Equivariant Fredholm Modules for the Full Quantum Flag Manifold of $\mathrm{SU}_q(3)$

Christian Voigt[1] and Robert Yuncken[2]

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Abstract. We introduce C^*-algebras associated to the foliation structure of a quantum flag manifold. We use these to construct $\mathrm{SL}_q(3, \mathbb{C})$-equivariant Fredholm modules for the full quantum flag manifold $\mathcal{X}_q = \mathrm{SU}_q(3)/T$ of $\mathrm{SU}_q(3)$, based on an analytical version of the Bernstein-Gelfand-Gelfand complex. As a consequence we deduce that the flag manifold $\mathcal{X}_q$ satisfies Poincaré duality in equivariant KK-theory. Moreover, we show that the Baum-Connes conjecture with trivial coefficients holds for the discrete quantum group dual to $\mathrm{SU}_q(3)$.

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Abstract

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1 Introduction

In noncommutative differential geometry [Con94], the notion of a smooth manifold is extended beyond its classical scope by adopting a spectral point of view. This is centred around the idea of constructing Dirac-type operators associated with possibly noncommutative algebras, capturing the underlying Riemannian structure of geometric objects for which ordinary differential geometry breaks down. The key concept in this theory, introduced by Connes, is the notion of a spectral triple [Con96].

Quantum groups provide provide a large class of examples of noncommutative spaces, and they have been studied extensively within the framework of noncommutative differential geometry. Among the many contributions in this direction let us only mention a few. Chakraborty and Pal [CP03] defined an equivariant spectral triple on $\mathrm{SU}_q(2)$, which was studied in detail by Connes [Con04]. Later Dabrowski, Landi, Sitarz, van Suijlekom and Várilly [DLS$^+$05], [vSDL$^+$05] defined and studied a deformation of the classical Dirac operator on $\mathrm{SU}(2)$, thus obtaining a different spectral triple on $\mathrm{SU}_q(2)$. The techniques used in these papers rely on explicit estimates involving Clebsch-Gordan coefficients. In a different direction, Neshveyev and Tuset exhibited a general mechanism for transporting the Dirac operator on an arbitrary compact simple Lie group to its quantum deformation, based on Drinfeld twists and properties of the Drinfeld associator [NT10]. The resulting spectral triples inherit various desirable properties from their classical counterparts, although unfortunately they are difficult to study directly since this requires a certain amount of control of the twisting procedure.

This article is concerned with the quantized full flag manifolds associated to the q-deformations of compact semisimple Lie groups, and in particular the flag manifold of $\mathrm{SU}_q(3)$, the simplest example in rank greater than one. In the rank-one case, that is for $\mathrm{SU}_q(2)$, the flag manifold $\mathrm{SU}_q(2)/T$ is known as the standard Podleś sphere, and Dirac operators on it have been defined and studied by several authors [Owc01, DS03, SW04]. A version of the local index formula for the Podleś sphere is exhibited in [NT05, Wag09, RS], although slight modifications must be made to Connes' original formalism.

The higher rank situation has proven to be considerably more difficult. Krähmer [Krä04] gave an algebraic construction of Dirac operators on quantized *irreducible* flag manifolds in higher rank. These retain a certain rank-one character in their geometry. In particular, the construction in [Krä04] does not cover the case of full flag manifolds. On the other hand, the Dirac operator defined by Neshveyev and Tuset can be used to write down spectral triples for arbitrary full quantum flag manifolds. However, the most direct way to do so, which was indicated already in [NT10], does not suffice to describe the equivariant K-homology group of the quantum flag manifold using Poincaré duality. More precisely, one only obtains certain multiples of the class of the Dirac operator in this way.

In this paper, we describe a construction of a Dirac-type class in equivariant K-homology for the full flag manifold $\mathcal{X}_q = \mathrm{SU}_q(3)/T$ of $\mathrm{SU}_q(3)$ as a *bounded* Fredholm module. This does not give the full "noncommutative Riemannian" structure on $\mathcal{X}_q$ that a Connes-type spectral triple would give. In fact, a key philosophical point behind our construction is that the natural geometric structure on quantized flag manifolds in higher rank is not Riemannian but parabolic, in the sense of [ČS00].

Correspondingly, the construction of our Dirac-type class is based not upon the Dirac or Dolbeault operator but upon the Bernstein-Gelfand-Gelfand (BGG) complex, see [BGG75, BE89, ČSS01]. The quantum version of the BGG complex for $\mathrm{SL}_q(n, \mathbb{C})$, in its algebraic form, first appeared in [Ros91]; see also [HK07a, HK07b]. It has not been much studied from an analytical point of view so far. In fact, developing a complete unbounded noncommutative version of parabolic geometries seems to be difficult. For instance, the BGG complex is neither elliptic nor order 1, although it does exhibit a kind of subellipticity. In the present work, we convert the BGG complex into a bounded K-homology cycle. Such a construction was achieved for a classical flag manifold in [Yun10, Yun11a]. A major goal of the present work is to demonstrate that the necessary analysis can also be carried out for a quantized flag manifold. In particular, our K-homology class is equivariant not only with respect to $\mathrm{SU}_q(3)$, but with respect to the complex quantum group $\mathrm{SL}_q(3, \mathbb{C}) = \mathbf{D}(\mathrm{SU}_q(3))$, the Drinfeld double of $\mathrm{SU}_q(3)$. Drinfeld doubles play an important role in the definition of equivariant Poincaré duality [NV10] and the proof of the Baum-Connes conjecture for the dual of $\mathrm{SU}_q(2)$, see [Voi11]. It is worth pointing out that the verification of $\mathrm{SL}_q(3, \mathbb{C})$-equivariance of our cycle is somewhat simpler than in the classical situation. We also remark that in the construction of our K-homology class we use some properties of principal series representations of $\mathrm{SL}_q(3, \mathbb{C})$ which will be discussed in a separate paper [VY]. Our main result can be formulated as follows.

Theorem 1.1. *The BGG complex for the full flag manifold $\mathcal{X}_q = \mathrm{SU}_q(3)/T$ of $\mathrm{SU}_q(3)$ can be normalized to give a bounded equivariant K-homology cycle in the Kasparov group $KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q), \mathbb{C})$. The equivariant index of this element with respect to $\mathrm{SU}_q(3)$ is the class of the trivial representation in $KK^{\mathrm{SU}_q(3)}(\mathbb{C}, \mathbb{C}) = R(\mathrm{SU}_q(3))$.*

We refer to Theorem 10.6 for the precise statement of this result. The main idea behind our construction can be sketched as follows. Firstly, corresponding to each of the two simple roots of $\mathrm{SU}_q(3)$ there is a fibration of the quantized flag manifold whose fibres are Podleś spheres. These fibrations carry families of Dirac-type operators analogous to the operators constructed by Dabrowski-Sitarz. As is common in Kasparov's KK-theory, we replace these longitudinal operators by their bounded transforms. We then use a variant of the Kasparov product, inspired by the BGG complex, to assemble them into a single $\mathrm{SL}_q(3, \mathbb{C})$-equivariant K-homology cycle for $\mathcal{X}_q$.

At present, there is only one ingredient which prevents us from carrying out our construction for the full flag manifold of $\mathrm{SU}_q(n)$ for any $n \geq 2$, namely the operator partition of unity in Lemma 10.5. We develop all the harmonic analysis in the generality of $\mathrm{SU}_q(n)$.

Using Theorem 1.1 we derive two consequences regarding equivariant KK-theory. Firstly, we conclude that the quantum flag manifold $\mathcal{X}_q$ satisfies equivariant Poincaré duality in KK-theory in the sense of [NV10].

Corollary 1.2. *The flag manifold $\mathcal{X}_q$ is $\mathrm{SU}_q(3)$-equivariantly Poincaré dual to itself. That is, there is a natural isomorphism*

$$KK_*^{\mathbf{D}(\mathrm{SU}_q(3))}(C(\mathcal{X}_q) \boxtimes A, B) \cong KK_*^{\mathbf{D}(\mathrm{SU}_q(3))}(A, C(\mathcal{X}_q) \boxtimes B)$$

for all $\mathbf{D}(\mathrm{SU}_q(3))$-$C^*$-algebras A and B, where $\boxtimes$ denotes the braided tensor product with respect to $\mathrm{SU}_q(3)$.

For the definition and properties of braided tensor products we refer to [NV10]. We note that it is crucial here that the class obtained in Theorem 1.1 is equivariant with respect to $\mathbf{D}(\mathrm{SU}_q(3))$ and not just $\mathrm{SU}_q(3)$. Let us also remark that we have chosen to write $\mathbf{D}(\mathrm{SU}_q(3))$ instead of $\mathrm{SL}_q(3,\mathbb{C})$ in Corollary 1.2 in order to make notation consistent with [NV10].
Secondly, we discuss an analogue of the Baum-Connes conjecture for the discrete quantum group dual to $\mathrm{SU}_q(3)$. In [MN06], Meyer and Nest have developed an approach to the Baum-Connes conjecture [BCH94] which allows one to construct assembly maps in rather general circumstances, and which applies in particular to duals of q-deformations. As already mentioned above, the simplest case of $\mathrm{SU}_q(2)$ was studied in [Voi11], and here we show how to go one step further as follows.

Corollary 1.3. *The Baum-Connes conjecture with trivial coefficients $\mathbb{C}$ holds for the discrete quantum group dual to $\mathrm{SU}_q(3)$.*

This result is significantly weaker than the analogous statement for the dual of $\mathrm{SU}_q(2)$ in [Voi11]. However, let us point out that one cannot hope to carry over the arguments used in [Voi11] to the higher rank situation. Indeed, according to work of Arano [Ara], the Drinfeld double of $\mathrm{SU}_q(3)$ has property (T). This forbids the existence of continuous homotopies along the complementary series representations to the trivial representation in the unitary dual. Such homotopies are at the heart of the arguments in [Voi11]. In other words, the problem is similar to well-known obstacles to proving the Baum-Connes conjecture with coefficients for the classical groups $\mathrm{SL}(n,\mathbb{C})$ in higher rank.

Let us now explain how the paper is organized. In Section 2 we collect some preliminaries on quantum groups and fix our notation. Sections 3 and 4 contain the definition and basic properties of certain ideals of C^*-algebras associated to the canonical fibrations of a quantum flag manifold. These C^*-ideals are defined in terms of the harmonic analysis of the block diagonal quantum subgroups of $\mathrm{SU}_q(n)$, and are the basis of all the analysis that follows.

In Section 5 we formulate the main technical results about these ideals. These results are parallel to classical facts from the calculus of longitudinally elliptic pseudodifferential operators. The proofs are deferred to subsequent sections, which may be skipped on a first reading. Specifically, Section 6 collects some facts about Gelfand-Tsetlin bases, and in particular the effect of reversing the order of roots used in their definition. Section 7 introduces the notion of essentially orthotypical quantum subgroups, in analogy with the considerations in [Yun10]. In Section 8 the analytic properties of longitudinal pseudodifferential-type operators are established.
Section 9 contains some definitions and facts related to complex quantum groups and their representations, and it is checked that our constructions are compatible with the natural action of $\mathrm{SL}_q(n,\mathbb{C})$. In section 10 we describe the analytical quantum BGG complex for the flag manifold of $\mathrm{SU}_q(3)$, and we prove our main theorem.
The final section 11 contains the corollaries stated above. That is, we show that $\mathcal{X}_q$ is equivariantly Poincaré dual to itself, and we verify the Baum-Connes conjecture with trivial coefficients for the dual of $\mathrm{SU}_q(3)$.

Let us conclude with some remarks on notation. The dual of a vector space V is denoted V^*. We write $\mathbb{L}(H,H')$ for the space of bounded operators between Hilbert spaces H and H', and $\mathbb{K}(H,H')$ denotes the space of compact operators. When $H = H'$ we abbreviate these as $\mathbb{L}(H)$ and $\mathbb{K}(H)$. Depending on the context, the symbol $\otimes$ denotes either an algebraic tensor product, the tensor product of Hilbert spaces or the minimal tensor product of C^*-algebras. All Hilbert spaces in this paper are separable.

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2 Preliminaries

In this section we discuss some preliminaries on quantum groups in general and q-deformations in particular. For more details and background we refer to the literature [CP95], [KS97], [Maj05].

2.1 Some notation

Let $K = \mathrm{SU}(n)$ or $\mathrm{U}(n)$ with $n \geq 2$. We write T for the standard maximal torus of K, that is, the diagonal subgroup, and $\mathfrak{t}$ for its Lie algebra. We write $\mathfrak{sl}_n$ for $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{gl}_n$ for $\mathfrak{gl}(n,\mathbb{C})$. In either case we denote by $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$ the Cartan subalgebra. We write $\mathbf{P}$ for the set of weights of K, viewed as a lattice in $\mathfrak{h}^*$. If V is a K-representation and $\mu \in \mathbf{P}$, the subspace of vectors of weight μ in V will be denoted V_μ.
It will be convenient to identify the weight lattice of $\mathrm{U}(n)$ with $\mathbb{Z}^n$, where an element $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ corresponds to the weight $\mu \in \mathfrak{h}^*$ given by

$$\mu(\mathrm{diag}(t_1, \ldots, t_n)) = \mu_1 t_1 + \cdots + \mu_n t_n.$$

The corresponding character of T will be denoted by $e^\mu \in C(T)$. We equip $\mathfrak{h}^*$ with the bilinear form which extends the standard pairing on $\mathbf{P} \cong \mathbb{Z}^n$:

$$(\,(\mu_1, \ldots, \mu_n),\, (\mu'_1, \ldots, \mu'_n)\,) = \sum_i \mu_i \mu'_i.$$

For $\mathrm{SU}(n)$, the weight lattice identifies with the quotient $\mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$, and the bilinear form on $\mathfrak{h}^*$ is obtained from that above by identifying $\mathfrak{h}^*$ with the orthogonal complement of $\mathbb{C}(1, \ldots, 1)$ in $\mathbb{C}^n$.
We write Δ for the set of roots of $\mathrm{SU}(n)$ or $\mathrm{U}(n)$; they are the same in both cases. We fix the set of simple roots $\Sigma = \{\alpha_1, \ldots, \alpha_{n-1}\}$ where $\alpha_i : \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i - t_{i+1}$.

2.2 Quantized universal enveloping algebras

We shall use the quantized universal enveloping algebras which are denoted $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ and $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ in [KS97] (pages 212 and 164, respectively), since these are the versions used in the literature on Gelfand-Tsetlin theory. We briefly recall their definitions.
Fix $q \in (0, 1)$. For any $a \in \mathbb{C}$ we write $[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$, and for $a \in \mathbb{N}$,

$$[a]_q! = \prod_{k=1}^{a} [k]_q, \qquad \begin{bmatrix} a \\ m \end{bmatrix}_q = \frac{[a]_q!}{[a-m]_q!\,[m]_q!}.$$

Often, we shall drop the subscript q in the notation.
The Hopf $*$-algebra $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$is generated by elements E_i, F_i $(i = 1, \ldots, n-1)$ and G_j, G_j^{-1} $(j = 1, \ldots, n)$ with the relations

$$G_j G_k = G_k G_j, \qquad G_j^{-1} G_j = 1 = G_j G_j^{-1}$$

$$G_j E_i G_j^{-1} = \begin{cases} q^{\frac{1}{2}} E_i, & j = i, \\ q^{-\frac{1}{2}} E_i, & j = i+1, \\ E_i, & \text{otherwise}, \end{cases} \qquad G_j F_i G_j^{-1} = \begin{cases} q^{-\frac{1}{2}} F_i, & j = i, \\ q^{\frac{1}{2}} F_i, & j = i+1, \\ F_i, & \text{otherwise}, \end{cases}$$

$$[E_i, F_j] = \delta_{ij} \frac{G_i^2 G_{i+1}^{-2} - G_i^{-2} G_{i+1}^2}{q - q^{-1}},$$

$$E_i^2 E_{i\pm1} - [2]_q E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0 = F_i^2 F_{i\pm1} - [2]_q F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2$$

$$[E_i, E_j] = 0 = [F_i, F_j], |i - j| \geq 2.$$

The formulas for the coproduct $\hat{\Delta} : \breve{\mathcal{U}}_q(\mathfrak{gl}_n) \to \breve{\mathcal{U}}_q(\mathfrak{gl}_n) \otimes \breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ are

$$\begin{aligned} \hat{\Delta}(E_i) &= E_i \otimes G_i G_{i+1}^{-1} + G_i^{-1} G_{i+1} \otimes E_i, \\ \hat{\Delta}(F_i) &= F_i \otimes G_i G_{i+1}^{-1} + G_i^{-1} G_{i+1} \otimes F_i, \\ \hat{\Delta}(G_i) &= G_i \otimes G_i, \end{aligned}$$

the counit $\hat{\epsilon} : \breve{\mathcal{U}}_q(\mathfrak{gl}_n) \to \mathbb{C}$ is given by

$$\hat{\varepsilon}(E_i) = 0, \qquad \hat{\varepsilon}(F_i) = 0, \qquad \hat{\varepsilon}(G_i) = 1,$$

and the antipode is determined by

$$\hat{S}(E_i) = -qE_i, \qquad \hat{S}(F_i) = -q^{-1}F_i, \qquad \hat{S}(G_i) = G_i^{-1}.$$

Finally, the $*$-structure is given by

$$E_i^* = F_i, \qquad G_i^* = G_i.$$

Throughout, we will use the Sweedler notation $\hat{\Delta}(X) = X_{(1)} \otimes X_{(2)}$ for the coproduct. We note that with this definition of $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$, weight spaces are defined by saying that G_i acts on vectors of weight $\mu = (\mu_1, \dots, \mu_n)$ by multiplication by $q^{\frac{1}{2}\mu_i}$.

The Hopf $*$-algebra $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ is the Hopf $*$-subalgebra of $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ generated by the elements E_i, F_i, $K_i = G_i G_{i+1}^{-1}$ and K_i^{-1}, for $i = 1, \dots, n-1$. The element K_i acts on vectors of weight $\mu \in \mathbf{P}$ by multiplication by $q^{\frac{1}{2}(\alpha_i, \mu)}$.

2.3 Quantized algebras of functions

Fix $K_q = \mathrm{SU}_q(n)$ for $n \geq 2$. The quantized algebra of functions $\mathcal{O}(K_q)$ is the space of matrix coefficients of finite-dimensional type 1 representations of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$; see [KS97] for more details. If σ is a type 1 representation of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ and $\xi \in V^\sigma$, $\xi^* \in V^{\sigma *}$, we denote the associated matrix coefficient by the bra-ket notation

$$\langle \xi^* | \cdot | \xi \rangle : X \mapsto (\xi^*, \sigma(X)\xi), \qquad \text{for } X \in \breve{\mathcal{U}}_q(\mathfrak{g}).$$

We shall use the $*$-Hopf algebra structure on $\mathcal{O}(K_q)$ which makes the evaluation map $\breve{\mathcal{U}}_q(\mathfrak{sl}_n) \times \mathcal{O}(\mathrm{SU}_q) \to \mathbb{C}$ into a *skew*-pairing of $*$-Hopf algebras, or equivalently, a Hopf pairing of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)^{\mathrm{cop}}$ and $\mathcal{O}(K_q)$. Thus, for all $X, Y \in \breve{\mathcal{U}}_q(\mathfrak{sl}_n)$, $f, g \in \mathcal{O}(K_q)$,

$$\begin{aligned}(XY, f) &= (X, f_{(1)})(Y, f_{(2)}),\\ (X, fg) &= (X_{(1)}, g)(X_{(2)}, f)\\ (\hat{S}(X), f) &= (X, S^{-1}(f)),\end{aligned}$$

where we use Sweedler notation $\Delta(f) = f_{(1)} \otimes f_{(2)}$ for $f \in \mathcal{O}(K_q)$. In terms of matrix coefficients the multiplication is given by

$$\langle \xi^* | \cdot | \xi \rangle \, \langle \eta^* | \cdot | \eta \rangle = \langle \eta^* \otimes \xi^* | \cdot | \eta \otimes \xi \rangle. \tag{2.1}$$

where $\xi \in V^\sigma$, $\xi^* \in V^{\sigma *}$, $\eta \in V^\tau$, $\eta^* \in V^{\tau *}$ for type 1 representations σ, τ.

The comultiplication of $\mathcal{O}(K_q)$ defines left and right corepresentations of $\mathcal{O}(K_q)$ on itself. They will play very different roles in what follows: the left regular

corepresentation will be used to define representations of K_q, while the right regular representation will be used to define K_q-invariant differential operators and carry out their harmonic analysis.
The right regular corepresentation of $\mathcal{O}(K_q)$ gives rise to a left action of $\check{\mathcal{U}}_q(\mathfrak{sl}_n)$ according to the formula

$$X \rightharpoonup f = f_{(1)}(X, f_{(2)}), \qquad \text{for } X \in \check{\mathcal{U}}_q(\mathfrak{gl}_n), f \in \mathcal{O}(\mathrm{U}_q(n)). \tag{2.2}$$

We shall usually write this simply as Xf.
The Hilbert space $L^2(K_q)$ is the completion of $\mathcal{O}(K_q)$ with respect to the inner product

$$\langle f, g \rangle = \phi(f^* g),$$

where ϕ is the Haar state of $\mathcal{O}(K_q)$.
The left and right multiplication action of $f \in \mathcal{O}(K_q)$ on $L^2(K_q)$ will be denoted by $\mathsf{M}_l(f)$ and $\mathsf{M}_r(f)$, respectively. The left multiplication action defines a $*$-homomorphism $\mathcal{O}(K_q) \to \mathbb{L}(L^2(K_q))$. By definition, the C^*-completion $C(K_q)$ of $\mathcal{O}(K_q)$ is the norm closure of the image of $\mathcal{O}(K_q)$ under this representation. In this way one obtains the compact quantum group structure of K_q.
The algebra $\mathcal{O}(\mathrm{U}_q(n))$ is defined analogously, as matrix coefficients of type 1 representations of $\check{\mathcal{U}}_q(\mathfrak{gl}_n)$. All the above constructions carry over to $\mathrm{U}_q(n)$.

2.4 Representations and duality

Let $K_q = \mathrm{SU}_q(n)$. By definition, a unitary representation of K_q on a Hilbert space H is a unitary element $U \in M(C(K_q) \otimes \mathbb{K}(H))$ such that $(\Delta \otimes \mathrm{Id})(U) = U_{13}U_{23}$. Here we are using leg numbering notation. We shall often designate unitary K_q-representations simply by the Hilbert spaces underlying them. If H, H' are unitary representations of K_q we write $\mathrm{Hom}_{K_q}(H, H')$ for the space of intertwiners, that is, for the set of all bounded linear maps $T : H \to H'$ satisfying $(\mathrm{Id} \otimes T)U = U'(\mathrm{Id} \otimes T)$.
A unitary representation H of K_q is irreducible if and only if $\mathrm{Hom}_{K_q}(H, H) = \mathbb{C}$. All irreducible unitary representations of K_q are finite dimensional, and we write $\mathrm{Irr}(K_q)$ for the set of their equivalence classes. In the context of harmonic analysis, elements of $\mathrm{Irr}(K_q)$ will be referred to as K_q-types. We shall usually blur the distinction between a specific irreducible representation and its class in $\mathrm{Irr}(K_q)$. Unless otherwise stated, the Hilbert space underlying a K_q-representation $\sigma \in \mathrm{Irr}(K_q)$ will be denoted V^σ.
We use $\mathbb{1}_{K_q}$ to denote the trivial representation of K_q. For $\sigma \in \mathrm{Irr}(K_q)$, we denote by σ^c the (unitary) conjugate representation. If a K_q-representation π contains σ as an irreducible subrepresentation, we write $\sigma \leq \pi$.
We define $C_c(\hat{K}_q)$ as the algebraic direct sum

$$C_c(\hat{K}_q) = \bigoplus_{\sigma \in \mathrm{Irr}(K_q)} \mathbb{L}(V^\sigma).$$

Its enveloping C^*-algebra is denoted $C_0(\hat{K}_q)$, this identifies with the C^*-algebra of functions on the dual discrete quantum group $\hat{K}_q$.
We will also work with the algebraic direct product

$$C(\hat{K}_q) = \prod_{\sigma \in \mathrm{Irr}(K_q)} \mathbb{L}(V^\sigma),$$

which can be identified with the algebraic dual space $\mathcal{O}(K_q)^*$ of $\mathcal{O}(K_q)$. In particular, the quantized universal enveloping algebra $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ is naturally a $*$-subalgebra of $C(\hat{K}_q)$, and we will routinely use the same notation for elements of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ and their images in $C(\hat{K}_q)$.
In our context, the main reason to consider the algebra $C(\hat{K}_q)$ is that it contains some elements outside $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ which we shall need. In particular, the universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n$ embeds into $C(K_q)$ if we identify $X \in \mathfrak{h}$ with the operator which acts as $\mu(X)$ on the weight space $(V^\sigma)_\mu$ for each $\sigma \in \mathrm{Irr}(K_q)$, $\mu \in \mathbf{P}$.

