Equivariant Cycles and Cancellation for Motivic Cohomology

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Abstract. We introduce a Bredon motivic cohomology theory for smooth schemes defined over a field and equipped with an action by a finite group. These cohomology groups are defined for finite dimensional representations as the hypercohomology of complexes of equivariant correspondences in the equivariant Nisnevich topology. We generalize the theory of presheaves with transfers to the equivariant setting and prove a Cancellation Theorem.

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Contents

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1. Introduction

The theory of motivic cohomology for smooth schemes over a base field is a well established one. It is a powerful computational tool with ramifications for many branches of algebra, algebraic and arithmetic geometry: quadratic forms, algebraic K-theory, special values of L-functions, to name a few. The success of this theory is best exemplified by its fundamental role in Voevodsky's resolution of Milnor's conjecture on Galois cohomology [Voe03].

The purpose of this article is to generalize Suslin and Voevodsky's construction of motivic cohomology [VSF00], especially Voevodsky's machinery of presheaves with transfers [Voe00], to the equivariant setting of smooth schemes over a field k equipped with an action of a finite group G (and $|G|$ coprime to $\mathrm{char}(k)$).

Using Totaro's construction [Tot99] of an algebro-geometric version of the classifying space of an algebraic group, Edidin-Graham [EG98] have constructed an equivariant version of Bloch's higher Chow groups. This theory has proved to be interesting, amongst other reasons, for its connection to equivariant algebraic K-theory and the equivariant Riemann-Roch theorem [EG00, EG08]. These equivariant higher Chow groups are an algebro-geometric version of topological Borel cohomology. Our construction follows a different route altogether and results in a more refined *Bredon style* cohomology theory. These Bredon motivic cohomology groups form a new set of invariants for smooth schemes with G-action. As a first indication that the Bredon motivic cohomology theory we construct is a refinement of equivariant higher Chow groups, we note that the former is equipped with a grading by the representations of G while the latter is graded by integers. In fact in Section 5.3 we construct a comparison map from our Bredon motivic cohomology to the equivariant higher Chow groups.

Topological Bredon cohomology has recently experienced a surge of interest in part because of its appearence in the work of Hill-Hopkins-Ravenel [HHR], specifically through the equivariant slice spectral sequence for certain spectra. The $\mathbb{Z}/2$-equivariant case of this spectral sequence was first constructed by Dugger in [Dug05] where he constructed a spectral sequence relating Bredon cohomology groups (with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$) and Atiyah's KR-theory. The motivic analog of Atiyah's KR-theory is Hermitian K-theory, constructed as a $\mathbb{Z}/2$-equivariant motivic spectrum $K\mathbb{R}^{alg}$ by Hu-Kriz-Ormsby [HKO11]. Our construction of Bredon motivic cohomology is motivated in part by a program to construct and use a $\mathbb{Z}/2$-equivariant motivic generalization of Dugger's spectral sequence as a tool for studying these Hermitian K-theory groups.

We define our theory via hypercohomology in the equivariant Nisnevich topology, introduced in [Del09] and [HKO11]. A surjective, equivariant étale map $f : Y \to X$ is an equivariant Nisnevich cover if for each point $x \in X$ there is a point $y \in Y$ such that f induces an isomorphism $k(x) \cong k(y)$ on residue fields and an isomorphism $G_y \cong G_x$ on *set-theoretic* stabilizers. A different generalization of the Nisnevich topology was introduced in [Her13], the fixed point Nisnevich topology. A cover in this topology is as above but with the requirement that f induce an isomorphism $I_y \cong I_x$ on scheme-theoretic stabilizers rather than the set-theoretic ones. (An equivalent formulation is that for each subgroup H, the map on fixed points $f^H : Y^H \to X^H$ is a Nisnevich cover.) We focus on the equivariant Nisnevich topology in the present work for two reasons. One is that equivariant algebraic K-theory fulfills descent in the equivariant Nisnevich topology but not in the fixed point Nisnevich topology. The second is that the fixed point Nisnevich topology does not behave well with respect to transfers, see Example 4.11.

A presheaf with equivariant transfers is a presheaf F of abelian groups on smooth G-schemes which are equipped with functorial maps $\mathcal{Z}^* : F(Y) \to F(X)$ for finite equivariant correspondences $\mathcal{Z}$ from X to Y. The presheaf with equivariant transfers freely generated by X is denoted by $\mathbb{Z}_{tr,G}(X)$. We define the equivariant motivic complexes $\mathbb{Z}_G(n)$ by forming the $\mathbb{A}^1$-singular chain complex on $\mathbb{Z}_{tr,G}(\mathbb{P}(k[G]^{\oplus n} \oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(k[G]^{\oplus n})$, where $k[G]$ is the regular representation. Bredon motivic cohomology is defined as equivariant Nisnevich hypercohomology with coefficients in the complex $\mathbb{Z}_G(n)$. In light of the equivariant Dold-Thom theorem [dS03], our construction is analogous to the topological Bredon cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. Our setup is however flexible enough to allow for more sophisticated coefficient systems. The benefit of allowing Mackey functor coefficients is well understood in topology. As explained in Section 5.2, we have an embedding of cohomological Mackey functors into our category.

Over the complex numbers there is a topological realization functor which relates our Bredon motivic cohomology groups and topological Bredon cohomology groups. This comparison map is the subject of an equivariant Beilinson-Lichtenbaum type conjecture, predicting that a range of these groups should be isomorphic with torsion coefficients. The ordinary Beilinson-Lichtenbaum conjecture, relating motivic cohomology and étale cohomology (or singular cohomology), is equivalent to the Milnor and Bloch-Kato conjectures which have been proved in work of Voevodsky and Rost. In a sequel paper [HVØ] we show that this equivariant conjecture is true for $G = \mathbb{Z}/2$ and any torsion coefficients. The key new ingredient in that work is a $(\mathbb{Z}/2)^n$-equivariant generalization of Voevodsky's Cancellation Theorem, which is the main result of this present paper. The Cancellation Theorem is an algebro-geometric version of the familiar suspension isomorphism in singular cohomology of topological spaces. It asserts that in motivic cohomology, we can cancel suspension factors of the algebro-geometric sphere $\mathbb{G}_m$. Besides the usual $\mathbb{G}_m$, our equivariant version also allows for $\mathbb{G}_m$ equipped with an action by $(\mathbb{Z}/2)^n$. Write ϵ_i for the generator of the ith factor of $(\mathbb{Z}/2)^n$. For a subset $I \subseteq \{1, \dots, n\}$, write $\mathbb{G}_m^{\sigma_I}$ for the $(\mathbb{Z}/2)^n$-scheme which is $\mathbb{G}_m$ equipped with the action specified by letting $\epsilon_i \in (\mathbb{Z}/2)^n$ act by $x \mapsto 1/x$ if $i \in I$ and by the identity otherwise.

Theorem 1.1 (Equivariant Cancellation). *Let X be a smooth $(\mathbb{Z}/2)^n$-scheme over a perfect field k,* $\mathrm{char}(k) \neq 2$. *Then*

$$H^n_{GNis}(X, C_*\mathbb{Z}_{tr,G}(Y)) = H^n_{GNis}(X \wedge \mathbb{G}_m^{\sigma_I}, C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I}))$$

for all $I \subseteq \{1, \dots n\}$.

We establish the equivariant Cancellation Theorem as Theorem 9.7 below. Its proof uses equivariant modifications of Voevodsky's arguments in [Voe10a] and relies on equivariant versions of the main results of Voevodsky's techniques for analyzing the cohomological behaviour of presheaves with transfers. Most of the paper is focused on the generalization of this machinery.

As mentioned, our motivation is to study $\mathbb{Z}/2$-equivariant phenomena, but where possible we establish our results in a greater generality. Everything works best under the assumption that the irreducible representations of G (which are defined over the base field) are all one-dimensional. Any group which satisfies this condition is necessarily abelian. An abelian group G over a field k will satisfy this condition if k contains a primitive dth roots of unity, where d is the least common multiple of the orders of elements of G. In particular over an algebraically closed field every finite abelian group satisfies this condition. However, abelian groups over fields without enough roots of unity can fail this condition, e.g., $G = \mathbb{Z}/3$ has an irreducible two dimensional representation over $k = \mathbb{R}$. See the beginning of Section 5 for more details. A main source of this condition on the group G arises from the question of existence of equivariant triples. We refer to Section 7 for a precise definition of an equivariant triple, but point out here that it is in particular a smooth equivariant relative curve $X \to S$ between smooth G-schemes. A typical step in several key arguments used in the course of establishing the main homotopy invariance result (see Theorem 8.12) is that in order to establish an isomorphism of sheaves with equivariant transfers, it suffices to show the isomorphism on the generic points of smooth G-schemes. To establish this reduction step we need a good supply of equivariant triples. More precisely, if $x \in X$ is a point, there should exist an invariant open neighborhood $U \subseteq X$ of x and a smooth G-scheme S such that $U \to S$ is a smooth equivariant curve. As a simple illustrative example, consider a representation V viewed as a smooth G-scheme. Suppose there is a smooth equivariant curve $f : U \to C$, where U is an invariant neighborhood of the origin in V. Then there is an equivariant surjection $df : V \cong T_0U \to T_{f(0)}C$. This surjection splits (as $|G|$ and $\mathrm{char}(k)$ are coprime) and so V contains a one-dimensional summand. It could however happen that V admits no one-dimensional summand and thus there is no neighborhood of the origin in V fitting into a smooth equivariant curve. We establish the existence of triples around an arbitrary point of a smooth quasi-projective G-schemes under the assumption that all irreducible representations of G are one-dimensional.

Other main results are as follows. Theorem 7.18 provides a Mayer-Vietoris exact sequence for certain special equivariant Nisnevich covers. This has important consequences. It in particular allows for the computation of the equivariant Nisnevich cohomology of open invariant subsets of G-line bundles over

smooth zero dimensional G-schemes, see Theorem 7.18, which is the precursor to the homotopy invariance theorem.
In Theorem 8.12 we establish our homotopy invariance result.

Theorem 1.2. *Suppose that all irreducible $k[G]$-modules are one-dimensional. Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Then $H^n_{GNis}(-, F_{GNis})$ is also a homotopy invariant presheaf with equivariant transfers.*

Lastly we mention that several other constructions of equivariant cohomology theories related to algebraic cycles exist. There is for example work of Lawson, Lima-Filho, and Michelson [LLFM96], Joshua [Jos07], and Levine-Serpe [LS08]. It would be interesting to see how these different constructions relate to the one carried out here.
An outline of this paper is as follows. In Section 2 we record results about G-schemes, equivariant divisors, bundles, and cohomology that we need in later sections. In Section 3 we recall the equivariant Nisnevich topology and establish some of its basic properties not already appearing in the literature. In Section 4 we introduce equivariant finite correspondences and presheaves with equivariant transfers. We formally define Bredon motivic cohomology in Section 5 and using the machinery developed in later sections we establish properties and some computations. We relate equivariant transfers and equivariant divisors in Section 6, in particular in Theorem 6.12 we compute the equivariant Suslin homology of equivariant affine curves. In Section 7 we study equivariant triples and establish a Mayer-Vietoris sequence in Theorem 7.18. The homotopy invariance of cohomology is established in Section 8. Finally in Section 9 we establish a $\mathbb{Z}/2$-equivariant generalization of Voevodsky's Cancellation Theorem.
Notations and conventions. Throughout k is a field, which is assumed to be perfect starting in Section 7 and G is a finite group whose order is coprime to char(k). The finite group G is viewed as a group scheme over k via $G_k := \coprod_G \mathrm{Spec}(k)$. Usually we simply write again G for this group scheme. Write $G\mathrm{Sch}/k$ (resp. $G\mathrm{Sm}/k$) for the category whose objects are seperated schemes of finite type (resp. smooth schemes) over k equipped with a left action by G and morphisms are equivariant morphisms.
We write $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ for the affine scheme associated to a vector space over k and $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee))$ for the associated projective scheme. Sometimes we write V for both the vector space as well as its associated scheme $\mathbb{A}(V)$.
It is important to distinguish between two types of stabilizer groups of $x \in X$,

(1) the *set-theoretic stabilizer* G_x of x is $G_x = \{g \in G \,|\, gx = x\}$
(2) the *inertia group* of x is $I_x = \ker(G_x \to \mathrm{Aut}(k(x)/k))$.

