Equivariant Operational Chow Rings of T-Linear Schemes

Richard P. Gonzales[1]

Received: May 14, 2014
Revised: December 26, 2014

Communicated by Alexander Merkurjev

Abstract. We study T-linear schemes, a class of objects that includes spherical and Schubert varieties. We provide a localization theorem for the equivariant Chow cohomology of these schemes that does not depend on resolution of singularities. Furthermore, we give an explicit presentation of the equivariant Chow cohomology of possibly singular complete spherical varieties admitting a smooth equivariant envelope (e.g., group embeddings).

2010 Mathematics Subject Classification: 14M27, 14L30, 20M32
Keywords and Phrases: spherical varieties, Chow cohomology, Kronecker duality, intersection theory.

1 Introduction and motivation

Let $\Bbbk$ be an algebraically closed field. Let G be a connected reductive linear algebraic group (over $\Bbbk$). Let B be a Borel subgroup of G and $T \subset B$ be a maximal torus of G. An algebraic variety X, equipped with an action of G, is *spherical* if it contains a dense orbit of B. (Usually spherical varieties are assumed to be normal but this condition is not needed here.) Spherical varieties have been extensively studied in the works of Akhiezer, Brion, Knop, Luna, Pauer, Vinberg, Vust and others. For an up-to-date discussion of spherical varieties, as well as a comprehensive bibliography, see [Ti] and the references therein. If X is spherical, then it has a finite number of B-orbits, and thus, also a finite number of G-orbits [Ti]. In particular, T acts on X with a finite number of fixed points. These properties make spherical varieties particularly

[1]Supported by the Institut des Hautes Études Scientifiques, TÜBİTAK Project No. 112T233, and DFG Research Grant PE2165/1-1.

well suited for applying the methods of Goresky-Kottwitz-MacPherson [GKM], nowadays called GKM theory, in the topological setup, and Brion's extension of GKM theory [Br2] to the algebraic setting of equivariant Chow groups, as defined by Totaro, Edidin and Graham [EG1]. Through this method, substantial information about the topology and geometry of a spherical variety can be obtained by restricting one's attention to the induced action of T.

Examples of spherical varieties include $G\times G$-equivariant embeddings of G (e.g., toric varieties are spherical) and the regular symmetric varieties of De Concini-Procesi [DP]. The equivariant cohomology and equivariant Chow groups of smooth complete spherical varieties have been studied by Bifet, De Concini and Procesi [BCP], De Concini-Littelmann [LP], Brion [Br2] and Brion-Joshua [BJ2]. In these cases, there is a comparison result relating equivariant cohomology with equivariant Chow groups: for a smooth complete spherical variety, the equivariant cycle map yields an isomorphism from the equivariant Chow group to the equivariant (integral) cohomology (Proposition 2.11). As for the study of the equivariant Chow groups of possibly singular spherical varieties, some progress has been made by Payne [P] and the author [G2]-[G4].

The problem of developing intersection theory on singular varieties comes from the fact that the Chow groups $A_*(-)$ do not admit, in general, a natural ring structure or intersection product. But when singularities are mild, for instance when X is a quotient of a smooth variety Y by a finite group F, then $A_*(X)\otimes\mathbb{Q}\simeq(A_*(Y)\otimes\mathbb{Q})^F$, and so $A_*(X)\otimes\mathbb{Q}$ inherits the ring structure of $A_*(Y)\otimes\mathbb{Q}$. To simplify notation, if A is a $\mathbb{Z}$-module, we shall write hereafter $A_{\mathbb{Q}}$ for the rational vector space $A\otimes\mathbb{Q}$.

In order to study more general singular schemes, Fulton and MacPherson [Fu] introduced the notion of operational Chow groups or Chow cohomology. Similarly, Edidin and Graham defined the *equivariant operational Chow groups* [EG1], which we briefly recall. (For our conventions on varieties and schemes, see Section 2.1.) Let X be a T-scheme. The i-th T-equivariant operational Chow group of X, denoted $A^i_T(X)$, is defined as follows. An element $c\in A^i_T(X)$ is a collection of homomorphisms $c_f^{(m)}: A_m^T(Y)\to A_{m-i}^T(Y)$, written $z\mapsto f^*c\cap z$, for every T-equivariant map $f:Y\to X$ and all integers m (the underlying category is the category of T-schemes). Here $A_*^T(Y)$ denotes the equivariant Chow group of Y (Section 2.1). As in the case of ordinary operational Chow groups, these homomorphisms must satisfy three conditions of compatibility: with proper pushforward (resp. flat pull-back, resp. intersection with a Cartier divisor) for T-equivariant maps $Y'\to Y\to X$, with $Y'\to Y$ proper (resp. flat, resp. determined by intersection with a Cartier divisor); see [Fu, Chapter 17] for precise statements. The homomorphism $c_f^{(m)}$ determined by an element $c\in A^i_T(X)$ is usually denoted simply by c, with an indication of where it acts. For any X, the ring structure on $A^*_T(X):=\oplus_i A^i_T(X)$ is given by composition of such homomorphisms. The ring $A^*_T(X)$ is graded, and $A^i_T(X)$ can be non-zero for any $i\geq 0$. The most salient functorial properties of equivariant operational Chow groups are summarized below:

(i) Cup products $A_T^p(X) \otimes A_T^q(X) \to A_T^{p+q}(X)$, $a \otimes b \mapsto a \cup b$, making $A_T^*(X)$ into a graded associative ring (commutative when resolution of singularities is known).

(ii) Contravariant graded ring maps $f^* : A_T^i(X) \to A_T^i(Y)$ for arbitrary equivariant morphisms $f : Y \to X$.

(iii) Cap products $A_T^i(X) \otimes A_m^T(X) \to A_{m-i}^T(X)$, $c \otimes z \mapsto c \cap z$, making $A_*^T(X)$ into an $A_T^*(X)$-module and satisfying the projection formula.

(iv) If X is a nonsingular n-dimensional T-variety, then the Poincaré duality map from $A_T^i(X)$ to $A_{n-i}^T(X)$, taking c to $c \cap [X]$, is an isomorphism, and the ring structure on $A_T^*(X)$ is that determined by intersection products of cycles on the mixed spaces X_T [EG1, Proposition 4].

(v) Equivariant vector bundles on X have equivariant Chern classes in $A_T^*(X)$.

(vi) Localization theorems of Borel-Atiyah-Segal type and GKM theory (with rational coefficients) for possibly singular complete T-varieties in characteristic zero. See [G3] or the Appendix for details.

In [FMSS], Fulton, MacPherson, Sottile and Sturmfels succeed in describing the non-equivariant operational Chow groups of complete spherical varieties. Indeed, they show that the Kronecker duality homomorphism

$$\mathcal{K} : A^i(X) \longrightarrow \mathrm{Hom}(A_i(X), \mathbb{Z}), \qquad \alpha \mapsto (\beta \mapsto \deg(\beta \cap \alpha))$$

is an isomorphism for complete spherical varieties. Here $\deg(-)$ is the degree homomorphism $A_0(X) \to \mathbb{Z}$. Moreover, they prove that $A_*(X)$ is finitely generated by the classes of B-orbit closures, and with the aid of the map $\mathcal{K}$, they provide a combinatorial description of $A^*(X)$ and the structure constants of the cap and cup products [FMSS]. In addition, if X is nonsingular and complete, they show that the cycle map $cl_X : A_*(X) \to H_*(X)$ is an isomorphism. Although we stated their results in the case of spherical varieties, these hold more generally for complete schemes with a finite number of orbits of a solvable group. In particular, the conclusions of [FMSS] hold for Schubert varieties. Later on, Totaro [To] extended these results to the broader class of linear schemes, a class first studied in work of Jannsen [Ja]. The results of [FMSS] and [To] are quite marvelous in that they give a presentation of a rather abstract ring, namely $A^*(X)$, in a very combinatorial manner.

In this article, we extend the results of the previous paragraph to the equivariant Chow cohomology of T-linear schemes (Definition 2.3). By Theorem 2.5, spherical varieties are T-linear (this fact does not follow directly from [To] and [R], see the comments before Theorem 2.5). Also, we obtain a localization theorem for the equivariant Chow cohomology of complete T-linear schemes that does not depend on resolution of singularities (Theorem 3.9). Last, and most important, we give a presentation of the rational equivariant Chow cohomology of complete possibly singular spherical varieties admitting an equivariant smooth envelope (Theorem 4.8). The latter vastly increases the applicability of Brion's techniques [Br2, Section 7] from the smooth to the singular setup.

Here is an outline of the paper. Section 2 reviews the necessary background material. Section 3 is the conceptual core of this article. In Subsection 3.1 we obtain the equivariant versions of the results of [FMSS] and [To] that concern us. We start by defining *equivariant Kronecker duality schemes*. These are complete T-schemes X which satisfy two conditions: (i) $A_*^T(X)$ is finitely generated over $S = A_T^*(pt)$, and (ii) the equivariant Kronecker duality map $\mathcal{K}_T : A_T^*(X) \longrightarrow \mathrm{Hom}_S(A_*^T(X), S)$ is an isomorphism of S-modules (Definition 3.1). As an example, we show that complete T-linear schemes satisfy equivariant Kronecker duality (Proposition 3.5). This is deduced from the equivariant Künneth formula (Proposition 3.4). In Subsection 3.2 we prove our second main result, namely, a localization theorem for equivariant Kronecker duality schemes (Theorem 3.9). We conclude Section 3 by showing that projective group embeddings in arbitrary characteristic satisfy equivariant localization (Theorem 3.11). This extends well-known results on torus embeddings [P] to more general compactifications of connected reductive groups. Finally, in Section 4, we apply the machinery just developed to spherical varieties in characteristic zero and prove the most important result of this paper, namely, Theorem 4.8. It asserts that if X is a complete, possibly singular, G-spherical variety, then the image of the injective map $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is fully described by congruences involving pairs, triples or quadruples of T-fixed points. Remarkably, this extends [Br2, Theorem 7.3] to the singular setting.

Acknowledgments

Most of the research in this paper was done during my first visit to the Institute des Hautes Études Scientifiques (IHES). I am deeply grateful to IHES for its support, outstanding hospitality and excellent atmosphere. A very special thank you goes to Michel Brion for carefully reading previous drafts of this article, and for offering detailed comments and suggestions on various parts. I would also like to thank Sabancı Üniversitesi, the Scientific and Technological Research Council of Turkey (TÜBİTAK), and the German Research Foundation (DFG) for their support during the final stages of completing this article. I am further grateful to the referee for very helpful comments and suggestions that improved the presentation and scope of the article.

2 Definitions and basic properties

2.1 Conventions and notation

Throughout this paper, we fix an algebraically closed field $\Bbbk$ (of arbitrary characteristic, unless stated otherwise). All schemes and algebraic groups are assumed to be defined over $\Bbbk$. By a scheme we mean a separated scheme of finite type over $\Bbbk$. A variety is a reduced and irreducible scheme. A subvariety is a closed subscheme which is a variety. A point on a scheme will always be a closed point. The additive and multiplicative groups over $\Bbbk$ are denoted by $\mathbb{G}_a$ and $\mathbb{G}_m$.

We denote by T an algebraic torus. We write Δ for the character group of T, and S for the symmetric algebra over $\mathbb{Z}$ of the abelian group Δ. We denote by $\mathcal{Q}$ the quotient field of S. A scheme X provided with an algebraic action of T is called a *T-scheme*. For a T-scheme X, we denote by X^T the fixed point subscheme and by $i_T : X^T \to X$ the natural inclusion. If H is a closed subgroup of T, we similarly denote by $i_H : X^H \to X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \to X^H$ for the natural (T-equivariant) inclusion.

A T-scheme X is called *locally linearizable* (and the T-action is called *locally linear*) if X is covered by invariant affine open subsets. For instance, T-stable subschemes of normal T-schemes are locally linearizable [Su]. A T-scheme is called *T-quasiprojective* if it has an ample T-linearized invertible sheaf. This assumption is satisfied, e.g. for T-stable subschemes of normal quasiprojective T-schemes [Su]. Recall that an *envelope* $p : \tilde{X} \to X$ is a proper map such that for any subvariety $W \subset X$ there is a subvariety $\tilde{W}$ mapping birationally to W via p [Fu, Definition 18.3]. In the case of T-actions, we say that $p : \tilde{X} \to X$ is an *equivariant envelope* if p is T-equivariant, and if we can take $\tilde{W}$ to be T-invariant for T-invariant W. If there is a dense open set $U \subset X$ over which p is an isomorphism, then we say that $p : \tilde{X} \to X$ is a *birational* envelope. By [Su, Theorem 2], if X is a T-scheme, then there exists a T-equivariant birational envelope $p : \tilde{X} \to X$, where $\tilde{X}$ is a T-quasiprojective scheme. Moreover, if $\mathrm{char}(\Bbbk) = 0$, then we may choose $\tilde{X}$ to be smooth [EG2, Proposition 7.5]. If $p : \tilde{X} \to X$ is a T-equivariant envelope, and $H \subset T$ is a closed subgroup, then the induced map $\tilde{X}^H \to X^H$ is a T-equivariant envelope [EG2, Lemma 7.2].