2.5 Quantum subgroups

Let $K_q = \mathrm{SU}_q(n)$. Given a set $I \subseteq \Sigma$ of simple roots, we let $\mathfrak{h}^{I\perp}$ denote the subspace of $\mathfrak{h}$ annihilated by the $\alpha_i \in I$, and let $\mathfrak{h}^I$ be its orthocomplement with respect to the invariant bilinear form. We let $\mathfrak{g}^I$ denote the following block-diagonal Lie subalgebra of $\mathfrak{sl}_n$:

$$\mathfrak{g}^I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbf{\Delta} \cap \mathbb{Z}I} \mathfrak{g}_\alpha.$$

This subalgebra admits the decomposition $\mathfrak{g}^I = \mathfrak{s}^I \oplus \mathfrak{h}^{I\perp}$ where $\mathfrak{s}^I = \mathfrak{h}^I \oplus \bigoplus_{\alpha \in \mathbf{\Delta} \cap \mathbb{Z}I} \mathfrak{g}_\alpha$ is semisimple and $\mathfrak{h}^{I\perp}$ is central. The subalgebra $\mathfrak{k}^I = \mathfrak{g}^I \cap \mathfrak{su}_n$ is the Lie algebra of a block-diagonal subgroup $K^I \subseteq \mathrm{SU}(n)$.
The analogous families of closed quantum subgroups of $\mathrm{SU}_q(n)$ are defined as follows. Here, we use the notation $\langle x_j \rangle$ to denote the $\sigma(C(\hat{K}_q), \mathcal{O}(K_q))$-closed subalgebra of $C(\hat{K}_q)$ generated by a collection of elements $x_j \in C(\hat{K}_q)$. For each $I \subseteq \Sigma$, we define

$$C(\hat{K}_q^I) = \langle X \in \mathcal{U}(\mathfrak{h}),\ E_i, F_i\ (i \in I) \rangle, \quad C(\hat{S}_q^I) = \langle X \in \mathcal{U}(\mathfrak{h}^I),\ E_i, F_i\ (i \in I) \rangle,$$
$$C(\hat{T}^I) = \langle X \in \mathcal{U}(\mathfrak{h}^I) \rangle, \qquad C(\hat{T}^{I\perp}) = \langle X \in \mathcal{U}(\mathfrak{h}^{I\perp}) \rangle.$$

We then define $\mathcal{O}(K_q^I)$, $\mathcal{O}(S_q^I)$, $\mathcal{O}(T^I)$ and $\mathcal{O}(T^{I\perp})$ to be the images of $\mathcal{O}(K_q)$ under the induced surjection of $C(\hat{K}_q)^*$ onto $C(\hat{K}_q^I)^*$, $C(\hat{S}_q^I)^*$, $C(\hat{T}^I)^*$ and $C(\hat{T}^{I\perp})^*$, respectively. They are Hopf *-algebras under the induced operations. In particular, $\mathcal{O}(K_q^\emptyset)$ is isomorphic to $\mathcal{O}(T)$. We write π_T for the projection homomorphism $\mathcal{O}(K_q) \twoheadrightarrow \mathcal{O}(T)$, and for its extension to the C^*-algebras. At the other extreme, we have $\mathcal{O}(K_q^\Sigma) = \mathcal{O}(K_q)$.

The quantum subgroups corresponding to the singleton subsets $I = \{\alpha_i\}$ with $i = 1, \ldots, n-1$ will play a particularly important role. In this case, we will write $K^i_q, S^i_q, T^i, T^{i\perp}$ for the above quantum groups. Note that $S^i_q \cong \mathrm{SU}_q(2)$. We will also write $\breve{\mathcal{U}}_q(\mathfrak{s}^i_q)$ for the Hopf subalgebra of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ generated by E_i, F_i, K_i and K_i^{-1}.

2.6 The quantized flag manifold

Here, we summarize the basic definitions and properties of quantum flag manifolds. For more details see [CP95], [SD99], [HK04], [Sto03].
The full flag manifold of $K_q = \mathrm{SU}_q(n)$ is the quantum space $\mathcal{X}_q = K_q/T$, defined via its algebra of functions as follows. The algebra $\mathcal{O}(K_q)$ is a right $\mathcal{O}(T)$-comodule algebra by restriction of the canonical right coaction of $\mathcal{O}(K_q)$ along the projection homomorphism $\pi_T : \mathcal{O}(K_q) \to \mathcal{O}(T)$. By definition, the algebra $\mathcal{O}(\mathcal{X}_q)$ is the $*$-subalgebra of $\mathcal{O}(T)$-coinvariant elements, that is,

$$\begin{aligned}\mathcal{O}(\mathcal{X}_q) &= \{f \in \mathcal{O}(K_q) \mid (\mathrm{Id} \otimes \pi_T)\Delta(f) = f \otimes 1\} \\ &= \{f \in \mathcal{O}(K_q) \mid K_i f = f \text{ for all } i = 1, \ldots, n-1\}.\end{aligned}$$

More generally, for any $\mu = (m_1, \ldots, m_n) \in \mathbf{P}$ we define the section space of the induced line bundle $\mathcal{E}_\mu$ over $\mathcal{X}_q$ by

$$\begin{aligned}\mathcal{O}(\mathcal{E}_\mu) &= \{f \in \mathcal{O}(K_q) \mid (\mathrm{Id} \otimes \pi_T)\Delta(f) = f \otimes e^\mu\} \\ &= \{f \in \mathcal{O}(K_q) \mid K_i f = q^{\frac{1}{2}(m_i - m_{i+1})} f \text{ for all } i = 1, \ldots, n-1\}.\end{aligned}$$

In other words, $\mathcal{O}(\mathcal{E}_\mu)$ is the μ-weight space of the right regular action of T. Similarly, $L^2(\mathcal{E}_\mu)$ and $C(\mathcal{E}_\mu)$ are the right μ-weight spaces of $L^2(\mathrm{U}_q(n))$ and $C(\mathrm{U}_q(n))$, respectively. They are the closures of $\mathcal{O}(\mathcal{E}_\mu)$ in $L^2(\mathrm{U}_q(n))$ and $C(\mathrm{U}_q(n))$, respectively. We will abbreviate direct sums of the form $\mathcal{O}(\mathcal{E}_\mu) \oplus \mathcal{O}(\mathcal{E}_\nu)$ as $\mathcal{O}(\mathcal{E}_\mu \oplus \mathcal{E}_\nu)$, and use analogous notation for their completions.
Multiplication in $\mathcal{O}(K_q)$ restricts to a map $\mathcal{O}(\mathcal{E}_\mu) \otimes \mathcal{O}(\mathcal{E}_\nu) \to \mathcal{O}(\mathcal{E}_{\mu+\nu})$ for any $\mu, \nu \in \mathbf{P}$. In particular, each $\mathcal{O}(\mathcal{E}_\mu)$ is a bimodule over $\mathcal{O}(\mathcal{E}_0) = \mathcal{O}(\mathcal{X}_q)$. These modules are projective as either left or right $\mathcal{O}(\mathcal{X}_q)$-modules since $\mathcal{O}(K_q/T) \subset \mathcal{O}(K_q)$ is a faithfully flat Hopf-Galois extension [MS99].
Later on we will need an analogue of trivializing partitions of unity for the line bundles $\mathcal{E}_\mu$. These are described in the following lemma, which is an immediate consequence of Hopf-Galois theory, see [Sch04].

Lemma 2.1. *For any $\mu \in \mathbf{P}$, there exists a finite collection of sections $f_1, \ldots, f_k \in \mathcal{O}(\mathcal{E}_\mu)$ and $g_1, \ldots, g_k \in \mathcal{O}(\mathcal{E}_{-\mu})$ such that $\sum_{j=1}^k f_j g_j = 1 \in \mathcal{O}(\mathcal{X}_q)$.*

We will be interested in operators arising from the action of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ on the above line bundles. Let $X \in \breve{\mathcal{U}}_q(\mathfrak{g})$ be of weight $\nu = (k_1, \ldots, k_n)$ for the left adjoint action, *i.e.*, $K_i X K_i^{-1} = q^{\frac{1}{2}(k_i - k_{i+1})} X$ for all i. Then the right regular action $X : \mathcal{O}(K_q) \to \mathcal{O}(K_q)$ given by $Xf = X \rightharpoonup f$ restricts to a map

$X : \mathcal{O}(\mathcal{E}_\mu) \to \mathcal{O}(\mathcal{E}_{\mu+\nu})$ for every $\mu \in \mathbf{P}$. In this way, X defines an unbounded operator from $L^2(\mathcal{E}_\mu)$ to $L^2(\mathcal{E}_{\mu+\nu})$ with dense domain $\mathcal{O}(\mathcal{E}_\mu)$, such that X^*X is essentially self-adjoint. It should be thought of as a K_q-invariant differential operator.

3 Isotypical decompositions and associated C^*-categories

In this section and the next, we introduce the fundamental analytical structures which will be used throughout the remainder of the paper. The idea is to describe the behaviour of certain linear operators with respect to a decomposition into isotypical subspaces.

3.1 Isotypical decompositions

Let K_q be a compact quantum group. Any unitary representation π of K_q on a Hilbert space H can be decomposed into a direct sum of its isotypical components,

$$H = \bigoplus_{\sigma \in \mathrm{Irr}(K_q)} H_\sigma,$$

where $H_\sigma \cong \mathrm{Hom}_{K_q}(V^\sigma, H) \otimes V^\sigma$. We denote by p_σ the orthogonal projection onto H_σ. More generally, for any set $S \subseteq \mathrm{Irr}(K_q)$, we write $p_S = \sum_{\sigma \in S} p_\sigma$, so that p_S is the orthogonal projection onto $H_S = \bigoplus_{\sigma \in S} H_\sigma$.
An important observation for what follows is that certain sufficiently nice subspaces of H, such as weight spaces, still admit a K_q-isotypical decomposition even though they may not be K_q-subrepresentations. This is the point of the following definition.

Definition 3.1. A *K_q-harmonic space* is a Hilbert space of the form $\mathcal{H} = PH$, where H is a unitary K_q-representation space and $P : H \to H$ is an orthogonal projection which commutes with every isotypical projection p_σ for $\sigma \in \mathrm{Irr}(K_q)$. In this case each p_σ restricts to a projection on $\mathcal{H}$, and we call $\mathcal{H}_\sigma = p_\sigma PH$ the *σ-isotypical subspace* of $\mathcal{H}$.

For us, the key example of a K_q-harmonic space will be the L^2-section space of a homogeneous line bundle over the quantized flag manifold. The corresponding Hilbert space is not a subrepresentation of the right regular representation of $\mathrm{SU}_q(n)$, but it is a $\mathrm{SU}_q(n)$-harmonic space with respect to the right regular representation, see Example 4.2 below.

3.2 Harmonically finite and harmonically proper operators

Let $\mathcal{H}$, $\mathcal{H}'$ be K_q-harmonic spaces and let $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ be a bounded linear operator between them. We denote by $T_{\sigma\tau} = p_\sigma T p_\tau$ for $\sigma, \tau \in \mathrm{Irr}(K_q)$ the matrix components of T with respect to the K_q-isotypical decompositions.

Definition 3.2. Let $\mathcal{H}$, $\mathcal{H}'$ be K_q-harmonic spaces. With the notation above, we say that

a) T is *K_q-harmonically finite* if $T_{\sigma\tau} = 0$ for all but finitely many pairs $\sigma, \tau \in \mathrm{Irr}(K_q)$;

b) T is *K_q-harmonically proper* if the matrix of T is row- and column-finite, that is, if for each fixed σ we have $T_{\sigma\tau} = 0$ for all but finitely many $\tau \in \mathrm{Irr}(K_q)$ and $T_{\tau\sigma} = 0$ for all but finitely many $\tau \in \mathrm{Irr}(K_q)$.

Definition 3.3. Let $\mathcal{H}$, $\mathcal{H}'$ be K_q-harmonic spaces.

a) We define $\mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}')$ to be the norm-closure of the set of K_q-harmonically finite operators in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$

b) We define $\mathbb{A}_{K_q}(\mathcal{H}, \mathcal{H}')$ to be the norm-closure of the set of K_q-harmonically proper operators in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$.

If $\mathcal{H} = \mathcal{H}'$ we will simply write $\mathbb{K}_{K_q}(\mathcal{H})$ and $\mathbb{A}_{K_q}(\mathcal{H})$, respectively.

These definitions can be thought of as defining the Hom-sets of C^*-categories $\mathbb{K}_{K_q}$ and $\mathbb{A}_{K_q}$ whose objects are K_q-harmonic spaces. This observation will serve us as a notational convenience, since it allows us to write statements such as $T \in \mathbb{K}_{K_q}$ if the domain and target spaces of the operator T are understood.

Remark 3.4. The above definitions can be reinterpreted in the language of coarse geometry. A K_q-harmonic space $\mathcal{H}$ is a geometric $|\mathrm{Irr}(K_q)|$-Hilbert space, which is merely to say that it admits a representation of $C_0(\mathrm{Irr}(K_q)) = Z(C_0(\hat{K}_q))$. The algebra $\mathbb{A}_{K_q}(\mathcal{H})$ is basically the Roe algebra with respect to the indiscrete coarse structure on the discrete space $\mathrm{Irr}(K_q)$, see [Roe03] for more information. The fact that Roe algebras over dual spaces enter into our K-homology construction is no surprise: see the discussions in [Roe97], [Luu05], [Yun11b].

3.3 Alternative characterizations

We write $S \subset\subset \mathrm{Irr}(K_q)$ if S is a finite set of K_q-types. Recall that we write $p_S = \sum_{\sigma \in S} p_\sigma$. It is convenient to regard $(p_S)_{S \subset\subset \mathrm{Irr}(K_q)}$ as a net of projections, where the indexing set is ordered by inclusion of subsets.
The following two lemmas are exact analogues of Lemmas 3.4 and 3.5 of [Yun10], with essentially the same proofs.

Lemma 3.5. *Let $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, where $\mathcal{H}$, $\mathcal{H}'$ are K_q-harmonic spaces. The following conditions are equivalent:*

a) $T \in \mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}')$,

b) $\lim_{S \subset\subset \mathrm{Irr}(K_q)} (1 - p_S)T = 0 = \lim_{S \subset\subset \mathrm{Irr}(K_q)} T(1 - p_S)$ *in the norm topology.*

c) $\lim_{S \subset\subset \mathrm{Irr}(K_q)} p_S T p_S = T$ *in the norm topology.*

Lemma 3.6. *Let $A \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, where $\mathcal{H}$, $\mathcal{H}'$ are K_q-harmonic spaces. The following conditions are equivalent:*

a) $A \in \mathbb{A}_{K_q}(\mathcal{H}, \mathcal{H}')$,

b) For any finite set $S' \subset\subset \mathrm{Irr}(K_q)$,

$$\lim_{S \subset\subset \mathrm{Irr}(K_q)} (1 - p_S) A p_{S'} = 0 = \lim_{S \subset\subset \mathrm{Irr}(K_q)} p_{S'} A (1 - p_S)$$

in the norm topology,

c) For any finite set $S \subset\subset \mathrm{Irr}(K_q)$, $Ap_S \in \mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}')$ *and* $p_S A \in \mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}')$.

d) A *is a two-sided multiplier of* $\mathbb{K}_{K_q}$, *that is, for any* K_q*-harmonic space* $\mathcal{H}''$, $TA \in \mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}'')$ *for all* $T \in \mathbb{K}_{K_q}(\mathcal{H}', \mathcal{H}'')$ *and* $AT \in \mathbb{K}_{K_q}(\mathcal{H}'', \mathcal{H}')$ *for all* $T \in \mathbb{K}_{K_q}(\mathcal{H}'', \mathcal{H})$.

3.4 BASIC PROPERTIES

If the K_q-isotypical components of a K_q-harmonic space $\mathcal{H}$ are all finite dimensional, we shall say that $\mathcal{H}$ *has finite* K_q*-multiplicities*. In this case, the family $(p_\sigma)_{\sigma \in \mathrm{Irr}(K_q)}$ is a complete system of mutually orthogonal finite-rank projections on $\mathcal{H}$, so the following result follows from Lemma 3.5.

LEMMA 3.7. *If either* $\mathcal{H}$ *or* $\mathcal{H}'$ *has finite* K_q*-multiplicities then* $\mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}') = \mathbb{K}(\mathcal{H}, \mathcal{H}')$, *the set of compact operators from* $\mathcal{H}$ *to* $\mathcal{H}'$.

If H is a K_q-representation and K_q' is a closed quantum subgroup of K_q, then H is a K_q'-representation by restriction. We thus have projections $p_{S'}$ on H for every $S' \subset \mathrm{Irr}(K_q')$. The following result is a straightforward consequence of considering successive isotypical decompositions.

LEMMA 3.8. *Let* $K_q' \subseteq K_q$ *be a closed quantum subgroup. For any* $S \subseteq \mathrm{Irr}(K_q)$, $S' \subseteq \mathrm{Irr}(K_q')$, *the projections* p_S *and* $p_{S'}$ *commute. In particular, if* H *is a unitary* K_q*-representation space and* $\tau \in \mathrm{Irr}(K_q')$ *then* $p_\tau H$ *is a* K_q*-harmonic space.*

LEMMA 3.9. *Let* $K_q' \subseteq K_q$ *be a closed quantum subgroup. Suppose that* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are simultaneously* K_q*-harmonic and* K_q'*-harmonic spaces, in the sense that* $\mathcal{H}_i = P_i H_i$ *for* $i = 1, 2$ *where* H_i *is a unitary* K_q*-representation, and* $P_i : H_i \to H_i$ *is an orthogonal projection which commutes with both the* K_q*- and the* K_q'*-isotypical projections. Then* $\mathbb{K}_{K_q}(\mathcal{H}_1, \mathcal{H}_2) \subseteq \mathbb{K}_{K_q'}(\mathcal{H}_1, \mathcal{H}_2)$.

Proof. Let $T \in \mathbb{L}(\mathcal{H}_1, \mathcal{H}_2)$ be K_q-harmonically finite, so $p_S T p_S = T$ for some finite set $S \subset\subset \mathrm{Irr}(K_q)$. Only finitely many K_q'-types occur in each $\sigma \in S$, so T is also K_q'-harmonically finite. The claim follows. $\square$

3.5 Commuting generating quantum subgroups

Definition 3.10. Let $K_{1,q}$, $K_{2,q}$ be closed quantum subgroups of a compact quantum group K_q, defined by projections $\pi_i : \mathcal{O}(K_q) \twoheadrightarrow \mathcal{O}(K_{i,q})$ for $i = 1, 2$. We shall say that $K_{1,q}$ and $K_{2,q}$ are *commuting and generating* if $(\pi_1 \otimes \pi_2)\Delta = (\pi_1 \otimes \pi_2)\Delta^{\mathrm{cop}}$ holds, and this map is an injection of $\mathcal{O}(K)$ into $\mathcal{O}(K_1) \otimes \mathcal{O}(K_2)$.

The subgroups $K_{i,q}$ give rise to injections $\pi_i^* : C(\hat{K}_{1,q}) \to C(\hat{K}_q)$. One can check that $K_{1,q}$ and $K_{2,q}$ are commuting and generating if and only if $\pi_1^*(C(\hat{K}_{1,q}))$ and $\pi_2^*(C(\hat{K}_{2,q}))$ commute and generate a subalgebra of $C(\hat{K}_q)$ which is separating for $\mathcal{O}(K_q)$. The latter condition is often easier to check.
Consider the direct product $K_{1,q} \times K_{2,q}$ defined by the tensor product $\mathcal{O}(K_{1,q} \times K_{2,q}) = \mathcal{O}(K_{1,q}) \otimes \mathcal{O}(K_{2,q})$. Note that $\mathrm{Irr}(K_{1,q} \times K_{2,q}) = \mathrm{Irr}(K_{1,q}) \times \mathrm{Irr}(K_{2,q})$, where a pair $(\sigma_1, \sigma_2) \in \mathrm{Irr}(K_{1,q}) \times \mathrm{Irr}(K_{2,q})$ is identified with the obvious corepresentation $\sigma_1 \times \sigma_2$ of $\mathcal{O}(K_{1,q}) \otimes \mathcal{O}(K_{2,q})$ on $\mathbb{K}(V^{\sigma_1} \otimes V^{\sigma_2})$. Thanks to the embedding $(\pi_1 \otimes \pi_2)\Delta : \mathcal{O}(K_q) \to \mathcal{O}(K_{1,q}) \otimes \mathcal{O}(K_{2,q})$, any corepresentation σ of K_q defines a corepresentation $\tilde{\sigma}$ of $K_{1,q} \times K_{2,q}$. If σ is irreducible, an application of Schur's Lemma shows that $\tilde{\sigma} = \sigma_1 \times \sigma_2$ for some $\sigma_i \in \mathrm{Irr}(K_{i,q})$, and moreover σ is uniquely determined by (σ_1, σ_2). We therefore have an injection $\mathrm{Irr}(K_q) \hookrightarrow \mathrm{Irr}(K_{1,q}) \times \mathrm{Irr}(K_{2,q})$.

Lemma 3.11. *Let $K_{1,q}, K_{2,q}$ be commuting and generating closed quantum subgroups of a compact quantum group K_q. Then for any K_q-representations H, H' we have*

$$\mathbb{K}_{K_q}(\mathcal{H}, \mathcal{H}') = \mathbb{K}_{K_{1,q}}(\mathcal{H}, \mathcal{H}') \cap \mathbb{K}_{K_{2,q}}(\mathcal{H}, \mathcal{H}'), \tag{3.1}$$

and

$$\mathbb{A}_{K_q}(\mathcal{H}, \mathcal{H}') \supseteq \mathbb{A}_{K_{1,q}}(\mathcal{H}, \mathcal{H}') \cap \mathbb{A}_{K_{2,q}}(\mathcal{H}, \mathcal{H}'). \tag{3.2}$$

Proof. For $\sigma \in \mathrm{Irr}(K_q)$, let $\tilde{\sigma} = \sigma_1 \times \sigma_2$ be the associated representation of $K_{1,q} \times K_{2,q}$. The isotypical projection for σ is given by $p_\sigma = p_{\sigma_1} p_{\sigma_2}$. It follows that an operator $T : H \to H'$ is K_q-harmonically finite if and only if it is both $K_{1,q}$- and $K_{2,q}$-harmonically finite. This proves Equation (3.1). Equation (3.2) follows from the characterization of $\mathbb{A}_{K_q}$ as multipliers of $\mathbb{K}_{K_q}$, as in Lemma 3.6. □

3.6 Harmonic properties of tensor products

If $\mathcal{H}_1 = P_1 H_1$ and $\mathcal{H}_2 = P_2 H_2$ are K_q-harmonic spaces, following the notation of Definition 3.1, then the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 = (P_1 \otimes P_2) H_1 \otimes H_2$ is naturally a K_q-harmonic space with respect to the tensor product representation of K_q on $H_1 \otimes H_2$.

Lemma 3.12. *Let K_q be a compact quantum group. Then $\mathbb{K}_{K_q} \otimes \mathbb{K}_{K_q} \subseteq \mathbb{K}_{K_q}$, in the sense that for any K_q-harmonic spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1', \mathcal{H}_2'$ and any $T_1 \in \mathbb{K}_{K_q}(\mathcal{H}_1, \mathcal{H}_1')$ and $T_2 \in \mathbb{K}_{K_q}(\mathcal{H}_2, \mathcal{H}_2')$ we have $T_1 \otimes T_2 \in \mathbb{K}_{K_q}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_1' \otimes \mathcal{H}_2')$. Similarly, $\mathbb{A}_{K_q} \otimes \mathbb{K}_{K_q} \subseteq \mathbb{A}_{K_q}$ and $\mathbb{K}_{K_q} \otimes \mathbb{A}_{K_q} \subseteq \mathbb{A}_{K_q}$.*

Proof. Suppose T_1 and T_2 are K_q-harmonically finite, so that for $i = 1, 2$ there are finite sets $S_1,\, S_2 \subset \mathrm{Irr}(K_q)$ such that $p_{S_i} T_i p_{S_i} = T_i$. If S denotes the set of all irreducible K_q-types which occur in some $\sigma_1 \otimes \sigma_2$ with $\sigma_i \in S_i$, then $T_1 \otimes T_2 = p_S(T_1 \otimes T_2)p_S$. From this we deduce $\mathbb{K}_{K_q} \otimes \mathbb{K}_{K_q} \subseteq \mathbb{K}_{K_q}$.
Now suppose A is K_q-harmonically proper and T is K_q-harmonically finite. Fix $S \subset \mathrm{Irr}(K_q)$ a finite set of K_q-types such that $p_S T p_S = T$. Take $\sigma \in \mathrm{Irr}(K_q)$ arbitrary. Then

$$(A \otimes T)p_\sigma = (A \otimes T)(1 \otimes p_S)p_\sigma.$$

Let $\tau \in S$. For any $\tau' \in \mathrm{Irr}(K_q)$, we have $\sigma \leq \tau' \otimes \tau$ if and only if $\tau' \leq \sigma \otimes \tau^c$. This implies that there are only finitely many $\tau' \in \mathrm{Irr}(K_q)$ for which $(p_{\tau'} \otimes p_S)p_\sigma \neq 0$. Letting $S' \subset\subset \mathrm{Irr}(K_q)$ denote the set of such τ', we have

$$(A \otimes T)p_\sigma = (A \otimes T)(p_{S'} \otimes p_S)p_\sigma,$$

From the K_q-harmonic properness of A and T we can deduce that $(A \otimes T)p_\sigma \in \mathbb{K}_{K_q}$. A similar argument shows $p_\sigma(A \otimes T) \in \mathbb{K}_{K_q}$ for all $\sigma \in \mathrm{Irr}(K_q)$, whence Lemma 3.6 shows that $A \otimes T \in \mathbb{A}_{K_q}$. Clearly, a similar argument works for $T \otimes A$. ☐

Recall that $\mathbb{1}_{K_q}$ denotes the trivial representation of K_q. Later we shall make much use of the following trick, which allows us to replace an arbitrary isotypical projection by the trivial one.

