Invariant Blocks Under Coprime Actions

Gunter Malle, Gabriel Navarro, Britta Späth[1]

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Abstract. If a finite group A acts coprimely as automorphisms on a finite group G, then the A-invariant Brauer p-blocks of G are exactly those that contain A-invariant irreducible characters.

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1. Introduction

One of the most basic situations in group theory is when a group A acts by automorphisms on another group G. If we further assume that A and G are finite of coprime orders, it is well-known that most of the representation theory of G admits a version in which only the A-invariant structure is taken into account. For instance, it is true (and not trivial) that the number of irreducible complex characters of G which are fixed by A equals the number of conjugacy classes of G fixed by A.

Although it is fair to say that the ordinary A-representation theory of G is mostly well developed, we cannot say the same about A-invariant modular representation theory (that is, of prime characteristic p). For instance, it is suspected that the number of A-invariant irreducible p-Brauer characters of G is the number of A-invariant p-regular classes of G, but this conjecture continues to be open. This, together with some other problems, was proposed more than 20 years ago in [13].

The extensive research during these years on Brauer p-blocks allows us now to give a solution to Problem 6 of [13].

Theorem 1.1. *Suppose that the finite group A acts by automorphisms on the finite group G with $(|A|, |G|) = 1$. Let p be a prime, and let B be a Brauer p-block of G. Then the following statements are equivalent:*

(i) *B is A-invariant,*
(ii) *B contains some A-invariant character $\chi \in \mathrm{Irr}(G)$,*
(iii) *B contains some A-invariant Brauer character $\phi \in \mathrm{IBr}(G)$.*

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Of course, (ii) and (iii) easily imply that B is A-invariant, so all the work is concentrated in proving that (i) implies (ii) and (iii).
The paper is structured in the following way. In Section 2 a version of the main theorem is proven in the case where G is quasi-simple. Afterwards in Section 3 we present a Gallagher type theorem for blocks. In connection with Dade's ramification group from [2], revisited in Section 4, we show the existence of character triple isomorphisms having crucial properties with respect to coprime action and blocks (see Section 5). We conclude in the final section with the reductions proving how the results on quasi-simple groups imply our main statement.

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2. Quasi-simple groups and their central products

The aim of this section is the proof of a strengthened version of Theorem 1.1 in cases where the group $G/\mathbf{Z}(G)$ is the direct product of r isomorphic non-abelian simple groups. First we deal with the case where G is the universal covering group of a simple group and hence G is a quasi-simple group.
In the following we use the standard notation around characters and blocks as introduced in [7] and [14]. Let p be a prime. If A acts on G, we denote by $\mathrm{Bl}_A(G)$ the set of A-invariant p-blocks of G. If $Z \lhd G$, $B \in \mathrm{Bl}(G)$ and $\nu \in \mathrm{Irr}(Z)$ we denote by $\mathrm{Irr}(B|\nu)$ the set $\mathrm{Irr}(B) \cap \mathrm{Irr}(G|\nu)$. Also, sometimes we will work in GA, the semidirect product of G with A.

Theorem 2.1. *Let G be the universal covering group of a non-abelian simple group S, A a group acting on G with $(|G|, |A|) = 1$, B an A-invariant p-block of G, Z the Sylow p-subgroup of $\mathbf{Z}(G)$ and $\nu \in \mathrm{Irr}(Z)$ A-invariant. Then B contains some A-invariant character $\chi \in \mathrm{Irr}(G|\nu)$.*

This theorem is true whenever B is the principal block and $Z = 1$, since then χ can be chosen to be the trivial character. On the other hand when B is a block of central defect, $\mathrm{Irr}(B|\nu)$ contains exactly one character and hence this one is A-invariant.
Note that neither alternating nor sporadic simple groups possess coprime automorphisms, and that for groups of Lie type the only coprime order automorphisms are field automorphisms (up to conjugation). Thus, for the proof of the theorem, we can assume that S is simple of Lie type. We consider the following setup. Let $\mathbf{G}$ be a simple algebraic group of simply connected type over an algebraic closure of a finite field of characteristic r, and let $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism, with group of fixed points $G := \mathbf{G}^F$. It is well known that all finite simple groups of Lie type occur as $G/\mathbf{Z}(G)$ with G as before, except for the Tits group ${}^2F_4(2)'$. Since the latter does not possess coprime automorphisms, we need not consider it here. Furthermore, in all but finitely many cases, the group G is the universal covering group of $S = G/\mathbf{Z}(G)$. None of the exceptions, listed for example in [11, Table 24.3], has coprime automorphisms, apart from the Suzuki group ${}^2B_2(8)$. But for $2^2.{}^2B_2(8)$, the outer automorphism of order three permutes the three non-trivial central characters, hence our claim holds. For the proof of Theorem 2.1 we may thus assume that G

and S are as above, and that A induces a (necessarily cyclic) group of coprime field automorphisms on G.

We first discuss the action of such automorphisms on the Lusztig series $\mathcal{E}(G,s) \subseteq \mathrm{Irr}(G)$ of irreducible characters of G, where s runs over semisimple elements of a dual group $G^* = \mathbf{G}^{*F}$ of G. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding, that is, $\tilde{\mathbf{G}}$ is connected reductive with connected center and with derived subgroup $\mathbf{G}$. Corresponding to this there exists a surjection $\tilde{\mathbf{G}}^* \to \mathbf{G}^*$ of dual groups. Note that all field automorphisms of $\mathbf{G}$ are induced by those of $\tilde{\mathbf{G}}$. Let γ be a field automorphisms of $\tilde{\mathbf{G}}$. We denote the corresponding field automorphism of $\tilde{\mathbf{G}}^*$ also by γ. Let $\tilde{s} \in \tilde{G}^* := \tilde{\mathbf{G}}^{*F}$ be semisimple. Now by [10, Prop. 3.5], γ acts trivially on $\mathcal{E}(\tilde{G},\tilde{s})$ whenever it stabilises $\mathcal{E}(\tilde{G},\tilde{s})$. Let $s \in G^*$ with preimage $\tilde{s}$. Then $\mathcal{E}(G,s)$ consists of the constituents of the restrictions of characters in $\mathcal{E}(\tilde{G},\tilde{s})$ to G, see e.g. [1, Prop. 15.6(i)].

Lemma 2.2. *In the above setting let γ be a coprime (field) automorphism of G. If γ stabilises $\mathcal{E}(G,s)$, then it fixes $\mathcal{E}(G,s)$ pointwise.*

Proof. According to what we said before, γ can only permute the G-constituents of a fixed character $\chi \in \mathcal{E}(\tilde{G},\tilde{s})$. We have $\tilde{\mathbf{G}} = \mathbf{G}\mathbf{Z}(\tilde{\mathbf{G}})$, so $|\tilde{G}| = |G|\,|\mathbf{Z}(\tilde{G})|$ and hence

$$|\tilde{G} : G\mathbf{Z}(\tilde{G})| = \frac{|\tilde{G}|\,|G \cap \mathbf{Z}(\tilde{G})|}{|G|\,|\mathbf{Z}(\tilde{G})|} = |G \cap \mathbf{Z}(\tilde{G})| = |\mathbf{Z}(G)|.$$

As irreducible characters of the central product $G\mathbf{Z}(\tilde{G})$ restrict irreducibly to G, any irreducible character of $\tilde{G}$ has at most $|\mathbf{Z}(G)|$ irreducible constituents upon restriction to G. It is easily checked that all primes not larger than $|\mathbf{Z}(G)|$ divide $|G|$, so that all prime divisors of the order of γ are larger than the number of such constituents. Thus the action has to be trivial. □