Given a subset $Z \subseteq X$ we write $G{\cdot}Z$ or GZ for the *orbit* $\cup_{g\in G} gZ$ of Z. For a nonclosed point $x \in X$, $G{\cdot}x$ is given a scheme structure via $G{\cdot}x = (G\times\{x\})/G_x$.

If F is a presheaf on $G\mathrm{Sm}/k$ and $S = \lim_i S_i$ is an inverse limit of smooth G-schemes over k, with equivariant transition maps, then we set $F(S) = \mathrm{colim}_i F(S_i)$.

2. Schemes with G-action

In this section we collect several useful facts used throughout this paper about schemes with an action of a finite group.

2.1. Quotients by group actions. We first recall some basic facts about quotients of schemes by finite groups, for full details see e.g., [SGA03] or [MFK94]. By a quotient $\pi : X \to X/G$ we simply mean a categorical quotient. In particular quotients are unique when they exist. If $X = \mathrm{Spec}(A)$ then $X/G = \mathrm{Spec}(A^G)$. More generally if X is quasi-projective then a quotient $\pi : X \to X/G$ exists. The categorical quotient of a scheme by a finite group satisfies the following additional properties:

(1) π is finite and surjective,
(2) The fibers of π are the G-orbits of the G-action on X,
(3) $\mathcal{O}_{X/G} = \pi_*(\mathcal{O}_X)^G$,
(4) if $Y \to X/G$ is flat, then $X \times_{X/G} Y \to Y$ is a quotient, and
(5) if $|G|$ and $\mathrm{char}(k)$ are coprime and $W \subseteq X$ is a closed and invariant, then $W \to \pi(W)$ is a quotient.

Definition 2.1. Say that X is *equivariantly irreducible* or *G-irreducible* provided there is an irreducible component X_0 of X such that $G{\cdot}X_0 = X$.

Let $H \subseteq G$ be a subgroup and X an H-scheme. The scheme $G \times X$ becomes an H-scheme under the action $h(g, x) = (gh^{-1}, hx)$ and we define

$$G \times^H X = (G \times X)/H.$$

The scheme $G \times^H X$ has a left G-action through the action of G on itself. Concretely $G \times^H X$ has the following description. Let g_i be a complete set of left coset representatives. Then $G \times^H X = \coprod_{g_i} X_i$, each X_i is a copy of X and $g \in G$ acts as $k : X_i \to X_j$ where $k \in H$ satisfies $gg_i = g_j k$. The functor $G \times^H - : H\mathrm{Sch}/k \to G\mathrm{Sch}/k$ is left adjoint to the restriction functor $G\mathrm{Sch}/k \to H\mathrm{Sch}/k$.

The H-action on $G \times X$ is free and so $\pi : G \times X \to G \times^H X$ is a principle H-bundle. In particular, π is étale and surjective. It follows that if X is smooth, then so is $G \times^H X$. This defines a left adjoint to the restriction functor $G\mathrm{Sm}/k \to H\mathrm{Sm}/k$,

$$G \times^H - : H\mathrm{Sm}/k \to G\mathrm{Sm}/k.$$

2.2. G-sheaves and cohomology. Let X be a G-scheme. Write $\sigma : G\times X \to X$ for the action map and $pr_2 : G \times X \to X$ for the projection. We write τ for any one of the Zariski, Nisnevich, or étale Grothendieck topologies on X.

Definition 2.2. Let $\mathcal{F}$ be a sheaf of abelian groups on X.

(1) A *G-linearization* of $\mathcal{F}$ is an isomorphism $\phi : \sigma^*\mathcal{F} \to pr_2^*\mathcal{F}$ of sheaves on $G \times X$ which satisfies the cocyle condition

$$[pr_{23}^*(\phi)] \circ [(1 \times \sigma)^*(\phi)] = (m \times 1)^*(\phi)$$

on $G \times G \times X$. Here $m : G \times G \to G$ is multiplication and $pr_{23} : G \times G \times X \to G \times X$ is the projection to second and third factors.

(2) A *G-sheaf* (in the τ-topology) on X is a pair $(\mathcal{F}, \phi)$ consisting of a sheaf $\mathcal{F}$ on X and a G-linearization ϕ of $\mathcal{F}$.

(3) A *G-module* on X is a G-sheaf $(\mathcal{M}, \phi)$ where $\mathcal{M}$ is an $\mathcal{O}_X$-module and the G-linearization $\phi : \sigma^*\mathcal{M} \cong pr_2^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_{G\times X}$-modules. Similarly a *G-vector bundle* on X consists of a G-module $(\mathcal{V}, \phi)$ such that $\mathcal{V}$ is a locally free $\mathcal{O}_X$-module on X.

We usually write $\mathcal{F}$ rather than $(\mathcal{F}, \phi)$ for a G-sheaf or module, leaving the G-linearization implicit.

Remark 2.3. The previous definition works for any algebraic group G. Our groups are always finite, in which case a G-linearization of $\mathcal{F}$ is equivalent to the data of isomorphisms $\phi_g : \mathcal{F} \xrightarrow{\cong} g_*\mathcal{F}$ for each $g \in G$ which are subject to the conditions that $\phi_e = id$ and $\phi_{gh} = h_*(\phi_g) \circ \phi_h$ for all $g, h \in G$.

An equivariant morphism $f : (\mathcal{E}, \phi_{\mathcal{E}}) \to (\mathcal{F}, \phi_{\mathcal{F}})$ of G-sheaves is a morphism f of sheaves which is compatible with the G-linearizations in the sense that $\phi_{\mathcal{F}} \circ \sigma^* f = pr_2^* f \circ \phi_{\mathcal{E}}$. Write $\mathrm{Ab}_\tau(G, X)$ for the category whose objects are G-sheaves on X and morphisms are the equivariant morphisms. The category of G-sheaves on X has enough injectives. We have similarly the category $\mathrm{Mod}_G(X)$ of G-modules on X and $\mathrm{Vec}_G(X)$ of G-vector bundles on X.

Remark 2.4. Recall that if G acts on the ring R, the skew-group ring $R^\#[G]$ is defined as follows. As a (left) R-module it is free with basis $\{[g] \,|\, g \in G\}$. Multiplication is defined by setting $(r_g[g])(r_h[h]) = r_g(g{\cdot}r_h)[gh]$ and extending linearly. If G acts trivially on R, then $R^\#[G]$ is the usual group ring $R[G]$.
If $X = \mathrm{Spec}(R)$, then the category of G-modules on X is equivalent to the category of modules over the skew-group ring $R^\#[G]$.

Given a G-sheaf $\mathcal{M}$, the group G acts on the global sections $\Gamma(X, \mathcal{M})$. Define the *invariant global sections* functor $\Gamma_X^G : \mathrm{Ab}_\tau(G, X) \to \mathrm{Ab}$ by $\Gamma_X^G(\mathcal{M}) = \Gamma(X, \mathcal{M})^G$. The τ-G-cohomology groups $H_\tau^p(G; X, \mathcal{M})$ are defined as the right derived functors

$$H_\tau^p(G; X, \mathcal{M}) := R^p\Gamma_X^G(\mathcal{M}).$$

The functor Γ_X^G can be expressed as a composition $\Gamma_X^G = (-)^G \circ \Gamma(X, -)$. The functor $\Gamma(X, -)$ sends injective G-sheaves to injective G-modules and so the Grothendieck spectral sequence for this composition yields the convergent spectral sequence

$$E_2^{p,q} = H^p(G, H_\tau^q(X, \mathcal{M})) \Rightarrow H_\tau^{p+q}(G; X, \mathcal{M}), \tag{2.5}$$

where $H^*(G, -)$ is group cohomology.

Definition 2.6 ([MFK94]). The equivariant Picard group $\mathrm{Pic}^G(X)$ is the group of G-line bundles on X, with group operation given by tensor product.

The equivariant Picard group has the following well known cohomological interpretation.

Theorem 2.7 (Equivariant Hilbert 90). *Let X be a G-scheme.*

(1) *There is a natural isomorphism*

$$\mathrm{Pic}^G(X) \xrightarrow{\cong} H^1_{et}(G; X, \mathcal{O}^*_X).$$

(2) *There are natural isomorphisms*

$$H^1_{Zar}(G; X, \mathcal{O}^*_X) \cong H^1_{Nis}(G; X, \mathcal{O}^*_X) \cong H^1_{et}(G; X, \mathcal{O}^*_X).$$

Proof. The second item follows from the first together with the spectral sequence (2.5). We sketch a proof of the first item. Consider the quotient stack $[X/G]$. The map $X \to [X/G]$ is an étale cover and $X \times_{[X/G]} X$ is represented by $G \times X$, see e.g.,[LMB00, Exemple 4.6.1]. We consider the ringed topos $([X/G]_{et}, \mathcal{O}_{[X/G]})$. A G-linearization of a sheaf $\mathcal{F}$ on X_{et} is exactly the descent data necessary to descend $\mathcal{F}$ to a sheaf on $[X/G]$. In particular the categories $\mathrm{Ab}_{et}([X/G])$ and $\mathrm{Ab}_{et}(G, X)$ are equivalent and the category of line bundles on $([X/G]_{et}, \mathcal{O}_{[X/G]})$ is equivalent to the category of G-line bundles on X. Additionally, $\Gamma([X/G], \mathcal{F}) = \Gamma(X, \mathcal{F})^G$. We thus have $H^1_{et}(G; X, \mathcal{O}^*_X) = H^1_{et}([X/G], \mathcal{O}^*_{[X/G]})$. Now by [Gir71, Proposition III.2.4.5, Remarque III.3.5.4], $H^1_{et}([X/G], \mathcal{O}^*)$ is the group of $\mathcal{O}^*_{[X/G]}$-torsors over $[X/G]_{et}$. By [Gir71, Corollaire III.2.5.2] this is isomorphic to the group of isomorphism classes of line bundles on $([X/G]_{et}, \mathcal{O}_{[X/G]})$, which in turn is isomorphic to $\mathrm{Pic}^G(X)$.

□

Theorem 2.7 and the spectral sequence (2.5) yield the exact sequence

(2.8)

$$0 \to H^1(G, H^0(X, \mathcal{O}^*_X)) \to \mathrm{Pic}^G(X) \to (\mathrm{Pic}(X))^G \to H^2(G, H^0(X, \mathcal{O}^*_X)).$$

Lemma 2.9. *If X is a reduced G-scheme then $p^* : \mathrm{Pic}^G(X) \to \mathrm{Pic}^G(X \times \mathbb{A}^1)$ is injective, where $p : X \times \mathbb{A}^1 \to X$ is the projection. If X is normal then p^* is an isomorphism.*

Proof. If X is reduced then $H^0(X, \mathcal{O}^*_X) = H^0(X \times \mathbb{A}^1, \mathcal{O}^*_{X \times \mathbb{A}^1})$ and $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(X \times \mathbb{A}^1)$ is injective. If X is normal then it is an isomorphism. The lemma then follows from (2.8) together with the five lemma. □

2.3. Divisors on G-schemes. The notion of Cartier divisor and rational equivalence of Cartier divisors admits a straightforward equivariant generalization.

Definition 2.10. (1) An *equivariant Cartier divisor* on X is an element of $\Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X)^G$. Write $\mathrm{Div}^G(X)$ for this group with the group law written additively.

(2) A *principal equivariant Cartier divisor* is defined to be an invariant rational function on X; that is, an element in the image of $\Gamma(X, \mathcal{K}^*)^G$.
(3) Two equivariant Cartier divisors are *equivariantly rationally equivalent*, written $D \sim D'$, if $D - D'$ is principal. Write $\mathrm{Div}^G_{rat}(X)$ for the group of equivariant Cartier divisors modulo rational equivalence.