Lemma 3.13. *Let $\sigma \in \mathrm{Irr}(K_q)$ and let V be any finite dimensional representation of K_q which contains σ as a subrepresentation. There exist intertwiners $\iota : \mathbb{C} \to V^c \otimes V$ and $\bar{\iota} : V^c \otimes V \to \mathbb{C}$ such that on any unitary K_q-representation H, the isotypical projection p_σ factorizes as*

$$p_\sigma : H \xrightarrow{\mathrm{Id}_H \otimes \iota} H \otimes V^c \otimes V \xrightarrow{p_{\mathbb{1}_{K_q}} \otimes \mathrm{Id}_V} H \otimes V^c \otimes V \xrightarrow{\mathrm{Id}_H \otimes \bar{\iota}} H.$$

Proof. If $V = V^\sigma$, this follows from standard facts about the contragredient representation. If V^σ is merely a subrepresentation of V, then we can use the inclusion map $V^\sigma \to V$ and the projection $V \to V^\sigma$, as well as the corresponding maps for the contragredient representation, to reduce to the previous situation.

☐

4 The lattice of C^*-ideals

We now specialize to the quantum group $K_q = \mathrm{SU}_q(n)$, although we note that the constructions and results of this section translate naturally to more general q-deformed compact semisimple Lie groups. We recall the family of quantum subgroups $K_q^I \subseteq K_q$ defined in Section 2.5.

Definition 4.1. A *fully K_q-harmonic space* is a Hilbert space of the form $\mathcal{H} = PH$, where H is a unitary representation of K_q and P is an orthogonal projection which commutes with all isotypical projections of each K_q^I, $I \subseteq \Sigma$.

Thus, a fully K_q-harmonic space is simultaneously a K_q^I-harmonic space for every $I \subseteq \Sigma$. Between fully K_q-harmonic spaces $\mathcal{H}$ and $\mathcal{H}'$, we have the spaces $\mathbb{K}_{K_q^I}(\mathcal{H},\mathcal{H}')$ and $\mathbb{A}_{K_q^I}(\mathcal{H},\mathcal{H}')$ for every $I \subseteq \Sigma$. To avoid unwieldy subscripts, we shall write $\mathbb{K}_I$ and $\mathbb{A}_I$ for $\mathbb{K}_{K_q^I}$ and $\mathbb{A}_{K_q^I}$ in the sequel. When $I = \{i\}$ is a singleton, we shall write $\mathbb{K}_i$ and $\mathbb{A}_i$.
The only examples of fully K_q-harmonic spaces we shall actually need are the following.

Example 4.2. *a*) Any unitary representation of K_q is a fully K_q-harmonic space.

b) By Lemma 3.8, any weight space of a K_q-representation is a fully K_q-harmonic space.

c) In particular, the L^2-section space $L^2(\mathcal{E}_\mu)$ of a homogeneous line bundle over the quantized flag manifold of K_q is a fully K_q-harmonic space. Note that the harmonic structure here comes from the *right* regular corepresentation.

Lemma 3.9 shows that we have a whole lattice of C^*-categories $(\mathbb{K}_I)_{I\subseteq\Sigma}$ for the fully K_q-harmonic spaces. Note that the ordering is reversed: $\mathbb{K}_{I_1} \subseteq \mathbb{K}_{I_2}$ if $I_1 \supseteq I_2$.
We point out, however, that this is typically not a lattice of C^*-ideals, that is, given sets $I_1 \supset I_2$ of simple roots and a fully K_q-harmonic space $\mathcal{H}$ we typically do not have $\mathbb{K}_{I_1}(\mathcal{H}) \triangleleft \mathbb{K}_{I_2}(\mathcal{H})$. To obtain a lattice of ideals, we must reduce $\mathbb{K}_I$ slightly, by restricting the class of operators we are working with.

Definition 4.3. For fully K_q-harmonic spaces $\mathcal{H}$, $\mathcal{H}'$, we define

$$\mathbb{A}(\mathcal{H},\mathcal{H}') = \bigcap_{I\subseteq\Sigma} \mathbb{A}_I(\mathcal{H},\mathcal{H}').$$

We also define

$$\mathbb{J}_I(\mathcal{H},\mathcal{H}') = \mathbb{K}_I(\mathcal{H},\mathcal{H}') \cap \mathbb{A}(\mathcal{H},\mathcal{H}')$$

for each $I \subseteq \Sigma$.

In other words, $\mathbb{A}$ is the simultaneous multiplier category of all the C^*-categories $\mathbb{K}_I$. Again, we view the spaces defined in definition 4.3 as the morphism sets of C^*-categories $\mathbb{A}$ and $\mathbb{J}_I$ whose objects are fully K_q-harmonic spaces. It is immediate from Lemma 3.9 that the $\mathbb{J}_I$ form a lattice of ideals, as we record in the following lemma.

Lemma 4.4. *If* $I_1 \supseteq I_2$ *then* $\mathbb{J}_{I_1} \triangleleft \mathbb{J}_{I_2}$.

In particular, we have $\mathbb{J}_{I_1\cup I_2} \subseteq \mathbb{J}_{I_1} \cap \mathbb{J}_{I_2}$ for any $I_1, I_2 \subseteq \Sigma$. In fact, it will be shown later that $\mathbb{J}_{I_1\cup I_2} = \mathbb{J}_{I_1} \cap \mathbb{J}_{I_2}$, see Theorem 5.1 and its proof in Section 7.3.

Lemma 4.5. *Let $\mathcal{H}$, $\mathcal{H}'$ be fully K_q-harmonic spaces. Then $\mathbb{J}_\Sigma(\mathcal{H},\mathcal{H}') = \mathbb{K}_\Sigma(\mathcal{H},\mathcal{H}')$. In particular, if either $\mathcal{H}$ or $\mathcal{H}'$ has finite K_q-multiplicities then $\mathbb{J}_\Sigma(\mathcal{H},\mathcal{H}') = \mathbb{K}(\mathcal{H},\mathcal{H}')$, the set of compact operators from $\mathcal{H}$ to $\mathcal{H}'$.*

Proof. By Lemma 3.9, we have $\mathbb{K}_\Sigma \subseteq \bigcap_{I\subseteq\Sigma} \mathbb{K}_I \subseteq \bigcap_{I\subseteq\Sigma} \mathbb{A}_I = \mathbb{A}$. This proves the first statement. The second follows using Lemma 3.7. □

The spaces of interest to us will have finite K_q-multiplicities, however they will usually not have finite K_q^I-multiplicities for $I \neq \Sigma$.

5 Longitudinal pseudodifferential operators: statement of results

In this section, we give statements of the necessary results concerning the lattice of C^*-categories $(\mathbb{J}_I)_{I\subseteq\Sigma}$ and the "pseudodifferential" operators $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$. All of these results will be discussed at the generality of $\mathrm{SU}_q(n)$. The proofs of the theorems below will be deferred until Sections 7.3 and 8. The reader willing to accept their veracity may safely skip forward to Section 9 after this section.

We begin with general results on the lattice of ideals $(\mathbb{J}_I)_{I\subseteq\Sigma}$.

Theorem 5.1. *Let $K_q = \mathrm{SU}_q(n)$ for $n \geq 2$ and let $\mathcal{H}$, $\mathcal{H}'$ be fully K_q-harmonic spaces.*

a) $\mathbb{A}(\mathcal{H},\mathcal{H}') = \bigcap_{i\in\Sigma} \mathbb{A}_i(\mathcal{H},\mathcal{H}')$.

b) For any $I \subseteq \Sigma$ and any $\sigma \in \mathrm{Irr}(K_q^I)$, $p_\sigma \in \mathbb{J}_I(\mathcal{H})$.

c) For any $I, I' \subseteq \Sigma$, $\mathbb{J}_I(\mathcal{H},\mathcal{H}') \cap \mathbb{J}_{I'}(\mathcal{H},\mathcal{H}') = \mathbb{J}_{I\cup I'}(\mathcal{H},\mathcal{H}')$.

d) If either $\mathcal{H}$ or $\mathcal{H}'$ has finite K_q-multiplicities then $\mathbb{J}_\Sigma(\mathcal{H},\mathcal{H}') = \mathbb{K}(\mathcal{H},\mathcal{H}')$, the compact operators from $\mathcal{H}$ to $\mathcal{H}'$, and hence $\bigcap_{i\in\Sigma} \mathbb{J}_i(\mathcal{H},\mathcal{H}') = \mathbb{K}(\mathcal{H},\mathcal{H}')$.

Next we consider longitudinal pseudodifferential operators along various fibrations of the quantum flag manifold. The guiding philosophy is that for $\mu,\nu \in \mathbf{P}$, $\mathbb{A}_I(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_\nu))$ should be thought of as containing the order zero longitudinal pseudodifferential operators along the leaves of the fibration $K_q/T \twoheadrightarrow K_q/K_q^I$, while $\mathbb{K}_I(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_\nu))$ should be thought of as the ideal of negative order longitudinal pseudodifferential operators. For instance, in the case $q = 1$ the space $\mathbb{A}_I(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_\nu))$ contains all order zero longitudinal pseudodifferential operators along the fibration, although it also contains many other operators, such as translations by the group action. Nevertheless, the reader should keep the analogy in mind when interpreting the next theorem. Let $\mu \in \mathbf{P}$ and $i \in \Sigma$. We introduce the unbounded operator

$$D_i = \begin{pmatrix} 0 & F_i \\ E_i & 0 \end{pmatrix} \qquad \text{on } L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i})$$

which is defined by the right regular representation on the dense domain $\mathcal{O}(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i})$. It is to be thought of as a longitudinal differential operator along the leaves of the fibration $\mathcal{X}_q \twoheadrightarrow K_q/K_{i,q}$. Notice that these operators are essentially families of Dirac operators of the type considered by Dabrowski-Sitarz in [DS03], over the base space $K_q/K_{i,q}$. We denote the operator phase of D_i by

$$\mathrm{ph}(D_i) = \begin{pmatrix} 0 & \mathrm{ph}(F_i) \\ \mathrm{ph}(E_i) & 0 \end{pmatrix},$$

where again $\mathrm{ph}(E_i) : L^2(\mathcal{E}_\mu) \to L^2(\mathcal{E}_{\mu+\alpha_i})$ and $\mathrm{ph}(F_i) : L^2(\mathcal{E}_\mu) \to L^2(\mathcal{E}_{\mu+\alpha_i})$ are acting by the right regular representation. These operators should be thought of as longitudinal pseudodifferentials operator of order 0.

Recall that if $f \in C(\mathcal{E}_\nu)$ for some $\nu \in \mathbf{P}$, then the left and right multiplication actions $\mathsf{M}_l(f)$, $\mathsf{M}_r(f)$ define operators in $\mathbb{L}(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu}))$.

Theorem 5.2. *Let $K_q = \mathrm{SU}_q(n)$ for $n \geq 2$. Let $\mu \in \mathbf{P}$, $i \in \Sigma$ and $f \in \mathcal{O}(\mathcal{E}_\nu)$ for some $\nu \in \mathbf{P}$. Then the following hold.*

a) $\mathsf{M}_l(f)$ *and* $\mathsf{M}_r(f)$ *are in* $\mathbb{A}(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu}))$.

b) $\mathrm{ph}(D_i) \in \mathbb{A}(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$

c) For any $\psi \in C_0(\mathbb{R})$, we have $\psi(D_i) \in \mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$, or equivalently, D_i has resolvent in $\mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$.

d) The following diagram

$$\begin{tikzcd} L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}) \arrow[r, "\mathrm{ph}(D_i)"] \arrow[d, "\mathsf{M}_l(f)"'] & L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}) \arrow[d, "\mathsf{M}_l(f)"] \\ L^2(\mathcal{E}_{\mu+\nu} \oplus \mathcal{E}_{\mu+\nu+\alpha_i}) \arrow[r, "\mathrm{ph}(D_i)"'] & L^2(\mathcal{E}_{\mu+\nu} \oplus \mathcal{E}_{\mu+\nu+\alpha_i}) \end{tikzcd}$$

commutes up to an element of $\mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_{\mu+\nu} \oplus \mathcal{E}_{\mu+\nu+\alpha_i}))$.

Remark 5.3. By slight abuse of notation, we will usually abbreviate part *d)* of Theorem 5.2 by writing $[\mathrm{ph}(D_i), \mathsf{M}_l(f)] \in \mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_{\mu+\nu} \oplus \mathcal{E}_{\mu+\nu+\alpha_i}))$ in the sequel. Notice that above statements about $\mathrm{ph}(D_i)$ can be restated as results about $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$. In particular, part d) is equivalent to the commutativity of the diagrams

$$\begin{tikzcd} L^2(\mathcal{E}_\mu) \arrow[r, "\mathrm{ph}(E_i)"] \arrow[d, "\mathsf{M}_l(f)"'] & L^2(\mathcal{E}_{\mu+\alpha_i}) \arrow[d, "\mathsf{M}_l(f)"] \\ L^2(\mathcal{E}_{\mu+\nu}) \arrow[r, "\mathrm{ph}(E_i)"'] & L^2(\mathcal{E}_{\mu+\nu+\alpha_i}) \end{tikzcd} \qquad \begin{tikzcd} L^2(\mathcal{E}_{\mu+\alpha_i}) \arrow[r, "\mathrm{ph}(F_i)"] \arrow[d, "\mathsf{M}_l(f)"'] & L^2(\mathcal{E}_\mu) \arrow[d, "\mathsf{M}_l(f)"] \\ L^2(\mathcal{E}_{\mu+\nu+\alpha_i}) \arrow[r, "\mathrm{ph}(F_i)"'] & L^2(\mathcal{E}_{\mu+\nu}) \end{tikzcd}$$

modulo $\mathbb{J}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu+\alpha_i}))$ and $\mathbb{J}_i(L^2(\mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_{\mu+\nu}))$, respectively.

6 Comparisons of Gelfand-Tsetlin bases

This section and the next provide the technical results from harmonic analysis which will be used to prove Theorems 5.1 and 5.2. It will be convenient to work with the quantum group $\mathrm{U}_q(n)$ rather than $\mathrm{SU}_q(n)$.

6.1 Quantum subgroups of $\mathrm{U}_q(n)$

We first introduce notation for the block diagonal quantum subgroups of $\mathrm{U}_q(n)$. These definitions follow the notation and conventions of Sections 2.2 and 2.5. We define $C(\widehat{\mathrm{U}_q(n)}) = \mathcal{O}(\mathrm{U}_q(n))^*$. For $I \subseteq \Sigma$, we define the $\sigma(C(\widehat{\mathrm{U}_q(n)}), \mathcal{O}(\mathrm{U}_q(n)))$-closed subalgebra

$$C(\widehat{\mathrm{U}_q^I(n)}) = \langle E_i, F_i \ (i \in I),\ G_j \ (j = 1, \ldots, n)\rangle.$$

and denote the associated closed quantum subgroup of $\mathrm{U}_q(n)$ by $\mathrm{U}_q^I(n)$.
In the particular cases $I = \{1, \ldots, k-1\}$, we will decompose $\mathrm{U}_q^I(n)$ as follows. Let $C(\widehat{\mathrm{U}_q^{\uparrow}(k)}) = \langle E_i, F_i \ (i = 1, \ldots, k-1), G_j \ (j = 1, \ldots, k)\rangle$ and $C(\widehat{Z_k^{\uparrow}}) = \langle G_j (j = k+1, \ldots, n)\rangle$, and let $\mathrm{U}_q^{\uparrow}(k)$ and $Z_k^{\uparrow}$ be the dual closed quantum subgroups of $\mathrm{U}_q(n)$. Then $\mathrm{U}_q^{\{1,\ldots,k-1\}}(n) = \mathrm{U}_q^{\uparrow}(k) \times Z_k^{\uparrow}$. The superscript $\uparrow$ refers to the fact that $\mathrm{U}_q^{\uparrow}(k) \cong \mathrm{U}_q(k)$ is embedded in the "upper-left corner" of $\mathrm{U}_q(n)$. We likewise decompose $\mathrm{U}_q^{\{n-k+1,\ldots,n-1\}}(n) = \mathrm{U}_q^{\downarrow}(k) \times Z_k^{\downarrow}$ where the two components are dual to $C(\widehat{\mathrm{U}_q^{\downarrow}(k)}) = \langle E_i, F_i \ (i = n-k+1, \ldots, n-1), G_j \ (j = n-k+1, \ldots, n)\rangle$ and $C(\widehat{Z_k^{\downarrow}}) = \langle G_j \ (j = 1, \ldots, n-k)\rangle$, respectively.

6.2 Upper and lower Gelfand-Tsetlin bases

Consider the nested family of quantum groups

$$T = \mathrm{U}_q^{\emptyset}(n) \subset \mathrm{U}_q^{\{1\}}(n) \subset \mathrm{U}_q^{\{1,2\}}(n) \subset \cdots \subset \mathrm{U}_q^{\Sigma}(n) = \mathrm{U}_q(n).$$

The isotypical projections of these quantum subgroups are mutually commuting. Gelfand-Tsetlin theory is based upon the observation that the simultaneous isotypical decomposition for all of these subgroups yields components of dimension one, and thus provides a basis which is well-adapted for all of them. We shall refer to this as the *upper* Gelfand-Tsetlin basis. We recall the main facts about the Gelfand-Tsetlin basis here, and refer to [KS97, §7.3] for the details.
The highest weights of type 1 representations of $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ are given by those $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. We denote the irreducible representation with highest weight μ by σ^μ. The Gelfand-Tsetlin basis for V^{σ^μ}

is indexed by tableaux of integers of the form

$$(M) = \begin{pmatrix} m_{n,1} & & m_{n,2} & & \cdots & & m_{n,n-1} & & m_{nn} \\ & m_{n-1,1} & & & \cdots & & & m_{n-1,n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & m_{21} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix},$$

where the top row is equal to μ and the lower rows satisfy the interlacing conditions $m_{i+1,j} \leq m_{ij} \leq m_{i+1,j+1}$ for all i, j. The corresponding basis element, which will be denoted $|(M)^{\uparrow}\rangle$, is determined up to phase by the fact that for each $k = 1, \ldots, n$, the vector $|(M)^{\uparrow}\rangle$ belongs to a $\mathrm{U}_q^{\uparrow}(k)$-subrepresentation with highest weight $(m_{k1}, \ldots, m_{kk})$. Moreover, $|(M)^{\uparrow}\rangle$ is a weight vector with weight

$$(s_1 - s_0, s_2 - s_1, \ldots, s_n - s_{n-1}), \tag{6.1}$$

where $s_i = \sum_{j=1}^{i} m_{ij}$ is the sum of the ith row and $s_0 = 0$ by convention. There is an alternative basis of V^{σ^μ} adapted to the lower-right inclusions

$$T = \mathrm{U}_q^{\emptyset}(n) \subset \mathrm{U}_q^{\{n-1\}}(n) \subset \mathrm{U}_q^{\{n-2,n-1\}}(n) \subset \cdots \subset \mathrm{U}_q^{\Sigma}(n) = \mathrm{U}_q(n).$$

This basis is most easily introduced by invoking the Hopf $*$-automorphism Ψ of $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ defined by:

$$\Psi(G_j) = G_{n+1-j}^{-1}, \qquad \Psi(E_i) = E_{n-i}, \qquad \Psi(F_i) = F_{n-i}. \tag{6.2}$$

Note that a highest weight vector for σ^μ is also a highest weight vector for the irreducible representation $\sigma^\mu \circ \Psi$, but with weight $\mu' = (-\mu_n, \ldots, -\mu_1)$. By Schur's Lemma, there is a unitary $\psi_\mu : V^{\sigma^{\mu'}} \to V^{\sigma^\mu}$ (unique up to scalar multiple) which intertwines $\sigma^{\mu'} \circ \Psi$ and σ^μ. We define the *lower* Gelfand-Tsetlin basis vectors by $|(M)^{\downarrow}\rangle = \psi_\mu |(M)^{\uparrow}\rangle$, where (M) is a Gelfand-Tsetlin tableau for the representation $\sigma^{\mu'}$.

6.3 Class 1 representations

Often, we will only be interested in the irreducible $\mathrm{U}_q(n)$-representations which contain a trivial $\mathrm{U}_q^{\{1,\ldots,n-2\}}(n)$-subrepresentation. These are a special case of the *class* 1 *representations* (see [KS97, §7.3.4]). A Gelfand-Tsetlin vector is contained in a trivial subrepresentation of $\mathrm{U}_q^{\uparrow}(n-1)$ if and only if it is of the form

$$\xi_m = \left| \begin{pmatrix} m & & 0 & & \cdots & & 0 & & -m' \\ & 0 & & & 0 \cdots 0 & & & 0 & \\ & & \ddots & & & & \iddots & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{pmatrix}^{\uparrow} \right\rangle$$

for some $m, m' \in \mathbb{N}$. To be contained in a trivial $\mathrm{U}_q^{\{1,\dots,n-2\}}(n)$-representation, it must additionally be of weight 0, which is to say $m = m'$. Thus, the representations of interest are precisely those with highest weight of the form $\mu = (m, 0, \dots, 0, -m)$. Note that in this case, $\sigma^\mu \cong \sigma^\mu \circ \Psi$, so that the upper and lower Gelfand-Tsetlin bases are indexed by the same set of tableaux.

We state the Gelfand-Tsetlin formulae for such representations; compare [KS97, §7.3.4]. The generic basis vector is

$$|(M)^\uparrow\rangle = \left| \begin{pmatrix} m_n & 0 & \cdots & 0 & -m'_n \\ & m_{n-1} & 0\cdots 0 & m'_{n-1} & \\ & \ddots & & \iddots & \\ & & m_2 \quad m'_2 & & \\ & & m_1 & & \end{pmatrix}^{\uparrow} \right\rangle ,$$

where we are putting $m_n = m'_n = m$ for ease of notation. We write $(M \pm \delta_{ij})$ to denote the Gelfand-Tsetlin tableau obtained from (M) by adding ± 1 to the (i,j)-entry. The action of the generators of $\breve{\mathcal{U}}_q(\mathfrak{gl}_n)$ is given by

$$\begin{aligned} &E_{k-1}|(M)^\uparrow\rangle \\ &= \left(\frac{[m_k - m_{k-1}][m_{k-1} - m'_k + k-1][m_{k-1} - m_{k-2} + 1][m_{k-1} - m'_{k-2} + k-2]}{[m_{k-1} - m'_{k-1} + k-1][m_{k-1} - m'_{k-1} + k-2]}\right)^{\frac{1}{2}} |(M + \delta_{k-1,1})^\uparrow\rangle \\ &+ \left(\frac{[m_k - m'_{k-1} + k-2][m'_{k-1} - m'_k + 1][m_{k-2} - m'_{k-1} + k-3][m'_{k-2} - m'_{k-1}]}{[m_{k-1} - m'_{k-1} + k-2][m_{k-1} - m'_{k-1} + k-3]}\right)^{\frac{1}{2}} |(M + \delta_{k-1,k-1})^\uparrow\rangle, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &F_{k-1}|(M)^\uparrow\rangle \\ &= \left(\frac{[m_k - m_{k-1} + 1][m_{k-1} - m'_k + k-2][m_{k-1} - m_{k-2}][m_{k-1} - m'_{k-2} + k-3]}{[m_{k-1} - m'_{k-1} + k-2][m_{k-1} - m'_{k-1} + k-3]}\right)^{\frac{1}{2}} |(M - \delta_{k-1,1})^\uparrow\rangle \\ &+ \left(\frac{[m_k - m'_{k-1} + k-1][m'_{k-1} - m'_k][m_{k-2} - m'_{k-1} + k-2][m'_{k-2} - m'_{k-1} + 1]}{[m_{k-1} - m'_{k-1} + k-1][m_{k-1} - m'_{k-1} + k-2]}\right)^{\frac{1}{2}} |(M - \delta_{k-1,k-1})^\uparrow\rangle, \end{aligned} \tag{6.4}$$

$$G_i|(M)^\uparrow\rangle = q^{\frac{1}{2}(s_i - s_{i-1})}|(M)^\uparrow\rangle, \tag{6.5}$$

where, as before, s_i is the sum of the ith row of (M) and $s_0 = 0$.

6.4 Change of basis formula

We now describe certain cases of the change of basis transformation between the upper and lower Gelfand-Tsetlin bases introduced in Section 6.2. We shall concentrate entirely on the family of representations of highest weight $\mu = (m, 0, 0, \dots, 0, -m)$ for $m \in \mathbb{N}$. In either choice of Gelfand-Tsetlin basis, the zero-weight subspace of σ^μ is spanned by the vectors $|(M_{\mathbf{m}})^\uparrow\rangle$ or $|(M_{\mathbf{m}})^\downarrow\rangle$ with tableaux

$$M_{\mathbf{m}} = \begin{pmatrix} m & & 0 & & \cdots & & 0 & & -m \\ & m_{n-1} & & & 0\cdots 0 & & & -m_{n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & & m_2 \; -m_2 & & & & \\ & & & & 0 & & & & \end{pmatrix}.$$

Here we use $\mathbf{m}$ to denote the increasing n-tuple $\mathbf{m} = (m_1 = 0, m_2, \ldots, m_n = m)$.

