p}}) \quad \text{and} \quad \psi_2 = |\mathfrak{q}|^{-1}\Big[(|\mathfrak{p}|+1)f_1|U_{\mathfrak{q}} - f_1|U_{\mathfrak{q}}(U_{\mathfrak{p}}+W_{\mathfrak{p}})B_{\mathfrak{p}}\Big].$$

Since Lemma 3.8 (2) implies that $E_{(\mathfrak{p},\mathfrak{q})}|(U_{\mathfrak{p}}+W_{\mathfrak{p}}) = 0$, we get $\psi_1 = f|(U_{\mathfrak{p}}+W_{\mathfrak{p}})\in\mathcal{H}_{00}(\mathfrak{q},R)$ and

$$\begin{aligned}\psi_2 \;=\;& |\mathfrak{q}|^{-1}\Big[(|\mathfrak{p}|+1)f|U_{\mathfrak{q}} - f|U_{\mathfrak{q}}(U_{\mathfrak{p}}+W_{\mathfrak{p}})B_{\mathfrak{p}}\Big]\\ & -|\mathfrak{q}|^{-1}(|\mathfrak{p}|+1)f^*(1)\nu(\mathfrak{p},\mathfrak{q})^{-1}E_{(\mathfrak{p},\mathfrak{q})}\in\mathcal{H}(\mathfrak{q},R).\end{aligned}$$

The constant term $\psi_2^0(1)$ is equal to

$$-|\mathfrak{q}|^{-1}(|\mathfrak{p}|+1)f^*(1)\cdot\frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{q(1-q^2)} = -|\mathfrak{q}|^{-1}(|\mathfrak{p}|+1)f^*(1)\cdot\frac{1-|\mathfrak{q}|}{\nu(\mathfrak{p})}E^0_{\mathfrak{p}}(1)$$

and $(\psi_2|W_\mathfrak{p})^0(1) = -\psi_2^0(1)$. Let

$$\psi_2' := \psi_2 + |\mathfrak{q}|^{-1}(|\mathfrak{p}|+1)f^*(1)\cdot\frac{1-|\mathfrak{q}|}{\nu(\mathfrak{p})}E_\mathfrak{p} \in \mathcal{H}(\mathfrak{p},R).$$

Since $q-1$ is invertible in R by assumption, Lemma 2.7 and 2.19 show that

$$\psi_2' \in \mathcal{H}_0(\mathfrak{p},R) = \mathcal{H}_{00}(\mathfrak{p},R).$$

Note that for every prime $\mathfrak{p}' \lhd A$ different from $\mathfrak{p}$ and $\mathfrak{q}$, we have

$$\psi_1|T_{\mathfrak{p}'} = (|\mathfrak{p}'|+1)\psi_1 \quad \text{and} \quad \psi_2'|T_{\mathfrak{p}'} = (|\mathfrak{p}'|+1)\psi_2'.$$

Lemma 3.7 implies that $\psi_1 \in \mathcal{E}_{00}(\mathfrak{q},R)$ and $\psi_2' \in \mathcal{E}_{00}(\mathfrak{p},R)$. By Theorem 6.6 in [38], we can find $a_1,\ a_2 \in R$ such that $\psi_1 = a_1E_\mathfrak{q}$ and $\psi_2' = a_2E_\mathfrak{p}$ (as 2 is invertible in R by our assumption). We conclude that

$$\begin{aligned}(|\mathfrak{p}|+1)f &= \psi_1|B_\mathfrak{p} + \psi_2|B_\mathfrak{q} + f^*(1)\nu(\mathfrak{p},\mathfrak{q})^{-1}E_{(\mathfrak{p},\mathfrak{q})}\\ &= a_2E_\mathfrak{q}|B_\mathfrak{p} + \Big(a_2 - |\mathfrak{q}|^{-1}(|\mathfrak{p}|+1)f^*(1)\cdot\frac{1-|\mathfrak{q}|}{\nu(\mathfrak{p})}\Big)E_\mathfrak{p}|B_\mathfrak{q}\\ &\qquad + f^*(1)\nu(\mathfrak{p},\mathfrak{q})^{-1}E_{(\mathfrak{p},\mathfrak{q})}\end{aligned}$$

is in $\mathcal{E}'(\mathfrak{pq},R)$. Similarly, we also get $(|\mathfrak{q}|+1)f \in \mathcal{E}'(\mathfrak{pq},R)$ by interchanging $\mathfrak{p}$ and $\mathfrak{q}$. Since $s_{\mathfrak{p},\mathfrak{q}}$ is invertible in R, f must be in $\mathcal{E}'(\mathfrak{pq},R)$, which completes the proof of Part (2). □

Lemma 3.10. *The following holds:*

(1) *Suppose $\nu(\mathfrak{p},\mathfrak{q})$ is invertible in R. The R-module $\mathcal{E}_0'(\mathfrak{pq},R)$ is torsion and isomorphic to the submodule of R^3 consisting of elements (a,b,c) with*

$$\begin{aligned}&\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{q^2-1}\nu(\mathfrak{p})a = 0,\\ &\frac{(|\mathfrak{p}|+1)(|\mathfrak{q}|-1)}{q^2-1}\nu(\mathfrak{q})b = 0,\\ &\frac{1-|\mathfrak{p}|}{1-q^2}\nu(\mathfrak{p})a + \frac{1-|\mathfrak{q}|}{1-q^2}\nu(\mathfrak{q})b + \frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{1-q^2}\nu(\mathfrak{p},\mathfrak{q})c = 0.\end{aligned}$$

(2) *Suppose further that 2 is also invertible in R. Then $\mathcal{E}_0'(\mathfrak{pq},R)$ is isomorphic to*

$$R\left[\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{q^2-1}\nu(\mathfrak{p})\right]\oplus R\left[\frac{(|\mathfrak{p}|+1)(|\mathfrak{q}|-1)}{q^2-1}\nu(\mathfrak{q})\right]\oplus R\left[\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|-1)}{q^2-1}\right].$$

Proof. Let $f = aE_\mathfrak{p} + bE_\mathfrak{q} + cE_{(\mathfrak{p},\mathfrak{q})}$ with $a,\ b,\ c \in R$. By Lemma 2.19, $f \in \mathcal{E}_0'(\mathfrak{pq},R)$ if and only if

$$f^0(1) = (f|W_\mathfrak{p})^0(1) = (f|W_\mathfrak{q})^0(1) = (f|W_{\mathfrak{pq}})^0(1) = 0.$$

This gives us the following equations:

$$\frac{1-|\mathfrak{p}|}{1-q^2}\nu(\mathfrak{p})a+\frac{1-|\mathfrak{q}|}{1-q^2}\nu(\mathfrak{q})b+\frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{1-q^2}\nu(\mathfrak{p},\mathfrak{q})c = 0, \tag{3.6}$$

$$-\frac{1-|\mathfrak{p}|}{1-q^2}\nu(\mathfrak{p})a+|\mathfrak{p}|\frac{1-|\mathfrak{q}|}{1-q^2}\nu(\mathfrak{q})b-\frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{1-q^2}\nu(\mathfrak{p},\mathfrak{q})c = 0, \tag{3.7}$$

$$|\mathfrak{q}|\frac{1-|\mathfrak{p}|}{1-q^2}\nu(\mathfrak{p})a-\frac{1-|\mathfrak{q}|}{1-q^2}\nu(\mathfrak{q})b-\frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{1-q^2}\nu(\mathfrak{p},\mathfrak{q})c = 0, \tag{3.8}$$

$$-|\mathfrak{q}|\frac{1-|\mathfrak{p}|}{1-q^2}\nu(\mathfrak{p})a-|\mathfrak{p}|\frac{1-|\mathfrak{q}|}{1-q^2}\nu(\mathfrak{q})b+\frac{(1-|\mathfrak{p}|)(1-|\mathfrak{q}|)}{1-q^2}\nu(\mathfrak{p},\mathfrak{q})c = 0. \tag{3.9}$$

We remark that

$$\text{Equation } (3.9) = -\Big(\text{ Equation } (3.6)+(3.7)+(3.8)\Big).$$

Considering the equation (3.6)+(3.7), (3.6)+(3.8), and (3.6)+(3.9), we get

$$\frac{(|\mathfrak{p}|+1)(|\mathfrak{q}|-1)}{q^2-1}\nu(\mathfrak{q})b = 0, \tag{3.10}$$

$$\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{q^2-1}\nu(\mathfrak{p})a = 0, \tag{3.11}$$

$$\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|-1)}{q^2-1}\big(\nu(\mathfrak{p})a+\nu(\mathfrak{q})b+2\nu(\mathfrak{p},\mathfrak{q})c\big) = 0. \tag{3.12}$$

Since the conditions of (a,b,c) described by Equation (3.6)$\sim$(3.9) is equivalent to those described by Equation (3.6), (3.10), and (3.11) combined, the proof of Part (1) is complete.

To prove Part (2), note that $2\nu(\mathfrak{p},\mathfrak{q})$ is invertible in R by assumption. Therefore the conditions of (a,b,c) described by Equation (3.6)$\sim$(3.9) is equivalent to those described by Equation (3.10)$\sim$(3.12). Let $E' := \nu(\mathfrak{p})E_{\mathfrak{p}}+\nu(\mathfrak{q})E_{\mathfrak{q}}+2\nu(\mathfrak{p},\mathfrak{q})E_{(\mathfrak{p},\mathfrak{q})}$. By Theorem 3.9 (1), we also have $\mathcal{E}'(\mathfrak{p}\mathfrak{q},R)=RE_{\mathfrak{p}}\oplus RE_{\mathfrak{q}}\oplus RE'$. Therefore Equation (3.10)$\sim$(3.12) assures the result. □

In fact, when $q-1$ and $s_{\mathfrak{p},\mathfrak{q}}$ are invertible in the coefficient ring R, one can show that $\mathcal{E}_{00}(\mathfrak{p}\mathfrak{q},R)=\mathcal{E}'_0(\mathfrak{p}\mathfrak{q},R)$; see Remark 7.4.

3.3. Special case. In this subsection we give a concrete description of $\mathcal{E}_0(xy,R)$ and $\mathcal{E}_{00}(xy,R)$ for an arbitrary coefficient ring R. Recall that E_0 is the function on $E(\mathscr{T})$ satisfying

$$E_0\left(\begin{pmatrix}\pi_\infty^k & u\\ 0 & 1\end{pmatrix}\right) = -E_0\left(\begin{pmatrix}\pi_\infty^k & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ \pi_\infty & 0\end{pmatrix}\right) = q^{-k}.$$

Lemma 3.11. *Given $a\in R[q+1]$, we have $aE_0=-aE_{\mathfrak{n}}\in\mathcal{H}(1,R)$ if $\deg(\mathfrak{n})$ is odd. Moreover, $aE_0|B_{\mathfrak{m}}=(-1)^{\deg(\mathfrak{m})}aE_0$.*

Proof. The proof is straightforward. □

According to Examples 3.4 and 3.5 we have:

Lemma 3.12. *$\mathcal{H}(x,R)=RE_x$ and $\mathcal{H}(y,R)=RE_y\oplus R[q+1]E_0$.*

Note that $\nu(x,y) = \nu(y) = 1$ and $\nu(x) = q+1$. Given $f \in \mathcal{E}(xy,R)$, let $f' := f - f^*(1)E_{(x,y)}$. Then $f'^*(\mathfrak{m}) = 0$ unless x or y divides $\mathfrak{m}$. By Theorem 2.26 and Lemma 3.12, there exists $a,\ b \in R$ and $b' \in R[q+1]$ such that

$$2f' = (bE_y + b'E_0)|B_x + aE_x|B_y.$$

When q is even, 2 is invertible in R and we have $f \in \mathcal{E}'(xy,R)$. Suppose q is odd. Note that $b'E_0|B_x = b'E_x$ by Lemma 3.11. Hence

$$2f = (a+b')E_x|B_y + bE_y|B_x + 2f^*(1)E_{(x,y)} \in \mathcal{E}'(xy,R).$$

In fact, there exists $a'',\ b'' \in R$ such that $a+b' = 2a''$ and $b = 2b''$. Indeed, by Lemma 2.24 we have

$$2f^*(x) = b+2f^*(1) \text{ and } 2f^0(\pi_\infty) = 2q^{-1}f^0(1) = |y|(a+b')+|x|b+2f^*(1)(q-1).$$

Set $f'' := f - \big(a''E_x|B_y + b''E_y|B_x + f^*(1)E_{(x,y)}\big) \in \mathcal{E}(xy,R)$. Then the above discussion shows that $2f'' = 0$. Take $f''' := f'' - (f'')^0(1)E_0$. Then

$$f''' \in \mathcal{E}(xy,R)[2] \quad \text{ and } \quad (f''')^*(1) = (f''')^0(1) = 0.$$

The following lemma shows that $f''' \in \mathcal{E}'(xy,R)$, which also implies that $f \in \mathcal{E}'(xy,R)$.

LEMMA 3.13. *Suppose q is odd. Given $\varphi \in \mathcal{E}(xy,R)[2]$ with $\varphi^*(1) = \varphi^0(1) = 0$, there exists $\alpha \in R[2]$ such that*

$$\varphi = \alpha(E_x + E_y|B_x).$$

Proof. We use the notation in the proof of Proposition 2.21. Given $\varphi \in \mathcal{E}(xy,R)[2]$ with $\varphi^0(1) = 0$, we have that for each prime $\mathfrak{p}$ with $\mathfrak{p} \nmid xy$,

$$\varphi|T_\mathfrak{p} = (|\mathfrak{p}|+1)\varphi = 0.$$

Therefore for every prime $\mathfrak{p}$ with $\mathfrak{p} \nmid xy$ and a non-zero ideal $\mathfrak{m} \lhd A$,

$$|\mathfrak{p}|\varphi^*(\mathfrak{p}\mathfrak{m}) + \varphi^*(\mathfrak{m}/\mathfrak{p}) = (\varphi|T_\mathfrak{p})^*(\mathfrak{m}) = 0. \tag{3.13}$$

By Equation (3.13) and the Fourier expansion of φ, we get

$$\varphi\left(\begin{pmatrix} \varpi_x^k & 0 \\ 0 & 1 \end{pmatrix}\right) = 0, \quad \forall k \in \mathbb{Z},$$

$$\varphi(a_1) = \varphi(a_2) = 0,$$

$$\varphi(b_u) = \varphi(a_4) = \varphi^*(x), \quad \forall u \in \mathbb{F}_q^\times,$$

$$\varphi(c_3) = \varphi^*(y), \quad \varphi(a_3) = \varphi^*(x) + \varphi^*(x^2), \quad \varphi(a_6) = \varphi^*(x^2).$$

The harmonicity of φ gives us that

$$0 = \varphi(a_4) + \sum_{u \in \mathbb{F}_q^\times} \varphi(b_u) - \varphi(a_3) = \varphi^*(x^2),$$

$$0 = \varphi(a_6) + \varphi(c_3) + (1-q)\varphi(a_2) = \varphi^*(x^2) + \varphi^*(y).$$
$$0 = \varphi(a_6) + (q-1)\varphi(a_3) - \varphi(c_4) = \varphi^*(x^2) + \varphi(c_4).$$

Hence

$$\varphi(e) = 0 \quad \text{ for } e = c_1, c_2, c_3, c_4, a_1, a_2, a_5, a_6, \text{ and}$$

$$\varphi(a_3) = \varphi(a_4) = \varphi(b_u) = \varphi^*(x) \in R[2], \quad \text{for } u \in \mathbb{F}_q^\times.$$

On the other hand, for $\alpha \in R[2]$,

$$\alpha(E_x + E_y|B_x)(e) = 0 \quad \text{for } e = c_1, c_2, c_3, c_4, a_1, a_2, a_5, a_6, \text{ and}$$

$\alpha(E_x+E_y|B_x)(a_3) = \alpha(E_x+E_y|B_x)(a_4) = \alpha(E_x+E_y|B_x)(b_u) = \alpha,$ for $u \in \mathbb{F}_q^\times$.
Therefore $\varphi = \varphi(a_3) \cdot (E_x + E_y|B_x)$ and the proof is complete. □

From the above discussion, we conclude that

Corollary 3.14. *$\mathcal{E}(xy, R) = \mathcal{E}'(xy, R)$ for every coefficient ring R. In other words, every Eisenstein harmonic cochain of level xy can be generated by Eisenstein series.*

By Lemma 3.10 and Corollary 3.14, we immediately get

Corollary 3.15. *The space $\mathcal{E}_0(xy, R)$ is isomorphic to the torsion R-module*

$$\{(a, b, c) \in R^3 \mid (q^2 + 1)a = (q + 1)b = a + b + (1 - q)c = 0\}.$$

In particular,

$$\mathcal{E}_0(xy, R) \cong R\Big[\frac{(q-1)}{2}(q^2+1)(q+1)\Big] \oplus R[2].$$

From the graph in Figure 4, an alternating R-valued function f on $E(\Gamma_0(xy)\backslash\mathscr{T})$ is in $\mathcal{H}_0(xy, R)$ if and only if f vanishes on the cusps c_1, c_2, c_3, c_4 and

$$\begin{aligned} &(q-1)f(a_1) + f(a_5) = 0, \quad (q-1)f(a_2) - f(a_6) = 0, \\ &(q-1)f(a_3) + f(a_6) = 0, \quad (q-1)f(a_4) - f(a_5) = 0, \\ &f(a_2) + \sum_{u \in \mathbb{F}_q^\times} f(b_u) = f(a_1), \quad f(a_3) - \sum_{u \in \mathbb{F}_q^\times} f(b_u) = f(a_4). \end{aligned}$$

Moreover, f is in $\mathcal{H}_{00}(xy, R)$ if and only if f satisfies an extra equation:

$$f(a_1) + f(a_4) = 0.$$

In particular, every harmonic cochain $f \in \mathcal{H}_{00}(xy, R)$ is determined uniquely by the values

$$f(a_1),\ f(b_u) \ \text{ for } u \in \mathbb{F}_q^\times.$$

Let $f = aE_x + bE_y + cE_{(x,y)} \in \mathcal{E}_0(xy, R)$. By Corollary 3.15 we have

$$(q^2 + 1)a = (q + 1)b = a + b + (1 - q)c = 0.$$

It is observed that

$$\begin{aligned} &E_x(a_1) = -1, && E_x(a_4) = -q, \\ &E_y(a_1) = 0, && E_y(a_4) = -1, \\ &E_{(x,y)}(a_1) = -1, && E_{(x,y)}(a_4) = -1. \end{aligned}$$

We then get

$$f(a_1) + f(a_4) = -\big((q + 1)a + b + 2c\big).$$

Hence $f \in \mathcal{E}_{00}(xy, R)$ if and only if

$$c \in R[(q^2+1)(q+1)],\ a = -q^{-1}(q+1)c,\ b = q^{-1}(q^2+1)c.$$

We conclude that

Proposition 3.16. *The module $\mathcal{E}_{00}(xy, R)$ is isomorphic to $R[(q^2+1)(q+1)]$. More precisely, every harmonic cochain in $\mathcal{E}_{00}(xy, R)$ must be of the form*

$$c \cdot \Big(-(q+1)E_x + (q^2+1)E_y + qE_{(x,y)} \Big)$$

where $c \in R[(q^2+1)(q+1)]$.

Corollary 3.17. *For every natural number n relatively prime to p the module $\mathcal{E}_{00}(xy, \mathbb{Z}/n\mathbb{Z})$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}[(q^2+1)(q+1)]$.*

Proof. The difference between this claim and Proposition 3.16 is that $\mathbb{Z}/n\mathbb{Z}$ is not a coefficient ring in general. Still, one can deduce this from Proposition 3.16 by arguing as in the proof of Corollary 6.9 in [38]. First, one easily reduces to the case when n is a power of some prime $\ell \neq p$, and applies Proposition 3.16 with $R = \mathbb{Z}_\ell[\zeta_p]/n\mathbb{Z}_\ell[\zeta_p]$, where ζ_p is the primitive pth root of unity. The claim follows by observing that R is a free $\mathbb{Z}/n\mathbb{Z}$-module. □

Corollary 3.18. $\mathbb{T}(xy)/\mathfrak{E}(xy) \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$.

Proof. Let $(\mathbb{T}(\mathfrak{n})^0)^{\mathrm{new}}$ be the quotient of $\mathbb{T}(\mathfrak{n})^0$ with which it acts on $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})^{\mathrm{new}}$; cf. Definition 2.13. By Lemma 4.2 and Theorem 4.5 in [39], for any *square-free* $\mathfrak{n} \lhd A$ the quotient ring $(\mathbb{T}(\mathfrak{n})^0)^{\mathrm{new}}/\mathfrak{E}(\mathfrak{n})$ is a finite cyclic group of order coprime to p; here with abuse of notation $\mathfrak{E}(\mathfrak{n})$ denotes the ideal generated by the images of $T_\mathfrak{p} - |\mathfrak{p}| - 1$ in $(\mathbb{T}(\mathfrak{n})^0)^{\mathrm{new}}$. Since $\mathcal{H}_0(xy, \mathbb{Q}) = \mathcal{H}_0(xy, \mathbb{Q})^{\mathrm{new}}$, we have $\mathbb{T}(xy)^0 = (\mathbb{T}(xy)^0)^{\mathrm{new}}$. On the other hand, by Proposition 2.21, $\mathbb{T}(xy)^0 = \mathbb{T}(xy)$. Hence $\mathbb{T}(xy)/\mathfrak{E}(xy) \cong \mathbb{Z}/n\mathbb{Z}$ for some n coprime to p.
The perfectness of the pairing (2.5) implies

$$\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{T}(xy) \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathcal{H}_{00}(xy, \mathbb{Z}/n\mathbb{Z}).$$

Hence

$$\begin{aligned}\mathcal{E}_{00}(xy, \mathbb{Z}/n\mathbb{Z}) &\cong \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{T}(xy) \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})[\mathfrak{E}(xy)] \\ &\cong \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}((\mathbb{T}(xy)/\mathfrak{E}(xy)) \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.\end{aligned}$$

Applying Corollary 3.17, we conclude that n must divide $(q+1)(q^2+1)$. Later in this paper we will prove that the component group $\Phi_\infty \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$ of $J_0(xy)$ is annihilated by $\mathfrak{E}(xy)$ (see Lemma 8.16). This implies that n is divisible by $(q^2+1)(q+1)$. Therefore, $n = (q^2+1)(q+1)$. □

Remark 3.19. In [44], we extended our calculation of $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n})$ to arbitrary $\mathfrak{n}$ of degree 3. Up to an affine transformation $T \mapsto aT + b$ with $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$, there are 5 different cases, namely

(1) If $\mathfrak{n} = T^3$, then $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n}) \cong \mathbb{Z}/q^2\mathbb{Z}$;
(2) If $\mathfrak{n} = T^2(T-1)$, then $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n}) \cong \mathbb{Z}/q(q^2-1)\mathbb{Z}$;

(3) If $\mathfrak{n}$ is irreducible, then $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n}) \cong \mathbb{Z}/(q^2+q+1)\mathbb{Z}$;
(4) If $\mathfrak{n} = xy$, then $\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n}) \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$;
(5) If $\mathfrak{n} = T(T-1)(T-c)$, where $c \in \mathbb{F}_q$, $c \neq 0, 1$ (here we must have $q > 2$), then

$$\mathbb{T}(\mathfrak{n})/\mathfrak{E}(\mathfrak{n}) \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q-1)^2(q+1)\mathbb{Z}.$$

4. Drinfeld modules and modular curves

In this section we collect some facts about Drinfeld modules and their moduli schemes that will be used later in the paper.

Let S be an A-scheme and $\mathcal{L}$ a line bundle over S. Let $\mathcal{L}\{\tau\}$ be the noncommutative ring $\oplus_{i\geq 0}\mathcal{L}^{\otimes(1-q^i)}(S)\tau^i$, where τ stands for the qth power Frobenius mapping. The multiplication in this ring is given by $\alpha_i\tau^i \cdot \alpha_j\tau^j = (\alpha_i \otimes \alpha_j^{\otimes q^j})\tau^{i+j}$. A *Drinfeld A-module of rank r (in standard form)* over S is given by a line bundle $\mathcal{L}$ over S together with a ring homomorphism $\phi^{\mathcal{L}} : A \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{L}) = \mathcal{L}\{\tau\}$, $a \mapsto \phi^{\mathcal{L}}_a$, such that $\phi^{\mathcal{L}}_a = \sum_{i=0}^{m(a)} \alpha_i(a)\tau^i$, where $m(a) = -r \cdot \mathrm{ord}_\infty(a)$, $\alpha_{m(a)}(a)$ is a nowhere vanishing section of $\mathcal{L}^{\otimes(1-m(a))}$, and α_0 coincides with the map $\partial : A \to H^0(S, \mathcal{O}_S)$ giving the structure of an A-scheme to S; cf. [6, p. 575]. The kernel of ∂ is called the *A-characteristic* of S. A Drinfeld A-module over S is clearly an A-module scheme over S, and a *homomorphism* of Drinfeld modules is a homomorphism of these A-module schemes. A homomorphism of Drinfeld modules over a connected scheme is either the zero homomorphism, or it has finite kernel, in which case it is usually called an *isogeny*. When S is the spectrum of a field K, we will omit mention of $\mathcal{L}$ and write $\phi : A \to K\{\tau\}$.

Let $\mathfrak{n} \lhd A$ be a non-zero ideal. A *cyclic subgroup of order* $\mathfrak{n}$ of $\phi^{\mathcal{L}}$ is an A-submodule scheme $C_\mathfrak{n}$ of $\mathcal{L}$ which is finite and flat over S, and such that there is a homomorphism of A-modules $\iota : A/\mathfrak{n} \to \mathcal{L}(S)$ giving an equality of relative effective Cartier divisors $\sum_{a\in A/\mathfrak{n}} \iota(a) = C_\mathfrak{n}$. Denote $\phi^{\mathcal{L}}[\mathfrak{n}] = \ker(\mathcal{L} \xrightarrow{\phi^{\mathcal{L}}_n} \mathcal{L})$, where n is a generator of the ideal $\mathfrak{n}$. It is clear that the A-submodule scheme $\phi^{\mathcal{L}}[\mathfrak{n}]$ of $\mathcal{L}$ does not depend on the choice of n, and $C_\mathfrak{n} \subset \phi^{\mathcal{L}}[\mathfrak{n}]$. To each $C_\mathfrak{n} \subset \phi$ one can associate a unique, up to isomorphism, Drinfeld module $\phi' := \phi/C_\mathfrak{n}$ such that there is an isogeny $\phi \to \phi'$ whose kernel is $C_\mathfrak{n}$.

Now assume $S = \mathrm{Spec}(K)$, where K is a field. Explicitly, a homomorphism of Drinfeld modules $u : \phi \to \psi$ is $u \in K\{\tau\}$ such that $\phi_a u = u\psi_a$ for all $a \in A$, and u is an isomorphism if $u \in K^\times$. Let $\mathrm{End}(\phi)$ denote the centralizer of $\phi(A)$ in $\bar{K}\{\tau\}$, i.e., the ring of all homomorphisms $\phi \to \phi$ over $\bar{K}$. The automorphism group $\mathrm{Aut}(\phi)$ is the group of units $\mathrm{End}(\phi)^\times$. It is known that $\mathrm{End}(\phi)$ is a free A-module of rank $\leq r^2$; cf. [6]. From now on we also assume that $r = 2$. Note that ϕ is uniquely determined by the image of T:

$$\phi_T = \partial(T) + g\tau + \Delta\tau^2,$$

where $g \in K$ and $\Delta \in K^\times$. The *j-invariant* of ϕ is $j(\phi) = g^{q+1}/\Delta$. It is easy to check that if K is algebraically closed, then $\phi \cong \psi$ if and only if $j(\phi) = j(\psi)$.

It is also easy to check that

$$\mathrm{Aut}(\phi) = \begin{cases} \mathbb{F}_q^\times & \text{if } j(\phi) \neq 0; \\ \mathbb{F}_{q^2}^\times & \text{if } j(\phi) = 0. \end{cases} \tag{4.1}$$

If $\mathfrak{n}$ is coprime to the A-characteristic of K, then $\phi[\mathfrak{n}](\bar{K}) \cong (A/\mathfrak{n})^2$. On the other hand, if $\mathfrak{p} = \ker(\partial) \neq 0$, then $\phi[\mathfrak{p}](\bar{K}) \cong (A/\mathfrak{p})$ or 0; when $\phi[\mathfrak{p}](\bar{K}) = 0$, ϕ is called *supersingular*. The following is Theorem 5.9 in [8]:

Theorem 4.1. *Let $\mathfrak{p} \lhd A$ be a prime ideal. The number of isomorphism classes of supersingular rank-2 Drinfeld A-modules over $\overline{\mathbb{F}}_\mathfrak{p}$ is*

$$\begin{cases} \frac{|\mathfrak{p}|-1}{q^2-1} & \textit{if } \deg(\mathfrak{p}) \textit{ is even}; \\ \frac{|\mathfrak{p}|-q}{q^2-1} + 1 & \textit{if } \deg(\mathfrak{p}) \textit{ is odd}. \end{cases}$$

The Drinfeld module with $j(\phi) = 0$ is supersingular if and only if $\deg(\mathfrak{p})$ is odd.