There are two quite different types of behaviour now. Either p is the defining characteristic of G, then coprime field automorphisms fix all p-blocks (but certainly not all irreducible characters); or p is different from the defining characteristic, in which case all characters in an invariant block are fixed individually (but not all p-blocks are invariant):

Proposition 2.3. *In the above situation, Theorem 2.1 holds when p is the defining characteristic of G.*

Proof. In this case the p-blocks of positive defect of a group of Lie type G are in bijection with the characters of $\mathbf{Z}(G)$, by a result of Humphreys [5]. Since the claim is certainly true for the principal block, we may assume that $\mathbf{Z}(G) \neq 1$, and so in particular G is not a Suzuki or Ree group. For each type of group and each γ-stable $1 \neq \nu \in \mathbf{Z}(G)$ we give in Table 1 a semisimple element s of the dual group G^* of G with the following properties: the Lusztig series $\mathcal{E}(G,s)$ of irreducible characters lies in $\mathrm{Irr}(G|\nu)$, and the class of s is γ-invariant (since s corresponds to the γ-stable central character ν). It then follows that $\mathcal{E}(G,s)$ is stable under all field automorphisms of G that stabilise ν, and this implies by Lemma 2.2 that the characters in $\mathcal{E}(G,s)$ are individually stable, hence provide characters as claimed. □

Let us now turn to the case where p is not the defining characteristic of G.

Table 1. Semisimple elements

G	$C_{G^*}(s)$	$o(\nu)$	conditions
$\mathrm{SL}_n(q)$	$\mathrm{GL}_{n-1}(q)$	divides $(n, q-1)$	$(n, q-1) > 1$
$\mathrm{SU}_n(q)$	$\mathrm{GU}_{n-1}(q)$	divides $(n, q+1)$	$(n, q+1) > 1$
$\mathrm{Spin}_{2n+1}(q)$	C_{n-1}	2	q odd
$\mathrm{Sp}_{2n}(q)$	$\mathrm{GO}_{2n}^{\pm}(q)$	2	q odd
$\mathrm{Spin}_{2n}^{\pm}(q)$	B_{n-1}	divides 4	q odd
$E_6(q)$	D_5	3	$q \equiv 1 \pmod 3$
${}^2E_6(q)$	${}^2D_5$	3	$q \equiv 2 \pmod 3$
$E_7(q)$	E_6	2	$q \equiv 1 \pmod 4$
$E_7(q)$	${}^2E_6$	2	$q \equiv 3 \pmod 4$

Proposition 2.4. *In the above situation, when p is different from the defining characteristic of G, if B is a γ-invariant p-block of G, then all $\chi \in \mathrm{Irr}(B)$ are fixed by γ. In particular Theorem 2.1 holds in this case.*

Proof. Let B be a p-block of G, $Z = \mathbf{Z}(G)_p$ and $\nu \in \mathrm{Irr}(Z)$. Let γ be a coprime (field) automorphism of G fixing B and ν. By a result of Broué and Michel there exists a semisimple p'-element $s \in G^*$ such that $B \subseteq \mathcal{E}_p(G, s)$. Let $G \hookrightarrow \tilde{G}$ be a regular embedding, with corresponding epimorphism $\tilde{G}^* \to G^*$ of dual groups. Let $\tilde{s} \in \tilde{G}^*$ be a preimage of s. Since the class of s is γ-stable, the same argument as in the proof of Lemma 2.2 shows that the class of $\tilde{s}$ is also γ-stable. Since the centraliser of $\tilde{s}$ is connected, this means that we may assume without loss of generality that $\tilde{s}$ itself is γ-stable, and so is $C_{\tilde{G}^*}(\tilde{s})$. Now consider $H := (\tilde{G}^*)^\gamma$, the fixed point subgroup of $\tilde{G}^*$ under γ. This is again a group of Lie type, of the same type as $\tilde{G}^*$. Since p divides $|\mathbf{Z}(G)|$ by assumption, it also divides the order of the Weyl group of G. By [10, Prop. 3.12], then $C_H(\tilde{s}) = C_{\tilde{G}^*}(\tilde{s})^\gamma$ contains a Sylow p-subgroup of $C_{\tilde{G}^*}(\tilde{s})$. In particular, every semisimple p-element in $C_{\tilde{G}^*}(\tilde{s})$ has a γ-stable conjugate $\tilde{t} \in \tilde{G}^*$. Then $\mathcal{E}(G, \tilde{s}\tilde{t})$ is γ-stable, which implies that $\mathcal{E}(G, st)$ is γ-stable, and hence fixed pointwise by γ, by Lemma 2.2. Thus, all elements of $B \cap \mathcal{E}_p(G, s)$ are fixed by γ, as claimed. □

We next prove an analogous result for Brauer characters:

Theorem 2.5. *Let G be the universal covering group of a non-abelian simple group S, A a group acting on G with $(|G|, |A|) = 1$, and $B \in \mathrm{Bl}_A(G)$. Then there exists an A-invariant Brauer character $\phi \in \mathrm{IBr}(B)$.*

Proof. As argued in the proof of Theorem 2.1 we may assume that S is of Lie type and A induces coprime field automorphisms. Moreover, G is not an exceptional covering group of S. First assume that p is the defining characteristic of G. Let B be an A-invariant block of G, corresponding to the central character ν of G. Then there is a faithful irreducible Brauer character ϕ of $G/\ker(\nu)$ corresponding to a suitable fundamental weight ω of the underlying algebraic group as given in Table 2. Note that we only need to consider cases when $Z(G) \neq 1$, which explains the restrictions in the last column of the table. If G is untwisted, defined over $\mathbb{F}_q$ with $q = p^f$, then a generator γ of A has order a with $f = ka$. By Steinberg's tensor product theorem (see [11, Thm. 16.12]) then $\phi' := \bigotimes_{i=0}^{a-1} \gamma^i(\phi)$ is an irreducible Brauer character of G corresponding to the weight $\sum_{i=0}^{a-1} p^{ik}\omega$, which is γ-invariant. It lies over the character ν^d of $\mathbf{Z}(G)$, with $d = \sum_{i=0}^{a-1} p^{ik}$. Since $|\mathbf{Z}(G)|$ divides $p^k - 1$ and a is prime to $|\mathbf{Z}(G)|$, this is again a faithful character of $\mathbf{Z}(G)$. Thus, this construction yields an invariant Brauer character in the p-block lying above ν^d. Starting instead with ϕ in the p-block above ν^c, with $cd \equiv 1 \pmod{o(\nu)}$, we find an invariant character in B.