Our first goal is to compute the coefficients of the $\mathrm{U}_q^{\downarrow}(n-1)$-invariant vector $|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle$ with respect to the upper Gelfand-Tsetlin basis. We write

$$|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle = \sum_{\mathbf{m}} a_{\mathbf{m}} |(M_{\mathbf{m}})^{\uparrow}\rangle. \tag{6.6}$$

Let us apply E_k to this. The Gelfand-Tsetlin formula (6.3) shows that the coefficient of $|(M_{\mathbf{m}} + \delta_{k,1})^{\uparrow}\rangle$ in $E_k|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle$ is

$$\begin{aligned} &\Big(\tfrac{[m_{k+1}-m_k][m_k+m_{k+1}+k][m_k-m_{k-1}+1][m_k+m_{k-1}+k-1]}{[2m_k+k][2m_k+k-1]}\Big)^{\frac{1}{2}} a_{\mathbf{m}} \\ &\quad + \Big(\tfrac{[m_{k+1}+m_k+k][-m_k+m_{k+1}][m_{k-1}+m_k+k-1][-m_{k-1}+m_k+1]}{[2m_k+k+1][2m_k+k]}\Big)^{\frac{1}{2}} a_{\mathbf{m}+\delta_k}, \end{aligned} \tag{6.7}$$

where $\mathbf{m} + \delta_k$ denotes the n-tuple obtained by adding 1 to the kth entry of $\mathbf{m}$. Since $E_k|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle = 0$ for all $2 \le k \le n-1$, we obtain the recurrence relation

$$a_{\mathbf{m}+\delta_k} = -\frac{[2m_k+k+1]^{\frac{1}{2}}}{[2m_k+k-1]^{\frac{1}{2}}} a_{\mathbf{m}} \tag{6.8}$$

for all $\mathbf{m}$ and all $2 \le k \le n-1$. This multi-parameter recurrence relation has the solution

$$a_{\mathbf{m}} = (-1)^{|\mathbf{m}|} A \prod_{k=2}^{n-1} [2m_k + k - 1]^{\frac{1}{2}}, \tag{6.9}$$

where $|\mathbf{m}| = m_1 + \cdots + m_n$ and $A \in \mathbb{C}$ is some overall constant. This constant is determined up to a phase by the fact that $|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle$ has norm one. We will assume a choice of phases for the Gelfand-Tsetlin bases such that $\langle (M_{(m,0,\ldots,0)})^{\downarrow} | (M_{(m,0,\ldots,0)})^{\uparrow}\rangle$ is positive. Then A is positive. From (6.9), we calculate

$$1 = \sum_{\mathbf{m}} |a_{\mathbf{m}}|^2 = A^2 \sum_{\mathbf{m}} \prod_{k=2}^{n-1} [2m_k + k - 1],$$

where the sum is over all n-tuples $\mathbf{m}$ with $0 = m_1 \le m_2 \le \cdots \le m_n = m$. An inductive argument shows that

$$\sum_{\mathbf{m}} \prod_{k=2}^{n-1} [2m_k + k - 1] = [n-2]! \begin{bmatrix} m_n + n - 2 \\ n-2 \end{bmatrix}^2,$$

and one obtains that $A = [n-2]!^{-\frac{1}{2}} \begin{bmatrix} m+n-2 \\ n-2 \end{bmatrix}^{-1}$. In summary, we have proved the following formula.

Proposition 6.1. *In the irreducible representation of* $\mathrm{U}_q(n)$ *with highest weight* $(m, 0, \ldots, 0, -m)$,

$$|(M_{(m,0,\ldots,0)})^{\downarrow}\rangle = \sum_{\mathbf{m}} \frac{(-1)^{|\mathbf{m}|} \prod_{k=2}^{n-1} [2m_k + k - 1]^{\frac{1}{2}}}{[n-2]!^{\frac{1}{2}} \begin{bmatrix} m+n-2 \\ n-2 \end{bmatrix}} |(M_{\mathbf{m}})^{\uparrow}\rangle,$$

where the sum is over all n*-tuples* $\mathbf{m}$ *with* $0 = m_1 \leq m_2 \leq \cdots \leq m_n = m$. $\square$

6.5 Change of basis formula for $\mathrm{U}_q(3)$

In the case of $\mathrm{U}_q(3)$, the above calculation gives the following change-of-basis coefficients for the trivial $\mathrm{U}_q^{\{2\}}(3)$-type:

$$\left\langle \begin{pmatrix} m & & 0 & & -m \\ & 0 & & 0 & \\ & & 0 & & \end{pmatrix}^{\downarrow} \middle| \begin{pmatrix} m & & 0 & & -m \\ & j & & -j & \\ & & 0 & & \end{pmatrix}^{\uparrow} \right\rangle = (-1)^{j+m} \frac{[2j+1]^{\frac{1}{2}}}{[m+1]}. \tag{6.10}$$

The complete change-of-basis coefficients between the two Gelfand-Tsetlin bases of any $\mathrm{U}_q(3)$-representation were computed in [MSK95]. They are given by q-Racah coefficients. We will only need the following special cases.

Proposition 6.2. *In the representation of* $\mathrm{U}_q(3)$ *with highest weight* $(m, 0, -m)$*, consider the vectors*

$$|\mathrm{x}_j\rangle = [2j+1]^{-\frac{1}{2}} \left| \begin{pmatrix} m & & 0 & & -m \\ & j & & -j & \\ & & 0 & & \end{pmatrix}^{\uparrow} \right\rangle,$$

$$|\mathrm{y}_k\rangle = [2k+1]^{-\frac{1}{2}} \left| \begin{pmatrix} m & & 0 & & -m \\ & k & & -k & \\ & & 0 & & \end{pmatrix}^{\downarrow} \right\rangle.$$

Then

$$\langle \mathrm{y}_k | \mathrm{x}_j \rangle = \frac{(-1)^{j+k+m}}{[m+1]} {}_4\overline{\phi}_3 \left(\begin{matrix} q^{-2k}, q^{2(k+1)}, q^{-2j}, q^{2(j+1)} \\ q^{-2m}, q^{2(m+2)}, q^2 \end{matrix} \middle| q^2; q^2 \right), \tag{6.11}$$

where ${}_4\overline{\phi}_3$ *denotes the* q*-hypergeometric function.*

In the q-Racah notation of [KLS10, §14.2], this translates as

$$\langle \mathrm{y}_k | \mathrm{x}_j \rangle = (-1)^{j+k+m} [m+1]^{-1} R_k(\mu(j); q^{2(m+1)}, q^{-2(m+1)}, 1, 1 | q^2),$$

though we shall not actually use this.

It is rather cumbersome to reconcile the notation and terminology of [MSK95] with ours. For this reason, we outline a short proof of Proposition 6.2 in Appendix A.1.

6.6 Action of the phase of E_1 on the lower Gelfand-Tsetlin basis

The final task of this section is to compute the action of $\mathrm{ph}(E_1)$ with respect to the lower Gelfand-Tsetlin basis. Obtaining an explicit formula is difficult. Instead, we compute the action asymptotically as the highest weight μ goes to infinity, which is all that will be necessary for our purposes.

We shall make use of the following basic estimate for products of values near 1, whose proof we leave to the reader.

Lemma 6.3. *Fix $q \in (0,1)$ and $N \in \mathbb{N}$. There is a constant c_N with the following property: For any real numbers $0 \leq d_1, \ldots, d_N \leq q$ and $-1 \leq r_1, \ldots, r_N \leq 1$,*

$$\left| 1 - \prod_{i=1}^{N} (1-d_i)^{r_i} \right| \leq c_N \sum_{i=1}^{N} |r_i| d_i .$$

Next we prove an estimate on the change of basis coefficients from Proposition 6.2 which will allow us to reduce the q-hypergeometric function from ${}_4\overline{\phi}_3$ to ${}_2\overline{\phi}_1$.

Lemma 6.4. *Fix $k \in \mathbb{N}$. There is a constant $C(k)$ such that for all $j, m \in \mathbb{N}$*

$$\left| (-1)^{j+k+m} \tfrac{[2j+1]}{[j]^{\frac{1}{2}} [j+1]^{\frac{1}{2}}} \langle \mathrm{y}_k | \mathrm{x}_j \rangle \; - \; q^{m-j+\frac{1}{2}} (q - q^{-1}) \mathrm{p}_k (q^{2(m-j)} | q^2) \right| \\ \leq \; C(k) q^{m+j}, \quad (6.12)$$

where $|\mathrm{x}_j\rangle$ and $|\mathrm{y}_k\rangle$ are as in Proposition 6.2, and $\mathrm{p}_k(\,\cdot\,|q^2)$ is the little q^2-Legendre polynomial:

$$\mathrm{p}_k(x|q^2) = {}_2\overline{\phi}_1 \left(\begin{matrix} q^{-2k}, q^{2(k+1)} \\ q^2 \end{matrix} \middle| q^2 ; q^2 x \right).$$

Proof. Proposition 6.2 says

$$(-1)^{j+k+m} \frac{[2j+1]}{[j]^{\frac{1}{2}} [j+1]^{\frac{1}{2}}} \langle \mathrm{y}_k | \mathrm{x}_j \rangle \\ = \sum_{l=0}^{k} \frac{1}{[m+1]} \frac{[2j+1]}{[j]^{\frac{1}{2}} [j+1]^{\frac{1}{2}}} \frac{(q^{-2k}, q^{2(k+1)}, q^{-2j}, q^{2(j+1)}; q^2)_l}{(q^{-2m}, q^{2(m+2)}, q^2, q^2; q^2)_l} q^{2l}. \quad (6.13)$$

On the other hand

$$q^{m-j+\frac{1}{2}} (q - q^{-1}) \mathrm{p}_k (q^{2(m-j)} | q^2) \\ = \sum_{l=0}^{k} (q - q^{-1}) \frac{(q^{-2k}, q^{2(k+1)}; q^2)_l}{(q^2, q^2; q^2)_l} q^{(2l+1)(m-j) + (2l+\frac{1}{2})}. \quad (6.14)$$

Denote by A_l and B_l the lth summand of (6.13) and (6.14), respectively. Note that $|B_l| \leq C_1(k) q^{m-j}$ for some constant $C_1(k)$.

If $j < l$, then the Pochammer symbol in the numerator of $|A_l|$ is zero, so

$$|A_l - B_l| = |B_l| \le C_1(k) q^{m-j} \le C_1(k) q^{-2k} q^{m+j}. \tag{6.15}$$

If $j \ge l$, we have

$$\begin{aligned}
&|A_l - B_l| \\
&\quad = |B_l| \left| 1 - \frac{q^{-(2l+1)(m-j)-\frac{1}{2}}}{(q-q^{-1})} \frac{1}{[m+1]} \frac{[2j+1]}{[j]^{\frac{1}{2}}[j+1]^{\frac{1}{2}}} \frac{(q^{-2j}, q^{2(j+1)}; q^2)_l}{(q^{-2m}, q^{2(m+2)}; q^2)_l} \right| \\
&\quad = |B_l| \left| 1 - \frac{1}{(1-q^{2(m+1)})} \frac{(1-q^{2(2j+1)})}{(1-q^{2j})^{\frac{1}{2}}(1-q^{2(j+1)})^{\frac{1}{2}}} \right. \\
&\qquad\qquad \left. \times \frac{\prod_{i=1}^{l}(1-q^{2j-2i+2}) \prod_{i=1}^{l}(1-q^{2j+2i})}{\prod_{i=1}^{l}(1-q^{2m-2i+2}) \prod_{i=1}^{l}(1-q^{2m+2i+2})} \right|.
\end{aligned}$$

In the latter expression, all the exponents of q are positive and bounded below by $2(j-l)$. The estimate of Lemma 6.3 yields

$$\begin{aligned}
|A_l - B_l| &\le C_1(k) q^{m-j}(4l+3) q^{2(j-l)} \\
&\le C_1(k) q^{m-j}(4k+3) q^{2(j-k)} \\
&\le C_1(k)(4k+3) q^{-2k} q^{m+j}.
\end{aligned} \tag{6.16}$$

Taken together, the estimates (6.15) and (6.16) yield a constant $C_2(k)$ such that $|A_l - B_l| \le C_2(k) q^{m+j}$ for all l, j, m. The left hand side of (6.12) is then bounded by

$$\sum_{l=0}^{k} |A_l - B_l| \le k C_2(k) q^{m+j}.$$

This yields the claim. □

Finally, we describe the coefficients of $\mathrm{ph}(E_1) \left| \begin{pmatrix} m & 0 & -m \\ & 0 \quad 0 & \\ & 0 & \end{pmatrix}^{\downarrow} \right\rangle$ asymptotically as $m \to \infty$.

PROPOSITION 6.5. *For any $k \in \mathbb{N}$,*

$$\lim_{m\to\infty} \left\langle \begin{pmatrix} m & 0 & -m \\ & k \quad -k{+}1 & \\ & 0 & \end{pmatrix}^{\downarrow} \middle| \mathrm{ph}(E_1) \middle| \begin{pmatrix} m & 0 & -m \\ & 0 \quad 0 & \\ & 0 & \end{pmatrix}^{\downarrow} \right\rangle$$
$$= (-1)^k \frac{[k]}{[2k]^{\frac{1}{2}}} \left(\left[\frac{2k-1}{2}\right]^{-1} - \left[\frac{2k+1}{2}\right]^{-1} \right). \tag{6.17}$$

Proof. From Equation (6.10) we have

$$\left|\begin{pmatrix} m & 0 & -m \\ & 0 \quad 0 & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle = \sum_{j=0}^{m} (-1)^{j+m} \frac{[2j+1]}{[m+1]} |\mathrm{x}_j\rangle,$$

where the $|\mathrm{x}_j\rangle$ are as defined in Proposition 6.2. Now $|\mathrm{x}_j\rangle$ belongs to a $\mathrm{U}^{\uparrow}_q(2)$-subrepresentation of highest weight $(j,-j)$, and it has weight 0. By the standard formulae for $\breve{\mathcal{U}}_q(\mathfrak{sl}_2)$-representations, $|\mathrm{x}_j\rangle$ is an eigenvector of $|E_1| = (F_1E_1)^{\frac{1}{2}}$ with eigenvalue $[j]^{\frac{1}{2}}[j+1]^{\frac{1}{2}}$. Thus,

$$\mathrm{ph}(E_1)\left|\begin{pmatrix} m & 0 & -m \\ & 0 \quad -0 & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle = E_1 \cdot \left(\sum_{j=1}^{m} \frac{(-1)^{j+m}}{[m+1]} \frac{[2j+1]}{[j]^{\frac{1}{2}}[j+1]^{\frac{1}{2}}} |\mathrm{x}_j\rangle \right).$$

Hence the inner product in the statement of the proposition is equal to

$$\sum_{j=1}^{m} \frac{(-1)^{j+m}}{[m+1]} \frac{[2j+1]}{[j]^{\frac{1}{2}}[j+1]^{\frac{1}{2}}} \left\langle \begin{pmatrix} m & 0 & -m \\ & k \quad -k+1 & \\ & 0 & \end{pmatrix}^{\downarrow} \middle| E_1 \middle| \mathrm{x}_j \right\rangle. \tag{6.18}$$

Now $E_1^* = F_1$ acts on the lower Gelfand-Tsetlin basis by Formula (6.4) for $\Psi(F_1) = F_2$:

$$\begin{aligned} E_1^* \left|\begin{pmatrix} m & 0 & -m \\ & k \quad -k+1 & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle &= \tfrac{[m-k+1]^{\frac{1}{2}}[m+k+1]^{\frac{1}{2}}[k]}{[2k-1]^{\frac{1}{2}}[2k]^{\frac{1}{2}}} \left|\begin{pmatrix} m & 0 & -m \\ & k-1 \quad -k+1 & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle \\ &\quad + \tfrac{[m-k+1]^{\frac{1}{2}}[m+k+1]^{\frac{1}{2}}[k]}{[2k+1]^{\frac{1}{2}}[2k]^{\frac{1}{2}}} \left|\begin{pmatrix} m & 0 & -m \\ & k \quad -k & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle \\ &= \tfrac{[m-k+1]^{\frac{1}{2}}[m+k+1]^{\frac{1}{2}}[k]}{[2k]^{\frac{1}{2}}} \left(|\mathrm{y}_{k-1}\rangle + |\mathrm{y}_k\rangle \right). \end{aligned}$$

Putting this into (6.18) yields

$$\begin{aligned} &\left\langle \begin{pmatrix} m & 0 & -m \\ & k \quad -k+1 & \\ & 0 & \end{pmatrix}^{\downarrow} \middle| \mathrm{ph}(E_1) \middle| \begin{pmatrix} m & 0 & -m \\ & 0 \quad -0 & \\ & 0 & \end{pmatrix}^{\downarrow} \right\rangle \\ &= \tfrac{[m-k+1]^{\frac{1}{2}}[m+k+1]^{\frac{1}{2}}}{[m+1]} \tfrac{[k]}{[2k]^{\frac{1}{2}}} \left(\sum_{j=1}^{m} (-1)^{j+m} \tfrac{[2j+1]}{[j]^{\frac{1}{2}}[j+1]^{\frac{1}{2}}} \left(\langle \mathrm{y}_{k-1}|\mathrm{x}_j\rangle + \langle \mathrm{y}_k|\mathrm{x}_j\rangle \right) \right). \end{aligned} \tag{6.19}$$

Now we let $m \to \infty$. Since $\lim_{m\to\infty} \frac{[m-k+1]^{\frac12}[m+k+1]^{\frac12}}{[m+1]} = 1$, it only remains to estimate the sum in (6.19). Lemma 6.4 gives us the estimate

$$\begin{aligned}\sum_{j=1}^{m}(-1)^{j+m}&\tfrac{[2j+1]}{[j]^{\frac12}[j+1]^{\frac12}}\langle \mathrm{y}_k|\mathrm{x}_j\rangle \\ &= \sum_{j=1}^{m}(-1)^k q^{m-j+\frac12}(q-q^{-1})\,\mathrm{p}_k(q^{2(m-j)}|q^2) \;+\; R_m, \qquad (6.20)\end{aligned}$$

where

$$|R_m| \le \sum_{j=1}^{m} C(k)q^{m+j} \longrightarrow 0 \qquad \text{as } m \to \infty.$$

Also,

$$\begin{aligned}\sum_{j=1}^{m}(-1)^k q^{m-j+\frac12}&(q-q^{-1})\mathrm{p}_k(q^{2(m-j)}|q^2) \\ &= (-1)^{k+1}q^{-\frac12}(1-q^2)\sum_{i=0}^{m-1} q^i\mathrm{p}_k(q^{2i}|q^2)\end{aligned}$$

which is a partial sum for the q-integral

$$(-1)^{k+1}q^{-\frac12}\int_0^1 x^{-\frac12}\mathrm{p}_k(x|q^2)\, d_{q^2}x = (-1)^{k+1}\left[k+\tfrac12\right]_q^{-1}.$$

The calculation of this q-integral is explained in Appendix A.2. Putting all this into (6.19), we arrive at

$$\begin{aligned}\lim_{m\to\infty}\left\langle \begin{pmatrix} m & 0 & -m \\ & k \quad -k{+}1 & \\ & 0 & \end{pmatrix}^{\downarrow}\middle|\,\mathrm{ph}(E_1)\,\middle|\begin{pmatrix} m & 0 & -m \\ & 0 \quad -0 & \\ & 0 & \end{pmatrix}^{\downarrow}\right\rangle \\ = (-1)^k\frac{[k]}{[2k]^{\frac12}}\left(\left[k-\tfrac12\right]_q^{-1} - \left[k+\tfrac12\right]_q^{-1}\right).\end{aligned}$$

This finishes the proof. ∎

7 Essential orthotypicality

The notion of essential orthotypicality was introduced in [Yun10] as a tool for studying harmonic analysis on manifolds with multiple fibrations. In brief, two closed subgroups of a compact group are essentially orthotypical if their isotypical subspaces are approximately mutually orthogonal. In [Yun13] it is shown that in a compact Lie group two connected subgroups are essentially orthotypical if and only if they generate the entire group. Since we do not have an analogous characterization in the quantum case, we shall prove essential orthotypicality for quantum subgroups of $\mathrm{SU}_q(n)$ by direct calculation.

7.1 Definitions and basic properties

Definition 7.1. Two closed quantum subgroups $K_{1,q}$, $K_{2,q}$ of a compact quantum group K_q are *essentially orthotypical* if for any $\tau_1 \in \mathrm{Irr}(K_{1,q})$, $\tau_2 \in \mathrm{Irr}(K_{2,q})$ and any $\epsilon > 0$ there are only finitely many $\sigma \in \mathrm{Irr}(K_q)$ for which

$$\sup\{|\langle p_{\tau_1}\xi, p_{\tau_2}\eta\rangle| \mid \xi, \eta \in V^\sigma,\ \|\xi\| = \|\eta\| = 1\} \geq \epsilon.$$

Lemma 7.2. *Let $K_{1,q}$, $K_{2,q}$ be closed quantum subgroups of a compact quantum group K_q. The following conditions are equivalent.*

a) $K_{1,q}$ and $K_{2,q}$ are essentially orthotypical.

b) For any $\tau_1 \in \mathrm{Irr}(K_{1,q})$ and any $\epsilon > 0$, there are only finitely many irreducible unitary K_q-representations $\sigma \in \mathrm{Irr}(K_q)$ for which

$$\sup\{|\langle p_{\tau_1}\xi, p_{\mathbb{1}_{K_{2,q}}}\eta\rangle| \mid \xi, \eta \in V^\sigma,\ \|\xi\| = \|\eta\| = 1\} \geq \epsilon,$$

where $\mathbb{1}_{K_{2,q}}$ denotes the trivial representation of $K_{2,q}$.

c) For any finite sets $S_1 \subset\subset \mathrm{Irr}(K_{1,q})$ and $S_2 \subset\subset \mathrm{Irr}(K_{2,q})$, $p_{S_2}p_{S_1} \in \mathbb{K}_{K_q}(H)$ on any unitary K_q-representation space H.

Proof. This is essentially Lemma 5.1 from [Yun10]. Here, we will only prove the implication $b) \Rightarrow c)$. The other implications can easily be adapted from the proof in [Yun10].

Let $\tau_1 \in \mathrm{Irr}(K_{1,q})$. Fix $\epsilon > 0$ and let $S \subset\subset \mathrm{Irr}(K_q)$ be the finite set of K_q-types which satisfy the condition in $b)$. Then $(1 - p_S)p_{\mathbb{1}_{K_{2,q}}}p_{\tau_1} = p_{\mathbb{1}_{K_{2,q}}}p_{\tau_1}(1 - p_S)$ has norm at most ϵ. By Lemma 3.5, we therefore have $p_{\mathbb{1}_{K_{2,q}}}p_{\tau_1} \in \mathbb{K}_{K_q}(H)$. From this, we obtain $p_{\mathbb{1}_{K_{2,q}}}p_{S_1} \in \mathbb{K}_{K_q}(H)$ for any finite set $S_1 \subset\subset \mathrm{Irr}(K_{1,q})$.

Now let $\tau_2 \in \mathrm{Irr}(K_{2,q})$ be arbitrary. Choose a finite-dimensional K_q-representation V in which τ_2 occurs as a $K_{2,q}$-type. By Lemma 3.13 there are linear maps $\iota : \mathbb{C} \to V^c \otimes V$ and $\bar\iota : V^c \otimes V \to \mathbb{C}$ so that p_{τ_2} factorizes as

$$p_{\tau_2} : H \xrightarrow{\mathrm{Id}_H \otimes \iota} H \otimes V^c \otimes V \xrightarrow{p_{\mathbb{1}_{K_{2,q}}} \otimes \mathrm{Id}_V} H \otimes V^c \otimes V \xrightarrow{\mathrm{Id}_H \otimes \bar\iota} H.$$

Let S_1 be the finite set of all $K_{1,q}$-types occurring in $p_{\tau_1}(H) \otimes V^c$. We get the factorization

$$p_{\tau_2}p_{\tau_1} : H \xrightarrow{p_{\tau_1} \otimes \iota} H \otimes V^c \otimes V \xrightarrow{p_{\mathbb{1}_{K_{2,q}}} p_{S_1} \otimes \mathrm{Id}_V} H \otimes V^c \otimes V \xrightarrow{\mathrm{Id}_H \otimes \bar\iota} H.$$

But $p_{\mathbb{1}_{K_{2,q}}}p_{S_1} \in \mathbb{K}_{K_q}(H \otimes V^c)$ by the preceding paragraph, and since V is finite-dimensional, we obtain $p_{\mathbb{1}_{K_{2,q}}}p_{S_1} \otimes \mathrm{Id}_V \in \mathbb{K}_{K_q}(H \otimes V^c \otimes V)$. We also have $p_{\tau_1} \otimes \iota \in \mathbb{A}_{K_q}(H, H \otimes V^c \otimes V)$ and $\mathrm{Id}_H \otimes \bar\iota \in \mathbb{A}_{K_q}(H \otimes V^c \otimes V, H)$, since they preserve K_q-types. We deduce that $p_{\tau_2}p_{\tau_1} \in \mathbb{K}_{K_q}(H, H)$. ☐

COROLLARY 7.3. *Let $K_{1,q}$ and $K_{2,q}$ be essentially orthotypical quantum subgroups of K_q. Suppose H, H', H'' are unitary K_q-representations and that H' has finite K_q-multiplicities. Then $\mathbb{K}_{K_{2,q}}(H', H'')\mathbb{K}_{K_{1,q}}(H, H') \subseteq \mathbb{K}(H, H'') \subseteq \mathbb{K}_{K_q}(H, H'')$.*

Proof. Suppose that $A \in \mathbb{K}_{K_{1,q}}(H, H')$ is $K_{1,q}$-harmonically finite and that $B \in \mathbb{K}_{K_{2,q}}(H', H'')$ is $K_{2,q}$-harmonically finite. Then there are finite sets $S_1 \subset\subset \mathrm{Irr}(K_{1,q})$, $S_2 \subset\subset \mathrm{Irr}(K_{2,q})$ such that $A = p_{S_1}A$ and $B = Bp_{S_2}$. By essential orthotypicality and Lemma 3.7 we have $p_{S_2}p_{S_1} \in \mathbb{K}_{K_q}(H') = \mathbb{K}(H')$. Thus $BA = Bp_{S_2}p_{S_1}A$ is compact. The result follows. □

Remark 7.4. It is important that H' has finite K_q-multiplicities in the above statement. Corollary 7.3 can fail when H' has infinite K_q-multiplicities.