The functor from the category of A-schemes to the category of sets, which associates to an A-scheme S the set of isomorphism classes of pairs $(\phi^{\mathcal{L}}, C_\mathfrak{n})$, where $\phi^{\mathcal{L}}$ is a Drinfeld module of rank 2 and $C_\mathfrak{n}$ is a cyclic subgroup of order $\mathfrak{n}$ has a coarse moduli scheme $Y_0(\mathfrak{n})$. The scheme $Y_0(\mathfrak{n})$ is affine, finite type, of relative dimension 1 over $\mathrm{Spec}(A)$, and smooth over $\mathrm{Spec}(A[\mathfrak{n}^{-1}])$. This is well-known and can be deduced from the results in [6]. The rigid-analytic uniformization of $Y_0(\mathfrak{n})$ over F_∞ is given by (3.1). The scheme $Y_0(\mathfrak{n})$ has a canonical compactification over $\mathrm{Spec}(A)$:

Theorem 4.2. *There is a proper normal geometrically irreducible scheme $X_0(\mathfrak{n})$ of pure relative dimension 1 over $\mathrm{Spec}(A)$ which contains $Y_0(\mathfrak{n})$ as an open dense subscheme. The complement $X_0(\mathfrak{n}) - Y_0(\mathfrak{n})$ is a disjoint union of irreducible schemes. Finally, $X_0(\mathfrak{n})$ is smooth over $\mathrm{Spec}(A[\mathfrak{n}^{-1}])$.*

Proof. See [6, §9] and [29, Prop. V.3.5]. □

Denote the Jacobian variety of $X_0(\mathfrak{n})_F$ by $J_0(\mathfrak{n})$. Let $\mathfrak{p} \lhd A$ be prime. There are two natural degeneracy morphisms $\alpha, \beta : Y_0(\mathfrak{np}) \to Y_0(\mathfrak{n})$ with moduli-theoretic interpretation:

$$\alpha : (\phi, C_{\mathfrak{np}}) \mapsto (\phi, C_\mathfrak{n}), \qquad \beta : (\phi, C_{\mathfrak{np}}) \mapsto (\phi/C_\mathfrak{p}, C_{\mathfrak{np}}/C_\mathfrak{p}),$$

where $C_\mathfrak{n}$ and $C_\mathfrak{p}$ are the subgroups of $C_{\mathfrak{np}}$ of order $\mathfrak{n}$ and $\mathfrak{p}$, respectively. These morphisms are proper, and hence uniquely extend to morphisms $\alpha, \beta : X_0(\mathfrak{np}) \to X_0(\mathfrak{n})$. By Picard functoriality, α and β induce two homomorphisms $\alpha_*, \beta_* : J_0(\mathfrak{n}) \to J_0(\mathfrak{pn})$. The *Hecke endomorphism* of $J_0(\mathfrak{n})$ is $T_\mathfrak{p} := \alpha^* \circ \beta_*$, where $\alpha^* : J_0(\mathfrak{pn}) \to J_0(\mathfrak{n})$ is the dual of α_*. The $\mathbb{Z}$-subalgebra of $\mathrm{End}(J_0(\mathfrak{n}))$ generated by all Hecke endomorphisms is canonically isomorphic to $\mathbb{T}(\mathfrak{n})$. This is a consequence of Drinfeld's reciprocity law [6, Thm. 2].

The Jacobian $J_0(\mathfrak{n})$ has a rigid-analytic uniformization over F_∞ as a quotient of a multiplicative torus by a discrete lattice. To simplify the notation denote $\Gamma := \Gamma_0(\mathfrak{n})$ and let $\overline{\Gamma}$ be the maximal torsion-free abelian quotient of Γ. In [18] and [13], Gekeler and Reversat associate a meromorphic theta function

$\theta(\omega,\eta,\cdot)$ on Ω with each pair $\omega,\eta\in\overline{\Omega}:=\Omega\cup\mathbb{P}^1(F)$. The theta function $\theta(\omega,\eta,\cdot)$ satisfies a functional equation

$$\theta(\omega,\eta,\gamma z)=c(\omega,\eta,\gamma)\theta(\omega,\eta,z),\qquad \forall\gamma\in\Gamma,$$

where $c(\omega,\eta,\cdot):\Gamma\to\mathbb{C}_\infty^\times$ is a homomorphism that factors through $\overline{\Gamma}$. The divisor of $\theta(\omega,\eta,\cdot)$ is Γ-invariant and, as a divisor on $X_0(\mathfrak{n})(\mathbb{C}_\infty)$, equals $[\omega]-[\eta]$, where $[\omega]$ is the class of $\omega\in\overline{\Omega}$ in $\Gamma\setminus\overline{\Omega}=X_0(\mathfrak{n})(\mathbb{C}_\infty)$.
For a fixed $\alpha\in\Gamma$, the function $u_\alpha(z)=\theta(\omega,\alpha\omega,z)$ is holomorphic and invertible on Ω. Moreover, u_α is independent of the choice of $\omega\in\overline{\Omega}$, and depends only on the class $\bar{\alpha}$ of α in $\overline{\Gamma}$. Let $c_\alpha(\cdot)=c(\omega,\alpha\omega,\cdot)$ be the multiplier of u_α. It induces a pairing

$$\begin{aligned}\overline{\Gamma}\times\overline{\Gamma}&\to F_\infty^\times\\ (\alpha,\beta)&\mapsto c_\alpha(\beta)\end{aligned}$$

which is bilinear, symmetric, and

$$\begin{aligned}\langle\cdot,\cdot\rangle:\overline{\Gamma}\times\overline{\Gamma}&\to\mathbb{Z}\\ \langle\alpha,\beta\rangle&=\mathrm{ord}_\infty(c_\alpha(\beta))\end{aligned}\tag{4.2}$$

is positive definite. One of the main result of [18] is that there is an exact sequence

$$0\to\overline{\Gamma}\xrightarrow{\alpha\mapsto c_\alpha(\cdot)}\mathrm{Hom}(\overline{\Gamma},\mathbb{C}_\infty^\times)\to J_0(\mathfrak{n})(\mathbb{C}_\infty)\to 0.\tag{4.3}$$

One can define Hecke operators $T_\mathfrak{p}$ as endomorphisms of $\overline{\Gamma}$ in purely group-theoretical terms as some sort of Verlagerung (see [18, (9.3)]). These operators then also act on the torus $\mathrm{Hom}(\overline{\Gamma},\mathbb{C}_\infty^\times)$ through their action on the first argument $\overline{\Gamma}$. By [18, (3.3.3)] and [17], there is a canonical isomorphism

$$j:\overline{\Gamma}\xrightarrow{\sim}\mathcal{H}_0(\mathfrak{n},\mathbb{Z})\tag{4.4}$$

which is compatible with the action of Hecke operators. Through this construction, the Hecke algebra $\mathbb{T}(\mathfrak{n})$ in Definition 2.11 acts faithfully on $\overline{\Gamma}$ and $\mathrm{Hom}(\overline{\Gamma},\mathbb{C}_\infty^\times)$. The sequence (4.3) is compatible with the action of $\mathbb{T}(\mathfrak{n})^0$ on its three terms; see [18, (9.4)].
Assume $\mathfrak{n}$ is square-free. The matrix (2.3) representing the Atkin-Lehner involution $W_\mathfrak{m}$ for $\mathfrak{m}|\mathfrak{n}$ is in the normalizer of Γ in $\mathrm{GL}_2(F_\infty)$, and the induced involution of $X_0(\mathfrak{n})_{F_\infty}$ does not depend on the choice of this matrix. In terms on the moduli problem, the involution $W_\mathfrak{m}$ on $X_0(\mathfrak{n})$ is given by

$$W_\mathfrak{m}:(\phi,C_\mathfrak{n})\mapsto(\phi/C_\mathfrak{m},(\phi[\mathfrak{m}]+C_{\mathfrak{n}/\mathfrak{m}})/C_\mathfrak{m}),$$

where $C_\mathfrak{m}$ and $C_{\mathfrak{n}/\mathfrak{m}}$ are the subgroups of $C_\mathfrak{n}$ of order $\mathfrak{m}$ and $\mathfrak{n}/\mathfrak{m}$, respectively.

5. Component groups

Let $\mathfrak{n}\lhd A$ be a non-zero ideal. Let $J:=J_0(\mathfrak{n})$ and $\mathcal{J}$ denote the Néron model of J over $\mathbb{P}^1_{\mathbb{F}_q}$. Let $\mathcal{J}^0$ denote the relative connected component of the identity of $\mathcal{J}$, that is, the largest open subscheme of $\mathcal{J}$ containing the identity section which has connected fibres. The *group of connected components* (or *component* *group*) of J at a place v of F is $\Phi_v := \mathcal{J}_{\mathbb{F}_v}/\mathcal{J}^0_{\mathbb{F}_v}$. This is a finite abelian group equipped with an action of the absolute Galois group $G_{\mathbb{F}_v}$. The homomorphism $\wp_v : J(F_v^{\mathrm{nr}}) \to \Phi_v$ obtained from the composition

$$\wp_v : J(F_v^{\mathrm{nr}}) = \mathcal{J}(\mathcal{O}_v^{\mathrm{nr}}) \to \mathcal{J}_{\mathbb{F}_v}(\overline{\mathbb{F}}_v) \to \Phi_v$$

will be called the *canonical specialization map.*

Assume $\mathfrak{p}\lhd A$ is a prime not dividing $\mathfrak{m}$. Then the curve $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ is smooth. We call a point $P \in X_0(\mathfrak{m})(\overline{\mathbb{F}}_\mathfrak{p})$ *supersingular* if it corresponds to the isomorphism class of a pair $(\phi, C_\mathfrak{m})$ with ϕ supersingular over $\overline{\mathbb{F}}_\mathfrak{p}$. For a Drinfeld A-module ϕ over $\overline{\mathbb{F}}_\mathfrak{p}$ given by $\phi_T = \partial(T) + g\tau + \Delta\tau^2$, let $\phi^{(\mathfrak{p})} : A \to \overline{\mathbb{F}}_\mathfrak{p}\{\tau\}$ be the Drinfeld module given by $\phi_T^{(\mathfrak{p})} = \partial(T) + g^{|\mathfrak{p}|}\tau + \Delta^{|\mathfrak{p}|}\tau^2$. Since $\partial(A) \subseteq \mathbb{F}_\mathfrak{p}$, we see that $\tau^{|\mathfrak{p}|}\phi_a = \phi_a^{(\mathfrak{p})}\tau^{|\mathfrak{p}|}$ for all $a \in A$, so $\tau^{|\mathfrak{p}|}$ is an isogeny $\phi \to \phi^{(\mathfrak{p})}$. Denote the image of $C_\mathfrak{m}$ in $\phi^{(\mathfrak{p})}$ under $\tau^{|\mathfrak{p}|}$ by $C_\mathfrak{m}^{(\mathfrak{p})}$. The map from $X_0(\mathfrak{m})(\overline{\mathbb{F}}_\mathfrak{p})$ to itself given by $(\phi, C_\mathfrak{m}) \mapsto (\phi^{(\mathfrak{p})}, C_\mathfrak{m}^{(\mathfrak{p})})$ restricts to an involution on the finite set of supersingular points; cf. [8, Thm. 5.3].

Theorem 5.1. *Assume $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, with $\mathfrak{p}$ prime not dividing $\mathfrak{m}$. The curve $X_0(\mathfrak{n})_{F_\mathfrak{p}}$ is smooth and extends to a proper flat scheme $X_0(\mathfrak{n})_{\mathcal{O}_\mathfrak{p}}$ over $\mathcal{O}_\mathfrak{p}$ such that the special fibre $X_0(\mathfrak{n})_{\mathbb{F}_\mathfrak{p}}$ is geometrically reduced and consists of two irreducible components, both isomorphic to $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$, intersecting transversally at the supersingular points. More precisely, a supersingular point $(\phi, C_\mathfrak{m})$ on the first copy of $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ is glued to $(\phi^{(\mathfrak{p})}, C_\mathfrak{m}^{(\mathfrak{p})})$ on the second copy. The curve $X_0(\mathfrak{n})_{\mathbb{F}_\mathfrak{p}}$ is smooth outside of the locus of supersingular points. Denote by $\mathrm{Aut}(\phi, C_\mathfrak{m})$ the subgroup of automorphisms of ϕ which map $C_\mathfrak{m}$ to itself. For a supersingular point $P \in X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ corresponding to $(\phi, C_\mathfrak{m})$, let $m(P) := \frac{1}{q-1}\#\mathrm{Aut}(\phi, C_\mathfrak{m})$. Then, locally at P for the étale topology, $X_0(\mathfrak{n})_{\mathcal{O}_\mathfrak{p}}$ is given by the equation $XY = \mathfrak{p}^{m(P)}$.*

Proof. This is proven in [10, §5] for $\mathfrak{n} = \mathfrak{p}$, but the proof easily extends to this more general case. □

We will compute the component group $\Phi_\mathfrak{p}$ using a classical theorem of Raynaud, but first we need to determine the number of singular points on $X_0(\mathfrak{n})_{\mathbb{F}_\mathfrak{p}}$, and the integers $m(P) \geq 1$ defined in Theorem 5.1. We call $m(P)$ the *thickness* of P.

Let $\mathfrak{m} = \prod_{1\leq i\leq s} \mathfrak{p}_i^{r_i}$ be the prime decomposition of $\mathfrak{m}$. Define

$$R(\mathfrak{m}) = \begin{cases} 1 & \text{if } \deg(\mathfrak{p}_i) \text{ is even for all } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases}$$

$$L(\mathfrak{m}) = \#\mathbb{P}^1(A/\mathfrak{m}) = \prod_{1\leq i\leq s} |\mathfrak{p}_i|^{r_i-1}(|\mathfrak{p}_i| + 1).$$

If $\mathfrak{m} = A$, we put $s = 0$ and $L(\mathfrak{m}) = R(\mathfrak{m}) = 1$.

Lemma 5.2. *The number of supersingular points on* $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ *is*

$$S(\mathfrak{p},\mathfrak{m}) = \begin{cases} \frac{|\mathfrak{p}|-1}{q^2-1}L(\mathfrak{m}) & \text{if } \deg(\mathfrak{p}) \text{ is even;} \\ \frac{|\mathfrak{p}|-q}{q^2-1}L(\mathfrak{m}) + \frac{L(\mathfrak{m})+q2^sR(\mathfrak{m})}{q+1} & \text{if } \deg(\mathfrak{p}) \text{ is odd.} \end{cases}$$

The thickness of a supersingular point on $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ *is either* 1 *or* $q+1$. *Supersingular points with thickness* $q+1$ *can exist only if* $\deg(\mathfrak{p})$ *is odd and their number is* $2^sR(\mathfrak{m})$.

Proof. Let ϕ be a fixed Drinfeld module of rank 2 over $\overline{\mathbb{F}}_\mathfrak{p}$. Since $\mathfrak{m}$ is assumed to be coprime to $\mathfrak{p}$, the number of distinct cyclic subgroups $C_\mathfrak{m} \subset \phi[\mathfrak{m}]$ is $L(\mathfrak{m})$. If $\mathrm{Aut}(\phi) = \mathbb{F}_q^\times$, then all pairs $(\phi, C_\mathfrak{m})$ are non-isomorphic, since $\mathbb{F}_q^\times$ fixes each of them. Therefore, all these pairs correspond to distinct points on $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$. We know from (4.1) that $\mathrm{Aut}(\phi) \neq \mathbb{F}_q^\times$ if and only if $j(\phi) = 0$. Since Theorem 4.1 gives the number of isomorphism classes of supersingular Drinfeld modules, the claim of the lemma follows if we exclude the case with $j(\phi) = 0$.
Now assume $j(\phi) = 0$. Then $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^2}^\times$. We can identify the set of cyclic subgroups of ϕ of order $\mathfrak{m}$ with $\mathbb{P}^1(A/\mathfrak{m})$. From this perspective, the action of $\mathrm{Aut}(\phi)$ on this set is induced from an embedding

$$\mathbb{F}_{q^2}^\times \hookrightarrow \mathrm{GL}_2(A/\mathfrak{m}) \cong \mathrm{Aut}(\phi[\mathfrak{m}]),$$

with $\mathrm{GL}_2(A/\mathfrak{m})$ acting on $\mathbb{P}^1(A/\mathfrak{m})$ in the usual manner. We can decompose

$$\mathbb{P}^1(A/\mathfrak{m}) \cong \prod_{1\leq i\leq s} \mathbb{P}^1(A/\mathfrak{p}_i^{r_i})$$

with $\mathrm{GL}_2(A/\mathfrak{m})$ acting on $\mathbb{P}^1(A/\mathfrak{p}_i^{r_i})$ via its quotient $\mathrm{GL}_2(A/\mathfrak{p}_i^{r_i})$. The image of $\mathbb{F}_{q^2}^\times$ in $\mathrm{GL}_2(A/\mathfrak{m})$ is a maximal non-split torus in $\mathrm{GL}_2(\mathbb{F}_q)$. If the stabilizer $\mathrm{Stab}_{\mathbb{F}_{q^2}^\times}(P)$ of $P \in \mathbb{P}^1(A/\mathfrak{p}_i^{r_i})$ is strictly larger than $\mathbb{F}_q^\times$, then it is easy to see that in fact $\mathrm{Stab}_{\mathbb{F}_{q^2}^\times}(P) = \mathbb{F}_{q^2}^\times$. Moreover, this is possible if and only if $\mathbb{F}_{q^2} \hookrightarrow A/\mathfrak{p}_i^{r_i}$, in which case there are exactly 2 fixed points in $\mathbb{P}^1(A/\mathfrak{p}_i^{r_i})$ under the action of $\mathbb{F}_{q^2}$; cf. [17, p. 695]. Note that the existence of an embedding $\mathbb{F}_{q^2} \hookrightarrow A/\mathfrak{p}_i^{r_i}$ is equivalent to $\deg(\mathfrak{p}_i)$ being even. We conclude that $\mathbb{F}_{q^2}^\times$ acting on the set of $L(\mathfrak{m})$ pairs $(\phi, C_\mathfrak{m})$ has $2^sR(\mathfrak{m})$ fixed elements, and the orbit of any other element has length $q+1$. This implies that there are

$$\frac{L(\mathfrak{m}) - 2^sR(\mathfrak{m})}{q+1} + 2^sR(\mathfrak{m}) = \frac{L(\mathfrak{m}) + q2^sR(\mathfrak{m})}{q+1}$$

points on $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$ corresponding to ϕ with $j(\phi) = 0$. Finally, by Theorem 4.1, the Drinfeld module with $j(\phi) = 0$ is supersingular if and only if $\deg(\mathfrak{p})$ is odd. $\square$

Theorem 5.3. *Assume* $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, *with* $\mathfrak{p}$ *prime not dividing* $\mathfrak{m}$. *If* $\deg(\mathfrak{p})$ *is odd and* $\mathfrak{m} = A$, *then*

$$\Phi_\mathfrak{p} \cong \mathbb{Z}/\left((q+1)(S(\mathfrak{p}, A) - 1) + 1\right)\mathbb{Z}.$$

If $\deg(\mathfrak{p})$ *is even or* $\mathfrak{m}$ *has a prime divisor of odd degree, then*

$$\Phi_{\mathfrak{p}} \cong \mathbb{Z}/S(\mathfrak{p}, \mathfrak{m})\mathbb{Z}.$$

Finally, if $\deg(\mathfrak{p})$ *is odd,* $\mathfrak{m} \neq A$*, and all prime divisors of* $\mathfrak{m}$ *have even degrees, then*

$$\Phi_{\mathfrak{p}} \cong \mathbb{Z}/\left((q+1)^2 S(\mathfrak{p}, \mathfrak{m}) - q(q+1)2^s\right)\mathbb{Z} \bigoplus_{1 \leq i \leq 2^s - 2} \mathbb{Z}/(q+1)\mathbb{Z}.$$

(*The isomorphisms above are meant only as isomorphisms of groups, not group schemes, so* $\Phi_{\mathfrak{p}}$ *can have non-trivial* $G_{\mathbb{F}_{\mathfrak{p}}}$*-action.*)

Proof. This follows from Corollary 11 on page 285 in [3], combined with Lemma 5.2 and Theorem 5.1. This result for $\mathfrak{n} = \mathfrak{p}$ is also in [10]. $\square$

Proposition 5.4. *Assume* $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$ *is a product of two distinct primes. Denote*

$$n(\mathfrak{p}, \mathfrak{q}) = \frac{(|\mathfrak{p}| - 1)(|\mathfrak{q}| + 1)}{q^2 - 1}.$$

Let $\Phi_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}})$ *be the subgroup of* $\Phi_{\mathfrak{p}}$ *fixed by* $G_{\mathbb{F}_{\mathfrak{p}}}$*. If* $\deg(\mathfrak{p})$ *is odd and* $\deg(\mathfrak{q})$ *is even, then*

$$\Phi_{\mathfrak{p}} \cong \mathbb{Z}/(q+1)^2 n(\mathfrak{p}, \mathfrak{q})\mathbb{Z}, \qquad \Phi_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}}) \cong \mathbb{Z}/(q+1)n(\mathfrak{p}, \mathfrak{q})\mathbb{Z}.$$

Otherwise,

$$\Phi_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}}) = \Phi_{\mathfrak{p}} \cong \mathbb{Z}/n(\mathfrak{p}, \mathfrak{q})\mathbb{Z}.$$

Proof. To simplify the notation, denote $X = X_0(\mathfrak{p}\mathfrak{q})_{\mathcal{O}_{\mathfrak{p}}}$ and $k = \mathbb{F}_{\mathfrak{p}}$. Let $\widetilde{X} \to X$ be the minimal resolution of X. We know that X_k consists of two irreducible components, both isomorphic to $X_0(\mathfrak{q})_k$. If $\deg(\mathfrak{p})$ is even or $\deg(\mathfrak{q})$ is odd, then $X = \widetilde{X}$. On the other hand, if $\deg(\mathfrak{p})$ is odd and $\deg(\mathfrak{q})$ is even, then there are two points P and Q of thickness $q + 1$. To obtain the minimal regular model one performs a sequence of q blows-ups at P and Q. With the notation of [41, §4.2], let $E_1, \dots, E_q$ and $G_1, \dots, G_q$ be the chain of projective lines resulting from these blow-ups at P and Q, respectively. Let Z and Z' denote the two copies of $X_0(\mathfrak{q})_k$, with the convention that E_1 and G_1 intersect Z. Let $B(\widetilde{X}_k)$ be the free abelian group generated by the set of geometrically irreducible components of $\widetilde{X}_k$. Let $B^0(\widetilde{X}_k)$ be the subgroup of degree-0 elements in $B(\widetilde{X}_k)$. The theorem of Raynaud that we mentioned earlier gives an explicit description of $\Phi_{\mathfrak{p}}$ as a quotient of $B^0(\widetilde{X}_k)$; cf. [41, §4.2]. Let $\mathrm{Frob}_{\mathfrak{p}} : \alpha \to \alpha^{|\mathfrak{p}|}$ be the usual topological generator of G_k. The Frobenius $\mathrm{Frob}_{\mathfrak{p}}$ naturally acts on the geometrically irreducible components of $\widetilde{X}_k$, and therefore on $B^0(\widetilde{X}_k)$. Since $X_0(\mathfrak{q})_k$ is defined over k, the action of $\mathrm{Frob}_{\mathfrak{p}}$ fixes $z := Z - Z' \in B^0(\widetilde{X}_k)$. If $\deg(\mathfrak{p})$ is even or $\deg(\mathfrak{q})$ is odd, then z generates $B^0(\widetilde{X}_k)$, so G_k acts trivially on $\Phi_{\mathfrak{p}}$, which by Theorem 5.3 is isomorphic to $\mathbb{Z}/n(\mathfrak{p}, \mathfrak{q})\mathbb{Z}$.

From now on we assume $\deg(\mathfrak{p})$ is odd and $\deg(\mathfrak{q})$ is even. The claim $\Phi_{\mathfrak{p}} \cong \mathbb{Z}/(q+1)^2 n(\mathfrak{p}, \mathfrak{q})\mathbb{Z}$ follows from Theorem 5.3. Let ϕ be given by $\phi_T = T + \tau^2$.

Note that $j(\phi) = 0$ and ϕ is defined over k, so $\mathrm{Frob}_\mathfrak{p}$ acts on the set of cyclic subgroups $C_\mathfrak{q} \subset \phi$. Denote $A' = \mathbb{F}_{q^2}[T]$. It is easy to see that $A' \subseteq \mathrm{End}(\phi)$. Since $\mathfrak{q} = \mathfrak{q}_1\mathfrak{q}_2$ splits in A' into a product of two irreducible polynomials of the same degree, we have

$$\phi[\mathfrak{q}] \cong A'/\mathfrak{q} \cong A'/\mathfrak{q}_1 \oplus A'/\mathfrak{q}_2 \cong A/\mathfrak{q} \oplus A/\mathfrak{q} \cong \phi[\mathfrak{q}_1] \oplus \phi[\mathfrak{q}_2].$$

The above decomposition is preserves under the action of $\mathbb{F}_{q^2}$. In particular, $\mathrm{Aut}(\phi, \phi[\mathfrak{q}_i]) \cong \mathbb{F}_{q^2}^\times$. On the other hand, since $\mathfrak{p}$ has odd degree, this decomposition is not defined over k and $\mathrm{Frob}_\mathfrak{p}(\phi[\mathfrak{q}_1]) = \phi[\mathfrak{q}_2]$. We conclude that $P = (\phi, \phi[\mathfrak{q}_1])$, $Q = (\phi, \phi[\mathfrak{q}_2])$ and $\mathrm{Frob}_\mathfrak{p}(P) = Q$. Since the action of $\mathrm{Frob}_\mathfrak{p}$ preserves the incidence relations of the irreducible components of $\widetilde{X}_k$, we have $\mathrm{Frob}_\mathfrak{p}(E_i) = G_i$, $1 \leq i \leq q$. Following the notation of Theorem 4.1 in [41], let $e_q = E_q - Z'$ and $g_q = G_q - Z'$. According to that theorem, the image of e_q in $\Phi_\mathfrak{p}$ generates $\Phi_\mathfrak{p}$, and in the component group we have $g_q = -\left((q+1)(S(\mathfrak{p},\mathfrak{q}) - 2) + 1\right) e_q$. Thus, the action of $\mathrm{Frob}_\mathfrak{p}$ on $\Phi_\mathfrak{p}$ in terms of the generator e_q is given by $\mathrm{Frob}_\mathfrak{p}(e_q) = -\left((q+1)(S(\mathfrak{p},\mathfrak{q}) - 2) + 1\right) e_q$. This implies that $ae_q \in \Phi_\mathfrak{p}(\mathbb{F}_\mathfrak{p})$ if and only if

$$ae_q = \mathrm{Frob}_\mathfrak{p}(ae_q) = -a\left((q+1)(S(\mathfrak{p},\mathfrak{q}) - 2) + 1\right) e_q,$$

which is equivalent to $a\left((q+1)S(\mathfrak{p},\mathfrak{q}) - 2q\right) e_q = 0$. Hence a must be a multiple of $q+1$, and

$$\Phi_\mathfrak{p}(\mathbb{F}_\mathfrak{p}) = \langle (q+1) \rangle \cong \mathbb{Z}/(q+1)n(\mathfrak{p},\mathfrak{q})\mathbb{Z}.$$

□

The existence of rigid-analytic uniformization (4.3) of J implies that J has totally degenerate reduction at ∞, i.e., $\mathcal{J}^0_{\mathbb{F}_\infty}$ is a split algebraic torus over $\mathbb{F}_\infty$. The problem of explicitly describing Φ_∞ is closely related to the problem of describing $\Gamma_0(\mathfrak{n}) \setminus \mathscr{T}$ together with the stabilizers of its edges; cf. [41, §5.2]. Since the graph $\Gamma_0(\mathfrak{n}) \setminus \mathscr{T}$ becomes very complicated as $|\mathfrak{n}|$ grows, no explicit description of Φ_∞ is known in general. On the other hand, Φ_∞ has a description in terms of the uniformization of J. Let $\langle \cdot, \cdot \rangle : \overline{\Gamma} \times \overline{\Gamma} \to \mathbb{Z}$ be the pairing (4.2). This pairing is bilinear, symmetric, and positive-definite, so it induces an injection $\iota : \overline{\Gamma} \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{Z})$, $\gamma \mapsto \langle \gamma, \cdot \rangle$. Identifying $\overline{\Gamma}$ with $\mathcal{H}_0(\mathfrak{n}, \mathbb{Z})$ via (4.4), we get an injection $\iota : \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_0(\mathfrak{n}, \mathbb{Z}), \mathbb{Z})$. The Hecke algebra $\mathbb{T}(\mathfrak{n})$ acts on $\mathrm{Hom}(\mathcal{H}_0(\mathfrak{n}, \mathbb{Z}), \mathbb{Z})$ through its action on the first argument. The action of $\mathbb{T}(\mathfrak{n})$ on J, by the Néron mapping property, canonically extends to an action on $\mathcal{J}$. Thus, $\mathbb{T}(\mathfrak{n})$ functorially acts on Φ_∞.