Let X be a T-scheme of dimension n (not necessarily equidimensional). Let V be a finite dimensional T-module, and let $U \subset V$ be an invariant open subset such that a principal bundle quotient $U \to U/T$ exists. Then T acts freely on $X \times U$ and the quotient scheme $X_T := (X \times U)/T$ exists. Following Edidin and Graham [EG1], we define the i-th equivariant Chow group $A_i^T(X)$ by $A_i^T(X) := A_{i+\dim U - \dim T}(X)$, if $V \setminus U$ has codimension more than $n - i$. Such pairs (V, U) always exist, and the definition is independent of the choice of (V, U), see [EG1]. Finally, $A_*^T(X) := \oplus_i A_i^T(X)$. Unlike ordinary Chow groups, $A_i^G(X)$ can be non-zero for any $i \leq n$, including negative i. If X is a T-scheme, and $Y \subset X$ is a T-stable closed subscheme, then Y defines a class $[Y]$ in $A_*^T(X)$. If X is smooth and equidimensional, then so is X_T, and $A_*^T(X)$ admits an intersection pairing; in this case, the corresponding ring graded by codimension is isomorphic to the equivariant operational Chow ring $A_T^*(X)$ [EG1, Proposition 4]. The equivariant Chow ring of a point $A_T^*(pt)$ identifies to S, and $A_*^T(X)$ is a S-module, where Δ acts on $A_*^T(X)$ by homogeneous maps of degree -1. This module structure is induced by pullback through the flat map $p_{X,T}: X_T \to U/T$. Restriction to a fiber of $p_{X,T}$ gives a canonical map $A_*^T(X) \to A_*(X)$, and this map is surjective (Theorem 2.6). If X is complete, we denote by $p_{X,T*}(\alpha)$ (or simply $p_{X*}(\alpha)$) the proper pushforward to a point of a class $\alpha \in A_*^T(X)$. We may also write $\int_X(\alpha)$ or $\deg(\alpha)$ for this pushforward. Note that $A_T^*(pt)$ is isomorphic to $A_*^T(pt)$ with the opposite grading.

Let X be a T-scheme. For any mixed space X_T we construct a map $r: A_T^i(X) \to A^i(X_T)$. Let $c \in A_T^*(X)$. For a map $Y \to X_T$ and $\alpha \in A_*(Y)$, we define $r(c) \cap \alpha$ as follows. Let $Y_U \to Y$ be the pullback of the principal T-bundle $X \times U \to X_T$. Since $Y_U \to Y$ is a principal bundle, we identify $A_*(Y)$ with $A_*^T(Y_U)$. Let $\alpha_U \in A_*^T(Y_U)$ correspond to $\alpha \in A_*(Y)$. Now simply define $r(c) \cap \alpha$ to be the class corresponding to $c \cap \alpha_U$. See [EG1, pages 620-621] for more information on the functorial properties of the map r. On the other hand, we also have a map $\rho: A^i(X_T) \to A_T^i(X)$. Indeed, let $c \in A^i(X_T)$, $Y \to X$ a T-equivariant map, and $\beta \in A_*^T(Y)$. For any representation, there are maps $Y_T \to X_T$. The class β is represented by a class $\beta_U \in A_{*+\dim U - \dim T}(Y_T)$ for some mixed space $Y \times U/T$. Define $\rho(c) \cap \beta = c \cap \beta_U$. This is an element of $A_{*+\dim U-\dim T-i}(Y_T) \simeq A_{*-i}^T(Y)$. Note that if X has a T-equivariant smooth envelope (e.g. X is a group embedding or $\mathrm{char}(\Bbbk) = 0$), and $V \setminus U$ has codimension more than i, then ρ and r are inverse functions; so in this case we get $A_T^i(X) \simeq A^i(X_T)$ [EG1, Theorem 2].

Finally, for any T-scheme X, restriction to a fiber of $p_{X,T}: X_T \to U/T$ induces a canonical map $A^*(X_T) \to A^*(X)$. Precomposing this map with $r: A_T^*(X) \to A^*(X_T)$ gives a natural map $\iota^*: A_T^*(X) \to A^*(X)$. In general, unlike its counterpart in equivariant Chow groups, the map ι^* is not surjective and its kernel is not necessarily generated in degree one, not even for toric varieties [KP]. This becomes an issue when translating results from equivariant to non-equivariant Chow cohomology. In Corollary 3.8 we give some conditions under which ι^* is surjective and yields an isomorphism $A_T^*(X)_{\mathbb{Q}}/\Delta A_T^*(X)_{\mathbb{Q}} \simeq A^*(X)_{\mathbb{Q}}$. Such conditions are fulfilled, among others, by complete $\mathbb{Q}$-filtrable spherical varieties [G4].

2.2 The Białynicki-Birula decomposition

Let X be a T-scheme. Let $X^T = \bigsqcup_{i=1}^m F_i$ be the decomposition of X^T into connected components. A one-parameter subgroup $\lambda: \mathbb{G}_m \to T$ is called *generic* if $X^{\mathbb{G}_m} = X^T$, where $\mathbb{G}_m$ acts on X via λ. Generic one-parameter subgroups always exist (when X is locally linearizable this certainly holds; the general case follows from this by considering the normalization of X). Now fix a generic one-parameter subgroup λ of T. For each F_i, we define the subset

$$X_+(F_i,\lambda) := \{x \in X \mid \lim_{t\to 0} \lambda(t)\cdot x \text{ exists and is in } F_i\}.$$

We denote by $\pi_i: X_+(F_i,\lambda) \to F_i$ the map $x \mapsto \lim_{t\to 0}\lambda(t)\cdot x$. Then $X_+(F_i,\lambda)$ is a locally closed T-invariant subscheme of X, and π_i is a T-equivariant morphism. The (disjoint) union of the $X_+(F_i, \lambda)$ might not cover all of X, but when it does (e.g. when X is complete), the decomposition $\{X_+(F_i, \lambda)\}_{i=1}^m$ is called the Białynicki-Birula decomposition, or *BB-decomposition*, of X associated to λ. Each $X_+(F_i, \lambda)$ is referred to as a *stratum* of the decomposition. If, moreover, all fixed points of the given T-action on X are isolated (i.e. X^T is finite), the corresponding $X_+(F_i, \lambda)$ are simply called *cells* of the decomposition.

Definition 2.1. Let X be a T-scheme endowed with a BB-decomposition $\{X_+(F_i, \lambda)\}$, for some generic one-parameter subgroup λ of T. The decomposition is said to be *filtrable* if there exists a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \ldots \subset \Sigma_m$ of T-invariant closed subschemes of X such that:

a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,

b) $\Sigma_j \setminus \Sigma_{j-1}$ is a stratum of the decomposition $\{X_+(F_i, \lambda)\}$, for each $j = 1, \ldots, m$.

In this context, it is common to say that X is *filtrable*. If, moreover, X^T is finite and the cells $X_+(F_i, \lambda)$ are isomorphic to affine spaces $\mathbb{A}^{n_i}$, then X is called *T-cellular*. The following result is due to Białynicki-Birula ([B1], [B2]).

Theorem 2.2. *Let X be a complete T-scheme, and let λ be a generic one-parameter subgroup. If X admits an ample T-linearized invertible sheaf, then the associated BB-decomposition $\{X_+(F_i, \lambda)\}$ is filtrable. Furthermore, if X is smooth, then X^T is also smooth, and for any component F_i of X^T, the map $\pi_i : X_+(F_i, \lambda) \to F_i$ makes $X_+(F_i, \lambda)$ into a T-equivariant locally trivial bundle in affine spaces over F_i.* $\square$

Hence, smooth projective T-schemes with isolated fixed points are T-cellular.

2.3 T-linear schemes

We introduce here the main objects of our study and outline some of their relevant features.

Definition 2.3. Let T be an algebraic torus and let X be a T-scheme.

1. We say that X is *T-equivariantly 0-linear* if it is either empty or isomorphic to $\mathrm{Spec}\,(\mathrm{Sym}(V^*))$, where V is a finite-dimensional rational representation of T.

2. For a positive integer n, we say that X is *T-equivariantly n-linear* if there exists a family of T-schemes $\{U, Y, Z\}$, such that $Z \subseteq Y$ is a T-invariant closed immersion with U its complement, Z and one of the schemes U or Y are T-equivariantly $(n-1)$-linear and X is the other member of the family $\{U, Y, Z\}$.

3. We say that X is *T-equivariantly linear* (or simply, *T-linear*) if it is T-equivariantly n-linear for some $n \geq 0$. T-linear varieties are varieties that are T-linear schemes.

It follows from the inductive definition that if X is T-equivariantly n-linear, then X^H is T-equivariantly n-linear, for any subtorus $H \subset T$. Moreover, if $T \to T'$ is a morphism of algebraic tori, then every T'-linear scheme is also T-linear. Observe that T-linear schemes are *linear schemes* in the sense of Jannsen [Ja] and Totaro [To]. While Totaro's class of linear schemes is slightly narrower than that of Jannsen, the difference is nevertheless immaterial for our purposes. In fact, one easily checks that the main result of Totaro used here, namely [To, Proposition 1], holds for the larger class. The following result is recorded in [JK].

Proposition 2.4. *Let T be an algebraic torus and let T' be a quotient of T. Let T act on T' via the quotient map. Then the following hold:*

(i) T' is T-linear.

(ii) A T-cellular scheme is T-linear.

(iii) Every T-scheme with finitely many T-orbits is T-linear. In particular, a toric variety with dense torus T is T-linear. □

It is well-known that flag varieties, partial flag varieties and Schubert varieties come with a paving by affine spaces (due to the Bruhat decomposition), so they are all T-cellular and hence T-linear.

Let B be a connected solvable linear algebraic group with maximal torus T. A result of Rosenlicht [R, Theorem 5] shows that a homogeneous space for B is isomorphic *as a variety* to $\mathbb{G}_a^r \times \mathbb{G}_m^s$, for some r, s. As observed by Totaro [To], this implies that a B-scheme with finitely many B-orbits (e.g. a spherical variety) is linear. Nevertheless, this does not readily imply that such a scheme is T-linear, as Rosenlicht does not show that the isomorphism above may be chosen T-equivariant. Presumably his arguments can be adjusted to achieve this. In any case, we shall give a direct proof of this fact, to keep the exposition self contained.

Theorem 2.5. *Let B be a connected solvable linear algebraic group with maximal torus T. Let X be a B-scheme. If B acts on X with finitely many orbits, then X is T-linear. In particular, spherical varieties are T-linear.*

Proof. The following argument was shown to the author by M. Brion (personal communication). Since X is a disjoint union of B-orbits and these are T-stable, it suffices to show that every B-orbit is T-linear. Write this orbit as B/H where H is a closed subgroup of B. Let U be the unipotent radical of B. Then, we have a natural map $f : B/H \to B/UH$ and the right-hand side is a torus (for it is a homogeneous space under the torus $T = B/U$). Moreover, f is a B-equivariant fibration with fiber $UH/H = U/(U \cap H)$, which is an affine space (as it is homogeneous under U).

We will show that the fibration f is T-equivariantly trivial by factoring it into T-equivariantly trivial fibrations with fiber the affine line. For this, we argue by induction on the dimension of UH/H. If this dimension is zero, then there is nothing to prove. If it is positive, then U acts non-trivially on B/H; replacing U and B with suitable quotients, we may assume that U acts faithfully. Because U is a unipotent group normalized by T, we may find a one-dimensional unipotent subgroup V of the center of U, which is normalized by T [Sp, Lemma 6.3.4]. So V is isomorphic to $\mathbb{G}_a$, and T acts linearly on V with some weight α. By construction, V acts freely on B/H via left multiplication, and the quotient map is the natural map $B/H \to B/VH$, which is a principal V-bundle. Since the variety B/VH is affine (e.g by the induction assumption) and $V \simeq \mathbb{G}_a$, this bundle is trivial. The isomorphism $B/H = B/VH \times V$, equivariant for the action of V, yields a regular function g on B/H such that $g(vx) = v + g(x)$ for all x in B/H and v in V (identified to $\mathbb{G}_a$). Let T act on the ring of regular functions on B/H via its action on B/H by left multiplication. We claim that g may be chosen an eigenvector of T. Indeed, write g as a sum of weight vectors g_λ. Then for any t in T, we obtain $(tg)(vx) = g(tvx) = \alpha(t)v + (tg)(x)$ which yields $\sum_\lambda \lambda(t)g_\lambda(vx) = \alpha(t)v + \sum_\lambda \lambda(t)g_\lambda(x)$. By viewing both sides as functions of t and using linear independence of characters, one gets $g_\alpha(vx) = v + g_\alpha(x)$, and $g_\lambda(vx) = g_\lambda(x)$ for all $\lambda \neq \alpha$. So these g_λ are invariant under V, i.e. they are regular functions on B/VH, and one may subtract them from g to get $g = g_\alpha$. Now that the claim has been verified, it follows that the product map $B/H \to B/VH \times V$, where the second map is g_α, yields the desired T-equivariant isomorphism, and we conclude by induction. □

2.4 Description of equivariant Chow groups

Next we state Brion's presentation of the equivariant Chow groups of schemes with a torus action in terms of invariant cycles [Br2, Theorem 2.1]. It also shows how to recover usual Chow groups from equivariant ones.