LEMMA 7.5. *Let $K_{1,q}$, $K'_{1,q}$, $K_{2,q}$, $K'_{2,q}$ be closed quantum subgroups of K_q, with $K_{1,q} \subseteq K'_{1,q}$ and $K_{2,q} \subseteq K'_{2,q}$. If $K_{1,q}$ and $K_{2,q}$ are essentially orthotypical then $K'_{1,q}$ and $K'_{2,q}$ are essentially orthotypical.*

Proof. For $i = 1, 2$, let τ_i be an irreducible $K'_{i,q}$-type, and let $S_i \subseteq \mathrm{Irr}(K_{i,q})$ be the finite collection of $K_{i,q}$-types which occur non-trivially in τ_i. Then on any K_q-representation H, we have

$$p_{\tau_1}p_{\tau_2} = p_{\tau_1}p_{S_1}p_{S_2}p_{\tau_2}.$$

The product $p_{S_1}p_{S_2}$ belongs to $\mathbb{K}_{K_q}(H)$ by Lemma 7.2, and the other projections belong to $\mathbb{A}_{K_q}(H)$ since they commute with K_q-isotypical projections. Hence the claim follows from Lemma 7.2. □

7.2 ESSENTIAL ORTHOTYPICALITY OF SUBGROUPS OF $\mathrm{U}_q(n)$

We now specialize to $\mathrm{U}_q(n)$. Recall that we defined block-diagonal quantum subgroups $\mathrm{U}_q^I(n)$ for any $I \subseteq \Sigma$ in section 6.1.

LEMMA 7.6. *The quantum subgroups $\mathrm{U}_q^{\{1,\ldots,n-2\}}(n)$ and $\mathrm{U}_q^{\{2,\ldots,n-1\}}(n)$ are essentially orthotypical in $\mathrm{U}_q(n)$.*

Proof. By Lemma 7.5, it suffices to prove that the quantum subgroups $\mathrm{U}_q^{\uparrow}(n-1)$ and $\mathrm{U}_q^{\{2,\ldots,n-1\}}(n)$ are essentially orthotypical. To show this, we will verify condition *b*) of Lemma 7.2.
Fix τ_1 an irreducible representation of $\mathrm{U}_q^{\uparrow}(n-1)$ and $\epsilon > 0$. Let σ be an irreducible $\mathrm{U}_q(n)$-representation. Arguing as in Section 6.3, we observe that the trivial $\mathrm{U}_q^{\{2,\ldots,n-1\}}(n)$-type does not occur in σ unless σ has highest weight of the form $\mu = (m, 0, \ldots, 0, -m)$ for some $m \in \mathbb{N}$, in which case the trivial $\mathrm{U}_q^{\{2,\ldots,n-1\}}(n)$-isotypical subspace is spanned by the lower Gelfand-Tsetlin

vector

$$|(M_{(m,0,\dots,0)})^{\downarrow}\rangle = \left| \begin{pmatrix} m & & 0 & & \cdots & & 0 & & -m \\ & 0 & & & 0\cdots 0 & & & 0 & \\ & & \ddots & & & & \iddots & & \\ & & & & 0 & & 0 & & \\ & & & & & 0 & & & \end{pmatrix}^{\downarrow} \right\rangle .$$

By proposition 6.1,

$$|(M_{(m,0,\dots,0)})^{\downarrow}\rangle = \sum_{\substack{0\le m_2\le\cdots \\ \cdots\le m_{n-1}\le m_n=m}} \frac{(-1)^{|\mathbf{m}|}\prod_{k=2}^{n-1}[2m_k+k-1]^{\frac12}}{[n-2]!^{\frac12}\begin{bmatrix} m+n-2 \\ n-2 \end{bmatrix}}$$
$$\times \left| \begin{pmatrix} m & & 0 & & \cdots & & 0 & & -m \\ & m_{n-1} & & & 0\cdots 0 & & & -m_{n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & & m_2 & & -m_2 & & \\ & & & & & 0 & & & \end{pmatrix}^{\uparrow} \right\rangle . \tag{7.1}$$

We see that the τ_1-isotypical subspace of σ will be orthogonal to $|(M_{(m,0,\dots,0)})^{\downarrow}\rangle$ unless τ_1 has highest weight of the form $(m_{n-1},0,\dots,0,-m_{n-1})$ for some $m_{n-1}\in\mathbb{N}$.
So, let τ_1 be the $\mathrm{U}_q^{\uparrow}(n-1)$-type with highest weight $(m_{n-1},0,\dots,0,-m_{n-1})$. Regardless of m, the sum in (7.1) contains at most a fixed finite number of vectors of $\mathrm{U}_q^{\uparrow}(n-1)$-type τ_1, since they all have to verify $m_2\le\cdots\le m_{n-2}\le m_{n-1}$. Moreover, the coefficient of each of these terms is bounded in absolute value by

$$\frac{[2m_{n-1}+(n-1)-1]^{\frac12(n-2)}}{[n-2]!^{\frac12}\begin{bmatrix} m+n-2 \\ n-2 \end{bmatrix}},$$

which tends to zero as $m\to\infty$. It follows that there are only finitely many $m\in\mathbb{N}$ for which there exists a unit vector ξ of $\check{\mathcal{U}}_q(\mathfrak{gl}_{n-1}^{\uparrow})$-type τ_1 such that $|\langle\xi|(M_{m,0,\dots,0})^{\downarrow}\rangle|>\epsilon$. This completes the proof. $\square$

Proposition 7.7. *Let $n\ge 2$ and let I_1, I_2 be sets of simple roots of $\mathrm{U}_q(n)$. If $I_1\cup I_2=\Sigma$ then $\mathrm{U}_q^{I_1}(n)$ and $\mathrm{U}_q^{I_2}(n)$ are essentially orthotypical in $\mathrm{U}_q(n)$.*

Proof. The result is trivial for $n=2$. Suppose now that $n>2$ and that the result has been proven for $\mathrm{U}_q(n-1)$.
We claim first that $\mathrm{U}_q^{J_1}(n)$ and $\mathrm{U}_q^{J_2}(n)$ are essentially orthotypical in $\mathrm{U}_q^{\{1,\dots,n-2\}}(n)$ whenever $J_1\cup J_2=\{1,\dots,n-2\}$. Recall that $\mathrm{U}_q^{\{1,\dots,n-2\}}(n)=\mathrm{U}_q^{\uparrow}(n-1)\times Z_{n-1}^{\uparrow}$. Moreover, we have $\mathrm{U}_q^{J}(n)=\mathrm{U}_q^{J}(n-1)\times Z_{n-1}^{\uparrow}$ for any

$J \subseteq \{1, \ldots, n-2\}$. Note that $\mathrm{U}_q^J(n-1)$ and $Z_{n-1}^{\uparrow}$ are commuting and generating quantum subgroups of $\mathrm{U}_q^J(n)$. Using Lemma 3.11, the inductive hypothesis implies that the quantum subgroups $\mathrm{U}_q^{J_1}(n), \mathrm{U}_q^{J_2}(n) \subseteq \mathrm{U}_q^{\{1,\ldots,n-2\}}(n)$ satisfy condition c) of Lemma 7.2 whenever $J_1 \cup J_2 = \{1, \ldots, n-2\}$. This proves the claim.

Suppose now that $I_1 \cup I_2 = \{1, \ldots, n-1\}$. Assume without loss of generality that $n-1 \in I_1$. Let $\tau_i \in \mathrm{Irr}(\mathrm{U}_q^{I_i}(n))$ for $i = 1, 2$. Moreover, let S_i denote the set of $\mathrm{U}_q^{I_i \setminus \{n-1\}}(n)$-types that occur in τ_i, so that we have $p_{\tau_i} = p_{\tau_i} p_{S_i}$ on any $\mathrm{U}_q(n)$-representation H. By the claim above, $p_{S_1} p_{S_2}$ is in $\mathbb{K}_{\mathrm{U}_q^{\{1,\ldots,n-2\}}(n)}(H)$, so for any $\epsilon > 0$ there exists a finite set $F_1 \subset\subset \mathrm{Irr}(\mathrm{U}_q^{\{1,\ldots,n-2\}}(n))$ such that $\|(1 - p_{F_1}) p_{S_1} p_{S_2}\| < \epsilon$. Note that p_{F_1} commutes with both p_{S_1} and p_{S_2}. We therefore obtain

$$\|p_{\tau_1} p_{\tau_2} - p_{\tau_1} p_{F_1} p_{\tau_2}\| = \|p_{\tau_1} p_{S_1} (1 - p_{F_1}) p_{S_2} p_{\tau_2}\| < \epsilon. \tag{7.2}$$

Now we repeat this trick, this time removing the first simple root instead of the last. Let T_1 denote the finite collection of $\mathrm{U}_q^{I_1 \setminus \{1\}}(n)$-types which occur in τ_1, and let T_2 denote the finite collection of $\mathrm{U}_q^{\{2,\ldots,n-2\}}(n)$-types which occur in any of the $\mathrm{U}_q^{\{1,\ldots,n-2\}}(n)$-types in F_1. Since we assumed that $n-1 \in I_1$, we have $(I_1 \setminus \{1\}) \cup \{2, \ldots, n-2\} = \{2, \ldots, n-1\}$. Another application of the above claim implies that $p_{T_1} p_{T_2} \in \mathbb{K}_{\mathrm{U}_q^{\{2,\ldots,n-1\}}(n)}(H)$. Thus, there is a finite subset $F_2 \subset\subset \mathrm{Irr}(\mathrm{U}_q^{\{2,\ldots,n-1\}}(n))$ such that $\|(1 - p_{F_2}) p_{T_1} p_{T_2}\| < \epsilon$, and we obtain

$$\|p_{\tau_1} p_{F_1} p_{\tau_2} - p_{\tau_1} p_{F_2} p_{F_1} p_{\tau_2}\| = \|p_{\tau_1} p_{T_1} (1 - p_{F_2}) p_{T_2} p_{F_1} p_{\tau_2}\| < \epsilon. \tag{7.3}$$

Combining Equations (7.2) and (7.3) gives

$$\|p_{\tau_1} p_{\tau_2} - p_{\tau_1} p_{F_2} p_{F_1} p_{\tau_2}\| < 2\epsilon.$$

By Lemma 7.6, $p_{F_2} p_{F_1} \in \mathbb{K}_{\mathrm{U}_q(n)}(H)$, so $p_{\tau_1} p_{F_2} p_{F_1} p_{\tau_2} \in \mathbb{K}_{\mathrm{U}_q(n)}(H)$. Since ϵ was arbitrary, $p_{\tau_1} p_{\tau_2} \in \mathbb{K}_{\mathrm{U}_q(n)}(H)$. This completes the proof. ☐

Lemma 7.8. *For any $I_1, I_2 \subseteq \Sigma$, $\mathrm{U}_q^{I_1}(n)$ and $\mathrm{U}_q^{I_2}(n)$ are essentially orthotypical as quantum subgroups of $\mathrm{U}_q^{I_1 \cup I_2}(n)$.*

Proof. Write $I = I_1 \cup I_2$. The quantum group $\mathrm{U}_q^I(n)$ has a block diagonal decomposition, which we shall write as $\mathrm{U}_q^I(n) = \prod_k \mathrm{U}_q(n_k)$. Let $\Sigma_k \subseteq \Sigma$ be the set simple roots of the block $\mathrm{U}_q(n_k)$, and put $I_{i,k} = I_i \cap \Sigma_k$ for $i = 1, 2$. We obtain decompositions $\mathrm{U}_q^{I_i}(n) = \prod_k \mathrm{U}_q^{I_{i,k}}(n_k)$. For each k, $I_{1,k} \cup I_{2,k} = \Sigma_k$, so Proposition 7.7 says that the subgroups $\mathrm{U}_q^{I_{1,k}}(n_k)$ and $\mathrm{U}_q^{I_{2,k}}(n_k)$ are essentially orthotypical in $\mathrm{U}_q(n_k)$. A repeated application of Lemma 3.11 completes the proof. ☐

We finish this section with the analogous result for quantum subgroups of $K_q = \mathrm{SU}_q(n)$.

Proposition 7.9. *For any $I_1, I_2 \subseteq \Sigma$, $K_q^{I_1}$ and $K_q^{I_2}$ are essentially orthotypical as quantum subgroups of $K_q^{I_1\cup I_2}$.*

Proof. Let T be the diagonal maximal torus of $\mathrm{SU}(n)$ and let Z be the centre of $\mathrm{U}(n)$. Both T and Z can be naturally identified with subgroups of the maximal torus of $\mathrm{U}_q(n)$.
Fix $\tau_1 \in \mathrm{Irr}(K_q^{I_1})$ and let $\mathbb{1}_2$ denote the trivial representation of $K_q^{I_2}$. Suppose $\sigma \in \mathrm{Irr}(K_q^{I_1\cup I_2})$ contains both of these as subrepresentations. Then in particular, T acts trivially on the trivial $K_q^{I_2}$-isotypical subspace, and by Schur's Lemma $T\cap Z$ acts trivially on all of V^σ. By comparison with the representation theory of the classical groups, we therefore know that σ and τ_1 extend uniquely to representations $\tilde\sigma \in \mathrm{Irr}(\mathrm{U}_q^{I_1\cup I_2}(n))$ and $\tilde\tau_1 \in \mathrm{Irr}(\mathrm{U}_q^{I_1}(n))$, respectively, in which Z acts trivially. Denote by $\tilde{\mathbb{1}}_2$ the trivial representation of $\mathrm{U}_q^{I_2}(n)$. By Lemma 7.8, for any $\epsilon > 0$, there are only finitely many $\tilde\sigma \in \mathrm{Irr}(\mathrm{U}_q^{I_1\cup I_2}(n))$ for which

$$\sup\{|\langle p_{\tilde\tau_1}\xi, p_{\tilde{\mathbb{1}}_2}\eta\rangle| \mid \xi,\eta \in V^{\tilde\sigma},\ \|\xi\| = \|\eta\| = 1\} \geq \epsilon.$$

The result therefore follows from Lemma 7.2. ☐

Corollary 7.10. *Let $I \subseteq \Sigma$ and let H be a unitary representation of K_q. For any $\tau \in \mathrm{Irr}(K_q^I)$ the isotypical projection p_τ belongs to $\mathbb{A}(H)$.*

Proof. Let $I' \subseteq \Sigma$. For any $\tau' \in \mathrm{Irr}(K_q^{I'})$, Proposition 7.9 implies that $p_\tau p_{\tau'}$ and $p_{\tau'}p_\tau$ are in $\mathbb{K}_{I\cup I'}(H) \subseteq \mathbb{K}_{I'}(H)$. From Lemma 3.6 we deduce that $p_\tau \in \mathbb{A}_{I'}(H)$. Since I' was arbitrary $p_\tau \in \mathbb{A}(H)$. ☐

7.3 Application to the lattice of ideals

Essential orthotypicality is the crucial property for proving Theorem 5.1.

Lemma 7.11. *Let $K_q = \mathrm{SU}_q(n)$ and let $I_1, I_2 \subseteq \Sigma$. Then*

a) $\mathbb{K}_{I_1} \cap \mathbb{K}_{I_2} = \mathbb{K}_{I_1\cup I_2}$,

b) $\mathbb{A}_{I_1} \cap \mathbb{A}_{I_2} \subseteq \mathbb{A}_{I_1\cup I_2}$.

Proof. From Lemma 3.9 we have $\mathbb{K}_{I_1\cup I_2} \subseteq \mathbb{K}_{I_1} \cap \mathbb{K}_{I_2}$. For the reverse inclusion, suppose $T \in \mathbb{K}_{I_1}(\mathcal{H},\mathcal{H}') \cap \mathbb{K}_{I_2}(\mathcal{H},\mathcal{H}')$ for some fully K_q-harmonic spaces $\mathcal{H}$ and $\mathcal{H}'$. Thus for any $\epsilon > 0$, there are finite sets $S_i \subset\subset \mathrm{Irr}(K_q^{I_i})$ such that $\|T - p_{S_i}Tp_{S_i}\| < \epsilon$ for $i = 1, 2$, and we obtain $\|T - p_{S_1}p_{S_2}Tp_{S_2}p_{S_1}\| < 2\epsilon$. By Proposition 7.9, there is a finite subset $F \subseteq \mathrm{Irr}(K_q^{I_1\cup I_2})$ such that $\|p_{S_1}p_{S_2} - p_Fp_{S_1}p_{S_2}p_F\| < \epsilon/\|T\|$, from which

$$\|T - p_Fp_{S_1}p_{S_2}p_FTp_Fp_{S_2}p_{S_1}p_F\| < 4\epsilon.$$

This proves the first statement. The second claim follows by using the characterization of $\mathbb{A}_I$ as multipliers of $\mathbb{K}_I$ in Lemma 3.6. ☐

Now we are ready to assemble the above results in order to prove Theorem 5.1. Indeed, parts *a*) and *c*) of the theorem now follow as a corollary of Lemma 7.11, and part *d*) is contained in Lemma 4.5. To prove part *b*), note that if $\sigma \in \mathrm{Irr}(K_q^I)$ for some $I \subseteq \Sigma$ then p_σ is in $\mathbb{K}_I(\mathcal{H})$ for any fully K_q-harmonic space $\mathcal{H}$, so the result follows from Corollary 7.10. This completes the proof of Theorem 5.1.

8 Longitudinal pseudodifferential operators

In this section we prove Theorem 5.2.

8.1 Multiplication operators

We shall begin with Theorem 5.2 *a*), which is a consequence of the next proposition. Let us recall once again that we are equipping $L^2(K_q)$, and its weight spaces $L^2(\mathcal{E}_\mu)$ for $\mu \in \mathbf{P}$, with the structure of a fully K_q-harmonic space coming from the *right* regular representation. Thus, if $\tau \in \mathrm{Irr}(K_q^I)$ for some $I \subseteq \Sigma$ and $g = \langle \eta^* | \cdot | \eta \rangle$ is a matrix coefficient, then $p_\tau g = \langle \eta^* | \cdot | p_\tau \eta \rangle$.

Proposition 8.1. *For any $f \in \mathcal{O}(K_q)$, the left and right multiplication operators $\mathsf{M}_l(f)$ and $\mathsf{M}_r(f)$ belong to $\mathbb{A}(L^2(K_q))$.*

Proof. Fix $I \subseteq \Sigma$. We may assume that $f = \langle \xi^* | \cdot | \xi \rangle$ is a matrix coefficient of an irreducible K_q-representation. Moreover we may assume that ξ belongs to a K_q^I subrepresentation, say of type σ.
Let $\tau \in \mathrm{Irr}(K_q^I)$. From the formula (2.1) for the product of matrix coefficients, one sees that $\mathsf{M}_l(f)p_\tau = p_S \mathsf{M}_l(f) p_\tau$ where S is the finite set of K_q^I-types which occur in $\tau \otimes \sigma$. This means $\mathsf{M}_l(f)p_\tau \in \mathbb{K}_i(L^2(K_q))$.
We therefore obtain $\mathsf{M}_l(\mathcal{O}(K_q))p_\tau \subseteq \mathbb{K}_i(L^2(K_q))$. Taking adjoints shows that $p_\tau \mathsf{M}_l(\mathcal{O}(K_q)) \subseteq \mathbb{K}_i(L^2(K_q))$. By Lemma 3.6, this implies $\mathsf{M}_l(\mathcal{O}(K_q)) \in \mathbb{A}_I(L^2(K_q))$. Since I was arbitrary this yields the claim for left multiplication operators. The proof for right multiplication operators is similar. $\square$

8.2 Basic properties of the phase of E_i, F_i

Before specializing to the section spaces of bundles over the quantum flag manifold, we will first consider the abstract properties of the operators

$$D_i = \begin{pmatrix} 0 & F_i \\ E_i & 0 \end{pmatrix} \qquad \text{on } H_\mu \oplus H_{\mu+\alpha_i}$$

for any K_q-representation H and any $\mu \in \mathbf{P}$. This operator D_i is essentially self-adjoint with domain the linear span of the K_q-isotypical components.

Lemma 8.2. *With the notation above, $\psi(D_i) \in \mathbb{K}_i(H_\mu \oplus H_{\mu+\alpha_i})$ for any function $\psi \in C_0(\mathbb{R})$, and $\phi(D_i) \in \mathbb{A}_i(H_\mu \oplus H_{\mu+\alpha_i})$ for any bounded function $\phi \in C_b(\mathbb{R})$.*

Proof. Recall from Section 2.5 the subgroup $S_q^i \cong \mathrm{SU}_q(2)$ which is associated to the simple root α_i. The standard formulae for irreducible $\breve{\mathcal{U}}_q(\mathfrak{sl}(2))$-representations show that the S_q^i-isotypical subspaces of $H_\mu \oplus H_{\mu+\alpha_i}$ are precisely the eigenspaces for D_i^2, and that the spectrum of D_i^2 is discrete. It follows that if the function ψ has compact support, then $\psi(D_i)$ annihilates all but finitely many S_q^i-isotypical subspaces. On a weight space, the S_q^i-isotypical and the K_q^i-isotypical decompositions are identical, so $\psi(D_i) \in \mathbb{K}_i(H_\mu \oplus H_{\mu+\alpha_i})$. By density, $\psi(D_i) \in \mathbb{K}_i(H_\mu \oplus H_{\mu+\alpha_i})$ for any $\psi \in C_0(\mathbb{R})$.
If ϕ is a bounded function then $\phi(D_i)$ is bounded, and it preserves K_q^i-types. This proves the second statement. □

Remark 8.3. It follows from this proof that if $\psi \in C_c(\mathbb{R})$, then $\psi(D_i^2)$ is a finite linear combination of isotypical projections for $K_{i,q}$. By Corollary 7.10, these isotypical projections are in $\mathbb{A}(H)$. We can therefore deduce that $\psi(D_i) \in \mathbb{J}_i(H)$ for any even function $\psi \in C_0(\mathbb{R})$. Unfortunately, showing that $\psi(D_i) \in \mathbb{J}_i(H)$ for an odd $\psi \in C_0(\mathbb{R})$, as required for Theorem 5.2 *c*), is more difficult.

Now let V be a finite dimensional unitary representation of K_q. Then D_i acts on $H \otimes V$, and in particular on $(H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i}$, as

$$\hat{\Delta}(D_i) = \begin{pmatrix} 0 & F_i \otimes K_i \\ E_i \otimes K_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & K_i^{-1} \otimes F_i \\ K_i^{-1} \otimes E_i & 0 \end{pmatrix}.$$

We will abbreviate this expression as $D_i \otimes K_i + K_i^{-1} \otimes D_i$.

Lemma 8.4. *Let H, V be unitary K_q-representations with V finite dimensional and let $\mu \in \mathbf{P}$. As operators on $(H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i}$, we have $\mathrm{ph}(\hat{\Delta}(D_i)) \equiv \mathrm{ph}(D_i) \otimes \mathrm{Id}_V$ modulo $\mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$.*

Proof. Since K_i is strictly positive, we have $\mathrm{ph}(D_i \otimes K_i) = \mathrm{ph}(D_i) \otimes \mathrm{Id}_V$. Let us set $A = \hat{\Delta}(D_i)$, $B = D_i \otimes K_i$. Then $A - B = K_i^{-1} \otimes D_i$, which is bounded on $(H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i}$ since V is finite dimensional, and so $A - B \in \mathbb{A}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$.
Let $\phi \in C_b(\mathbb{R})$ be the function $\phi(x) = x(1+x^2)^{-\frac{1}{2}}$. Lemma 8.2 implies that $\mathrm{ph}(A) \equiv \phi(A)$ modulo $\mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$. We claim that also $\mathrm{ph}(B) \equiv \phi(B)$ modulo $\mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$. To see this, note that $(H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i} = \bigoplus_\nu (H_{\mu-\nu} \oplus H_{\mu+\alpha_i-\nu}) \otimes V_\nu$, where the sum is over all weights of V. This decomposition is invariant for B, and on each summand B acts as $q^{\frac{1}{2}(\alpha_i,\nu)} D_i \otimes \mathrm{Id}_{V_\nu}$. Lemma 8.2 implies that $\psi(B) \in \mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$ for any $\psi \in C_0(\mathbb{R})$, and the claim follows.
Therefore, it suffices to prove that $\phi(A) - \phi(B) \in \mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$. Now,

$$\phi(A) - \phi(B) = (A - B)(1 + A^2)^{-\frac{1}{2}} + B((1 + A^2)^{-\frac{1}{2}} - (1 + B^2)^{-\frac{1}{2}}).$$

The first term $(A - B)(1 + A^2)^{-\frac{1}{2}}$ is contained in $\mathbb{K}_i((H \otimes V)_\mu \oplus (H \otimes V)_{\mu+\alpha_i})$ by Lemma 8.2. For the second term we use the integral formula

$$(1 + x^2)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty t^{-\frac{1}{2}} (1 + x^2 + t)^{-1} \, dt,$$

which gives

$$
\begin{aligned}
&B((1+A^2)^{-\frac{1}{2}} - (1+B^2)^{-\frac{1}{2}}) \\
&\quad = \frac{1}{\pi} B \int_0^\infty t^{-\frac{1}{2}}((1+A^2+t)^{-1} - (1+B^2+t)^{-1})\,dt \\
&\quad = \frac{1}{\pi} B \int_0^\infty t^{-\frac{1}{2}}(1+B^2+t)^{-1}(B^2-A^2)(1+A^2+t)^{-1}\,dt \\
&\quad = \frac{1}{\pi} B \int_0^\infty t^{-\frac{1}{2}}(1+B^2+t)^{-1}B(B-A)(1+A^2+t)^{-1}\,dt \\
&\qquad + \frac{1}{\pi} B \int_0^\infty t^{-\frac{1}{2}}(1+B^2+t)^{-1}(B-A)A(1+A^2+t)^{-1}\,dt. \qquad (8.1)
\end{aligned}
$$

By Lemma 8.2, we have $B(1+B^2+t)^{-1}B \in \mathbb{A}_i$ with norm at most 1, $B-A \in \mathbb{A}_i$, and $(1+A^2+t)^{-1} \in \mathbb{K}_i$ with norm at most $(1+t)^{-1}$, so the first integral on the right hand side of equation (8.1) converges in norm in $\mathbb{K}_i((H\otimes V)_\mu \oplus (H\otimes V)_{\mu+\alpha_i})$. For the second integral, we can write

$$
\begin{aligned}
&B(1+B^2+t)^{-1}(B-A)A(1+A^2+t)^{-1} \\
&\quad = B(1+B^2+t)^{-\frac{1}{2}}(1+B^2+t)^{-\frac{1}{2}}(B-A)A(1+A^2+t)^{-\frac{1}{2}}(1+A^2+t)^{-\frac{1}{2}},
\end{aligned}
$$

where $B(1+B^2+t)^{-\frac{1}{2}}$ and $A(1+A^2+t)^{-\frac{1}{2}}$ are in $\mathbb{A}_i$ with norm at most 1, $B-A \in \mathbb{A}_i$, and $(1+B^2+t)^{-\frac{1}{2}}$ and $(1+A^2+t)^{-\frac{1}{2}}$ are in $\mathbb{K}_i$ with norm at most $(1+t)^{-\frac{1}{2}}$, so we again have norm convergence in $\mathbb{K}_i((H\otimes V)_\mu \oplus (H\otimes V)_{\mu+\alpha_i})$. This completes the proof. □

Considering the matrix entries of the operators in Lemma 8.4 gives the following result for $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$.