Theorem 5.5. *The absolute Galois group $G_{\mathbb{F}_\infty}$ acts trivially on Φ_∞, and there is a $\mathbb{T}(\mathfrak{n})^0$-equivariant exact sequence*

$$0 \to \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \xrightarrow{\iota} \mathrm{Hom}(\mathcal{H}_0(\mathfrak{n}, \mathbb{Z}), \mathbb{Z}) \to \Phi_\infty \to 0.$$

Proof. The exact sequence is the statement of Corollary 2.11 in [11]. The $\mathbb{T}(\mathfrak{n})^0$-equivariance of this sequence follows from the $\mathbb{T}(\mathfrak{n})^0$-equivariance of (4.3). The fact that $G_{\mathbb{F}_\infty}$ acts trivially on Φ_∞ is a consequence of $\mathcal{J}^0_{\mathbb{F}_\infty}$ being a split torus. □

6. Cuspidal divisor group

The *cuspidal divisor group* $\mathcal{C}(\mathfrak{n})$ of $J := J_0(\mathfrak{n})$ is the subgroup of J generated by the classes of divisors $[c] - [c']$, where c, c' run through the set of cusps of $X_0(\mathfrak{n})_F$.

Theorem 6.1. *$\mathcal{C}(\mathfrak{n})$ is finite and if $\mathfrak{n}$ is square-free then $\mathcal{C}(\mathfrak{n}) \subset J(F)_{\mathrm{tor}}$.*

Proof. This theorem is due to Gekeler [15], where the finiteness of $\mathcal{C}(\mathfrak{n})$ is proven for general congruence subgroups over general function fields. Since the proof is fairly short, for the sake of completeness, we give a sketch. The space $\mathcal{H}_0(\mathfrak{n}, \mathbb{C})$ has an interpretation as a space of automorphic cusp forms (cf. [18]), so the Ramanujan-Petersson conjecture over function fields (proved by Drinfeld) implies that the eigenvalues of $T_\mathfrak{p}$ are algebraic integers of absolute value $\leq 2\sqrt{|\mathfrak{p}|}$ for any prime $\mathfrak{p} \nmid \mathfrak{n}$. This implies that the operator $\eta_\mathfrak{p} = T_\mathfrak{p} - |\mathfrak{p}| - 1$ is invertible on $\mathcal{H}_0(\mathfrak{n}, \mathbb{C})$. Hence $\eta_\mathfrak{p} : \mathcal{H}_0(\mathfrak{n}, \mathbb{Z}) \to \mathcal{H}_0(\mathfrak{n}, \mathbb{Z})$ is injective with finite cokernel. Now using the $T_\mathfrak{p}$ equivariance of (4.3) and (4.4), we see that $\eta_\mathfrak{p} : J \to J$ is an isogeny. Assume for simplicity that $\mathfrak{n}$ is square-free. In the proof of Lemma 3.1 we showed that $\eta_\mathfrak{p}([c] - [c']) = 0$. Hence $\mathcal{C}(\mathfrak{n}) \subseteq \ker(\eta_\mathfrak{p})$, which is finite. Finally, when $\mathfrak{n}$ is square-free, all the cusps of $X_0(\mathfrak{n})_F$ are rational over F, so the cuspidal divisor group is in $J(F)$; see [16, Prop. 6.7]. □

Proposition 6.2. *With notation of* §3.1, *let $\mu(\mathfrak{n})$ be the largest integer ℓ such that there exists an ℓ-th root of $\Delta/\Delta_\mathfrak{n}$ in $\mathcal{O}(\Omega)^\times$. Then $\mu(\mathfrak{n}) = (q-1)^2$ if $\deg \mathfrak{n}$ is odd and $\mu(\mathfrak{n}) = (q-1)(q^2-1)$ if $\deg \mathfrak{n}$ is even. Let $D_\mathfrak{n}$ be a $\mu(\mathfrak{n})$-th root of $\Delta/\Delta_\mathfrak{n}$. There exists a character $\omega_\mathfrak{n} : \Gamma_0(\mathfrak{n}) \to \mathbb{F}_q^\times$ such that for each $\gamma \in \Gamma_0(\mathfrak{n})$,*

$$D_\mathfrak{n}(\gamma z) = \omega_\mathfrak{n}(\gamma) D_\mathfrak{n}(z).$$

Proof. See Corollary 3.5 and 3.21 in [14]. □

If $\mathfrak{n} = \prod_i \mathfrak{p}_i^{r_i} \lhd A$ is the prime decomposition of $\mathfrak{n}$, define a character $\chi_\mathfrak{n} : \Gamma_0(\mathfrak{n}) \to \mathbb{F}_q^\times$ by the following: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{n})$,

$$\chi_\mathfrak{n}(\gamma) := \prod_i N_i(d \bmod \mathfrak{p}_i)^{-r_i},$$

where $N_i : (A/\mathfrak{p}_i A)^\times \to \mathbb{F}_q^\times$ is the norm map. Then $\omega_\mathfrak{n} = \chi_\mathfrak{n} \cdot \det^{\deg(\mathfrak{n})/2}$ if $\deg(\mathfrak{n})$ is even and $\omega_\mathfrak{n} = \chi_\mathfrak{n}^2 \cdot \det^{\deg(\mathfrak{n})}$ if $\deg(\mathfrak{n})$ is odd. In particular, the order of $\omega_\mathfrak{n}$ is $q-1$ when $\mathfrak{n}$ is square-free (cf. [14] Proposition 3.22). By Proposition 6.2, we immediately get:

Corollary 6.3.

(1) $D'_\mathfrak{n} := D_\mathfrak{n}^{q-1}$ *is a meromorphic function on the Drinfeld modular curve $X_0(\mathfrak{n})$ satisfying*

$$(D'_\mathfrak{n})^{\frac{\mu(\mathfrak{n})}{q-1}} = \frac{\Delta}{\Delta_\mathfrak{n}}.$$

(2) *Given two ideals* $\mathfrak{m}$ *and* $\mathfrak{n}$ *of* A*, we set* $D_{\mathfrak{n},\mathfrak{m}}(z) := D_{\mathfrak{n}}(z)/D_{\mathfrak{n}}(\mathfrak{m}z)$*,* $\forall z \in \Omega$*. Then*

$$(D_{\mathfrak{n},\mathfrak{m}})^{\mu(\mathfrak{n})} = \frac{\Delta\Delta_{\mathfrak{m}\mathfrak{n}}}{\Delta_{\mathfrak{n}}\Delta_{\mathfrak{m}}} = (D_{\mathfrak{m},\mathfrak{n}})^{\mu(\mathfrak{m})}$$

and for every $\gamma \in \Gamma_0(\mathfrak{m}\mathfrak{n})$*, we have*

$$D_{\mathfrak{n},\mathfrak{m}}(\gamma z) = D_{\mathfrak{n},\mathfrak{m}}(z).$$

In other words, $D_{\mathfrak{n},\mathfrak{m}}$ *and* $D_{\mathfrak{m},\mathfrak{n}}$ *can be viewed as meromorphic functions on* $X_0(\mathfrak{m}\mathfrak{n})$*.*

Remark 6.4. Take two coprime ideals $\mathfrak{m}$ and $\mathfrak{n}$ of A. By Corollary 6.3 (1), the $(q-1)$-th roots of $(\Delta\Delta_{\mathfrak{m}})/(\Delta_{\mathfrak{n}}\Delta_{\mathfrak{m}\mathfrak{n}})$ always exist in the function field of $X_0(\mathfrak{m}\mathfrak{n})_{\mathbb{C}_\infty}$. In fact, we can find a (q^2-1)-th root of $(\Delta\Delta_{\mathfrak{m}})/(\Delta_{\mathfrak{n}}\Delta_{\mathfrak{m}\mathfrak{n}})$ when $\deg \mathfrak{n}$ or $\deg \mathfrak{m}\mathfrak{n}$ is even. Indeed, we notice that

$$\frac{\Delta(z)\Delta_{\mathfrak{m}}(z)}{\Delta_{\mathfrak{n}}(z)\Delta_{\mathfrak{m}\mathfrak{n}}(z)} = \frac{\Delta(z)}{\Delta_{\mathfrak{n}}(z)} \cdot \frac{\Delta(z')}{\Delta_{\mathfrak{n}}(z')} = \frac{\Delta(z)}{\Delta_{\mathfrak{m}\mathfrak{n}}(z)} \cdot \frac{\Delta(z'')}{\Delta_{\mathfrak{m}\mathfrak{n}}(z'')} \cdot \mathfrak{m}^{-(q^2-1)}.$$

Here

$$z' = \mathfrak{m}z, \quad z'' = \frac{\mathfrak{m}z+1}{b\mathfrak{m}\mathfrak{n}z + a\mathfrak{m}} = \begin{pmatrix} \mathfrak{m} & 1 \\ b\mathfrak{m}\mathfrak{n} & a\mathfrak{m} \end{pmatrix} \cdot z,$$

and $a, b \in A$ such that $a\mathfrak{m} - b\mathfrak{n} = 1$. In particular, when q is odd,

$$D''_{\mathfrak{n}}(z) := (D_{\mathfrak{n}}(z) \cdot D_{\mathfrak{n}}(mz))^{\frac{q-1}{2}}$$

is a $2(q^2-1)$-th (resp. $2(q-1)$-th) root of $(\Delta\Delta_{\mathfrak{m}})/(\Delta_{\mathfrak{n}}\Delta_{\mathfrak{m}\mathfrak{n}})$ in the function field of $X_0(\mathfrak{m}\mathfrak{n})_{\mathbb{C}_\infty}$ when $\deg \mathfrak{n}$ is even (resp. odd).

The following lemma is immediate from the definitions.

Lemma 6.5. *Let* r *be the van der Put derivative (3.3) and* $E_{\mathfrak{n}} \in \mathcal{H}(\mathfrak{n}, \mathbb{Z})$ *the Eisenstein series (3.4). Then* $r(D_{\mathfrak{n}}) = E_{\mathfrak{n}}$*, and for any two distinct prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$*,*

$$r(D_{\mathfrak{p},\mathfrak{q}}) = \begin{cases} (q+1) \cdot E_{(\mathfrak{p},\mathfrak{q})}, & \textit{if } \deg \mathfrak{p} \textit{ is odd and } \deg \mathfrak{q} \textit{ is even,} \\ E_{(\mathfrak{p},\mathfrak{q})}, & \textit{otherwise.} \end{cases}$$

Take two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, let ℓ be the largest number such that there exists an ℓ-th root ξ of $(\Delta\Delta_{\mathfrak{p}\mathfrak{q}})/(\Delta_{\mathfrak{p}}\Delta_{\mathfrak{q}})$ in the function field of $X_0(\mathfrak{p}\mathfrak{q})_{\mathbb{C}_\infty}$. Comparing the first Fourier coefficient $r(\xi)^*(1)$ with $r\big((\Delta\Delta_{\mathfrak{p}\mathfrak{q}})/(\Delta_{\mathfrak{p}}\Delta_{\mathfrak{q}})\big)^*(1)$, we must have $\ell|(q-1)(q^2-1)$. By Corollary 3.17 in [14], for every non-zero ideal $\mathfrak{n}$ of A

$$\frac{\Delta}{\Delta_{\mathfrak{n}}} = \text{const.} \frac{G_{\mathfrak{n}}^{(q-1)(q^2-1)}}{\Delta^{(q^{\deg \mathfrak{n}}-1)}},$$

where $G_{\mathfrak{n}}$ is a Drinfeld modular form on Ω such that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{n})$

$$G_{\mathfrak{n}}(\gamma z) = \chi_{\mathfrak{n}}(\gamma)(cz+d)^{(|\mathfrak{n}|-1)/(q-1)} G_{\mathfrak{n}}(z).$$

Then

$$\xi^\ell = \frac{\Delta(z)\Delta_{\mathfrak{pq}}(z)}{\Delta_{\mathfrak{p}}(z)\Delta_{\mathfrak{q}}(z)} = \text{const.}\left(\frac{G_{\mathfrak{p}}(z)}{G_{\mathfrak{p}}(\mathfrak{q}z)}\right)^{(q-1)(q^2-1)} \cdot \left(\frac{\Delta_{\mathfrak{q}}(z)}{\Delta(z)}\right)^{|\mathfrak{p}|-1}$$
$$= \text{const.}\left(\frac{G_{\mathfrak{p}}(z)}{G_{\mathfrak{p}}(\mathfrak{q}z)}\right)^{(q-1)(q^2-1)} \cdot D_{\mathfrak{q}}(z)^{-\mu(\mathfrak{q})(|\mathfrak{p}|-1)}.$$

Since $G_{\mathfrak{p}}(z)/G_{\mathfrak{p}}(\mathfrak{q}z)$ is a meromorphic function on $X_0(\mathfrak{pq})_{\mathbb{C}_\infty}$,

$$D_{\mathfrak{q}}(z)^{\frac{\mu(\mathfrak{q})(|\mathfrak{p}|-1)}{\ell}} = \text{const.}\left(\frac{G_{\mathfrak{p}}(z)}{G_{\mathfrak{p}}(\mathfrak{q}z)}\right)^{\frac{(q-1)(q^2-1)}{\ell}} \cdot \xi^{-1}$$

is also in the function field of $X_0(\mathfrak{pq})_{\mathbb{C}_\infty}$. Note that the character $\omega_{\mathfrak{q}}$ has order $q-1$. Set

$$\mu(\mathfrak{p},\mathfrak{q}) := \begin{cases} (q-1)(q^2-1), & \text{if } \deg(\mathfrak{p}) \text{ or } \deg(\mathfrak{q}) \text{ is even,} \\ (q-1)^2, & \text{otherwise.} \end{cases}$$

We then have:

LEMMA 6.6. *The largest number ℓ such that there exists an ℓ-th roots of $\frac{\Delta\Delta_{\mathfrak{pq}}}{\Delta_{\mathfrak{p}}\Delta_{\mathfrak{q}}}$ in the function field of $X_0(\mathfrak{pq})_{\mathbb{C}_\infty}$ is $\mu(\mathfrak{p},\mathfrak{q})$.*

From now on we assume that $\mathfrak{n} = \mathfrak{pq}$ is a product of two distinct primes. In this case, $X_0(\mathfrak{n})_F$ has 4 cusps, which in the notation of Lemma 2.14 are $[\infty]$, $[1]$, $[\mathfrak{p}]$, $[\mathfrak{q}]$. Let $c_1, c_{\mathfrak{p}}, c_{\mathfrak{q}} \in J(F)$ be the classes of divisors $[1]-[\infty]$, $[\mathfrak{p}]-[\infty]$, and $[\mathfrak{q}]-[\infty]$, respectively. To simplify the notation, we put $\mathcal{C} := \mathcal{C}(\mathfrak{pq})$. The cuspidal divisor group $\mathcal{C}$ is generated by c_1, $c_{\mathfrak{p}}$, and $c_{\mathfrak{q}}$.

PROPOSITION 6.7. *The order of c_1 and $c_{\mathfrak{q}}$ is divisible by*

$$\begin{cases} \frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{q-1} & \text{if } \deg(\mathfrak{p}) \text{ is odd and } \deg(\mathfrak{q}) \text{ is even,} \\ \frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{q^2-1} & \text{otherwise.} \end{cases}$$

Proof. We use the notation in the proof of Proposition 5.4. Let $\wp_{\mathfrak{p}} : J(F_{\mathfrak{p}}) \to \Phi_{\mathfrak{p}}$ be the canonical specialization map. This map can be explicitly described as follows. Let $D = \sum_i n_i P_i$ be a degree-0 divisor, where all $P_i \in X(F_{\mathfrak{p}})$. Denote by D also the linear equivalence class of D in $J(F_{\mathfrak{p}})$. Since X and $\widetilde{X}$ are proper, $X(F_{\mathfrak{p}}) = X(\mathcal{O}_{\mathfrak{p}}) = \widetilde{X}(\mathcal{O}_{\mathfrak{p}})$. Since $\widetilde{X}$ is regular, each P_i specializes to a unique irreducible component $c(P_i)$ of $\widetilde{X}_k$. Then $\wp_{\mathfrak{p}}(D)$ is the image of $\sum_i n_i c(P_i) \in B^0(\widetilde{X}_k)$ in $\Phi_{\mathfrak{p}}$. By Theorem 4.2, the cusps reduce to distinct points in the smooth locus of X_k. The Atkin-Lehner involution $W_{\mathfrak{p}}$ interchanges the two irreducible components Z and Z' of X_k. Since $W_{\mathfrak{p}}([\infty]) = [\mathfrak{q}]$, the reductions of $[\infty]$ and $[\mathfrak{q}]$ lie on distinct components. On the other hand, $W_{\mathfrak{q}}$ acts on X_k by acting on each component Z and Z' separately, without interchanging them. Since $W_{\mathfrak{q}}([\infty]) = [\mathfrak{p}]$, the reductions of $[\infty]$ and $[\mathfrak{p}]$ lie on the same component of X_k. Let Z' be the component containing $[\infty]$ and $[\mathfrak{p}]$. Let z be the image of $Z - Z'$ in $\Phi_{\mathfrak{p}}$. Then $\wp_{\mathfrak{p}}(c_1) = \wp_{\mathfrak{p}}(c_{\mathfrak{q}}) = z$ and $\wp_{\mathfrak{p}}(c_{\mathfrak{p}}) = 0$.

By Theorem 4.1 in [41], $\Phi_\mathfrak{p}/\langle z\rangle \cong \mathbb{Z}/(q+1)\mathbb{Z}$. Now the claim follows from Theorem 5.3. $\square$

Remark 6.8. Suppose $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$, with $\mathfrak{p}$ prime not dividing $\mathfrak{m}$. Let $\mathcal{C}(\mathfrak{n})(F)$ denote the F-rational cuspidal subgroup of $J_0(\mathfrak{n})$. Generalizing the argument in the proof of Proposition 6.7 and using Theorem 5.3, it is not hard to show that the exponent of the group $\Phi_\mathfrak{p}/\wp_\mathfrak{p}(\mathcal{C}(\mathfrak{n})(F))$ divides $(q+1)$. In particular, the map

$$\wp_\mathfrak{p} : \mathcal{C}(\mathfrak{n})(F)_\ell \to (\Phi_\mathfrak{p})_\ell$$

is surjective for $\ell \nmid (q+1)$.

The orders of Δ at the cusps are known (cf. [14, (3.10)]):

$$\mathrm{ord}_{[\infty]}\Delta = 1, \quad \mathrm{ord}_{[1]}\Delta = |\mathfrak{p}||\mathfrak{q}|, \quad \mathrm{ord}_{[\mathfrak{p}]}\Delta = |\mathfrak{p}|, \quad \mathrm{ord}_{[\mathfrak{q}]}\Delta = |\mathfrak{q}|.$$

Using the action of the Aktin-Lehner involutions on the cusps (Lemma 2.14) and on the functions Δ, $\Delta_\mathfrak{p}$, and $\Delta_{\mathfrak{p}\mathfrak{q}}$, one obtains the divisors of the following functions on $X_0(\mathfrak{n})_{\mathbb{C}_\infty}$ (cf. [41]):

$$\begin{aligned}
\mathrm{div}(\Delta/\Delta_\mathfrak{p}) &= (|\mathfrak{p}|-1)\Big(|\mathfrak{q}|([1]-[\infty]) - |\mathfrak{q}|([\mathfrak{p}]-[\infty]) + ([\mathfrak{q}]-[\infty])\Big),\\
\mathrm{div}(\Delta/\Delta_\mathfrak{q}) &= (|\mathfrak{q}|-1)\Big(|\mathfrak{p}|([1]-[\infty]) + ([\mathfrak{p}]-[\infty]) - |\mathfrak{p}|([\mathfrak{q}]-[\infty])\Big),\\
\mathrm{div}(\Delta/\Delta_{\mathfrak{p}\mathfrak{q}}) &= (|\mathfrak{p}\mathfrak{q}|-1)([1]-[\infty]) + (|\mathfrak{q}|-|\mathfrak{p}|)([\mathfrak{p}]-[\infty]) +\\
&\qquad + (|\mathfrak{p}|-|\mathfrak{q}|)([\mathfrak{q}]-[\infty]).
\end{aligned}$$

Hence

$$\mathrm{div}\left(\frac{\Delta\Delta_{\mathfrak{p}\mathfrak{q}}}{\Delta_\mathfrak{p}\Delta_\mathfrak{q}}\right) = (|\mathfrak{p}|-1)(|\mathfrak{q}|-1)\Big([\infty]+[1]-[\mathfrak{p}]-[\mathfrak{q}]\Big), \tag{6.1}$$

$$\mathrm{div}\left(\frac{\Delta\Delta_{\mathfrak{q}}}{\Delta_\mathfrak{p}\Delta_{\mathfrak{p}\mathfrak{q}}}\right) = (|\mathfrak{p}|-1)(|\mathfrak{q}|+1)\Big([\infty]+[1]-[\mathfrak{p}]+[\mathfrak{q}]\Big), \tag{6.2}$$

$$\mathrm{div}\left(\frac{\Delta\Delta_{\mathfrak{p}}}{\Delta_\mathfrak{q}\Delta_{\mathfrak{p}\mathfrak{q}}}\right) = (|\mathfrak{p}|+1)(|\mathfrak{q}|-1)\Big([\infty]+[1]+[\mathfrak{p}]-[\mathfrak{q}]\Big). \tag{6.3}$$

From these equations we get:

LEMMA 6.9.

$$0 = (|\mathfrak{p}|^2-1)(|\mathfrak{q}|^2-1)c_1 = (|\mathfrak{p}|^2-1)(|\mathfrak{q}|-1)c_\mathfrak{p} = (|\mathfrak{p}|-1)(|\mathfrak{q}|^2-1)c_\mathfrak{q}.$$

In particular, the order of $\mathcal{C}$ is not divisible by p.

Let

$$c_{(-,-)} := c_1 - c_\mathfrak{p} - c_\mathfrak{q}, \quad c_{(-,+)} := c_1 - c_\mathfrak{p} + c_\mathfrak{q}, \quad c_{(+,-)} := c_1 + c_\mathfrak{p} - c_\mathfrak{q}.$$

The subgroup $\mathcal{C}'$ of $\mathcal{C}$ generated by $c_{(-,-)}$, $c_{(-,+)}$, and $c_{(+,-)}$ has index 1, 2, or 4. Set

$$\epsilon(\mathfrak{p},\mathfrak{q}) := \begin{cases} q-1, & \text{if } q \text{ is even, } \deg\mathfrak{p} \text{ is odd and } \deg\mathfrak{q} \text{ is even,} \\ 2(q-1), & \text{if } q \text{ is odd, } \deg\mathfrak{p} \text{ is odd, and } \deg\mathfrak{q} \text{ is even,} \\ q^2-1, & \text{if (i) } \deg\mathfrak{p} \text{ and } \deg\mathfrak{q} \text{ are both odd or} \\ & \quad \text{(ii) } q \text{ is even and } \deg\mathfrak{p} \text{ is even,} \\ 2(q^2-1), & \text{otherwise,} \end{cases}$$

and

$$N_{(-,-)} := \frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|-1)}{\mu(\mathfrak{p},\mathfrak{q})},$$

$$N_{(-,+)} := \frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|+1)}{\epsilon(\mathfrak{p},\mathfrak{q})}, \qquad N_{(+,-)} := \frac{(|\mathfrak{p}|+1)(|\mathfrak{q}|-1)}{\epsilon(\mathfrak{q},\mathfrak{p})}.$$

Proposition 6.10. *The orders of $c_{(-,-)}$, $c_{(-,+)}$, and $c_{(+,-)}$ are $N_{(-,-)}$, $N_{(-,+)}$, and $N_{(+,-)}$, respectively. The group $\mathcal{C}'$ is isomorphic to*

$$\frac{\mathbb{Z}}{N_{(-,-)}\mathbb{Z}} \times \frac{\mathbb{Z}}{N_{(-,+)}\mathbb{Z}} \times \frac{\mathbb{Z}}{N_{(+,-)}\mathbb{Z}}.$$

Proof. Lemma 6.6 and Equation (6.1) tell us immediately that the order of $c_{(-,-)}$ is $N_{(-,-)}$. In Remark 6.4, we have constructed $\epsilon(\mathfrak{p},\mathfrak{q})$-th root of $\frac{\Delta\Delta_\mathfrak{q}}{\Delta_\mathfrak{p}\Delta_{\mathfrak{p}\mathfrak{q}}}$ in the function field of $X_0(\mathfrak{p}\mathfrak{q})_{\mathbb{C}_\infty}$. Therefore Equation (6.2) and (6.3) imply that

$$N_{(-,+)}c_{(-,+)} = N_{(+,-)}c_{(+,-)} = 0.$$

Recall from the proof of Proposition 6.7 that $\wp_\mathfrak{p}(c_1) = \wp_\mathfrak{p}(c_\mathfrak{q}) = z$ and $\wp_\mathfrak{p}(c_\mathfrak{p}) = 0$, where $\wp_\mathfrak{p} : J(F_\mathfrak{p}) \to \Phi_\mathfrak{p}$ is the canonical specialization map. Then $\wp_\mathfrak{p}(c_{(-,+)}) = 2z$, and the order of $c_{(-,+)}$ must be divisible by $N_{(-,+)}$. Therefore the order of $c_{(-,+)}$ is $N_{(-,+)}$, which is also the order of $\wp_\mathfrak{p}(c_{(-,+)})$. By interchanging $\mathfrak{p}$ and $\mathfrak{q}$ we obtain that the order of $c_{(+,-)}$ is $N_{(+,-)}$. This proves the first claim.

Now, suppose $c = \alpha_1 c_{(-,-)} + \alpha_2 c_{(-,+)} + \alpha_3 c_{(+,-)} = 0$ for $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{Z}$. Then $\wp_\mathfrak{p}(c) = \alpha_2\wp_\mathfrak{p}(c_{(-,+)}) = 0$, which implies that $N_{(-,+)} \mid \alpha_2$. Similarly, we have $N_{(+,-)} \mid \alpha_3$. Hence $c = \alpha_1 c_{(-,-)} = 0$, and $N_{(-,-)} \mid \alpha_1$. This implies the second claim. □

When q is even, Lemma 6.9 implies that $\mathcal{C} = \mathcal{C}'$. Suppose q is odd. Note that the quotient group $\mathcal{C}/\mathcal{C}'$ is generated by $c_\mathfrak{p}$ and $c_\mathfrak{q}$, where $2c_\mathfrak{p} \equiv 2c_\mathfrak{q} \equiv 0 \bmod \mathcal{C}'$. When $\deg(\mathfrak{p})$ and $\deg(\mathfrak{q})$ are both odd, we can find meromorphic functions $\varphi_\mathfrak{p}$ and $\varphi_\mathfrak{q}$ on $X_0(\mathfrak{p}\mathfrak{q})_{\mathbb{C}_\infty}$ such that

$$\mathrm{div}(\varphi_\mathfrak{p}) = \frac{(|\mathfrak{p}|^2-1)(|\mathfrak{q}|-1)}{(q-1)(q^2-1)}([\mathfrak{p}]-[\infty]),$$

$$\mathrm{div}(\varphi_\mathfrak{q}) = \frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|^2-1)}{(q-1)(q^2-1)}([\mathfrak{q}]-[\infty]).$$

Indeed, let

$$\varphi_{\mathfrak{p}}(z) := \frac{D_{\mathfrak{q}}(z)^{\frac{|\mathfrak{p}|-q}{q^2-1}} \cdot D_{\mathfrak{q}}(\mathfrak{p}z)^{\frac{|\mathfrak{p}|q-1}{q^2-1}}}{\Big(D_{\mathfrak{pq}}(z)\cdot D_{\mathfrak{pq}}(W_{\mathfrak{p}}z)\Big)^{\frac{|\mathfrak{p}|-1}{q-1}}}, \quad \varphi_{\mathfrak{q}}(z) := \frac{D_{\mathfrak{p}}(z)^{\frac{|\mathfrak{q}|-q}{q^2-1}} \cdot D_{\mathfrak{p}}(\mathfrak{q}z)^{\frac{|\mathfrak{q}|q-1}{q^2-1}}}{\Big(D_{\mathfrak{pq}}(z)\cdot D_{\mathfrak{pq}}(W_{\mathfrak{q}}z)\Big)^{\frac{|\mathfrak{q}|-1}{q-1}}}$$

(Proposition 6.2 implies that $\varphi_{\mathfrak{p}}$ and $\varphi_{\mathfrak{q}}$ are $\Gamma_0(\mathfrak{pq})$-invariant.) We conclude that the orders of $c_{\mathfrak{p}}$ and $c_{\mathfrak{q}}$ are odd, and therefore $\mathcal{C} = \mathcal{C}'$.

