Topology and Purity for Torsors

Benjamin Antieau and Ben Williams

Received: February 7, 2014
Revised: March 1, 2015

Communicated by Alexander Merkurjev

Abstract. We study the homotopy theory of the classifying space of the complex projective linear groups to prove that purity fails for PGL_p-torsors on regular noetherian schemes when p is a prime. Extending our previous work when $p = 2$, we obtain a negative answer to a question of Colliot-Thélène and Sansuc, for all PGL_p. We also give a new example of the failure of purity for the cohomological filtration on the Witt group, which is the first example of this kind of a variety over an algebraically closed field.

Contents

1. Introduction

Let X be a regular noetherian integral scheme, let G be a smooth reductive group scheme over X, and let K be the function field of X. Consider the injective map

$$\text{(1)}\quad \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(X, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right) \to \\ \to \bigcap_{x \in X^{(1)}} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right),$$

where the intersection is over all codimension-1 points of X. Colliot-Thélène and Sansuc ask in [13, Question 6.4] whether this map is surjective. When it is, we say that *purity* holds for $\mathrm{H}^1_{\text{ét}}(X, G)$.

Purity trivially holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ when G is special in the sense of Serre, for example if $G = \mathrm{SL}_n$, since $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$ is a single point in this case. It holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ where G is a finite type X-group scheme of multiplicative type by [13, Corollaire 6.9]. It is also known to hold in many cases when X is the spectrum of a regular local ring containing a field of characteristic 0. With this assumption, purity was proven for $\mathrm{H}^1_{\text{ét}}(X, G)$ when G is a split group of type A_n, a split orthogonal or special orthogonal group, or a split spin group by Panin [32] and also when $G = \mathrm{G}_2$ by Chernousov and Panin [12]. The local purity conjecture asserts that purity holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ whenever X is the spectrum of a regular noetherian integral semi-local ring and G is a smooth reductive algebraic X-group scheme. Finally, purity holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ if the Krull dimension of X is at most 2 by [13, Theorem 6.13].

Purity is often considered along with another property, the so-called injectivity property, which is said to hold when $\mathrm{H}^1_{\text{ét}}(X, G) \to \mathrm{H}^1_{\text{ét}}(U, G)$ has trivial kernel for all $U \subseteq X$ containing $X^{(1)}$. In fact, Grothendieck and Serre conjectured that this map is always injective when X is the spectrum of a regular local ring R and G is a reductive X-group scheme. This has been proved recently using affine Grassmannians by Fedorov and Panin [19] when R contains an infinite field following partial progress by many other mathematicians. They prove more strongly that $\mathrm{H}^1_{\text{ét}}(X, G) \to \mathrm{H}^1_{\text{ét}}(U, G)$ is injective. The injectivity property for torsors is usually only sensible when X is in fact the spectrum of a local ring: otherwise it typically fails, even for $G = \mathbb{G}_m$.

When X is neither local nor low-dimensional and G is a non-special semisimple algebraic group, no results were known about purity for torsors until our paper [3], which showed that purity fails for PGL_2-torsors on smooth affine complex 6-folds in general. It is the purpose of this paper to use p-local homotopy theory to extend our previous result to PGL_p for all p.

Theorem 1.1. *Let p be a prime. Then, there exists a smooth affine complex variety X of dimension $2p+2$ such that purity fails for $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_p)$.*

We outline the proof. Recall first that PGL_p-torsors correspond to degree-p Azumaya algebras, and write $\mathrm{Br}_{\text{top}}(X(\mathbb{C}))$ for the topological Brauer group, which classifies topological Azumaya algebras up to Brauer equivalence [25]. Let X be a smooth complex variety such that $\mathrm{H}^2(X(\mathbb{C}),\mathbb{Z}) = 0$. In this case, by [2, Lemma 6.3], there is an isomorphism $\mathrm{Br}(X) \cong \mathrm{Br}_{\mathrm{top}}(X(\mathbb{C})) = \mathrm{H}^3(X(\mathbb{C}),\mathbb{Z})_{\mathrm{tors}}$. Because $\mathrm{H}^2(X(\mathbb{C}),\mathbb{Z}) = 0$, topological Azumaya algebras of degree n and exponent m on $X(\mathbb{C})$ are classified by homotopy classes of maps $X(\mathbb{C}) \to \mathrm{BP}(m,n)$, where $\mathrm{P}(m,n) = \mathrm{SL}_n(\mathbb{C})/\mu_m$.

In order to prove the theorem, we must construct a complex affine variety, X. First, following Totaro [35], we take a high dimensional algebraic approximation, X, to the classifying space $\mathrm{BP}(p,pq)$, where $q > 1$ is prime to p. This X is equipped with an SL_{pq}/μ_p-torsor, which induces a PGL_{pq}-torsor and therefore an Azumaya algebra A. Let α be the Brauer class of A on X. The exponent of α is p. Comparing the p-local homotopy type of $\mathrm{BP}(p,pq)$ to that of $\mathrm{BPGL}_p(\mathbb{C})$, we find that there is a non-vanishing topological obstruction in $\mathrm{H}^{2p+2}(X(\mathbb{C}),\mathbb{Z}/p)$ to the existence of a degree-p Azumaya algebra on X with the same Brauer class as A. Second, we replace X by a homotopy-equivalent smooth affine variety using Jouanolou's device [28]. Third, we use the affine Lefschetz theorem [24, Introduction, Section 2.2] to cut down to a smooth affine $2p+2$-dimensional variety where the obstruction in $\mathrm{H}^{2p+2}(X(\mathbb{C}),\mathbb{Z}/p)$ persists. By using an unpublished preprint of Ekedahl [18], it is possible to construct smooth projective complex examples of this nature as well, although we will not emphasize this last point in our paper.

Let K be the function field of the $2p+2$–dimensional affine variety X alluded to in the previous paragraph. The theorem is deduced from the properties of X as follows. The Brauer class $\alpha_K \in \mathrm{Br}(K)$ has exponent p and index dividing pq. Its index is therefore p by a result of Brauer [23, Proposition 4.5.13], and it is represented by a division algebra D of degree p over K. If $P \in X^{(1)}$, then α restricts to a class $\alpha_P \in \mathrm{Br}(\mathscr{O}_{X,p})$. Since D is unramified along $\mathscr{O}_{X,P}$ and since $\mathscr{O}_{X,P}$ is a discrete valuation ring, it follows that any maximal order in D over $\mathscr{O}_{X,P}$ is in fact an Azumaya algebra (see the proof of [7, Proposition 7.4]). Thus, the class of D is in the target of the map of (1), but by our choice of X, the class of D is not in the source of that map.

We make the following conjecture.

CONJECTURE 1.2. *Let G be a non-special semisimple k-group scheme. Then there exists a smooth affine k-variety X such that purity fails for $\mathrm{H}^1_{\text{ét}}(X,G)$.*

Our theorem proves the conjecture for $G = \mathrm{PGL}_p$ over $\mathbb{C}$, and since the schemes in question may all be defined over $\mathbb{Q}$, the conjecture is actually settled for PGL_p over any field of characteristic 0.

We explore three other points in the paper. First, in Section 3.3 we show that, in contrast to the global case, purity holds for PGL_n-torsors over regular noetherian integral semi-local rings R, at least if we restrict our attention to those torsors whose Brauer class has exponent invertible in R. This is a generalization of equivalent results of Ojanguren [30] and Panin [32] in characteristic 0.

Second, in Section 3.4, we give a topological perspective that explains why we expect purity to fail for $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$ for all n.
Third, in Section 3.5, we give examples where purity fails for $\mathrm{I}^2(X)/\mathrm{I}^3(X)$ where $\mathrm{I}^\bullet$ is the filtration on the Witt group induced by the cohomological filtration on $\mathrm{W}(\mathbb{C}(X))$ and X is a certain smooth affine complex 5-fold. Previous examples of a different, arithmetic nature were produced by Parimala and Sridharan [33], but these were explained by Auel [5] as failing to take into account quadratic modules with coefficients in line bundles. Our examples have $\mathrm{Pic}(X) = 0$.
The authors would like to thank Asher Auel, Eric Brussel, Roman Fedorov, Henri Gillet, Max Lieblich, and Manuel Ojanguren for conversations related to the present work. We would also like to thank Burt Totaro for several useful comments on an early version of this paper.