A global section of $\Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$ is specified by giving an open covering U_i and $f_i \in \Gamma(U_i, \mathcal{K}^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*_X)$ for all i, j. This section is G-invariant if $\{(U_i, f_i)\}$ and $\{(gU_i, gf_i)\}$ determine the same global section for all $g \in G$, where gf_i is the rational function $gf_i(x) = f_i(g^{-1}x))$. This means that $\{(U_i, f_i)\}$ is G-invariant if and only if $gf_i/f_j \in \Gamma(gU_i \cap U_j, \mathcal{O}^*_X)$ for all i, j and $g \in G$.
Write $\mathcal{Z}^1(X)$ for the group of codimension one cycles on X. The homomorphism $\mathrm{cyc} : \mathrm{Div}(X) \to \mathcal{Z}^1(X)$ is defined by $\mathrm{cyc}(D) = \sum_{Z \in X^{(1)}} \mathrm{ord}_Z(D)Z$ where $X^{(1)}$ is the set of closed integral codimension one subschemes.

LEMMA 2.11. *Let X be a smooth G-scheme. Then*

$$\mathrm{cyc} : \mathrm{Div}(X) \to \mathcal{Z}^1(X)$$

is an equivariant isomorphism.

Proof. Note that if $D \in \mathrm{Div}(X)$ then $\mathrm{ord}_Z(gD) = \mathrm{ord}_{g^{-1}Z}(D)$. It follows that cyc is equivariant. Since X is smooth, cyc is an isomorphism. □

Given an equivariant Cartier divisor the usual construction yields a G-line bundle. Recall that if $D = \{(U_i, f_i)\}$ is a Cartier divisor, then the associated line bundle $\mathcal{O}(D)$ is defined by $\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i} f_i^{-1}$. It is straightforward to check that when D is an equivariant Cartier divisor then the associated line bundle $\mathcal{O}(D)$ has a canonical G-linearization. We write again $\mathcal{O}(D)$ for the G-line bundle defined by this choice of linearization.

PROPOSITION 2.12. *Let X be a regular G-scheme.*
(1) *The association $D \mapsto \mathcal{O}(D)$ induces an injective homomorphism*

$$\mathrm{Div}^G_{rat}(X) \hookrightarrow \mathrm{Pic}^G(X),$$

whose image consists of G-line bundles $\mathcal{L}$ which admit an equivariant injection $\mathcal{L} \hookrightarrow \mathcal{K}_X$.
(2) *If G acts faithfully on X then every G-line bundle admits such an injection into $\mathcal{K}_X$. In particular* $\mathrm{Div}^G_{rat}(X) = \mathrm{Pic}^G(X)$.

Proof. The first part is straightforward from the definitions. When X has faithful action, then $H^1_{Zar}(G; X, \mathcal{K}^*_X) = 0$ which implies that $\mathrm{Div}^G(X) \to \mathrm{Pic}^G(X)$ is surjective. □

When the action isn't faithful, $\mathrm{Div}^G_{rat}(X) \subseteq \mathrm{Pic}^G(X)$ can be a proper subgroup. For example, $\mathrm{Pic}^G(k)$ is isomorphic to the character group of G over k while $\mathrm{Div}^G_{rat}(k) = 0$.

PROPOSITION 2.13. *If X is regular then* $\mathrm{Div}^G_{rat}(X \times \mathbb{A}^1) = \mathrm{Div}^G_{rat}(X)$.

Proof. Let $K = \ker(G \to \mathrm{Aut}(X))$. Then G/K acts faithfully on X and $\mathrm{Div}^G_{rat}(X) = \mathrm{Div}^{G/K}_{rat}(X)$. Since $\mathrm{Pic}^{G/K}(X \times \mathbb{A}^1) = \mathrm{Pic}^{G/K}(X)$ the proposition follows from Proposition 2.12. $\square$

3. Equivariant Nisnevich topology

In this section we introduce the equivariant Nisnevich topology and list some of its properties. The equivariant Nisnevich topology on quasiprojective G-schemes was defined by Voevodsky [Del09] in order to extend the functor of taking quotients by group actions to motivic spaces. More recently, versions of the equivariant Nisnevich topology (on not necessarily quasiprojective smooth G-schemes) have been considered by Hu-Kriz-Ormsby [HKO11] and Krishna-Østvær [KØ12] (for Deligne-Mumford stacks). A related topology, the fixed point Nisnevich topology, was defined and studied by Herrmann in [Her13]. The fixed point Nisnevich topology has pleasant homotopical properties but unfortunately does not seem well suited for our constructions involving presheaves with equivariant transfers.

3.1. Basic properties. A *cd-structure* on a category $\mathcal{C}$ is a collection $\mathcal{P}$ of commutative squares of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "e"] & X \end{tikzcd}$$

which are closed under isomorphism. The Grothendieck topology associated to $\mathcal{P}$ is the Grothendieck topology generated by declaring all pairs $(A \to X, Y \to X)$ to be coverings.

Definition 3.1. (1) A Cartesian square in $G\mathrm{Sch}/k$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "e"] & X \end{tikzcd}$$

is said to be an *equivariant distinguished square* if p is étale, $e : A \subseteq X$ is an invariant open embedding and p induces an isomorphism $(Y - B)_{red} \xrightarrow{\cong} (X - A)_{red}$.

(2) The *equivariant Nisnevich topology* on $G\mathrm{Sm}/k$ (resp. $G\mathrm{Sch}/k$) is the Grothendieck topology associated to the *cd*-structure defined by the equivariant distinguished squares and we write $(G\mathrm{Sm}/k)_{GNis}$ (resp. $(G\mathrm{Sch}/k)_{GNis}$) for the associated site.

Lemma 3.2. *A presheaf of sets F is a sheaf in the equivariant Nisnevich topology if and only if $F(\emptyset) = *$ and for any distinguished square Q as above the* *square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{array}$$

is a pullback square.

Proof. This follows from [Voe10b, Lemma 2.9]. □

EXAMPLE 3.3. Let V be a representation. Consider the equivariant open covering of $\mathbb{P}(V \oplus 1)$ given by $\mathbb{P}(V \oplus 1) - \mathbb{P}(V) = \mathbb{A}(V)$ and $\mathbb{P}(V \oplus 1) - \mathbb{P}(1)$. The intersection of these opens is identified with $\mathbb{A}(V) - 0$. We thus have an equivariant distinguished square

$$\begin{array}{ccc} \mathbb{A}(V) - 0 & \longrightarrow & \mathbb{A}(V) \\ \downarrow & & \downarrow \\ \mathbb{P}(V \oplus 1) - \mathbb{P}(1) & \longrightarrow & \mathbb{P}(V \oplus 1). \end{array}$$

The standard characterizations of a Nisnevich cover in the nonequivariant setting admit an equivariant generalization.

DEFINITION 3.4. Let $f : Y \to X$ be an equivariant morphism. An *equivariant splitting sequence* for $f : Y \to X$ is a sequence of invariant closed subvarieties

$$\emptyset = Z_{m+1} \subseteq Z_m \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = X$$

such that $f|_{Z_i - Z_{i+1}} : f^{-1}(Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ has an equivariant section. The integer m is called the length of this splitting sequence.

PROPOSITION 3.5. *Let $f : Y \to X$ be an equivariant étale map between G-schemes. The following are equivalent.*

(1) *The map f is an equivariant Nisnevich cover.*
(2) *The map f has an equivariant splitting sequence.*
(3) *For every point $x \in X$, there is a point $y \in Y$ such that f induces an isomorphism $k(x) \cong k(y)$ of residue fields and an isomorphism $G_y \cong G_x$ of set-theoretic stabilizers.*

Proof. The proof follows along the lines of the nonequivariant arguments in [MV99, Lemma 3.1.5] and [Voe10c, Proposition 2.17].

(1)⇔(2) Suppose that $\{V_i \to X\}$ is an equivariant Nisnevich cover. Note that there is a dense invariant open subscheme $U \subseteq X$ on which $f : \coprod V_i \to X$ has a splitting. Indeed, this is true by definition for covers coming from distinguished squares and this property is preserved by pullbacks and by compositions. Restricting to the complement of this open and repeating the argument we construct an equivariant splitting sequence, which must stop at a finite stage because X is Noetherian.

For the converse, we proceed by induction on the length of a splitting sequence. The case $m = 0$ is immediate. Suppose that we have an equivariant splitting sequence of length m. The restriction of f to $Z_m \times_X Y \to Z_m$ has an equivariant section s. Since s is equivariant and étale, $s(Z_m) \subseteq Z_m \times_X Y$ is an invariant open. Let D be its closed complement, equipped with the induced reduced structure. Consider the map $\widetilde{Y} := Y - D \to X$. Then $\{\widetilde{Y} \to X, X - Z_m\}$ forms an equivariant distinguished covering of X. The pullback of $f : Y \to X$ along $X - Z_m$ has an equivariant splitting sequence of length less than m and so by induction is an equivariant Nisnevich cover. Similarly the pullback of f along $\widetilde{Y} \to X$ equivariantly splits and is thus also an equivariant Nisnevich cover. It follows that f itself is an equivariant Nisnevich covering.

(2)⇔(3) Suppose that f has an equivariant splitting sequence. Then $x \in U_k = Z_k - Z_{k+1}$ for some k. Let s be a section of f over U_k and let $y = s(x)$. Then one immediately verifies that f induces an isomorphism $k(x) \cong k(y)$ and $G_y \cong G_x$.

For the other direction, by Noetherian induction it suffices to show that if for each generic point $\eta \in X$ there is $\eta' \in Y$ so that f induces $k(\eta) \cong k(\eta')$ and $G_\eta \cong G_{\eta'}$ then there is an equivariant dense open $U \subset X$ such that $Y \times_X U \to U$ has an equivariant splitting. To show this it suffices to assume that X is equivariantly irreducible. Let $\eta \in X$ be a generic point. Then there is an $\eta' \in Y$ such that $f : \eta' \cong \eta$ and $G_{\eta'} \cong G_\eta$. This implies that $G{\cdot}\eta' \to G{\cdot}\eta$ is an equivariant isomorphism. We have that $G{\cdot}\eta' = \cap W'$ (resp. $G{\cdot}\eta_i$) is the intersection over all invariant opens W' in Y containing η' (resp. all invariant opens in X) and so there is some invariant open $W' \subseteq Y$ such that $W' \to f(W')$ is an equivariant isomorphism. Setting $U = f(W')$ we obtain our equivariant splitting.

□

Changing the condition above on stabilizers leads to the variant of the equivariant Nisnevich topology defined in [Her13].

Definition 3.6. An equivariant étale map $f : Y \to X$ is a *fixed point Nisnevich cover* if for each point point $x \in X$, there is a point $y \in Y$ such that f induces an isomorphism $k(x) \cong k(y)$ of residue fields and an isomorphism $I_y \cong I_x$ of inertia groups.

By [Her13, Lemma 2.12], a map $Y \to X$ is a fixed point Nisnevich cover if and only if for every subgroup $H \subseteq G$, the map on fixed points $f^H : Y^H \to X^H$ is a Nisnevich cover. The following simple example illustrates an important difference between these two topologies.

Example 3.7. Let X be a smooth G-scheme with free action. Consider the action map $G \times X^{triv} \to X$, where X^{triv} is the G-scheme X considered with trivial action. This is a fixed point Nisnevich cover. However, it is not equivariantly locally split and so is not an equivariant Nisnevich cover.

We refer to [Voe10b] for the definition of a complete, regular, and bounded cd-structure.