Suppose q is odd and $\deg(\mathfrak{p})\cdot\deg(\mathfrak{q})$ is even. Then Proposition 6.7 tells us that the order of $c_{\mathfrak{p}}$ and $c_{\mathfrak{q}}$ are both even. Thus from the canonical specialization maps $\wp_{\mathfrak{p}}$ and $\wp_{\mathfrak{q}}$, we have the following exact sequence:

$$0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0.$$

Since $2c_{\mathfrak{p}} = c_{(+,-)} - c_{(-,-)}$ and $2c_{\mathfrak{q}} = c_{(-,+)} - c_{(-,-)}$, the order of $c_{\mathfrak{p}}$ is $2 \cdot \mathrm{lcm}(N_{(-,-)}, N_{(+,-)})$, and the order of $c_{\mathfrak{q}}$ is $2\cdot\mathrm{lcm}(N_{(-,-)}, N_{(-,+)})$. From the above discussion, we finally conclude that:

Theorem 6.11.

(1) *The order of $c_{\mathfrak{p}}$ is $\frac{(|\mathfrak{p}|^2-1)(|\mathfrak{q}|-1)}{(q-1)(q^2-1)}$ and the order of $c_{\mathfrak{q}}$ is $\frac{(|\mathfrak{p}|-1)(|\mathfrak{q}|^2-1)}{(q-1)(q^2-1)}$.*

(2) *The odd part of $\mathcal{C}$ is isomorphic to*

$$\frac{\mathbb{Z}}{N^{\mathrm{odd}}_{(-,-)}\mathbb{Z}} \times \frac{\mathbb{Z}}{N^{\mathrm{odd}}_{(-,+)}\mathbb{Z}} \times \frac{\mathbb{Z}}{N^{\mathrm{odd}}_{(+,-)}\mathbb{Z}},$$

where N^{odd} is the odd part of a positive integer N.

(3) *When q is even or $q\cdot\deg(\mathfrak{p})\cdot\deg(\mathfrak{q})$ is odd, we have $\mathcal{C} = \mathcal{C}'$.*

(4) *Suppose q is odd and $\deg(\mathfrak{p})\cdot\deg(\mathfrak{q})$ is even. Let $\mathcal{C}_2$ (resp. $\mathcal{C}_2'$) be the 2-primary part of $\mathcal{C}$ (resp. $\mathcal{C}'$). Let $r_1,\ r_2,\ r_3 \in \mathbb{Z}_{\geq 0}$ with $r_1 \geq r_2 \geq r_3$ such that*

$$\mathcal{C}_2' \cong \mathbb{Z}/2^{r_1}\mathbb{Z} \times \mathbb{Z}/2^{r_2}\mathbb{Z} \times \mathbb{Z}/2^{r_3}\mathbb{Z}.$$

Then

$$\mathcal{C}_2 \cong \mathbb{Z}/2^{r_1+1}\mathbb{Z} \times \mathbb{Z}/2^{r_2+1}\mathbb{Z} \times \mathbb{Z}/2^{r_3}\mathbb{Z}.$$

Example 6.12. If q is even, then $\mathcal{C}'(xy) = \mathcal{C}(xy) \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z}$. If q is odd, then $\mathcal{C}'(xy) \cong \frac{\mathbb{Z}}{\frac{q+1}{2}\mathbb{Z}} \times \frac{\mathbb{Z}}{\frac{q^2+1}{2}\mathbb{Z}}$ and $\mathcal{C}(xy) \cong \mathbb{Z}/(q+1)\mathbb{Z}\times\mathbb{Z}/(q^2+1)\mathbb{Z}$.

Example 6.13. Assume $\deg(\mathfrak{p}) = \deg(\mathfrak{q}) = 2$. If q is even, then

$$\mathcal{C}'(\mathfrak{pq}) = \mathcal{C}(\mathfrak{pq}) \cong \frac{\mathbb{Z}}{(q^2+1)\mathbb{Z}} \times \frac{\mathbb{Z}}{(q+1)(q^2+1)\mathbb{Z}}.$$

If q is odd,

$$\mathcal{C}'(\mathfrak{pq}) \cong \frac{\mathbb{Z}}{(q+1)\mathbb{Z}} \times \frac{\mathbb{Z}}{\frac{q^2+1}{2}\mathbb{Z}} \times \frac{\mathbb{Z}}{\frac{q^2+1}{2}\mathbb{Z}},$$

$$\mathcal{C}(\mathfrak{pq}) \cong \frac{\mathbb{Z}}{(q^2+1)\mathbb{Z}} \times \frac{\mathbb{Z}}{(q+1)(q^2+1)\mathbb{Z}}.$$

7. Rational torsion subgroup

7.1. Main theorem. Let $\mathfrak{n} \lhd A$ be a non-zero ideal. To simplify the notation in this section we denote $J = J_0(\mathfrak{n})$. Let $\mathcal{J}$ denote the Néron model of J over $\mathbb{P}^1_{\mathbb{F}_q}$. Let $\mathcal{T}(\mathfrak{n})$ the torsion subgroup of $J(F)$.

Lemma 7.1. *Let ℓ be a prime not equal to p. Then $\mathcal{T}(\mathfrak{n})_\ell$ is annihilated by the Eisenstein ideal $\mathfrak{E}(\mathfrak{n})$, i.e., $(T_\mathfrak{p} - |\mathfrak{p}| - 1)P = 0$ for every prime ideal $\mathfrak{p}$ not dividing $\mathfrak{n}$ and $P \in \mathcal{T}(\mathfrak{n})_\ell$.*

Proof. It follows from Theorem 4.2 that J has good reduction at $\mathfrak{p} \nmid \mathfrak{n}$. By the Néron mapping property, $\mathcal{T}(\mathfrak{n})_\ell$ extends to an étale subgroup scheme of $\mathcal{J}$. This implies that there is a canonical injective homomorphism $\mathcal{T}(\mathfrak{n})_\ell \hookrightarrow \mathcal{J}_{\mathbb{F}_\mathfrak{p}}(\mathbb{F}_\mathfrak{p})$. The action of $\mathbb{T}(\mathfrak{n})$ on J canonically extends to an action on $\mathcal{J}$. Since the reduction map commutes with the action of $\mathbb{T}(\mathfrak{n})$, it is enough to show that $(T_\mathfrak{p} - |\mathfrak{p}| - 1)$ annihilates $\mathcal{T}(\mathfrak{n})_\ell$ over $\mathbb{F}_\mathfrak{p}$. Let $\mathrm{Frob}_\mathfrak{p}$ be the Frobenius endomorphism of $J_{\mathbb{F}_\mathfrak{p}}$. The Hecke operator $T_\mathfrak{p}$ satisfies the Eichler-Shimura relation:

$$\mathrm{Frob}_\mathfrak{p}^2 - T_\mathfrak{p} \cdot \mathrm{Frob}_\mathfrak{p} + |\mathfrak{p}| = 0.$$

Since $\mathrm{Frob}_\mathfrak{p}$ acts trivially on $\mathcal{T}(\mathfrak{n})_\ell$, the claim follows. □

Lemma 7.2. *Suppose ℓ is a prime not dividing $q(q-1)$. There is a natural injective homomorphism $\mathcal{T}(\mathfrak{n})_\ell \hookrightarrow \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z})$ for any $r \in \mathbb{Z}_{\geq 0}$ with $\ell^r \geq \#(\Phi_{\infty,\ell})$.*

Proof. Since J has split toric reduction at ∞, $\mathcal{J}^0_{\mathbb{F}_\infty}(\mathbb{F}_\infty) \cong \prod_{i=1}^g \mathbb{F}_q^\times$, where $g = \dim(J)$. Under the assumption that ℓ does not divide $q(q-1)$, we see that $\mathcal{T}(\mathfrak{n})_\ell$ has trivial intersection with $\mathcal{J}^0_{\mathbb{F}_\infty}$, so the canonical specialization $\mathcal{T}(\mathfrak{n})_\ell \to (\Phi_\infty)_\ell$ is injective. Since this map is $\mathbb{T}(\mathfrak{n})$-equivariant, by Lemma 7.1 we get an injection $\mathcal{T}(\mathfrak{n})_\ell \to (\Phi_\infty)_\ell[\mathfrak{E}]$. Fix some $r \in \mathbb{Z}_{\geq 0}$ with $\ell^r \cdot (\Phi_\infty)_\ell = 0$. Multiplying the sequence in Theorem 5.5 by ℓ^r and applying the snake lemma, we get an injection $(\Phi_\infty)_\ell \hookrightarrow \mathcal{H}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z})$. Since this map is again $\mathbb{T}(\mathfrak{n})$-equivariant, restricting to the kernels of $\mathfrak{E}(\mathfrak{n})$, we get $(\Phi_\infty)_\ell[\mathfrak{E}(\mathfrak{n})] \hookrightarrow \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z})$. Therefore, $\mathcal{T}(\mathfrak{n})_\ell \hookrightarrow \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z})$ as was required to show. □

Theorem 7.3. *Suppose $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$ is a product of two distinct primes $\mathfrak{p}$ and $\mathfrak{q}$. Let $s_{\mathfrak{p},\mathfrak{q}} = \gcd(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$. If ℓ does not divide $q(q-1)s_{\mathfrak{p},\mathfrak{q}}$, then $\mathcal{C}(\mathfrak{n})_\ell = \mathcal{T}(\mathfrak{n})_\ell$.*

Proof. First of all, since $\mathfrak{n}$ is square-free, $\mathcal{C}(\mathfrak{n})$ is rational over F, so $\mathcal{C}(\mathfrak{n})_\ell \subseteq \mathcal{T}(\mathfrak{n})_\ell$; see Theorem 6.1. Next, note that ℓ is odd by our assumption, so by Proposition 6.10

$$\mathcal{C}(\mathfrak{n})_\ell = \mathcal{C}(\mathfrak{n})_\ell^{\mathrm{odd}} \cong \mathbb{Z}/\ell^{r_{(-,-)}}\mathbb{Z} \times \mathbb{Z}/\ell^{r_{(-,+)}}\mathbb{Z} \times \mathbb{Z}/\ell^{r_{(+,-)}}\mathbb{Z},$$

where $r_{(\pm,\pm)} := \mathrm{ord}_\ell(N_{(\pm,\pm)})$. Since ℓ is coprime to $q(q-1)s_{\mathfrak{p},\mathfrak{q}}$, Theorem 3.9 and Lemma 3.10 give the inclusion

$$\begin{aligned} \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z}) &= \mathcal{E}'_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z}) \\ &\subseteq \mathcal{E}'_0(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z}) \cong \mathbb{Z}/\ell^{r_{(-,-)}}\mathbb{Z} \times \mathbb{Z}/\ell^{r_{(-,+)}}\mathbb{Z} \times \mathbb{Z}/\ell^{r_{(+,-)}}\mathbb{Z}. \end{aligned}$$

Finally, by Lemma 7.2 we have an injection $\mathcal{C}(\mathfrak{n})_\ell \hookrightarrow \mathcal{T}(\mathfrak{n})_\ell \hookrightarrow \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z})$. Comparing the orders of these groups, we conclude that

$$\mathcal{C}(\mathfrak{n})_\ell = \mathcal{T}(\mathfrak{n})_\ell = \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z}) = \mathcal{E}_0'(\mathfrak{n}, \mathbb{Z}/\ell^r\mathbb{Z}).$$

□

Remark 7.4. The previous proof shows that $\mathcal{E}_{00}(\mathfrak{pq}, \mathbb{Z}/\ell^r\mathbb{Z}) = \mathcal{E}_0'(\mathfrak{pq}, \mathbb{Z}/\ell^r\mathbb{Z})$. A generalization of this argument gives the equality $\mathcal{E}_{00}(\mathfrak{pq}, R) = \mathcal{E}_0'(\mathfrak{pq}, R)$ for any coefficient ring in which $q-1$ and $s_{\mathfrak{p},\mathfrak{q}}$ are invertible.

Corollary 7.5. *If ℓ does not divide $q(|\mathfrak{p}|^2-1)(|\mathfrak{q}|^2-1)$, then $\mathcal{T}(\mathfrak{pq})_\ell = 0$.*

Lemma 7.6. *Assume $\mathfrak{n} = \mathfrak{mp}$ is square-free, $\mathfrak{p}$ is prime, and $\deg(\mathfrak{m}) \leq 2$. Then $\mathcal{T}(\mathfrak{n})_p = 0$.*

Proof. If $\deg(\mathfrak{m}) \leq 2$, then $X_0(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \cong \mathbb{P}^1_{\mathbb{F}_\mathfrak{p}}$. It follows from Theorem 5.1 and [3, p. 246] that $\mathcal{J}^0_{\mathbb{F}_\mathfrak{p}}$ is a torus. Since J has toric reduction at $\mathfrak{p}$, the p-primary torsion subgroup $\mathcal{T}(\mathfrak{n})_p$ injects into $\Phi_\mathfrak{p}$; see Lemma 7.13 in [38]. Finally, as is easy to see from Theorem 5.3, if $\mathfrak{n}$ is square-free, the order of $\Phi_\mathfrak{p}$ is coprime to p. Thus, $\mathcal{T}(\mathfrak{n})_p = 0$. □

7.2. Special case. Here we focus on the case $\mathfrak{n} = xy$ and prove that $\mathcal{C}(\mathfrak{n}) = \mathcal{T}(\mathfrak{n})$. To simplify the notation, let $\mathcal{C} := \mathcal{C}(\mathfrak{n})$ and $\mathcal{T} := \mathcal{T}(\mathfrak{n})$. By Theorem 7.3 and Lemma 7.6, we know that $\mathcal{C}_\ell = \mathcal{T}_\ell$ for any $\ell \nmid (q-1)$. Let

$$N = (q+1)(q^2+1).$$

By Corollary 3.18, $\mathbb{T}(xy)/\mathfrak{E}(xy) \cong \mathbb{Z}/N\mathbb{Z}$, so $N \in \mathfrak{E}(xy)$. On the other hand, $\mathfrak{E}(xy)$ annihilates $\mathcal{T}_\ell$ for $\ell \neq p$. Therefore, the exponent of $\mathcal{T}_\ell$ divides N. Since $\gcd(q-1, N)$ divides 4, $\mathcal{T}_\ell = 0$ when $\ell \mid (q-1)$ is an odd prime. Therefore, we are reduced to showing that $\mathcal{C}_2 = \mathcal{T}_2$ in the case when q is odd. To prove this we will use the fact that $X_0(xy)_F$ is hyperelliptic.

Let C be a hyperelliptic curve of genus g over a field F of characteristic not equal to 2. Let $B \subset C(\bar{F})$ be the set of fixed points of the hyperelliptic involution of C. The cardinality of B is $2g+2$. Let J be the Jacobian variety of C. Let $\mathcal{G}$ be the set of subsets of even cardinality of B modulo the equivalence relation defined by $S_1 \sim S_2$ if $S_1 = S_2$ or $S_1 = B - S_2$ (= the complement of S_2). Denote the equivalence class of $S \subset B$ by $[S]$. Define a binary operation on $\mathcal{G}$ by

$$[S_1] \circ [S_2] = [(S_1 \cup S_2) - (S_1 \cap S_2)].$$

Then $\mathcal{G}$ is an abelian group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$ (the identity is $[\emptyset]$). There is an obvious action of the absolute Galois group G_F on $\mathcal{G}$, induced from the action on B.

Theorem 7.7. *There is a canonical isomorphism $J[2] \cong \mathcal{G}$ of Galois modules.*

Proof. Follows from Lemma 2.4 in [36], page 3.32. □

Now let $F = \mathbb{F}_q(T)$ with q odd. Let $f(T)$ be a monic square-free polynomial of degree 3. Let $\mathfrak{n}$ be the ideal in A generated by $f(T)$. The Drinfeld modular curve $X_0(\mathfrak{n})_F$ is hyperelliptic with the Atkin-Lehner involution $W_\mathfrak{n}$ being the hyperelliptic involution; see [49]. Let $e \in \mathbb{F}_q^\times$ be a non-square. Denote $K_1 = F(\sqrt{f(T)})$, $K_2 = F(\sqrt{ef(T)})$, $\mathcal{O}_1 = A[\sqrt{f(T)}]$, $\mathcal{O}_2 = A[\sqrt{ef(T)}]$. Note that since ∞ does not split in K_i/F, $\mathcal{O}_i$ is a maximal order in K_i $(i = 1, 2)$.

Theorem 7.8. *The fixed points of $W_\mathfrak{n}$ on $X_0(\mathfrak{n})_F$ correspond to the isomorphism classes of Drinfeld modules with complex multiplication by $\mathcal{O}_1$ or $\mathcal{O}_2$.*

Proof. See (3.5) in [10]. □

Theorem 7.9. *Let K be an imaginary quadratic extension of F, i.e., ∞ does not split in K/F. Let $\mathcal{O}$ be the integral closure of A in K. Let $\mathcal{H}$ be the Hilbert class field of K, i.e., the maximal abelian unramified extension of K in which ∞ splits completely.*

(1) $\mathrm{Gal}(\mathcal{H}/K) \cong \mathrm{Pic}(\mathcal{O})$.
(2) *There is a unique irreducible monic polynomial $H_K(z) \in A[z]$ whose roots are the j-invariants of various non-isomorphic rank-2 Drinfeld A-modules over $\mathbb{C}_\infty$ with CM by $\mathcal{O}$.*
(3) *The degree of $H_K(z)$ is the class number of $\mathcal{O}$, and $\mathcal{H}$ is its splitting field.*
(4) *The field $F' = F[z]/H_K(z)$ is linearly disjoint from K, and $\mathcal{H}$ is the composition of F' and K.*

Proof. Follows from Corollary 2.5 in [8]. □

Denote $H_i(z) = H_{K_i}(z)$, $F_i = F[z]/H_i(z)$, $h_i = \#\mathrm{Pic}(\mathcal{O}_i)$, and $\mathcal{H}_i$ the Hilbert class field of K_i $(i = 1, 2)$. Let B be the set of fixed points of $W_\mathfrak{n}$ on $X_0(\mathfrak{n})$. Since the action of G_F on $X_0(\mathfrak{n})(\bar{F})$ commutes with the action of $W_\mathfrak{n}$, the set B is stable under the action of G_F. The previous two theorems imply that B can be identified with the set of roots of the polynomial $H_1(z)H_2(z)$ compatibly with the action of G_F. Let $J := J_0(\mathfrak{n})$.

Lemma 7.10. *If $f(T)$ is irreducible, then $J(F)[2] = 0$.*

Proof. Let X be the smooth, projective curve over $\mathbb{F}_q$ with function field K_1. It is well-known that there is an exact sequence

$$0 \to \mathrm{Jac}_X(\mathbb{F}_q) \to \mathrm{Pic}(\mathcal{O}_1) \to \mathbb{Z}/d_\infty\mathbb{Z} \to 0,$$

where d_∞ is the degree of ∞ on X. Since $f(T)$ has degree 3, ∞ ramifies in K_1/F, so $d_\infty = 1$ and X is isomorphic to the elliptic curve E_1 defined by the equation $Y^2 = f(T)$. Thus, $\mathrm{Pic}(\mathcal{O}_1) \cong E_1(\mathbb{F}_q)$. Similarly, $\mathrm{Pic}(\mathcal{O}_2) \cong E_2(\mathbb{F}_q)$, where E_2 is defined by $Y^2 = ef(T)$. In particular,

$$h_1 + h_2 = \#E_1(\mathbb{F}_q) + \#E_2(\mathbb{F}_q) = 2q + 2.$$

The last equality follows from the observation that for any $\alpha \in \mathbb{F}_q$ either $f(\alpha) = 0$, in which case we get one $\mathbb{F}_q$-rational point on both E_1 and E_2, or exactly one of $f(\alpha)$, $ef(\alpha)$ is a square in $\mathbb{F}_q^\times$, in which case we get two $\mathbb{F}_q$-rational on one of the elliptic curves and no points on the other.

Now suppose $f(T)$ is irreducible. Then $f(T)$ has no $\mathbb{F}_q$-rational roots, and therefore $E_1(\mathbb{F}_q)[2] = O$. This implies that h_1 and h_2 are both odd. Denote the set of roots of H_1 (resp. H_2) by B_1 (resp. B_2), so that $B = B_1 \cup B_2$. Note that B_1 and B_2 are stable under the action of G_F, but have no non-trivial G_F stable subsets. Since $\#B_1$ and $\#B_2$ are odd, by Mumford's theorem the only possibility for having an F-rational 2-torsion on J is to have a subset $S \subset B$ of order $q+1$ such that $\sigma S = S$ or $\sigma S = B - S$ for any $\sigma \in G_F$. Denote $S_1 = S \cap B_1$. Without loss of generality we can assume that $S_1 \neq \emptyset, B_1$. We must have $\sigma S_1 = S_1$ or $\sigma S_1 = B_1 - S_1$ for any $\sigma \in G_F$. But $\#S_1$ and $\#(B_1 - S_1)$ have different parity, and therefore $\sigma S_1 = S_1$ for any $\sigma \in G_F$, which is a contradiction. $\square$

Remark 7.11. Lemma 7.10 also follows from Theorem 1.2 in [38]. Indeed, if $f(T)$ is irreducible, then according to [38], $J(F)_{\mathrm{tor}} \cong \mathbb{Z}/(q^2+q+1)\mathbb{Z}$, so $J(F)[2] = 0$.

LEMMA 7.12. *If $f(T)$ decomposes into a product $f_1(T)f_2(T)$, where $f_2(T)$ is irreducible of degree 2, then $J(F)[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. We retain the notation of the proof of Lemma 7.10. The first part of the proof of that lemma implies that $\mathrm{Pic}(\mathcal{O}_i)[2] \cong \mathbb{Z}/2\mathbb{Z}$, since both elliptic curves E_1 and E_2 have exactly one non-trivial $\mathbb{F}_q$-rational 2-torsion point corresponding to $(\alpha, 0)$, where $f_1(\alpha) = 0$. In particular, h_1 and h_2 are even, and $\mathcal{H}_i/K_i$ has a unique quadratic subextension. We conclude that $[B_1] = [B_2]$ is F-rational.

The other F-rational 2-torsion points on J are in bijection with the disjoint decompositions $B_1 = R_1 \cup R_2$ and $B_2 = R_1' \cup R_2'$ such that $\#R_1 = \#R_2$, $\#R_1' = \#R_2'$, and for any $\sigma \in G_F$ either

$$\sigma R_1 = R_1 \text{ and } \sigma R_1' = R_1'$$

or

$$\sigma R_1 = R_2 \text{ and } \sigma R_1' = R_2'.$$

In that case, $P = [R_1 \cup R_1'] = [R_2 \cup R_2']$ and $Q = [R_1 \cup R_2'] = [R_2 \cup R_1']$ give two distinct non-trivial 2-torsion points on J which are rational over F. Note that the subgroup generated by P, Q and $[B_1]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, such disjoint decompositions are in bijection with the quadratic extensions L of F which simultaneously embed into both F_1 and F_2. Note that over a quadratic subextension L of F_1/F the polynomial H_1 decomposes into a product $G_1(z)G_2(z)$ of two irreducible polynomials in $L[z]$ of the same degree. In that case, R_1 and R_2 are the sets of roots of G_1 and G_2, respectively. Now if there are two distinct quadratic subextensions L and L' of F_1, then $\mathcal{H}_1 = F_1K_1$ also contains two distinct quadratic subextensions. As we indicated above, this is not the case, hence there is at most one L.

Consider $L = F(\sqrt{f_2})$. Since f_2 is monic of degree 2, ∞ splits in L/F. Since the only place that ramifies in L/F is the place corresponding to f_2, which also ramifies in K/F with the same ramification index, LK is a quadratic subextension of $\mathcal{H}_1/K$. Hence the composition LF_1 is a subfield of $\mathcal{H}_1$, and if L and F_1 are linearly disjoint over F, then by comparing the degrees we see that $LF_1 = \mathcal{H}_1$. Since $H_1(z)$ has even degree, F_1 embeds into the completion F_∞. The same is true for L. Thus, if $LF_1 = \mathcal{H}_1$, then $\mathcal{H}_1$ embeds into F_∞, which is not the case as K/F is imaginary quadratic. We conclude that L and F_1 cannot be linearly disjoint, and therefore L embeds into F_1. The same argument works also with F_2, and this finishes the proof of the lemma. □

Theorem 7.13. $\mathcal{C} = \mathcal{T} \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z}$.

Proof. As we already discussed, it is enough to show that $\mathcal{C}_2 = \mathcal{T}_2$ when q is odd. By Example 6.12,

$$\mathcal{C} = \langle c_x \rangle \oplus \langle c_y \rangle \cong \mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/(q^2+1)\mathbb{Z},$$

and $c_1 = c_x + c_y$. The component group Φ_∞ (for the case $\mathfrak{n} = xy$) and the canonical specialization map $\wp_\infty : \mathcal{C} \to \Phi_\infty$ are computed in [41, Thm. 5.5]:

$$\Phi_\infty = \Phi_\infty(\mathbb{F}_\infty) \cong \mathbb{Z}/N\mathbb{Z}, \quad \wp_\infty(c_x) = q^2+1, \quad \wp_\infty(c_y) = -q(q+1).$$

In particular, if we denote $\mathcal{C}^0 = \ker(\wp_\infty : \mathcal{C} \to \Phi_\infty)$, then $\mathcal{C}^0 \cong \mathbb{Z}/2\mathbb{Z}$ when q is odd. On the other hand, by Lemma 7.12, $\mathcal{T}[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence $\mathcal{C}[2] = \mathcal{T}[2]$.

Let $\mathcal{T}^0 := \ker(\wp_\infty : \mathcal{T} \to \Phi_\infty)$. As we discussed at the beginning of this section, $\mathcal{T}^0$ is a subgroup of $(\mathbb{Z}/(q-1)\mathbb{Z})^{\oplus q}$ annihilated by N. We have

$$\gcd((q-1), N) = \begin{cases} 2 & \text{if } q \equiv 3 \bmod 4 \\ 4 & \text{if } q \equiv 1 \bmod 4. \end{cases}$$

Assume $q \equiv 3 \bmod 4$. Then $\mathcal{T}^0 \subset \mathcal{T}[2] = \mathcal{C}[2]$. This implies $\mathcal{T}^0 = \mathcal{C}^0 \cong \mathbb{Z}/2\mathbb{Z}$, and is generated by

$$c := \frac{(q+1)}{2} c_x + \frac{(q^2+1)}{2} c_y.$$

In this case $\wp_\infty$ is injective on $\langle c_1 \rangle$, and $\wp_\infty(c_1)$ generates $\wp_\infty(\mathcal{C})$. If $\mathcal{T}_2 \neq \mathcal{C}_2$, then there is an element $t \in \mathcal{T}$ such that $2\wp_\infty(t) = \wp_\infty(c_1)$. Thus, $2t = c_1 + c$ or $2t = c_1$. We know from the proof of Proposition 6.7 that $\wp_y(c_y) = 0$, and $\wp_y(c_x)$ generates $\Phi_y(\mathbb{F}_y) \cong \Phi_y \cong \mathbb{Z}/(q+1)\mathbb{Z}$. If $2t = c_1$, then $2\wp_y(t)$ generates Φ_y. Since 2 divides $q+1$, the multiplication by 2 map is not surjective on Φ_y, so we get a contradiction. If $2t = c_1 + c$, then $2\wp_y(t) = \wp_y(c_x) + \frac{(q+1)}{2}\wp_y(c_x)$, which is still a generator of Φ_y, and we again arrive at a contradiction.