2. TOPOLOGY

In [3], we used knowledge of both the low-degree singular cohomology of $\mathrm{BPGL}_2(\mathbb{C})$ and of the low-degree Postnikov tower of $\mathrm{BPGL}_2(\mathbb{C})$ to produce counterexamples to the existence of Azumaya maximal orders in unramified division algebras. This is equivalent to showing that purity fails for PGL_2 over $\mathbb{C}$. At the time we wrote [3], we did not know how to extend our results to other primes, because our argument relied on the accessibility of the low-degree Postnikov tower of $\mathrm{BPGL}_2(\mathbb{C})$. While remarkable calculations have been made by Vezzosi [38] and Vistoli [39] on the cohomology of $\mathrm{BPGL}_p(\mathbb{C})$ for odd primes p, the problem of determining the Postnikov tower up the necessary level, $2p+1$, was beyond us. By using a p-local version of our arguments in [3] we bypass our ignorance to prove similar results.
We prove a result in this section about self-maps of $\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$, the $2p+1$ stage in the Postnikov tower of $\mathrm{BPGL}_p(\mathbb{C})$. Our theorem is in some sense related to the important results of Jackowski, McClure, and Oliver [27] about maps $\mathrm{BG} \to \mathrm{BH}$ when G and H are compact Lie groups, and especially about self-maps of BG. For the applications to algebraic geometry we have in mind, one must use finite approximations to $\mathrm{BPGL}_p(\mathbb{C}) \simeq \mathrm{BPU}_p$, where the results of [27] do not immediately apply. For more on the relationship of our work to [27], see Section 3.4.
The group $\mathrm{PGL}_p(\mathbb{C})$ and other classical groups are always equipped with the classical topology.

2.1. THE p-LOCAL COHOMOLOGY OF SOME EILENBERG-MACLANE SPACES.

We fix a prime number p. The p-local cohomology of a space X is the singular cohomology of X with coefficients in $\mathbb{Z}_{(p)}$. In the next few lemmas, we compute the low-degree p-local cohomology of $K(\mathbb{Z}, n)$, up to the first p-torsion. These results are both straightforward and classical, being corollaries of the all-encompassing calculations of Cartan [11] for instance. We include proofs here for the sake of completeness.

Recall that $K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty$ and that $\mathrm{H}^*(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z}[\iota_2]$, where $\deg(\iota_2) = 2$. In general, there is a canonical class $\iota_n \in \mathrm{H}^n(K(\mathbb{Z},n),\mathbb{Z})$ representing the identity map. We will use the Serre spectral sequences for the fiber sequences $K(\mathbb{Z},n) \to * \to K(\mathbb{Z},n+1)$ as well as the multiplicative structure in the spectral sequences.

Lemma 2.1. *For $1 \le k \le 2p+4$, the p-local cohomology of $K(\mathbb{Z},3)$ is*

$$\mathrm{H}^k(K(\mathbb{Z},3),\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = 3, \\ \mathbb{Z}/p & \text{if } k = 2p+2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can choose ι_3 so that $d_3(\iota_2) = \iota_3$ in the Serre spectral sequence for $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$:

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(K(\mathbb{Z},3),\mathrm{H}^t(K(\mathbb{Z},2),\mathbb{Z}_{(p)})) \Rightarrow \mathrm{H}^{s+t}(*,\mathbb{Z}_{(p)}).$$

Then, $d_3(\iota_2^n) = n\iota_2^{n-1}\iota_3$ and it follows that $d_3(\iota_2^n)$ is a generator of $\mathrm{E}_3^{3,2n-2}$ for $1 \le n < p$. For $4 \le k \le 2p+1$ the cohomology group $\mathrm{H}^k(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$ vanishes since there are no possible non-zero differentials hitting it. The first point on the t-axis where d_3 is not surjective is $d_3 : \mathrm{E}_3^{0,2p} \to \mathrm{E}_3^{3,2p-2}$ where the cokernel is $\mathbb{Z}/p$, see Figure 1. In order for the sequence to converge to

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \cdot \iota_2^{p+1} \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z}_{(p)} \cdot \iota_2^{p+1}\iota_3 \\
0 & 0 & 0 & 0 \\
\mathbb{Z}_{(p)} \cdot \iota_2^{p} \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z}_{(p)} \cdot \iota_2^{p}\iota_3 \\
0 & 0 & 0 & 0 \\
\mathbb{Z}_{(p)} \cdot \iota_2^{p-1} & 0 & 0 & \mathbb{Z}_{(p)} \cdot \iota_2^{p-1}\iota_3 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\mathbb{Z}_{(p)} \cdot \iota_2 \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z}_{(p)} \cdot \iota_2\iota_3 \\
0 & 0 & 0 & 0 \\
\mathbb{Z}_{(p)} & 0 & 0 & \mathbb{Z}_{(p)} \cdot \iota_3
\end{tikzcd}$$

Figure 1. The E_3-page of the Serre spectral sequence associated to $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$.

zero, this non-zero cokernel must support a non-zero departing differential;

since $\mathrm{H}^k(K(\mathbb{Z},2),\mathbb{Z}_{(p)}) = 0$ for $4 \leq k \leq 2p+1$, the differential d_{2p-1} induces an isomorphism $\mathbb{Z}/p \to \mathrm{H}^{2p+2}(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$. Let j_p be a generator of $\mathrm{H}^{2p+2}(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$. In terms of total degree, the next non-zero term in the spectral sequence is $\mathrm{E}_3^{2,2p+2} = \mathbb{Z}/p \cdot \iota_2 j_p$. Thus, the next potentially non-zero p-local cohomology group of $K(\mathbb{Z},3)$ is $\mathrm{H}^{2p+5}(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$. □

The next two lemmas have proofs conceptually similar to the preceding proof.

LEMMA 2.2. *For $0 \leq k \leq 2p+5$, the p-local cohomology of $K(\mathbb{Z},4)$ is*

$$\mathrm{H}^k(K(\mathbb{Z},4),\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } k = 0 \bmod 4, \\ \mathbb{Z}/p & \text{if } k = 2p+3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Again, we may assume that $d_3(\iota_3) = \iota_4$ in the Serre spectral sequence. Then, $d_3(\iota_3\iota_4^n) = \iota_4^{n+1}$. Moreover, the powers of ι_4 are non-zero because the d_3 differential leaving $\mathrm{E}_3^{4n,3} = \mathbb{Z}_{(p)} \cdot \iota_3\iota_4^n$ cannot have a kernel as all cohomology of $K(\mathbb{Z},3)$ in degrees higher than 3 is torsion. For this reason the group $\mathrm{H}^{2p+2}(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$ survives to the E_{2p+3}-page of the spectral sequence, and the differential

$$d_{2p+3} : \mathbb{Z}/p \cong \mathrm{H}^{2p+2}(K(\mathbb{Z},3),\mathbb{Z}_{(p)}) \to \mathrm{H}^{2p+3}(K(\mathbb{Z},4),\mathbb{Z}_{(p)})$$

is an isomorphism. The next potential non-zero torsion class in the spectral sequence is in $\mathrm{H}^{2p+5}(K(\mathbb{Z},3),\mathbb{Z}_{(p)})$, which shows that the other cohomology groups vanish in the range indicated. □

LEMMA 2.3. *The cohomology groups $\mathrm{H}^k(K(\mathbb{Z},n),\mathbb{Z}_{(p)})$ are torsion-free for $0 \leq k \leq 2p+3$ and $n \geq 5$. In this range they are isomorphic to a polynomial algebra over $\mathbb{Z}_{(p)}$ with a single generator ι_n in degree n if n is even or an exterior algebra over $\mathbb{Z}_{(p)}$ with a single generator ι_n in degree n if n is odd.*

Proof. This follows inductively as in the previous two lemmas. The important point is that the first p-torsion in $\mathrm{H}^*(K(\mathbb{Z},n-1),\mathbb{Z}_{(p)})$ is in degree $2p+(n-1)-1$. No differential exiting it can be non-zero until the differential $d_{2p+(n-1)}$, which produces p-torsion in $\mathrm{H}^{2p+n-1}(K(\mathbb{Z},n),\mathbb{Z}_{(p)})$. If $n \geq 5$, then $2p+n-1 \geq 2p+4$. □

2.2. THE p-LOCAL HOMOTOPY TYPE OF $\mathrm{BSL}_p(\mathbb{C})$. Now we harness the computations of the previous section to study the p-local homotopy type of truncations of $\mathrm{BPGL}_p(\mathbb{C})$. If X is a connected topological space, we will write $\tau_{\leq n}X$ for the nth stage in the Postnikov tower of a path-connected space X. Thus, $\tau_{\leq n}X$ is a topological space such that

$$\pi_i(\tau_{\leq n}X) \cong \begin{cases} \pi_i(X) & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The Postnikov tower is the sequence of natural maps

$$\begin{tikzcd}
X \arrow[dr] \arrow[ddr] \arrow[ddddr, bend right=15] \arrow[dddddr, bend right=30] & \vdots \arrow[d] \\
& \tau_{\leq n}X \arrow[d] \\
& \tau_{\leq n-1}X \arrow[d] \\
& \vdots \arrow[d] \\
& \tau_{\leq 1}X \arrow[d] \\
& \tau_{\leq 0}X = *
\end{tikzcd}$$

with the fiber of $\tau_{\leq n}X \to \tau_{\leq n-1}X$ identified with $K(\pi_n X, n)$. In good cases, such as when the action of $\pi_1 X$ on $\pi_n X$ for $n \geq 1$ is trivial, the extension

$$\begin{tikzcd}
K(\pi_n X, n) \arrow[r] & \tau_{\leq n}X \arrow[d] \\
& \tau_{\leq n-1}X
\end{tikzcd}$$

is classified by the k-invariant

$$k_{n-1} : \tau_{\leq n-1}X \to K(\pi_n X, n+1)$$

in the sense that $\tau_{\leq n}X$ is the homotopy fiber of k_{n-1}. This k-invariant is a cohomology class in $\mathrm{H}^{n+1}(\tau_{\leq n-1}X, \pi_n X)$. When it vanishes, the fibration is trivial.