Theorem 3.8. *The equivariant Nisnevich cd-structure on* $G\mathrm{Sm}/k$ *is complete, regular and bounded.*

Proof. The argument is similar to that of [Voe10c, Theorem 2.2] for the usual Nisnevich topology. We provide a brief sketch of the details. First, since the equivariant distinguished squares are closed under pullback, it follows from [Voe10b, Lemma 2.4] that the equivariant Nisnevich cd-structure is complete. For regularity, one needs to see that for an equivariant distinguished square the square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ B\times_A B \arrow[r] & Y\times_X Y \end{tikzcd}$$

is also distinguished, where the horizontal arrows are the diagonal. Because an equivariant distinguished square is a square whose maps are equivariant and which is nonequivariantly a distinguished square, this follows immediately from the nonequivariant case which is verified in [Voe10c, Lemma 2.14].
It remains to see that the cd-structure is bounded. For this we use the equivariant analogue of the standard density structure. That is for a smooth G-scheme X, let $D_q(X)$ be the set of open, invariant embeddings $U\to X$ whose complement has codimension at least q. The arguments of [Voe10c, Proposition 2.10] carry over to the equivariant case to show that equivariant cd-structure is bounded by this density structure. □

Corollary 3.9. *Let* $\mathcal{F}$ *be a sheaf of abelian groups on* $G\mathrm{Sm}/k$ *in the equivariant Nisnevich topology and let* X *be a smooth* G*-scheme over* k*. Then*

$$H^i_{GNis}(X,\mathcal{F})=0$$

for $i>\dim(X)$*.*

3.2. Points. Let A be a commutative ring and $I\subseteq A$ an ideal contained in the Jacobson radical of A. Recall that (A,I) is said to be a *Henselian pair* if for every étale ring map $f:A\to B$ and any $p:B\to A/I$ such that the composition $pf:A\to A/I$ equals the quotient map, there is a lifting of p to an A-homomorphism $\overline{p}:B\to A$. We say that the pair (A,I) has a G-action, if A has a G-action and the ideal I is invariant. There is a functorial Henselization of the pair (A,I), consisting of a ring map $A\to A^h$ such that (A^h,IA^h) is a Hensel pair and $A/I\cong A^h/IA^h$, see e.g., [Ray70]. If (A,I) has G-action then G acts on (A^h,IA^h) as well because $(-)^h$ is functorial.

Definition 3.10. Say that S is a *semilocal Henselian affine* G*-scheme over* k if $S=\mathrm{Spec}(A^h)$, where A^h is the Henselization of a pair (A,I) where A is a semilocal ring with G-action which is essentially of finite type over k, and I is the Jacobson radical. We say that S is *smooth over* k if A is essentially smooth over k.

Remark 3.11. Let S be a semilocal Henselian affine G-scheme over k. Let $Z \subseteq S$ be the set of closed points and suppose that $f : Y \to S$ is an equivariant étale map which admits an equivariant splitting over Z. Then f admits an equivariant splitting. Indeed, since S is Henselian there exists a splitting $s : S \to Y$ extending the one over Z. Then s is both an open and closed immersion. Thus s is an isomorphism of S onto its image and determines a decomposition $Y = s(S) \coprod Y'$. Note that $s(S)$ is invariant, otherwise $gz \in Y'$ for some $g \in G$ and $z \in Z$ but s is equivariant on Z. It follows that $s(S) \subseteq Y$ is invariant and so s is equivariant, being the inverse of the equivariant isomorphism $f|_{s(S)}$.
In particular if S is a semilocal Henselian affine G-scheme with a single closed orbit and $Y \to S$ is an equivariant Nisnevich cover, then it can be refined by the trivial covering.

Suppose that X is a G-scheme over k and $x \in X$ has an invariant open affine neighborhood. Then $\mathcal{O}_{X,Gx}$ is a semilocal ring with G-action and $\mathrm{Spec}(\mathcal{O}^h_{X,Gx})$ is a semilocal Henselian affine G-scheme over k with a single closed orbit. Any semilocal Henselian affine G-scheme over k with a single orbit is equivariantly isomorphic to $\mathrm{Spec}(\mathcal{O}^h_{A,Gx})$, for some affine G-scheme A and $x \in A$.
In general, a point $x \in X$ might not be contained in any G-invariant affine neighborhood. We can however still consider $G \times^{G_x} \mathrm{Spec}(\mathcal{O}^h_{X,x})$. Additionally, it is always the case that $Gx = G \times^{G_x} \{x\} \subseteq G \times^{G_x} X$ has a G-invariant affine neighborhood. The canonical map $\pi : G \times^{G_x} \mathcal{O}^h_{X,x} \to \mathcal{O}^h_{X,x}$ is étale and $G \times^{G_x} \{x\} \to G{\cdot}x$ is an isomorphism, so π is equivariantly split over $G{\cdot}x$.
For $x \in X$ write $N(Gx)$ for the filtering category whose objects are pairs $(p : U \to X, s)$ where p is étale and $s : Gx \to U$ is a section of p over Gx, and U is the union of its connected components which contain an element of $s(Gx)$. A morphism $(U \to X, s)$ to $(V \to X, s')$ is a map $f : U \to V$ making the evident triangles commute. Write $N_G(Gx)$ for the filtering category whose objects are pairs $(p : U \to X, s)$ where U is an equivariantly irreducible G-scheme, p is an equivariant étale map, and $s : Gx \to U$ is an equivariant section of p over Gx. A morphism $(U \to X, s)$ to $(V \to X, s')$ in $N_G(Gx)$ is a map $f : U \to V$ making the evident triangles commute. We sometimes write $N(X; Gx)$ and $N_G(X; Gx)$ for these indexing categories if we need to be explicit about the ambient G-scheme containing Gx.

Remark 3.12. Let U be a G_x-invariant affine neighborhood of $x \in X$. The transition maps of $N_G(G \times^{G_x} U, Gx)$ are affine, so $\lim_{V \in N_G(G \times^{G_x} U, Gx)} V$ exists in the category of k-schemes. The map $G \times^{G_x} U \to X$ is an equivariant étale neighborhood of Gx. In particular the map $N_G(G \times^{G_x} U, Gx) \to N_G(X; Gx)$ is initial and so $\lim_{N_G(X;Gx)} V$ exists as well (and equals $\lim_{N_G(G \times^{G_x} U, Gx)} V$).

Proposition 3.13. *The forgetful functor $\phi : N_G(Gx) \to N(Gx)$ is initial. In particular*
$$\lim_{U\in N_G(Gx)} U \cong \operatorname{Spec}(\mathcal{O}^h_{G\times^{G_x}X,x}) \cong G\times^{G_x}\operatorname{Spec}(\mathcal{O}^h_{X,x}).$$
If $x\in X$ has a G-invariant affine neighborhood then additionally we have a canonical G-isomorphism $G\times^{G_x}\operatorname{Spec}(\mathcal{O}^h_{X,x}) \xrightarrow{\cong} \operatorname{Spec}(\mathcal{O}^h_{X,Gx})$.

Proof. We need to show that the comma category $(\phi/(p,s))$ is nonempty and connected for any $(p : U \to X, s) \in N(Gx)$. It suffices to check that it is nonempy since if $\phi(q_1,s_1) \to (p,s)$ and $\phi(q_2,s_2) \to (p,s)$ are two arrows in $(\phi/(p,s))$, there is $(q_3 : V_3 \to X, s_3)$ in $N_G(Gx)$ which maps to (q_1,s_1) and (q_2,s_2). The two maps $(q_3,s_3) \to (p,s)$ obtained from composition agree on each point of $s(Gx)$ and induce the same map on the residue fields of these points. Each connected component of V_3 contains a point of $s(Gx)$ and so both maps $(q_3,s_3) \to (p,s)$ are equal.
Let $(p : U \to X, s) \in N(Gx)$. For $g\in G$, define $p^g : g_*U \to X$ to be the étale X-scheme given by $g_*U = U$ and $p^g := gp$. The identity $U = U$ can be viewed as a map $g_*U \to (hg)_*U$ over $h : X \to X$.
Label the elements of G by $e = g_0, g_1, \dots g_n$. Define W to be the $(n+1)$-fold fiber product
$$W = U \times_X (g_1)_*U \times_X \cdots \times_X (g_n)_*U.$$
Write $\pi_{g_i} : W \to (g_i)_*U$ for the projection and consider W as an X-scheme via the composite $p\pi_e : W \to U \to X$. Note that $hp\pi_h = p\pi_e$.
Now W has a G-action given by permuting the factors. In other words we define $h : W \to W$ to be the map determined by the formula $\pi_{g_i}h = \pi_{h^{-1}g_i}$. This determines a map since $p^{g_i}\pi_{h^{-1}g_i} = p^{g_j}\pi_{h^{-1}g_j}$. Moreover $p^{g_i}\pi_{h^{-1}g_i} = hp\pi_e$ and thus $p\pi_e : W \to X$ is an equivariant étale map.
Define now $s' : Gx \to W$ to be the map determined in the g_ith coordinate by $sg_i^{-1} : Gx \to (g_i)_*U$. This defines an equivariant section of $p\pi_e : W \to X$ over $Gx \subseteq X$. and so $(W \to X, s') \in N_G(Gx)$. Now π_e determines a map $(W \to X, s') \to (U \to X, s)$ in $N(Gx)$ and so $(\phi/(p,s))$ is nonempty, which completes the proof. □

For a G-scheme X and $x \in X$ write $p_x^*F = F(\mathcal{O}^h_{G\times^{G_x}X,Gx}) = \operatorname{colim}_{U\in N_G(Gx)} F(U)$. This defines a fiber functor from the category of sheaves to sets, i.e., it commutes with colimits and finite products and so determines a point of the equivariant Nisnevich topos. Every affine semilocal G-scheme with a single closed orbit determines such a point. By the previous paragraphs, any such S is of the form $G\times^{G_x}\operatorname{Spec}(\mathcal{O}^h_{X,x}) = \operatorname{Spec}(\mathcal{O}^h_{G\times^{G_x}X,Gx})$ for an appropriate G-scheme X. We now verify that these points form a conservative set of points.

Theorem 3.14. *A map $\phi : F_1 \to F_2$ of sheaves of sets on $G\mathrm{Sch}/k$ (resp. $G\mathrm{Sm}/k$) is an isomorphism if and only if $F_1(S) \to F_2(S)$ is an isomorphism for all (resp. all smooth) semilocal affine G-schemes S over k with a single closed orbit.*

Proof. Let S be a semilocal affine G-scheme over k with a single closed orbit. If ϕ is an isomorphism of sheaves then it induces an isomorphism $F_1(S) \cong F_2(S)$ as S_{GNis} is trivial.

Suppose that ϕ induces isomorphisms $F_1(S) \cong F_2(S)$ for all S. We first show that ϕ is a monomorphism. Suppose that $\phi(\alpha) = \phi(\beta)$ for some $\alpha, \beta \in F_1(X)$. Then $[\alpha] = [\beta] \in F_1(\mathcal{O}^h_{G\times^{G_x}X,Gx})$ for all $x \in X$. Thus for each $x \in X$, there is some equivariant étale map $V_x \to X$ which admits an equivariant section over $Gx \to V$ and $\alpha|_V = \beta|_V$. Let $\eta_i \in X$ be generic points. Then $V_{\eta_i} \to X$ has an equivariant section over an invariant open $U_1 \subseteq X$. Consider V_{α_i} corresponding to generic points of $Z_1 = X - U_1$ and let $Z_2 \subseteq Z_1$ the complement of the set where $\coprod V_{\alpha_i}$ has a section. Proceeding in this way we obtain a finite number of equivariant étale maps $V_{x_i} \to X$ such that $V = \coprod V_{x_i} \to X$ is a Nisnevich cover. Moreover V has the property that $\alpha|_V = \beta|_V$ and because F_1 is a sheaf this means that $\alpha = \beta$ in $F_1(X)$.