TABLE 2. Faithful Brauer characters

G	$\phi(1)$	weight	conditions
$\mathrm{SL}_n(q), \mathrm{SU}_n(q)$	$\binom{n}{i}$	$\omega_i\ (1 \leq i \leq n-1)$	$(n, q \pm 1) > 1$
$\mathrm{Spin}_{2n+1}(q)$	2^n	ω_n	q odd
$\mathrm{Sp}_{2n}(q)$	$2n$	ω_1	q odd
$\mathrm{Spin}_{2n}^{\pm}(q)$	$2n, 2^{n-1}$	$\omega_1, \omega_{n-1}, \omega_n$	q odd
$E_6(q), {}^2E_6(q)$	27	ω_1	$3 \nmid q$
$E_7(q)$	56	ω_7	q odd

If G is twisted, we may assume that it is not of type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$ and that the twisting has order 2 (since else there is just one p-block of positive defect). So G is defined over $\mathbb{F}_{q^2}$, with $q = p^f$ and $a|f$. The same argument as before applies in this case as well.
Now assume that p is different from the defining characteristic of G, and let B be an A-invariant p-block. Then B is contained in $\mathcal{E}_p(G, s)$ for some semisimple element $s \in G^*$, and we showed in Proposition 2.4 that all elements in B are fixed by A. Since any irreducible Brauer character in B is an integral linear combination of ordinary irreducible characters in B restricted to p'-classes, it follows that $\mathrm{IBr}(B)$ is fixed point-wise by A as well. □

We conclude this section with the analogous result on central products of quasi-simple groups.

COROLLARY 2.6. *Let G be a finite group and let A act on G with $(|G|, |A|) = 1$. Assume that $G/\mathbf{Z}(G)$ is the direct product of r isomorphic non-abelian simple groups that are transitively permuted by A. Let $B \in \mathrm{Bl}(G)$ be A-invariant.*

(a) *Let $Z \in \mathrm{Syl}_p(\mathbf{Z}(G))$ and $\nu \in \mathrm{Irr}(Z)$. Then B contains some A-invariant $\chi \in \mathrm{Irr}(G|\nu)$.*
(b) *B contains some A-invariant $\phi \in \mathrm{IBr}(G)$.*

Proof. Both parts can be shown analogously. We give here the proof of part (a). Let S be the simple non-abelian group such that $G/\mathbf{Z}(G)$ is isomorphic to the r-fold direct product of groups isomorphic to S.

It suffices to prove the statement in the case where G is perfect. Indeed, we have $G = [G,G]\mathbf{Z}(G)$ by the given structure of G. Assume that the statement holds in the case of perfect groups. Hence the block $B' \in \mathrm{Bl}([G,G])$ covered by B has then an A-invariant character χ_0 lying over $\nu_{\mathbf{Z}(G)\cap[G,G]}$. The character $\chi_0 \cdot \nu$ defined as in [8, Section 5] as the unique character in $\mathrm{Irr}(G|\chi_0) \cap \mathrm{Irr}(G|\nu)$ is then necessarily A-invariant.

In the following we consider the case where G is perfect. Accordingly G has a universal covering group, namely

$$X := \widehat{S} \times \cdots \times \widehat{S} = \widehat{S}^r \text{ (r factors)} ,$$

where $\widehat{S}$ is the universal covering group of S. Let $\epsilon : X \to G$ be the associated canonical epimorphism. Because of [3, 5.1.4] there is a canonical action of A on X induced by the action of A on G, such that ϵ is A-equivariant. Note that the action of A on X is coprime, since the prime divisors of $|G|$ and $|X|$ coincide.

For $1 \leq i \leq r$ let $X_i := \{(1,\dots,1,x,1,\dots,1) \mid x \in \widehat{S}\}$, which is canonically isomorphic to $\widehat{S}$ via $\iota_i : x \mapsto (1,\dots,1,x,1,\dots,1)$. By assumption A acts transitively on the set of groups X_i. So for every $1 \leq i \leq r$ there exists an element $a_i \in A$ such that $X_1^{a_i} = X_i$.

The character $\nu \in \mathrm{Irr}(Z)$ can be uniquely extended to a character $\widetilde{\nu} \in \mathrm{Irr}(\mathbf{Z}(G))$ such that $\mathrm{Irr}(B|\widetilde{\nu}) \neq \emptyset$. Via ϵ the character $\widetilde{\nu}$ corresponds to some character $\widehat{\nu} \in \mathrm{Irr}(\mathbf{Z}(X))$ that can be written as

$$\widehat{\nu} = \widehat{\nu}_1 \times \cdots \times \widehat{\nu}_r$$

for suitable $\widehat{\nu}_i \in \mathrm{Irr}(\mathbf{Z}(\widehat{S}))$

The block B corresponds to a unique block $\widehat{B} \in \mathrm{Bl}(X)$, see [14, (9.9) and (9.10)]. Accordingly $\widehat{B}$ is A-invariant and can be written as $\widehat{B} = \widehat{B}_1 \times \cdots \times \widehat{B}_r$ where $\widehat{B}_i \in \mathrm{Bl}(\widehat{S})$. Let A_i be the stabiliser of X_i in A. The action of A_i on X_i induces via ι_i a coprime action of A_i on $\widehat{S}$ stabilising $\widehat{B}_i$. According to Theorem 2.1 there exists an A_1-invariant character $\psi_1 \in \mathrm{Irr}(\widehat{B}_1|\widehat{\nu}_1)$.

There exists a unique $\psi_i \in \mathrm{Irr}(\widehat{S})$ with $(\psi_1 \circ \iota_1^{-1})^{a_i} = \psi_i \circ \iota_i^{-1}$. Since $\widehat{B}$ and $\widehat{\nu}$ are A-invariant the character ψ_i belongs to $\mathrm{Irr}(\widehat{B}_i|\widehat{\nu}_i)$. Accordingly the character $\psi := \psi_1 \times \cdots \times \psi_r$ belongs to $\mathrm{Irr}(\widehat{B}|\widehat{\nu})$.

In the next step we prove that ψ is A-invariant. Let $\phi_i := \psi_i \circ \iota_i^{-1} \in \mathrm{Irr}(X_i)$. Then it is sufficient to prove $\phi_i^a = \phi_j$ for any $a \in A$ and every $1 \leq i,j \leq r$ with $X_i^a = X_j$. The equality $X_i^a = X_j$ implies $(X_1^{a_i})^a = X_1^{a_j}$, and hence $a_i a a_j^{-1} \in \mathbf{N}_A(X_1) = A_1$. On the other hand by the definition of ψ_i and ϕ_i we have

$$(\phi_i)^a = (\phi_1)^{a_i a} = (\phi_1)^{a_i a a_j^{-1} a_j} = \phi_1^{a_j},$$

since ϕ_1 is A_1-invariant. This proves that ψ is A-invariant as required.

Part (b) follows from these considerations by applying Theorem 2.5. $\square$

3. A Gallagher type theorem for blocks

Let G be a finite group and $N \triangleleft G$. P. X. Gallagher proved that if $\theta \in \mathrm{Irr}(N)$ has an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$, then the map $\mathrm{Irr}(G/N) \to \mathrm{Irr}(G|\theta)$ given by $\overline{\eta} \mapsto \eta\widetilde{\theta}$ is a

bijection, where $\eta \in \mathrm{Irr}(G)$ is the lift of $\overline{\eta}$. (See Corollary (6.17) of [7].) Now we need a similar theorem for blocks, see Theorem 3.4.
For a character θ of G we denote by $\mathrm{bl}(\theta)$ the p-block of G containing θ.

Lemma 3.1. *Let $N \lhd G$, $b \in \mathrm{Bl}(N)$ and $\theta \in \mathrm{Irr}(b)$. Assume there exists an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ. Then the map*

$$\upsilon : \mathrm{Bl}(G/N) \to \mathrm{Bl}(G|b) \text{ given by } \mathrm{bl}(\overline{\eta}) \mapsto \mathrm{bl}(\widetilde{\theta}\eta)$$

is surjective, where $\eta \in \mathrm{Irr}(G)$ is the lift of $\overline{\eta} \in \mathrm{Irr}(G/N)$.