There is a p-localization functor $\mathrm{L}_{(p)}$ that takes a topological space X and produces a space $\mathrm{L}_{(p)}X$ whose homotopy groups are $\mathbb{Z}_{(p)}$-modules. For the theory of localization of CW complexes, we refer to the monograph of Bousfield and Kan [10]. This functor takes fiber sequences to fiber sequences when the base is simply connected by the principal fibration lemma [10, Chapter II]. Since the $\mathbb{Z}_{(p)}$-localization of an Eilenberg-MacLane space $K(\pi, n)$ is $K(\pi \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, n)$, for which see [10, page 65], it follows that application of $\mathrm{L}_{(p)}$ commutes with the formation of Postnikov towers of simply-connected spaces.

Now, we consider the p-local homotopy type of certain stages in the Postnikov tower of $\mathrm{BSL}_n(\mathbb{C})$. By Bott periodicity [9, Theorem 5] the p-local homotopy groups of $\mathrm{BSL}_n(\mathbb{C})$ for $1 \leq i \leq 2n+1$ are

$$\begin{aligned}\pi_i\left(\mathrm{L}_{(p)}\,\mathrm{BSL}_n(\mathbb{C})\right) &\cong \pi_i\left(\mathrm{BSL}_n(\mathbb{C})\right) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \\ &\cong \begin{cases} \mathbb{Z}_{(p)} & \text{if } i \text{ is even and } i \geq 4, \\ \mathbb{Z}/(n!) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} & \text{if } i = 2n+1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Proposition 2.4. *The localization $\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_n(\mathbb{C})$, where $n \geq p$, is a generalized Eilenberg–MacLane space:*

$$\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_n(\mathbb{C}) \simeq K(\mathbb{Z}_{(p)},4) \times K(\mathbb{Z}_{(p)},6) \times \cdots \times K(\mathbb{Z}_{(p)},2p). \tag{2}$$

Proof. We prove the general statement

$$\mathrm{L}_{(p)}\tau_{\leq 2j}\,\mathrm{BSL}_n(\mathbb{C}) \simeq K(\mathbb{Z}_{(p)},4) \times K(\mathbb{Z}_{(p)},6) \times \cdots \times K(\mathbb{Z}_{(p)},2j), \quad \text{for } j \leq p$$

by induction on j. The base case when $j = 1$ is trivial. For the induction step, suppose that $\mathrm{L}_{(p)}\tau_{\leq 2j}\,\mathrm{BSL}_n(\mathbb{C})$ is

$$K(\mathbb{Z}_{(p)},4) \times \cdots \times K(\mathbb{Z}_{(p)},2j)$$

for some $1 \leq j < p$. The extension

$$K(\mathbb{Z}_{(p)},2j+2) \to \mathrm{L}_{(p)}\tau_{\leq 2j+2}\,\mathrm{BSL}_n(\mathbb{C}) \to \mathrm{L}_{(p)}\tau_{\leq 2j}\,\mathrm{BSL}_n(\mathbb{C})$$

is classified by the k-invariant

$$k_{2j} \in \mathrm{H}^{2j+3}(K(\mathbb{Z}_{(p)},4) \times \cdots \times K(\mathbb{Z}_{(p)},2j), \mathbb{Z}_{(p)}).$$

By Lemmas 2.2 and 2.3, this cohomology group must vanish, since $j < p$. Hence $k_{2j} = 0$ and the extension is trivial. □

Before we prove the next proposition, we need a well-known lemma. Recall that an n-equivalence is a map such that $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $0 \leq k < n$ and a surjection for $k = n$.

Lemma 2.5. *Let $f : X \to Y$ be an n-equivalence. Then, for any coefficient abelian group A, the induced map*

$$f^* : \mathrm{H}^k(Y, A) \to \mathrm{H}^k(X, A)$$

is an isomorphism for $0 \leq k \leq n-1$ and an injection for $k = n$.

Proof. This follows most easily from the Serre spectral sequence for the fibration sequence $F \to X \to Y$. Since the fiber is n-connected, the groups $\widetilde{\mathrm{H}}^k(F, A)$ vanish for $k < n$. The first nontrivial extension-problem in the spectral sequence takes the form

$$0 \to \mathrm{H}^n(Y, A) \to \mathrm{H}^n(X, A) \to \mathrm{H}^n(F, A),$$

which proves the result. □

The previous proposition asserts that $\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C})$ is a generalized Eilenberg–MacLane space, the following asserts that the $\tau_{\leq 2p}$ appearing there is sharp, and the nontriviality of the extension can be detected after pulling the extension back along an inclusion $K(\mathbb{Z}_{(p)},4) \to \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C})$.

Proposition 2.6. *Denote by i a map $i : K(\mathbb{Z}_{(p)},4) \to \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C})$ splitting the projection map. Write $k_{2p} \in \mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}),\mathbb{Z}/p)$ for the k–invariant of the extension*

$$\begin{tikzcd} K(\mathbb{Z}/p,2p+1) \arrow[r] & \mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \arrow[d] \\ & \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}). \end{tikzcd} \tag{3}$$

Then k_{2p} is of order p, and moreover $i^*(k_{2p})$ is a generator for $\mathrm{H}^{2p+2}(K(\mathbb{Z}_{(p)},4),\mathbb{Z}/p) \cong \mathbb{Z}/p$.

Proof. Note that $X \to \tau_{\leq n}X$ is an $(n+1)$–equivalence. By Lemma 2.5, the map of rings

$$\mathrm{H}^i(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p,\mathbb{Z}_{(p)}) \to \mathrm{H}^i(\mathrm{L}_{(p)}\,\mathrm{BSL}_p,\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[c_2,c_3,\dots,c_p]$$

is an isomorphism when $i \leq 2p$, and an injection, and hence an isomorphism, when $i = 2p+1$. By Lemmas 2.2 and 2.3, the ring $\mathrm{H}^i(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p,\mathbb{Z}_{(p)})$ is isomorphic to a polynomial ring on generators in degrees $4,6,8,\dots,2p$ in the range where $i \leq 2p+2$, so that it follows that

$$\mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p,\mathbb{Z}_{(p)}) \to \mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\,\mathrm{BSL}_p,\mathbb{Z}_{(p)})$$

is an isomorphism as well. We also deduce that

$$\mathrm{H}^{2p+3}(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}),\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p\cdot\rho,$$

where $i^*(\rho)$ is a generator of $\mathrm{H}^{2p+3}(K(\mathbb{Z},4),\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$.
Considering the long exact sequence in cohomology associated to the sequence

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0$$

we deduce the existence of a decomposition

$$\mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p,\mathbb{Z}/p) = \mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\,\mathrm{BSL}_p,\mathbb{Z}/p) \oplus \mathbb{Z}/p\cdot\sigma,$$

where $\beta_p(\sigma) = \rho$.
We observe two things. First that there is a quotient relationship arising from the Postnikov extensions

$$\mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p,\mathbb{Z}/p)/\langle k_{2p}\rangle \cong \mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p,\mathbb{Z}/p),$$

and second that the functorial map

$$\mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p,\mathbb{Z}/p) \hookrightarrow \mathrm{H}^{2p+2}(\mathrm{L}_{(p)}\,\mathrm{BSL}_p,\mathbb{Z}/p)$$

is injective by Lemma 2.5. It follows directly that $k_{2p} = u\sigma$ where u is a unit. By naturality, $\beta_{2p}(i^*(k_{2p})) = ui^*(\rho)$, and in particular, $i^*(k_{2p}) \neq 0$. □

COROLLARY 2.7. *A map* $h : \mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \to \mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C})$ *that induces an isomorphism on* $\pi_4\left(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C})\right) \cong \mathbb{Z}_{(p)}$*, also induces an isomorphism on*

$$\pi_{2p+1}\left(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C})\right) \cong \mathbb{Z}/p.$$