Now we show that ϕ is a surjection. Let $\alpha \in F_2(X)$. For any $x \in X$ there is $[\beta] \in F_1(\mathcal{O}^h_{G\times^{G_x}X,Gx})$ such that $\phi([\beta]) = [\alpha] \in F_2(\mathcal{O}^h_{G\times^{G_x}X,Gx})$. Thus for each $x \in X$ there is an equivariant étale map $f_x : V_x \to X$, which admits an equivariant section $Gx \to V_x$, and $\beta_x \in F_1(V_x)$ such that $\phi(\beta_x) = \alpha|_{V_x}$. As in the previous paragraph we can find a finite number of points $x_1, \ldots, x_n$ such that $V = \coprod V_{x_i} \to X$ is an equivariant Nisnevich cover. The elements β_{x_i} determine the element $\beta \in F_1(V)$ with the property that $\phi(\beta) = \alpha|_V$ and thus ϕ is surjective as well. □

Remark 3.15. By [Her13], the points of the fixed point Nisnevich topology are the semilocal affine G-schemes of the form $G/H \times \operatorname{Spec}(\mathcal{O}^h_{X,x})$ where $H \subseteq G$ is a subgroup, and x is a point of a smooth scheme X equipped with trivial action.

Since $(G\mathrm{Sm}/k)_{GNis}$ has enough points we can form the Godement resolution of a presheaf.

Definition 3.16. Let $F(-)$ be a presheaf of chain complexes of abelian groups on $G\mathrm{Sm}/k$. Let

$$G^0F(U) = \prod_{u\in U} F(\mathcal{O}^h_{G\times^{G_u}U,u}).$$

Define $G^nF = G^0 \circ \cdots \circ G^0F$ to be the $(n+1)$-fold composition of G^0. The inclusions and projections of the various factors give $n \mapsto G^nF(U)$ a cosimplicial structure. The *Godement resolution* $F(-) \to \mathcal{G}F(-)$ is defined by

$$\mathcal{G}F(U) := \operatorname{Tot} G^{\bullet}F(U).$$

Then $F \to \mathcal{G}F$ is a flasque resolution of F on $(G\mathrm{Sm}/k)_{GNis}$. Consequently $\mathcal{G}F(U)$ computes the hypercohomology with coefficients in F:

$$H^k\mathcal{G}F(U) = H^k_{GNis}(U, F).$$

3.3. CHANGE OF GROUPS. Let $H \subseteq G$ be a subgroup. We have an adjoint pair of functors $\epsilon : H\mathrm{Sch}/k \rightleftarrows G\mathrm{Sch}/k : \rho$ where $\epsilon(X) = G \times^H X$ and $\rho(W) = W$. These restrict to an adjoint pair

$$\epsilon : H\mathrm{Sm}/k \rightleftarrows G\mathrm{Sm}/k : \rho. \tag{3.17}$$

Both functors commute with fiber products and send covering families to covering families for the equivariant Nisnevich topologies. We thus have adjoint functors

$$\epsilon^* : \mathrm{Shv}_{HNis}(H\mathrm{Sm}/k) \rightleftarrows \mathrm{Shv}_{GNis}(G\mathrm{Sm}/k) : \epsilon_*$$

and

$$\rho^* : \mathrm{Shv}_{GNis}(G\mathrm{Sm}/k) \rightleftarrows \mathrm{Shv}_{HNis}(H\mathrm{Sm}/k) : \rho_*,$$

where $\epsilon_*F(X) = F(G \times^H X)$ and $\rho_*K(W) = K(W)$, and similarly for the categories of sheaves on $G\mathrm{Sch}/k$ and $H\mathrm{Sch}/k$. Additionally we have that $\rho^* = \epsilon_*$. It follows that ϵ_* is exact and so we have the following.

LEMMA 3.18. *Let X be an H-scheme. Then*

$$H^i_{GNis}(G \times^H X, F) = H^i_{HNis}(X, \epsilon_* F).$$

If we restrict our attention to the category GQP/k of quasiprojective schemes of finite type over k equipped with left G-action we have an adjoint pair

$$\lambda : GQP/k \rightleftarrows QP/k : \eta,$$

where $\lambda(X) = X/G$ is the quotient by the G-action and $\eta(W) = W^{triv}$, where W^{triv} is the scheme W equipped with the trivial action. The functor η commutes with fiber products and sends covering families to covering families. By [Del09, Proposition 43] the functor λ induces a continuous map of sites $(QP/k)_{Nis} \to (GQP/k)_{GNis}$, i.e., the presheaf $X \mapsto F(X/G)$ on $(GQP)_{GNis}$ is a sheaf whenever F is a sheaf on $(QP/k)_{Nis}$. We thus have adjoint functors

$$\eta^* : \mathrm{Shv}_{Nis}(QP/k) \rightleftarrows \mathrm{Shv}_{GNis}(GQP/k) : \eta_*$$

and

$$\lambda^* : \mathrm{Shv}_{GNis}(GQP/k) \rightleftarrows \mathrm{Shv}_{Nis}(QP/k) : \lambda_*,$$

where $\eta_*F(W) = F(W^{triv})$ and $\lambda_*K(X) = K(X/G)$. Additionally we have that $\eta^* = \lambda_*$. It follows that λ_* is exact and so we have the following.

LEMMA 3.19. *Let X be a quasiprojective G-scheme. Then*

$$H^i_{GNis}(X, \lambda_* F) = H^i_{Nis}(X/G, F).$$

4. PRESHEAVES WITH EQUIVARIANT TRANSFERS

Let $G\,\mathrm{Cor}_k$ denote the category whose objects are smooth G-varieties and morphisms are equivariant finite correspondences, that is

$$G\,\mathrm{Cor}_k(X, Y) := \mathrm{Cor}_k(X, Y)^G.$$

An *elementary equivariant correspondence* from X to Y is a correspondence of the form $\mathcal{Z} = Z + g_1 Z + \cdots + g_k Z$, where $Z \subseteq X \times Y$ is an elementary correspondence and g_i range over a set of coset representatives for $\mathrm{Stab}(Z) =$

$\{g \in G | g(Z) = Z\}$. The group $G\operatorname{Cor}_k(X,Y)$ is the free abelian group generated by the elementary equivariant correspondences.

There is an embedding of categories $G\mathrm{Sm}/k \subseteq G\operatorname{Cor}_k$ which sends an equivariant map $f : X \to Y$ to its graph $\Gamma_f \subseteq X \times Y$.

Definition 4.1. A *presheaf with equivariant transfers* on $G\mathrm{Sm}/k$ is an additive presheaf $F : G\operatorname{Cor}_k^{op} \to \mathrm{Ab}$.

Definition 4.2.

1. An *elementary equivariant* $\mathbb{A}^1$*-homotopy* between two maps in $G\mathrm{Sm}/k$ (resp. in $G\operatorname{Cor}_k$) $f_0, f_1 : X \to Y$ is a map $H : X \times \mathbb{A}^1 \to Y$ in $G\mathrm{Sm}/k$ (resp. in $G\operatorname{Cor}_k$) such that $H|_{X\times\{i\}} = f_i$.
2. A map $f : X \to Y$ is said to be an *elementary equivariant* $\mathbb{A}^1$*-homotopy equivalence* if there is a map $g : Y \to X$ such that both fg and gf are elementary equivariant $\mathbb{A}^1$-homotopic to the identity.
3. If F is a presheaf on $G\mathrm{Sm}/k$ or on $G\operatorname{Cor}_k$, we say that F is *homotopy invariant* if the projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p^* : F(X) \xrightarrow{\cong} F(X \times \mathbb{A}^1)$.

A simple but useful consequence of homotopy invariance is that all representations are contractible.

Proposition 4.3. *Let F be homotopy invariant presheaf of abelian groups and V a finite dimensional representation. Then $p^* : F(X \times \mathbb{A}(V)) \to F(X)$ is an isomorphism.*

Proof. The map $\mathbb{A}(V) \times \mathbb{A}^1 \to \mathbb{A}(V)$, $(v,t) \mapsto tv$ is an equivariant homotopy between the identity on $\mathbb{A}(V)$ and $\mathbb{A}(V) \to \{0\} \subseteq \mathbb{A}(V)$. □

Every smooth G-scheme Y represents a presheaf with equivariant transfers which we write as

$$\mathbb{Z}_{tr,G}(Y)(-) = G\operatorname{Cor}_k(-,Y).$$

Note that this is in fact a sheaf in the equivariant Nisnevich topology. If Y is quasi-projective, the nth symmetric power $\mathrm{Sym}^n(Y) = Y^{\times n}/\Sigma_n$ of Y exists as a scheme. We write $\mathrm{Sym}(Y) = \coprod_n \mathrm{Sym}^n(Y)$. When X is normal, the map

$$\operatorname{Cor}_k(X,Y) \to \mathrm{Hom}_{\mathrm{Sch}/k}(X, \mathrm{Sym}(Y))^+$$

becomes an isomorphism after inverting the exponential characteristic, see e.g., [SV96, Theorem 6.8] or [BV08, Proposition 2.1.3]. Here $(-)^+$ denotes the group completion of the displayed monoid and the monoid structure comes from the one on $\mathrm{Sym}(Y)$.

Example 4.4. The sheaf $(\mathcal{O}^*)^G$ of invariant invertible functions is a presheaf with equivariant transfers which can be seen using Lemma 6.13. Alternatively one may describe the transfer structure as follows. The sheaf $\mathcal{O}^*$ is represented by the group scheme $\mathbb{G}_m$. We have an induced monoid morphism $\rho : \mathrm{Sym}(\mathbb{G}_m) \to \mathbb{G}_m$. Let $\mathcal{W} : X \to \mathrm{Sym}(Y)$ be an effective finite correspondence. Define $\mathcal{W}^* : \mathbb{G}_m(Y) \to \mathbb{G}_m(X)$ by $\mathcal{W}^*(\phi) = \rho\mathrm{Sym}(\phi)\mathcal{W}$. It is immediate from this definition that $\mathcal{W}^*(\phi)$ is equivariant whenever $\mathcal{W}$ and ϕ are equivariant.

Lemma 4.5. *Let $H \subseteq G$ be a subgroup. There is an adjunction*

$$\epsilon : H\,\mathrm{Cor}_k \rightleftarrows G\,\mathrm{Cor}_k : \rho \tag{4.6}$$

where $\epsilon(X) = G \times^H X$ and $\rho(W) = W$.

Proof. We need to show that if X is a smooth H-scheme and W is a smooth G-scheme then $H\,\mathrm{Cor}_k(X, W) \cong G\,\mathrm{Cor}_k(G \times^H X, W)$. We have an H-equivariant map $i : X \to G \times^H X$, induced by $x \mapsto (e, x)$. This gives $i^* : G\,\mathrm{Cor}_k(G \times^H X, W) \to H\,\mathrm{Cor}_k(X, W)$. It is straightforward to check that this is an isomorphism. □

Definition 4.7. A presheaf F with equivariant transfers is said to be an *equivariant Nisnevich sheaf with transfers* provided that the restriction of F to $G\mathrm{Sm}/k$ is a sheaf in the equivariant Nisnevich topology.

We finish this section with a discussion of the relation between transfers and sheafification.

Theorem 4.8. *Let X be a smooth G-scheme and $p : Y \to X$ an equivariant Nisnevich cover. Then*

$$\cdots \xrightarrow{p_0 - p_1 + p_2} \mathbb{Z}_{tr,G}(Y \times_X Y) \xrightarrow{p_0 - p_1} \mathbb{Z}_{tr,G}(Y) \xrightarrow{p} \mathbb{Z}_{tr,G}(X) \to 0$$

is exact as a complex of equivariant Nisnevich sheaves.