Theorem 2.6. *Let X be a T-scheme. Then the S-module $A_*^T(X)$ is defined by generators $[Y]$, where Y is an invariant subvariety of X, and relations $[\mathrm{div}_Y(f)] - \chi[Y]$ where f is a rational function on Y which is an eigenvector of T of weight χ. Moreover, the map $A_*^T(X) \to A_*(X)$ vanishes on $\Delta A_*^T(X)$, and it induces an isomorphism $A_*^T(X)/\Delta A_*^T(X) \to A_*(X)$.* □

Now let Γ be a connected solvable linear algebraic group with maximal torus T. If X is a Γ-scheme, then the generators of $A_*^T(X)$ in Theorem 2.6 can be taken to be Γ-invariant [Br2, Proposition 6.1]. In particular, if X has finitely many Γ-orbits (e.g. X is spherical), then the S-module $A_*^T(X)$ is *finitely generated* by the classes of the Γ-orbit closures. More generally, one has the following lemma.

Lemma 2.7. *Let X be a T-linear scheme. Then the S-module $A_*^T(X)$ is finitely generated. In particular, $A_*(X)$ is a finitely generated abelian group. Moreover, if X is complete, then rational equivalence and algebraic equivalence coincide on X.*

Proof. The first two assertions are easy consequences of the inductive definition of T-linear schemes (see e.g. [To]). Regarding the last one, observe that if X is complete, then the kernel of the natural morphism $A_*(X) \to B_*(X)$ is divisible [Fu, Example 19.1.2], and thus trivial, for $A_*(X)$ is finitely generated. □

Recall that if X is a smooth equidimensional T-scheme, then $A_T^*(X)$ is isomorphic to the equivariant Chow group of X graded by codimension [EG1, Proposition 4].

Theorem 2.8 ([Br2], [VV]). *Let X be a smooth T-variety. If X is complete, then the $S_{\mathbb{Q}}$-module $A_*^T(X)_{\mathbb{Q}}$ is free. Moreover, the restriction homomorphism $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is injective, and its image is the intersection of all the images of the restriction homomorphisms $i_{T,H}^* : A_T^*(X^H)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$, where H runs over all subtori of codimension one in T.* □

A few comments are in order here. First, Brion showed that Theorem 2.8 holds in the special case that X is projective [Br2, Theorems 3.2 and 3.3]. Later on, Vezzosi and Vistoli [VV] generalized Brion's results to the setting of equivariant higher K-theory and established the corresponding analogue of Theorem 2.8, which holds for X complete [VV, Corollary 5.11]. From this, by making appropriate changes in the proofs of [VV, Proposition 5.13 and Theorem 5.4], one obtains Theorem 2.8 in its full form. The details can be found in a preprint version of [VV] (arXiv version 3). Alternatively, see [Kr, Sections 9 and 10], where the results of [VV, Section 5] have been generalized to equivariant higher Chow groups.

In characteristic zero Theorem 2.8 extends to all possibly singular complete varieties [G3, Section 7]. See the Appendix for a review of the main results in this regard.

The next lemma will become relevant later, when integrality of the equivariant operational Chow rings is discussed (cf. Lemma 3.3). It is essentially due to Brion, del Baño and Karpenko.

Lemma 2.9. *Let X be a smooth projective T-variety. Then the following are equivalent.*

(i) $A_*(X^T)$ is $\mathbb{Z}$-free.

(ii) $A_*^T(X)$ is S-free.

(iii) $A_*(X)$ is $\mathbb{Z}$-free.

If moreover X is T-linear, then any (and hence all) of these conditions hold.

Proof. The implication (i)⇒(ii) follows from [Br2, Corollary 3.2.1], as any smooth projective variety is filtrable (Theorem 2.2). That (ii) implies (iii) is a consequence of Theorem 2.6. To show that (iii) implies (i) we use a result of del Baño [dB, Theorem 2.4] and Karpenko [Ka, Section 6]. Namely, let λ

be a generic one-parameter subgroup of T, and let $X^T = \bigsqcup_i F_i$ be the decomposition of X^T into connected components. Then, for every non-negative integer $j \leq \dim(X)$, there is a natural isomorphism $\bigoplus_i A^{j-d_i}(F_i) \xrightarrow{\simeq} A^j(X)$, where d_i is the codimension of $X_+(F_i, \lambda)$ in X (all spaces involved are smooth, so there is an intersection product on the Chow groups graded by codimension). These isomorphisms yield the assertion (iii)⇒(i).
Finally, if X is a smooth projective T-linear variety, then, in particular, it is a projective linear variety, and so it satisfies the Künneth formula (see below). Now Theorem 2.10 (ii) implies that condition (iii) of the lemma holds for X. This concludes the argument. $\square$

For any schemes X and Y, one has a Künneth map

$$A_*(X) \otimes A_*(Y) \to A_*(X \times Y),$$

taking $[V] \otimes [W]$ to $[V \times W]$, where V and W are subvarieties of X and Y. This is an isomorphism only for very special schemes, e.g. linear schemes [To, Proposition 1]; but when it is, strong consequences can be derived from it, as we shall see below. Let us start with the following result due to Ellingsrud and Stromme [ES, Theorem 2.1].

Theorem 2.10. *Let X be a smooth complete variety. Assume that the rational equivalence class δ of the diagonal $\Delta(X) \subseteq X \times X$ is in the image of the Künneth map $A_*(X) \otimes A_*(X) \to A_*(X \times X)$. Let $\delta = \sum u_i \otimes v_i$ be a corresponding decomposition of δ, where $u_i, v_i \in A_*(X)$. Then*

(i) The v_i generate $A_*(X)$, i.e. any $z \in A_*(X)$ has the form $\sum (u_i \cdot z) v_i$.

(ii) Numerical and rational equivalence coincide on X. In particular, $A_*(X)$ is a free $\mathbb{Z}$-module.

(iii) If $k = \mathbb{C}$, then the cycle map $cl_X : A_*(X) \to H_*(X, \mathbb{Z})$ is an isomorphism. In particular, the homology and cohomology groups of X vanish in odd degrees. $\square$

Now consider a smooth complex algebraic variety X with an action of a complex algebraic torus T. Together with a cycle map $cl_X : A^*(X) \to H^*(X, \mathbb{Z})$ (which doubles degrees [Fu, Corollary 19.2]), there is also an equivariant cycle map $cl_X^T : A_T^*(X) \to H_T^*(X, \mathbb{Z})$ where $H_T^*(X, \mathbb{Z})$ denotes equivariant cohomology with integral coefficients, see [EG1, Section 2.8]. Next is a version of Theorem 2.10 (iii) for cl_X^T.

Proposition 2.11. *Let X be a smooth complete complex T-variety. If the class of the diagonal $\Delta(X) \subseteq X \times X$ is in the image of the Künneth map $A_*(X) \otimes A_*(X) \to A_*(X \times X)$, then the equivariant cycle map*

$$cl_X^T : A_T^*(X) \to H_T^*(X, \mathbb{Z})$$

is an isomorphism. In particular, this holds if X is T-linear.

Proof. In view of Theorem 2.10 (iii), the given hypothesis on X imply that cl_X is an isomorphism, and that X has no integral cohomology in odd degrees. Then the spectral sequence associated to the fibration $X \times_T ET \to BT$ collapses, where $ET \to BT$ is the universal T-bundle. So the S-module $H_T^*(X, \mathbb{Z})$ is free, and the map $H_T^*(X, \mathbb{Z})/\Delta H_T^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is an isomorphism. These results, together with the graded Nakayama Lemma, yield surjectivity of the equivariant cycle map $cl_X^T : A_T^*(X) \to H_T^*(X, \mathbb{Z})$. To show injectivity, we proceed as follows. First, choose a basis $z_1, .., z_n$ of $H^*(X, \mathbb{Z})$. Now identify that basis with a basis of $A^*(X)$ (via cl_X) and lift it to a generating system of the S-module $A_T^*(X)$. Then this generating system is a basis, since its image under the equivariant cycle map is a basis of $H_T^*(X, \mathbb{Z})$.
For the last assertion of the proposition, simply recall that if X is T-linear, then the Künneth map is an isomorphism [To, Proposition 1]. □

2.5 Equivariant Localization

Let T be an algebraic torus. The following is the localization theorem for equivariant Chow groups [Br2, Corollary 2.3.2].

THEOREM 2.12. *Let X be a T-scheme. If X is locally linearizable, then the S-linear map $i_{T*} : A_*^T(X^T) \to A_*^T(X)$ is an isomorphism after inverting all non-zero elements of Δ.* □

For later use, we prove a slightly more general statement.

PROPOSITION 2.13. *Let X be a T-scheme, and let $H \subset T$ be a closed subgroup. Then the S-linear map $i_{H*} : A_*^T(X^H) \to A_*^T(X)$ becomes an isomorphism after inverting finitely many characters of T that restrict non-trivially to H.*

Before proving this proposition, we need a technical lemma. We would like to thank M. Brion for suggesting a simplified proof of this fact.

LEMMA 2.14. *Let X be an affine T-scheme. Let H be a closed subgroup of T. Then the ideal of the fixed point subscheme X^H is generated by all regular functions on X which are eigenvectors of T with a weight that restricts non-trivially to H.*

Proof. Recall that X^H is the largest closed subscheme of X on which H acts trivially. In other words, the ideal I of X^H is the smallest H-stable ideal of $k[X]$ such that H acts trivially on the quotient $k[X]/I$. So I is T-stable and hence the direct sum of its T-eigenspaces. Moreover, if $f \in k[X]$ is a T-eigenvector of weight χ which restricts non-trivially to H, then $f \in I$. Indeed, let $\overline{f}$ be the image of f in $k[X]/I$. Notice that $\overline{f}$ is a T-eigenvector of the same weight χ as f. Since H acts trivially on $k[X]/I$, we obtain the identity $\overline{f} = h \cdot \overline{f} = \chi(h)\overline{f}$, valid for all $h \in H$. Nevertheless, there exists $h_0 \in H$ such that $\chi(h_0) \neq 1$, by our assumption on χ. Substituting this information into the above identity yields $\overline{f} = 0$, equivalently, $f \in I$. Thus, I contains the ideal J generated by all such functions f. But $k[X]/J$ is a trivial H-module by construction, and hence $I = J$ by minimality. □

Proof of Proposition 2.13. In virtue of Lemma 2.14, the proof is an easy adaptation of Brion's proof of [Br2, Corollary 2.3.2], so we provide only a sketch of the crucial points. First, assume that X is locally linearizable, i.e. X is a finite union of T-stable affine open subsets X_i. Lemma 2.14 implies that the ideal of each fixed point subscheme X_i^H is generated by all regular functions on X_i which are eigenvectors of T with a weight that restricts non-trivially to H. Choose a finite set of such generators (f_{ij}), with respective weights χ_{ij}. From Theorem 2.6 we know that the S-module $A_*^T(X)$ is generated by the classes of T-invariant subvarieties of X. Now let $Y \subset X$ be a T-invariant subvariety of positive dimension. If Y is not fixed pointwise by H, then one of the f_{ij} defines a non-zero rational function on Y. Then, in the Chow group, we have $\chi_{ij}[Y] = [\mathrm{div}_Y f_{ij}]$. So after inverting χ_{ij}, we get $[Y] = \chi_{ij}^{-1}[\mathrm{div}_Y f_{ij}]$. Arguing by induction on the dimension of Y, we obtain that i_* becomes surjective after inverting the χ_{ij}'s. A similar argument, using these χ_{ij}'s in the proof of [Br2, Corollary 2.3.2], shows that i^* is injective after localization.