Now assume $q \equiv 1 \bmod 4$. In this case

$$\mathcal{C}_2 \cong (\mathbb{Z}/(q+1)\mathbb{Z})_2 \times (\mathbb{Z}/(q^2+1)\mathbb{Z})_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathcal{C}[2]$$

is generated by $\frac{q+1}{2} c_x$ and $\frac{q^2+1}{2} c_y$. If $\mathcal{T} \neq \mathcal{C}$, then there is $t \in \mathcal{T}$ of order 4 such that $2t \in \mathcal{C}[2]$. If $2t = c$ or $2t = \frac{q+1}{2} c_x$, then applying $\wp_y$ we see that $2\wp_y(t) \neq 0$. On the other hand, $\wp_y(t) \in (\Phi_y)_2 \cong \mathbb{Z}/2\mathbb{Z}$, so $2\wp_y(t) = 0$, which is a contradiction. Finally, suppose $2t = \frac{q^2+1}{2} c_y$, which implies $2\wp_x(t) \neq 0$. Since t is a rational point, $\wp_x(t) \in \Phi_x(\mathbb{F}_x)_2$. On the other hand, by Proposition 5.4,

$\Phi_x(\mathbb{F}_x)_2 \cong (\mathbb{Z}/(q^2+1)\mathbb{Z})_2 \cong \mathbb{Z}/2\mathbb{Z}$, which again leads to a contradiction. (Note that $\Phi_x \cong \mathbb{Z}/(q+1)(q^2+1)\mathbb{Z}$, so here we cannot just rely on Theorem 5.3.) $\square$

Corollary 7.14. $J(F) \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z}$. *For any $\ell \neq p$, the ℓ-primary part of the Tate-Shafarevich group $Ш(J)$ is trivial.*

Proof. Denote by $L(J,s)$ the L-function of J; see [26] for the definition. Let $f \in \mathcal{H}_0(\mathfrak{n}, \mathbb{C})$ be an eigenform for $\mathbb{T}(\mathfrak{n})$, normalized by $f^*(1) = 1$. The L-function of f is the sum

$$L(f,s) = \sum_{\mathfrak{m} \text{ pos. div.}} f^*(\mathfrak{m}) q^{-s\cdot \deg(\mathfrak{m})}$$

over all positive divisors on $\mathbb{P}^1_{\mathbb{F}_q}$; here $s \in \mathbb{C}$. Drinfeld's fundamental result [6, Thm. 2] implies that $L(J,s) = \prod_f L(f,s)$, where the product is over normalized $\mathbb{T}(xy)$-eigenforms in $\mathcal{H}_0(xy, \mathbb{C})$. It is known that $L(f,s)$ is a polynomial in q^{-s} of degree $\deg(\mathfrak{n}) - 3$; cf. [54, p. 227]. Thus, $L(J,s) = 1$. From the main theorem in [26] we conclude that $J(F) = \mathcal{T}$, $Ш(J)$ is a finite group, and the Birch and Swinnerton-Dyer conjecture holds for J. The claim $J(F) \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z}$ follows from Theorem 7.13. The ℓ-primary part of the Birch and Swinnerton-Dyer formula becomes the equality

$$1 = \frac{(\#Ш(J)_\ell)(\#\Phi_x(\mathbb{F}_x)_\ell)(\#\Phi_y(\mathbb{F}_y)_\ell)(\#\Phi_\infty(\mathbb{F}_\infty)_\ell)}{(\#\mathcal{T}_\ell)^2}.$$

We know all entries of this formula from Theorem 7.13 and its proof, except $\#Ш(J)_\ell$. This implies that $Ш(J)_\ell = 1$. $\square$

8. Kernel of the Eisenstein ideal

8.1. Shimura subgroup. Let $\mathfrak{n} \lhd A$ be a non-zero ideal. Consider the subgroup $\Gamma_1(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ consisting of matrices

$$\Gamma_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; a \equiv 1 \bmod \mathfrak{n}, c \equiv 0 \bmod \mathfrak{n} \right\}.$$

The quotient $\Gamma_1(\mathfrak{n}) \setminus \Omega$ is the rigid-analytic space associated to a smooth affine curve $Y_1(\mathfrak{n})_{F_\infty}$ over F_∞; cf. (3.1). This curve is the modular curve of isomorphism classes of pairs (ϕ, P), where ϕ is a Drinfeld A-module of rank 2 and $P \in \phi[\mathfrak{n}]$ is an element of exact order $\mathfrak{n}$. Let $Y_1(\mathfrak{n})_F$ be the canonical model of $Y_1(\mathfrak{n})_{F_\infty}$ over F, and $X_1(\mathfrak{n})_F$ be the smooth projective curve containing $Y_1(\mathfrak{n})_F$ as a Zariski open subvariety. Denote by $J_1(\mathfrak{n})$ the Jacobian variety of $X_1(\mathfrak{n})_F$. To simplify the notation, in this section we denote $\Gamma := \Gamma_0(\mathfrak{n})$ and $\Delta := \Gamma_1(\mathfrak{n})$. The map $w : \Gamma \to (A/\mathfrak{n})^\times$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \bmod \mathfrak{n}$ is a surjective homomorphism whose kernel is Δ. Hence Δ is a normal subgroup of Γ and $\Gamma/\Delta \cong (A/\mathfrak{n})^\times$. One deduces from the action of Γ on $\Omega \cup \mathbb{P}^1(F)$ an action of Γ on $X_1(\mathfrak{n})_{\mathbb{C}_\infty}$. The group Δ and the scalar matrices act trivially on $X_1(\mathfrak{n})_{\mathbb{C}_\infty}$, hence we have an action of the group $(A/\mathfrak{n})^\times/\mathbb{F}_q^\times$ on $X_1(\mathfrak{n})_{\mathbb{C}_\infty}$. This implies that there is a natural morphism $X_1(\mathfrak{n})_{\mathbb{C}_\infty} \to X_0(\mathfrak{n})_{\mathbb{C}_\infty}$ which is a Galois covering with Galois group $(A/\mathfrak{n})^\times/\mathbb{F}_q^\times$. This covering is in fact defined over F, as in terms of the moduli problems it is induced by $(\phi, P) \mapsto (\phi, \langle P\rangle)$, where $\langle P\rangle$ is the order-$\mathfrak{n}$ cyclic subgroup of ϕ generated by P. By the Picard functoriality we get a homomorphism $\pi : J_0(\mathfrak{n}) \to J_1(\mathfrak{n})$ defined over F, whose kernel $\mathcal{S}(\mathfrak{n})$ is the *Shimura subgroup* of $J_0(\mathfrak{n})$.

DEFINITION 8.1. Let Q be the subgroup of $(A/\mathfrak{n})^\times$ generated by the elements which satisfy $a^2 - ta + \kappa = 0 \bmod \mathfrak{n}$ for some $t \in \mathbb{F}_q$ and $\kappa \in \mathbb{F}_q^\times$. Denote $U = (A/\mathfrak{n})^\times/Q$.

THEOREM 8.2. *The Shimura subgroup $\mathcal{S}(\mathfrak{n})$, as a group scheme over F_∞, is canonically isomorphic to the Cartier dual U^* of U viewed as a constant group scheme.*

Proof. Denote by Γ^{ab} the abelianization of Γ and let $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$ be the maximal abelian torsion-free quotient of Γ. The inclusion $\Delta \hookrightarrow \Gamma$ induces a homomorphism $I : \overline{\Delta} \to \overline{\Gamma}$ (which is not necessarily injective). There is also a homomorphism $V : \overline{\Gamma} \to \overline{\Delta}$, the transfer map, such that $I \circ V : \overline{\Gamma} \to \overline{\Gamma}$ is the multiplication by $[\Gamma : \Delta]/(q-1)$; see [18, p. 71].
First, we note that the homomorphism $V : \overline{\Gamma} \to \overline{\Delta}$ is injective with torsion-free cokernel. Indeed, by [18, p. 72], there is a commutative diagram

$$\begin{tikzcd} \overline{\Gamma} \arrow[r, "j_\Gamma"] \arrow[d, "V"] & \mathcal{H}_0(\mathcal{T}, \mathbb{Z})^\Gamma \arrow[d] \\ \overline{\Delta} \arrow[r, "j_\Delta"] & \mathcal{H}_0(\mathcal{T}, \mathbb{Z})^\Delta \end{tikzcd}$$

where the right vertical map is the natural injection. This last homomorphism obviously has torsion-free cokernel. Since by [17] j_Γ and j_Δ are isomorphisms, the claim follows.
Next, by the results in Sections 6 and 7 of [18], there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{\Gamma} \arrow[r] \arrow[d, "V"] & \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times) \arrow[r] \arrow[d, "I^*"] & J_0(\mathfrak{n}) \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & \overline{\Delta} \arrow[r] & \mathrm{Hom}(\overline{\Delta}, \mathbb{C}_\infty^\times) \arrow[r] & J_1(\mathfrak{n}) \arrow[r] & 0, \end{tikzcd}$$

where the top row is the uniformization in (4.3), the bottom row is a similar uniformization for $J_1(\mathfrak{n})$ constructed in [18] for an arbitrary congruence group, and the middle vertical map I^* is the dual of $I : \overline{\Delta} \to \overline{\Gamma}$. This diagram gives an exact sequence of group schemes

$$0 \to (\overline{\Gamma}/I\overline{\Delta})^* \to \mathcal{S}(\mathfrak{n}) \to \overline{\Delta}/V\overline{\Gamma}.$$

Since there are no non-trivial maps from an abelian variety to a discrete group, $\mathcal{S}(\mathfrak{n})$ is finite. On the other hand, by the previous paragraph, $\overline{\Delta}/V\overline{\Gamma}$ is torsion-free, so

$$\mathcal{S}(\mathfrak{n}) \cong (\overline{\Gamma}/I\overline{\Delta})^*.$$

Denote by Γ^c the commutator subgroup of Γ. The fact that Γ/Δ is abelian implies $\Gamma^c \subset \Delta$. Hence $\Gamma/\Delta \cong \Gamma^{\mathrm{ab}}/(\Delta/\Gamma^c)$. It follows from Corollary 1 on page 55 of [50] that $(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$ is generated by the images of finite order elements of Γ in Γ^{ab}. Since $\overline{\Gamma}$ is the quotient of Γ^{ab} by the torsion subgroup $(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$, we conclude that $\overline{\Gamma}/I\overline{\Delta}$ is the quotient of Γ/Δ by the subgroup generated by the images of finite order elements of Γ.

An element $\gamma \in \mathrm{GL}_2(A)$ has finite order if and only if $\mathrm{Tr}(\gamma) \in \mathbb{F}_q$. Therefore, if $\gamma \in \Gamma$ has finite order then $w(\gamma)$ satisfies the equation $a^2 - ta + \kappa = 0$, where $t = \mathrm{Tr}(\gamma) \in \mathbb{F}_q$ and $\kappa = \det(\gamma) \in \mathbb{F}_q^\times$. Conversely, suppose $\bar{a} \in (A/\mathfrak{n})^\times$ satisfies $\bar{a}^2 - t\bar{a} + \kappa = 0$ for some $t \in \mathbb{F}_q$ and $\kappa \in \mathbb{F}_q^\times$. Fix some $a \in A$ reducing to $\bar{a}$ modulo $\mathfrak{n}$. Since $a(t-a) \equiv \kappa \pmod{\mathfrak{n}}$, there exists $c \in A$ such that $a(t-a) = \kappa + c\mathfrak{n}$. The matrix $\gamma = \begin{pmatrix} a & 1 \\ c\mathfrak{n} & t-a \end{pmatrix}$ has determinant κ and trace t. It is clear that $\gamma \in \Gamma$ is a torsion element, and $w(\gamma) = \bar{a}$. We conclude that $\overline{\Gamma}/I\overline{\Delta} \cong (A/\mathfrak{n})^\times/Q$. □

Remark 8.3. The previous theorem is the function field analogue of Theorem 1 in [30].

Lemma 8.4. *Assume* $\mathfrak{n}$ *is square-free, so that the order of* $(A/\mathfrak{n})^\times$ *is not divisible by* p *and* $\mathcal{S}(\mathfrak{n})$ *is étale over* F_∞. *The Shimura subgroup* $\mathcal{S}(\mathfrak{n})$ *extends to a finite flat subgroup scheme of* $\mathcal{J}^0_{\mathcal{O}_\infty}$.

Proof. From the proof of Theorem 8.2, $\mathcal{S}(\mathfrak{n})$ is canonically a subgroup of $\mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)$. It is easy to see that $\mathcal{S}(\mathfrak{n})$ actually lies in $\mathrm{Hom}(\overline{\Gamma}, (\mathcal{O}_\infty^{\mathrm{nr}})^\times)$. On the other hand, as is implicit in the proof of Corollary 2.11 in [11], there is a canonical isomorphism $\mathcal{J}^0(\mathcal{O}_\infty^{\mathrm{nr}}) \cong \mathrm{Hom}(\overline{\Gamma}, (\mathcal{O}_\infty^{\mathrm{nr}})^\times)$. □

Proposition 8.5. *Assume* $\mathfrak{n}$ *is square-free. The Shimura subgroup* $\mathcal{S}(\mathfrak{n})$*, as a group scheme over* F*, is an étale group scheme whose Cartier dual is canonically isomorphic to* U *viewed as a constant group scheme. The endomorphisms* $T_\mathfrak{p} - |\mathfrak{p}| - 1$ *and* $W_\mathfrak{n} + 1$ *of* $J_0(\mathfrak{n})$ *annihilate* $\mathcal{S}(\mathfrak{n})$*; here* $\mathfrak{p} \lhd A$ *is any prime not dividing* $\mathfrak{n}$.

Proof. The covering $\pi : X_1(\mathfrak{n})_F \to X_0(\mathfrak{n})_F$ can ramify only at the elliptic points and the cusps of $X_0(\mathfrak{n})_F$. (By definition, an elliptic point on $X_0(\mathfrak{n})_{\mathbb{C}_\infty}$ is the image of $z \in \Omega$ whose stabilizer in $\mathrm{GL}_2(A)$ is strictly larger than $\mathbb{F}_q^\times$.) The proof of Theorem 8.2 shows that U is the Galois group of the maximal abelian unramified covering $X_F \to X_0(\mathfrak{n})_F$ through which π factorizes. Now [30, Prop. 6] implies that $\mathcal{S}(\mathfrak{n})$ as a group scheme over F is isomorphic to $\mathrm{Hom}(U, \bar{F}^\times)$.

The Jacobian $J_0(\mathfrak{n})$ has good reduction at $\mathfrak{p}$. Since $\mathcal{S}(\mathfrak{n})$ has order coprime to p and is unramified at $\mathfrak{p}$, the reduction map injects $\mathcal{S}(\mathfrak{n})$ into $J_0(\mathfrak{n})(\overline{\mathbb{F}}_\mathfrak{p})$. Let $\mathrm{Frob}_\mathfrak{p}$ be the Frobenius endomorphism of $J_0(\mathfrak{n})_{\mathbb{F}_\mathfrak{p}}$. The Hecke operator $T_\mathfrak{p}$ satisfies the Eichler-Shimura relation:

$$\mathrm{Frob}_\mathfrak{p}^2 - T_\mathfrak{p} \cdot \mathrm{Frob}_\mathfrak{p} + |\mathfrak{p}| = 0.$$

Since $\mathcal{S}(\mathfrak{n})^*$ is constant, $\mathrm{Frob}_\mathfrak{p}$ acts on $\mathcal{S}(\mathfrak{n})$ by multiplication by $|\mathfrak{p}|$. Therefore, the endomorphisms $|\mathfrak{p}|(T_\mathfrak{p} - |\mathfrak{p}| - 1)$ annihilates $\mathcal{S}(\mathfrak{n})$. Since the reduction map commutes with the action of Hecke algebra and the multiplication by $|\mathfrak{p}|$ is an automorphism of $\mathcal{S}(\mathfrak{n})$, we conclude that $T_\mathfrak{p} - |\mathfrak{p}| - 1$ annihilates $\mathcal{S}(\mathfrak{n})$.

Note that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$\begin{pmatrix} 0 & -1 \\ \mathfrak{n} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1/\mathfrak{n} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c/\mathfrak{n} \\ -b\mathfrak{n} & a \end{pmatrix}.$$

Since d, up to an element of $\mathbb{F}_q^\times$, is the inverse of a modulo $\mathfrak{n}$, this calculation shows that $W_\mathfrak{n}$ acts on the group $(\Gamma/\Delta)/\mathbb{F}_q^\times$ by $u \mapsto u^{-1}$, so it acts as -1 on $\mathcal{S}(\mathfrak{n})$. $\square$

Theorem 8.6. *Assume $\mathfrak{n}$ is square-free, and let $\mathfrak{n} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$ be its prime decomposition. As an abelian group, $\mathcal{S}(\mathfrak{n})$ is isomorphic to*

$$\begin{cases} \prod_{i=1}^s \left(\mathbb{Z}\Big/\frac{|\mathfrak{p}_i|-1}{q-1}\mathbb{Z}\right), & \textit{if some } \mathfrak{p}_i \textit{ has odd degree} \\ \prod_{i=1}^s \left(\mathbb{Z}\Big/\frac{|\mathfrak{p}_i|-1}{q^2-1}\mathbb{Z}\right), & \textit{if } q \textit{ is even and all } \mathfrak{p}_i \textit{ have even degrees} \\ \prod_{i=1}^s \left(\mathbb{Z}\Big/\frac{2(|\mathfrak{p}_i|-1)}{q^2-1}\mathbb{Z}\right)/\mathrm{diag}\,(\mathbb{Z}/2\mathbb{Z}), & \textit{if } q \textit{ is odd and all } \mathfrak{p}_i \textit{ have even degrees} \end{cases}$$

where $\mathrm{diag} : \mathbb{Z}/2\mathbb{Z} \to \prod_{i=1}^s \left(\mathbb{Z}\Big/\frac{2(|\mathfrak{p}_i|-1)}{q^2-1}\mathbb{Z}\right)$ *is the diagonal embedding.*

Proof. First, we claim that the image of Q under the isomorphism $(A/\mathfrak{n})^\times \cong \prod_{i=1}^s \mathbb{F}_{\mathfrak{p}_i}^\times$ given by

$$a \mapsto (a \bmod \mathfrak{p}_1, \cdots, a \bmod \mathfrak{p}_s)$$

contains the subgroup $\prod_{i=1}^s \mathbb{F}_q^\times$. Indeed, let $\alpha \in \mathbb{F}_q^\times$ be arbitrary, and consider $(1, \dots, \alpha, \dots, 1)$ with α in the ith position. This element is in the image of Q since $a \in (A/\mathfrak{n})^\times$ mapping to it satisfies the quadratic equation $(x-\alpha)(x-1) = 0$ modulo $\mathfrak{n}$.

Suppose $a \in Q$ is a zero of $f(x) := x^2 - tx + \kappa$. Then a satisfies the same equation modulo $\mathfrak{p}_i$. If $\deg(\mathfrak{p}_i)$ is odd, then $f(x)$ must be reducible over $\mathbb{F}_q$. This implies that the image of a in $\mathbb{F}_{\mathfrak{p}_i}^\times$ lies in the subfield $\mathbb{F}_q$. On the other hand, since $a \bmod \mathfrak{p}_j$ $(1 \leq j \leq s)$ satisfies the same reducible quadratic equation, the image of a in $\mathbb{F}_{\mathfrak{p}_j}$ also lies in its subfield $\mathbb{F}_q$, and we conclude that $Q \cong \prod_{i=1}^s \mathbb{F}_q^\times$.

Now suppose all $\mathfrak{p}_i$'s have even degrees. Let $\alpha \in \mathbb{F}_{q^2}^\times$ be arbitrary. The elements $(\alpha, \dots, \alpha)$ and $(\alpha, \dots, \alpha^q, \dots, \alpha)$, with α^q in the ith position $(1 \leq i \leq s)$, are in the image of Q. Indeed, $a \in (A/\mathfrak{n})^\times$ which maps to such an element satisfies $(x-\alpha)(x-\alpha^q) = 0$, and this polynomial has coefficients in $\mathbb{F}_q$. Therefore, $(1, \dots, \alpha^{q-1}, \dots, 1) \in Q$. Since $\alpha^{q+1} \in \mathbb{F}_q^\times$, we get that $(1, \dots, \alpha^2, \dots, 1) \in Q$. Thus, $\mathrm{diag}(\mathbb{F}_{q^2}^\times) \cdot \prod_{i=1}^s (\mathbb{F}_{q^2}^\times)^2$ is a subgroup of Q, where $(\mathbb{F}_{q^2}^\times)^2 = \{\alpha^2 \mid \alpha \in \mathbb{F}_{q^2}^\times\}$ and $\mathrm{diag} : \mathbb{F}_{q^2}^\times \hookrightarrow \prod_{i=1}^s \mathbb{F}_{\mathfrak{p}_i}^\times$ is the diagonal embedding. If $f(x)$ is reducible, then $a \in \prod_{i=1}^s \mathbb{F}_q^\times$. If $f(x)$ is irreducible, then $a = (\alpha_{i_1}, \dots, \alpha_{i_s})$ with $\alpha_{i_1}, \dots, \alpha_{i_s} \in \{\alpha, \alpha^q\}$, where α, α^q are the roots of $f(x)$. We can write

$$a = (\alpha_{i_1}, \dots, \alpha_{i_1}) \cdot \left(1, \frac{\alpha_{i_2}}{\alpha_{i_1}}, \dots, \frac{\alpha_{i_s}}{\alpha_{i_1}}\right).$$

The first element in this product is in $\mathrm{diag}(\mathbb{F}_{q^2}^\times)$ and the second is in $\prod_{i=1}^s (\mathbb{F}_{q^2}^\times)^2$ since each $\alpha_{i_j}/\alpha_{i_1}$ is either 1 or $\alpha^{\pm(q-1)}$. We conclude that $\mathrm{diag}(\mathbb{F}_{q^2}^\times) \cdot \prod_{i=1}^s (\mathbb{F}_{q^2}^\times)^2 = Q$. Since $\mathcal{S}(\mathfrak{n})$ is isomorphic to $(A/\mathfrak{n})^\times/Q$, the claim follows. □

Example 8.7. If $\mathfrak{p} \lhd A$ is prime, then $\mathcal{S}(\mathfrak{p})$ is cyclic of order $\frac{|\mathfrak{p}|-1}{q-1}$ (resp. $\frac{|\mathfrak{p}|-1}{q^2-1}$) is $\deg(\mathfrak{p})$ if odd (resp. even). This is also proved in [38] by a different method.

Example 8.8. $\mathcal{S}(xy)$ is cyclic of order $q+1$.

Example 8.9. Assume $\mathfrak{p}$ and $\mathfrak{q}$ are two distinct primes of degree 2. Then $\mathcal{S}(\mathfrak{p}\mathfrak{q}) \cong \mathbb{Z}/2\mathbb{Z}$ (resp. 0) if q is odd (resp. even).

Definition 8.10. Let $\mathfrak{n} \lhd A$ be a non-zero ideal and denote $J = J_0(\mathfrak{n})$. The *kernel of the Eisenstein ideal* $J[\mathfrak{E}(\mathfrak{n})]$ is the intersection of the kernels of all elements of $\mathfrak{E}(\mathfrak{n})$ acting on $J(\bar{F})$. Given a prime $\ell \neq p$, the action of $\mathbb{T}(\mathfrak{n})$ on J induces an action on $J[\ell^n]$ $(n \geq 1)$ and the ℓ-divisible group $J_\ell := \varinjlim J[\ell^n]$, so one can also define $J_\ell[\mathfrak{E}(\mathfrak{n})]$. From the proof of Theorem 6.1, we see that $J[\mathfrak{E}(\mathfrak{n})]$ is a finite group scheme over F, as $J[\mathfrak{E}(\mathfrak{n})] \subseteq J[\eta_\mathfrak{p}]$ for any $\mathfrak{p} \nmid \mathfrak{n}$, where $\eta_\mathfrak{p} = T_\mathfrak{p} - |\mathfrak{p}| - 1$. By Lemma 7.1 and Proposition 8.5, if $\mathfrak{n}$ is square-free then $\mathcal{S}(\mathfrak{n})_\ell$ and $\mathcal{C}(\mathfrak{n})_\ell$ are in $J_\ell[\mathfrak{E}(\mathfrak{n})]$.

Definition 8.11. We say that a group scheme H over the base S is *μ-type* if it is finite, flat and its Cartier dual is a constant group scheme over S; H is *pure* if it is the direct sum of a constant and μ-type group schemes; H is *admissible* if it is finite, étale and has a filtration by groups schemes such that the successive quotients are pure.

Lemma 8.12. *Assume $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$ is a product of two distinct primes. Let $s_{\mathfrak{p},\mathfrak{q}} = \gcd(|\mathfrak{p}|+1, |\mathfrak{q}|+1)$. If ℓ does not divide $q(q-1)s_{\mathfrak{p},\mathfrak{q}}$, then $J_\ell[\mathfrak{E}(\mathfrak{n})]$ is unramified except possibly at $\mathfrak{p}$ and $\mathfrak{q}$, and there is an exact sequence of group schemes over F*

$$0 \to \mathcal{C}(\mathfrak{n})_\ell \to J_\ell[\mathfrak{E}(\mathfrak{n})] \to \mathcal{M}_\ell \to 0,$$

where $\mathcal{M}_\ell$ is μ-type and contains $\mathcal{S}(\mathfrak{n})_\ell$.

Proof. Since $\mathcal{C}(\mathfrak{n})_\ell$ is fixed by G_F and is invariant under the action of $\mathbb{T}(\mathfrak{n})$, the quotient $\mathcal{M}_\ell := J_\ell[\mathfrak{E}(\mathfrak{n})]/\mathcal{C}(\mathfrak{n})_\ell$ is naturally a $\mathbb{T}(\mathfrak{n}) \times G_F$-module . From the uniformization sequence (4.3) and the isomorphism (4.4), for any $n \geq 1$ we obtain the exact sequence of $\mathbb{T}(\mathfrak{n}) \times G_{F_\infty}$-modules

$$0 \to D[\ell^n] \to J[\ell^n] \to \mathcal{H}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^n\mathbb{Z}) \to 0,$$

where $D := \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)$. Considering the parts annihilated by $\mathfrak{E}(\mathfrak{n})$, we obtain the exact sequence

$$0 \to D[\ell^n, \mathfrak{E}(\mathfrak{n})] \to J[\ell^n, \mathfrak{E}(\mathfrak{n})] \to \mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

From the proof of Theorem 7.3 we know that if ℓ does not divide $q(q-1)s_{\mathfrak{p},\mathfrak{q}}$ and n is sufficiently large, then $\mathcal{C}(\mathfrak{n})_\ell \subseteq J[\ell^n, \mathfrak{E}(\mathfrak{n})]$ and $\mathcal{C}(\mathfrak{n})_\ell$ maps isomorphically onto $\mathcal{E}_{00}(\mathfrak{n}, \mathbb{Z}/\ell^n\mathbb{Z})$. Hence for $n \gg 0$ the above sequence is exact also on the right:

$$0 \to D[\ell^n, \mathfrak{E}(\mathfrak{n})] \to J[\ell^n, \mathfrak{E}(\mathfrak{n})] \to \mathcal{C}(\mathfrak{n})_\ell \to 0.$$

Since the map of $\mathbb{T}(\mathfrak{n}) \times G_{F_\infty}$-modules $J[\ell^n, \mathfrak{E}(\mathfrak{n})] \to \mathcal{C}(\mathfrak{n})_\ell$ has a retraction, we get the splitting $J_\ell[\mathfrak{E}(\mathfrak{n})] \cong D_\ell[\mathfrak{E}(\mathfrak{n})] \oplus \mathcal{C}(\mathfrak{n})_\ell$ over F_∞. This shows that $\mathcal{M}_\ell \cong D_\ell[\mathfrak{E}(\mathfrak{n})]$ is μ-type over F_∞, and $J_\ell[\mathfrak{E}(\mathfrak{n})]$ is unramified at ∞. Since $\mathcal{S}(\mathfrak{n})_\ell \subseteq D_\ell[\mathfrak{E}(\mathfrak{n})]$ (cf. Lemma 8.4), we see that $\mathcal{S}(\mathfrak{n})_\ell$ is a subgroup scheme of $\mathcal{M}_\ell$.

By the Néron-Ogg-Shafarevich criterion $J_\ell[\mathfrak{E}(\mathfrak{n})]$ is unramified at all finite places except possibly at $\mathfrak{p}$ and $\mathfrak{q}$. Let $\mathfrak{l} \lhd A$ be any prime not equal to $\mathfrak{p}$ or $\mathfrak{q}$. Since $T_\mathfrak{l}$ acts on $J_\ell[\mathfrak{E}(\mathfrak{n})]$ by multiplication by $|\mathfrak{l}|+1$, the Eichler-Shimura congruence relation implies that the action of $\mathrm{Frob}_\mathfrak{l}$ on $J_\ell[\mathfrak{E}(\mathfrak{n})]$ satisfies the relation $(\mathrm{Frob}_\mathfrak{l}-1)(\mathrm{Frob}_\mathfrak{l}-|\mathfrak{l}|) = 0$. Now one can use Mazur's argument [33, Prop. 14.1] to show that $J_\ell[\mathfrak{E}(\mathfrak{n})]$ is admissible over $U = \mathbb{P}^1_{\mathbb{F}_q} - \mathfrak{p} - \mathfrak{q}$; cf. the proof of Proposition 10.8 in [38]. Since the quotient map $J_\ell[\mathfrak{E}(\mathfrak{n})] \to \mathcal{M}_\ell$ is compatible with the action of G_F and $\mathbb{T}(\mathfrak{n})$, $\mathcal{M}_\ell$ is also admissible over $U = \mathbb{P}^1_{\mathbb{F}_q} - \mathfrak{p} - \mathfrak{q}$. On the other hand, $\mathcal{M}_\ell$ is μ-type over F_∞, so all Jordan-Hölder components of $\mathcal{M}_\ell$ over U must be isomorphic to μ_ℓ. We say that $\mathfrak{l}_1, \mathfrak{l}_2$ is a pair of good primes, if $\mathfrak{l}_i \neq \mathfrak{p}, \mathfrak{q}$, $|\mathfrak{l}_i| - 1$ is not divisible by ℓ, and the images of $(\mathfrak{l}_1, \mathfrak{l}_1)$ and $(\mathfrak{l}_2, \mathfrak{l}_2)$ in $(\mathbb{F}_\mathfrak{p}^\times/(\mathbb{F}_\mathfrak{p}^\times)^\ell \times \mathbb{F}_\mathfrak{q}^\times/(\mathbb{F}_\mathfrak{q}^\times)^\ell)/\mathbb{F}_q^\times$ generate this group; here $(\mathbb{F}_\mathfrak{p}^\times)^\ell$ is the subgroup of $\mathbb{F}_\mathfrak{p}^\times$ consisting of ℓth powers, and $\mathbb{F}_q^\times$ is embedded diagonally into $\mathbb{F}_\mathfrak{p}^\times \times \mathbb{F}_\mathfrak{q}^\times$. The Chebotarev density theorem shows that a pair of good primes exists. The operator $(\mathrm{Frob}_{\mathfrak{l}_i} - |\mathfrak{l}_i|)(\mathrm{Frob}_{\mathfrak{l}_i} - 1)$ annihilates $\mathcal{M}_\ell$. On the other hand, since the semi-simplification of $\mathcal{M}_\ell$ is isomorphic to $(\mu_\ell)^n$ for some n and ℓ does not divide $|\mathfrak{l}_i|-1$, the operator $\mathrm{Frob}_{\mathfrak{l}_i}-1$ must be invertible on $\mathcal{M}_\ell$. Therefore, $\mathcal{M}_\ell$ is annihilated by $\mathrm{Frob}_{\mathfrak{l}_i} - |\mathfrak{l}_i|$. This implies that $\mathcal{M}_\ell$ is μ-type over $F_{\mathfrak{l}_i}$. Finally, one can argue as in Proposition 10.7 in [38] to conclude that $\mathcal{M}_\ell$ is μ-type over F. $\square$

Remark 8.13. When $\mathfrak{n} = \mathfrak{p}$ is prime and ℓ does not divide $q-1$, then $J_\ell[\mathfrak{E}(\mathfrak{p})] = \mathcal{C}(\mathfrak{p})_\ell \oplus \mathcal{S}(\mathfrak{p})_\ell$; see [38]. As we will see in the next section, for a composite level, $\mathcal{M}_\ell$ can be larger than $\mathcal{S}(\mathfrak{n})_\ell$ and the sequence in Lemma 8.12 need not split over F.