Corollary 8.5. *Let H and V be unitary K_q-representations, with V finite dimensional. For any weight $\mu \in \mathbf{P}$ the operators $(\mathrm{ph}(\hat{\Delta}(E_i)) - \mathrm{ph}(E_i)\otimes \mathrm{Id}_V)p_\mu$ and $p_\mu(\mathrm{ph}(\hat{\Delta}(E_i)) - \mathrm{ph}(E_i)\otimes \mathrm{Id}_V)$ belong to $\mathbb{K}_i(H\otimes V)$. Likewise, $(\mathrm{ph}(\hat{\Delta}(F_i)) - \mathrm{ph}(F_i)\otimes \mathrm{Id}_V)p_\mu$ and $p_\mu(\mathrm{ph}(\hat{\Delta}(F_i)) - \mathrm{ph}(F_i)\otimes \mathrm{Id}_V)$ belong to $\mathbb{K}_i(H\otimes V)$.*

8.3 The phase of the longitudinal Dirac operators

In this section we prove Theorem 5.2 *b*) and *c*). It is easy to see that $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$ are multipliers of $\mathbb{K}_i$, but Theorem 5.2 *b*) claims a more subtle fact, namely that $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$ are multipliers of $\mathbb{K}_j$ for every $j \in \Sigma$. We will prove this fact in a series of Lemmas, beginning with the case of $\mathrm{SU}_q(3)$.
We use the notation for subgroups of $\mathrm{U}_q(3)$ which was introduced in Section 6.1.

Lemma 8.6. *Let $\mathbb{1}_2$ be the trivial representation of $\mathrm{U}_q^{\{2\}}(3)$. On any unitary $\mathrm{U}_q(3)$-representation H, the operators $\mathrm{ph}(E_1)p_{\mathbb{1}_2}$ and $\mathrm{ph}(F_1)p_{\mathbb{1}_2}$ belong to $\mathbb{K}_{\mathrm{U}_q^{\{2\}}(3)}(H)$.*

Proof. We will show that for any $\epsilon > 0$ there is a finite collection $S \subset\subset \mathrm{Irr}(\mathrm{U}_q^{\{2\}}(3))$ such that on every irreducible $\mathrm{U}_q(3)$-representation V^σ the estimate

$$\|(1 - p_S)\,\mathrm{ph}(E_1) p_{\mathbb{1}_2}\| < \epsilon \tag{8.2}$$

holds. Since S does not depend on σ in this statement, the lemma will follow by decomposing H into irreducibles for $\mathrm{U}_q(3)$.

As before, we write σ^μ for the irreducible $\mathrm{U}_q(3)$-representation with highest weight μ. It follows from Section 6.3 that the operator $p_{\mathbb{1}_2}$ is zero on V^{σ^μ} unless $\mu = (m, 0, -m)$ for some $m \in \mathbb{N}$, in which case $p_{\mathbb{1}_2} V^\sigma$ is spanned by the lower Gelfand-Tsetlin vector

$$|(M_{(m,0,0)})^\downarrow\rangle = \left| \begin{pmatrix} m & & 0 & & -m \\ & 0 & & 0 & \\ & & 0 & & \end{pmatrix}^{\downarrow} \right\rangle .$$

Note that $|(M_{(m,0,0)})^\downarrow\rangle$ has weight 0, so that $\mathrm{ph}(E_1)|(M_{(m,0,0)})^\downarrow\rangle$ is contained in the weight space $(V^{\sigma^\mu})_{\alpha_1}$, which is spanned by the vectors

$$\left| \begin{pmatrix} m & & 0 & & -m \\ & k & & -k{+}1 & \\ & & 0 & & \end{pmatrix}^{\downarrow} \right\rangle \tag{8.3}$$

for $k = 1, \ldots, m$.

Let us denote by τ_k the $\mathrm{U}_q^{\{2\}}(3)$-type of the vector (8.3), and let $S_l = \{\tau_1, \ldots, \tau_l\}$. On $V^{(m,0,-m)}$, the operator $p_{S_l}\,\mathrm{ph}(E_1) p_{\mathbb{1}_2}$ satisfies

$$\|p_{S_l}\,\mathrm{ph}(E_1) p_{\mathbb{1}_2}\|^2 = \sum_{k=1}^{l} \left\langle \begin{pmatrix} m & & 0 & & -m \\ & k & & -k{+}1 & \\ & & 0 & & \end{pmatrix}^{\downarrow} \middle| \mathrm{ph}(E_1) \middle| (M_{(m,0,0)})^\downarrow \right\rangle^2 .$$

Using Proposition 6.5, we have that

$$\begin{aligned}
\lim_{m\to\infty} &\|p_{S_l}\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 \\
&= \sum_{k=1}^{l} \frac{[k]^2}{[2k]} \left(\left[k-\frac{1}{2}\right]^{-1} - \left[k+\frac{1}{2}\right]^{-1} \right)^2 \\
&= \sum_{k=1}^{l} \frac{1}{[2k]\left[k-\frac{1}{2}\right]^2\left[k+\frac{1}{2}\right]^2} \left([k]\left[k+\frac{1}{2}\right] - [k]\left[k-\frac{1}{2}\right] \right)^2 \\
&= \sum_{k=1}^{l} \frac{1}{[2k]\left[k-\frac{1}{2}\right]^2\left[k+\frac{1}{2}\right]^2} \left(\left[\frac{1}{2}\right][2k] \right)^2 \\
&= \sum_{k=1}^{l} \left[\frac{1}{2}\right]^2 \left(\frac{1}{[k-\frac{1}{2}]^2} - \frac{1}{[k+\frac{1}{2}]^2} \right) \\
&= 1 - \frac{[\frac{1}{2}]^2}{[l+\frac{1}{2}]^2}.
\end{aligned}$$

Let l be sufficiently large that $\frac{[\frac{1}{2}]^2}{[l+\frac{1}{2}]^2} < \frac{1}{2}\epsilon$. Then

$$\lim_{m\to\infty} \|p_{S_l}\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 > 1 - \frac{1}{2}\epsilon.$$

This implies that for all m greater than some m_0 we have

$$\|p_{S_l}\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 > 1 - \epsilon$$

on the representation $V^{\sigma^{(m,0,-m)}}$. Therefore we obtain

$$\|(1-p_{S_l})\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 = \|\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 - \|p_{S_l}\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\|^2 < \epsilon$$

for all $m > m_0$.

Let $S \subset\subset \mathrm{Irr}(\mathrm{U}_q^{\{2\}}(3))$ be the finite set containing S_l as well as the finite collection of $\mathrm{U}_q^{\{2\}}(3)$-types which appear in any of the representations of highest weight $(m,0,-m)$ for $m = 0,\ldots,m_0$. By construction, we have:

- $(1-p_S)\,\mathrm{ph}(E_1)p_{\mathbb{1}_2} = 0$ on every $V^{\sigma^{\mu}}$ with μ not of the form $(m,0,-m)$;
- $(1-p_S)\,\mathrm{ph}(E_1)p_{\mathbb{1}_2} = 0$ on every $V^{\sigma^{(m,0,-m)}}$ with $m \leq m_0$;
- $\|(1-p_S)\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}\| < \sqrt{\epsilon}$ on every $V^{\sigma^{(m,0,-m)}}$ with $m > m_0$.

We conclude that on any unitary $\mathrm{U}_q(3)$-representation H, the operator $\mathrm{ph}(E_1)p_{\mathbb{1}_2}$ is approximated to within $\sqrt{\epsilon}$ by $p_S\,\mathrm{ph}(E_1)p_{\mathbb{1}_2}$. This proves that $\mathrm{ph}(E_1)p_{\mathbb{1}_2} \in \mathbb{K}_{\mathrm{U}_q^{\{2\}}(3)}(H)$. The proof that $\mathrm{ph}(F_1)p_{\mathbb{1}_2} \in \mathbb{K}_{\mathrm{U}_q^{\{2\}}(3)}(H)$ is similar. □

COROLLARY 8.7. *With $K_q = \mathrm{SU}_q(3)$, let $\mathbb{1}_2$ denote the trivial corepresentation of $K_q^{\{2\}}$. On any unitary K_q-representation H, the operators* $\mathrm{ph}(E_1)p_{\mathbb{1}_2}$ *and* $\mathrm{ph}(F_1)p_{\mathbb{1}_2}$ *belong to* $\mathbb{K}_2(H)$.

Proof. We reuse the notation from the proof of Proposition 7.9. Recall that $T \cap Z$ acts trivially on any irreducible K_q-representation which contains the trivial $K_q^{\{2\}}$-type. Putting $H' = p_{\mathbb{1}_{T\cap Z}} H$, we have $p_{\mathbb{1}_2} = 0$ on $H'^{\perp}$, so it suffices to prove the result with H' in place of H.
The K_q-representation on H' extends to a $\mathrm{U}_q(3)$-representation in which Z acts trivially. With this extension, $\mathbb{K}_Z(H') = \mathbb{L}(H')$. One can check that Z and $K_q^{\{2\}}$ are commuting and generating quantum subgroups of $\mathrm{U}_q^{\{2\}}(3)$, so the result follows from Lemma 8.6 and Lemma 3.11. □

LEMMA 8.8. *With $K_q = \mathrm{SU}_q(3)$, let H be a unitary K_q-representation. Then we have* $\mathrm{ph}(E_i) \in \mathbb{A}(H)$ *and* $\mathrm{ph}(F_i) \in \mathbb{A}(H)$ *for* $i = 1, 2$.

Proof. Let us first assume that H has finite K_q-multiplicities.
In order to prove $\mathrm{ph}(E_1) \in \mathbb{A}(H)$, we only need to show $\mathrm{ph}(E_1) \in \mathbb{A}_2(H)$ since $\mathrm{ph}(E_1) \in \mathbb{A}_1(H)$ is clear. Let $\tau \in \mathrm{Irr}(K_q^{\{2\}})$. Choose a finite dimensional K_q-representation V which contains τ as a $K_q^{\{2\}}$-type, and use Lemma 3.13 to factorize $\mathrm{ph}(E_1)p_\tau$ on H as

$$\begin{aligned} \mathrm{ph}(E_1)p_\tau &= \mathrm{ph}(E_1)(\mathrm{Id}_H \otimes \bar{\iota})(p_{\mathbb{1}_2} \otimes \mathrm{Id}_V)(\mathrm{Id}_H \otimes \iota) \\ &= (\mathrm{Id}_H \otimes \bar{\iota})(\mathrm{ph}(E_1) \otimes \mathrm{Id}_{V^c} \otimes \mathrm{Id}_V)(p_{\mathbb{1}_2} \otimes \mathrm{Id}_V)(\mathrm{Id}_H \otimes \iota). \end{aligned} \tag{8.4}$$

Write

$$(\mathrm{ph}(E_1) \otimes \mathrm{Id}_{V^c})p_{\mathbb{1}_2} = \mathrm{ph}(\hat{\Delta}(E_1))p_{\mathbb{1}_2} + (\mathrm{ph}(E_1) \otimes \mathrm{Id}_{V^c} - \mathrm{ph}(\hat{\Delta}(E_1)))p_{\mathbb{1}_2}.$$

We have $\mathrm{ph}(\hat{\Delta}(E_1))p_{\mathbb{1}_2} \in \mathbb{K}_2(H \otimes V)$ by Corollary 8.7. We also have that $(\mathrm{ph}(E_1) \otimes \mathrm{Id}_{V^c} - \mathrm{ph}(\hat{\Delta}(E_1)))p_{\mathbb{1}_2} \in \mathbb{K}_2(H \otimes V)\,\mathbb{K}_1(H \otimes V)$ by Corollary 8.5, and since $H \otimes V$ has finite K_q-multiplicities, Lemma 3.7 shows that this is in $\mathbb{K}(H \otimes V) \subseteq \mathbb{K}_2(H \otimes V)$. We conclude that $\mathrm{ph}(E_1)p_\tau \in \mathbb{K}_2(H)$.
One can similarly show that $\mathrm{ph}(F_1)p_\tau \in \mathbb{K}_2(H)$. Moreover, by taking adjoints, we obtain $p_\tau \,\mathrm{ph}(E_1)$, $p_\tau \,\mathrm{ph}(F_1) \in \mathbb{K}_2(H)$. Using Lemma 3.6, we conclude that $\mathrm{ph}(E_1)$ and $\mathrm{ph}(F_1)$ are in $\mathbb{A}_2(H)$.
Suppose now that H does not necessarily have finite K_q-multiplicities. We can embed H into the universal K_q-representation $H_0 = L^2(K_q) \otimes \ell^2(\mathbb{N})$, where $\ell^2(\mathbb{N})$ is equipped with the trivial K_q-representation. Since $\ell^2(\mathbb{N})$ contains only the trivial $K_q^{\{2\}}$-type we have $\mathrm{Id}_{\ell^2(\mathbb{N})} \in \mathbb{K}_2(\ell^2(\mathbb{N}))$. Now $\mathrm{ph}(E_1)$ acts on H_0 as $\mathrm{ph}(E_1) \otimes \mathrm{Id}_{\ell^2(\mathbb{N})}$. This operator belongs to $\mathbb{A}_2(L^2(K_q)) \otimes \mathbb{K}_2(\ell^2(\mathbb{N}))$, and hence to $\mathbb{A}_2(H_0)$ by Lemma 3.12. It follows that the restriction of $\mathrm{ph}(E_1)$ to H belongs to $\mathbb{A}_2(H)$. A similar argument shows that $\mathrm{ph}(F_1) \in \mathbb{A}_2(H)$.
To show that $\mathrm{ph}(E_2), \mathrm{ph}(F_2) \in \mathbb{A}_1(H)$ it suffices to use the automorphism Ψ of Equation (6.2) to interchange the simple roots. This completes the proof. □

Proposition 8.9. *Let $K_q = \mathrm{SU}_q(n)$ and let H be any unitary K_q-representation. For each $i = 1, \ldots, n-1$, the operators $\mathrm{ph}(E_i) : H \to H$ and $\mathrm{ph}(F_i) : H \to H$ belong to $\mathbb{A}(H)$.*

Proof. We need to prove $\mathrm{ph}(E_i) \in \mathbb{A}_j(H)$ for every $j \in \Sigma$. Note that S_q^j and $T^{j\perp}$ are commuting and generating quantum subgroups of K_q^j. If $v \in H$ is of $T^{j\perp}$-type λ, then $\mathrm{ph}(E_i)v$ is of $T^{j\perp}$-type $\lambda + \alpha_i|_{T^{j\perp}}$, where by abuse of notation we are identifying $\alpha_i \in \mathbf{P}$ with its exponential in $\hat{T}$. It follows that $\mathrm{ph}(E_i) \in \mathbb{A}_{T^{j\perp}}(H)$. By Lemma 3.11 it remains only to prove $\mathrm{ph}(E_i) \in \mathbb{A}_{S_q^j}(H)$. If $i = j$, this is immediate. If $|i-j| = 1$, then E_i, F_i, E_j, F_j belong to a subalgebra of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$ isomorphic to $\breve{\mathcal{U}}_q(\mathfrak{sl}_3)$, and the result follows from Lemma 8.8. Finally, if $|i-j| > 1$, then $\mathrm{ph}(E_i)$ commutes with $\breve{\mathcal{U}}_q(\mathfrak{s}^j)$ so it preserves S_q^j-types and the result follows.
By taking adjoints, we also obtain $\mathrm{ph}(F_i) \in \mathbb{A}(H)$. □

We can now prove parts *b*) and *c*) of Theorem 5.2. Consider $D_i = \begin{pmatrix} 0 & F_i \\ E_i & 0 \end{pmatrix}$ acting on $L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i})$ for some $\mu \in \mathbf{P}$. Theorem 5.2 *b*) follows directly from Proposition 8.9. In order to prove part *c*) let $\phi \in C_b(\mathbb{R})$ be a continuous odd function such that $\phi(D_i) = \mathrm{ph}(D_i)$. We know from Remark 8.3 that $(1+D_i^2)^{-1} \in \mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$. Since $\phi(D_i) \in \mathbb{A}(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$ we also have $\phi(D_i)(1+D_i^2)^{-1} \in \mathbb{J}_i(L^2(\mathcal{E}_\mu \oplus \mathcal{E}_{\mu+\alpha_i}))$. By the Stone-Weierstrass Theorem the functions $x \mapsto (1+x^2)^{-1}$ and $x \mapsto \phi(x)(1+x^2)^{-1}$ generate a dense subalgebra of $C_0(\mathbb{R})$, so Theorem 5.2 *c*) follows.

8.4 Commutator of functions with the phase of a longitudinal Dirac operator

In this subsection we prove Theorem 5.2 *d*).
For any $\lambda \in \mathfrak{h}^*$ one may define an element K_λ in $C(\hat{K}_q)$ by declaring that K_λ acts on the weight ν subspace of any irreducible K_q-representation by multiplication by $q^{\frac{1}{2}(\lambda,\mu)}$. If $\lambda = \alpha_i$ is a simple root, then K_{α_i} is the generator K_i of $\breve{\mathcal{U}}_q(\mathfrak{sl}_n)$.
The element $K_{2\rho} \in C(\hat{K}_q)$, where ρ is the half-sum of all positive roots, shows up in the Schur orthogonality relations. Specifically, the L^2-norms of the matrix coefficients of an irreducible unitary representation σ of K_q satisfy

$$\|\langle \xi^* | \cdot | \xi \rangle\| = \frac{1}{\dim_q(\sigma)^{\frac{1}{2}}} \|K_{2\rho} \cdot \xi^*\| \|\xi\| \tag{8.5}$$

for $\xi \in V^\sigma$ and $\xi^* \in V^{\sigma *}$, where $\dim_q$ denotes the quantum dimension. We remark that the Hilbert space structure on $V^{\sigma *}$ is induced from the canonical isometric isomorphism of $V^{\sigma *}$ with the conjugate Hilbert space of V^σ. Moreover $K_{2\rho} \in C(\hat{K}_q)$ acts by the transpose action on $V^{\sigma *}$.
Let us derive an estimate on slightly more general matrix coefficients.

Lemma 8.10. *Fix $\sigma \in \mathrm{Irr}(K_q)$. For any $\tau \in \mathrm{Irr}(K_q)$ and $\zeta \in V^\tau \otimes V^\sigma$, $\zeta^* \in V^{\tau *} \otimes V^{\sigma *}$, we have*

$$\|\langle \zeta^* | \cdot | \zeta \rangle\| \leq \frac{\dim_q(\sigma)^{\frac{1}{2}}}{\dim_q(\tau)^{\frac{1}{2}}} \|K_{2\rho} \cdot \zeta^*\| \, \|\zeta\|.$$

Here all norms are Hilbert space norms.

Proof. Take an orthogonal decomposition $\tau \otimes \sigma = \bigoplus_j \tau_j$ where the τ_j are irreducible K_q-subrepresentations of $\tau \otimes \sigma$. Correspondingly, we decompose $\zeta = \sum_j \zeta_j$ and $\zeta^* = \sum_j \zeta_j^*$ where $\zeta_j \in V^{\tau_j}$, $\zeta_j^* \in V^{\tau_j *}$. Since $\tau \leq \tau_j \otimes \sigma^c$ we have $\dim_q(\tau) \leq \dim_q(\tau_j) \dim_q(\sigma)$. We obtain

$$\|\langle \zeta^* | \cdot | \zeta \rangle\|^2 = \sum_j \frac{1}{\dim_q(\tau_j)} \|K_{2\rho} \cdot \zeta_j^*\|^2 \|\zeta_j\|^2 \leq \sum_j \frac{\dim_q(\sigma)}{\dim_q(\tau)} \|K_{2\rho} \cdot \zeta_j^*\|^2 \|\zeta_j\|^2.$$

The result follows. ☐

Let $\mu, \nu \in \mathbf{P}$, $i \in \Sigma$ and $f \in C(\mathcal{E}_\nu)$. We will use the bracket $[\mathrm{ph}(E_i), \mathsf{M}_l(f)]$ to denote the operator $\mathrm{ph}(E_i)\mathsf{M}_l(f) - \mathsf{M}_l(f)\,\mathrm{ph}(E_i) : L^2(\mathcal{E}_\mu) \to L^2(\mathcal{E}_{\mu+\nu+\alpha_i})$ of Theorem 5.2 *d*). From the other parts of Theorem 5.2, we know that this operator belongs to $\mathbb{A}(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu+\alpha_i}))$, so Theorem 5.2 *d*) is a consequence of the following lemma.

Lemma 8.11. *Fix $\mu, \nu \in \mathbf{P}$, $i \in \Sigma$ and $f \in C(\mathcal{E}_\nu)$. Then $[\mathrm{ph}(E_i), \mathsf{M}_l(f)] \in \mathbb{K}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu+\alpha_i}))$. Similarly, $[\mathrm{ph}(F_i), \mathsf{M}_l(f)] \in \mathbb{K}_i(L^2(\mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_{\mu+\nu}))$.*

Proof. Again, we write σ^λ for the irreducible K_q-representation with highest weight $\lambda \in \mathbf{P}^+$. Let $\mathrm{Wts}(\sigma^\lambda) \subset \mathbf{P}$ denote the set of weights occurring in σ^λ.
We will assume that $f = \langle \xi^* | \cdot | \xi \rangle$ is a matrix coefficient of $\sigma^\lambda \in \mathrm{Irr}(K_q)$, and that ξ has weight ν and K_q^i-type β for some $\beta \in \mathrm{Irr}(K_q^i)$. Such f span a dense subspace of $C(\mathcal{E}_\nu)$.
Let $\epsilon > 0$. By Corollary 8.5 we can find a finite set $S \subset\subset \mathrm{Irr}(K_q^i)$ such that for any unitary K_q-representation H we have the following estimates, where all operators are acting on $H \otimes V^{\sigma^\lambda}$:

$$\begin{aligned} &\|(\mathrm{ph}(\hat{\Delta} E_i) - \mathrm{ph}(E_i) \otimes \mathrm{Id}_{V^{\sigma^\lambda}}) p_{\mu+\nu} (\mathrm{Id} - p_S)\| \\ &\qquad\qquad < \epsilon / (|\mathrm{Wts}(\sigma^\lambda)| \dim_q(\sigma^\lambda)^{\frac{1}{2}} \|K_{2\rho} \cdot \xi^*\| \|\xi\|), \\ &\|(\mathrm{Id} - p_S)(\mathrm{ph}(\hat{\Delta} E_i) - \mathrm{ph}(E_i) \otimes \mathrm{Id}_{V^{\sigma^\lambda}}) p_{\mu+\nu}\| \\ &\qquad\qquad < \epsilon / (|\mathrm{Wts}(\sigma^\lambda)| \dim_q(\sigma^\lambda)^{\frac{1}{2}} \|K_{2\rho} \cdot \xi^*\| \|\xi\|). \end{aligned}$$

Let S' be the set of all $\gamma' \in \mathrm{Irr}(K_q^i)$ such that $\gamma' \otimes \beta$ contains some K_q^i-type γ belonging to S. This is a finite set, since any such γ' is a subrepresentation of $\gamma \otimes \beta^c$ with $\gamma \in S$.