Finally, if X is not locally linearizable, choose an equivariant birational envelope $\pi : \tilde{X} \to X$, where $\tilde{X}$ is normal (and possibly not irreducible). Let $U \subset X$ be the open subset where π is an isomorphism. Set $Z = X \setminus U$ and $E = \pi^{-1}(Z)$. Then, by [FMSS, Lemma 2] and [EG2, Lemma 7.2], there is a commutative diagram

$$\begin{tikzcd}
A_*^T(E^H) \arrow[r] \arrow[d, "i_{H*}"] & A_*^T(Z^H) \oplus A_*^T(\tilde{X}^H) \arrow[r] \arrow[d, "i_{H*}"] & A_*^T(X^H) \arrow[r] \arrow[d, "i_{H*}"] & 0 \\
A_*^T(E) \arrow[r] & A_*^T(Z) \oplus A_*^T(\tilde{X}) \arrow[r] & A_*^T(X) \arrow[r] & 0.
\end{tikzcd}$$

Observe that E and Z have strictly smaller dimension than X. Moreover, E and $\tilde{X}$ are locally linearizable. Applying Noetherian induction and the previous part of the proof, we get that the first two left vertical maps become isomorphisms after localization; hence so does the third one. ☐

3 Equivariant Kronecker duality and Localization

3.1 Equivariant Kronecker duality schemes

Definition 3.1. Let X be a complete T-scheme. We say that X satisfies *T-equivariant Kronecker duality* if the following conditions hold:

(i) $A_*^T(X)$ is a finitely generated S-module.

(ii) The equivariant Kronecker duality map

$$\mathcal{K}_T : A_T^*(X) \longrightarrow \mathrm{Hom}_S(A_*^T(X), S) \qquad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

is an isomorphism of S-modules.

Likewise, we say that X satisfies *rational* T-equivariant Kronecker duality if the $S_{\mathbb{Q}}$-modules $A_*^T(X)_{\mathbb{Q}}$ and $A_T^*(X)_{\mathbb{Q}}$ satisfy the conditions (i) and (ii) with rational coefficients.

Remark 3.2. The equivariant Kronecker duality map is functorial for morphisms between complete T-schemes. Indeed, let $f : \tilde{X} \to X$ be an equivariant (proper) morphism of complete T-schemes. For any $\xi \in A_T^*(X)$, we have

$$\int_{\tilde{X}} f^*(\xi) \cap z = \int_X f_*(f^*(\xi) \cap z) = \int_X (\xi \cap f_*(z)),$$

due to the projection formula [Fu]. This identity implies the commutativity of the diagram

$$\begin{tikzcd} A_T^*(X) \arrow[r, "f^*"] \arrow[d, "\mathcal{K}_T"'] & A_T^*(\tilde{X}) \arrow[d, "\mathcal{K}_T"] \\ \mathrm{Hom}_S(A_*^T(X), S) \arrow[r, "(f_*)^t"] & \mathrm{Hom}_S(A_*^T(\tilde{X}), S), \end{tikzcd}$$

where $(f_*)^t$ is the transpose of $f_* : A_*^T(\tilde{X}) \to A_*^T(X)$.

It follows from Definition 3.1 that if X satisfies T-equivariant Kronecker duality, then the S-module $A_T^*(X)$ is finitely generated and torsion free. In particular, if T is one dimensional, i.e. $T = \mathbb{G}_m$, then $A_{\mathbb{G}_m}^*(X)$ is a finitely generated free module over $A_{\mathbb{G}_m}^* = \mathbb{Z}[t]$. Moreover, if X is projective and smooth, then $A_*(X^{\mathbb{G}_m})$ is a finitely generated free abelian group (Lemma 2.9).

As it stems from the previous paragraph, not all smooth varieties with a torus action satisfy Equivariant Kronecker duality. For a more concrete example, consider the trivial action of T on a projective smooth curve. In this case, one checks that $\mathcal{K}_T$ is an extension of the non-equivariant Kronecker duality map $\mathcal{K}$. But, as pointed out in [FMSS], the kernel of $\mathcal{K}$ in degree one is the Jacobian of the curve, which is non-trivial if the curve has positive genus.

Lemma 3.3. *Let X be a smooth complete T-variety. Then X satisfies rational T-equivariant Kronecker duality if and only if it satisfies the rational non-equivariant Kronecker duality, i.e. $\mathcal{K} : A^i(X)_{\mathbb{Q}} \to \mathrm{Hom}(A_i(X), \mathbb{Q})$ is an isomorphism for all i. If, moreover, X is projective and $A_*(X^T)$ is $\mathbb{Z}$-free, then the equivalence holds over the integers.*

Proof. Both assertions are proved similarly, so we focus on the second one. Since X is smooth and projective, the assumption on $A_*(X^T)$ implies that $A_*^T(X)$ is a free S-module (Lemma 2.9; cf. Theorem 2.8). Now, by Poincaré duality [EG1, Proposition 4], $A_T^*(X)$ is isomorphic to $A_*^T(X)$; so it is also a free S-module. By the graded Nakayama lemma, $\mathcal{K}_T$ is an isomorphism if and only if

$$\overline{\mathcal{K}_T} : A_T^*(X)/\Delta A_T^*(X) \to \mathrm{Hom}_S(A_*^T(X), S)/\Delta \mathrm{Hom}_S(A_*^T(X), S)$$

is an isomorphism. But freeness of $A_*^T(X)$ yields an isomorphism

$$\mathrm{Hom}_S(A_*^T(X),S)/\Delta\mathrm{Hom}_S(A_*^T(X),S) \simeq \mathrm{Hom}(A_*^T(X)/\Delta A_*^T(X),\mathbb{Z})$$

and the later identifies to $\mathrm{Hom}(A_*(X),\mathbb{Z})$, by Theorem 2.6. On the other hand, by Theorem 2.6 again, the map $A_T^*(X)/\Delta A_T^*(X) \to A^*(X)$ is an isomorphism. These facts, together with the commutativity of the diagram below

$$\begin{tikzcd} A_T^*(X) \arrow[r, "\mathcal{K}_T"] \arrow[d] & \mathrm{Hom}_S(A_*^T(X),S) \arrow[d] \\ A^*(X) \arrow[r, "\mathcal{K}"] & \mathrm{Hom}(A_*(X),\mathbb{Z}), \end{tikzcd}$$

yield the content of the lemma. □

Next we show that complete T-linear schemes satisfy equivariant Kronecker duality. For this, the main ingredient is the following result, due to Totaro [To] in the non-equivariant case. Recall that a T-scheme X is said to satisfy the *equivariant Künneth formula* if the Künneth map (or exterior product [EG1])

$$A_*^T(X) \otimes_S A_*^T(Y) \to A_*^T(X \times Y)$$

is an isomorphism for *any* T-scheme Y.

Proposition 3.4. *Let X be a T-scheme. If X is T-linear, then it satisfies the equivariant Künneth formula.*

Proof. When X is T-linear, one can choose representations V of T so that X_T is linear, see e.g. [Br2, Section 2.2] and [P, Section 1]. Now the result follows from [To, Proposition 1]. □

Proposition 3.5. *If X is a complete T-linear scheme, then the equivariant Kronecker map*

$$\mathcal{K}_T : A_T^*(X) \to \mathrm{Hom}_S(A_*^T(X),S)$$

is an isomorphism.

This result follows quite formally from Proposition 3.4, as in the non-equivariant case [FMSS, Theorem 3], so we only sketch the proof. To define the inverse to $\mathcal{K}_T$, given a S-module homomorphism $\varphi : A_*^T(X) \to S$, we construct an element $c_\varphi \in A_T^*(X)$. Since the S-module $A_*^T(X)$ is finitely generated, we can assume, without loss of generality, that φ is homogeneous [Bo, Part II, Section 11.6]. Bearing this in mind, given a homomorphism $\varphi : A_*^T(X) \to S$ of degree $-\lambda$, we build $c_\varphi \in A_T^\lambda(X)$ as follows. For a T-map $f : Y \to X$, the corresponding homomorphism $f^*c_\varphi := c_\varphi(f) : A_*^T(Y) \to A_{*-\lambda}^T(Y)$ is defined to be the composite

$$A_*^T(Y) \xrightarrow{(\gamma_f)_*} A_*^T(X \times Y) \xrightarrow{\simeq} A_*^T(X) \otimes_S A_*^T(Y) \xrightarrow{\varphi\otimes\mathrm{id}} S \otimes_S A_*^T(Y) \simeq A_*^T(Y),$$

where $(\gamma_f)_*$ denotes the proper pushforward along the graph of f, and the second displayed map is the Künneth isomorphism (Proposition 3.4). The verification that c_φ satisfies the compatibility axioms, and that this construction indeed gives the inverse to $\mathcal{K}_T$, is the same as in [FMSS].

For $c \in A_T^\lambda(X)$ and $z \in A_*^T(X)$, we write $c(z)$ for $\deg(c \cap z)$. The next two corollaries describe the cap and cup product structures. They are easily deduced from the previous proposition, cf. [FMSS, Corollaries 1 and 2].

Corollary 3.6. *Let $f : Y \to X$, $c \in A_T^\lambda(X)$, $z \in A_m^T(Y)$. Suppose $(\gamma_f)_*(z) = \sum u_i \otimes v_i$ with $u_i \in A_{p(i)}^T(X)$ and $v_i \in A_{m-p(i)}^T(Y)$. Then $f^*c \cap z = \sum_{p(i)\leq\lambda} c(u_i)v_i$.* □

Corollary 3.7. *Let $c \in A_T^\lambda(X)$, $c' \in A_T^\mu(X)$, and $z \in A_m^T(X)$, where $m \leq \lambda+\mu$. Write $\delta_*(z) = \sum u_i \otimes v_i$ with $u_i \in A_{p(i)}^T(X)$ and $v_i \in A_{m-p(i)}^T(X)$. Then $(c \cup c')(z) = \sum_{m-\mu\leq p(i)\leq\lambda} c(u_i)c'(v_i)$.* □

For a T-scheme X, there is a natural map $\iota^* : A_T^*(X) \to A^*(X)$ (Section 2.1). In general, when X is singular, ι^* may not be surjective, and its kernel may not be generated in degree one [KP]. Next we describe a class of possibly singular T-schemes for which the map ι^* is well-behaved. This yields the compatibility of our product formulas with those of [FMSS].

Corollary 3.8. *Let X be a complete T-scheme. If X is T-linear and $A_*^T(X)$ is S-free, then the map $A_T^*(X)/\Delta A_T^*(X) \to A^*(X)$, induced by ι^*, is an isomorphism.*

Proof. Proposition 3.5 together with freeness of $A_*^T(X)$ yield

$$\begin{aligned} A_T^*(X)/\Delta A_T^*(X) &\simeq \mathrm{Hom}_S(A_*^T(X), S)/\Delta\mathrm{Hom}_S(A_*^T(X), S) \\ &\simeq \mathrm{Hom}_{\mathbb{Z}}(A_*^T(X)/\Delta A_*^T(X), \mathbb{Z}). \end{aligned}$$

Furthermore, by Theorem 2.6, the term on the right hand side corresponds to $\mathrm{Hom}(A_*(X), \mathbb{Z})$, which, in turn, is isomorphic to $A^*(X)$, due to the non-equivariant version of Kronecker duality [To, Proposition 1]. Considering this information alongside the commutative diagram

$$\begin{tikzcd} A_T^*(X) \arrow[r, "\mathcal{K}_T"] \arrow[d] & \mathrm{Hom}_S(A_*^T(X), S) \arrow[d] \\ A^*(X) \arrow[r, "\mathcal{K}"] & \mathrm{Hom}(A_*(X), \mathbb{Z}), \end{tikzcd}$$

produces the content of the corollary. □

The conditions of Corollary 3.8 are satisfied by possibly singular T-cellular varieties (e.g. Schubert varieties). With $\mathbb{Q}$-coefficients, the corresponding statement is satisfied by $\mathbb{Q}$-filtrable spherical varieties [G4]. This class includes all rationally smooth projective equivariant embeddings of reductive groups [G4].

3.2 Localization for T-equivariant Kronecker duality schemes

From the viewpoint of algebraic torus actions, the main attribute of equivariant Kronecker duality schemes is that they supply a somewhat more intrinsic background for establishing localization theorems on integral equivariant Chow cohomology.

Theorem 3.9. *Let X be a complete T-scheme satisfying T-equivariant Kronecker duality. Let $H \subset T$ be a subtorus of T and let $i_H : X^H \to X$ be the inclusion of the fixed point subscheme. If X^H also satisfies T-equivariant Kronecker duality, then the morphism*

$$i_H^* : A_T^*(X) \to A_T^*(X^H)$$

becomes an isomorphism after inverting finitely many characters of T that restrict non-trivially to H. In particular, i_H^* is injective over $\mathbb{Z}$.