Proof. According to [16, Lemma 2.2] we have for every $g \in G$ that

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(g)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(\overline{g})^+),$$

where L is defined by $L/N := \mathrm{C}_{G/N}(\overline{g})$ and $\overline{g} = gN$. This implies that the blocks $\mathrm{bl}(\widetilde{\theta}\eta)$ and $\mathrm{bl}(\widetilde{\theta}\eta')$ coincide for every two characters $\overline{\eta}, \overline{\eta}' \in \mathrm{Irr}(G/N)$ with $\mathrm{bl}(\overline{\eta}) = \mathrm{bl}(\overline{\eta}')$. Hence υ is well-defined.
On the other hand, every block of $\mathrm{Bl}(G|b)$ has a character in $\mathrm{Irr}(G|\theta)$ (by [14, (9.2)]) and such a character can be written as $\widetilde{\theta}\eta$ for some $\overline{\eta} \in \mathrm{Irr}(G/N)$, by Gallagher's theorem. This proves that υ is surjective. $\square$

In general, the map in Lemma 3.1 is not a bijection. (For instance, if b has a defect group D such that $\mathbf{C}_G(D) \subseteq N$, then it is well-known that there is a unique block of G covering b, see [17, Lemma 3.1]. On the other hand, G/N might have many p-blocks. Take, for instance, $G = \mathrm{SL}_2(3)$, $N = Q_8$, $p = 2$, and $\theta \in \mathrm{Irr}(N)$ the irreducible character of degree 2.) Our aim in this section is to find general conditions which guarantee that the map in Lemma 3.1 is a bijection.
The following statement follows also from Theorem 3.3(d) of [4], where a Morita equivalence between the involved blocks is proven. For completeness we nevertheless give here an alternative character theoretic proof.

Lemma 3.2. *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$ with trivial defect group. Let $\theta \in \mathrm{Irr}(b)$. Assume there exists an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ. Then the map*

$$\upsilon : \mathrm{Bl}(G/N) \to \mathrm{Bl}(G|b) \text{ given by } \mathrm{bl}(\overline{\eta}) \mapsto \mathrm{bl}(\widetilde{\theta}\eta)$$

is a bijection.

Proof. By Lemma 3.1, we know that υ is surjective. Let $\overline{\eta}, \overline{\eta}' \in \mathrm{Irr}(G/N)$ with $\mathrm{bl}(\eta) \neq \mathrm{bl}(\eta')$. By [14, Exercise (3.3)] there exists some $\overline{g} \in (G/N)^0$ with $\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(\overline{g})^+) \neq \lambda_{\overline{\eta}'}(\mathfrak{Cl}_{G/N}(\overline{g})^+)$. Assume there exists some $c \in G$ with $cN = \overline{g}$ with

$$\lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(c)^+)^* \neq 0,$$

where L is defined by $L/N := \mathrm{C}_{G/N}(\overline{g})$. Then $\mathrm{bl}(\widetilde{\theta}\eta) \neq \mathrm{bl}(\widetilde{\theta}\eta')$, since

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(c)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(c)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(\overline{g})^+).$$

Thus, let $g \in G$ with $gN = \overline{g}$. Note that gN is closed under L-conjugation, and let $\mathbb{S}$ be a representative set of the L-conjugacy classes contained in gN, i.e.

$$\dot{\bigcup_{s \in \mathbb{S}}} \mathfrak{Cl}_L(s) = gN.$$

By [7, Lemma (8.14)] we have

$$\sum_{c \in gN} \widetilde{\theta}(c)\widetilde{\theta}(c^{-1}) = |N|.$$

This implies

$$\sum_{s \in \mathbb{S}} |\mathfrak{Cl}_L(s)|\widetilde{\theta}(s)\widetilde{\theta}(s^{-1}) = |N|.$$

Dividing by $\theta(1)$ we obtain

$$\sum_{s \in \mathbb{S}} \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(s)^+)\widetilde{\theta}(s^{-1})^* = \Big(\sum_{s \in \mathbb{S}} \frac{|\mathfrak{Cl}_L(s)|\widetilde{\theta}(s)}{\theta(1)}\widetilde{\theta}(s^{-1})\Big)^* = \Big(\frac{|N|}{\theta(1)}\Big)^* \neq 0$$

since θ is of defect 0. This implies that for some $c \in \mathbb{S}$ we have $\lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(c)^+)^* \neq 0$ as required. $\square$

Next, we generalise Lemma 3.2. Recall Theorem 2.1 of [15]: for a finite group X, a p-subgroup $Y \lhd X$ and an X-invariant character $\nu \in \mathrm{Irr}(Y)$ there is a natural bijection $\mathrm{dz}(X/Y) \to \mathrm{rdz}(X|\nu)$, $\chi \mapsto \chi_\nu$, with $\frac{\chi(1)_p}{\nu(1)_p} = \frac{|X|_p}{|Y|_p}$, where $\mathrm{dz}(X)$ denotes the characters lying in a defect zero block and $\mathrm{rdz}(X|\nu)$ the set of characters $\chi \in \mathrm{Irr}(X|\nu)$, see [15] for the definition of this bijection.

Lemma 3.3. *Let $N \lhd G$, and suppose that $D \lhd G$ is contained in N. Let $\mu \in \mathrm{Irr}(D)$ be G-invariant. Let $\theta \in \mathrm{dz}(N/D)$. Then θ extends to G/D if θ_μ extends to G.*

Proof. Note that since θ_μ extends to G, the character μ is G-invariant. The bijection $\mathrm{dz}(N/D) \to \mathrm{rdz}(N|\mu)$, $\chi \mapsto \chi_\mu$ is G-equivariant. Hence θ is G/D-invariant.
Suppose that θ_μ extends to G. In order to show that θ extends to G it is enough to prove that θ extends to Q, whenever Q/N is a Sylow q-subgroup of G/N for some prime q. If $Q/N \in \mathrm{Syl}_p(G/N)$, then θ considered as a character of N/D has defect zero, and therefore it extends to Q/D in this case (see, for instance, Problem (3.10) of [14]). So θ as a character of N extends to Q. Now suppose that Q/N is a q-group for some $q \neq p$. We know that θ_μ extends to some $\eta \in \mathrm{Irr}(Q)$. According to its degree, $\eta \in \mathrm{rdz}(Q|\mu)$. Then $\eta = \gamma_\mu$ for some $\gamma \in \mathrm{dz}(Q/D)$. By the values of the functions γ_μ and using that $\hat{\mu}(g) \neq 0$ whenever $g_p \in D$ we check that $\gamma_N = \theta$. $\square$

Theorem 3.4. *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$ with a defect group D with $D\mathbf{C}_G(D) = G$. Let $\theta \in \mathrm{Irr}(b)$. Assume there exists an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ. Then the map*

$$\upsilon : \mathrm{Bl}(G/N) \to \mathrm{Bl}(G|b) \textit{ given by } \mathrm{bl}(\overline{\eta}) \mapsto \mathrm{bl}(\widetilde{\theta}\eta)$$

is a bijection.