Proof. Let $i : K(\mathbb{Z}_{(p)}, 4) \to \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C})$ again denote a map splitting the projection onto $K(\mathbb{Z}_{(p)}, 4)$ in Proposition 2.4. Let

$$\tau_{\leq 2p}h : \tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}) \to \tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C})$$

be the truncation of h. This map fits into a commutative diagram

(4)
$$\begin{tikzcd}
K(\mathbb{Z}_{(p)}, 4) \arrow[r, "i"] \arrow[d, "\simeq"] & \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}) \arrow[r, "k_{2p}"] \arrow[d, "\tau_{\leq 2p}h"] & K(\mathbb{Z}/p, 2p+2) \arrow[d, "\mathrm{B}h_*"] \\
K(\mathbb{Z}_{(p)}, 4) \arrow[r, "i"] & \mathrm{L}_{(p)}\tau_{\leq 2p}\,\mathrm{BSL}_p(\mathbb{C}) \arrow[r, "k_{2p}"] & K(\mathbb{Z}/p, 2p+2),
\end{tikzcd}$$

where the map $\mathrm{B}h_*$ is the result of applying a functorial classifying-space construction to the endomorphism of $K(\pi_{2p+1}\left(\mathrm{L}_{(p)}\,\mathrm{BSL}_p\right), 2p+1) \simeq K(\mathbb{Z}/p, 2p+1)$ arising from the map h_* on $\pi_{2p+1}\mathrm{L}_{(p)}\,\mathrm{BSL}_p$. The map $K(\mathbb{Z}_{(p)}, 4) \to K(\mathbb{Z}_{(p)}, 4)$ is the composition of i with h and the projection, and is a weak equivalence since i, h and the projection all induce isomorphisms on π_4, by hypothesis. Since $i^*(k_{2p}) \neq 0$ is a generator of $\mathrm{H}^{2p+2}(K(\mathbb{Z}_{(p)}, 4), \mathbb{Z}/p))$, commutativity of the diagram proves that h_* is an equivalence, as claimed. □

2.3. THE p-LOCAL HOMOTOPY TYPE OF $\mathrm{BPGL}_p(\mathbb{C})$. There is a fiber sequence, obtained by truncating a sequence associated to the defining quotient $\mathrm{SL}_p(\mathbb{C})/\mu_p = \mathrm{PGL}_p(\mathbb{C})$, of the form

$$\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \longrightarrow \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \longrightarrow K(\mathbb{Z}/p, 2).$$

The main theorem concerns itself with maps $f : \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \to \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$ that induce isomorphisms on $\pi_2(\tau_{2p+1}\mathrm{BPGL}_p(\mathbb{C}))$, these maps fit into diagrams

$$\begin{tikzcd}
\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \arrow[r] \arrow[d, dashed, "\tilde{f}"] & \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \arrow[r] \arrow[d] & K(\mathbb{Z}/p, 2) \arrow[d, dash, "\simeq"] \\
\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \arrow[r] & \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \arrow[r] & K(\mathbb{Z}/p, 2)
\end{tikzcd}$$

in which the right-hand square commutes up to homotopy. The map, $\tilde{f}$, making the left-hand square commute up to homotopy exists, but is not unique. We refer to such a map as a lift of the map f.
Since $\pi_{2p+1}\left(\mathrm{BSL}_p(\mathbb{C})\right) \cong \mathbb{Z}/(p!)$, it follows that $\pi_{2p+1}\left(\mathrm{BPGL}_p(\mathbb{C})\right) \cong \mathbb{Z}/(p!)$, and hence that

$$\pi_{2p+1}\left(\mathrm{L}_{(p)}\mathrm{BPGL}_p(\mathbb{C})\right) \cong \mathbb{Z}/(p!) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathbb{Z}/p.$$

The following lemma is a technical ingredient in Theorem 2.9.

LEMMA 2.8. *Let $f : \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \to \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$ be a map that induces an isomorphism*

$$\pi_2(f) : \pi_2\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) \to \pi_2\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) = \mathbb{Z}/p.$$

Any lift $\tilde{f} : \tau_{\leq 2p+1}\mathrm{BSL}_p(\mathbb{C}) \to \tau_{\leq 2p+1}\mathrm{BSL}_p(\mathbb{C})$ of f has the property that the p-localization

$$\pi_4(\mathrm{L}_{(p)}\tilde{f}_*) : \pi_4\left(\mathrm{BSL}_p(\mathbb{C})\right) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \pi_4\left(\mathrm{BSL}_p(\mathbb{C})\right) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

is an isomorphism.

Proof. The first nontrivial fiber sequence appearing in the Postnikov tower of $\mathrm{BPGL}_p(\mathbb{C})$ is

$$K(\mathbb{Z},4) \longrightarrow \tau_{\leq 4}\mathrm{BPGL}_p(\mathbb{C}) \longrightarrow K(\mathbb{Z}/p,2).$$

In [2] we proved that the $K(\mathbb{Z},4)$–bundle above is classified by a map $K(\mathbb{Z}/p,2) \to K(\mathbb{Z},5)$ that represents a generator of the group $\mathrm{H}^5(K(\mathbb{Z}/p,2),\mathbb{Z}) \cong \mathbb{Z}/p^{\epsilon}$, where $\epsilon = 1$ unless $p = 2$ in which case $\epsilon = 2$. If f induces an isomorphism on π_2, it must also induce a map f_* on $\pi_4\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) \cong \mathbb{Z}$ such that the functorially-derived diagram

$$\begin{tikzcd} K(\mathbb{Z}/p,2) \arrow[r] \arrow[d, "\cong"] & K(\mathbb{Z},5) \arrow[d, "\mathrm{B}f_*"] \\ K(\mathbb{Z}/p,2) \arrow[r] & K(\mathbb{Z},5) \end{tikzcd}$$

commutes. The class $\mathrm{B}f_* \in \mathrm{H}^5(K(\mathbb{Z},5),\mathbb{Z}) = \mathbb{Z}$ must be an integer that is relatively prime to p, and in turn the endomorphism induced by f on $\pi_4\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) \cong \mathbb{Z}$ must be multiplication by an integer that is relatively prime to p. The map induced by f on $\pi_4\left(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right)$ is consequently an isomorphism.
For any choice of $\tilde{f}$, the p-localized diagram

$$\begin{tikzcd} \mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BSL}_p(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{L}_{(p)}\tilde{f}"] & \mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \arrow[d, "\mathrm{L}_{(p)}f"] \\ \mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BSL}_p(\mathbb{C}) \arrow[r] & \mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \end{tikzcd}$$

commutes. Here the horizontal arrows induce isomorphisms on all homotopy groups, π_i, where $i \geq 3$, and the result follows. $\square$

We are now in a position to prove the main topological theorem of the paper.

THEOREM 2.9. *Let $f : \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C}) \to \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$ be a map that induces an isomorphism*

$$\pi_2(f) : \pi_2\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) \to \pi_2\left(\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right) \cong \mathbb{Z}/p.$$

Then

$$\pi_{2p+1}(\mathrm{L}_{(p)}f) : \pi_{2p+1}\Big(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\Big) \to \pi_{2p+1}\Big(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\Big)$$

is an isomorphism.

Proof. Suppose f is a map meeting the hypothesis of the theorem. Choose a lift, $\tilde{f} : \tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \to \tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C})$. By Lemma 2.8, the map $\tilde{f}_*$ is an isomorphism on $\pi_4\left(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C})\right)$, and therefore by Corollary 2.7, $\pi_{2p+1}(\mathrm{L}_{(p)}\tilde{f})$ is an isomorphism.
Since the projection $\mathrm{L}_{(p)}\tau_{\leq 2p+1}\,\mathrm{BSL}_p(\mathbb{C}) \to \mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$ induces an isomorphism on all higher homotopy groups π_i where $i \geq 3$, it follows that f_* is an isomorphism on $\pi_{2p+1}\left(\mathrm{L}_{(p)}\tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})\right)$, as claimed. $\square$

3. PURITY

We consider purity in this section, giving two applications of algebraic topology to algebraic purity questions. The first uses the machinery of Section 2 to show that purity fails in general for PGL_p torsors, while the second uses [2, Theorem D] to show that purity fails for the cohomological filtration on the Witt group.

3.1. DEFINITIONS. Let $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a presheaf on some category of schemes $\mathscr{C}$. We will suppress any mention of the category $\mathscr{C}$ throughout, and we will assume that all necessary localizations of an object X in $\mathscr{C}$ are also in $\mathscr{C}$. Suppose that X is a regular noetherian integral scheme in $\mathscr{C}$, and let K be the function field of X. If the natural map

$$\mathrm{im}(\mathscr{F}(X) \to \mathscr{F}(\operatorname{Spec} K)) \to \bigcap_{P \in X^{(1)}} \mathrm{im}(\mathscr{F}(\operatorname{Spec} \mathscr{O}_{X,P}) \to \mathscr{F}(\operatorname{Spec} K))$$

is a bijection, where $X^{(1)}$ denotes the set of codimension 1 points of X, then we say that *purity* holds for $\mathscr{F}(X)$.