Proof. By Proposition 2.13, the localized map $(i_H*)_{\mathcal{F}} : A_*^T(X^H)_{\mathcal{F}} \to A_*^T(X)_{\mathcal{F}}$ is an isomorphism, where $\mathcal{F}$ is a finite family of characters of T that restrict non-trivially to H.
Now consider the commutative diagram

$$\begin{tikzcd}
A_T^*(X) \arrow[r, "i_H^*"] \arrow[d] & A_T^*(X^H) \arrow[d] \\
\mathrm{Hom}_S\left(A_*^T(X), S\right) \arrow[r, "(i_{H*})^t"] & \mathrm{Hom}_S\left(A_*^T(X^H), S\right),
\end{tikzcd}$$

where $(i_{H*})^t$ represents the transpose of $i_{H*} : A_*^T(X^H) \to A_*^T(X)$ (commutativity follows from Remark 3.2, because i_H is proper). By our assumptions on X and X^H, both vertical maps are isomorphisms. Moreover, after localization at $\mathcal{F}$, the above commutative diagram becomes

$$\begin{tikzcd}
A_T^*(X)_{\mathcal{F}} \arrow[r, "(i_H^*)_{\mathcal{F}}"] \arrow[d] & A_T^*(X^H)_{\mathcal{F}} \arrow[d] \\
(\mathrm{Hom}_S\left(A_*^T(X), S\right))_{\mathcal{F}} \arrow[r, "((i_{H*})^t)_{\mathcal{F}}"] & (\mathrm{Hom}_S(A_*^T(X^H), S))_{\mathcal{F}}.
\end{tikzcd}$$

Since $A_*^T(X)$ is a finitely generated S-module (as X satisfies equivariant Kronecker duality), localization commutes with formation of Hom (see [Ei, Prop. 2.10, p. 69]), and so

$$A_T^*(X)_{\mathcal{F}} \simeq (\mathrm{Hom}_S\left(A_*^T(X), S\right))_{\mathcal{F}} \simeq \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X)_{\mathcal{F}}, S_{\mathcal{F}}).$$

Similarly, for X^H we obtain

$$A_T^*(X^H)_{\mathcal{F}} \simeq (\mathrm{Hom}_S(A_*^T(X^H), S))_{\mathcal{F}} \simeq \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X^H)_{\mathcal{F}}, S_{\mathcal{F}}).$$

In other words, the bottom map from the preceding diagram fits in the commutative square

$$\begin{tikzcd}
(\mathrm{Hom}_S\,(A_*^T(X),S))_{\mathcal{F}} \arrow[r, "((i_{H*})^t)_{\mathcal{F}}"] \arrow[d] & (\mathrm{Hom}_S(A_*^T(X^H),S))_{\mathcal{F}} \arrow[d] \\
\mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X)_{\mathcal{F}},S_{\mathcal{F}}) \arrow[r, "((i_{H*})_{\mathcal{F}})^t"] & \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X^H)_{\mathcal{F}},S_{\mathcal{F}}),
\end{tikzcd}$$

where the vertical maps are natural isomorphisms. But we already know that $(i_{H*})_{\mathcal{F}}$ is an isomorphism, hence so are $((i_{H*})_{\mathcal{F}})^t$, $((i_{H*})^t)_{\mathcal{F}}$ and $(i_H^*)_{\mathcal{F}}$.

Finally, to prove the last assertion of the theorem, recall that the S-module $A_T^*(X)$ is finitely generated and torsion free (Definition 3.1). Hence the natural map $A_T^*(X) \to A_T^*(X) \otimes_S \mathcal{Q}$ is injective, where $\mathcal{Q}$ is the quotient field of S. In particular, the (also natural) map $A_T^*(X) \to A_T^*(X)_{\mathcal{F}}$ is injective. This, together with the first part of the theorem, yields injectivity of i_H^*. We are done. □

COROLLARY 3.10. *Let X be a complete T-scheme. Let H be a codimension-one subtorus of T. If X is T-linear, then the pullback $i_H^* : A_T^*(X) \to A_T^*(X^H)$ is injective over $\mathbb{Z}$.*

Proof. If X is T-linear, then so is X^H. Now use Proposition 3.5 and Theorem 3.9. □

Let X be a complete T-linear scheme. It follows from Corollary 3.10 that the image of the *injective* map $i_T^* : A_T^*(X) \to A_T^*(X^T)$ is contained in the intersection of the images of the (also injective) maps $i_{T,H}^* : A_T^*(X^H) \to A_T^*(X^T)$, where H runs over all subtori of codimension one in T. When the image of i_T^* is exactly the intersection of the images of the maps $i_{T,H}^*$ we say, following [G3], that X has the *Chang-Skjelbred property* (or *CS property*). If the defining condition holds over $\mathbb{Q}$ rather than $\mathbb{Z}$, we say that X has the *rational CS property.* By Theorem 2.8, any smooth complete T-scheme has the rational CS property; by Theorem A.6, so does any complete T-scheme in characteristic zero. It would be interesting to determine, in arbitrary characteristic, which complete, possibly singular, T-linear schemes satisfy the CS property. For instance, toric varieties are known to have this property [P]. We anticipate that this also holds for projective embeddings of semisimple groups of adjoint type (this shall be pursued elsewhere). For the corresponding problem with rational coefficients, we provide an answer next.

THEOREM 3.11. *Let X be a complete T-linear scheme. If there exists an equivariant envelope $\pi : \tilde{X} \to X$ with $\tilde{X}$ smooth, then X has the rational CS property. In particular, projective embeddings of connected reductive linear algebraic groups have the rational CS property in arbitrary characteristic.*

Proof. Let $u \in A_T^*(X^T)_{\mathbb{Q}}$ be such that $u \in \bigcap_{H \subset T} \operatorname{Im}(i_{T,H}^*)$, where the intersection runs over all codimension-one subtori H of T. Our task is to show that $u \in \operatorname{Im}(i_T^*)_{\mathbb{Q}}$. First, observe that there is a commutative diagram

$$\begin{tikzcd}
 & A_T^*(X)_{\mathbb{Q}} \arrow[rr, "\pi^*"] \arrow[d, "i_T^*"] \arrow[dl, "i_H^*"'] & & A_T^*(\tilde{X})_{\mathbb{Q}} \arrow[d, "\widetilde{i_T^*}"] \arrow[dl, "\widetilde{i_H^*}"'] \\
 & A_T^*(X^T)_{\mathbb{Q}} \arrow[rr, "{\pi_T}^*"] & & A_T^*(\tilde{X}^T)_{\mathbb{Q}} \\
A_T^*(X^H)_{\mathbb{Q}} \arrow[ur, "i_{T,H}^*"'] \arrow[rr, "{\pi_H}^*"] & & A_T^*(\tilde{X}^H)_{\mathbb{Q}} \arrow[ur, "\widetilde{i_{T,H}^*}"'] &
\end{tikzcd}$$

obtained by combining and comparing the sequences that [EG2, Lemma 7.2] and Theorem A.1 assign to the envelopes $\pi : \tilde{X} \to X$, $\pi_H : \tilde{X}^H \to X^H$ and $p_T : \tilde{X}^T \to X^T$ (cf. [G3, proof of Theorem 4.4]. From the diagram it follows that ${\pi_T}^*(u)$ is in the image of $\widetilde{i_{T,H}^*}$. Hence, ${\pi_T}^*(u)$ is in the intersection of the images of all $\widetilde{i_{T,H}^*}$, where H runs over all codimension-one subtori of T. Since $\tilde{X}$ is known to have the rational CS property (Theorem 2.8), ${\pi_T}^*(u)$ is in the image of $\widetilde{i_T^*}$. So let $y \in A_T^*(\tilde{X})$ be such that $\tilde{i}_T^*(y) = \pi_T^*(u)$. To conclude the proof, we need to check that y is in the image of π^*. In view of equivariant Kronecker duality (Proposition 3.5), this is equivalent to checking that the dual of y, namely, $\tilde{\varphi} := \mathcal{K}_T(y)$ is in the image of π_*^t, the transpose of the surjective morphism $\pi_* : A_*^T(\tilde{X})_{\mathbb{Q}} \to A_*^T(X)_{\mathbb{Q}}$. Also, we should observe that the functor $\mathcal{K}_T(-)$ transforms the previous commutative diagram into another one involving the corresponding dual modules $\operatorname{Hom}(A_*^T(-)_{\mathbb{Q}}, S_{\mathbb{Q}})$. Now set $\varphi_u := \mathcal{K}_T(u)$. By construction, for every codimension-one subtorus H, there exists $\varphi_u^H : A_*^T(X^H)_{\mathbb{Q}} \to S_{\mathbb{Q}}$ such that $\varphi_u = \varphi_u^H \circ {i_{T,H}}_*$. In fact, we can place this information into a commutative diagram:

$$\begin{tikzcd}
A_*^T(\tilde{X^T})_{\mathbb{Q}} \arrow[r, "\widetilde{{i_{T,H}}_*}"] \arrow[d, "\pi_*^T"] & A_*^T(\tilde{X}^H)_{\mathbb{Q}} \arrow[r] \arrow[d, "\pi_*^H"] & A_*^T(\tilde{X})_{\mathbb{Q}} \arrow[d, "\pi_*"] \arrow[dl] \\
A_*^T(X^T)_{\mathbb{Q}} \arrow[r, "{i_{T,H}}_*"] \arrow[dr, "\varphi_u"] & A_*^T(X^H)_{\mathbb{Q}} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\varphi_u^H"] & A_*^T(X)_{\mathbb{Q}} \arrow[dl, dashed, "\varphi\ (\exists ?)"] \\
 & S_{\mathbb{Q}}. &
\end{tikzcd}$$

In this form, our task reduces to showing that there exists φ making the dotted arrow into a solid arrow. Bearing this in mind, we claim that $\tilde{\varphi}$ is zero on the kernel of π_*. Indeed, let $v \in A_*^T(\tilde{X})_{\mathbb{Q}}$ be such that $\pi_*(v) = 0$. By the localization theorem there exists a product of non-trivial characters $\chi_1 \cdots \chi_m$ such that $\chi_1 \cdots \chi_m \cdot v$ is in the image of $\tilde{i}_*^T : A_*^T(\tilde{X}^T)_{\mathbb{Q}} \to A_*^T(\tilde{X})_{\mathbb{Q}}$. As both of these S-modules are free, then, unless v is zero, we have $\chi_1 \cdots \chi_m \cdot v \neq 0$. Let w be such that $\tilde{i}^T_*(w) = \chi_1 \cdot \chi_m \cdot v$. By commutativity of the diagram, $i^T_*(\pi^T_*(w)) = 0$. But i^T_* is injective by Theorem 3.9, so $\pi^T_*(w) = 0$. Thus

$$\tilde{\varphi}(\chi_1 \cdots \chi_m \cdot v) = \chi_1 \cdots \chi_m \cdot \tilde{\varphi}(v) = \varphi_u(\pi^T_*(w)) = 0.$$

As $S_{\mathbb{Q}}$ has no torsion, we get $\tilde{\varphi}(v) = 0$, which proves the claim. Using this, one easily defines φ with the sought-after properties.
Finally, for the last assertion of the theorem, recall that group embeddings are a special class of spherical varieties known to have resolutions of singularities in arbitrary characteristic [BK, Chapter 6]. □

Remark 3.12. Theorem 3.11 and its proof can be readily translated into the language of equivariant operational K-theory with $\mathbb{Z}$-coefficients. See [G3, Section 6] for a presentation of the (integral) equivariant operational K-theory of projective group embeddings.

4 Rational equivariant Chow cohomology of spherical varieties

Throughout this section we work in characteristic zero. The aim is to describe the rational equivariant Chow cohomology of a spherical variety by comparing it with that of an equivariant resolution, using the rational CS property (Theorem 3.11) and equivariant Kronecker duality (Proposition 3.5). The main result (Theorem 4.8), inspired by [Br2, Theorem 7.3], is an extension of Brion's description to the setting of equivariant operational Chow groups.

In what follows, we denote by G a connected reductive linear algebraic group with Borel subgroup B and maximal torus $T \subset B$. We denote by W the Weyl group of (G, T). Observe that W is generated by reflections $\{s_\alpha\}_{\alpha \in \Phi}$, where Φ stands for the set of roots of (G, T). Recall that $S_{\mathbb{Q}}^W = (A^*_{T\mathbb{Q}})^W = A^*_G \otimes \mathbb{Q}$. For the purposes of this section, we shall assume that G-spherical varieties are *locally linearizable* for the induced T-action.

4.1 Preliminaries

Recall that any spherical G-variety contains only finitely many G-orbits; as a consequence, it contains only finitely many fixed points of T. Moreover, since $\mathrm{char}(\Bbbk) = 0$, any spherical G-variety X admits an equivariant resolution of singularities, i.e., there exists a smooth G-variety $\tilde{X}$ together with a proper birational G-equivariant morphism $\pi : \tilde{X} \to X$. Then the G-variety $\tilde{X}$ is also spherical; if moreover X is complete, we may arrange so that $\tilde{X}$ is projective. Notice that, in general, a resolution of singularities need not be an equivariant envelope. The next result gives an important class of spherical varieties for which equivariant resolutions *are* equivariant envelopes. We thank M. Brion for leading us to the following proof.