Proof. The argument is similar to the nonequivariant argument [MVW06, Proposition 6.12]. It suffices to check that the complex

$$\cdots \to \mathbb{Z}_{tr,G}(Y \times_X Y)(S) \to \mathbb{Z}_{tr,G}(Y)(S) \to \mathbb{Z}_{tr,G}(X)(S) \to 0 \tag{4.9}$$

is exact for every semilocal Henselian affine G-scheme S with a single closed orbit. Let $Z \subseteq X \times S$ be an invariant closed subscheme which is quasi-finite over S. Define $L(Z/S)$ to be the free abelian group generated by the irreducible components of Z which are finite and surjective over S. The assignment $Z \mapsto L(Z/S)^G$ is covariantly functorial for equivariant maps of quasi-finite G-schemes over S. The sequence (4.9) is a filtered colimit of sequences of the form

$$\cdots \to L(Z_Y \times_Z Z_Y/S)^G \to L(Z_Y/S)^G \to L(Z/S)^G \to 0 \tag{4.10}$$

where the colimit is taken over all invariant closed subschemes $Z \subseteq X \times S$ which are finite and surjective over S. It therefore suffices to show that (4.10) is exact. Since S is a semilocal Henselian affine G-scheme over k with a single closed orbit and Z is finite over S it is also Hensel semilocal. The equivariant Nisnevich covering $Z_Y \to Z$ therefore splits equivariantly (see Remark 3.11). Let $s_1 : Z \to Z_Y$ be a splitting. Set $(Z_Y)^k_Z := Z_Y \times_Z \cdots \times_Z Z_Y$ and let $s_k : L((Z_Y)^k_Z/S)^G \to L((Z_Y)^{k+1}_Z)^G$ be the map induced by $s_1 \times_Z id_{(Z_Y)^k_Z}$. This is a contracting homotopy, which completes the proof. □

The previous statement fails when we replace the equivariant Nisnevich topology with the fixed point Nisnevich topology. (The following is also an example of a fixed point Nisnevich covering for which equivariant K-theory does not satisfy descent).

Example 4.11. Consider the $\mathbb{Z}/2$-schemes $X = \mathrm{Spec}(\mathbb{C})$ over $\mathrm{Spec}(\mathbb{R})$ with conjugation action and X^{triv} the scheme X with trivial action. Let $Y = \mathbb{Z}/2 \times X^{triv}$. The action map $Y \to X$ is then a fixed point Nisnevich cover. In [Her13] it is shown that the points of the fixed point Nisnevich topology are of the form $G/H \times \mathcal{O}^h_{W^H,w}$. In particular, if

$$\cdots \to \mathbb{Z}_{tr,G}(Y \times_X Y) \to \mathbb{Z}_{tr,G}(Y) \to \mathbb{Z}_{tr,G}(X) \to 0$$

were to be exact in the fixed point Nisnevich topology, then its restriction to $\mathrm{Sm}/\mathbb{R}$ would be exact in the Nisnevich topology. But its restriction to $\mathrm{Sm}/\mathbb{R}$ is

$$\cdots \to \mathbb{Z}_{tr}(\mathbb{C} \times_{\mathbb{R}} \mathbb{C}) \to \mathbb{Z}_{tr}(\mathbb{C}) \to \mathbb{Z}_{tr}(\mathbb{R}) \to 0,$$

which is not exact in the Nisnevich topology. Indeed if it were exact then applying $\mathbb{Z}/2(n)$, the complex computing weight n motivic cohomology with mod-2 coefficients, would imply a quasi-isomorphism $\mathbb{Z}/2(n)(\mathbb{R}) = \mathbb{Z}/2(n)(\mathbb{C})^{h\mathbb{Z}/2}$ and then we would have

$$H^i_{\mathcal{M}}(\mathbb{R}, \mathbb{Z}/2(n)) = H^i\mathbb{Z}/2(n)(\mathbb{R}) = H^i\mathbb{Z}/2(n)(\mathbb{C})^{h\mathbb{Z}/2} = H^i_{et}(\mathbb{R}, \mathbb{Z}/2)$$

for all $i \geq 0$, which is not true.

Lemma 4.12. *Let $p : U \to Y$ be an equivariant Nisnevich cover and $f : X \to Y$ an equivariant finite correspondence. Then there is an equivariant Nisnevich covering $p' : V \to X$ and an equivariant finite correspondence $f' : V \to U$ which fit into the following commutative square in $G\,\mathrm{Cor}_k$,*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Proof. We may assume that f is an equivariant elementary correspondence. Write $Z = \mathrm{Supp}(f)$ and consider the pullback $Z' = Z \times_Y U \subseteq X \times U$. Then $Z' \to Z$ is an equivariant Nisnevich cover and $\pi : Z \to X$ is finite. We can find an equivariant Nisnevich cover $V \to X$ such that $V \times_X Z' \to V \times_X Z$ has a section.
Let s be such a section. Then $s(V \times_X Z) \subseteq V \times U$ is finite and equivariant over V and its associated equivariant correspondence defines the required $f' : V \to U$. □

Theorem 4.13. *Let F be a presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Then F_{GNis} has a unique structure of a presheaf with equivariant transfers such that $F \to F_{GNis}$ is a morphism of presheaves with equivariant transfers.*

Proof. The proof is parallel to the proof of [MVW06, Theorem 6.17]. We begin with uniqueness. Let F_1 and F_2 be two presheaves with transfers with a map of presheaves with equivariant transfers $F \to F_i$ whose underlying map of presheaves is the canonical map $F \to F_{GNis}$. Let $f : X \to Y$ be a map in $G\,\mathrm{Cor}_k$ and $y \in F_1(Y) = F_2(Y) = F_{GNis}(Y)$. Choose an equivariant Nisnevich covering $U \to Y$ such that $y|_U$ is in the image of $u \in F(U)$. Applying Lemma 4.12 we have a commutative square in $G\,\mathrm{Cor}_k$

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $p' : V \to X$ is an equivariant Nisnevich cover. It is straightforward to verify, using this square, that $F_1(f)(y) = F_2(f)(y)$ and so $F_1 = F_2$ as presheaves with equivariant transfers.

For existence, we first define a map $F_{GNis}(Y) \to \mathrm{Hom}_{Sh}(\mathbb{Z}_{tr,G}(Y), F_{GNis})$, natural for maps in $G\,\mathrm{Cor}_k$ and such that the following square commutes

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Pre}(G\,\mathrm{Cor}_k)}(\mathbb{Z}_{tr,G}(Y), F) \arrow[d] \\ F_{GNis}(Y) \arrow[r] & \mathrm{Hom}_{\mathrm{Shv}(G\,\mathrm{Cor}_k)}(\mathbb{Z}_{tr,G}(Y), F_{GNis}). \end{tikzcd}$$

Given $y \in F_{GNis}(Y)$ there is an equivariant Nisnevich cover $U \to Y$ such that $y|_U$ is the image of $u \in F(U)$. The element u determines a morphism $\mathbb{Z}_{tr,G}(U) \to F$ of presheaves with equivariant transfers. By shrinking U we may assume that u restricts to the zero map $\mathbb{Z}_{tr,G}(U \times_Y U) \to F$ under the difference map $(p_1)_* - (p_2)_* : \mathbb{Z}_{tr,G}(U \times_Y U) \to \mathbb{Z}_{tr,G}(U)$. This in turn implies that the induced morphism of sheaves $\mathbb{Z}_{tr,G}(U) \to F_{GNis}$ also restricts to zero under the difference map.

Now Theorem 4.8 implies that that u determines a map $[y] : \mathbb{Z}_{tr,G}(Y) \to F_{GNis}$ and it is straightforward to verify that this is independent of the choice of U and u. We now define $G\,\mathrm{Cor}_k(X,Y) \otimes F_{GNis}(Y) \to F_{GNis}(X)$ as follows. Let $f : X \to Y$ be a finite equivariant correspondence and $y \in F_{GNis}(Y)$. Consider the composition $[y]f : \mathbb{Z}_{tr,G}(X) \to \mathbb{Z}_{tr,G}(Y) \to F_{GNis}$ and define the pairing by sending (f, y) to the image of the identity in $\mathbb{Z}_{tr,G}(X)(X)$ in $F_{GNis}(X)$ under $[y]f$. □

A presheaf F with equivariant transfers is said to be an *equivariant Nisnevich sheaf* with equivariant transfers if the restriction of F to $G\mathrm{Sm}/k$ is a sheaf in the equivariant Nisnevich topology. We write $\mathrm{Shv}(G\,\mathrm{Cor}_k)$ for the category of sheaves with equivariant transfers in the equivariant Nisnevich topology.

COROLLARY 4.14. *The category* $\mathrm{Shv}(G\,\mathrm{Cor}_k)$ *is an abelian category with enough injectives and the inclusion* $i : \mathrm{Shv}(G\,\mathrm{Cor}_k) \to \mathrm{Pre}(G\,\mathrm{Cor}_k)$ *has a left adjoint* a_{GNis} *which is exact and commutes with the forgetful functor to (pre)sheaves on* $G\mathrm{Sm}/k$.

THEOREM 4.15. *Let F be an equivariant Nisnevich sheaf with transfers. Then*

(1) *the cohomology presheaves* $H^n_{GNis}(-,F)$ *are presheaves with equivariant transfers,*
(2) *for any smooth* X*,* $F(X) = \mathrm{Hom}_{\mathrm{Shv}(G\,\mathrm{Cor}_k)}(\mathbb{Z}_{tr}(X), F)$*, and*
(3) *for any smooth* X*,*

$$\mathrm{Ext}^n_{\mathrm{Shv}(G\,\mathrm{Cor}_k)}(\mathbb{Z}_{tr,G}(X), F) = H^n_{GNis}(X, F).$$

Proof. Suppose that F is an equivariant Nisnevich sheaf with transfers. Then the Godement resolution $F \to \mathcal{G}F$ in Definition 3.16 is a resolution of sheaves with equivariant transfers by the same reasoning as in [MVW06, Example 6.20]. This implies the first statement. The second statement follows from the previous corollary together with the Yoneda lemma. For the third item, note that if F is an injective equivariant Nisnevich sheaf with transfers then $F \to G^0F$ is split and so $H^n_{GNis}(X,F)$ is a summand of $H^n_{GNis}(X, G^0F) = 0$. It follows that $H^n_{GNis}(X,F) = 0$ whenever F is an injective and the result follows. □

5. Bredon motivic cohomology

The rest of the paper is devoted to developing the machinery for presheaves with equivariant transfers, the proofs of the homotopy invariance theorem, and equivariant cancellation. Before delving into this rather technical material, we pause to illustrate the utility of the theory and discuss our main application, Bredon motivic cohomology. In this section we introduce Bredon motivic cohomology, explain how Mackey functors naturally appear in this setting, and give some examples. In the process we will use material from Sections 6-8 (but not from Section 9). This is mainly through the use of the homotopy invariance theorem, Theorem 8.12, which we restate below. Except for the statement of Condition 5.1, the material from this section is not used or referred to in Sections 6-8. Some of the material from this section is used in Section 9.
We will often require that G satisfies the following condition.

Condition 5.1. All irreducible $k[G]$-modules are one dimensional.

Theorem 8.12 (Homotopy Invariance). Assume that Condition 5.1 holds. Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Then $H^n_{GNis}(-, F_{GNis})$ is also a homotopy invariant presheaf with equivariant transfers.

Groups satisfying Condition 5.1 are necessarily abelian. Indeed, in this case the regular representation is a sum of one-dimensional representations, each of which corresponds to a character $G \to k^*$. Therefore, if G satisfies Condition 5.1 we obtain an injective group homomorphism $G \hookrightarrow k^* \times \cdots \times k^*$ into an abelian group and so G itself is abelian. The converse does not hold, for example over the field $\mathbb{R}$, the group $\mathbb{Z}/3$ has an irreducible two dimensional representation. However, note that if G satisfies Condition 5.1 over k then it also does so over every field extension of k as do all of its subgroups.

Lemma 5.2. *Let G be a finite abelian group and suppose that k contains a primitive nth-root of unity where n is the exponent of G (i.e., the least common multiple of the orders its elements). Then Condition 5.1 is satisfied.*

Proof. For an abelian group G, Condition 5.1 is equivalent to the condition that k is a splitting field for G (i.e.,if W is a simple $k[G]$-module then W_L is a simple $L[G]$-module for any field extension L/k). The lemma is thus a special case of a theorem of Brauer [CR62, Theorem 41.1,Corollary 70.24]. □

5.1. Definition and first properties. If F is a presheaf of abelian groups on $G\mathrm{Sm}/k$, write $C_nF(X) = F(X \times \Delta^n_k)$, where Δ^n_k is the standard algebraic simplex. This gives a presheaf of simplicial abelian groups $n \mapsto C_nF(X)$ and thus yields an associated chain complex C_*F. We write C^*F for the associated cochain complex, $C^{-k}F = C_kF$. If A is a cochain complex then the shifted complex $A[n]$ is the complex $A[n]^i = A^{i+n}$.