8.2. Special case. Let $\mathfrak{p} \lhd A$ be a prime of degree 3. Then

(1) The rational torsion subgroup $\mathcal{T}(\mathfrak{p})$ of $J_0(\mathfrak{p})$ is equal to the cuspidal divisor group $\mathcal{C}(\mathfrak{p}) \cong \mathbb{Z}/(q^2+q+1)\mathbb{Z}$;
(2) The maximal μ-type étale subgroup scheme of $J_0(\mathfrak{p})$ is the Shimura subgroup $\mathcal{S}(\mathfrak{p}) \cong (\mathbb{Z}/(q^2+q+1)\mathbb{Z})^*$;
(3) The kernel $J[\mathfrak{E}(\mathfrak{p})]$ is everywhere unramified and has order $(q^2+q+1)^2$.

Here (1) and (2) follow from the main results in [38], and (3) follows from [38] and [40].

We proved that (see Theorem 7.13)

$$\mathcal{T}(xy) = \mathcal{C}(xy) \cong \mathbb{Z}/(q+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z},$$

so the analogue of (1) is true for $\mathfrak{n} = xy$, although $\mathcal{C}(xy)$ is not cyclic if q is odd. Interestingly, even for small composite levels such as $\mathfrak{n} = xy$, the analogues of (2) and (3) are no longer true. For simplicity, we will omit the level xy from notation. First of all, $J_\ell[\mathfrak{E}]$ can be ramified. To see this, let $q = 2^s$ with s odd, $x = T + 1$, $y = T^2 + T + 1$. Consider the elliptic curve E over F given by the Weierstrass equation

$$E : Y^2 + TXY + Y = X^3 + X^2 + T.$$

This curve embeds into J according to [41, Prop. 7.10]. There is an exact sequence of G_F-modules

$$0 \to \mathbb{Z}/5\mathbb{Z} \to E[5] \to \mu_5 \to 0.$$

The component group of E at y is trivial, so $E[5]$ is ramified at y and the above sequence does not split over F. On the other hand, from the Eichler-Shimura congruence relations we see that $E[5] \subseteq J_5[\mathfrak{E}]$. Thus, $J_5[\mathfrak{E}]$ is ramified at y.
Next, the Shimura subgroup $\mathcal{S} \cong (\mathbb{Z}/(q+1)\mathbb{Z})^*$ has smaller order than the cuspidal divisor group, and $\mathcal{S}$ is not the maximal μ-type étale subgroup scheme of J when q is odd. Indeed, $\mathcal{C}[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is μ-type but is not cyclic.
In the remainder of this section we will show that even though (2) and (3) fail for the xy-level, these properties do not fail "too much".

Proposition 8.14. *Assume q is odd. Then $\mathcal{S}_2 + \mathcal{C}[2] \cong \mathcal{S}_2 \times \mathbb{Z}/2\mathbb{Z}$ is the maximal 2-primary μ-type subgroup of $J(\bar{F})$.*

Proof. First, we show that the canonical specialization map $\wp_y : \mathcal{S} \to \Phi_y$ is an isomorphism. Since $\mathcal{S}$ and Φ_y have the same order, it is enough to show that the induced morphism $\mathcal{J}_0(xy)^0_{\overline{\mathbb{F}}_y} \to \mathcal{J}_1(xy)^0_{\overline{\mathbb{F}}_y}$ on the connected components of the identity of the special fibres of the Néron models of $J_0(xy)$ and $J_1(xy)$ at y is injective. Here we argue as in Proposition 11.9 in [33]. Let Gr_0 be the dual graph of $X_0(xy)_{\overline{\mathbb{F}}_y}$; this graph has two vertices corresponding to the two irreducible components of $X_0(xy)_{\overline{\mathbb{F}}_y}$ joined by $q+1$ edges corresponding to the supersingular points. Let Gr_1 be the dual graph of the special fibre of the model of $X_1(xy)$ over $\mathcal{O}_y$ constructed in [51]. The underlying reduced subscheme $X_1(xy)^{\mathrm{red}}_{\overline{\mathbb{F}}_y}$ has two irreducible components intersecting at the supersingular points. Since the Jacobian $J_0(xy)$ has toric reduction at y, there is a canonical isomorphism (cf. [3, p. 246])

$$\mathcal{J}_0(xy)^0_{\overline{\mathbb{F}}_y} \cong H^1(\mathrm{Gr}_0, \mathbb{Z}) \otimes \mathbb{G}_{m,\overline{\mathbb{F}}_y}.$$

Similarly, the toric part of $\mathcal{J}_1(xy)^0_{\overline{\mathbb{F}}_y}$ is canonically isomorphic to $H^1(\mathrm{Gr}_1, \mathbb{Z}) \otimes \mathbb{G}_{m,\overline{\mathbb{F}}_y}$. There is an obvious map $\mathrm{Gr}_1 \to \mathrm{Gr}_0$ and our claim reduces to showing that this map induces a surjection $H_1(\mathrm{Gr}_1, \mathbb{Z}) \to H_1(\mathrm{Gr}_0, \mathbb{Z})$. This is clear from the description of the graphs Gr_0 and Gr_1.
Let $\mathcal{M}_2$ be the maximal 2-primary μ-type subgroup of $J(\bar{F})$. It is clear that $\mathcal{S}_2 + \mathcal{T}[2] \subseteq \mathcal{M}_2$ and $\mathcal{M}_2[2] = \mathcal{T}[2]$. Similar to the notation in the proof of Theorem 7.13, let $\mathcal{M}_2^0$ denote the intersection of $\mathcal{M}_2$ with $\mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)$.

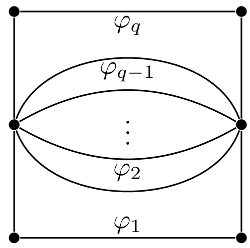

Figure 5. $(\Gamma_0(xy)\setminus\mathscr{T})^0$

Note that $\mathcal{S}_2\subseteq\mathcal{M}_2^0$. Moreover, we must have $\mathcal{S}[2]=\mathcal{M}_2^0[2]$ since $\mathcal{M}_2^0[2]\subseteq\mathcal{T}^0[2]=\mathcal{C}^0[2]=\mathcal{S}[2]$. Thus, if $\mathcal{S}_2\neq\mathcal{M}_2^0$, then there exists $P\in\mathcal{M}_2^0$ such that $2P$ generates $\mathcal{S}_2\cong(\mathbb{Z}/(q+1)\mathbb{Z})_2^*$. Since $\wp_y:\mathcal{S}\to\Phi_y$ is an isomorphism, $2\wp_y(P)$ generates $\mathbb{Z}/(q+1)\mathbb{Z}$. On the other hand, 2 divides $q+1$, so we get a contradiction. We conclude that there is an exact sequence

$$0\to\mathcal{S}_2\to\mathcal{M}_2\to\mathbb{Z}/2^n\mathbb{Z}\to 0$$

for some $n\geq 1$, where $\mathbb{Z}/2^n\mathbb{Z}$ is the image of $\mathcal{M}_2$ in $\Phi_\infty\cong\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$ (see Corollary 8.15). Since $\mathcal{M}_2[2]=\mathcal{T}[2]\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, the above sequence splits as a sequence of abelian groups. Moreover, since $\mathcal{S}_2$ is G_F-invariant and $\mathcal{M}_2^*$ is constant, the sequence splits as a sequence of F-groups schemes: $\mathcal{M}_2\cong\mathcal{S}_2\oplus\mathbb{Z}/2^n\mathbb{Z}$. If $q\equiv 3 \pmod 4$, then $\mathbb{Z}/2^n\mathbb{Z}$ is μ-type if and only if $n\leq 1$, which implies $\mathcal{M}_2=\mathcal{S}_2+\mathcal{T}[2]$. If $q\equiv 1 \pmod 4$ and $n\geq 2$, then $\mathcal{T}=\mathcal{C}$ contains a point of order 4, which is a contradiction. $\square$

The point of the previous proposition is that even though $\mathcal{S}_2$ is not the maximal 2-primary μ-type subgroup of J, it is not far from it. Next, we will show that under a reasonable assumption $\mathcal{S}_\ell$ is the maximal ℓ-primary μ-type subgroup of J for any odd $\ell\neq p$, and the order of $J_\ell[\mathfrak{E}]$ is $(\#\mathcal{C}_\ell)^2$ for any $\ell\neq p$, which is of course very similar to (2) and (3).

We start by showing that Φ_∞ is annihilated by $\mathfrak{E}$. This almost follows from Theorem 5.5 in [41], since according to that theorem the canonical specialization map $\wp_\infty:\mathcal{C}\to\Phi_\infty$ is an isomorphism if q is even and has cokernel $\mathbb{Z}/2\mathbb{Z}$ if q is odd. On the other hand, $\wp_\infty$ is $\mathbb{T}$-equivariant and $\mathcal{C}$ is annihilated by $\mathfrak{E}$. Below we give an alternative direct argument which works for any characteristic and does not use the cuspidal divisor group $\mathcal{C}$.

Choose the cycles $\varphi_1,...,\varphi_q$ as a basis of $H_1(\Gamma_0(xy)\ \setminus\mathscr{T},\mathbb{Z})\cong\mathcal{H}_0(xy,\mathbb{Z})$; see Figure 5. We assume that all these cycles are oriented counterclockwise. Let $\psi_1,...,\psi_q$ be the dual basis of $\mathrm{Hom}(\mathcal{H}_0(xy,\mathbb{Z}),\mathbb{Z})$. The map ι in Theorem 5.5

sends $\varphi \in \mathcal{H}_0(xy, \mathbb{Z})$ to $\sum_{i=1}^{q}(\varphi, \varphi_i)\psi_i$, where for $1 \leq j \leq i \leq q$

$$(\varphi_i, \varphi_j) = \begin{cases} q+2, & \text{if } i=j=1 \text{ or } q, \\ 2, & \text{if } 1 < i = j < q, \\ -1, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

This follows from [11, Thm. 2.8] and a calculation of the stabilizers of edges of $(\Gamma_0(xy) \setminus \mathscr{T})^0$ (see the proof of Proposition 2.21). Hence the component group Φ_∞ can be explicitly described as follows: It is the quotient of the free abelian group $\bigoplus_{i=1}^q \mathbb{Z}\psi_i$ by the relations

$$\begin{aligned} (q+2)\psi_q - \psi_{q-1} &= 0, \\ -\psi_{i+1} + 2\psi_i - \psi_{i-1} &= 0, \quad \text{for } 1 \leq i \leq q-1, \\ -\psi_2 + (q+2)\psi_1 &= 0. \end{aligned}$$

From the last $q-1$ relations we get

$$\psi_i = \big((i-1)q + i\big)\psi_1, \quad 1 \leq i \leq q.$$

On the other hand, by the first relation, $\psi_{q-1} = (q+2)\psi_q = q^2(q+2)\psi_1$. Therefore $q^2(q+2)\psi_1 = (q^2 - q - 1)\psi_1$ and

$$(q^2+1)(q+1)\psi_1 = 0.$$

We conclude that

Corollary 8.15. *$\Phi_\infty \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$ is generated by the image of ψ_1.*

Multiplication by $N = (q^2+1)(q+1)$ on the exact sequence in Theorem 5.5 and the snake lemma give an embedding $\delta_N : \Phi_\infty \hookrightarrow \mathcal{H}_{00}(xy, \mathbb{Z}/N\mathbb{Z})$.

Lemma 8.16. *Let $\bar{\psi}_1$ be the image of ψ_1 in Φ_∞. Then $\delta_N(\bar{\psi}_1) = -(q+1)E_x + (q^2+1)E_y + qE_{(x,y)}$, the generator of $\mathcal{E}_{00}(xy, \mathbb{Z}/N\mathbb{Z})$. In particular, the component group Φ_∞ is annihilated by the Eisenstein ideal.*

Proof. Note that

$$N\psi_1 = \sum_{i=1}^{q} \big((i-1)q + i\big)\iota(\varphi_i) \in \mathrm{Hom}(\mathcal{H}_0(xy, \mathbb{Z}), \mathbb{Z}).$$

Hence $\delta_N(\bar{\psi}_1) = \sum_{i=1}^q \big((i-1)q + i\big)\varphi_i \bmod N \in \mathcal{H}_{00}(xy, \mathbb{Z}/N\mathbb{Z})$. Viewing the cycles $\varphi_1, ..., \varphi_q$ as harmonic cochains in $\mathcal{H}_0(xy, \mathbb{Z})$, it is observed that

(i)

$$\varphi_1(a_1) = 1,\ \varphi_1(a_4) = -1,\ \varphi_1(a_6) = 1-q,\ \varphi_1(b_1) = 1,$$
$$\varphi_1(a_2) = \varphi_1(a_3) = \varphi_1(a_5) = \varphi_1(b_u) = 0,\ 2 \leq u \leq q-1;$$

(ii)

$$\varphi_q(a_1) = \varphi_q(a_4) = \varphi_q(a_6) = \varphi_q(b_u) = 0,\ 1 \leq u \leq q-2,$$
$$\varphi_q(a_2) = 1,\ \varphi_q(a_3) = -1,\ \varphi_q(a_5) = q-1,\ \varphi_1(b_{q-1}) = -1.$$

(iii) for $2 \leq j \leq q-1$,

$$\varphi_j(a_i) = \varphi_j(b_u) = 0,\ 1 \leq i \leq 6,\ u \neq j-1, j; \quad \varphi_j(b_j) = -\varphi_j(b_{j-1}) = 1.$$

Here $a_i,\ b_u$ are the oriented edges of the graph $(\Gamma_0(xy) \setminus \mathscr{T})^0$ in Figure 4. Therefore $\delta_N(\bar{\psi}_1)(a_1) = 1$ and $\delta_N(\bar{\psi}_1)(b_u) = -(q+1)$, $1 \leq u \leq q-1$.
On the other hand, let $f = -(q+1)E_x + (q^2+1)E_y + qE_{(x,y)} \in \mathcal{E}_{00}(xy, \mathbb{Z}/N\mathbb{Z})$. From the Fourier expansion of E_x, E_y, and $E_{(x,y)}$, we also get

$$f(a_1) = 1 \quad \text{and} \quad f(b_u) = -(q+1) \ \text{ for } 1 \leq u \leq q-1.$$

Since every harmonic cochain in $\mathcal{H}_{00}(xy, \mathbb{Z}/N\mathbb{Z})$ is determined uniquely by its values at a_1 and b_u for $1 \leq u \leq q-1$, we get $\delta_N(\bar{\psi}_1) = f$ and the proof is complete. □

Definition 8.17. Let $\mathfrak{n} \lhd A$ be a non-zero ideal and ℓ a prime number. Let $\mathbb{T}(\mathfrak{n})_\ell := \mathbb{T}(\mathfrak{n}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Given a maximal ideal $\mathfrak{M} \lhd \mathbb{T}(\mathfrak{n})_\ell$, let $\mathbb{T}(\mathfrak{n})_{\mathfrak{M}}$ be the completion of $\mathbb{T}(\mathfrak{n})_\ell$ at $\mathfrak{M}$. We say that $\mathbb{T}(\mathfrak{n})_{\mathfrak{M}}$ is a *Gorenstein* $\mathbb{Z}_\ell$-algebra if $\mathbb{T}(\mathfrak{n})_{\mathfrak{M}}^\vee := \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{T}(\mathfrak{n})_{\mathfrak{M}}, \mathbb{Z}_\ell)$ is a free $\mathbb{T}(\mathfrak{n})_{\mathfrak{M}}$ module of rank 1. In [38], following Mazur [33], Pál proved that $\mathbb{T}(\mathfrak{p})_{\mathfrak{M}}$ is Gorenstein for the maximal ideals containing $\mathfrak{E}(\mathfrak{p})$.

Let $\mathfrak{E}_\ell$ be the ideal generated by $\mathfrak{E}(xy)$ in $\mathbb{T}_\ell$. We know that $\mathbb{T}_\ell/\mathfrak{E}_\ell \cong \mathbb{Z}_\ell/N\mathbb{Z}_\ell$; see Corollary 3.18. Thus, N annihilates $J_\ell[\mathfrak{E}]$.

Proposition 8.18. *Assume $\ell \mid N$. The finite group-scheme $J_\ell[\mathfrak{E}]$ is unramified at ∞. If $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein for $\mathfrak{M} = (\mathfrak{E}_\ell, \ell)$, then there is an exact sequence of G_{F_∞}-modules*

$$0 \to (\mathbb{Z}_\ell/N\mathbb{Z}_\ell)^* \to J_\ell[\mathfrak{E}] \to \mathbb{Z}_\ell/N\mathbb{Z}_\ell \to 0.$$

Proof. The argument in the proof of Lemma 8.12 and the isomorphism of Hecke modules $(\Phi_\infty)_\ell \cong \mathcal{E}_{00}(xy, \mathbb{Z}/\ell^n\mathbb{Z})$ that we just proved ($n \gg 0$) give the exact sequence

$$0 \to D_\ell[\mathfrak{E}] \to J_\ell[\mathfrak{E}] \to (\Phi_\infty)_\ell \to 0.$$

Since $J[\ell^n]^{I_\infty} \cong D[\ell^n] \times (\Phi_\infty)[\ell^n]$, we see that $J_\ell[\mathfrak{E}]$ is unramified at ∞. Next, using Proposition 2.21, (4.3) and (4.4), we have

$$\begin{aligned} D_\ell[\mathfrak{E}] &\cong \mathrm{Hom}(\mathcal{H}_0(xy, \mathbb{Z}_\ell)/\mathfrak{E}_\ell\mathcal{H}_0(xy, \mathbb{Z}_\ell), \mathbb{C}_\infty^\times) \\ &\cong \mathrm{Hom}(\mathbb{T}_\ell^\vee/\mathfrak{E}_\ell\mathbb{T}_\ell^\vee, \mathbb{C}_\infty^\times) \cong \mathrm{Hom}(\mathbb{T}_{\mathfrak{M}}^\vee/\mathfrak{E}_\ell\mathbb{T}_{\mathfrak{M}}^\vee, \mathbb{C}_\infty^\times). \end{aligned}$$

If $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein, then

$$\mathrm{Hom}(\mathbb{T}_{\mathfrak{M}}^\vee/\mathfrak{E}_\ell\mathbb{T}_{\mathfrak{M}}^\vee, \mathbb{C}_\infty^\times) \cong \mathrm{Hom}(\mathbb{T}_{\mathfrak{M}}/\mathfrak{E}_\ell, \mathbb{C}_\infty^\times) \cong \mathrm{Hom}(\mathbb{Z}_\ell/N\mathbb{Z}_\ell, \mathbb{C}_\infty^\times).$$

□

If ℓ is odd and divides $q+1$ then, under the assumption that $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein, Proposition 8.18 implies that $\mathcal{S}_\ell$ is the maximal μ-type subgroup scheme of J_ℓ and $J_\ell[\mathfrak{E}] \cong \mathcal{C}_\ell \oplus \mathcal{S}_\ell$ is pure. Indeed, since Φ_∞ is constant and $\gcd(q+1, q-1)$ divides 2, the maximal μ-type subgroup scheme of J_ℓ specializes to the connected component $\mathcal{J}^0_{\mathbb{F}_\infty}$ of the Néron model, so must be isomorphic to $(\mathbb{Z}_\ell/(q+1)\mathbb{Z}_\ell)^*$. Since $\mathcal{S}_\ell$ is μ-type, it must be maximal by comparing the orders. The fact that $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein in this case can be proved by Mazur's Eisenstein descent discussed in Section 10 of [38]. Since this fact is not central for our paper, and the proof closely follows the argument in [38], we omit the details.

Now assume ℓ is odd and divides q^2+1. Assume $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein. Since $s_{x,y}$ divides 2, Lemma 8.12 and Proposition 8.18 imply that there is an exact sequence of G_F-modules

$$0 \to \mathcal{C}_\ell \to J_\ell[\mathfrak{E}] \to \mathcal{M}_\ell \to 0, \tag{8.1}$$

where $\mathcal{M}_\ell \cong (\mathbb{Z}_\ell/(q^2+1)\mathbb{Z}_\ell)^*$. If $J_\ell[\mathfrak{E}]^{I_y}$ is strictly larger than $\mathcal{C}_\ell$, then the Galois representation $\rho_{\mathfrak{M}} : G_F \to \mathrm{Aut}(J_\ell[\mathfrak{M}]) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ is unramified at y. On the other hand, $(\Phi_y)_\ell = 0$, so $J_\ell[\mathfrak{M}]$ is isomorphic to a $\mathbb{T}_\ell \times G_{\mathbb{F}_y}$-submodule of the torus $\mathcal{J}^0_{\mathbb{F}_y}$. Now the same argument that proves Proposition 3.8 in [47] implies that Frob_y acts on $J_\ell[\mathfrak{M}]$ as a scalar $\pm|y|^2$. Hence $\det(\rho_{\mathfrak{M}}) = q^4$. On the other hand, the sequence (8.1) shows that the eigenvalues of Frob_y are 1 and $|y| = q^2$. Thus, $\det(\rho_{\mathfrak{M}}) = q^2$. This forces $q^4 \equiv q^2 \pmod{\ell}$, which is a contradiction since $\gcd(q^2+1, q^2(q^2-1))$ divides 2. We conclude that if $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein then the ℓ-primary μ-type subgroup scheme of J is trivial. This "explains" why $\mathcal{S}$ is smaller than $\mathcal{C}$ – the missing part corresponds to the ramified portion of $J[\mathfrak{E}]$.

9. Jacquet-Langlands isogeny

9.1. Modular curves of $\mathscr{D}$-elliptic sheaves. Let D be a division quaternion algebra over F. Assume D is split at ∞, i.e., $D \otimes_F F_\infty \cong \mathrm{Mat}_2(F_\infty)$; this is the analogue of "indefinite" over $\mathbb{Q}$. Let $\mathfrak{n} \lhd A$ be the discriminant of D. It is known that $\mathfrak{n}$ is square-free with an even number of prime factors. Moreover, any $\mathfrak{n}$ with this property is a discriminant of some D, and, up to isomorphism, $\mathfrak{n}$ determines D; cf. [56].

Let $C := \mathbb{P}^1_{\mathbb{F}_q}$. Fix a locally free sheaf $\mathscr{D}$ of $\mathcal{O}_C$-algebras with stalk at the generic point equal to D and such that $\mathscr{D}_v := \mathscr{D} \otimes_{\mathcal{O}_C} \mathcal{O}_v$ is a maximal order in $D_v := D \otimes_F F_v$ for every place v. Let S be an $\mathbb{F}_q$-scheme. Denote by Frob_S its Frobenius endomorphism, which is the identity on the points and the qth power map on the functions. Denote by $C \times S$ the fiber product $C \times_{\mathrm{Spec}(\mathbb{F}_q)} S$. Let $z : S \to \mathrm{Spec}(A[\mathfrak{n}^{-1}])$ be a morphism of $\mathbb{F}_q$-schemes. A *$\mathscr{D}$-elliptic sheaf over S*, with pole ∞ and zero z, is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where each $\mathcal{E}_i$ is a locally free sheaf of $\mathcal{O}_{C\times S}$-modules of rank 4 equipped with a right action of $\mathscr{D}$ compatible with the $\mathcal{O}_C$-action, and where

$$\begin{aligned} &j_i : \mathcal{E}_i \to \mathcal{E}_{i+1} \\ &t_i : {}^\tau\mathcal{E}_i := (\mathrm{Id}_C \times \mathrm{Frob}_S)^*\mathcal{E}_i \to \mathcal{E}_{i+1} \end{aligned}$$

are injective $\mathcal{O}_{C\times S}$-linear homomorphisms compatible with the $\mathscr{D}$-action. The maps j_i and t_i are sheaf modifications at ∞ and z, respectively, which satisfy certain conditions, and it is assumed that for each closed point P of S, the Euler-Poincaré characteristic $\chi(\mathcal{E}_0|_{C\times P})$ is in the interval $[0, 2)$; we refer to [28,

§2] for the precise definition. Denote by $\mathcal{E}\ell\ell^{\mathscr{D}}(S)$ the set of isomorphism classes of $\mathscr{D}$-elliptic sheaves over S. The following theorem can be deduced from (4.1), (5.1) and (6.2) in [28]:

Theorem 9.1. *The functor $S \mapsto \mathcal{E}\ell\ell^{\mathscr{D}}(S)$ has a coarse moduli scheme $X^{\mathfrak{n}}$, which is proper and smooth of pure relative dimension* 1 *over* $\mathrm{Spec}(A[\mathfrak{n}^{-1}])$.

For each prime $\mathfrak{p} \lhd A$ not in R, there is a Hecke correspondence $T_{\mathfrak{p}}$ on $X^{\mathfrak{n}}$; we refer to [28, §7] for the definition.

9.2. Rigid-analytic uniformization. Let $\mathcal{D}$ be a maximal A-order in D. Let $\Gamma^{\mathfrak{n}} := \mathcal{D}^{\times}$ be the group of units of $\mathcal{D}$. By fixing an isomorphism $D\otimes_F F_\infty \cong \mathrm{Mat}_2(F_\infty)$, we get an embedding $\Gamma^{\mathfrak{n}} \hookrightarrow \mathrm{GL}_2(F_\infty)$. Hence $\Gamma^{\mathfrak{n}}$ acts on the Drinfeld half-plane $\Omega = \mathbb{C}_\infty - F_\infty$. We have the following uniformization theorem

$$\Gamma^{\mathfrak{n}} \setminus \Omega \cong (X^{\mathfrak{n}}_{F_\infty})^{\mathrm{an}}, \tag{9.1}$$

which follows by applying Raynaud's "generic fibre" functor to Theorem 4.4.11 in [2]. The proof of this theorem is only outlined in [2]. Nevertheless, as is shown in [52, Prop. 4.28], (9.1) can be deduced from Hausberger's version [23] of the Cherednik-Drinfeld theorem for $X^{\mathfrak{n}}$. An alternative proof of (9.1) is given in [53, Thm. 4.6].

Assume for simplicity that $\mathfrak{n}$ is divisible by a prime of even degree. In this case, the normal subgroup $\Gamma^{\mathfrak{n}}_f$ of $\Gamma^{\mathfrak{n}}$ generated by torsion elements is just the center $\mathbb{F}_q^\times$ of $\Gamma^{\mathfrak{n}}$, cf. [42, Thm. 5.6]. This implies that the image of $\Gamma^{\mathfrak{n}}$ in $\mathrm{PGL}_2(F_\infty)$ is a discrete, finitely generated free group, and (9.1) is a Mumford uniformization [20], [35].