PROPOSITION 4.1. *Let X be a normal simply-connected spherical G-variety (i.e. the B-isotropy group of its dense orbit is connected). Let $f : \tilde{X} \to X$ be a proper birational morphism. Then f is an equivariant envelope.*

Proof. It suffices to show that every B-orbit in X is the isomorphic image via f of a B-orbit in $\tilde{X}$. So let $\mathcal{O} = (B) \cdot x = B/B_x$ be an orbit in X. It follows from [BJ1] that $\mathcal{O}$ has a connected isotropy group. The preimage $f^{-1}(\mathcal{O}) \subset \tilde{X}$ is of the form $B \times^{B_x} F$, where F denotes the fiber $p^{-1}(x)$. Since F is connected and complete (by Zariski's main theorem), it contains a fixed point y of the connected solvable group B_x. Then the orbit $B \cdot y$ in $\tilde{X}$ is mapped isomorphically to $B \cdot x$. □

REMARK 4.2. Examples of simply-connected spherical varieties include (normal) $G \times G$-equivariant embeddings of G.

Now we record a few notions and results from [Br2, Section 7] needed in our task. A subtorus $H \subset T$ is called *regular* if its centralizer $C_G(H)$ is equal to T; otherwise H is called *singular*. A subtorus of codimension one is singular if and only if it is the kernel of some positive root α. In this case, α is unique and the group $C_G(H)$ is the product of H with a subgroup Γ isomorphic to SL_2 or PSL_2. Furthermore, if X is a G-variety, then X^H inherits an action of $C_G(H)/H$, a quotient of Γ.

PROPOSITION 4.3 ([Br2, Proposition 7.1]). *Let X be a spherical G-variety. Let $H \subset T$ be a subtorus of codimension one. Then each irreducible component of X^H is a spherical $C_G(H)$-variety. Moreover,*

(i) If H is regular, then X^H is at most one-dimensional.

(ii) If H is singular, then X^H is at most two-dimensional. If moreover X is complete and smooth, then any two-dimensional connected component of X^H is (up to a finite, purely inseparable equivariant morphism) either a rational ruled surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, where $C_G(H)$ acts through the natural action of SL_2, or the projective plane, where $C_G(H)$ acts through the projectivization of a non-trivial SL_2-module of dimension three. □

For later use, we record Brion's presentation of the rational equivariant Chow rings of nonsingular ruled surfaces. We follow closely the notation and conventions of [Br2, Section 7]. Let D be the torus of diagonal matrices in SL_2, and let α be the character of D given by

$$\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2.$$

We identify the character ring of D with $\mathbb{Q}[\alpha]$. Now consider a rational ruled surface $\mathbb{F}_n$ with ruling $\pi : \mathbb{F}_n \to \mathbb{P}^1$. Notice that $\mathbb{F}_n$ has exactly four fixed points x, y, z, t of D, where x, y (resp. z, t) are mapped to 0 (resp. ∞) by π.

Moreover, we may assume that x and z lie in one G-invariant section of π, and that y and t lie in the other G-invariant section. With this ordering of the fixed points, we identify $A_T^*(\mathbb{F}_n^D)_{\mathbb{Q}}$ with $\mathbb{Q}[\alpha]^4$. In contrast, denote by $\mathbb{P}(V)$ the projectivization of a nontrivial SL_2-module V of dimension three. The weights of D in V are either -2α, 0, 2α (in the case when $V = sl_2$) or $-\alpha$, 0, α (in the case when $V = k^2 \oplus k$). We denote by x, y, z the corresponding fixed points of D in $\mathbb{P}(V)$, and we identify $A_T^*(\mathbb{P}(V)^D)_{\mathbb{Q}}$ with $\mathbb{Q}[\alpha]^3$.

Proposition 4.4. *Notation being as above, the image of*

$$i_D^* : A_D^*(\mathbb{F}_n)_{\mathbb{Q}} \to \mathbb{Q}[\alpha]^4$$

consists of all (f_x, f_y, f_z, f_t) *such that* $f_x \equiv f_y \equiv f_z \equiv f_t \mod \alpha$ *and* $f_x - f_y + f_z - f_t \equiv 0 \mod \alpha^2$. *On the other hand, the image of*

$$i_D^* : A_D^*(\mathbb{P}(V))_{\mathbb{Q}} \to \mathbb{Q}[\alpha]^3$$

consists of all (f_x, f_y, f_z) *such that* $f_x \equiv f_y \equiv f_z \mod \alpha$ *and* $f_x - 2f_y + f_z \equiv 0 \mod \alpha^2$. □

4.2 T-equivariant Chow cohomology

Let X be a complete possibly singular spherical G-variety. Let $H \subset T$ be a singular subtorus of codimension one. In order to obtain an explicit description of $A_T^*(X)_{\mathbb{Q}}$ out of Theorem 3.11 we need to determine which $C_G(H)$-spherical surfaces could appear as irreducible components of X^H. We will do this by means of Proposition 4.3. (This is the only case of interest to us, for if H is regular, then X^H is T-skeletal, and GKM theory applies, see Appendix.) So let Y be a two-dimensional irreducible component of X^H. By [EG2, Proposition 7.5] we may find a proper birational equivariant morphism $X' \to X$ with X' smooth and G-spherical. Thus Y is the image of some irreducible component Y' of X'^H (by Borel's fixed point theorem). Given that X is complete, so is X' and, under our considerations, Y' is a two-dimensional complete $C_G(H)$-spherical variety. Hence Y' is either the projective plane or a rational ruled surface (up to a finite, purely inseparable equivariant morphism). We inspect these two cases in more detail.

(a) If $Y' = \mathbb{P}^2$, then the normalization $\tilde{Y}$ of Y is also $\mathbb{P}^2$ (up to a finite, purely inseparable equivariant morphism, which is, in particular, bijective).
(b) If Y' is a rational ruled surface, then the normalization $\tilde{Y}$ of Y is either (i) Y' or (ii) the surface obtained by contracting the unique section $\mathcal{C}$ of negative self-intersection in Y' (this is a very special weighted projective plane).

Notice that, except for case (b)-(ii), the normalization $\tilde{Y}$ of Y is a smooth projective surface with finitely many T-fixed points. In such cases, it readily follows that $A_T^*(\tilde{Y})_{\mathbb{Q}}$ is free of rank $|\tilde{Y}^T|$ (Theorem 2.8). We show that this property also holds in case (b)-(ii).

Lemma 4.5. *Let $P_n = \mathbb{F}_n/\mathcal{C}$ be the weighted projective plane obtained by contracting the unique section $\mathcal{C}$ of negative self-intersection in $\mathbb{F}_n$. Then $A_*^T(P)_{\mathbb{Q}}$ is a free $S_{\mathbb{Q}}$-module of rank three. Hence, $A_T^*(P)_{\mathbb{Q}}$ is also $S_{\mathbb{Q}}$-free of rank three.*

Proof. Clearly, $|P_n^T| = 3$. The associated BB-decomposition of P_n consists of three cells: a point, a copy of $\mathbb{A}^1$ and an open cell, say U, isomorphic to $\mathbb{A}^2/\mu_n$, where $\mu_n \subset D$ is the cyclic group with eigenvalues (ξ, ξ^{-1}), where ξ is a n-th root of unity. Note that $A_*(U)_{\mathbb{Q}} \simeq A_*(\mathbb{A}^2)_{\mathbb{Q}}^{\mu_n}$, and the latter identifies to $A_*(\mathbb{A}^2)_{\mathbb{Q}}$, because the action of μ_n on $\mathbb{A}^2$ is induced by the action of D (a connected group). So $A_*(U)_{\mathbb{Q}} \simeq \mathbb{Q}$. This yields the isomorphism $A_*^T(U) \simeq S_{\mathbb{Q}}$ (see e.g. [G4]). From this, and the fact that the BB-decomposition is filtrable, it easily follows that the $S_{\mathbb{Q}}$-module $A_*^T(P_n)_{\mathbb{Q}}$ is free of rank 3. Finally, the second assertion of the lemma follows from Proposition 3.5. □

Corollary 4.6. *Notation being as above, assume that $\mathcal{C}$ joins the fixed points y and t of $\mathbb{F}_n$, so that the fixed points of P_n are identified with x, y, z. Then the image of $i_D^* : A_D^*(P_n)_{\mathbb{Q}} \to \mathbb{Q}[\alpha]^3$ consists of all (f_x, f_y, f_z) such that $f_x \equiv f_y \equiv f_z \mod (\alpha)$ and $f_x - 2f_y + f_z \equiv 0 \mod (\alpha^2)$.*

Proof. Observe that $q : \mathbb{F}_n \to P_n$ is an envelope. By Theorem A.1 and Proposition A.2, the problem reduces to find the image of q^*. By Theorem A.1 again, an element $(f_x, f_y, f_z, f_t) \in A_T^*(\mathbb{F}_n)_{\mathbb{Q}}$ is in the image of q^* if and only if it satisfies the usual relations $f_x \equiv f_y \equiv f_z \equiv f_t \mod \alpha$ and $f_x - f_y + f_z - f_t \equiv 0 \mod (\alpha^2)$, plus the extra relation $f_y = f_t$ (which accounts for the fact that $\mathcal{C}$ is collapsed to a fixed point in P_n). Hence, the relation $f_x - f_y + f_z - f_t \equiv 0 \mod (\alpha^2)$ reduces to $f_x - 2f_y + f_z \equiv 0 \mod (\alpha^2)$, finishing the argument. □

Back to the general setup, let X be a G-spherical variety and let H be a singular subtorus of codimension one. Let Y be an irreducible component of X^H, and let $\pi : \tilde{Y} \to Y$ be the normalization map. By the previous analysis, we know the relations that define the image of $i_T^* : A_T^*(\tilde{Y})_{\mathbb{Q}} \to A_T^*(\tilde{Y}^T)_{\mathbb{Q}}$. We claim that $\pi^* : A_T^*(Y)_{\mathbb{Q}} \to A_T^*(\tilde{Y})_{\mathbb{Q}}$ is in fact an isomorphism. First, consider the commutative diagram

$$\begin{array}{ccc} A_T^*(Y)_{\mathbb{Q}} & \xrightarrow{\pi^*} & A_T^*(\tilde{Y})_{\mathbb{Q}} \\ \downarrow{\scriptstyle \mathcal{K}_T} & & \downarrow{\scriptstyle \mathcal{K}_T} \\ \mathrm{Hom}_{S_{\mathbb{Q}}}(A_*^T(Y), S_{\mathbb{Q}}) & \xrightarrow{(\pi_*)^t} & \mathrm{Hom}_{S_{\mathbb{Q}}}(A_T^*(\tilde{Y})_{\mathbb{Q}}, S_{\mathbb{Q}}). \end{array}$$

where the vertical maps are isomorphisms because of equivariant Kronecker duality (Proposition 3.5), and $(\pi_*)^t$ represents the transpose of the surjective map $\pi_* : A_*^T(\tilde{Y})_{\mathbb{Q}} \to A_*^T(Y)_{\mathbb{Q}}$ (commutativity follows from the projection formula). Thus, to prove our claim, it suffices to show that π_* is injective. In fact, since π_* is a surjective map of free $S_{\mathbb{Q}}$-modules, the problem reduces to comparing the ranks of $A_*^T(\tilde{Y})_{\mathbb{Q}}$ and $A_*^T(Y)_{\mathbb{Q}}$. If these ranks agree, we are done, for a surjective map of free $S_{\mathbb{Q}}$-modules of the same rank is an isomorphism.

Bearing this in mind, we invoke the localization theorem (Theorem 2.12): the ranks of $A_*^T(\tilde{Y})_{\mathbb{Q}}$ and $A_*^T(Y)_{\mathbb{Q}}$ are $|\tilde{Y}^T|$ and $|Y^T|$ respectively. But $|\tilde{Y}^T| = |Y^T|$ by Lemma 4.7. This yields the claim.

Lemma 4.7. *Let Y be a complete T-variety with finitely many fixed points. Let $p : \tilde{Y} \to Y$ be the normalization. If Y is locally linearizable and $\tilde{Y}$ is projective, then the normalization p induces a bijection $p_T : \tilde{Y}^T \to Y^T$ of the fixed point sets.*

Proof. Clearly, p induces a surjection $p_T : \tilde{Y}^T \to Y^T$. Arguing by contradiction, suppose that p_T is not injective. Then there are at least two different fixed points $x, y \in \tilde{Y}$ such that $p(x) = p(y)$. Now choose a T-invariant curve $\ell \subset \tilde{Y}$ passing through x and y. It follows that the image $\pi(\ell)$ is an invariant curve on Y with exactly one fixed point. But this is impossible, for the action on Y is locally linearizable (cf. [Ti, Example 4.2]). $\square$

With all the ingredients at our disposal, we are now ready to state the main result of this section. This builds on and extends Brion's result ([Br2, Theorem 7.3]) to the rational equivariant Chow cohomology of possibly singular complete spherical varieties. Our findings complement Brion's deepest results [Br2].