Let $g \in L^2(\mathcal{E}_\mu)$ be a matrix coefficient $g = \langle \eta^* | \cdot | \eta \rangle$ of an irreducible K_q-representation σ^κ. Using Lemma 8.10 we obtain

$$\begin{aligned}
&\|[\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'})g\| \\
&\quad = \|\langle \eta^* \otimes \xi^* | \cdot |(\mathrm{ph}(\hat{\Delta} E_i) - \mathrm{ph}(E_i) \otimes \mathrm{Id})(((\mathrm{Id} - p_{S'})\eta) \otimes \xi)\rangle\| \\
&\quad = \|\langle \eta^* \otimes \xi^* | \cdot |(\mathrm{ph}(\hat{\Delta} E_i) - \mathrm{ph}(E_i) \otimes \mathrm{Id}) p_{\mu+\nu}(\mathrm{Id} - p_S)((\mathrm{Id} - p_{S'})\eta) \otimes \xi\rangle\| \\
&\quad \leq \frac{\dim_q(\sigma^\lambda)^{\frac{1}{2}}}{\dim_q(\sigma^\kappa)^{\frac{1}{2}}} \|K_{2\rho} \cdot \eta^*\| \|K_{2\rho} \cdot \xi^*\| \\
&\qquad\qquad\qquad \times \|(\mathrm{ph}(\hat{\Delta} E_i) - \mathrm{ph}(E_i) \otimes \mathrm{Id}) p_{\mu+\nu}(\mathrm{Id} - p_S)\| \|\eta\| \|\xi\| \\
&\quad < \frac{\epsilon}{|\mathrm{Wts}(\sigma^\lambda)| \, \dim_q(\sigma^\kappa)^{\frac{1}{2}}} \|K_{2\rho} \cdot \eta^*\| \|\eta\| \\
&\quad = \frac{\epsilon}{|\mathrm{Wts}(\sigma^\lambda)|} \|g\|. \qquad (8.6)
\end{aligned}$$

A similar calculation shows that

$$\|(\mathrm{Id} - p_S)[\mathrm{ph}(E_i), \mathsf{M}_l(f)]g\| < \frac{\epsilon}{|\mathrm{Wts}(\sigma^\lambda)|} \|g\|. \qquad (8.7)$$

If $g \in L^2(\mathcal{E}_\mu)$ is instead a sum of irreducible matrix coefficients, then the inequalities (8.6) and (8.7) do not necessarily hold. To resolve this, we will take advantage of the fact that the operator $[\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'})$ is band-diagonal with respect to K_q-types in the following sense. The operators $\mathrm{ph}(E_i)$ and $(\mathrm{Id} - p_S)$ commute with the K_q-isotypical projections, while $\mathsf{M}_l(f)$ satisfies $p_{\sigma^{\kappa'}} \mathsf{M}_l(f) p_{\sigma^\kappa} = 0$ unless $\sigma^{\kappa'}$ occurs as an irreducible subrepresentation of $\sigma^\kappa \otimes \sigma^\lambda$. We note that if $\sigma^{\kappa'} \leq \sigma^\kappa \otimes \sigma^\lambda$ then $\kappa' = \kappa + \omega$ for some weight ω of σ^λ. Therefore we have a decomposition

$$\begin{aligned}
&[\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'}) \\
&\qquad = \sum_{\omega \in \mathrm{Wts}(\sigma^\lambda)} \left(\sum_{\kappa \in \mathbf{P}^+} p_{\sigma^{\kappa+\omega}} [\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'}) p_{\sigma^\kappa} \right), \qquad (8.8)
\end{aligned}$$

where we take the convention that $p_{\sigma^{\kappa+\omega}} = 0$ if $\kappa + \omega$ is not dominant. By the calculation (8.6) for irreducible matrix coefficients, $p_{\sigma^{\kappa+\omega}}[\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'}) p_{\sigma^\kappa}$ is norm bounded by $\epsilon/|\mathrm{Wts}(\sigma^\lambda)|$ for each κ, ω. Since the projections p_{σ^κ} are mutually orthogonal, the sum in parentheses in (8.8) is bounded by $\epsilon/|\mathrm{Wts}(\sigma^\lambda)|$ for each fixed ω. We conclude that $\|[\mathrm{ph}(E_i), \mathsf{M}_l(f)](\mathrm{Id} - p_{S'})\| < \epsilon$. Similarly, one obtains $\|(\mathrm{Id} - p_S)[\mathrm{ph}(E_i), \mathsf{M}_l(f)]\| < \epsilon$. This completes the proof that $[\mathrm{ph}(E_i), \mathsf{M}_l(f)] \in \mathbb{K}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\nu+\alpha_i}))$ for all $f \in C(\mathcal{E}_\mu)$.
Finally, one can obtain $[\mathrm{ph}(F_i), \mathsf{M}_l(f)] \in \mathbb{K}_i(L^2(\mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_{\mu+\nu}))$ by taking adjoints. □

9 The action of $\mathrm{SL}_q(n, \mathbb{C})$

In this section we recall the definition of the quantum group $\mathrm{SL}_q(n, \mathbb{C})$ and its principal series representations, and prove some estimates that will be used later.

9.1 The complex quantum group $\mathrm{SL}_q(n, \mathbb{C})$

The quantized complex semisimple Lie group $G_q = \mathrm{SL}_q(n, \mathbb{C})$ is defined as the quantum double of $K_q = \mathrm{SU}_q(n)$. More precisely, the C^*-algebra of functions on G_q is given by

$$C_0(G_q) = C(K_q) \otimes C_0(\hat{K}_q)$$

with the comultiplication

$$\Delta_{G_q} = (\mathrm{Id} \otimes \sigma \otimes \mathrm{Id})(\mathrm{Id} \otimes \mathrm{ad}(W) \otimes \mathrm{Id})(\Delta \otimes \hat{\Delta}),$$

where $\mathrm{ad}(W)$ is conjugation with the multiplicative unitary $W \in M(C(K_q) \otimes C^*(K_q))$ of K_q and σ denotes the flip map. In the special case $n = 2$ this quantum group has been studied in detail by Podleś and Woronowicz [PW90]. The unitary representations of G_q are in one-to-one correspondence with unitary Yetter-Drinfeld modules for K_q, compare [NV10]. Passing to the subspace of K_q-finite vectors, one can study such representations algebraically, namely in terms of Yetter-Drinfeld modules over the Hopf algebra $\mathcal{O}(K_q)$. We recall that a Yetter-Drinfeld module over $\mathcal{O}(K_q)$ is a vector space V equipped with both a left action and a left coaction of $\mathcal{O}(K_q)$ in the purely algebraic sense, satisfying the compatibility condition

$$f_{(1)}\xi_{(-1)}S(f_{(3)}) \otimes f_{(2)} \cdot \xi_{(0)} = (f \cdot \xi)_{(-1)} \otimes (f \cdot \xi)_{(0)}$$

for $f \in \mathcal{O}(K_q)$ and $\xi \in V$. Here we use the Sweedler notation $\xi \mapsto \xi_{(-1)} \otimes \xi_{(0)}$ for the coaction $V \to \mathcal{O}(K_q) \otimes V$, and we write $f \cdot v$ for the action of $f \in \mathcal{O}(K_q)$ on $v \in V$.

9.2 Principal series representations of $\mathrm{SL}_q(n, \mathbb{C})$

As mentioned before, For $\lambda \in \mathfrak{h}^*$ one may define an element K_λ of $C(\hat{K}_q)$ by declaring that K_λ acts on the weight ν subspace of any irreducible K_q-representation by multiplication by $q^{\frac{1}{2}(\lambda,\nu)}$. We note that $K_\lambda = K_{\lambda'}$ if $\lambda \equiv \lambda'$ modulo $2i\hbar^{-1}\mathbf{Q}$, where $\hbar = \frac{\log(q)}{2\pi}$ and $\mathbf{Q}$ is the root lattice. We write $\mathfrak{h}_q^* = \mathfrak{h}^*/2i\hbar^{-1}\mathbf{Q}$ and $i\mathfrak{t}_q^* = i\mathfrak{t}^*/2i\hbar^{-1}\mathbf{Q}$. Our conventions here are adjusted to the quantized enveloping algebra $\check{\mathcal{U}}_q(\mathfrak{sl}_n)$; recall that in the notation of [KS97, §6.1.2] the element K_i in $\mathcal{U}_q(\mathfrak{sl}_n)$ corresponds to K_i^2 in $\check{\mathcal{U}}_q(\mathfrak{sl}_n)$.

The principal series Yetter-Drinfeld modules are parametrized by pairs $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. We will denote the principal series Yetter-Drinfeld module with parameter (μ, λ) by $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$. As a K_q-representation it is just $\mathcal{O}(\mathcal{E}_\mu)$ with the left regular coaction. The action of $\mathcal{O}(K_q)$ is given by

$$\pi_{\mu,\lambda}(a)g = a_{(1)}\, g\,(K^2_{\lambda+\rho} \rightharpoonup S(a_{(2)})) \tag{9.1}$$

for $a \in \mathcal{O}(K_q)$ and $g \in \mathcal{O}(\mathcal{E}_\mu)$.

For $\lambda \in i\mathfrak{t}^*_q$ the representations $\pi_{\mu,\lambda}$ are $*$-representations with respect to the standard inner product on $\mathcal{O}(\mathcal{E}_\mu)$, and the resulting unitary representations of G_q on the completion $L^2(\mathcal{E}_\mu)$ of $\mathcal{O}(\mathcal{E}_\mu)$ are called unitary principal series representations. We will write $L^2(\mathcal{E}_{\mu,\lambda}) = L^2(\mathcal{E}_\mu)$ if we want to emphasize the corresponding Yetter-Drinfeld structure.

We will only need the unitary principal series representations with parameters $(\mu, 0)$, and we shall make use of the following properties.

Theorem 9.1. *a) The representations $L^2(\mathcal{E}_{\mu,0})$ are irreducible.*

b) The representations $L^2(\mathcal{E}_{\mu,0})$ and $L^2(\mathcal{E}_{\nu,0})$ are equivalent if and only if μ and ν belong to the same orbit of the Weyl group action on $\mathbf{P}$. In particular, if $\nu = w_i\mu$ where w_i is the reflection associated to a simple root α_i, then the representations are intertwined by the operator

$$\begin{aligned} \mathrm{ph}(E_i)^n &: L^2(\mathcal{E}_{\mu,0}) \xrightarrow{\cong} L^2(\mathcal{E}_{w_i\mu,0}), \qquad \text{if } w_i\mu - \mu = n\alpha_i \text{ with } n > 0, \\ \mathrm{ph}(F_i)^n &: L^2(\mathcal{E}_{\mu,0}) \xrightarrow{\cong} L^2(\mathcal{E}_{w_i\mu,0}), \qquad \text{if } w_i\mu - \mu = n\alpha_i \text{ with } n < 0. \end{aligned}$$

The above facts are at least partially "known to experts," although they do not appear in this form in the literature. They are quantum analogues of results about the principal series representations of classical complex semisimple Lie groups as described, for instance, in [Duf75, Ch. III & IV]; see also [Yun11a, Section 4.2] for a formulation along the present lines. For $\mathrm{SL}_q(2,\mathbb{C})$ the above results are essentially contained in Theorems 5.2 and 6.3 of [PW00]. The results for $\mathrm{SL}_q(n,\mathbb{C})$ can be deduced from this case. We refer to [VY] for a detailed exposition.

9.3 Almost $\mathrm{SL}_q(n,\mathbb{C})$-equivariance of the phases of E_i and F_i

A straightforward computation shows that the multiplication operators on $L^2(\mathcal{E}_\mu)$ satisfy the following covariance property with respect to principal series representations.

Lemma 9.2. *Let $\mu, \nu \in \mathbf{P}$ and $f \in \mathcal{O}(\mathcal{E}_\nu)$, so that $\mathsf{M}_l(f)$ defines an operator from $L^2(\mathcal{E}_\mu)$ to $L^2(\mathcal{E}_{\mu+\nu})$. Then for any $a \in \mathcal{O}(K_q)$,*

$$\pi_{\mu+\nu,0}(a)\, \mathsf{M}_l(f) = \mathsf{M}_l(a_{(1)} f S(a_{(2)}))\, \pi_{\mu,0}(a_{(3)}).$$

The next result will be used in the proof of the equivariance properties of our K-homology cycle.

Theorem 9.3. *Let $K_q = \mathrm{SU}_q(n)$ for $n \geq 2$. Moreover let $\mu \in \mathbf{P}$ and $i \in \Sigma$.*

a) For any $a \in \mathcal{O}(K_q)$, we have $\pi_{\mu,0}(a) \in \mathbb{A}(L^2(\mathcal{E}_\mu))$.

b) The operators $\mathrm{ph}(E_i) : L^2(\mathcal{E}_{\mu,0}) \to L^2(\mathcal{E}_{\mu+\alpha_i,0})$ and $\mathrm{ph}(F_i) : L^2(\mathcal{E}_{\mu+\alpha_i,0}) \to L^2(\mathcal{E}_{\mu,0})$ are K_q-equivariant. Moreover, they are $\mathrm{SL}_q(n,\mathbb{C})$-equivariant modulo $\mathbb{J}_i$, in the sense that for any $a \in \mathcal{O}(K_q)$,

$$\pi_{\mu+\alpha_i,0}(a)\,\mathrm{ph}(E_i) - \mathrm{ph}(E_i)\pi_{\mu,0}(a) \in \mathbb{J}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\alpha_i})), \tag{9.2}$$
$$\pi_{\mu,0}(a)\,\mathrm{ph}(F_i) - \mathrm{ph}(F_i)\pi_{\mu+\alpha_i,0}(a) \in \mathbb{J}_i(L^2(\mathcal{E}_{\mu+\alpha_i}), L^2(\mathcal{E}_\mu)). \tag{9.3}$$

Proof. The Yetter-Drinfeld action of a on $L^2(\mathcal{E}_{\mu,0})$ can be written as $\pi_{\mu,0}(a) = \mathsf{M}_l(a_{(1)})\,\mathsf{M}_r(K_\rho \rightharpoonup S(a_{(2)}))$, so $a)$ follows from Proposition 8.1
The K_q-equivariance of $\mathrm{ph}(E_i)$ and $\mathrm{ph}(F_i)$ is immediate from the fact that they are defined via the right regular representation.
Let $i \in \Sigma$. We have $w_i\rho = \rho - \alpha_i$, so according to Theorem 9.1 the operator $\mathrm{ph}(E_i) : L^2(\mathcal{E}_{\rho-\alpha_i,0}) \to L^2(\mathcal{E}_{\rho,0})$ is an intertwiner. Thus, the differences in (9.2) and (9.3) are zero when $\mu = \rho - \alpha_i$. For general μ, we use Lemma 2.1 to obtain $f_1, \dots, f_k \in \mathcal{O}(\mathcal{E}_{\mu+\alpha_i-\rho})$ and $g_1, \dots, g_k \in \mathcal{O}(\mathcal{E}_{\rho-\alpha_i-\mu})$ such that $\sum_j g_j f_j = 1$. We can then use Theorem 5.2 and Lemma 9.2 to compute

$$\begin{aligned}
&\pi_{\mu+\alpha_i,0}(a)\,\mathrm{ph}(E_i) \\
&\qquad = \sum_j \pi_{\mu+\alpha_i,0}(a)\,\mathrm{ph}(E_i)\mathsf{M}_l(f_j)\mathsf{M}_l(g_j) \\
&\qquad \equiv \sum_j \pi_{\mu+\alpha_i,0}(a)\mathsf{M}_l(f_j)\,\mathrm{ph}(E_i)\mathsf{M}_l(g_j) \qquad \pmod{\mathbb{J}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\alpha_i}))} \\
&\qquad = \sum_j \mathsf{M}_l(a_{(1)} f_j S(a_{(2)}))\pi_{\rho,0}(a_{(3)})\,\mathrm{ph}(E_i)\mathsf{M}_l(g_j),
\end{aligned}$$

noting that all operators involved belong to $\mathbb{A}$. A similar computation yields

$$\sum_j \mathrm{ph}(E_i)\pi_{\mu,0}(a) \equiv \sum_j \mathsf{M}_l(a_{(1)} f_j S(a_{(2)}))\,\mathrm{ph}(E_i)\pi_{\rho-\alpha_i,0}(a_{(3)})\mathsf{M}_l(g_j) \qquad \pmod{\mathbb{J}_i(L^2(\mathcal{E}_\mu), L^2(\mathcal{E}_{\mu+\alpha_i}))}.$$

Thus Equation (9.2) is reduced to the case $\mu = \rho - \alpha_i$ which we have just proved. Equation (9.3) follows by taking adjoints. □

10 BGG elements in K-homology

In [Yun11a] it was shown how an equivariant Fredholm module can be constructed from the geometric BGG complex for the full flag manifold of SU(3). Given the results of the previous sections, that construction can now be applied also to the quantized flag manifold of $\mathrm{SU}_q(3)$. The construction carries over almost word for word, so we shall merely give an outline of the steps involved here.

10.1 The normalized BGG complex

The reader can consult [BGG75] or [BE89] for the general combinatorial structure underlying the BGG complex of a complex semisimple Lie group. Since we only need a bounded version of the BGG complex for $\mathrm{SL}_q(3,\mathbb{C})$, we will proceed in an *ad hoc* manner.

LEMMA 10.1. *The following is a commuting diagram of intertwining operators between* $\mathrm{SL}_q(3,\mathbb{C})$ *principal series representations:*

$$\begin{tikzcd}
 & L^2(\mathcal{E}_{-\alpha_2,0}) \arrow[rr, "\mathrm{ph}(E_2)^2"] & & L^2(\mathcal{E}_{\alpha_1,0}) \arrow[dr, "\mathrm{ph}(E_2)"] & \\
L^2(\mathcal{E}_{-\rho,0}) \arrow[ur, "\mathrm{ph}(E_1)"] \arrow[dr, "\mathrm{ph}(E_2)"'] & & & & L^2(\mathcal{E}_{\rho,0}) \\
 & L^2(\mathcal{E}_{-\alpha_1,0}) \arrow[uurr, "\mathrm{ph}(E_1)^2"] & & L^2(\mathcal{E}_{\alpha_2,0}) \arrow[ur, "\mathrm{ph}(E_1)"'] &
\end{tikzcd} \tag{10.1}$$

Proof. That these operators are intertwiners results from Theorem 9.1. By Schur's Lemma, the diagram commutes up to a scalar. By checking on the minimal K_q-type, one can verify that the diagram commutes on the nose. □

To define the normalized BGG complex, we displace all the weights in the above construction by $\rho = \alpha_1 + \alpha_2$.

LEMMA 10.2. *The following diagram commutes modulo* $\mathbb{J}_1 + \mathbb{J}_2$*:*

$$\begin{tikzcd}
 & L^2(\mathcal{E}_{\alpha_1,0}) \arrow[ddrr, "\mathrm{ph}(E_2)^2"] & & L^2(\mathcal{E}_{2\alpha_1+\alpha_2,0}) \arrow[dr, "\mathrm{ph}(E_2)"] & \\
L^2(\mathcal{E}_{0,0}) \arrow[ur, "\mathrm{ph}(E_1)"] \arrow[dr, "\mathrm{ph}(E_2)"'] & & & & L^2(\mathcal{E}_{2\rho,0}) \\
 & L^2(\mathcal{E}_{\alpha_2,0}) \arrow[uurr, "\mathrm{ph}(E_1)^2"] & & L^2(\mathcal{E}_{\alpha_1+2\alpha_2,0}) \arrow[ur, "\mathrm{ph}(E_1)"'] &
\end{tikzcd}$$

Proof. According to Lemma 2.1 we find $f_1,\ldots,f_k \in C(\mathcal{E}_\rho)$, $g_1,\ldots,g_k \in C(\mathcal{E}_{-\rho})$ such that $\sum_j f_j g_j = 1$. Consider the composition $\mathrm{ph}(E_1)\,\mathrm{ph}(E_2)^2\,\mathrm{ph}(E_1) : L^2(\mathcal{E}_{0,0}) \to L^2(\mathcal{E}_{2\rho,0})$. By Theorem 5.2,

$$\begin{aligned}
\mathrm{ph}(E_1)\,\mathrm{ph}(E_2)^2\,\mathrm{ph}(E_1) &= \sum_i \mathsf{M}_l(f_i)\mathsf{M}_l(g_i)\,\mathrm{ph}(E_1)\,\mathrm{ph}(E_2)^2\,\mathrm{ph}(E_1) \\
&\equiv \sum_i \mathsf{M}_l(f_i)\,\mathrm{ph}(E_1)\,\mathrm{ph}(E_2)^2\,\mathrm{ph}(E_1)\mathsf{M}_l(g_i) \\
&\qquad \mod \mathbb{J}_1(L^2(\mathcal{E}_0), L^2(\mathcal{E}_{2\rho})) + \mathbb{J}_2(L^2(\mathcal{E}_0), L^2(\mathcal{E}_{2\rho})),
\end{aligned}$$

where the operators in the last expression are the intertwiners of Lemma 10.1. By a similar calculation, we obtain

$$\begin{aligned}
\mathrm{ph}(E_2)\,\mathrm{ph}(E_1)^2\,\mathrm{ph}(E_2) &\equiv \sum_i \mathsf{M}_l(f_i)\,\mathrm{ph}(E_2)\,\mathrm{ph}(E_1)^2\,\mathrm{ph}(E_2)\mathsf{M}_l(g_i) \\
&\qquad \mod \mathbb{J}_1(L^2(\mathcal{E}_0), L^2(\mathcal{E}_{2\rho})) + \mathbb{J}_2(L^2(\mathcal{E}_0), L^2(\mathcal{E}_{2\rho})),
\end{aligned}$$

and the result then follows from Lemma 10.1. $\square$

LEMMA 10.3. *Let $\mu \in \mathbf{P}$, $i \in \Sigma$ and $n \in \mathbb{N}$. Then* $\mathrm{ph}(F_i)^n\,\mathrm{ph}(E_i)^n - \mathrm{Id} \in \mathbb{J}_i(L^2(\mathcal{E}_\mu))$ *and* $\mathrm{ph}(E_i)^n\,\mathrm{ph}(F_i)^n - \mathrm{Id} \in \mathbb{J}_i(L^2(\mathcal{E}_\mu))$.

Proof. Let $\mu_{\alpha_i} \in \frac{1}{2}\mathbb{N}$ be the restriction of μ to a weight of $S_q^i \cong \mathrm{SU}_q(2)$. The operator $\mathrm{ph}(E_i)^n : L^2(\mathcal{E}_\mu) \to L^2(\mathcal{E}_{\mu+n\alpha_i})$ is a partial isometry, and its kernel is the span of those S_q^i-isotypical subspaces whose highest weight $l \in \frac{1}{2}\mathbb{N}$ satisfies $l < \mu_{\alpha_i} + n$. Therefore $\mathrm{ph}(F_i)^n\,\mathrm{ph}(E_i)^n - \mathrm{Id}$ is a projection onto a finite number of S_q^i-types, and hence K_q^i-types, in $L^2(\mathcal{E}_\mu)$. A similar statement can be made for $\mathrm{ph}(E_i)^n\,\mathrm{ph}(F_i)^n - \mathrm{Id}$. The result then follows from Theorem 5.1 *b*). $\square$

We now augment the diagram of Lemma 10.2 by adding two more operators:

$$\begin{tikzcd}
 & L^2(\mathcal{E}_{\alpha_1,0}) \arrow[r, "-A_1"] \arrow[ddr, "\mathrm{ph}(E_2)^2"] & L^2(\mathcal{E}_{2\alpha_1+\alpha_2,0}) \arrow[dr, "\mathrm{ph}(E_2)"] & \\
L^2(\mathcal{E}_{0,0}) \arrow[ur, "\mathrm{ph}(E_1)"] \arrow[dr, "\mathrm{ph}(E_2)"'] & & & L^2(\mathcal{E}_{2\rho,0}) \\
 & L^2(\mathcal{E}_{\alpha_2,0}) \arrow[r, "-A_2"'] \arrow[uur, "\mathrm{ph}(E_1)^2"'] & L^2(\mathcal{E}_{\alpha_1+2\alpha_2,0}) \arrow[ur, "\mathrm{ph}(E_1)"'] &
\end{tikzcd} \tag{10.2}$$

where

$$A_1 = \mathrm{ph}(E_1)^2\,\mathrm{ph}(E_2)\,\mathrm{ph}(F_1)$$
$$A_2 = \mathrm{ph}(E_2)^2\,\mathrm{ph}(E_1)\,\mathrm{ph}(F_2).$$

By Lemma 10.3, $A_1^*A_1 \equiv \mathrm{Id}$ modulo $\mathbb{J}_1(L^2(\mathcal{E}_{\alpha_1})) + \mathbb{J}_2(L^2(\mathcal{E}_{\alpha_1}))$ and $A_1A_1^* \equiv \mathrm{Id}$ modulo $\mathbb{J}_1(L^2(\mathcal{E}_{2\alpha_1+\alpha_2})) + \mathbb{J}_2(L^2(\mathcal{E}_{2\alpha_1+\alpha_2}))$. Similar statements hold for A_2. The inclusion of the minus signs before A_1 and A_2 in the diagram (10.2) ensures that all squares in the diagram *anti*-commute modulo $\mathbb{J}_1 + \mathbb{J}_2$. Thus (10.2) is a complex modulo $\mathbb{J}_1 + \mathbb{J}_2$.
The combinatorial structure underlying the diagram (10.2) is the Bruhat graph of the group $G = \mathrm{SL}(3, \mathbb{C})$. Rather than detail this in generality, let us simply introduce some convenient notation.

DEFINITION 10.4. Let Γ be the set of arrows in the diagram (10.2) and $\Gamma^{(0)}$ the set of six vertices. Denote by T_γ the operator corresponding to $\gamma \in \Gamma$ in (10.2). Also, to each arrow γ we associate a set of simple roots, denoted $\mathrm{supp}(\gamma)$ and called the *support of* γ, according to the Weyl reflection underlying it as follows:

$$\begin{tikzcd}
 & \cdot \arrow[r, "\{\alpha_1,\alpha_2\}"] \arrow[ddr, "\{\alpha_2\}"] & \cdot \arrow[dr, "\{\alpha_2\}"] & \\
\cdot \arrow[ur, "\{\alpha_1\}"] \arrow[dr, "\{\alpha_2\}"'] & & & \cdot \\
 & \cdot \arrow[r, "\{\alpha_1,\alpha_2\}"'] \arrow[uur, "\{\alpha_1\}"'] & \cdot \arrow[ur, "\{\alpha_1\}"'] &
\end{tikzcd}$$

10.2 Construction of the Fredholm module

Let $\mathcal{H}_{\mathrm{BGG}}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space which is the direct sum of the six section spaces in the BGG diagram (10.2) graded by even and odd Bruhat length, namely $L^2(\mathcal{E}_0)$, $L^2(\mathcal{E}_{2\alpha_1+\alpha_2})$ and $L^2(\mathcal{E}_{\alpha_1+2\alpha_2})$ have degree 0 and the other three summands have degree 1. The sum $\mathbf{T} = \sum_\gamma (T_\gamma + T_\gamma^*)$ is an odd $\mathrm{SU}_q(3)$-equivariant operator on $\mathcal{H}_{\mathrm{BGG}}$. It verifies all the axioms of an equivariant Kasparov K-homology cycle, but modulo $\mathbb{J}_1(\mathcal{H}_{\mathrm{BGG}}) + \mathbb{J}_2(\mathcal{H}_{\mathrm{BGG}})$ instead of modulo $\mathbb{K}(\mathcal{H}_{\mathrm{BGG}})$. To refine this into a genuine Kasparov cycle we use the operator partition of unity constructed in [Yun11a], which is described in the following lemma.