Proof. By Lemma 3.1 it is sufficient to prove that $|\operatorname{Bl}(G/N)| = |\operatorname{Bl}(G|b)|$. Because of $D\mathbf{C}_G(D) = G$ the defect group D is normal in N. By [14, Thm. (9.12)], there exists a unique character $\theta_1 \in \operatorname{Irr}(N)$ of b with $D \subseteq \ker\theta_1$ that is a lift of some $\overline{\theta}_1 \in \operatorname{dz}(N/D)$. By Lemma 3.3 the character $\overline{\theta}_1$ extends to G/D. Write $\overline{N} = N/D$ and $\overline{G} = G/D$. Then Lemma 3.2 applies to the character $\overline{\theta}_1 \in \operatorname{Irr}(N/D)$ associated to θ_1. Then $|\operatorname{Bl}(\overline{G}/\overline{N})| = |\operatorname{Bl}(\overline{G}|\overline{b})|$, where $\overline{b} = \operatorname{bl}(\overline{\theta}_1)$. Now by [14, Thm. (9.10)] there is a canonical bijection between the blocks of $\overline{G}$ and the blocks of G, given by covering. Using [14, Thm. (9.2)] with Brauer characters, we easily check that under this bijection, a block $\overline{B}$ of $\overline{G}$ covers $\overline{b}$ if and only if B covers b. This proves the statement. □

4. Dade's Ramification Group

In order to further generalise Theorem 3.4, we need to go deeper and use a subgroup with remarkable properties introduced by E. C. Dade in 1973, see [2]. This subgroup is key in the remainder of this paper. We shall use M. Murai's version of it (see [12]). Suppose that $M \triangleleft T$ and that $\theta \in \operatorname{Irr}(M)$ is T-invariant. If $x, y \in T$ are such that $[x, y] \in M$, then Dade and Isaacs defined a complex number $\langle\langle x, y\rangle\rangle_\theta$, in the following way: since $M\langle y\rangle/M$ is cyclic, it follows that θ extends to some $\psi \in \operatorname{Irr}(M\langle y\rangle)$. Now, ψ^x is some other extension and by Gallagher's theorem, there exists some $\lambda \in \operatorname{Irr}(M\langle y\rangle)$ such that $\psi^x = \lambda\psi$. Now $\langle\langle x, y\rangle\rangle_\theta = \lambda(y)$. The properties of this (well-defined) number are listed in Lemma (2.1) and Theorem (2.3) of [6] which essentially assert that $\langle\langle , \rangle\rangle_\theta$ is multiplicative (in both arguments). Thus, if H, K are subgroups of T with $[H, K] \subseteq M$, then we have uniquely defined a subgroup

$$H^\perp \cap K := \{k \in K \mid \langle\langle k, h\rangle\rangle_\theta = 1 \text{ for all } h \in H\}\,.$$

Note that by the definition of $\langle\langle , \rangle\rangle_\theta$ this group always contains $M \cap K$.

Lemma 4.1. *Suppose that $M \triangleleft T$ and that $\theta \in \operatorname{Irr}(M)$ is T-invariant. Suppose that H, K are subgroups of T with $[H, K] \subseteq M$, and $M \subseteq K$. If $\rho \in \operatorname{Irr}(K)$ extends θ, then $\rho_{H^\perp\cap K}$ is H-invariant.*

Proof. Notice that $M \subseteq H^\perp \cap K$. Let $\nu = \rho_{H^\perp\cap K}$. We claim that ν is H-invariant. (Since $[H, K] \subseteq M$, notice that H normalises every subgroup between M and K.) Now, let $h \in H$ and $x \in H^\perp \cap K$. We want to show that $\nu^h(x) = \nu(x)$. Let $J = M\langle m\rangle$, and write $(\nu_J)^h = \lambda\nu_J$, where $\lambda \in \operatorname{Irr}(J/M)$. Then $\lambda(x) = \langle\langle h, x\rangle\rangle_\theta = 1$, and $\nu^h(x) = \nu(x)$. □

For any block $b \in \operatorname{Bl}(N)$ with defect group D, we call a character $\eta \in \operatorname{Irr}(D\mathbf{C}_N(D))$ canonical character of b if $D \subseteq \ker\eta$ and $\operatorname{bl}(\eta)^N = b$, see [14, p. 204].

Theorem 4.2. *Let $N \triangleleft G$, and let $b \in \operatorname{Bl}(N)$ be G-invariant. Then there exists a subgroup $N \subseteq G[b] \triangleleft G$, uniquely determined by G and b, satisfying the following properties.*

(a) *Suppose that D is any defect group of b, and let $\eta \in \operatorname{Irr}(D\mathbf{C}_N(D))$ be a canonical character of b. Let K be the stabiliser of η in $D\mathbf{C}_G(D)$, and let H be the stabiliser of η in $\mathbf{N}_N(D)$. Then $G[b] = N(H^\perp \cap K)$.*

(b) *If $B \in \mathrm{Bl}(G)$ covers $B' \in \mathrm{Bl}(G[b])$ and B' covers b, then B is the only block of G that covers B'.*

Proof. We have that $\mathbf{N}_N(D)$ and $D\mathbf{C}_G(D)$ are normal subgroups of $\mathbf{N}_G(D)$ whose intersection is $D\mathbf{C}_N(D)$. In particular, $[H,K] \subseteq D\mathbf{C}_N(D)$ for every pair of subgroups $H \subseteq \mathbf{N}_N(D)$ and $K \subseteq D\mathbf{C}_G(D)$. Now, part (a) follows from [12, Thm. 3.13], while part (b) follows from [12, Thm. 3.5]. $\square$

The subgroup $G[b]$ has many further properties, but we restrict ourselves to those that shall be used in this paper. For our later considerations we mention the following.

Corollary 4.3. *Let $N \lhd G$, and let $b \in \mathrm{Bl}(N)$ be G-invariant. For $Z \lhd G$ with $N \cap Z = 1$ let $\overline{b} \in \mathrm{Bl}(NZ/Z)$ be the induced block. Then $G[b]/Z = \overline{G}[\overline{b}]$ for $\overline{G} := G/Z$.*

Proof. ¿From the assumptions we see $Z \subseteq \mathbf{C}_G(N)$ and so $Z \subseteq G[b]$ by Theorem 4.2(a). Let D be a defect group of b, η a canonical character of b, K the stabiliser of η in $D\mathbf{C}_G(D)$ and H the stabiliser of η in $\mathbf{N}_N(D)$. By Theorem 4.2(a) we have $G[b] = N(H^\perp \cap K)$.
Now DZ/Z is a defect group of b, $\overline{\eta}$ is the canonical character of b induced by η, $\overline{K} := KZ/Z$ the stabiliser of $\overline{\eta}$ in $\overline{D}\mathbf{C}_{\overline{G}}(\overline{D})$ and $\overline{H} := HZ/Z$ the stabiliser of $\overline{\eta}$ in $\mathbf{N}_{NZ/Z}(\overline{D})$. For $k \in K$ and $h \in H$ we have

$$\langle\langle k,h\rangle\rangle_\eta = \langle\langle kZ,hZ\rangle\rangle_{\overline{\eta}}.$$

This leads to $\overline{H}^\perp \cap \overline{K} = (H^\perp \cap K)Z/Z$. Together with Theorem 4.2(a) this implies the statement. $\square$

The properties of $G[b]$ allow us to generalise Theorem 3.4 to the following situation.

Theorem 4.4. *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$. Let $\theta \in \mathrm{Irr}(b)$. Assume there exists an extension $\widetilde{\theta} \in \mathrm{Irr}(G)$ of θ. If $G[b] = G$, then the map*

$$\upsilon : \mathrm{Bl}(G/N) \to \mathrm{Bl}(G|b) \textit{ given by } \mathrm{bl}(\overline{\eta}) \mapsto \mathrm{bl}(\widetilde{\theta}\eta)$$

is a bijection.