EXAMPLE 3.1. If X is a regular noetherian integral scheme with an ample line bundle such that $\mathbb{Q} \subseteq \Gamma(X, \mathscr{O}_X)$, then purity holds for $\mathrm{Br}(X)$. In particular, purity holds for $\mathrm{Br}(X)$ for smooth quasi-projective schemes over field of characteristic 0. This follows from two facts. First, it is a theorem of Gabber and de Jong [16] that if X has an ample line bundle, then $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$. Second, Gabber has shown (see Fujiwara [20]) that $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ satisfies purity when X is a regular scheme and when each positive integer is invertible in X. The case of smooth affine schemes over fields had been handled previously by Hoobler [26], following Auslander and Goldman's work on the 2-dimensional affine situation [7, Proposition 6.1], while Gabber [22] had proved the result in characteristic 0 with an added excellence condition. Gabber [21] proved purity for $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ without the excellence hypothesis when $\dim X \leq 3$; hence, in combination with the $\mathrm{Br} = \mathrm{Br}'$ result above, purity holds for the Brauer group when $\dim X \leq 3$ and X has an ample line bundle. If X is an arbitrary regular noetherian integral scheme, then purity holds for $\mathrm{Br}(X)'$, the part of the Brauer group containing the m-torsion for all $m > 0$ invertible in X. This follows from purity for $\mathrm{H}^2_{\text{ét}}(X, \mu_n)$ when n is prime to p. See Fujiwara [20] together with [4, Exposée XIV, Section 3] or [14, Theorem 3.8.2].

Currently unknown is whether purity holds for $\mathrm{Br}(X)$ for every regular noetherian integral scheme X. The results above should be contrasted to what happens for degree 3 cohomology classes: for any integer $n > 1$, there are smooth projective complex varieties X such that purity fails for $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/n)$. See [15, Section 5] for an overview, or Totaro [36] and Schoen [34] for examples. It is not hard to see that unramified cohomology is homotopy invariant [37, Theorem 1.3], so it follows by using Jouanolou's device [28] that there are smooth affine complex varieties where purity fails for $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/n)$ as well.

3.2. PURITY FOR TORSORS. Let X be a regular noetherian integral scheme, and let G be a smooth reductive group scheme over X. In [13, Question 6.4], Colliot-Thélène and Sansuc ask whether purity holds for $\mathrm{H}^1_{\text{ét}}(X, G)$. As stated in the introduction, many examples are known where purity holds in the special case where $X = \operatorname{Spec} R$ is the spectrum of a regular noetherian local ring R. But, as far as the authors are aware, except for our negative results [3] for $G = \mathrm{PGL}_2$, no results are known in the non-local case, either for or against purity, except in some trivial cases such as for special groups like SL_n and in the following two theorems.

THEOREM 3.2 ([13, Corollaire 6.9]). *Purity holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ for all regular noetherian integral schemes X and all finite type X-group schemes of multiplicative type G.*

THEOREM 3.3 ([13, Théorème 6.13]). *Purity holds for $\mathrm{H}^1_{\text{ét}}(X, G)$ for all regular noetherian integral 2-dimensional schemes X and all smooth reductive X-group schemes G.*

Before we prove our main theorem, we need a standard result.

LEMMA 3.4. *Let R be a discrete valuation ring, and let $\alpha \in \mathrm{Br}(R) \subseteq \mathrm{Br}(K)$ be a Brauer class. If D is a central simple algebra over K, the fraction field of R, with Brauer class α, then every maximal order A in D is Azumaya over R.*

Proof. A maximal order A is in particular reflexive. Since a reflexive module on a regular domain of dimension at most 2 is projective, A is projective. The lemma now follows from the argument in the second paragraph of the proof of [7, Proposition 7.4]. □

The goal of this paper is to show that Theorem 3.3 does not extend to higher-dimensional schemes. The method is based on [3], augmented by the results of Section 2.

Let a, b be positive integers and let $\mathrm{P}(a, ab)$ denote the complex algebraic group $\mathrm{SL}_{ab}(\mathbb{C})/\mu_a$. There is a commutative diagram of short exact sequences of groups

$$\begin{tikzcd}
1 \arrow[r] & \mu_a \arrow[r] \arrow[d] & \mathrm{SL}_{ab}(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{P}(a, ab) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}_{ab}(\mathbb{C}) \arrow[r] & \mathrm{PGL}_{ab}(\mathbb{C}) \arrow[r] & 1.
\end{tikzcd}$$

and therefore, for any topological space X, a commutative square

$$\begin{tikzcd}
\mathrm{H}^1(X, \mathrm{P}(a, ab)) \arrow[r] \arrow[d] & \mathrm{H}^2(X, \mathbb{Z}/a) \arrow[d] \\
\mathrm{H}^1(X, \mathrm{PGL}_{ab}(\mathbb{C})) \arrow[r] & \mathrm{H}^2(X, \mathbb{C}^*) \cong \mathrm{H}^3(X, \mathbb{Z}).
\end{tikzcd} \tag{5}$$

If we have a principal $\mathrm{P}(a, ab)$-bundle on a topological space X, then the quotient map $\mathrm{P}(a, ab) \to \mathrm{PGL}_{ab}(\mathbb{C})$ gives rise to a principal $\mathrm{PGL}_{ab}(\mathbb{C})$-bundle and therefore a degree ab topological Azumaya algebra. Diagram 5 implies that this Azumaya algebra is of exponent dividing a.

Similarly, in the category of schemes over $\mathbb{C}$, an SL_{ab}/μ_a-torsor (for the étale topology) gives rise to a degree ab Azumaya algebra, and the exponent of this Azumaya algebra divides a.

We rely on the following argument repeatedly: If X is a simply connected topological space, then $\pi_2(X) \cong \mathrm{H}_2(X, \mathbb{Z})$ by the Hurewicz theorem. Then, by the universal coefficient theorem, the torsion $\mathrm{Br}(X) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ is naturally the dual of the torsion subgroup of $\mathrm{H}_2(X, \mathbb{Z}) \cong \pi_2(X)$. In the cases we consider, $\pi_2(X)$ is itself a torsion abelian group, and therefore $\mathrm{Br}(X)$ is naturally the dual of $\pi_2(X)$.

LEMMA 3.5. *Suppose $f : X \to \mathrm{BP}(a, ab)$ is a 3-equivalence of topological spaces. Denote the Azumaya algebra associated to the $\mathrm{PGL}_{ab}(\mathbb{C})$ bundle classified by the composite $X \xrightarrow{f} \mathrm{BP}(a, ab) \to \mathrm{BPGL}_{ab}(\mathbb{C})$ by $\mathscr{A}(\mathbb{C})$. The exponent of $\mathscr{A}(\mathbb{C})$ is a.*

Proof. Since f^* is a 3-equivalence, the Hurewicz and universal coefficients theorems imply that it induces an isomorphism on $\mathrm{Br}_{\mathrm{top}}(\cdot) \cong \mathrm{H}^3(\cdot, \mathbb{Z})_{\mathrm{tors}}$. It suffices therefore to show that the map $\phi^* : \mathrm{Br}_{\mathrm{top}}(\mathrm{BPGL}_{ab}(\mathbb{C})) \to \mathrm{Br}_{\mathrm{top}}(\mathrm{BP}(a, ab))$ takes a generator to a class of order a.

The following is a diagram of short exact sequences of groups

$$\begin{tikzcd}
1 \arrow[r] & \mu_a \arrow[r] \arrow[d, "i"] & \mathrm{SL}_{ab}(\mathbb{C}) \arrow[r] \arrow[d, equal] & \mathrm{P}(a, ab) \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & \mu_a \arrow[r] & \mathrm{SL}_{ab}(\mathbb{C}) \arrow[r] & \mathrm{PGL}_{ab}(\mathbb{C}) \arrow[r] & 1.
\end{tikzcd}$$

Here i denotes the inclusion $\mu_a \subset \mu_{ab}$. This gives rise to a map of fiber sequences:

$$\begin{tikzcd}
\mathrm{BSL}_{ab}(\mathbb{C}) \arrow[r] \arrow[d, equal] & \mathrm{BP}(a, ab) \arrow[r] \arrow[d, "\phi"] & \mathrm{B}^2\mu_a \arrow[d, "\mathrm{B}^2 i"] \\
\mathrm{BSL}_{ab}(\mathbb{C}) \arrow[r] & \mathrm{BPGL}_{ab}(\mathbb{C}) \arrow[r] & \mathrm{B}^2\mu_{ab}.
\end{tikzcd}$$