Theorem 4.8. *Let X be a complete G-spherical variety. The image of the injective map*

$$i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$$

consists of all families $(f_x)_{x \in X^T}$ satisfying the relations:

(i) $f_x \cong f_y \mod \chi$, whenever x, y are connected by a T-invariant curve with weight χ.

(ii) $f_x - 2f_y + f_z \equiv 0 \mod \alpha^2$ whenever α is a positive root, x, y, z lie in an irreducible component of $X^{\ker \alpha}$ whose normalization is isomorphic to $\mathbb{P}^2$ or the weighted projective plane P_n, and x, y, z are ordered as in Section 4.1.

(iii) $f_x - f_y + f_z - f_t \equiv 0 \mod \alpha^2$ whenever α is a positive root, x, y, z, t lie in an irreducible component of $X^{\ker \alpha}$ whose normalization is isomorphic to $\mathbb{F}_n$, and x, y, z, t are ordered as in Section 4.1.

Proof. In light of Theorem 3.11 and Theorem A.9, it suffices to consider the case when H is a singular codimension one subtorus, i.e. $H = \ker \alpha$, for some positive root α. Let $X^H = \bigcup_j X_j$ be the decomposition into irreducible components. Notice that each X_j is either a fixed point, a T-invariant curve or a possibly singular rational surface (by Proposition 4.3 and our previous analysis). Now, by Remark A.4, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_T^*(X^{\ker \pi})_{\mathbb{Q}} \arrow[r] \arrow[d, "i_{T,H}^*"'] & \bigoplus_i A_T^*(X_j)_{\mathbb{Q}} \arrow[r] \arrow[d] & \bigoplus_{i,j} A_T^*(X_{i,j})_{\mathbb{Q}} \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & A_T^*(X^T)_{\mathbb{Q}} \arrow[r] & \bigoplus_i A_T^*(X_j^T)_{\mathbb{Q}} \arrow[r] & \bigoplus_{i,j} A_T^*(X_{i,j}^T)_{\mathbb{Q}},
\end{tikzcd}$$

where each $X_{i,j}$ is at worst a union of fixed points and T-invariant curves. The image of the middle vertical map is completely characterized by our previous analysis, Lemma 4.5 and Corollary 4.6. Hence so is the image of $i^*_{T,H}$, as it follows from the conclusion of Remark A.4. Now apply Theorem 3.11 to conclude the proof. $\square$

Observe that cases (ii) and (iii) do not occur if X is a $G \times G$-equivariant projective embedding of G, for then X is $T \times T$-skeletal [G2]. In this situation, Theorem 4.8 yields an explicit description of $A^*_{T\times T}(X)_{\mathbb{Q}}$; this is obtained by appealing to the results of [G2], which identifies all the characters involved in case (i), and proceeding as in [G3, Section 6]. Moreover, when X is $\mathbb{Q}$-filtrable [G4], the $S_{\mathbb{Q}}$-module $A_*^{T\times T}(X)_{\mathbb{Q}}$ is free, and there are some criteria for Poincaré duality in equivariant Chow cohomology. See [G4] for details.

4.3 G-equivariant Chow cohomology

To adapt the definition of G-equivariant operational Chow groups, in this subsection we work in the category of G-quasiprojective schemes, i.e. G-schemes having an ample G-linearized invertible sheaf. This assumption is fulfilled, e.g., by G-stable subschemes of normal quasiprojective G-schemes [Su].

The following result is a synthesis of [EG1, Proposition 6] and [Vi, Note 2.5].

Proposition 4.9. *For a G-scheme X, we have $A_*^G(X)_{\mathbb{Q}} \simeq A_*^T(X)_{\mathbb{Q}}^W$ and $A^*_G(X)_{\mathbb{Q}} \simeq A^*_T(X)^W_{\mathbb{Q}}$.* $\square$

Now we further describe $A_*^G(X)_{\mathbb{Q}}$ and $A^*_G(X)_{\mathbb{Q}}$ when X is a G-spherical variety.

Proposition 4.10. *Let X be a G-scheme with finitely many B-orbits. Then for any G-scheme Y the Künneth map $A_*^G(X) \otimes_{S_{\mathbb{Q}}^W} A_*^G(Y) \to A_*^G(X \times Y)$ is an isomorphism.*

Proof. Simply argue as in [G3, Theorem A.2], using Proposition 3.4 and the fact that $S_{\mathbb{Q}}$ is a free $S_{\mathbb{Q}}^W$-module of rank $|W|$. $\square$

Proposition 4.11. *Let X be a projective G-scheme with a finite number of B-orbits. Then the G-equivariant Kronecker map*

$$\mathcal{K}_G : A^*_G(X)_{\mathbb{Q}} \longrightarrow Hom_{S_{\mathbb{Q}}^W}(A_*^G(X)_{\mathbb{Q}}, S_{\mathbb{Q}}^W) \qquad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

is a $S_{\mathbb{Q}}^W$-linear isomorphism. Moreover, we have $A^*_T(X)_{\mathbb{Q}} \simeq A^*_G(X)_{\mathbb{Q}} \otimes_{S_{\mathbb{Q}}^W} S_{\mathbb{Q}}$.

Proof. The first part is formally deduced from Proposition 4.10, as in the T-equivariant case (Proposition 3.5). The second one is obtained by adapting the proof of [G3, Corollary A.5]. $\square$

Arguing as in [Br2, Corollary 6.7.1], one obtains the next result.

Corollary 4.12. *Let X be a projective G-spherical variety. If $A_*^T(X)_{\mathbb{Q}}$ is $S_{\mathbb{Q}}$-free, then $A_G^*(X)_{\mathbb{Q}}$ is $S_{\mathbb{Q}}^W$-free and restriction to the fiber induces an isomorphism $A_G^*(X)_{\mathbb{Q}}/S_{\mathbb{Q}}^W{}_+A_G^*(X)_{\mathbb{Q}} \simeq A^*(X)_{\mathbb{Q}}$.* □

Corollary 4.12 is satisfied by $\mathbb{Q}$-filtrable spherical G-varieties [G4].

5 Further remarks

(1) *Description of the image of restriction to the fiber $i^* : A_T^*(X) \to A^*(X)$ by using equivariant multiplicities.* So far, this has been carried out for singular toric varieties [KP]. It would be interesting to obtain similar formulas for more general, possibly singular, projective group embeddings.

(2) *Understand the action of $PP_T^*(X)$ on $A_*^T(X)$ for T-skeletal spherical varieties*, in light of Brion's description of the intersection pairing between curves and divisors on spherical varieties [Br1]. This should also provide a geometric interpretation of the coefficients arising from the cap and cup product formulas (Corollaries 3.6 and 3.7). This will be pursued elsewhere.

Appendix A: Localization theorem and GKM theory for rational equivariant Chow cohomology

Here we translate the results of [G3] into the language of equivariant Chow cohomology. In that paper we studied equivariant operational K-theory, but as stated in [G3, Section 7] the results readily extend to equivariant Chow cohomology with rational coefficients. The purpose of this appendix is to supply a detailed proof of this claim, merely for the sake of completeness. From now on, we assume $\mathrm{char}(\Bbbk) = 0$.

Let $p : \tilde{X} \to X$ be a T-equivariant birational envelope which is an isomorphism over an open set $U \subset X$. Let $\{Z_i\}$ be the irreducible components of $Z = X - U$, and let $E_i = p^{-1}(Z_i)$, with $p_i : E_i \to Z_i$ denoting the restriction of p. The next theorem is Kimura's fundamental result adapted to our setup.

Theorem A.1 ([Ki, Theorem 3.1]). *Let $p : \tilde{X} \to X$ be a T-equivariant envelope. Then the induced map $p^* : A_T^*(X) \to A_T^*(\tilde{X})$ is injective. Furthermore, if p is birational (and notation is as above), then the image of p^* is described inductively as follows: a class $\tilde{c} \in A_T^*(\tilde{X})$ equals $p^*(c)$, for some $c \in A_T^*(X)$ if and only if, for all i, we have $\tilde{c}|_{E_i} = p_i^*(c_i)$ for some $c_i \in A_T^*(Z_i)$.* □

Since E_i and Z_i have smaller dimension than X, we can apply this result to compute $A_T^*(X)$ using a resolution of singularities and induction on dimension. In fact, if $\tilde{X}$ is chosen to be smooth, then $A_T^*(\tilde{X})$ corresponds to the Chow group of $\tilde{X}$ graded by codimension [EG1, Proposition 4]; thus $A_T^*(X) \subset A_T^*(\tilde{X})$ sits inside a more geometric object. Theorem A.1 is one of the reasons why Kimura's results [Ki] make operational Chow groups more computable.

In Proposition A.2, we state another crucial consequence of Kimura's work. Put in perspective, it asserts that the rational equivariant operational Chow ring $A_T^*(X)_{\mathbb{Q}}$ of *any* complete T-scheme X is a subring of $A_T^*(X^T)_{\mathbb{Q}}$. Moreover, there is a natural isomorphism $A_T^*(X^T) \simeq A^*(X^T) \otimes_{\mathbb{Z}} S$. Indeed, for a fixed degree j, [EG1, Theorem 2] yields the identifications $A_T^j(X^T) \simeq A^j((X^T \times U)/T) \simeq A^j(X^T \times (U/T))$, where U is an open T-invariant subset of a T-module V, so that the quotient $U \to U/T$ exists and is a principal T-bundle, and the codimension of $V \setminus U$ is large enough. Additionally, we can find U such that U/T is a product of projective spaces (see e.g. [EG1]). It follows that $A^*(X^T \times U/T) \simeq A^*(X^T) \otimes A^*(U/T)$, by the projective bundle formula [Fu, Example 17.5.1 (b)]. In many cases of interest, X^T is finite (e.g. for spherical varieties) and so one has $A_T^*(X)_{\mathbb{Q}} \subseteq \bigoplus_1^\ell A_T^*(X^T)_{\mathbb{Q}} = S_{\mathbb{Q}}^\ell$, where $\ell = |X^T|$. This motivated our introduction of localization techniques, and ultimately GKM theory, into the study of rational equivariant operational Chow rings and integral equivariant operational K-theory [G3].

Proposition A.2. *Let X be a T-scheme. If X is complete, then the pull-back $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is injective.*

Proof. The argument is essentially that of [G3, Proposition 3.7]. We include it for convenience. Choose a T-equivariant envelope $p : \tilde{X} \to X$, with $\tilde{X}$ smooth. It follows that $p^* : A_T^*(X) \to A_T^*(\tilde{X})$ is injective (Theorem A.1). Since $\tilde{X}$ is smooth, $\tilde{i}_T^* : A_T^*(\tilde{X})_{\mathbb{Q}} \to A_T^*(\tilde{X}^T)_{\mathbb{Q}}$ is injective (by Theorem 2.8). Now the chain of inclusions $\tilde{X}^T \subset p^{-1}(X^T) \subset \tilde{X}$ indicate that $\tilde{i}_T^*$ factors through $\iota^* : A_T^*(\tilde{X}) \to A_T^*(p^{-1}(X^T))$, where $\iota : p^{-1}(X^T) \hookrightarrow \tilde{X}$ is the natural inclusion. Thus, ι^* is injective over $\mathbb{Q}$ as well. Finally, adding this information to the commutative diagram below

$$\begin{tikzcd} A_T^*(X)_{\mathbb{Q}} \arrow[r, "p^*"] \arrow[d, "i_T^*"] & A_T^*(\tilde{X})_{\mathbb{Q}} \arrow[d, "\iota^*"] \\ A_T^*(X^T)_{\mathbb{Q}} \arrow[r, "p^*"] & A_T^*(p^{-1}(X^T))_{\mathbb{Q}}. \end{tikzcd}$$

renders $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ injective. □

Corollary A.3 ([G3, Corollary 3.8]). *Let X be a complete T-scheme. Let Y be a T-invariant closed subscheme containing X^T. Let $\iota : Y \to X$ be the natural inclusion. Then the pullback $\iota^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(Y)_{\mathbb{Q}}$ is injective. In particular, if H is a closed subgroup of T, then $i_H^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^H)_{\mathbb{Q}}$ is injective.* □

Remark A.4. Let Y be a complete T-scheme with irreducible components $Y_1, \ldots, Y_n$. Let $Y_{ij} = Y_i \cap Y_j$. By Theorem A.1 the following sequence is exact

$$0 \to A_T^*(Y)_{\mathbb{Q}} \to \bigoplus_i A_T^*(Y_i)_{\mathbb{Q}} \to \bigoplus_{i,j} A_T^*(Y_{ij})_{\mathbb{Q}}.$$

This sequence yields the commutative diagram [G3, Corollary 3.6]:

$$\begin{tikzcd}
0 \arrow[r] & A_T^*(Y)_{\mathbb{Q}} \arrow[r] \arrow[d, "i^*_{T,Y}"] & \bigoplus_i A_T^*(Y_i)_{\mathbb{Q}} \arrow[r] \arrow[d, "\oplus_i i^*_{T,Y_i}"] & \bigoplus_{i,j} A_T^*(Y_{ij})_{\mathbb{Q}} \arrow[d, "i^*_{T,Y_{i,j}}"] \\
0 \arrow[r] & A_T^*(Y^T)_{\mathbb{Q}} \arrow[r, "p"] & \bigoplus_i A_T^*(Y_i^T)_{\mathbb{Q}} \arrow[r, "q"] & \bigoplus_{i,j} A_T^*(Y_{ij}^T)_{\mathbb{Q}}
\end{tikzcd}$$

Since all vertical maps are injective (Proposition A.2), we can describe the image of the first vertical map in terms of the image of the second vertical map and the kernel of q. Indeed, $p(\mathrm{Im}(i^*_{T,Y})) \simeq \mathrm{Im}(\oplus_i i^*_{T,Y_i}) \cap \mathrm{Ker}(q)$. Moreover, if Y^T is finite, then the kernel of q consists of all families $(f_i)_{i=1}^n$ such that $f_i(x) = f_j(x)$ whenever $x \in Y_{ij}^T$.