Proof. By Lemma 3.1 it is enough to show that $|\mathrm{Bl}(G|b)| = |\mathrm{Bl}(G/N)|$. By Theorem 4.2(a), we have that $G = N\mathbf{C}_G(D)$, where D is any defect group of b. Let b' be any block of $D\mathbf{C}_N(D)$ with defect group D inducing b. By [9, Thm. C(a.2)], there exists some $\theta' \in \mathrm{Irr}(b')$ that extends to $K = D\mathbf{C}_G(D)$. By Lemma 3.3 the unique (canonical) character $\eta \in \mathrm{Irr}(b')$ that has D in its kernel extends to some $\widetilde{\eta} \in \mathrm{Irr}(K)$. Let T be the stabiliser of η in $\mathbf{N}_G(D)$, hence $D\mathbf{C}_G(D) = K \subseteq T \subseteq \mathbf{N}_G(D)$. Let $H = T \cap N$. Accordingly $[H,K] \subseteq D\mathbf{C}_N(D)$ and $T = KH$. By Theorem 3.4, we have that

$$|\mathrm{Bl}(G/N)| = |\mathrm{Bl}(D\mathbf{C}_G(D)/D\mathbf{C}_N(D))| = |\mathrm{Bl}(D\mathbf{C}_G(D)|b')|.$$

By applying Lemma 4.1 (with $DC_N(D)$ as the normal subgroup M of $T = HK$ and the T-invariant character η), we have that $\widetilde{\eta}$ is H-invariant. Hence, we conclude that T is the stabiliser of every block of H covering b' (using Gallagher's theorem). Now, let $\tilde{b} = (b')^{\mathbf{N}_N(D)}$. By the Harris-Knörr correspondence [14, Thm. (9.28)] we have

$|\operatorname{Bl}(G|b)| = |\operatorname{Bl}(\mathbf{N}_G(D)|\tilde{b})|$. If $\{e_1, \dots, e_s\}$ are the blocks of $D\mathbf{C}_G(D)$ covering b', then $\{e_1^{\mathbf{N}_G(D)}, \dots, e_s^{\mathbf{N}_G(D)}\}$ are all the blocks of $\mathbf{N}_G(D)$ covering $\widetilde{b}$. Now, if $(e_i)^{\mathbf{N}_G(D)} = (e_j)^{\mathbf{N}_G(D)}$, then it follows that $(e_i)^x = e_j$ for some $x \in \mathbf{N}_G(D)$. Since e_i and e_j only cover the block b', it follows that $(b')^x = b'$, and $x \in T$. However e_i is T-invariant, and therefore $e_i = e_j$. □

5. CHARACTER TRIPLE ISOMORPHISMS UNDER COPRIME ACTIONS AND BLOCKS

In considerations using character triples the existence of an isomorphic character triple whose character is linear and faithful plays an important role. We give here an A-version of this statement that will be used later. Furthermore we analyse how blocks behave under the bijections of characters. If A acts on G, then $\operatorname{Irr}_A(G)$ denotes the set of the irreducible complex characters of G which are invariant under A. Let $\operatorname{IBr}_A(G)$ be defined analogously.

PROPOSITION 5.1. *Let A act on G with $(|G|, |A|) = 1$ and $N \lhd G$ be A-stable.*

(a) *Let $\theta \in \operatorname{Irr}_{GA}(N)$. Then there exists a character triple (G^*, N^*, θ^*), an action of A on G^* stabilising N^* and θ^* and an isomorphism*

$$(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*),$$

such that $N^* \subseteq \mathbf{Z}(G^*)$ and both ι and σ_G are A-equivariant.

(b) *Let $\theta \in \operatorname{IBr}_{GA}(N)$. Then there exists a modular character triple (G^*, N^*, θ^*), an action of A on G^* stabilising N^* and θ^* and an isomorphism*

$$(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*),$$

such that $N^* \subseteq \mathbf{Z}(G^*)$ is a p'-group and both ι and σ_G are A-equivariant.

Proof. Since the proof of both statements is based on the same ideas, we give here only the proof of (a).

First note that θ extends to NA because of [7, Cor. (8.16)]. By the assumptions it is clear that (GA, N, θ) is a character triple. Let $\mathcal{P}$ be a projective representation of GA with the following properties:

(i) $\mathcal{P}_{NA}$ affords an extension of θ to NA,
(ii) the values of the associated factor set $\alpha : GA \times GA \to \mathbb{C}$ are roots of unity and
(iii) α is constant on $N \times N$-cosets.

Let $X := GA$ and E be the finite cyclic group generated by the values of α. For the construction of (G^*, N^*, θ^*) and the A-action on G^* we follow the proof of Theorem (8.28) in [14]. Let $\widetilde{X}$ be constructed as there using $\mathcal{P}$: the group $\widetilde{X}$ consists of pairs (x, ϵ) with $x \in X$ and $\epsilon \in E$ and multiplication in $\widetilde{X}$ is given by

$$(x_1, \epsilon_1)(x_2, \epsilon_2) = (x_1 x_2, \alpha(x_1, x_2)\epsilon_1\epsilon_2).$$

The projective representation $\mathcal{P}$ lifts to a representation of $\widetilde{X}$. Let τ be the character afforded by that representation. The groups $\widetilde{N} := \{(n, \epsilon) \mid n \in N, \epsilon \in E\}$ and $\widetilde{G} := \{(g, \epsilon) \mid g \in G, \epsilon \in E\}$ are normal in $\widetilde{X}$. Since $\mathcal{P}_{NA}$ is a representation the set $\{(x, 1) \mid x \in NA\}$ forms a group isomorphic to NA. Via this isomorphism A can be

identified with a group of $\widetilde{X}$ and acts on $\widetilde{G}$ and $\widetilde{N}$. Let $E_0 := 1 \times E$ and $N_0 := N \times 1$. Identifying N and N_0 we set $\widetilde{\theta} = \theta \times 1_E$. This character is A-invariant.
Via the epimorphism $\iota_1 : \widetilde{X} \to X$, $(x, \epsilon) \mapsto x$, the character triples (X, N, θ) and $(\widetilde{X}, \widetilde{N}, \widetilde{\theta})$ are isomorphic.
The map $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{N})$ with $\widetilde{\lambda}(n, \epsilon) = \epsilon^{-1}$ is a linear character with kernel N_0. By the construction of $\widetilde{X}$ the character $\widetilde{\lambda}$ is $\widetilde{X}$-invariant. We see that $\tau_{\widetilde{N}} = \widetilde{\lambda}^{-1}\widetilde{\theta}$.
Now one can argue that $(\widetilde{X}, \widetilde{N}, \widetilde{\lambda})$ and $(\widetilde{X}, \widetilde{N}, \widetilde{\theta})$ are isomorphic character triples. Analogously $(\widetilde{X}, \widetilde{N}, \widetilde{\lambda})$ and $(\widetilde{X}/N_0, \widetilde{N}/N_0, \lambda)$ are isomorphic character triples, where $\lambda \in \mathrm{Irr}(\widetilde{N}/N_0)$ is the character induced by $\widetilde{\lambda}$.
With $G^* := \widetilde{G}/N_0$, $N^* := \widetilde{N}/N_0$ and $\theta^* := \lambda$ we obtain the required isomorphism

$$(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*).$$

Let $\iota_2 : \widetilde{X} \to \widetilde{X}/N_0$ be the canonical epimorphism. Because of $\ker \iota_1 = E$ and $\ker \iota_2 = N_0$ the isomorphism ι is given by $gN \mapsto \iota_2 \circ \iota_1^{-1}(gN)$.
For $\chi \in \mathrm{Irr}(G|\theta)$ the character $\sigma_G(\chi)$ is obtained in the following way: the character χ lifts to some $\tau_{\widetilde{G}}\mu$ for some $\mu \in \mathrm{Irr}(\widetilde{G})$ with $\ker \mu \geq N$ and $\mu \in \mathrm{Irr}(\widetilde{G}|\widetilde{\lambda})$. Hence $\mu \circ \iota_2^{-1}$ is a character of $\mathrm{Irr}(\widetilde{G}/N_0|\lambda)$. By its construction θ^* is A-invariant and the maps ι and σ_G are A-equivariant, since $\tau_{\widetilde{G}}$ is A-invariant. $\square$

In order to include blocks in the above result, additional assumptions are required.