Since $\widetilde{\mathrm{H}}^*(\mathrm{SL}_{ab}(\mathbb{C}), \mathbb{Z})$ vanishes below degree 4, the natural map of Serre spectral sequences for $\mathrm{H}^*(\cdot, \mathbb{Z})$ yields to a commutative square

$$\begin{tikzcd}
\mathrm{Br}_{\mathrm{top}}(\mathrm{BP}(a, ab)) \arrow[r, equal] & \mathrm{H}^3(\mathrm{BP}(a, ab), \mathbb{Z}) & \mathrm{H}^3(\mathrm{B}^2\mu_a, \mathbb{Z}) \cong \mathbb{Z}/a \arrow[l, "\cong"'] \\
\mathrm{Br}_{\mathrm{top}}(\mathrm{BPGL}_{ab}(\mathbb{C})) \arrow[r, equal] & \mathrm{H}^3(\mathrm{BPGL}_{ab}(\mathbb{C}), \mathbb{Z}) \arrow[u, "(\mathrm{B}\phi)^*"] & \mathrm{H}^3(\mathrm{B}^2\mu_a b, \mathbb{Z}) \cong \mathbb{Z}/(ab), \arrow[l, "\cong"'] \arrow[u, "(\mathrm{B}^2 i)^*"']
\end{tikzcd}$$

where $(\mathrm{B}^2 i)^*$, by means of the Hurewicz and universal coefficient theorems, is seen to be the dual of the inclusion $\mathbb{Z}/a = \mu_a \subset \mu_{ab} = \mathbb{Z}/(ab)$. Namely, it is a surjection $\mathbb{Z}/(ab) \to \mathbb{Z}/a$, as required. □

We now can prove our main theorem:

Theorem 3.6. *Let p be a prime. There exists a smooth affine complex variety X of dimension $2p + 2$ such that purity fails for $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_p)$.*

Proof. Let $q > 1$ be an integer prime to p. Let V be an algebraic representation of the complex algebraic group $G = \mathrm{SL}_{pq} / \mu_p$ such that there is a G-invariant closed subvariety S of codimension at least $p + 2$ with the following properties: the complement $V - S$ is contained in the stable locus of the G-action on V (in the sense of [29]) for some G-linearization of $\mathscr{O}_V$, and G acts freely on $V - S$. Such a representation is constructed in [35, Remark 1.4] by taking a large direct sum of any faithful G-representation. There is a universal geometric quotient $q : (V - S) \to (V - S)/G$ with $(V - S)/G$ a quasi-projective variety [29]. Moreover, $(V - S) \to (V - S)/G$ is an algebraic principal G bundle, and $(V - S)/G$ is smooth, since $(V - S) \to (V - S)/G$ is a smooth surjective morphism with $(V - S)$ smooth. We can replace $(V - S)/G$ by an affine scheme using Jouanolou's device [28], and then we can use the affine Lefschetz theorem [24, Introduction, Section 2.2] to cut down to a $2p + 2$-dimensional closed subscheme X. Pulling q back along $X \to (V - S)/G$ gives an algebraic G-torsor $E \to X$, and therefore an induced PGL_{pq}-torsor $E \times_{\mathrm{P}(p,pq)} \mathrm{PGL}_{pq}$, and finally an associated (algebraic) Azumaya algebra $\mathscr{A}$ on X. Write $\alpha \in \mathrm{Br}(X)$ for the class of $\mathscr{A}$; since $\mathscr{A}$ is induced from a principal SL_{pq}/μ_p-bundle, the exponent of α divides p.

The map $X \to (V-S)/G$ is an affine vector bundle, and upon complex realization, yields a homotopy equivalence. The realization $G(\mathbb{C})$ is the group $\mathrm{P}(p, pq)$, and by construction $((V-S)/G)(\mathbb{C}) \to \mathrm{BP}(p,pq)$ is a $2p+3$ equivalence. The topological Azumaya algebra classified by the composite $X(\mathbb{C}) \to \mathrm{BP}(p,pq) \to \mathrm{BPGL}_{pq}(\mathbb{C})$ is $\mathscr{A}(\mathbb{C})$, and by Lemma 3.5 it has exponent p. Since there is a homomorphism $\mathrm{Br}(X) \to \mathrm{Br}(X(\mathbb{C}))$ taking the class, α, of $\mathscr{A}$ to that of $\mathscr{A}(\mathbb{C})$, it follows that the exponent of α is exactly p.

Returning to algebra, let K be the function field of X. There is an inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(K)$, and the class $\alpha \in \mathrm{Br}(K)$ corresponds to a central simple algebra $\mathscr{A} \otimes_{\mathcal{O}_X} K$ of degree pq and exponent p. From the theory of the index of a Brauer class of a field, [23, Proposition 4.5.13], we know that there is an Azumaya algebra A' of degree p (in fact, a division algebra) over K in the class of α. By Lemma 3.4, therefore, every codimension 1 local ring $\mathscr{O}_{X,x}$ of X has the property that there is some Azumaya algebra of degree p representing the class of α in $\mathrm{Br}(\mathscr{O}_{X,x})$, which is to say that the class of A' in $\mathrm{H}^1_{\text{ét}}(K, \mathrm{PGL}_p)$ lies in the intersection

$$\bigcap_{x \in X^{(1)}} \mathrm{im}\left(\mathrm{H}_{\text{ét}}(\mathrm{Spec}\,\mathscr{O}_{X,x}, \mathrm{PGL}_p) \to \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\,K, \mathrm{PGL}_p)\right).$$

To show that purity does not hold for PGL_p, therefore, it suffices to show that $\alpha \in \mathrm{Br}(X)$ is not represented by any Azumaya algebra of degree p. By comparison, it is sufficient to show that the class of $\mathscr{A}(\mathbb{C})$ in $\mathrm{Br}(X(\mathbb{C}))$, is not represented by any topological Azumaya algebra of degree p.

Suppose for the sake of contradiction that such a topological Azumaya algebra exists. Let $f : X(\mathbb{C}) \to \mathrm{BPGL}_p(\mathbb{C})$ be a map classifying it. Since the class of $\mathscr{A}(\mathbb{C})$ in $\mathrm{Br}(X(\mathbb{C}))$ is of exponent p, it follows that the map $f^* : \mathrm{Br}(\mathrm{BPGL}_p(\mathbb{C})) \to \mathrm{Br}(X(\mathbb{C}))$ is nonzero, and by the universal coefficients and Hurewicz theorems, it follows that the map $f_* : \mathbb{Z}/p \cong \pi_2(X(\mathbb{C})) \to \pi_2(\mathrm{BPGL}_p(\mathbb{C})) \cong \mathbb{Z}/p$ is nonzero, and in particular is an isomorphism.

We consider the composition

$$\tau_{\leq 2p+2}\mathrm{BPGL}_p(\mathbb{C}) \to \tau_{\leq 2p+2}\mathrm{BP}(p,pq) \to \tau_{\leq 2p+2}X \to \tau_{\leq 2p+2}\mathrm{BPGL}_p(\mathbb{C}),$$

where the first arrow is the $2p+2$-truncation of the q-fold block sum map $\mathrm{BPGL}_p(\mathbb{C}) \to \mathrm{BP}(p,pq)$, the second arrow is a homotopy inverse to the homotopy equivalence $\tau_{\leq 2p+2}X \to \tau_{\leq 2p+2}\mathrm{BP}(p,pq)$, and the third arrow is the truncation of f. This composition induces an isomorphism on π_2, and hence on π_{2p+1}, by Theorem 2.9 (applied to the further truncation $\tau_{\leq 2p+1}$ of the composition). But $\pi_{2p+1}\mathrm{BP}(p,pq) = 0$, which is a contradiction. □

The theorem implies in particular that on X there is an unramified degree-p division algebra over K that does not extend to an Azumaya algebra on X. The case $p=2$ was proved first in [3].

Corollary 3.7. *Let p be a prime There exists a smooth affine complex variety X of dimension $2p+2$ and an unramified division algebra D over $\mathbb{C}(X)$ of degree p that contains no Azumaya maximal order on X.*

Scholium 3.8. *Let p be a prime, and let $n_1, \ldots, n_k$ be integers greater than p such that $\gcd_i\{n_i\} = p$. There is a smooth affine complex variety X of dimension $2p+2$ and a Brauer class $\alpha \in \mathrm{Br}(X)$ of exponent p such that there are Azumaya algebras of degrees $n_1, \ldots, n_k$ in the class α, but no Azumaya algebra of degree p.*

Proof. The proof is largely the same as that of the theorem, but using the algebraic group

$$\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_k} / \mu_p,$$

where μ_p is embedded diagonally in each of the groups SL_{n_i}. □

3.3. Local purity. In contrast to the global failure of purity for PGL_p-torsors exhibited above, in this section, we give a proof that purity holds for $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$ when X is the spectrum of a regular local ring R and the Brauer class has exponent invertible in X. Our result is a minor generalization of a recent theorem of Ojanguren [30] and of the local purity result for PGL_n in characteristic 0 due to Panin [32].