Denote $\Gamma = \Gamma^{\mathfrak{n}}/\mathbb{F}_q^\times$ and let $\overline{\Gamma}$ be the abelianization of Γ. The group Γ acts without inversions on the Bruhat-Tits tree $\mathscr{T}$, and the quotient graph $\Gamma^{\mathfrak{n}}\setminus\mathscr{T} = \Gamma\setminus\mathscr{T}$ is finite; cf. [42, Lem. 5.1]. Fix a vertex $v_0 \in \mathscr{T}$. For any $\gamma \in \Gamma$ there is a unique path in $\mathscr{T}$ without backtracking from v_0 to γv_0. The map $\Gamma \to H_1(\Gamma\setminus\mathscr{T},\mathbb{Z})$ which sends γ to the homology class of the image of this path in $\Gamma\setminus\mathscr{T}$ does not depend on the choice of v_0, and induces an isomorphism

$$\overline{\Gamma} \cong H_1(\Gamma\setminus\mathscr{T},\mathbb{Z}). \tag{9.2}$$

Since Γ is a free group and $\Gamma\setminus\mathscr{T}$ is a finite graph,

$$\begin{aligned}\mathcal{H}(\mathfrak{n},\mathbb{Z})' := \mathcal{H}_0(\mathscr{T},\mathbb{Z})^{\Gamma^{\mathfrak{n}}} = \mathcal{H}_0(\mathscr{T},\mathbb{Z})^{\Gamma} = \mathcal{H}(\mathscr{T},\mathbb{Z})^{\Gamma}\\ \cong \mathcal{H}(\Gamma\setminus\mathscr{T},\mathbb{Z}) \cong H_1(\Gamma\setminus\mathscr{T},\mathbb{Z}),\end{aligned}$$

cf. Definitions 2.1 and 2.2. The space $\mathcal{H}(\mathfrak{n},\mathbb{Z})'$ is equipped with a natural action of Hecke operators $T_{\mathfrak{m}}$ ($\mathfrak{m} \lhd A$), which generate a commutative $\mathbb{Z}$-algebra; cf. [34, §5.3].

The map $\langle\cdot,\cdot\rangle : E(\mathscr{T}) \times E(\mathscr{T}) \to \mathbb{Z}$

$$\langle e, f\rangle = \begin{cases} 1 & \text{if } f = e\\ -1 & \text{if } f = \bar{e}\\ 0 & \text{otherwise}\end{cases}$$

induces a $\mathbb{Z}$-valued bilinear symmetric positive-definite pairing on $H_1(\Gamma\setminus\mathscr{T},\mathbb{Z})$, which we denote by the same symbol. Using (9.2) we get a pairing

$$\begin{aligned}\overline{\Gamma}\times\overline{\Gamma}&\to\mathbb{Z}\\ \gamma,\delta&\mapsto\langle\gamma,\delta\rangle.\end{aligned}\tag{9.3}$$

Let $J^{\mathfrak{n}}$ denote the Jacobian variety of $X_F^{\mathfrak{n}}$, and $\mathcal{J}^{\mathfrak{n}}$ denote the Néron model of $J^{\mathfrak{n}}$ over $\mathcal{O}_\infty$. Since $X^{\mathfrak{n}}$ is a Mumford curve over F_∞, $(\mathcal{J}^{\mathfrak{n}})^0_{\mathbb{F}_\infty}$ is a split algebraic torus. Let $M:=\mathrm{Hom}((\mathcal{J}^{\mathfrak{n}})^0_{\overline{\mathbb{F}}_\infty},\mathbb{G}_{m,\overline{\mathbb{F}}_\infty})$ be the character group of this torus. By the mapping property of Néron models, each endomorphism of $J^{\mathfrak{n}}$ defined over F_∞ canonically extends to $\mathcal{J}^{\mathfrak{n}}$, hence acts functorially on M. Since $J^{\mathfrak{n}}$ has purely toric reduction, $\mathrm{End}_{F_\infty}(J^{\mathfrak{n}})$ operates faithfully on M. By [35, p. 132], the graph $\Gamma\setminus\mathscr{T}$ is the dual graph of the special fibre of the minimal regular model of $X^{\mathfrak{n}}$ over $\mathcal{O}_\infty$. On the other hand, by [3, p. 246], there is a canonical isomorphism $M\cong H_1(\Gamma\setminus\mathscr{T},\mathbb{Z})$. The Hecke correspondence $T_{\mathfrak{p}}$ induces an endomorphism of $J^{\mathfrak{n}}$ by Picard functoriality, which we denote by the same symbol. Via the isomorphisms mentioned in this paragraph and (9.2), we get a canonical action of $T_{\mathfrak{p}}$ on $\overline{\Gamma}$. This action agrees with the action of $T_{\mathfrak{p}}$ on $\mathcal{H}(\mathfrak{n},\mathbb{Z})'$.

Using rigid-analytic theta functions, one constructs a symmetric bilinear pairing

$$\begin{aligned}\overline{\Gamma}\times\overline{\Gamma}&\to F_\infty^\times\\ \gamma,\delta&\mapsto[\gamma,\delta],\end{aligned}$$

such that $\mathrm{ord}_\infty[\gamma,\delta]=\langle\gamma,\delta\rangle$; see [31, Thm. 5]. The main result of [31] gives a rigid-analytic uniformization of $J^{\mathfrak{n}}$:

$$0\to\overline{\Gamma}\xrightarrow{\gamma\mapsto[\gamma,\cdot]}\mathrm{Hom}(\overline{\Gamma},\mathbb{C}_\infty)\to J^{\mathfrak{n}}(\mathbb{C}_\infty)\to 0.\tag{9.4}$$

This sequence is equivariant with respect to the action of $T_{\mathfrak{p}}$ ($\mathfrak{p}\not\in R$); this was proven by Ryan Flynn [7] following the method in [18].

Finally, we have the following quaternionic analogue of Theorem 5.5. Let Φ'_∞ be the component group of $J^{\mathfrak{n}}$ at ∞. After identifying $\overline{\Gamma}$ with the character group M, the pairing (9.3) becomes Grothendieck's monodromy pairing on M, so there is an exact sequence (see [22, §§11-12] and [43])

$$0\to\overline{\Gamma}\xrightarrow{\gamma\mapsto\langle\gamma,\cdot\rangle}\mathrm{Hom}(\overline{\Gamma},\mathbb{Z})\to\Phi'_\infty\to 0.\tag{9.5}$$

This sequence is equivariant with respect to the action of $T_{\mathfrak{p}}$.

9.3. EXPLICIT JACQUET-LANGLANDS ISOGENY CONJECTURE. Let $\mathfrak{n}\lhd A$ be a product of an even number of distinct primes. The space $\mathcal{H}_0(\mathfrak{n},\mathbb{Q})$ can be interpreted as a space of automorphic forms on $\Gamma_0(\mathfrak{n})$; see [18, §4]. A similar argument can be used to show that $\mathcal{H}(\mathfrak{n},\mathbb{Q})'$ has an interpretation as a space of automorphic forms on $\Gamma^{\mathfrak{n}}$; cf. [43, §8]. The Jacquet-Langlands correspondence over F implies that there is an isomorphism

$$\mathrm{JL}:\mathcal{H}_0(\mathfrak{n},\mathbb{Q})^{\mathrm{new}}\cong\mathcal{H}(\mathfrak{n},\mathbb{Q})'$$

	$\deg(\mathfrak{q})$ is even	$\deg(\mathfrak{q})$ is odd
$\deg(\mathfrak{p})=1$	$\mathbb{Z}/(q+1)M(\mathfrak{q})\mathbb{Z}$	$\mathbb{Z}/M(\mathfrak{q})\mathbb{Z}$
$\deg(\mathfrak{p})=2$	$\mathbb{Z}/M(\mathfrak{q})\mathbb{Z}$	$\mathbb{Z}/(q+1)M(\mathfrak{q})\mathbb{Z}$

Table 1. Component group $\Phi'_{\mathfrak{q}}$ of $J^{\mathfrak{pq}}$

which is compatible with the action of Hecke operators $T_{\mathfrak{p}}$, $\mathfrak{p}\nmid\mathfrak{n}$; cf. [25, Ch. III].

Remark 9.2. In fact, JL is $U_{\mathfrak{p}}$-equivariant also for $\mathfrak{p}\mid\mathfrak{n}$. The analogous statement over $\mathbb{Q}$ follows from the results of Ribet [47, §4] (cf. [24, Thm. 2.3]), which rely on the geometry of the Jacobians of modular curves. The Néron models of $J_0(\mathfrak{n})$ and $J^{\mathfrak{n}}$ have the necessary properties for Ribet's arguments to apply. Since this fact will not be essential for our purposes, we do not discuss the details further. The $U_{\mathfrak{p}}$-equivariance of JL can also be deduced from [25]; the necessary input is contained in [25, Thm. 4.2] and its proof.

Let $J_0(\mathfrak{n})^{\mathrm{old}}$ be the abelian subvariety of $J_0(\mathfrak{n})$ generated by the images of $J_0(\mathfrak{m})$ under the maps $J_0(\mathfrak{m})\to J_0(\mathfrak{n})$ induced by the degeneracy morphisms $X_0(\mathfrak{n})_F\to X_0(\mathfrak{m})_F$ for all $\mathfrak{m}\supsetneq\mathfrak{n}$. Let $J_0(\mathfrak{n})^{\mathrm{new}}$ be the quotient of $J_0(\mathfrak{n})$ by $J_0(\mathfrak{n})^{\mathrm{old}}$. Combining JL with the main results in [6] and [28], and Zarhin's isogeny theorem, one concludes that there is a $\mathbb{T}(\mathfrak{n})^0$-equivariant isogeny $J_0(\mathfrak{n})^{\mathrm{new}}\to J^{\mathfrak{n}}$ defined over F, which we call a *Jacquet-Langlands isogeny*; cf. [41, Cor. 7.2]. (This isogeny is in fact $\mathbb{T}(\mathfrak{n})$-equivariant by the previous remark; see [24, Cor. 2.4] for the corresponding statement over $\mathbb{Q}$.) Zarhin's isogeny theorem provides no information about the Jacquet-Langlands isogenies beyond their existence. One possible approach to making these isogenies more explicit is via the study of component groups. More precisely, since $J_0(\mathfrak{n})^{\mathrm{new}}$ and $J^{\mathfrak{n}}$ have purely toric reduction at the primes dividing $\mathfrak{n}$ and at ∞, one can deduce restrictions on possible kernels of isogenies $J_0(\mathfrak{n})^{\mathrm{new}}\to J^{\mathfrak{n}}$ from the component groups of these abelian varieties using [41, Thm. 4.3]. Unfortunately, the explicit structure of these component groups is not known in general.

From now on we restrict to the case when $\mathfrak{n}=\mathfrak{pq}$ is a product of two distinct primes and $\deg(\mathfrak{p})\le 2$. In this case, the structure of $X^{\mathfrak{n}}_{\mathbb{F}_{\mathfrak{q}}}$ is fairly simple and can be deduced from [41, Prop. 6.2]. Using this and Raynaud's theorem, one computes that the component group $\Phi'_{\mathfrak{q}}$ of $J^{\mathfrak{n}}$ at $\mathfrak{q}$ is given by Table 1, where

$$M(\mathfrak{q})=\begin{cases}|\mathfrak{q}|+1, & \text{if } \deg(\mathfrak{q}) \text{ is even;}\\ \frac{|\mathfrak{q}|+1}{q+1}, & \text{if } \deg(\mathfrak{q}) \text{ is odd.}\end{cases}$$

The cuspidal divisor group of $J_0(\mathfrak{q})$ is generated by $[1]-[\infty]$, which has order

$$N(\mathfrak{q})=\begin{cases}\frac{|\mathfrak{q}|-1}{q^2-1}, & \text{if } \deg(\mathfrak{q}) \text{ is even;}\\ \frac{|\mathfrak{q}|-1}{q-1}, & \text{if } \deg(\mathfrak{q}) \text{ is odd.}\end{cases}$$

	$\deg(\mathfrak{q})$ is even	$\deg(\mathfrak{q})$ is odd
$\deg(\mathfrak{p})=1$	$\mathbb{Z}/(q+1)\mathbb{Z}$	0
$\deg(\mathfrak{p})=2$	$\mathbb{Z}/(q^2+1)\mathbb{Z}$	$\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$

Table 2. $\tilde{\Phi}_{\mathfrak{q}}$

Let $\alpha : X_0(\mathfrak{p}\mathfrak{q})_F \to X_0(\mathfrak{q})_F$ be the degeneracy morphism discussed in Section 4. The image of $\mathcal{C}(\mathfrak{q})$ in $J_0(\mathfrak{p}\mathfrak{q})$ under the induced map $J_0(\mathfrak{q}) \to J_0(\mathfrak{p}\mathfrak{q})$ is generated by

$$c := \alpha^*([1]-[\infty]) = |\mathfrak{p}|[1]+[\mathfrak{p}]-|\mathfrak{p}|[\mathfrak{q}]-[\infty].$$

By examining the specializations of the cusps in $X_0(\mathfrak{p}\mathfrak{q})_{\mathbb{F}_\mathfrak{q}}$, we see that

$$\wp_\mathfrak{q}(c) = (|\mathfrak{p}|+1)z,$$

where $z \in \Phi_\mathfrak{q}$ is the element from the proof of Proposition 5.4. Let $\tilde{\Phi}_\mathfrak{q} := \Phi_\mathfrak{q}/\wp_\mathfrak{q}(c)$. The order of z in $\Phi_\mathfrak{q}$ is given in [41, Thm. 4.1], and the order of $\Phi_\mathfrak{q}$ itself is given in Proposition 5.4. From this one easily computes that $\tilde{\Phi}_\mathfrak{q}$ is the group in Table 2.

Since $c \in J_0(\mathfrak{n})^{\mathrm{old}}$, the map of component groups $\Phi_\mathfrak{q} \to \Phi_\mathfrak{q}^{\mathrm{new}}$ induced by the quotient $J_0(\mathfrak{n}) \to J_0(\mathfrak{n})^{\mathrm{new}}$ must factor through $\tilde{\Phi}_\mathfrak{q}$ (here $\Phi_\mathfrak{q}^{\mathrm{new}}$ denotes the component group of $J_0(\mathfrak{n})^{\mathrm{new}}$ at $\mathfrak{q}$).

Assume $\deg(\mathfrak{p})=1$. Then the cuspidal divisor $c_\mathfrak{q} := [\mathfrak{q}]-[\infty] \in \mathcal{C}(\mathfrak{p}\mathfrak{q})$ has order $N(\mathfrak{q})M(\mathfrak{q})$ (see Theorem 6.11) and specializes to the connected component of identity $\mathcal{J}^0_{\mathbb{F}_\mathfrak{q}}$ of the Néron model of $J_0(\mathfrak{n})$. Theorem 4.3 in [41] describes how the component groups of abelian varieties with toric reduction over a local field change under isogenies, depending on the specialization of the kernel of the isogeny in the closed fibre. After comparing the groups $\tilde{\Phi}_\mathfrak{q}$ and $\Phi'_\mathfrak{q}$, and the orders of c and $c_\mathfrak{q}$, this theorem suggests that there is an isogeny $J_0(\mathfrak{n})^{\mathrm{new}} \to J^\mathfrak{n}$ whose kernel is isomorphic to $\mathbb{Z}/M(\mathfrak{q})\mathbb{Z}$ and is generated by the image of $c_\mathfrak{q}$ in $J_0(\mathfrak{n})^{\mathrm{new}}$.

The case $\deg(\mathfrak{p})=2$ can be analysed similarly. The cuspidal divisor $c_\mathfrak{q}$ has order $(q+1)N(\mathfrak{q})M(\mathfrak{q})$. The image of $c_\mathfrak{p}$ in $\tilde{\Phi}_\mathfrak{q}$ generates its cyclic subgroup of order q^2+1. Hence there might be an isogeny $J_0(\mathfrak{n})^{\mathrm{new}} \to J^\mathfrak{n}$ whose kernel is isomorphic to $\frac{\mathbb{Z}}{M(\mathfrak{q})\mathbb{Z}} \times \frac{\mathbb{Z}}{(q^2+1)\mathbb{Z}}$.

Conjecture 9.3. *Assume* $\mathfrak{n}=\mathfrak{p}\mathfrak{q}$, *where* $\mathfrak{p},\mathfrak{q}$ *are distinct primes and* $\deg(\mathfrak{p}) \leq 2$. *There is an isogeny* $J_0(\mathfrak{p}\mathfrak{q})^{\mathrm{new}} \to J^{\mathfrak{p}\mathfrak{q}}$ *whose kernel* K *is generated by cuspidal divisors and*

$$K \cong \begin{cases} \frac{\mathbb{Z}}{M(\mathfrak{q})\mathbb{Z}} & \textit{if } \deg(\mathfrak{p})=1; \\ \frac{\mathbb{Z}}{M(\mathfrak{q})\mathbb{Z}} \times \frac{\mathbb{Z}}{(q^2+1)\mathbb{Z}} & \textit{if } \deg(\mathfrak{p})=2. \end{cases}$$

This conjecture is the function field analogue of Ogg's conjectures about Jacquet-Langlands isogenies over $\mathbb{Q}$; see [37, pp. 212-216].

There are only two cases when $\mathfrak{n}$ is square-free, $J_0(\mathfrak{n})$ is non-trivial, and $J_0(\mathfrak{n})^{\mathrm{new}} = J_0(\mathfrak{n})$. The first case is $\mathfrak{n}=xy$; Conjecture 9.3 then specializes

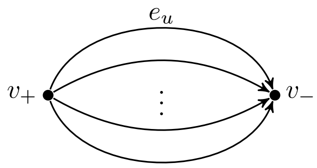

Figure 6. $\Gamma^{xy} \setminus \mathscr{T}$

to the conjecture in [41]. We will prove this conjecture in §9.4 under certain assumptions, and unconditionally for some small q. The second case is $\mathfrak{n} = \mathfrak{p}\mathfrak{q}$, where $\mathfrak{p} \neq \mathfrak{q}$ are primes of degree 2. In this case the conjecture predicts that there is a Jacquet-Langlands isogeny whose kernel is isomorphic to $\mathbb{Z}/(q^2+1)\mathbb{Z} \times \mathbb{Z}/(q^2+1)\mathbb{Z}$; cf. Example 6.13. The method of this paper should be possible to adapt to this latter case, and prove the conjecture for some small q.

9.4. Special case. Assume $\mathfrak{n} = xy$. To simplify the notation we put

$$J := J_0(xy),\ J' := J^{xy},\ \mathcal{H} := \mathcal{H}_0(xy, \mathbb{Z}),\ \mathcal{H}' := \mathcal{H}(xy, \mathbb{Z})',\ \mathbb{T} := \mathbb{T}(xy).$$

First, we prove the analogue of Lemma 8.16 for J'.

Lemma 9.4. *Let Φ'_∞ be the component group of J' at ∞. Let $\mathfrak{p} \lhd A$ be any prime not dividing xy. Then $T_\mathfrak{p} - |\mathfrak{p}| - 1$ annihilates $\Phi'_\infty \cong \mathbb{Z}/(q+1)\mathbb{Z}$.*

Proof. The quotient graph $\Gamma^{xy} \setminus \mathscr{T}$ has two vertices joined by $q+1$ edges; see [41, Prop. 6.5]. We label the vertices by v_+ and v_-, and label the edges by the elements of $\mathbb{P}^1(\mathbb{F}_q)$; see Figure 6. Let $\gamma \in \mathrm{GL}_2(F_\infty)$ be an arbitrary element. Then $\gamma v_\pm \equiv v_\pm \mod \Gamma^{xy}$ if $\mathrm{ord}_\infty(\det\gamma)$ is even, and $\gamma v_\pm \equiv v_\mp \mod \Gamma^{xy}$ if $\mathrm{ord}_\infty(\det\gamma)$ is odd. Consider the free $\mathbb{Z}$-module with generators $\{e_u, \overline{e_u} \mid u \in \mathbb{P}^1(\mathbb{F}_q)\}$, modulo the relations $\overline{e_u} = -e_u$. The action of $T_\mathfrak{p}$ on this module satisfies

$$T_\mathfrak{p} \sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u = (|\mathfrak{p}|+1) \begin{cases} \sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u & \text{if } \deg(\mathfrak{p}) \text{ is even,} \\ -\sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u & \text{if } \deg(\mathfrak{p}) \text{ is odd.} \end{cases}$$

$\mathcal{H}'$ is generated by the cycles $\varphi_u = e_u - e_\infty$, $u \in \mathbb{F}_q$. Let φ_u^* be the dual basis of $\mathrm{Hom}(\mathcal{H}', \mathbb{Z})$. The map in (9.5) sends φ_u to $\varphi_u^* + \sum_{w \in \mathbb{F}_q} \varphi_w^*$. It is easy to see from this that Φ'_∞ is cyclic of order $q+1$ and is generated by $\sum_{w \in \mathbb{F}_q} \varphi_w^*$. Note that $|\mathfrak{p}|+1 \equiv 0 \mod (q+1)$ if $\deg(\mathfrak{p})$ is odd. Hence

$$T_\mathfrak{p} \sum_{w \in \mathbb{F}_q} \varphi_w^* = T_\mathfrak{p} \left(\sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u^* - (q+1) e_\infty^* \right) = \pm(|\mathfrak{p}|+1) \sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u^* - (q+1) T_\mathfrak{p} e_\infty^*$$

$$\equiv (|\mathfrak{p}|+1) \sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} e_u^* - (q+1)(|\mathfrak{p}|+1) e_\infty^* = (|\mathfrak{p}|+1) \sum_{w \in \mathbb{F}_q} \varphi_w^* \mod (q+1).$$

This implies that $T_\mathfrak{p}$ acts by multiplication by $|\mathfrak{p}|+1$ on Φ'_∞. $\square$

Theorem 9.5. *Assume $\mathcal{H} \cong \mathcal{H}'$ as $\mathbb{T}$-modules. There is an isogeny $J \to J'$ defined over F whose kernel is cyclic of order q^2+1 and annihilated by the Eisenstein ideal.*

Proof. By (4.3) and (4.4), the rigid-analytic uniformization of J over F_∞ is given by the $\mathbb{T}^0$-equivariant sequence

$$0 \to \mathcal{H} \to \mathrm{Hom}(\mathcal{H}, \mathbb{C}_\infty^\times) \to J(\mathbb{C}_\infty) \to 0.$$

By Proposition 2.21, $\mathrm{Hom}(\mathcal{H}, \mathbb{Z}) \cong \mathbb{T} = \mathbb{T}^0$, so we can write this sequence as the $\mathbb{T}$-equivariant sequence

$$0 \to \mathcal{H} \to \mathbb{T} \otimes \mathbb{C}_\infty^\times \to J(\mathbb{C}_\infty) \to 0.$$

By Theorem 5.5, the sequence derived from this using the valuation homomorphism ord_∞ is

$$0 \to \mathcal{H} \to \mathbb{T} \to \Phi_\infty \to 0,$$

where Φ_∞ is the component group of the Néron model of J at ∞. Now we can consider $\mathcal{H}$ as ideal in $\mathbb{T}$. We know from Lemma 8.16 that the Eisenstein ideal $\mathfrak{E}$ annihilates $\Phi_\infty \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$. Hence Φ_∞ is a quotient of $\mathbb{T}/\mathfrak{E}$. On the other hand, by Corollary 3.18, $\mathbb{T}/\mathfrak{E} \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$. Comparing the orders of Φ_∞ and $\mathbb{T}/\mathfrak{E}$, we conclude that $\Phi_\infty \cong \mathbb{T}/\mathfrak{E}$ and $\mathcal{H} \cong \mathfrak{E}$.
From the discussion in §9.2, if we assume $\mathcal{H} \cong \mathcal{H}'$ as $\mathbb{T}$-modules, the rigid-analytic uniformization of J' over F_∞ is given by the $\mathbb{T}$-equivariant sequence

$$0 \to \mathcal{H}' \to \mathbb{T} \otimes \mathbb{C}_\infty^\times \to J'(\mathbb{C}_\infty) \to 0.$$

The argument in the previous paragraph allows us to identify $\mathcal{H}'$ in the above sequence with the annihilator $\mathfrak{E}' \lhd \mathbb{T}$ of Φ'_∞ in $\mathbb{T}$. On the other hand, by Lemma 9.4, $T_\mathfrak{p} - |\mathfrak{p}| - 1 \in \mathfrak{E}'$ for any $\mathfrak{p} \nmid xy$. Therefore, $\mathfrak{E} \subset \mathfrak{E}'$ and

$$\mathbb{T}/\mathfrak{E}' \cong \Phi'_\infty \cong \mathbb{Z}/(q+1)\mathbb{Z}.$$

Hence $\mathfrak{E}'/\mathfrak{E} \cong \mathbb{Z}/(q^2+1)\mathbb{Z}$.
We have identified the uniformizing tori of J and J' with $\mathbb{T} \otimes \mathbb{C}_\infty^\times$, and the uniformizing lattices $\mathcal{H}$ and $\mathcal{H}'$ with $\mathfrak{E}$ and $\mathfrak{E}'$, respectively. Now specializing a theorem of Gerritzen [19] to this situation, we get a natural bijection

$$\mathrm{Hom}_\mathbb{T}(\mathbb{T} \otimes \mathbb{C}_\infty^\times, \mathfrak{E}; \mathbb{T} \otimes \mathbb{C}_\infty^\times, \mathfrak{E}') \xrightarrow{\sim} \mathrm{Hom}_\mathbb{T}(J_{F_\infty}, J'_{F_\infty}),$$

where on the left hand-side is the group of homomorphisms $\mathbb{T} \otimes \mathbb{C}_\infty^\times \to \mathbb{T} \otimes \mathbb{C}_\infty^\times$ which map $\mathfrak{E}$ to $\mathfrak{E}'$ and are compatible with the action of $\mathbb{T}$. It is clear that identity map on $\mathbb{T} \otimes \mathbb{C}_\infty^\times$ is in this set. The snake lemma applied to the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{E} \arrow[r] \arrow[d, hook] & \mathbb{T} \otimes \mathbb{C}_\infty^\times \arrow[r] \arrow[d, equal] & J(\mathbb{C}_\infty) \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathfrak{E}' \arrow[r] & \mathbb{T} \otimes \mathbb{C}_\infty^\times \arrow[r] & J'(\mathbb{C}_\infty) \arrow[r] & 0
\end{tikzcd}$$

shows that there is an isogeny $\pi : J \to J'$ with $\ker(\pi) \cong \mathfrak{E}'/\mathfrak{E}$. Moreover, since $\mathrm{Hom}_\mathbb{T}(\mathbb{T}, \mathbb{T}) \cong \mathbb{T}$, every $\mathbb{T}$-equivariant homomorphism $J \to J'$ can be obtained as a composition of π with an element of $\mathbb{T}$. We know that there is an isogeny

$J \to J'$ defined over F. Since the endomorphisms of J induced by the Hecke operators are also defined over F, we conclude that π is defined over F. $\square$

Theorem 9.6. *In addition to the assumption in Theorem 9.5, assume $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein for all maximal Eisenstein ideals $\mathfrak{M}$ of residue characteristic dividing q^2+1. Then there is an isogeny $J \to J'$ whose kernel is $\langle c_y \rangle$.*

Proof. By Theorem 9.5, there is an isogeny $J \to J'$ defined over F whose kernel $H \subset J[\mathfrak{E}]$ is cyclic of order q^2+1. Assume $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein for all maximal Eisenstein ideals $\mathfrak{M}$ of residue characteristic dividing q^2+1. First, by Theorem 7.13 and Proposition 8.18, this implies $J[2,\mathfrak{E}] = \mathcal{C}[2]$. Next, let $\ell | (q^2+1)$ be an odd prime. From the discussion after Proposition 8.18 we know that the action of inertia at y on $J_\ell[\mathfrak{E}]$ is unipotent and the I_y-invariant subgroup of $J_\ell[\mathfrak{E}]$ is $\mathcal{C}_\ell = \langle c_y \rangle_\ell$. Since 4 does not divide q^2+1, these two observations imply that there is an isogeny $J \to J'$ whose kernel is cyclic of order q^2+1 and is contained in $\mathcal{C}[2] + \langle c_y \rangle$.
If q is even, then $\mathcal{C}[2] + \langle c_y \rangle = \langle c_y \rangle$, and we are done. If q is odd, then the kernel is generated by c_y, or $c_y + \frac{q+1}{2} c_x$, or $2c_y + \frac{q+1}{2} c_x$. We know that $\wp_y(c_y) = 0$ and $\wp_y(c_x) = 1$ (cf. Proposition 6.7). If H is generated by $c_y + \frac{q+1}{2} c_x$ or $2c_y + \frac{q+1}{2} c_x$, then the specialization map $\wp_y$ gives the exact sequence

$$0 \to \frac{\mathbb{Z}}{\frac{q^2+1}{2}\mathbb{Z}} \to H \xrightarrow{\wp_y} \mathbb{Z}/2\mathbb{Z} \to 0.$$

It is a consequence of the uniformization theorem in [23] that $X^{\mathfrak{n}}_{F_{\mathfrak{p}}}$ is a twisted Mumford curve for any $\mathfrak{p}|\mathfrak{n}$ (here $\mathfrak{n}$ is arbitrary). This implies that $J^{\mathfrak{n}}$ has toric reduction at $\mathfrak{p}$. In particular, J' has toric reduction at y. Now we can apply Theorem 4.3 in [41] to get an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \Phi_y \to \Phi'_y \to \frac{\mathbb{Z}}{\frac{q^2+1}{2}\mathbb{Z}} \to 0,$$

where Φ'_y is the component group of J' at y. This implies that the order of Φ'_y is $(q+1)(q^2+1)/4$. But according to [41, Thm. 6.4], $\Phi'_y \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}$, which leads to a contradiction. $\square$

To be able to verify the assumptions in Theorem 9.6 computationally, it is crucial to be able to compute the action of $\mathbb{T}$ on $\mathcal{H}$ and $\mathcal{H}'$. The methods for doing this will be discussed in Section 10. Our calculations lead to the following:

Proposition 9.7. *The assumptions of Theorem 9.6 hold for $q = 2$, and for the 12 cases listed in Table 4. In particular, in these cases there is an isogeny $J \to J'$ whose kernel is $\langle c_y \rangle$.*

Remark 9.8. We believe that the assumptions in Theorem 9.6 hold in general. Our method for verifying that $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic $\mathbb{T}$-modules relies on finding a perfect $\mathbb{T}$-equivariant pairing $\mathbb{T} \times \mathcal{H}' \to \mathbb{Z}$. Unlike the case of $\mathcal{H}$,

q	y	$\mathrm{disc}(\mathbb{T})$
2	T^2+T+1	4
3	T^2+1	80
3	T^2+T+2	68
3	T^2+2T+2	68
5	T^2+T+1	265216
5	T^2+T+2	278800
7	T^2+1	7372800000
7	T^2+T+4	6567981056

Table 3. Discriminant of $\mathbb{T}(xy)$

there is no natural pairing between these modules, so our method is by trial-and-error. We essentially construct some $\mathbb{T}$-equivariant pairings, and check if one of those is perfect (see the discussion after (10.2)). This method is very inefficient, and our computer calculations terminated in a reasonable time only in the cases listed in Table 4.