Lemma 10.5. *Let $K_q = \mathrm{SU}_q(3)$. There exist mutually commuting operators $N_\gamma \in \mathbb{L}(\mathcal{H}_{\mathrm{BGG}})$, indexed by the arrows $\gamma \in \Gamma$ above, with the following properties:*

a) For each γ, $N_\gamma \mathbb{J}_i(\mathcal{H}_{\mathrm{BGG}}) \subseteq \mathbb{K}(\mathcal{H}_{\mathrm{BGG}})$ for all $\alpha_i \in \mathrm{supp}(\gamma)$.

b) For any vertex $v \in \Gamma^{(0)}$, $\sum_{\gamma \ni v} N_\gamma^2 = \mathrm{Id}_{\mathcal{H}_{\mathrm{BGG}}}$, where the sum is over all arrows entering or leaving v.

c) Whenever vertices $v, v' \in \Gamma^{(0)}$ are at distance two in the graph we have $N_{\gamma_1} N_{\gamma_2} = N_{\gamma_1'} N_{\gamma_2'}$ where (γ_1, γ_2) and (γ_1', γ_2') are the unique two (undirected) paths of length two joining v, v'.

d) Each N_γ is K_q-equivariant.

e) Each N_γ commutes modulo compact operators with the left multiplication action of $C(\mathcal{X}_q)$, the Yetter-Drinfeld action of $\mathcal{O}(K_q)$ and all of the normalized BGG operators $T_{\gamma'}$.

Proof. Firstly, one applies the Kasparov technical theorem of Baaj-Skandalis [BS89, Theorem 4.3] with respect to the Drinfeld double $G_q = \mathbf{D}(K_q)$. This yields an almost G_q-invariant operator $M_0 \in L^2(\mathcal{H}_{\mathrm{BGG}})$, in the sense that $(C_0(G_q) \otimes 1)(\delta(M_0) - 1 \otimes M_0) \in C_0(G) \otimes \mathbb{K}(\mathcal{H}_{\mathrm{BGG}})$, which satisfies $M_0 \mathbb{J}_1(\mathcal{H}_{\mathrm{BGG}}), (1 - M_0)\mathbb{J}_2(\mathcal{H}_{\mathrm{BGG}}) \subseteq \mathbb{K}(\mathcal{H}_{\mathrm{BGG}})$ and such that M_0 commutes modulo compacts with the left multiplication action of $C(\mathcal{X}_q)$ and the normalized BGG operators. Here δ denotes the adjoint action of G_q induced from $\mathcal{H}_{\mathrm{BGG}}$. From almost invariance it follows that M_0 commutes modulo compacts with the Yetter-Drinfeld action of $\mathcal{O}(K_q)$. Moreover, if M denotes the operator obtained by averaging the K_q-action on M_0 using the Haar functional of $C(K_q)$, then $M - M_0 \in \mathbb{K}(\mathcal{H}_{\mathrm{BGG}})$. Thus M satisfies the same properties as M_0 but is additionally K_q-invariant. Now, putting $N_1 = M^{\frac{1}{2}}$, $N_2 = (1 - M)^{\frac{1}{2}}$, the remainder of the construction is carried out as in [Yun11a, Lemma 4.14]. ☐

Theorem 10.6. *The operator $\mathbf{F} = \sum_\gamma N_\gamma (T_\gamma + T_\gamma^*)$ is an odd Fredholm operator on $\mathcal{H}_{\mathrm{BGG}}$ which defines an equivariant K-homology class $[\mathbf{F}] \in KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q), \mathbb{C})$. The $\mathrm{SU}_q(3)$-equivariant index of this class in $KK^{\mathrm{SU}_q(3)}(\mathbb{C}, \mathbb{C}) = R(\mathrm{SU}_q(3))$ is the class of the trivial representation.*

Proof. The fact that $\mathbf{F}$ defines a K-homology class in $KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q),\mathbb{C})$ can be proven as for Theorem 4.19 of [Yun11a]. Note that in order to prove $\mathrm{SL}_q(3,\mathbb{C})$-equivariance it suffices to check that the action of $\mathcal{O}(\mathrm{SU}_q(3))$ corresponding to the Yetter-Drinfeld structure commutes with $\mathbf{F}$ up to compact operators. This in turn follows using Theorem 9.3.
Finally, the $\mathrm{SU}_q(3)$-types occur in the spaces $L^2(\mathcal{E}_\mu)$ with the same multiplicities as their classical counterparts, so the index computation follows from the fact that the classical BGG complex is a resolution of the trivial representation. □

Remark 10.7. The same general construction can be used to make a Fredholm module in $KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q),\mathbb{C})$ with any desired $\mathrm{SU}_q(3)$-equivariant index. A BGG complex can be formed starting with an arbitrary weight μ (in the notation we have used here it should be an anti-dominant weight), where the weights appearing in the equivalent of the diagram (10.2) are those in the ρ-shifted Weyl orbit of μ. The procedure above applies, and the equivariant index of the resulting KK-cycle is the class of the irreducible representation with lowest weight μ.

11 Applications to Poincaré duality and the Baum-Connes conjecture

In this section we explain how our previous constructions imply Poincaré duality in equivariant KK-theory for the flag manifold $\mathcal{X}_q = \mathrm{SU}_q(3)/T$, and a certain analogue of the Baum-Connes conjecture for the dual of $\mathrm{SU}_q(3)$. Some of the arguments will only be sketched, and for more information and background we refer to [MN06], [MN10], [Mey08], [NV10], [Voi11].
Equivariant Poincaré duality in KK-theory with respect to quantum group actions was introduced in [NV10], where it was also shown that the standard Podleś sphere is equivariantly Poincaré dual to itself with respect to the natural action of $\mathrm{SU}_q(2)$. An important ingredient in the study of Poincaré duality with respect to quantum group actions is the use of braided tensor products, and we refer to [NV10] for definitions and more details.
Our aim here is to exhibit another example of equivariant Poincaré duality in the sense of [NV10], namely for the quantum flag manifold $\mathcal{X}_q = \mathrm{SU}_q(3)/T$. The key ingredient for this is the class $[\mathbf{F}] \in KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q),\mathbb{C})$ obtained in Theorem 10.6. It yields a class in $KK^{\mathrm{SL}_q(3,\mathbb{C})}(C(\mathcal{X}_q)\boxtimes C(\mathcal{X}_q),\mathbb{C})$ by precomposing the representation of $C(\mathcal{X}_q)$ with the $*$-homomorphism $C(\mathcal{X}_q)\boxtimes C(\mathcal{X}_q)\to C(\mathcal{X}_q)$ induced by multiplication. Here $\boxtimes$ denotes the braided tensor product over $\mathrm{SU}_q(3)$, and we write $\mathbf{D}(\mathrm{SU}_q(3)) = \mathrm{SL}_q(3,\mathbb{C})$ for the quantum double of $\mathrm{SU}_q(3)$.

Theorem 11.1. *The quantum flag manifold $\mathcal{X}_q$ is $\mathrm{SU}_q(3)$-equivariantly Poincaré dual to itself. That is, there is a natural isomorphism*

$$KK_*^{\mathbf{D}(\mathrm{SU}_q(3))}(C(\mathcal{X}_q)\boxtimes A, B) \cong KK_*^{\mathbf{D}(\mathrm{SU}_q(3))}(A, C(\mathcal{X}_q)\boxtimes B)$$

for all $\mathbf{D}(\mathrm{SU}_q(3))$-$C^*$-algebras A and B.

Proof. With the class $[\mathbf{F}] \in KK^{\mathrm{SL}_q(3)}(C(\mathcal{X}_q)\boxtimes C(\mathcal{X}_q), \mathbb{C})$ at hand, the argument is completely analogous to the proof of Theorem 6.5 in [NV10], reducing it to Poincaré duality for the classical flag manifold $\mathcal{X}_1$. We shall therefore not go into the details.

Let us remark that we do not need an explicit description of the element $\eta_q \in KK_*^{\mathbf{D}(\mathrm{SU}_q(3))}(\mathbb{C}, C(\mathcal{X}_q) \boxtimes C(\mathcal{X}_q))$ corresponding to the unit of the adjunction, since this element is uniquely determined from the unit $\eta_1 \in KK_*^{\mathrm{SU}(3)}(\mathbb{C}, C(\mathcal{X}_1) \otimes C(\mathcal{X}_1))$ of the classical SU(3)-equivariant Poincaré duality thanks to the continuous field structure of q-deformations, see [NT12], [Yam13]. We recall that η_1 is the image of the class of the trivial line bundle $1_{\mathcal{X}_1} \in KK_*^{\mathrm{SU}(3)}(\mathbb{C}, C(\mathcal{X}_1))$ under the wrong way map induced by the diagonal embedding $\mathcal{X}_1 \hookrightarrow \mathcal{X}_1 \times \mathcal{X}_1$. We refer the reader to [EM10b, EM10a] for a full discussion of equivariant KK-duality for manifolds, and [RS86] for more information on the case of flag varieties. □

Let us now come to the Baum-Connes conjecture. We continue to write $K_q = \mathrm{SU}_q(3)$ and denote by $\hat{K}_q$ the discrete quantum group dual to K_q. The starting point of the approach in [MN06] is to view equivariant Kasparov theory as a triangulated category. More precisely, if Γ is a discrete quantum group we consider the category KK^Γ which has as objects all separable Γ-C^*-algebras, and $KK^\Gamma(A, B)$ as the set of morphisms between two objects A and B. Composition of morphisms is given by the Kasparov product. For a description of the structure of KK^Γ as a triangulated category we refer to [NV10], [Voi11]. Suffice it to say that this extra structure allows one to do homological algebra in the context of Kasparov theory.

In fact, there is one further ingredient needed in the definition of the Baum-Connes assembly map. Namely, one has to identify the category $\mathcal{CI}_\Gamma$ of compactly induced actions within KK^Γ. Classically, the objects of $\mathcal{CI}_\Gamma$ are the C^*-algebras induced from finite subgroups of the discrete group Γ. If Γ is torsion-free the situation is particularly simple, in the sense that only the trivial subgroup has to be taken into account in this case.

It turns out that the dual of K_q behaves like a torsion-free group. More precisely, the quantum group $\hat{K}_q$ is torsion-free in the sense that any ergodic action of K_q on a finite dimensional C^*-algebra is K_q-equivariantly Morita equivalent to the trivial action on $\mathbb{C}$, see [Mey08], [Gof12].

For a torsion-free quantum group Γ we define the full subcategory $\mathcal{CI}_\Gamma$ of KK^Γ by

$$\mathcal{CI}_\Gamma = \{C_0(\Gamma) \otimes A | A \in KK\},$$

where the coaction on $C_0(\Gamma) \otimes A$ is given by comultiplication on the first tensor factor. Similarly, we let $\mathcal{CC}_\Gamma \subset KK^\Gamma$ be the full subcategory of all objects which become isomorphic to 0 in KK under the obvious forgetful functor. The subcategory $\mathcal{CC}_\Gamma$ is localising, and we denote by $\langle \mathcal{CI}_\Gamma \rangle$ the localising subcategory generated by $\mathcal{CI}_\Gamma$. Moreover, the pair of localising subcategories $(\langle \mathcal{CI}_\Gamma \rangle, \mathcal{CC}_\Gamma)$

in KK^Γ is complementary, compare [Mey08]. That is, $KK^\Gamma(P,N)=0$ for all $P\in\langle\mathcal{CI}_\Gamma\rangle$ and $N\in\mathcal{CC}_\Gamma$, and every object $A\in KK^\Gamma$ fits into an exact triangle

$$\Sigma N \longrightarrow \tilde{A} \longrightarrow A \longrightarrow N$$

with $\tilde{A}\in\langle\mathcal{CI}_\Gamma\rangle$ and $N\in\mathcal{CC}_\Gamma$. Such a triangle is called a Dirac triangle for A, it is uniquely determined up to isomorphism in KK^Γ and depends functorially on A.

Definition 11.2. Let Γ be a torsion-free discrete quantum group and let A be a Γ-C^*-algebra. The Baum-Connes assembly map for Γ with coefficients in A is the map

$$\mu_A : K_*(\Gamma\ltimes_{\mathrm{r}}\tilde{A}) \to K_*(\Gamma\ltimes_{\mathrm{r}} A)$$

induced from a Dirac triangle for A. If μ_A is an isomorphism we shall say that Γ satisfies the Baum-Connes conjecture with coefficients in A.

By the work of Meyer and Nest [MN06], this terminology is consistent with the usual definitions in the case that Γ is a torsion-free discrete group.
Using the Fredholm module for the quantum flag manifold $\mathrm{SU}_q(3)/T$ in Theorem 10.6 we obtain the following result.

Theorem 11.3. *The dual of* $\mathrm{SU}_q(3)$ *for* $q\in(0,1]$ *satisfies the Baum-Connes conjecture with trivial coefficients* $\mathbb{C}$.

Proof. We shall follow the arguments in [MN07]and show $\mathbb{C}\in\langle\mathcal{CI}_{\hat{K}_q}\rangle$. This clearly implies that $\mu_\mathbb{C}$ is an isomorphism. Using Baaj-Skandalis duality, it is enough to prove $C(K_q)\in\langle\mathcal{T}_{K_q}\rangle$, where $\mathcal{T}_{K_q}\subset KK^{K_q}$ denotes the category of all trivial K_q-C^*-algebras.
We have $C(T)\subset\langle\mathcal{T}_T\rangle$ by the Baum-Connes conjecture for the abelian group $\hat{T}$, where $\mathcal{T}_T\subset KK^T$ is the category of trivial T-C^*-algebras. This implies $C(K_q)=\mathrm{ind}_T^{K_q}(C(T))\in\langle C(K_q/T)\rangle$. Hence it suffices to show $C(K_q/T)\in\langle\mathbb{C}\rangle$. In the case $q=1$ one obtains inverse isomorphisms $\alpha_1 : C(K_1/T)\to\mathbb{C}^{|W|}$ and $\beta_1:\mathbb{C}^{|W|}\to C(K_1/T)$ in KK^{K_1} using Poincaré duality, where $|W|=6$ is the order of the Weyl group of $K_1=SU(3)$, see [RS86], [MN07]. For general q we could argue in a similar way by invoking Theorem 11.1. Alternatively we may proceed as follows, avoiding the use of braided tensor products.
The element β_1 is given by induced vector bundles over the flag manifold, and one obtains a corresponding class $\beta_q\in KK^{K_q}(\mathbb{C}^{|W|},C(K_q/T))$ for any $q\in(0,1]$ using the induction isomorphism $KK^{K_q}(\mathbb{C},C(K_q/T))\cong KK^T(\mathbb{C},\mathbb{C})$. Similarly, the element α_1 is given by twisted Dolbeault operators. Using theorem 10.6 we obtain a corresponding class α_q in $KK^{K_q}(C(K_q/T),\mathbb{C}^{|W|})$. From K_q-equivariance it is immediate that we have $\beta_q\circ\alpha_q=\mathrm{Id}$ in $KK^{K_q}(\mathbb{C}^{|W|},\mathbb{C}^{|W|})$ for the classes thus obtained. To check $\alpha_q\circ\beta_q=\mathrm{Id}$ in $KK^{K_q}(C(K_q/T),C(K_q/T))$ we may use the canonical isomorphism $KK^{K_q}(C(K_q/T),C(K_q/T))\cong KK^T(C(K_q/T),\mathbb{C})$ and the fact that the C^*-algebras $C(K_q/T)$ form a T-equivariant continuous field, implementing a KK^T-equivalence between $C(K_q/T)$ and $C(K/T)$, see [NT12], [Yam13]. It therefore

suffices to consider the effect of $\alpha_q \circ \beta_q$ on $K_*^T(C(K_q/T)) \cong R(K) \otimes_{R(T)} R(K)$, which is the same for all $q \in (0,1]$. $\square$

We remark that Theorem 11.3 is of rather theoretical value. In particular, it does not lead to K-theory computations similar to the ones for free orthogonal quantum groups in [Voi11].

A Some results from q-calculus

A.1 Proof of the change of basis formula of Proposition 6.2

The vectors

$$|\xi_j\rangle = \left| \begin{pmatrix} m & & 0 & & -m \\ & j & & -j & \\ & & 0 & & \end{pmatrix}^{\uparrow} \right\rangle, \qquad |\eta_k\rangle = \left| \begin{pmatrix} m & & 0 & & -m \\ & k & & -k & \\ & & 0 & & \end{pmatrix}^{\downarrow} \right\rangle,$$

are the basis vectors for the 0-weight space of $\mu = (m, 0, -m)$ in the upper and lower Gelfand-Tsetlin bases, respectively. Our calculation of the change-of-basis coefficients avoids the use of raising and lowering operators from [MSK95], instead using a recurrence relation which arises by considering the bracket

$$\langle \eta_k | E_1^* E_1 | \xi_j \rangle. \tag{A.1}$$

Letting $E_1^* E_1$ act on $|\xi_j\rangle$ first, the Gelfand-Tsetlin formulae (6.3), (6.4) give

$$\langle \eta_k | E_1^* E_1 | \xi_j \rangle = [j][j+1] \langle \eta_k | \xi_j \rangle. \tag{A.2}$$

On the other hand, in the lower Gelfand-Tsetlin basis E_1 acts according to the formula (6.3) for $\Psi(E_1) = E_2$ (see the definition of the lower basis in Section 6.2). We get

$$\begin{aligned} E_1^* E_1 |\eta_k\rangle \quad = \quad & \frac{[m+k+2][m-k][k+1]^2}{[2k+1]^{\frac12}[2k+2][2k+3]^{\frac12}} |\eta_{k+1}\rangle \\ & + \left(\frac{[m+k+2][m-k][k+1]^2}{[2k+1][2k+2]} + \frac{[m+k+1][m-k+1][k]^2}{[2k][2k+1]} \right) |\eta_k\rangle \\ & + \frac{[m+k+1][m-k+1][k]^2}{[2k-1]^{\frac12}[2k][2k+1]^{\frac12}} |\eta_k\rangle \end{aligned}$$

Taking the inner product of this with $|\xi_j\rangle$ and equating with (A.2) yields a three-term recurrence relation for $\langle \eta_k | \xi_j \rangle$. The result is simplified if we introduce the non-unit vectors

$$|\mathrm{x}_j\rangle = [2j+1]^{-\frac12} |\xi_j\rangle, \qquad |\mathrm{y}_k\rangle = [2k+1]^{-\frac12} |\eta_k\rangle. \tag{A.3}$$

One obtains

$$[j][j+1] \langle \mathrm{y}_k | \mathrm{x}_j \rangle = a(k) \langle \mathrm{y}_{k+1} | \mathrm{x}_j \rangle + (a(k) + c(k)) \langle \mathrm{y}_k | \mathrm{x}_j \rangle + c(k) \langle \mathrm{y}_{k-1} | \mathrm{x}_j \rangle, \tag{A.4}$$

where

$$
\begin{aligned}
a(k) &= \frac{[m+k+2][m-k][k+1]^2}{[2k+1][2k+2]}, \\
c(k) &= \frac{[m+k+1][m-k+1][k]^2}{[2k][2k+1]}.
\end{aligned}
$$

We claim that the solution of Equation (A.4) is given by q-Racah coefficients. Unfortunately, the q-Racah polynomials are typically written in terms of the non-symmetric q-numbers $[[n]] = \frac{1-q^n}{1-q}$, so we must rewrite the recurrence relation as

$$
\begin{aligned}
(1-q^{2j})&(1-q^{2(j+1)})\langle \mathrm{y}_k|\mathrm{x}_j\rangle \\
&= A(k)\langle \mathrm{y}_{k+1}|\mathrm{x}_j\rangle + (A(k)+C(k))\langle \mathrm{y}_k|\mathrm{x}_j\rangle + A(k)\langle \mathrm{y}_{k-1}|\mathrm{x}_j\rangle, \quad \text{(A.5)}
\end{aligned}
$$

where

$$
\begin{aligned}
A(k) &= \frac{(1-q^{2(m+k+2)})(1-q^{2(k-m)})(1-q^{2(k+1)})^2}{(1-q^{2(2k+1)})(1-q^{2(2k+2)})}, \\
C(k) &= \frac{q^2(1-q^{2(k+m+1)})(1-q^{2(k-m-1)})(1-q^{2(k)})^2}{(1-q^{2(2k)})(1-q^{2(2k+1)})}.
\end{aligned}
$$

These are precisely the coefficients in the recurrence relation for the q^2-Racah polynomials described in Equation (14.2.3) of [KLS10], with parameters $\alpha = q^{2(m+1)}$, $\beta = q^{-2(m+1)}$, $\gamma = \delta = 1$ and $N = m$. The initial condition for the recurrence relation is fixed by Equation (6.10), which gives

$$
\langle \mathrm{y}_0|\mathrm{x}_j\rangle = \frac{(-1)^{j+m}}{[m+1]},
$$

and formula of Proposition 6.2 follows.

A.2 A q-integral identity for little q-Legendre polynomials

We recall the definitions of the standard q-differentiation and q-integration operators:

$$
\begin{aligned}
D_q f(x) &= \frac{f(qx)-f(x)}{(qx-x)}, \\
\int_0^x f(y)d_q y &= x(1-q)\sum_{j=0}^{\infty} q^j f(q^j x).
\end{aligned}
$$

We also recall the following basic q-derivatives, where $[[n]] = \frac{1-q^n}{1-q}$:

$$
\begin{aligned}
D_q x^\alpha &= [[\alpha]]_q x^{\alpha-1}, \qquad \text{for all } \alpha \in \mathbb{R}, && \text{(A.6)} \\
D_q(x;q^{-1})_n &= -[[n]]_{q^{-1}}(x;q^{-1})_{n-1} && \text{(A.7)} \\
&= -q^{-(n-1)}[[n]]_q(x;q^{-1})_{n-1}, \qquad \text{for all } n \in \mathbb{N}.
\end{aligned}
$$

PROPOSITION A.1.

$$\int_0^1 x^{-\frac{1}{2}} \mathrm{p}_k(x|q^2)\, d_{q^2}x = q^{\frac{1}{2}} \left[\frac{2k+1}{2}\right]_q^{-1}.$$

Proof. Let us put $r = q^2$. The little q-Legendre polynomials satisfy the following Rodrigues-type Formula (see [KLS10]):

$$\mathrm{p}_k(x|r) = \frac{1}{[[k]]_r!} D_r^k \left[x^k (x; r^{-1})_k\right].$$

From Equations (A.6) and (A.7) one has that for all $0 \le i < k$, $D_q^i x^k = 0$ at $x = 0$ and $D_q^i (x; r^{-1})_k = 0$ at $x = 1$. Thus, by k applications of q-integration by parts, we get

$$\begin{aligned}
&\int_0^1 x^{-\frac{1}{2}} \mathrm{p}_k(x|r)\, d_r x \\
&= \frac{(-1)^k}{[[k]]_r!} \left[\left[-\tfrac{1}{2}\right]\right]_r r^{\frac{1}{2}} \left[\left[-\tfrac{3}{2}\right]\right]_r r^{\frac{3}{2}} \cdots \left[\left[-\tfrac{(2k-1)}{2}\right]\right]_r r^{\frac{(2k-1)}{2}} \int_0^1 x^{-\frac{(2k+1)}{2}} x^k (x; r^{-1})_k d_r x \\
&= \frac{1}{[[k]]_r!} \left[\left[\tfrac{1}{2}\right]\right]_r \left[\left[\tfrac{3}{2}\right]\right]_r \cdots \left[\left[\tfrac{(2k-1)}{2}\right]\right]_r \int_0^1 x^{-\frac{1}{2}} (x; r^{-1})_k d_r x
\end{aligned}$$

where in the last equality we have used $[[\alpha]]_r = -r^{\alpha}[[-\alpha]]_r$. The last q-integral can be computed by q-integrating by parts k more times, giving

$$\begin{aligned}
\frac{(-1)^k}{[[k]]_r!} r^{\frac{1}{2}} r^{\frac{3}{2}} \cdots r^{\frac{2k-1}{2}} r^k [[-k]]_r [[-k+1]]_r \cdots [[-1]]_r \int_0^1 x^{\frac{(2k-1)}{2}} d_r x& \\
= r^{\frac{k}{2}} \int_0^1 x^{\frac{(2k-1)}{2}} d_r x& \\
= q^{\frac{1}{2}} \left[\frac{2k+1}{2}\right]_q^{-1}.&
\end{aligned}$$

□

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Christian Voigt
School of Mathematics
and Statistics
University of Glasgow
15 University Gardens
Glasgow G12 8QW
United Kingdom

Robert Yuncken
(1) Université Clermont Auvergne
Université Blaise Pascal
Laboratoire de Mathématiques
BP 10448
F-63000 Clermont-Ferrand
France
(2) CNRS, UMR 6620, LM
F-63178 Aubiere
France
Robert.Yuncken
@math.univ-bpclermont.fr