Proposition 5.2. *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$. Suppose that A acts on G with $(|G|, |A|) = 1$, such that N and b are A-stable. Assume $G[b] = G$.*

(a) *Let $\theta \in \mathrm{Irr}_{GA}(N) \cap \mathrm{Irr}(b)$. Let (G^*, N^*, θ^*) and $(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*)$ be as in Proposition 5.1(a). Then two characters $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$ satisfy $\mathrm{bl}(\chi_1) = \mathrm{bl}(\chi_2)$ if and only if $\mathrm{bl}(\sigma_G(\chi_1)) = \mathrm{bl}(\sigma_G(\chi_2))$*
(b) *Let $\theta \in \mathrm{IBr}_{GA}(N) \cap \mathrm{IBr}(b)$. Let (G^*, N^*, θ^*) and $(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*)$ be as in Proposition 5.1(b). Then two characters $\phi_1, \phi_2 \in \mathrm{IBr}(G|\theta)$ satisfy $\mathrm{bl}(\phi_1) = \mathrm{bl}(\phi_2)$ if and only if $\mathrm{bl}(\sigma_G(\phi_1)) = \mathrm{bl}(\sigma_G(\phi_2))$.*

Proof. We continue using the notation introduced in the proof of Proposition 5.1. Let $N_0 := N \times 1 \subseteq \widetilde{X}$ and $\theta_0 := \theta \times 1_E \in \mathrm{Irr}(\widetilde{N})$. Let $b_0 := \mathrm{bl}(\theta_0)$. From Corollary 4.3 we see that $G[b] = \widetilde{G}[b_0]/E$, where $b_0 := \mathrm{bl}(\theta_0)$.
For the proof of (a) let μ_1 and $\mu_2 \in \mathrm{Irr}(\widetilde{G})$ with $N_0 \subseteq \ker \mu_1$ and $N_0 \subseteq \ker \mu_2$ such that $\tau_{\widetilde{G}}\mu_1$ is a lift of χ_1 and $\tau_{\widetilde{G}}\mu_2$ is a lift of χ_2. Note that $E \subseteq \mathbf{Z}(\widetilde{G})$. According to [14, (9.9) and (9.10)] $\mathrm{bl}(\chi_1) = \mathrm{bl}(\chi_2)$ if and only if $\tau_{\widetilde{G}}\mu_1$ and $\tau_{\widetilde{G}}\mu_2$ belong to the same block. According to Theorem 4.4 we see that the characters of $\widetilde{G}/N_0$ induced by μ_1 and μ_2 are in the same block if and only if $\mathrm{bl}(\tau_{\widetilde{G}}\mu_1) = \mathrm{bl}(\tau_{\widetilde{G}}\mu_2)$. This proves the statement of (a). Part (b) follows from similar considerations. $\square$

As a corollary that might be of independent interest we conclude the following.

Corollary 5.3. *Let $N \lhd G$ and $b \in \mathrm{Bl}(N)$. Assume there exists some G-invariant $\theta \in \mathrm{Irr}_A(N) \cap \mathrm{Irr}(b)$ or $\theta \in \mathrm{IBr}_A(N) \cap \mathrm{IBr}(b)$. Let (G^*, N^*, θ^*) and $(\iota, \sigma) : (G, N, \theta) \longrightarrow (G^*, N^*, \theta^*)$ be as in Proposition 5.1. Let $H := G[b]$ and H^* the group with $\iota(H/N) = H^*/N^*$.*

(a) *Then* $\mathrm{Bl}(G|b)$ *is in bijection with* $\mathrm{Bl}(H^*|b^*)$*, where* $b^* := \mathrm{bl}(\theta^*)$.
(b) *Two characters* $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$ *satisfy* $\mathrm{bl}(\chi_1) = \mathrm{bl}(\chi_2)$ *if and only if* $\mathrm{bl}(\sigma_G(\chi_1))$ *and* $\mathrm{bl}(\sigma_G(\chi_2))$ *cover the same block of* H^*.
(c) *Two characters* $\phi_1, \phi_2 \in \mathrm{IBr}(G|\theta)$ *satisfy* $\mathrm{bl}(\phi_1) = \mathrm{bl}(\phi_2)$ *if and only if* $\mathrm{bl}(\sigma_G(\phi_1))$ *and* $\mathrm{bl}(\sigma_G(\phi_2))$ *cover the same block of* H^*.

Proof. Part (a) follows directly from Proposition 5.2. Parts (b) and (c) are applications of Proposition 5.2 together with Theorem 4.2(b). $\square$

6. Reduction

In this section we show how Theorem 1.1 is implied by the analogous statement for the central product of quasi-simple groups given in Corollary 2.6. In fact, we will work with the following slightly more general statement.

Theorem 6.1. *Let the group* A *act on the group* G *with* $(|A|, |G|) = 1$.
(a) *Let* Z *be an* A*-invariant central* p*-subgroup of* G*, let* $\nu \in \mathrm{Irr}_A(Z)$ *and let* $B \in \mathrm{Bl}_A(G)$*. Then there exists an* A*-invariant character* $\chi \in \mathrm{Irr}(B|\nu)$.
(b) *Let* $B \in \mathrm{Bl}_A(G)$*. Then there exists an* A*-invariant character* $\phi \in \mathrm{IBr}(B)$.

The following well-known result will be used for part (a).

Theorem 6.2. *Let the group* A *act on the group* G *with* $(|A|, |G|) = 1$ *and let* $N \lhd G$ *be* A*-stable. Let* $\theta \in \mathrm{Irr}_A(N)$*. Then there exists an* A*-invariant character in* $\mathrm{Irr}(G|\theta)$.

Proof. This is Theorem (13.28) and Corollary (13.30) of [7]. $\square$

For the proof of Theorem 6.1(b) we need the following analogue for Brauer characters.

Theorem 6.3. *Let the group* A *act on the group* G *with* $(|A|, |G|) = 1$ *and let* $N \lhd G$ *be* A*-stable. Let* $\phi \in \mathrm{IBr}_A(N)$*. Then there exists an* A*-invariant character in* $\mathrm{IBr}(G|\phi)$.