To prove the theorem, we recall first a result of DeMeyer, which is also used by both Ojanguren and Panin.

Theorem 3.9 (DeMeyer [17, Corollary 1]). *Suppose that R is an integral semi-local ring and that $\alpha \in \mathrm{Br}(R)$. Then, there exists a unique Azumaya algebra A with class α having no idempotents besides 0 and 1. Moreover, any other Azumaya algebra with class α is of the form $\mathrm{M}_n(A)$ for some n.*

Now, we prove our local purity result. Define $\mathrm{H}^1(X, \mathrm{PGL}_n)'$ to be the set of PGL_n-torsors whose associated Brauer class in $\mathrm{Br}(X)$ has exponent invertible in X.

Theorem 3.10. *Suppose that R is a regular noetherian integral semi-local ring. Then, purity holds for $\mathrm{H}^1(\operatorname{Spec} R, \mathrm{PGL}_n)'$.*

Proof. Let K be the function field of R, and let D be a degree n central simple algebra in

$$\bigcap_{\operatorname{ht} P=1} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R_p, \mathrm{PGL}_n)' \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, \mathrm{PGL}_n)'\right).$$

Let m be the exponent of $[D] \in \mathrm{Br}(K)$. Because D lifts to every codimension 1 local ring, so does the Brauer class. Since m is invertible in R and hence in these local rings, this Brauer class lifts to a Brauer class $\alpha \in \mathrm{Br}(R)$, by purity for $\mathrm{Br}(R)'$ (see Example 3.1).

By DeMeyer's theorem, there exists an Azumaya algebra A with Brauer class α such that every other Azumaya algebra in the class α is isomorphic to $\mathrm{M}_r(A)$ for some r. In particular, $\operatorname{ind}(\alpha) = \deg(A)$, where, if X is a scheme and $\alpha \in \mathrm{Br}(X)$, we define $\operatorname{ind}(\alpha)$ to be the gcd of the degrees of all Azumaya algebras with class α. On the other hand, by [2, Proposition 6.1], the index of α can be computed either over R or over K. Thus, $\mathrm{ind}(\alpha)$ divides $\deg(D)$. Therefore, $D \cong \mathrm{M}_r(A_K)$ for some integer $r > 0$. It follows that $\mathrm{M}_r(A)$ is a class in $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathrm{PGL}_n)'$ that restricts to D, which shows that purity holds for $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathrm{PGL}_n)'$. □

3.4. Canonical factorization. We prove in this section a theorem we view as evidence for Conjecture 1.2 for all PGL_n.

Let $m > 1$ divide n. Both $\mathrm{BP}(m,n)$ and $\mathrm{BPGL}_m(\mathbb{C})$ are equipped with canonical maps to $K(\mathbb{Z}/m, 2)$. Moreover, a topological $\mathrm{PGL}_n(\mathbb{C})$ bundle, $P \to X$, may be lifted to a $\mathrm{P}(m,n)$ bundle if and only if the associated obstruction class $\delta_n(P)$ in $\mathrm{H}^2(X, \mathbb{Z}/n)$ is m-torsion. A canonical factorization of Azumaya algebras with structure group $\mathrm{P}(m,n)$ is a factorization $\mathrm{BP}(m,n) \to \mathrm{BPGL}_m \to K(\mathbb{Z}/m, 2)$. The existence of such a factorization would give, for every Azumaya algebra A of degree n and m-torsion obstruction class, a canonical Azumaya algebra B of degree m with the same obstruction class in $\mathrm{H}^2(X, \mathbb{Z}/m)$. Unsurprisingly, this cannot occur.

Theorem 3.11. *If $n > m$, then there is no canonical factorization $\mathrm{BP}(m,n) \to \mathrm{BPGL}_m(\mathbb{C}) \to K(\mathbb{Z}/m, 2)$.*

Proof. Suppose that $\mathrm{BP}(m,n) \to K(\mathbb{Z}/m, 2)$ factors through $\mathrm{BPGL}_m(\mathbb{C}) \to K(\mathbb{Z}/m, 2)$. Let $\mathrm{BPGL}_m(\mathbb{C}) \to \mathrm{BP}(m,n)$ be the map induced block-summation. Write $f : \mathrm{BP}(m,n) \to \mathrm{BP}(m,n)$ for the composition. This map induces an isomorphism $\mathrm{H}^2(\mathrm{BP}(m,n), \mathbb{Z}/m) \cong \mathbb{Z}/m$, and is in particular not nullhomotopic.

As $\mathrm{BP}(m,n)$ is homotopy equivalent to BSU_n/μ_m, there is a complete description of the homotopy-classes of self-maps $\mathrm{BP}(m,n) \to \mathrm{BP}(m,n)$ due to Jackowski, McClure, and Oliver [27, Theorem 2]. Their theorem says we can factor f as $\mathrm{B}\alpha \circ \psi^k$, where α is an outer automorphism of $P(m,n)$, and ψ^k is an unstable Adams operation on $\mathrm{BP}(m,n)$, for some $k \geq 0$ prime to the order of the Weyl group of $P(m,n)$, which is $n!$. The map ψ^k induces multiplication by k^i on $\mathrm{H}^{2i}(\mathrm{BP}(m,n), \mathbb{Q})$. In particular a map $\mathrm{BP}(m,n) \to \mathrm{BP}(m,n)$ is either nullhomotopic or induces an isomorphism on rational cohomology.

The rational cohomology of $\mathrm{BP}(m,n)$ is

$$\mathrm{H}^*(\mathrm{BP}(m,n), \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_n], \quad c_i \in \mathrm{H}^{2i}(\mathrm{BP}(m,n), \mathbb{Q}),$$

while that of $\mathrm{BPGL}_m(\mathbb{C})$ is

$$\mathrm{H}^*(\mathrm{BPGL}_m(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_m], \quad c_i \in \mathrm{H}^{2i}(\mathrm{BP}(m,n), \mathbb{Q}).$$

In particular,

$$\dim \mathrm{H}^{2m+2}(\mathrm{BP}(m,n), \mathbb{Q}) = \dim \mathrm{H}^{2m+2}(\mathrm{BPGL}_m(\mathbb{C}), \mathbb{Q}) + 1,$$

so that f cannot induce an isomorphism on rational cohomology, and must be nullhomotopic, a contradiction. □

The argument above has philosophically informed the authors' work both in this paper and in [3]. In order to construct algebraic counterexamples, however, we must use complex algebraic varieties X that approximate $\mathrm{BPGL}_m(\mathbb{C})$ in the sense that there exists a map $X(\mathbb{C}) \to \mathrm{BPGL}_m(\mathbb{C})$ induced by an algebraic PGL_m-torsor on X and inducing an isomorphism on homotopy groups in a range dimensions, and for these we cannot bring the strength of [27] to bear. We have made do with *ad hoc* arguments that furnish obstructions in known, bounded dimension to maps $\mathrm{BPGL}_m(\mathbb{C}) \to \mathrm{BPGL}_m(\mathbb{C})$. For instance, the topological plank in the argument proving that purity fails for $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_p)$ is an obstruction to a map

$$\tau_{\leq 2p+1}\mathrm{BP}(p, pq) \to \tau_{\leq 2p+1}\mathrm{BPGL}_p(\mathbb{C})$$

that induces an isomorphism on Brauer group. This obstruction depends on Theorem 2.9, which describes a restriction on maps

$$\tau_{\leq 2p+1}\mathrm{BPGL}_p \to \tau_{\leq 2p+1}\mathrm{BPGL}_p,$$

in that it says a map inducing an isomorphism on π_2 must necessarily also induce an isomorphism on the p–primary part of π_{2p+1}. To prove Conjecture 1.2 for all PGL_m, one might only have to find an obstruction to the existence of maps $X \to \mathrm{BPGL}_m$ where X approximates $\mathrm{BP}(m, mq)$, with $q > 1$ prime to m.

3.5. The Witt group. Our second application of topology to purity is to give a new example where purity fails for the cohomological filtration on the Witt group.

Example 3.12. Local purity is known for the Witt group $\mathrm{W}(\operatorname{Spec} R)$ whenever R is a regular noetherian local ring containing a field of characteristic not 2 by work of Ojanguren and Panin [31].

Given the positive results for the Brauer group, it is natural to ask the following question.

Question 3.13. Does purity hold for $\mathrm{W}(X)$ when X is an regular excellent noetherian integral scheme having no points of characteristic 2?

It is known that purity holds for $\mathrm{W}(X)$ when X is a regular noetherian separated integral scheme of Krull dimension at most 4 and 2 is invertible in $\Gamma(X, \mathscr{O}_X)$ by Balmer-Walter [8, Corollary 10.3]. However, Totaro [37] showed that the injectivity property fails for the Witt group: there is a smooth affine complex 5-fold such that $\mathrm{W}(X) \to \mathrm{W}(K)$ is not injective. Thus, it might be natural to guess that the purity property fails as well. For an extensive overview of results on purity for the Witt group, see Auel [6].