10. Computing the action of Hecke operators

10.1. Action on $\mathcal{H}$. Assume $\mathfrak{n} = xy$. To simplify the notation, we denote $\mathcal{H} = \mathcal{H}_0(xy, \mathbb{Z})$, $\mathcal{H}' = \mathcal{H}(xy, \mathbb{Z})'$, $\mathbb{T} = \mathbb{T}(xy)$. Assume $x = T$. Theorem 6.8 in [9] gives a recipe for computing a matrix by which T_{x-s} acts on $\mathcal{H}$ for any $s \in \mathbb{F}_q^\times$. Since by Proposition 2.21 the operators $\{1, T_{x-s} \mid s \in \mathbb{F}_q^\times\}$ form a $\mathbb{Z}$-basis of $\mathbb{T}$, this essentially gives a complete description of the action of $\mathbb{T}$ on $\mathcal{H}$. This also allows us to compute the discriminant of $\mathbb{T}$, an interesting invariant measuring the congruences between Hecke eigenforms. (Recall that $\mathrm{disc}(\mathbb{T})$ is the determinant of the $q \times q$ matrix $\big(\mathrm{Trace}(t_i t_j)\big)_{1\leq i,j\leq q}$, where $\{t_1, \dots, t_q\}$ is a $\mathbb{Z}$-basis of $\mathbb{T}$.) Table 3 lists the values of $\mathrm{disc}(\mathbb{T})$ in some cases.

Remark 10.1. The algebra $\mathbb{T}(\mathfrak{n})\otimes\mathbb{C}_\infty$ is isomorphic to the Hecke algebra acting on doubly cuspidal Drinfeld modular forms of weight 2 and type 1 on $\Gamma_0(\mathfrak{n})$; see [18, (6.5)]. The algebra $\mathbb{T}(\mathfrak{n}) \otimes \mathbb{C}_\infty$ has no nilpotent elements if and only if $p \nmid \mathrm{disc}(\mathbb{T})$. Table 3 indicates that $p \nmid \mathrm{disc}(\mathbb{T})$ for $q = 3$ and arbitrary y, but for $q = 5$ or 7, there exist y_1 and y_2 such that $p \mid \mathrm{disc}(\mathbb{T}(xy_1))$ and $p \nmid \mathrm{disc}(\mathbb{T}(xy_2))$. It seems like an interesting problem to investigate the frequency with which p divides $\mathrm{disc}(\mathbb{T})$.

Remark 10.2. For the sake of completeness, and also because [9] is in German, we give Gekeler's method for computing a matrix $G(x-s) \in \mathrm{Mat}_q(\mathbb{Z})$, $s \in \mathbb{F}_q^\times$, representing the action of T_{x-s} on $\mathcal{H}$. Let $y = T^2 + aT + b$. Label the rows and columns of $G(x-s)$ by $u, w \in \mathbb{F}_q$. Then the (u, w) entry of $G(x-s)$ is equal to

$$2 - Q(u,w) - (q+1)\delta_{w,s} + q\delta_{u,0}\delta_{w,b/s},$$

where δ is Kroneker's delta, and $Q(u,w)$ is the number of solutions $\beta \in \mathbb{F}_q$ (without multiplicities) of the equation

$$(u-\beta)(w-\beta)(s-\beta)+\beta(\beta^2+a\beta+b)=0$$

plus 1 if $u+w+s+a=0$.

By comparing the discriminant of the characteristic polynomial of T_{x-s} with $\mathrm{disc}(\mathbb{T})$, one can deduce that in some cases $\mathbb{T}$ is monogenic, i.e., is generated by a single element as a $\mathbb{Z}$-algebra:

EXAMPLE 10.3. For $q=2$ and $y=T^2+T+1$

$$\mathbb{T}\cong\mathbb{Z}[T_{x-1}]\cong\mathbb{Z}[X]/X(X+2).$$

For $q=3$ and $y=T^2+T+2$

$$\mathbb{T}\cong\mathbb{Z}[T_{x-2}]\cong\mathbb{Z}[X]/(X+1)(X^2-X-4).$$

If $\mathbb{T}$ is monogenic, then its localization at any maximal ideal is Gorenstein; see [32, Thm. 23.5]. This is stronger than what we need for Theorem 9.6, but it is also computationally harder to establish. A simpler test is based on the following lemma:

LEMMA 10.4. *Let ℓ be a prime number dividing $(q+1)(q^2+1)$. Suppose there is an element η in $\mathfrak{E}$ such that*

$$\dim_{\mathbb{F}_\ell}\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\eta]=1.$$

Then $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein, where $\mathfrak{M}=(\mathfrak{E},\ell)$.

Proof. Note that the dimension of $\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\eta]$ is at least 1 since $\mathcal{E}_{00}(xy,\mathbb{F}_\ell)\cong\mathbb{F}_\ell$ is a subspace. Now consider the ideal $\mathfrak{I}=(\ell,\eta)\mathbb{T}_\ell$. We have $\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\mathfrak{I}]=\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\eta]$. On the other hand, $\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\mathfrak{I}]$ is $\mathbb{F}_\ell$-dual to $\mathbb{T}_\ell/\mathfrak{I}$, cf. Proposition 2.21. Hence, if $\dim_{\mathbb{F}_\ell}\mathcal{H}_{00}(xy,\mathbb{F}_\ell)[\eta]=1$, then $\mathbb{T}_\ell/\mathfrak{I}\cong\mathbb{F}_\ell$. This implies $\mathbb{T}_\ell/\mathfrak{M}\cong\mathbb{T}_\ell/\mathfrak{I}$. Since $\mathfrak{I}\subset\mathfrak{M}$, we get $\mathfrak{M}=(\ell,\eta)$, which implies that $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein by Proposition 15.3 in [33]. □

Any $\mathbb{Z}$-linear combination η of the operators $\{T_{x-s}-(q+1)\}_{s\in\mathbb{F}_q^\times}$ is in $\mathfrak{E}$. We can compute the characteristic polynomial of such η acting on $\mathcal{H}$ using Gekeler's method. Fix ℓ dividing $(q+1)(q^2+1)$. If we find η whose characteristic polynomial modulo ℓ does not have 0 as a multiple root, then we can apply Lemma 10.4 to conclude that $\mathbb{T}_{\mathfrak{M}}$ is Gorenstein for Eisenstein $\mathfrak{M}$ of residue characteristic ℓ. Using this strategy, for each prime $q\le 7$ we found by computer calculations an appropriate η for any ℓ dividing $(q+1)(q^2+1)$.

10.2. ACTION ON $\mathcal{H}'$. Suppose $y=T^2+aT+b$. We denote the place x by ∞'. Let $T'=T^{-1}$, $A'=\mathbb{F}_q[T']$, and $y'=T'^2+ab^{-1}T'+b^{-1}$. We have the correspondence of places of F:

$$\begin{array}{ccc}\infty & \longleftrightarrow & T'\\ T & \longleftrightarrow & \infty'\\ y & \longleftrightarrow & y'\end{array}$$

Let D be the quaternion algebra over F ramified precisely at ∞' and y' (i.e. ramified at x and y). Take an Eichler A'-order $\mathcal{D}'$ in D of level T'. More precisely, $\mathcal{D}'_{\mathfrak{p}'} := \mathcal{D}' \otimes_{A'} \mathcal{O}_{\mathfrak{p}'}$ is a maximal $\mathcal{O}_{\mathfrak{p}'}$-order in $D_{\mathfrak{p}'} := D \otimes_F F_{\mathfrak{p}'}$ for each prime $\mathfrak{p}'$ of A' with $\mathfrak{p}' \neq T'$, and there exists an isomorphism $\iota_{T'} : D_{T'} := D \otimes_F F_{T'} \cong \mathrm{Mat}_2(F_{T'})$ such that

$$\iota_{T'}(\mathcal{D}'_{T'}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}_{T'}) \;\middle|\; c \equiv 0 \bmod T' \right\}.$$

Let $\mathcal{O}_{D_{\infty'}}$ be the maximal $\mathcal{O}_{\infty'}$-order in $D_{\infty'}$. Consider the double coset spaces

$$\mathcal{G}'_{xy} := D^\times \backslash D^\times(\mathbb{A}_F) / \left(\widehat{\mathcal{D}'}^\times \cdot (\mathcal{O}_{D_{\infty'}}^\times F_{\infty'}^\times) \right) \text{ and } \mathrm{Cl}(\mathcal{D}') := D^\times \backslash D^\times(\mathbb{A}_F^{\infty'}) / \widehat{\mathcal{D}'}^\times,$$

where:

- $\mathbb{A}_F$ is the adele ring of F, i.e. $\mathbb{A}_F$ is the restricted direct product $\prod'_v F_v$;
- $\mathbb{A}_F^{\infty'}$ is the finite (with respect to ∞') adele ring of F, i.e. $\mathbb{A}_F^{\infty'} = \prod_{v \neq \infty'} F_v$;
- $D^\times(\mathbb{A}_F)$ (resp. $D^\times(\mathbb{A}_F^{\infty'})$) denotes $(D \otimes_F \mathbb{A}_F)^\times$ (resp. $(D \otimes_F \mathbb{A}_F^{\infty'})^\times$);
- $\widehat{\mathcal{D}'} = \prod_{\mathfrak{p}' \lhd A'} \mathcal{D}'_{\mathfrak{p}'}$.

Then the strong approximation theorem (with respect to $\{\infty\}$) shows that

Lemma 10.5. *The double coset space $\mathcal{G}'_{xy}$ can be identified with the set of the oriented edges of the quotient graph $\Gamma^{xy} \backslash \mathscr{T}$ in Figure 6.*

Note that $\mathrm{Cl}(\mathcal{D}')$ can be identified with the locally-principal right ideals of $\mathcal{D}'$ in D, and $\#\mathrm{Cl}(\mathcal{D}') = q+1$. Moreover, if we take $i_u \in \mathrm{GL}_2(F_{T'})$, $u \in \mathbb{P}^1(\mathbb{F}_q)$, to be

$$i_u = \begin{cases} \begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } u \in \mathbb{F}_q, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } u = \infty, \end{cases}$$

then, via the natural embedding $\mathrm{GL}_2(F_{T'}) \cong D_{T'}^\times \hookrightarrow D^\times(\mathbb{A}_F^{\infty'})$, $\{i_u \mid u \in \mathbb{P}^1(\mathbb{F}_q)\}$ is a set of representatives of $\mathrm{Cl}(\mathcal{D}')$. Take an element $\varpi_{\infty'} \in D_{\infty'}$ such that its reduced norm $\mathrm{Nr}(\varpi_{\infty'}) = T$. From the natural surjection from $\mathcal{G}'_{xy}$ to $\mathrm{Cl}(\mathcal{D}')$, one observes that

$$\left\{ (i_u, \varpi_{\infty'}^c) \in D^\times(\mathbb{A}_F^{\infty'}) \times D_{\infty'}^\times = D^\times(\mathbb{A}_F) \mid u \in \mathbb{P}^1(\mathbb{F}_q),\ c = 0, 1 \right\}$$

is a set of representatives of $\mathcal{G}'_{xy}$. We may take $e_u := [i_u, 1] \in \mathcal{G}'_{xy}$. Then there exists a unique permutation $\gamma : \mathbb{P}^1(\mathbb{F}_q) \to \mathbb{P}^1(\mathbb{F}_q)$ of order 2 so that

$$\overline{e_u} := \left[i_u \begin{pmatrix} 0 & 1 \\ T' & 0 \end{pmatrix}, 1 \right] = [i_{\gamma(u)}, \varpi_{\infty'}].$$

Moreover, $\mathcal{H}'$ can be viewed as the set of $\mathbb{Z}$-valued functions f on $\mathcal{G}'_{x,y}$ satisfying

$$f(e_u) + f(\overline{e_u}) = 0,\ \forall u \in \mathbb{P}^1(\mathbb{F}_q) \quad \text{and} \quad \sum_{u \in \mathbb{P}^1(\mathbb{F}_q)} f(e_u) = 0.$$

Let I_u be the right ideal of $\mathcal{D}'$ in D corresponding to i_u, i.e.,

$$I_u := D \cap i_u \cdot \widehat{\mathcal{D}'}.$$

Then the reduced norm of I_u is trivial, i.e., the ideal of A' generated by the reduced norms of all elements in I_u is A'. For each ideal $\mathfrak{m}' \lhd A'$, the $\mathfrak{m}'$-*th Brandt matrix* $B(\mathfrak{m}') = \big(B_{u,u'}(\mathfrak{m}')\big)_{u,u'} \in \mathrm{Mat}_{q+1}(\mathbb{Z})$ is defined by

$$B_{u,u'}(\mathfrak{m}') := \frac{\#\{b \in I_u I_{u'}^{-1} : \mathrm{Nr}(b) \cdot A' = \mathfrak{m}'\}}{q-1}.$$

In Section 10.3, we give an explicit recipe of computing $B_{u,u'}(\mathfrak{m}')$ for $\deg(\mathfrak{m}') = 1$ when q is odd and the constant term b of y is not a square in $\mathbb{F}_q^\times$, and also when $q = 2$. By an analogue of Hecke's theory (cf. [5]) we obtain the following result:

Proposition 10.6.

(1) *Viewing γ as a permutation matrix in $\mathrm{Mat}_{q+1}(\mathbb{Z})$, one has*

$$B(T') = 2 \cdot J - \gamma,$$

where every entry of J is 1.

(2) *Identifying the place $x - s$ and $T' - s^{-1}$ of F, one has that for $s \in \mathbb{F}_q^\times$,*

$$T_{x-s} e_u = \sum_{u' \in \mathbb{P}^1(\mathbb{F}_q)} B_{u,\gamma(u')}(T' - s^{-1}) \overline{e_{u'}}.$$

Proof. By Lemma II.5 in [5], we observe that for every $u \in \mathbb{F}_q$,

$$[i_{\gamma(u)}] + \sum_{u' \in \mathbb{F}_q} B_{u,u'}(T')[i_{u'}] = 2 \sum_{u'' \in \mathbb{F}_q} [i_{u''}] \in \mathbb{Z}[\mathrm{Cl}(\mathcal{D}')].$$

This shows (1). To prove (2), notice that for every $g_v \in \mathrm{GL}_2(F_v)$ with $v \neq \infty'$ and $u \in \mathbb{F}_q$, there exists $u' \in \mathbb{F}_q$ such that

$$[i_u g_v, 1] = [i_{u'}, \varpi_{\infty'}^c],$$

where $c = \mathrm{ord}_v(\det g_v) \cdot \deg v$. By Proposition II.4 in [5] we have that for $s \in \mathbb{F}_q^\times$,

$$\begin{aligned} T_{x-s} e_u &= \sum_{u' \in \mathbb{P}^1(\mathbb{F}_q)} B_{u,u'}(T' - s^{-1})[i_{u'}, \varpi_{\infty'}] \\ &= \sum_{u' \in \mathbb{P}^1(\mathbb{F}_q)} B_{u,u'}(T' - s^{-1}) \overline{e_{\gamma(u')}} \\ &= \sum_{u' \in \mathbb{P}^1(\mathbb{F}_q)} B_{u,\gamma(u')}(T' - s^{-1}) \overline{e_{u'}}. \end{aligned}$$

$\square$

For $u \in \mathbb{F}_q$, let $f_u \in \mathcal{H}'$ such that $f_u(e'_u) = \delta_{u,u'}$ for $u' \in \mathbb{F}_q$ and $f_u(e_\infty) = -1$. We immediately get the following.

Corollary 10.7. *For $u, u' \in \mathbb{F}_q$ and $s \in \mathbb{F}_q^\times$, set*

$$B'_{u',u}(x-s) := B_{u',\gamma(\infty)}(T' - s^{-1}) - B_{u',\gamma(u)}(T' - s^{-1}).$$

Then

$$T_{x-s} f_u = \sum_{u' \in \mathbb{F}_q} B'_{u',u}(x-s) f_{u'}.$$

In other words, $B'(x-s) := \big(B'_{u',u}(x-s)\big)_{u',u} \in \mathrm{Mat}_q(\mathbb{Z})$ is a matrix representation of T_{x-s} acting on $\mathcal{H}'$ with respect to the basis $\{f_u \mid u \in \mathbb{F}_q\}$.

Remark 10.8. From (2.7) we know that $\sum_{s\in\mathbb{F}_q} T_{x-s} = -1$, as endomorphisms of $\mathcal{H}$. On the other hand, by Remark 9.2, $\mathrm{JL} : \mathcal{H}\otimes\mathbb{Q} \cong \mathcal{H}'\otimes\mathbb{Q}$ is T_x-equivariant. Hence the previous corollary allows us to obtain also the matrix representation of T_x acting on $\mathcal{H}'$.

Remark 10.2 and Corollary 10.7 give the matrices by which T_{x-s} ($s \in \mathbb{F}_q^\times$) acts on $\mathcal{H}$ and $\mathcal{H}'$. Since T_{x-s} generate $\mathbb{T}$, the condition that the $\mathbb{T}$-modules $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic in Theorem 9.5 is equivalent to the matrices $B'(x-s)$ being simultaneously $\mathbb{Z}$-conjugate to Gekeler's matrices $G(x-s)$, i.e., to the existence of a single matrix $C \in \mathrm{GL}_q(\mathbb{Z})$ such that

$$C^{-1} \cdot B'(x-s) \cdot C = G(x-s) \qquad \forall s \in \mathbb{F}_q^\times. \tag{10.1}$$

Remark 10.9. Due to Jacquet-Langlands correspondence, there does exist a matrix $C \in \mathrm{GL}_q(\mathbb{Q})$ satisfying (10.1), but the existence of an integral matrix is more subtle; cf. [27].

Example 10.10. Let $q = 2$ and $y = T^2 + T + 1$. By Remark 10.2

$$G(x-1) = \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}.$$

On the other hand, with respect to the basis $\{i_\infty, i_0, i_1\}$ of $\mathbb{Z}[\mathrm{Cl}(\mathcal{D}')]$ we calculate that (see Remark 10.12)

$$\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B(T'-1) = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore

$$B'(x-1) = \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}.$$

We can take $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note that the matrix $\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ is conjugate to $G(x-1)$ over $\mathbb{Q}$, but not over $\mathbb{Z}$; in fact, there are exactly two conjugacy classes of matrices in $\mathrm{Mat}_2(\mathbb{Z})$ with characteristic polynomial $X(X+2)$.

q	y	α
3	T^2+T+2	$[0,1,0]$
3	T^2+2T+2	$[0,0,1]$
5	T^2+T+2	$[-1,1,4,5,2]$
5	T^2+2T+3	$[-1,-3,-6,-5,-2]$
5	T^2+3T+3	$[-1,-6,-3,-2,-5]$
5	T^2+4T+2	$[-1,4,1,2,5]$
7	T^2+T+6	$[-8,0,-6,-5,-8,-7,5]$
7	T^2+2T+3	$[-8,-7,-7,2,3,-6,-6]$
7	T^2+3T+5	$[-5,-6,-6,-4,2,3,-5]$
7	T^2+4T+5	$[-8,-8,0,-7,-6,5,-5]$
7	T^2+5T+3	$[-5,-4,-5,-6,3,-6,2]$
7	T^2+6T+6	$[-5,-6,2,-5,-6,-4,3]$

Table 4. Available choice of α

There exists an algorithm for deciding whether, for two collections of integral matrices $\{X_1,\dots,X_m\}$ and $\{Y_1,\dots,Y_m\}$, there exists an integral and integrally invertible matrix C relating them via conjugation, i.e., such that $C^{-1}X_iC=Y_i$ for all i; see [48]. Unfortunately, this algorithm is complicated and does not seem to have been implemented into the standard computational programs, such as `Magma`. Instead of trying to find C, we take a different approach to proving that the $\mathbb{T}$-modules $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic. Note that the pairing in Proposition 2.21 gives an isomorphism $\mathcal{H}\cong \mathrm{Hom}(\mathbb{T},\mathbb{Z})$ of $\mathbb{T}$-modules. If one constructs a perfect $\mathbb{T}$-equivariant pairing

$$\mathcal{H}'\times\mathbb{T}\to\mathbb{Z}, \tag{10.2}$$

then the desired isomorphism $\mathcal{H}'\cong\mathrm{Hom}(\mathbb{T},\mathbb{Z})\cong\mathcal{H}$ follows. The absence of Fourier expansion in the quaternionic setting makes the construction of such a pairing ad hoc.

Note that $\mathrm{Hom}(\mathcal{H}',\mathbb{Z})=\bigoplus_{u\in\mathbb{F}_q}\mathbb{Z}e_u^*$, where $\langle f_u,e_{u'}^*\rangle:=f_u(e_{u'})=\delta_{u,u'}$. One way to construct (10.2) is to find a $\mathbb{Z}$-linear combination $\alpha=\sum_u a_ue_u^*$ such that

$$\det\left(\langle f_u,T_{x-s}\alpha\rangle\right)_{u,s\in\mathbb{F}_q}=\pm1.$$

We were able to find such α in several cases; see Table 4 where α is given as $[a_0,a_1,\dots,a_{p-1}]$.

10.3. Computation of Brandt matrices. Recall that we denote $y=T^2+aT+b\in A$ and $y'=T'^2+ab^{-1}T+b^{-1}$, where $T'=1/T$. Assume q is odd and b is not a square in $\mathbb{F}_q^\times$. Set $y_0:=bT'^2+aT'+1$. Let

$$D:=F+Fi+Fj+Fij$$

where $i^2=T'$, $j^2=y_0$, $ij=-ji$. Since we have the Hilbert quadratic symbols $(T',y_0)_T=(T',y_0)_y=-1$ and $(T',y_0)_v=1$ for every places $v\neq T$ or y, the quaternion algebra D is ramified precisely at T and y.

Let $\mathcal{D}' = A' + A'i + A'j + A'ij$, which is an Eichler A'-order in D of level $T'y'$. We choose the representatives of locally principal right ideal classes of $\mathcal{D}'$ as follows.

$$\begin{aligned} I_\infty &:= \mathcal{D}' = A' + A'i + A'j + A'ij, \\ I_u &:= A'(1-j) + A'(\frac{i+ij}{T'} + 2uj) + A'T'j + A'ij, \quad u \in \mathbb{F}_q. \end{aligned}$$

Then the reduced norm of I_u is trivial for every $u \in \mathbb{P}^1(\mathbb{F}_q)$. Moreover, for $u, u' \in \mathbb{F}_q$, $I_u I_{u'}^{-1}$ is equal to

$$\begin{cases} A' + A'T'i + A'(j - 2ui) + A\left(\frac{i(1+j)^2}{T'} + 4u^2 i\right), & \text{if } u = u', \\ A'T' + A\big(i - (u'-u)^{-1}\big) + A\left(j - \frac{u'+u}{u'-u}\right) + A\left(\frac{i(1+j)^2}{T'} + \frac{4uu'}{u'-u}\right), & \text{if } u \neq u'. \end{cases}$$

After tedious calculations, we obtain the following:

Lemma 10.11.

(1) *For* $s \in \mathbb{F}_q$,

$$\frac{\#\{z \in I_\infty \mid \mathrm{Nr}(z)A' = (x-s)A'\}}{q-1} = \begin{cases} 2 & \text{if } s \in (\mathbb{F}_q^\times)^2, \\ 1 & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For* $s, u \in \mathbb{F}_q$, *set*

$$\alpha_y(s,u) := 1 + s(4u^2 + a + sb) \quad \text{and} \quad \beta_y(s,u) = \big((a+4u^2)^2 - 4b\big)s + 16u^2.$$

Then

$$\frac{\#\{z \in I_u \mid \mathrm{Nr}(z)A' = (x-s)A'\}}{q-1} = \begin{cases} 2 & \text{if } \alpha_y(s,u) \in (\mathbb{F}_q^\times)^2, \\ 1 & \text{if } \alpha_y(s,u) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{\#\{z \in I_u I_u^{-1} \mid \mathrm{Nr}(z)A' = (x-s)A'\}}{q-1} = \begin{cases} 2 & \text{if } \beta_y(s,u) \in (\mathbb{F}_q^\times)^2, \\ 1 & \text{if } \beta_y(s,u) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(3) *For* $s, u, u' \in \mathbb{F}_q$ *with* $u \neq u'$, *set*

$$\begin{aligned} \xi_y(s,u,u') :=& \big(2u^2 + 2u'^2 + s(u'-u)a\big)^2 \\ & - \big(1 - s(u'-u)^2\big)\big(16u^2u'^2 - s(u'-u)^2(a^2-4b)\big). \end{aligned}$$

Then

$$\frac{\#\{z \in I_u I_{u'}^{-1} \mid \mathrm{Nr}(z)A' = (x-s)A'\}}{q-1} = \begin{cases} 2 & \text{if } \xi_y(s,u) \in (\mathbb{F}_q^\times)^2, \\ 1 & \text{if } \xi_y(s,u) = 0, \\ 0 & \text{otherwise;} \end{cases}$$

Since the Brandt matrices are symmetric under our settings, the above lemma gives us the recipe of computing the Brandt matrices $B(T'-s)$ for $s \in \mathbb{F}_q$. In particular, we can get

$$\gamma(\infty) = \infty, \quad \gamma(0) = 0, \quad \text{and} \quad \gamma(u) = -u \quad \forall u \in \mathbb{F}_q^\times.$$

Remark 10.12. In characteristic 2 the presentation of quaternion algebras has to be modified. Consider the case when $q = 2$ and $y = T^2 + T + 1$. In this case, we let $D = F + Fi + Fj + Fij$, where

$$i^2 + i = T', \quad j^2 = y_0 = T'^2 + T' + 1, \quad ji = (i+1)j.$$

Let $\mathcal{D}'_{\max} = A + Ai + Aj + Aij$, which is a maximal A'-order in D. We take the Eichler A'-order $\mathcal{D}' = A' + A'i + A'T'j + A'ij$ and the representatives of ideal classes of $\mathcal{D}'$ in the following:

$$\begin{aligned} I_\infty &= \mathcal{D}' = A' + A'i + A'T'j + A'ij, \\ I_0 &= A'T' + A'(1+i) + A'j + A'ij, \\ I_1 &= A'T' + A'(1+i) + A'(1+j) + A'ij. \end{aligned}$$

Then

$$\begin{aligned} I_0 I_0^{-1} &= A' + A'i + A'xj + A'(i+1)j, \\ I_0 I_1^{-1} &= A'T' + A'i + A'(1+j) + A'(1+ij), \\ I_1 I_1^{-1} &= A' + A'T'i + A'j + A'(i+ij). \end{aligned}$$

Since

$$\{z \in \mathcal{D}'_{\max} \mid \mathrm{Nr}(z) = T'\} = \{i, 1+i, 1+T'+j\}$$

and

$$\{z \in \mathcal{D}'_{\max} \mid \mathrm{Nr}(z) = 1+T'\} = \{T'+j, T'+i+j, 1+T'+i+j\},$$

we can get

$$B(T') = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B(T'-1) = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Acknowledgements. Part of this work was carried out while the first author was visiting Taida Institute for Mathematical Sciences in Taipei and National Center for Theoretical Sciences in Hsinchu. He heartily thanks Professor Winnie Li and Professor Jing Yu for their invitation, and the members at these institutions for creating a welcoming and productive atmosphere.
This research was completed while the second author was visiting Max Planck Institute for Mathematics. He wishes to thank the institute for kind hospitality and good working conditions.

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