Proof. We prove this statement by induction on $|G : N|$ and then on $|G : \mathbf{Z}(G)|$. We can assume that ϕ is G-invariant, since otherwise some $\phi' \in \mathrm{IBr}_A(G_\phi|\phi)$ exists by induction and hence $\phi'^G \in \mathrm{IBr}_A(G|\phi)$.
Assume there exists an A-stable subgroup $K \lhd G$ with $N \lneq K \lneq G$. Then by induction there exists some A-invariant character $\phi' \in \mathrm{IBr}(K|\phi)$ and one in $\mathrm{IBr}(G|\phi')$. Accordingly we can assume that G/N is a chief factor of GA and hence G/N is the direct product of isomorphic simple groups, such that A acts transitively on the factors of G/N.
Then (G, N, ϕ) forms a modular character triple that is isomorphic to some (G^*, N^*, ϕ^*) according to Proposition 5.1, such that $N^* \subseteq \mathbf{Z}(G^*)$ and $p \nmid |N^*|$. Since the isomorphism of the character triples is A-equivariant it is sufficient to prove the statement for (G^*, N^*, ϕ^*). For this it suffices to prove that there exists some A-invariant character in $\mathrm{IBr}(G^*|\theta^*)$. If G/N is a q-group for $p \neq q$, the group G^* is a p'-group and the statement follows immediately from Theorem 6.2. If G/N is a p-group, the set $\mathrm{IBr}(G^*|\phi^*)$ is a singleton.
Let $\nu \in \mathrm{Irr}(N^*)$ be the character with $\nu^0 = \phi^*$. (Note that ν is unique since N^* is a p'-group.) Since ν is A-invariant there exists some A-invariant $\chi \in \mathrm{Irr}(G^*|\nu)$

by Theorem 6.2. This character hence belongs to an A-invariant block $B \in \mathrm{Bl}(G^*)$. Further G^*/N^* is the direct product of isomorphic non-abelian simple groups that are permuted transitively by A. According to Corollary 2.6 there exists an A-invariant Brauer character in $\mathrm{IBr}(B)$. $\square$

We start proving Theorem 6.1 in a series of intermediate results, working by induction first on $|G/\mathbf{Z}(G)|$ and second on $|G|$. It is clear that we may assume that $Z \in \mathrm{Syl}_p(\mathbf{Z}(G))$.

Let $N \lhd G$ such that G/N is chief factor of GA. Then by Glauberman's Lemma, [7, Lemma (13.8)], there exists $b \in \mathrm{Bl}_A(N)$ such that $B \in \mathrm{Bl}(G|b)$. (Recall that by [14, Cor. (9.3)], G acts transitively on the set of blocks of N covered by b.) Another application of Glauberman's Lemma shows that b has an A-invariant defect group D. Next, notice that $Z \subseteq N$ in part (a) of Theorem 6.1. Otherwise, we have that $NZ = G$, and therefore G/N is a p-group. In particular B is the only block covering b ([14, Cor. (9.6)]). Let $\nu \in \mathrm{Irr}_A(Z)$. Since b has an A-invariant character $\chi_1 \in \mathrm{Irr}(N|\nu_{Z\cap N})$ by induction, the character $\chi_1 \cdot \nu$ defined as in [8, Section 5] has the required properties and we are done.
Analogously one can argue that $\mathbf{Z}(G) \subseteq N$ since there exists a unique $\mu \in \mathrm{Irr}(\mathbf{Z}(G)|\nu)$ in a block of $\mathbf{Z}(G)$ covered by B.

Lemma 6.4. *We can assume that $G_b = G$.*

Proof. By the Fong-Reynolds theorem [14, Thm. (9.14)], there exists a unique $\widetilde{B} \in \mathrm{Bl}(G_b|b)$ with $\widetilde{B}^G = B$. Since b is A-invariant, so is G_b. Also, $\widetilde{B}$ is A-invariant by uniqueness. Notice that $Z \subseteq \mathbf{Z}(G) \subseteq G_b$. If $G_b < G$, then by induction G_b has an A-invariant character $\chi_0 \in \mathrm{Irr}(\widetilde{B}) \cap \mathrm{Irr}_A(G_b|\nu)$ and $\phi_0 \in \mathrm{IBr}(\widetilde{B})$. Now, the characters χ_0^G and ϕ_0^G are irreducible, A-invariant and belong to B. $\square$

Lemma 6.5. *We can assume that $G[b] = G$.*

Proof. By Theorem 4.2(b) there exists a unique $B' \in \mathrm{Bl}(G[b]|b)$ with $(B')^G = B$. Since $G[b]$ is uniquely determined by b, we have that $G[b]$ is A-stable and by uniqueness that B' is A-invariant. Note that $\mathbf{Z}(G) \subseteq G[b]$ by Theorem 4.2(a). If $G[b] \neq G$ then we can conclude by induction that there exist some A-fixed $\chi_0 \in \mathrm{Irr}(G[b]|\nu) \cap \mathrm{Irr}(B')$ and $\phi_0 \in \mathrm{IBr}(B')$. By Theorem 6.2 and 6.3 there exist some $\chi \in \mathrm{Irr}(G|\chi_0)$ and $\phi \in \mathrm{IBr}(G|\phi_0)$, respectively that is A-fixed. Those characters belong to $\mathrm{Bl}(G|B') = \{B\}$. $\square$

Lemma 6.6. *We can assume that $N = \mathbf{Z}(G)$.*

Proof. By induction on $|G : \mathbf{Z}(G)|$ we see that b contains an A-invariant character $\theta \in \mathrm{Irr}(b|\nu)$, respectively $\theta \in \mathrm{IBr}(b)$. By the above we can assume $G[b] = G$.
Let (G^*, N^*, θ^*) be the character triple associated to (G, N, θ) and σ_G the A-equivariant bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ from Proposition 5.1(a), respectively the A-equivariant bijection $\mathrm{IBr}(G|\theta) \to \mathrm{IBr}(G^*|\theta^*)$ from Proposition 5.1(b). According to Proposition 5.2(b) there is some block $C \in \mathrm{Bl}(G^*)$ such that $\sigma_G(\mathrm{Irr}(B|\theta)) = \mathrm{Irr}(C|\theta^*)$ or $\sigma_G(\mathrm{IBr}(B|\theta)) = \mathrm{IBr}(C|\theta^*)$, respectively.

If $N \neq \mathbf{Z}(G)$ then $\mathrm{Irr}(C|\theta^*)$ and $\mathrm{IBr}(C|\theta^*)$ both contain A-invariant characters because $|G^* : \mathbf{Z}(G^*)| = |G : N| < |G : \mathbf{Z}(G)|$. Since σ_G is A-equivariant this proves the statement. □

Proof of Theorem 6.1. Now it remains to consider the case where $\mathbf{Z}(G) = N$. Since N was chosen such that G/N is a chief factor of GA, the quotient G/N is the direct product of isomorphic simple groups that are transitively permuted by A.
If G/N is non-abelian, Corollary 2.6 applies and proves the statement. Otherwise G/N is an elementary abelian p-group or a p'-group. In the first case $\mathrm{Bl}(G|b)$ is a singleton and the statement follows from Theorems 6.2 and 6.3. In the latter case B has a central defect group and the sets $\mathrm{Irr}(B|\nu)$ and $\mathrm{IBr}(B)$ are singletons. □

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Gunter Malle
FB Mathematik
TU Kaiserslautern
Postfach 3049
67653 Kaiserslautern
Germany.
malle@mathematik.uni-kl.de

Gabriel Navarro
Departament d'Àlgebra
Universitat de València
Dr. Moliner 50
46100 Burjassot
Spain.
gabriel.navarro@uv.es

Britta Späth
FB Mathematik
TU Kaiserslautern
Postfach 3049
67653 Kaiserslautern
Germany.
spaeth@mathematik.uni-kl.de