Definition 5.3. (1) Let V be a finite dimensional representation. Define $\mathbb{Z}_G(V)$ to be the complex of presheaves with equivariant transfers given by

$$\mathbb{Z}_G(V) := C^*\left(\mathbb{Z}_{tr,G}(\mathbb{P}(V \oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(V))\right)[-2\dim(V)].$$

(2) When $V = k[G]^{\oplus n}$ we adopt the notation

$$\mathbb{Z}_G(n) = \mathbb{Z}_G(k[G]^{\oplus n})$$

By virtue of their definition, the complexes $\mathbb{Z}_G(V)$ are acyclic in degrees larger than $2\dim(V)$. In particular $\mathbb{Z}_G(n)$ is acyclic in degrees larger than $2n|G|$.

Definition 5.4. Let X be a smooth G-variety. Define the *Bredon motivic cohomology* of X to be

$$H^n_G(X, \mathbb{Z}(m)) = H^n_{GNis}(X, \mathbb{Z}_G(m)).$$

More generally we write

$$H^n_G(X, \mathbb{Z}(V)) = H^n_{GNis}(X, \mathbb{Z}_G(V)).$$

Remark 5.5. By Corollary 3.9 all objects of $G\mathrm{Sm}/k$ have finite equivariant Nisnevich cohomological dimension. This implies that the displayed hypercohomology groups (whose coefficients are unbounded complexes) are well defined, see [Wei94, Corollary 10.5.11].

Lemma 5.6. (1) *If F is a presheaf and $f_0, f_1 : X \to Y$ are elementary equivariant $\mathbb{A}^1$-homotopic then the maps $f_0^*, f_1^* : C^*F(Y) \to C^*F(X)$ are chain homotopic.*
(2) *The cohomology presheaves $X \mapsto H^iC^*F(X)$ are homotopy invariant.*
(3) *If $f : X \to Y$ is an elementary $\mathbb{A}^1$-homotopy equivalence then the induced map of complexes $f_* : C_*\mathbb{Z}_{tr,G}(X) \to C_*\mathbb{Z}_{tr,G}(Y)$ is a chain homotopy equivalence.*

Proof. The proofs of all these statements are exactly as in the nonequivariant setting. See e.g., [MVW06, Lecture 2]. □

Proposition 5.7. (1) *Let $U\coprod Y \to X$ be an equivariant distinguished cover. There is a Mayer-Vietoris long exact sequence*

$$\cdots \to H^n_G(X,\mathbb{Z}(m)) \to H^n_G(U,\mathbb{Z}(m)) \oplus H^n_G(Y,\mathbb{Z}(m)) \to H^n_G(U\times_X Y,\mathbb{Z}(m)) \\ \to H^{n+1}_G(X,\mathbb{Z}(m)) \to \cdots$$

(and similarly for coefficients in $\mathbb{Z}_G(V)$).

(2) *If G satisfies Condition 5.1 then*

$$H^n_G(X\times\mathbb{A}^1,\mathbb{Z}(m)) \cong H^n_G(X,\mathbb{Z}(m))$$

(and similarly for coefficients in $\mathbb{Z}_G(V)$).

Proof. The first statement follows immediately from the fact that the Bredon motivic cohomology is defined as hypercohomology in the equivariant Nisnevich topology. The cohomology presheaves of $\mathbb{Z}(m)$ are homotopy invariant presheaves with transfers and so the second item follows from Theorem 8.12 together with the standard hypercohomology spectral sequence. □

Theorem 5.8. *Suppose that G satisfies Condition 5.1 and F is a homotopy invariant presheaf with equivariant transfers. If $F_{GNis} = 0$ then $(C_*F)_{GNis} \simeq 0$.*

Proof. Using Theorem 8.12, the argument is the same as in [MVW06, Theorem 13.12]. □

Proposition 5.9. *Suppose that G satisfies Condition 5.1. Let V be a finite dimensional representation. There is a quasi-isomorphism*

$$C_*\big(\mathbb{Z}_{tr,G}(\mathbb{A}(V))/\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)\big) \xrightarrow{\simeq} C_*\big(\mathbb{Z}_{tr,G}(\mathbb{P}(V\oplus 1)/\mathbb{Z}_{tr,G}(\mathbb{P}(V))\big)$$

of complexes of equivariant Nisnevich sheaves.

Proof. It follows from Example 3.3 that the map

$$\mathbb{Z}_{tr,G}(\mathbb{A}(V))/\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0) \to \mathbb{Z}_{tr,G}(\mathbb{P}(V\oplus 1)/\mathbb{Z}_{tr,G}(\mathbb{P}(V\oplus 1)-\mathbb{P}(1))$$

becomes an isomorphism after equivariant Nisnevich sheafification. The inclusion $\mathbb{P}(V\oplus 1)-\mathbb{P}(1) \subseteq \mathbb{P}(V\oplus 1)$ is equivariantly $\mathbb{A}^1$-homotopic to the inclusion $\mathbb{P}(V)\subseteq\mathbb{P}(V\oplus 1)$, the requisite deformation being given by $([x_0:\cdots:x_{n+1}],t)\mapsto[x_0:\cdots:x_n:tx_{n+1}]$.
The result now follows from an application of Theorem 5.8 and Lemma 5.6. □

Corollary 5.10. *Under the assumptions above, there is a quasi-isomorphism*

$$\operatorname{cone}\big(C_*\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)\to\mathbb{Z}\big) \simeq C_*\big(\mathbb{Z}_{tr,G}(\mathbb{P}(V\oplus 1)/\mathbb{Z}_{tr,G}(\mathbb{P}(V))\big).$$

Proof. The map $\mathbb{A}(V)\to\operatorname{Spec}(k)$ is an equivariant $\mathbb{A}^1$-homotopy equivalence. The result thus follows from the previous proposition together with Theorem 5.8 and Lemma 5.6. □

5.2. Coefficient systems. Let $\mathcal{O}_G$ denote the category of finite left G-sets together with equivariant maps. A *Bredon coefficient system* is an additive functor $M : \mathcal{O}_G^{op} \to \mathrm{Ab}$. In algebraic topology, Bredon's original construction [Bre67] allowed the coefficients to be an arbitrary Bredon coefficient system and was only integer graded. Later it was shown by Lewis-May-McClure [LMM81] that this theory is representable in the stable equivariant homotopy category (and hence has a grading by representations) exactly when the coefficient system is a Mackey functor. There are multiple descriptions of a Mackey functor. One definition is as follows. The Burnside category $\mathcal{B}_G$ of G, has the same objects as $\mathcal{O}_G$ and $\mathrm{Hom}_{\mathcal{B}_G}(A,B)$ is the group of maps between A_+ and B_+ in the stable equivariant homotopy category. Concretely it is the group completion of the monoid of isomorphism classes of diagrams of equivariant maps of finite G-sets of the form $A \leftarrow X \to B$ (the monoid structure is given by disjoint union). The composition of $A \leftarrow X \to B$ and $B \leftarrow X' \to C$ is given by $A \leftarrow X \times_B X' \to C$. A *Mackey functor* is then defined to be an additive functor $M : \mathcal{B}_G^{op} \to \mathrm{Ab}$. A related construction is the Hecke category $\mathcal{H}_G$ which has the same objects as $\mathcal{O}_G$ and morphisms are given by $\mathrm{Hom}_{\mathcal{H}_G}(S,T) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[S], \mathbb{Z}[T])$. A *cohomological Mackey functor* (also called a Hecke functor) is an additive functor $M : \mathcal{H}_G^{op} \to \mathrm{Ab}$. There is a Hurewicz functor $H : \mathcal{B}_G \to \mathcal{H}_G$ given by sending the map $S \xleftarrow{f} X \xrightarrow{g} T$ to the map $\mathbb{Z}[S] \to \mathbb{Z}[T]$ given by $s \mapsto \sum_{x\in f^{-1}(s)} g(x)$. Via the Hurewicz functor, a cohomological Mackey functor is viewed as a Mackey functor. By a theorem of Yoshida [Yos83], the cohomological Mackey functors are exactly the Mackey functors with the property that for a subgroup $K \subseteq H$, the composition f_*f^* is multiplication by $[H : K]$, where $f : (G/K)_+ \to (G/H)_+$.

A key reason that Mackey functors play a central role in defining equivariant generalizations of singular cohomology in algebraic topology is that Mackey functors are the heart of a t-structure on the stable equivariant homotopy category. A corresponding theory of motivic Mackey functors is not yet developed and is beyond the scope of this paper. Nonetheless we view our construction as having coefficients in the "constant motivic Mackey functor $\underline{\mathbb{Z}}$". This is justified by analogy with a topological construction for Bredon cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$, arising from the equivariant Dold-Thom theorem [dS03]. Additionally, over $\mathbb{C}$, the topological realization functor takes the theory we have constructed to the usual topological Bredon cohomology with coefficients in $\underline{\mathbb{Z}}$. The classical cohomological Mackey functors fit into our setting as follows. We have an embedding $\mathcal{O}_G \subseteq G\mathrm{Sm}/k$ given by $S \mapsto \coprod_S \mathrm{Spec}(k)$. The composition $\mathcal{O}_G \subseteq G\mathrm{Sm}/k \subseteq G\,\mathrm{Cor}_k$ factors through a faithful embedding $\mathcal{H}_G \subseteq G\,\mathrm{Cor}_k$. We thus have an embedding of the category of cohomological Mackey functors into the category of presheaves with equivariant transfers. The category of (pre)sheaves with transfers has a tensor structure and by tensoring the complexes $\mathbb{Z}_G(n)$ with a cohomological Mackey functor M, we obtain Bredon motivic cohomology theory with coefficients in M.

Example 5.11. In Section 9 we use certain complexes of presheaves with equivariant transfers that are associated to topological representation spheres for $G = (\mathbb{Z}/2)^n$.

(1) For simplicity, we first consider $G = \mathbb{Z}/2$. The topological sign representation sphere S^σ is the one-point compactification of the sign representation σ. It fits into a homotopy cofiber sequence $(\mathbb{Z}/2)_+ \to S^0 \to S^\sigma$ of based G-spaces. We define $\mathbb{Z}_{top}(\sigma)$ by

$$\mathbb{Z}_{top}(\sigma) := \operatorname{cone}(\mathbb{Z}_{tr,G}(\mathbb{Z}/2) \to \mathbb{Z}).$$

(2) Now consider $G = (\mathbb{Z}/2)^n$. Write ϵ_i for the generator of the ith factor of $\mathbb{Z}/2$ in G. For a subset $I \subseteq \{1, \dots n\}$ write σ_I for the $(\mathbb{Z}/2)^n$-representation specified by letting ϵ_i act by -1 on k if $i \in I$ and by the identity otherwise. We define $\mathbb{Z}_{top}(\sigma)$ as follows. Write $\mathcal{Z}_I$ for $\mathbb{Z}/2$ considered as a $(\mathbb{Z}/2)^n$-scheme with action specified by letting ϵ_i act nontrivially if $i \in I$ and as the identity otherwise. We set

$$\mathbb{Z}_{top}(\sigma_I) := \operatorname{cone}(\mathbb{Z}_{tr,G}(\mathcal{Z}_I) \to \mathbb{Z}).$$

When $G = \mathbb{Z}/2$, we write simply $\mathbb{Z}_{top}(\sigma) = \mathbb{Z}_{top}(\sigma_{\{1\}})$ (which agrees with the previous definition).