Let $\mathrm{I}^1(X)$ be the ideal of $\mathrm{W}(X)$ generated by even-dimensional quadratic spaces. There is a discriminant map $\mathrm{I}^1(X) \to \mathrm{H}^1_{\text{ét}}(X, \mu_2)$. Let $\mathrm{I}^2(X)$ be the kernel. There is a map $\mathrm{I}^2(X) \to {}_2\operatorname{Br}(X)$, called the Clifford invariant map. Denote by $\mathrm{I}^3(X)$ the kernel. It is known that purity fails for $\mathrm{I}^2(X)/\mathrm{I}^3(X)$. The first examples were due to Parimala and Sridharan [33], who showed that it fails for some affine bundle torsors over smooth projective p-adic curves. We include another example below, which uses a smooth affine variety we constructed in [2], giving the first examples over $\mathbb{C}$.

Example 3.14. Let X be the smooth affine 5-dimensional variety over $\mathbb{C}$ constructed in [2, Theorem D], having a Brauer class $\alpha \in \mathrm{Br}(X)$ of exponent 2 that is not in the image of the Clifford invariant map $\mathrm{I}^2(X) \to {}_2\mathrm{Br}(X)$. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathrm{I}^3(X) \arrow[r] \arrow[d] & \mathrm{I}^3(K) \arrow[r] \arrow[d] & \bigoplus_{p\in X^{(1)}} \mathrm{I}^2(k(p)) \arrow[d] \\
 & \mathrm{I}^2(X) \arrow[r] \arrow[d] & \mathrm{I}^2(K) \arrow[r] \arrow[d] & \bigoplus_{p\in X^{(1)}} \mathrm{I}^1(k(p)) \arrow[d] \\
0 \arrow[r] & {}_2\mathrm{Br}(X) \arrow[r] & {}_2\mathrm{Br}(K) \arrow[r] \arrow[d] & \bigoplus_{p\in X^{(1)}} \mathrm{H}^1(k(p),\mathbb{Z}/2) \arrow[d] \\
 & & 0 & 0,
\end{tikzcd}$$

where the columns and the bottom row are exact, and where $\mathrm{I}^2(X)$ (resp. $\mathrm{I}^3(X)$) maps into the kernel of the map $\mathrm{I}^2(K) \to \bigoplus \mathrm{I}^1(k(p))$ (resp $\mathrm{I}^3(K) \to \bigoplus \mathrm{I}^2(k(p))$). The image of α in ${}_2\mathrm{Br}(K)$ is in the image of the map $\mathrm{I}^2(K) \to {}_2\mathrm{Br}(K)$ by Merkurjev's theorem; say it is the Clifford invariant of $\sigma \in \mathrm{I}^2(K)$. Then, σ is unique up to an element of $\mathrm{I}^3(K)$. On the other hand, the ramification classes $\partial_p(\sigma)$ are all in $\mathrm{I}^2(k(p))$. Hence, $\overline{\sigma} \in \mathrm{I}^2(K)/\mathrm{I}^3(K)$ is unramified. But, by construction, it is not in the image of $\mathrm{I}^2(X)/\mathrm{I}^3(K) \to \mathrm{I}^2(K)/\mathrm{I}^3(K)$.

This is the first such example known for a variety over an algebraically closed field. It has the added advantage that it is not explained by the presence of line-bundle valued quadratic forms, as explained in [2, Section 7].

References

[1] B. Antieau and B. Williams, *The period-index problem for twisted topological K-theory*, Geom. Topol. 18 (2014), no. 2, 1115–1148.

[2] ———, *The topological period-index problem over 6-complexes*, J. Topol. 7 (2014), no. 3, 617–640.

[3] ———, *Unramified division algebras do not always contain Azumaya maximal orders*, Invent. Math. 197 (2014), no. 1, 47–56.

[4] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4); Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[5] A. Auel, *Surjectivity of the total Clifford invariant and Brauer dimension*, ArXiv e-prints (2011), available at `http://arxiv.org/abs/1108.5728`.

[6] ______, *Remarks on the Milnor conjecture over schemes*, Galois-Teichmller Theory and Arithmetic Geometry (Kyoto, 2010) (H. Nakamura, F. Pop, L. Schneps, and A. Tamagawa, eds.), Advanced Studies in Pure Mathematics, vol. 63, 2012, pp. 1–30.

[7] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409.

[8] P. Balmer and C. Walter, *A Gersten-Witt spectral sequence for regular schemes*, Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 1, 127–152.

[9] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. 5 (1958), 35–61.

[10] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972.

[11] H. Cartan, *Détermination des algèbres* $\mathrm{H}_*(\pi, n; \mathbb{Z})$, Séminaire H. Cartan 7 (1954/1955), no. 1, 11-01–11-24.

[12] V. Chernousov and I. Panin, *Purity of* G_2*-torsors*, C. R. Math. Acad. Sci. Paris 345 (2007), no. 6, 307–312.

[13] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. 244 (1979), no. 2, 105–134.

[14] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.

[15] J.-L. Colliot-Thélène and C. Voisin, *Cohomologie non ramifiée et conjecture de Hodge entière*, Duke Math. J. 161 (2012), no. 5, 735–801.

[16] A. J. de Jong, *A result of Gabber*, available at `http://www.math.columbia.edu/~dejong/`.

[17] F. R. DeMeyer, *Projective modules over central separable algebras*, Canad. J. Math. 21 (1969), 39–43.

[18] T. Ekedahl, *Approximating classifying spaces by smooth projective varieties*, ArXiv e-prints (2009), available at `http://arxiv.org/abs/0905.1538`.

[19] R. Fedorov and I. Panin, *Proof of Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields*, ArXiv e-prints (2012), available at `http://arxiv.org/abs/1211.2678`.

[20] K. Fujiwara, *A proof of the absolute purity conjecture (after Gabber)*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 153–183.

[21] O. Gabber, *Some theorems on Azumaya algebras*, The Brauer group (Sem., Les Plans-sur-Bex, 1980), Lecture Notes in Math., vol. 844, Springer, Berlin, 1981, pp. 129–209.

[22] ______, *A note on the unramified Brauer group and purity*, Manuscripta Math. 95 (1998), no. 1, 107–115.

[23] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006.

[24] M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 14, Springer-Verlag, Berlin, 1988.

[25] A. Grothendieck, *Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses*, Séminaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1995, pp. Exp. No. 290, 199–219.

[26] R. T. Hoobler, *A cohomological interpretation of Brauer groups of rings*, Pacific J. Math. 86 (1980), no. 1, 89–92.

[27] S. Jackowski, J. McClure, and B. Oliver, *Homotopy classification of self-maps of BG via G-actions. I*, Ann. of Math. (2) 135 (1992), no. 1, 183–226.

[28] J. P. Jouanolou, *Une suite exacte de Mayer-Vietoris en K-théorie algébrique*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 293–316. Lecture Notes in Math., Vol. 341.

[29] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[30] M. Ojanguren, *Wedderburn's theorem for regular local rings*, Linear Algebraic Groups Preprint Server (2013), available at `http://www.mathematik.uni-bielefeld.de/LAG/man/500.html`.

[31] M. Ojanguren and I. Panin, *A purity theorem for the Witt group*, Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 1, 71–86.

[32] I. A. Panin, *Purity conjecture for reductive groups*, Vestnik St. Petersburg Univ. Math. 43 (2010), no. 1, 44–48.

[33] R. Parimala and R. Sridharan, *Nonsurjectivity of the Clifford invariant map*, Proc. Indian Acad. Sci. Math. Sci. 104 (1994), no. 1, 49–56. K. G. Ramanathan memorial issue.

[34] C. Schoen, *Complex varieties for which the Chow group mod n is not finite*, J. Algebraic Geom. 11 (2002), no. 1, 41–100.

[35] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[36] ______, *Torsion algebraic cycles and complex cobordism*, J. Amer. Math. Soc. 10 (1997), no. 2, 467–493.

[37] ______, *Non-injectivity of the map from the Witt group of a variety to the Witt group of its function field*, J. Inst. Math. Jussieu 2 (2003), no. 3, 483–493.

[38] G. Vezzosi, *On the Chow ring of the classifying stack of* $\mathrm{PGL}_{3,\mathbf{C}}$, J. Reine Angew. Math. 523 (2000), 1–54.

[39] A. Vistoli, *On the cohomology and the Chow ring of the classifying space of* PGL_p, J. Reine Angew. Math. 610 (2007), 181–227.

Benjamin Antieau
UIC
antieau@math.uic.edu

Ben Williams
UBC
tbjw@math.ubc.ca