Back to the general case, let X be a complete T-scheme. One wishes to describe the image of the injective map $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_*^T(X^T)_{\mathbb{Q}}$. For this, let H be a subtorus of T of codimension one. Since i_T factors as $i_{T,H} : X^T \to X^H$ followed by $i_H : X^H \to X$, the image of i_T^* is contained in the image of $i^*_{T,H}$. Hence, $\mathrm{Im}(i_T^*) \subseteqq \bigcap_{H\subset T} \mathrm{Im}(i^*_{T,H})$, where the intersection runs over all codimension-one subtori H of T. This observation yields a complete description of the image of i_T^* over $\mathbb{Q}$. Before stating it, we recall a definition from [G3, Section 4].

Definition A.5. Let X be a complete T-scheme. We say that X has the *Chang-Skjelbred property* (or *CS property*, for short) if the pullback $i_T^* : A_T^*(X) \to A_T^*(X^T)$ is injective, and its image is exactly the intersection of the images of $i^*_{T,H} : A_T^*(X^H) \to A_T^*(X^T)$, where H runs over all subtori of codimension one in T. When the defining conditions hold over $\mathbb{Q}$ rather than $\mathbb{Z}$, we say that X has the *rational CS property*.

Some obstructions for the CS property to hold are e.g. (i) $A_*(X^T)$ could have $\mathbb{Z}$-torsion, (ii) $\dim T \geq 2$ and there exist a T-orbit on X whose stabilizer is not connected, for instance, if X is nonsingular, T-skeletal (Definition A.7) and the weights of the T-invariant curves are not primitive.

By Theorem 2.8, every nonsingular complete T-scheme has the rational CS property. We extend this result to include all possibly singular complete T-schemes. See [G3, Theorem 4.4] for the corresponding statement in equivariant operational K-theory with integral coefficients.

Theorem A.6. *If X is a complete T-scheme, then it has the rational CS property.*

Proof. Simply argue as in [G3, Theorem 4.4], using [EG2, Lemma 7.2], Theorem A.1 and Proposition A.2. ∎

Before stating our version of GKM theory, let us recall a few definitions from [GKM], [G1] and [G3].

DEFINITION A.7. Let X be a complete T-variety. Let $\mu : T \times X \to X$ be the action map. We say that μ is a *T-skeletal action* if the number of T-fixed points and one-dimensional T-orbits in X is finite. In this context, X is called a *T-skeletal variety.* The associated graph of fixed points and invariant curves is called the *GKM graph* of X. We shall denote this graph by $\Gamma(X)$.

Notice that, in principle, Definition A.7 allows for T-invariant irreducible curves with exactly one fixed point (i.e. the GKM graph $\Gamma(X)$ may have simple loops). In [G3, Proposition 5.3] we show that the functor $A_T^*(-)_{\mathbb{Q}}$ "contracts" such loops to a point. The proof there is given in the context of operational K-theory, but it easily extends to our current setup.

PROPOSITION A.8 ([G3, Proposition 5.3]). *Let X be a complete T-variety and let C be a T-invariant irreducible curve of X which is not fixed pointwise by T. Then the image of the injective map $i_T^* : A_T^*(C)_{\mathbb{Q}} \to A_T^*(C^T)_{\mathbb{Q}}$ is described as follows:*

(i) If C has only one fixed point, say x, then $i_T^* : A_T^*(C)_{\mathbb{Q}} \to A_T^*(x)_{\mathbb{Q}}$ is an isomorphism; that is, $A_T^*(C)_{\mathbb{Q}} \simeq S_{\mathbb{Q}}$.

(ii) If C has two fixed points, then

$$A_T^*(C)_{\mathbb{Q}} \simeq \{(f_0, f_\infty) \in S_{\mathbb{Q}} \oplus S_{\mathbb{Q}} \mid f_0 \cong f_\infty \mod \chi\},$$

where T acts on C via the character χ. □

Let X be a complete T-skeletal variety. It is possible to define a ring $PP_T^*(X)_{\mathbb{Q}}$ of (rational) *piecewise polynomial functions.* Indeed, let $A_T^*(X^T)_{\mathbb{Q}} = \bigoplus_{x\in X^T} S_{\mathbb{Q}}$. We then define $PP_T^*(X)_{\mathbb{Q}}$ as the subalgebra of $A_T^*(X^T)_{\mathbb{Q}}$ defined by

$$PP_T^*(X)_{\mathbb{Q}} = \{(f_1, ..., f_m) \in \bigoplus_{x\in X^T} S_{\mathbb{Q}} \mid f_i \equiv f_j \; mod(\chi_{i,j})\}$$

where x_i and x_j are the two (perhaps equal) fixed points in the closure of the one-dimensional T-orbit $\mathcal{C}_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $\mathcal{C}_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite). Invariant curves with only one fixed point do not impose any relation, and this is compatible with Proposition A.8. Now we are ready to state our version of GKM theory.

THEOREM A.9 ([G3, Theorem 5.4]). *Let X be a complete T-skeletal variety. Then the pullback $i_T^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ induces an algebra isomorphism between $A_T^*(X)_{\mathbb{Q}}$ and $PP_T^*(X)_{\mathbb{Q}}$.* □

REFERENCES

[B1] Bialynicki-Birula, A. *Some theorems on actions of algebraic groups.* Ann. of Math., 2nd Ser., Vol 98, No. 3, (Nov. 1973), 480-497.

[B2] Bialynicki-Birula, A. *Some properties of the decompositions of algebraic varieties determined by actions of a torus.* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 24 (1976), no. 9, 667-674.

[BCP] Bifet, E.; De Concini, C.; Procesi,C. *Cohomology of regular embeddings.* Adv. in Math., 82 (1990), 1-34.

[BJ1] Brion, M., Joshua, R. *Intersection cohomology of Reductive Varieties.* J. Eur. Math. Soc. 6 (2004).

[BJ2] Brion, M., Joshua, R. *Equivariant Chow Ring and Chern Classes of Wonderful Symmetric Varieties of Minimal Rank.* Transformation groups, Vol. 13, N. 3-4 (2008), 471-493.

[BK] Brion, M., Kumar, S. *Frobenius splitting methods in geometry and representation theory.* Progress in Mathematics, 231. Birkhäuser Boston.

[Bo] Bourbaki, N. *Elements of mathematics. Algebra I: Chapters 1-3.* Springer (1989).

[Br1] Brion, M. *Curves and divisors in spherical varieties.* Algebraic groups and Lie groups, 21-34, Austral. Math. Soc. Lect. Ser., 9, Cambridge Univ. Press, Cambridge (1997).

[Br2] Brion, M. *Equivariant Chow groups for torus actions.* Transf. Groups, vol. 2, No. 3 (1997), 225-267.

[Br3] Brion, M. *Equivariant cohomology and equivariant intersection theory.* Notes by Alvaro Rittatore. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 1-37, Kluwer Acad. Publ., Dordrecht (1998).

[C] Carrell, J. *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties.* Proc. Sympos. Pure Math., 56, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Da] Danilov, V. *The geometry of toric varieties.* Uspekhi Mat. Nauk 33 (1978), no. 2(200), 85-134, 247.

[DP] De Concini, C., Procesi, C. *Complete symmetric varieties.* Invariant theory (Montecatini, 1982), 1-44, Lecture Notes in Math., 996 (1983), Springer, Berlin.

[dB] del Baño, S. *On the Chow motive of some moduli spaces.* J. Reine Angew. Math. 532 (2001), 105-132.

[EG1] Edidin, D., Graham, W. *Equivariant Intersection Theory.* Invent. Math. 131 (1998), 595-634.

[EG2] Edidin, D., Graham, W. *Algebraic cycles and completions of equivariant K-theory*. Duke Math. Jour. Vol. 144, No. 3 (2008).

[Ei] Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry*. Springer-Verlag (1995).

[ES] Ellingsrud, G., Strømme, S. *Towards the Chow ring of the Hilbert scheme of* $\mathbb{P}^2$, J. reine angew. Math. 441 (1993), 33-44.

[FMSS] Fulton, W., MacPherson, R., Sottile, F., Sturmfels, B. *Intersection theory on spherical varieties*. J. of Alg. Geom., 4 (1994), 181-193.

[Fu] Fulton, W. *Intersection Theory*. Springer-Verlag, Berlin (1984).

[G1] Gonzales, R. *Rational smoothness, cellular decompositions and GKM theory*. Geometry & Topology, Vol. 18, No. 1 (2014), 291-326.

[G2] Gonzales, R. *Equivariant cohomology of rationally smooth group embeddings*. Submitted. arXiv:1112.1365

[G3] Gonzales, R. *Localization in equivariant operational K-theory and the Chang-Skjelbred property*. Submitted. arXiv:1403.4412.

[G4] Gonzales, R. *Algebraic rational cells, equivariant intersection theory, and Poincaré duality*. Submitted. arXiv:1404.2486.

[GKM] Goresky, M., Kottwitz, R., MacPherson, R. *Equivariant cohomology, Koszul duality, and the localization theorem*. Invent. Math. 131 (1998), 25-83.

[Ja] Jannsen, U. *Mixed motives and algebraic K-theory*. Lecture Notes in Mathematics 1400 (1990), Springer-Verlag, New York.

[JK] Joshua, R.; Krishna, A. *Higher K-theory of toric stacks*. Ann. Scoula Norm. Sup. Pisa Cl. Sci. To appear.

[Jo] Joshua, R. *Algebraic K-theory and higher Chow groups of linear varieties*. Math. Proc. Cambridge Phil. Soc., 130 (2001), 37-60.

[Ka] Karpenko, N.A. *Cohomology of relative cellular spaces and of isotropic flag varieties*. Algebra i Analiz. 12 (2000), 3-69.

[Ki] Kimura, S. *Fractional intersection and bivariant theory*. Comm. Algebra 20 (1992), no. 1, 285-302.

[KP] Katz, E.; Payne, S. *Piecewise polynomials, Minkowski weights, and localization on toric varieties*. Algebra Number Theory 2 (2008), no. 2, 135-155.

[Kr] Krishna, A. *Higher Chow groups of varieties with group action*. Algebra Number Theory 7 (2013), no. 2, 449-507.

[LP] Littelmann, P.; Procesi, C. *Equivariant Cohomology of Wonderful Compactifications*. Operator algebras, unitary representations, enveloping algebras, and invariant theory, Paris (1989).

[P] Payne, S. *Equivariant Chow cohomology of toric varieties.* Math. Res. Lett. 13 (2006), no. 1, 29-41.

[R] Rosenlicht, M. *Questions of rationality for solvable algebraic groups over nonperfect fields.* Annali di Mat. 61 (1963), 97-120.

[Sp] Springer, T. *Linear Algebraic Groups.* Birkhäuser Classics. 2nd Edition.

[Su] Sumihiro, H. *Equivariant completion.* J. Math. Kyoto Univ. 14 (1974), 1-28.

[Ti] Timashev, D. *Homogeneous spaces and equivariant embeddings.* Encyc. Math. Sci. 138 (2011), Springer-Verlag.

[To] Totaro, B. *Chow groups, Chow cohomology and linear varieties.* Forum of Mathematics, Sigma (2014), Vol. 2, e17. 1-25. (First appeared in 1996 in preprint form.)

[VV] Vezzosi, G., Vistoli, A. *Higher algebraic K-theory for actions of diagonalizable algebraic groups.* Invent. Math. 153 (2003), No. 1, 1-44.

[Vi] Vistoli, A. *Characteristic classes of principal bundles in algebraic intersection theory.* Duke Math. J. Vol. 58 (1989), n. 2, 299-315

Institut des Hautes Études
Scientifiques
35 Route de Chartres
F-91440
Bures-sur-Yvette
France
rgonzalesv@gmail.com

Faculty of Engineering and
Natural Sciences
Sabancı Üniversitesi
Orhanli, Tuzla
34956 Istanbul
Turkey