As in [MVW06, Section 8], there is a tensor product $\otimes_{tr}$ on $D^-(G\,\mathrm{Cor}_k)$ which is induced by $\mathbb{Z}_{tr,G}(X) \otimes_{tr} \mathbb{Z}_{tr,G}(Y) = \mathbb{Z}_{tr}(X \times Y)$. The following will be useful in Section 9.

Lemma 5.12. *The complex $\mathbb{Z}_{top}(\sigma_I)$ is invertible in $(D^-(G\,\mathrm{Cor}_k), \otimes_{tr})$.*

Proof. Write $q_i : \mathcal{Z}_I \times \mathcal{Z}_I \to \mathcal{Z}_I$ for the projection to the ith factor and write $p : \mathcal{Z}_I \to *$ for the projection to a point. The complex $\mathbb{Z}_{top}(\sigma_I)$ is the complex $0 \to \mathbb{Z}_{tr,G}(\mathcal{Z}_I) \xrightarrow{p} \mathbb{Z} \to 0$ (with $\mathbb{Z} = \mathbb{Z}_{tr,G}(*)$ in degree zero). We claim that the inverse $\mathbb{Z}_{top}(-\sigma_I)$ is given by $0 \to \mathbb{Z} \xrightarrow{p^t} \mathbb{Z}_{tr,G}(\mathcal{Z}_I) \to 0$ (again with $\mathbb{Z}$ in degree zero, and $(-)^t$ denotes the transpose). The tensor product $\mathbb{Z}_{top}(\sigma_I) \otimes^{\mathbb{L}}_{tr} \mathbb{Z}_{top}(-\sigma_I)$ is the complex

$$0 \to \mathbb{Z}_{tr,G}(\mathcal{Z}_I) \xrightarrow{(p,-q_1^t)} \mathbb{Z} \oplus \mathbb{Z}_{tr,G}(\mathcal{Z}_I \times \mathcal{Z}_I) \xrightarrow{p^t \oplus q_2} \mathbb{Z}_{tr,G}(\mathcal{Z}_I) \to 0.$$

Write E_* for this complex. We have a chain homotopy $s : E_* \to E_{*+1}$ between the identity on E_* and the composite $E_* \to \mathbb{Z} \to E_*$ (where $\mathbb{Z}$ is concentrated in degree zero) given by $s_0 = -\Delta$, $s_{-1} = \Delta$ and $s_i = 0$ for $i \neq 0, -1$, where $\Delta : \mathcal{Z}_I \to \mathcal{Z}_I \times \mathcal{Z}_I$ is the diagonal. □

5.3. Examples. We record a few simple examples. The first one is straightforward.

Lemma 5.13. *Suppose that G satisfies Condition 5.1. Then there is a quasi-isomorphism $\mathbb{Z}_G(0) \simeq \mathbb{Z}$ of complexes of equivariant Nisnevich sheaves, where $\mathbb{Z}$ is the complex consisting of the constant sheaf $\mathbb{Z}$ in degree zero.*

Proposition 5.14. *Let V be a one dimensional representation. Then we have an isomorphism*

$$C_*\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)\simeq(\mathcal{O}^*)^G\oplus\mathbb{Z},$$

in the derived category of equivariant Nisnevich sheaves on $G\mathrm{Sm}/k$, where $(\mathcal{O}^*)^G$ is the sheaf of invariant invertible functions viewed as a complex concentrated in degree zero.

Remark 5.15. This is not a decomposition of complexes of sheaves with equivariant transfers except in the case when V is the trivial representation.

Proof. The homology of $C_*\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)(X)$ is identified with equivariant Suslin homology defined in §6, i.e.,

$$H_i(C_*\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)(X))=H_i^{Sus}(G;X\times(\mathbb{A}(V)-0)/X).$$

By Theorem 6.12 we thus have

$$H_i(C_*\mathbb{Z}_{tr,G}(\mathbb{A}(V)-0)(X))=\begin{cases}\mathrm{Div}^G_{rat}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\}) & i=0\\ 0 & i\neq 0.\end{cases}$$

Write K for the kernel of the action of G on $X\times\mathbb{P}(V\oplus 1)$. Then G/K acts faithfully on $X\times\mathbb{P}(V\oplus 1)$,

$$\mathrm{Div}^G_{rat}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\})=\mathrm{Div}^{G/K}_{rat}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\}),$$

and by Proposition 6.8,

$$\mathrm{Div}^{G/K}_{rat}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\})=\mathrm{Pic}^{G/K}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\}).$$

Using the exact sequence (6.6) for the relative equivariant Picard group and that for X smooth $\mathrm{Pic}^G(X\times\mathbb{P}(V\oplus 1))=\mathrm{Pic}^G(X)\times\mathbb{Z}$, we have

$$\mathrm{Pic}^{G/K}(X\times\mathbb{P}(V\oplus 1),X\times\{0,\infty\})=\mathcal{O}^*(X)^{G/K}\oplus\mathbb{Z}=\mathcal{O}^*(X)^G\oplus\mathbb{Z},$$

from which the result follows. □

Proposition 5.16. *Let X be a smooth, quasi-projective G-scheme. Then*

$$H^i_{GNis}(X,(\mathcal{O}^*_X)^G)=\begin{cases}\Gamma(X,\mathcal{O}^*)^G & i=0\\ \mathrm{Pic}(X/G) & i=1\end{cases}$$

Proof. Since $(\mathcal{O}^*_X)^G)$ is the sheaf on X_{GNis} given by $U\mapsto\mathcal{O}^*_{U/G}$, the proposition follows from Lemma 3.19. □

Corollary 5.17. *Suppose that G satisfies Condition 5.1 and V is a one dimensional representation. Then $\mathbb{Z}_G(V)\simeq(\mathcal{O}^*)^G[-1]$. In particular if X is a smooth, quasi-projective G-scheme then*

$$H^i_G(X,\mathbb{Z}(V))=\begin{cases}\Gamma(X,\mathcal{O}^*)^G & i=1\\ \mathrm{Pic}(X/G) & i=2\end{cases}$$

Proof. By Proposition 5.14 and Corollary 5.10 we have that $\mathbb{Z}_G(V)\simeq(\mathcal{O}^*)^G[-1]$. The second statement follows immediately from the previous proposition. □

For one dimensional representations V, V', the chain complexes $\mathbb{Z}_G(V)$ and $\mathbb{Z}_G(V')$ are quasi-isomorphic as complexes of sheaves in the equivariant Nisnevich topology. The following example makes explicit that for higher dimensional representations, distinct representations give rise to distinct chain complexes. For a representation V, we write $\mathbb{Z}_{tr,G}(T^V) := \mathbb{Z}_{tr,G}(\mathbb{P}(V \oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(V))$.

Example 5.18. Let p be a prime and $G = \mathbb{Z}/p$ and k a field which admits resolution of singularities. For any representation V, we have that

$$H^i_G(k, \mathbb{Z}(V)) = H^{i-2pn}(C_*\mathbb{Z}_{tr,G}(T^V)(k)) = H^{i-2pn}(z_{equi}(\mathbb{A}(V), 0)(\Delta^\bullet_k)^G),$$

where $z_{equi}(X, 0)$ is the presheaf of equidimensional cycles of relative dimension zero. We compare the complexes $\mathbb{Z}_G(V)[2\dim(V)] = C_*\mathbb{Z}_{tr,G}(T^V)$ for $V = nk[G]$ and 1^{np}. We have that $H^i(C_*\mathbb{Z}_{tr,G}(1^{np})(k)) = H^{i+2np}_{\mathcal{M}}(k, \mathbb{Z}(np))$. On the other hand, $C_*z_{equi}(\mathbb{A}(nk[G]), 0)^G = \oplus_{j=n}^{np-1}\mathbb{Z}/p(j)[2j] \oplus \mathbb{Z}(np)[2np]$ in $DM(k)$ by [Nie08, Theorem 5.4]. Therefore we have that

$$H^iC_*\mathbb{Z}_{tr,G}(1^{np})(k) = H^{i+2np}_{\mathcal{M}}(k, \mathbb{Z}(np)).$$

while

$$H^iC_*\mathbb{Z}_{tr,G}(T^{nk[G]})(k) = H^{i+2np}_{\mathcal{M}}(k, \mathbb{Z}(np)) \oplus (\oplus_{j=n}^{np-1} H^{i+2j}_{\mathcal{M}}(k, \mathbb{Z}/p(j))).$$

We see that $C_*\mathbb{Z}_{tr,G}(1^{np})$ and $C_*\mathbb{Z}_{tr,G}(T^{nk[G]})$ are not quasi-isomorphic in general because there are values of i so that the group $\oplus_{j=n}^{np-1} H^{i+2j}_{\mathcal{M}}(k, \mathbb{Z}/p(j))$ is nonzero (e.g.,$H^0_{\mathcal{M}}(k, \mathbb{Z}/p(n)) = \mathbb{Z}/p$ and so the above group is nonzero whenever $i + 2j = 0$).

We finish this section by relating our construction to Edidin-Graham's equivariant higher Chow groups [EG98]. Recall that these are constructed as follows. Consider a pair (V, U) where V is a faithful representation and $U \subseteq \mathbb{A}(V)$ is an open subscheme on which G acts freely. Then U/G exists as a scheme and it is an algebro-geometric approximation to BG. Such pairs always exist, moreover one can arrange that $\dim V$ and $\mathrm{codim}_{\mathbb{A}(V)}(\mathbb{A}(V) - U)$ are arbitrarily large. The equivariant higher Chow group of a quasi-projective G-scheme X in bidegree (n, m) is defined to be $CH^n_G(X, m) = CH^n(X \times^G U, m)$ for a pair (V, U) with $\mathbb{A}(V) - U$ of sufficiently large codimension. We refer to loc. cit. for full details.

Theorem 5.19. *Let X be a smooth quasi-projective G-scheme. There is a natural map*

$$H^i_G(X, \mathbb{Z}(1^q)) \to CH^q_G(X, 2q - i).$$

Proof. We have a natural isomorphism $H^n_{\mathcal{M}}(X, \mathbb{Z}(q)) \cong CH^k(X, 2q - n)$. The complex $\mathbb{Z}(q)$ on Sm/k is $C_*(\mathbb{Z}_{tr}(\mathbb{P}^q)/\mathbb{Z}_{tr}(\mathbb{P}^{q-1}))[-2n]$. If Y has trivial action then $G\,\mathrm{Cor}_k(X, Y) = \mathrm{Cor}_k(X/G, Y)$ for a G-scheme X. Therefore we have the natural identification $\mathbb{Z}_G(1^q)(X) = \mathbb{Z}(q)(X/G)$. Using this identification, Lemma 3.19, and the projection $X \times U \to X$ we thus have the comparison map

$$H^i_G(X, \mathbb{Z}(1^q)) \to H^i_G(X \times U, \mathbb{Z}(1^q))) \\ = H^i_{\mathcal{M}}(X \times^G U, \mathbb{Z}(q)) = CH^k(X \times^G U, 2q-i).$$

Taking (V, U) such that $\mathbb{A}(V) - U$ has sufficiently large codimension yields the result.

□

Remark 5.20. The map of the previous theorem can be seen to be an isomorphism when X has free action. It is not an isomorphism in general. For example, if X has trivial action then $H^i_G(X, \mathbb{Z}(1^q))$ is isomorphic to ordinary motivic cohomology groups, which in turn is isomorphic to Bloch's higher Chow groups. Under these isomorphisms, the comparison map just constructed is identified in this case with the comparison map $CH^q(X, 2q-i) \to CH^q_G(X, 2q-i)$ between ordinary higher Chow groups and equivariant higher Chow groups, which is not an isomorphism.

6. Relative equivariant Cartier divisors

In this section we introduce an equivariant version of Suslin homology and relate it to the group of relative equivariant Cartier divisors.

Let $f : X \to S$ be smooth. Recall that $C_0(X/S)$ denotes the group of cycles on X which are finite and surjective over a component of S. If $f : X \to S$ is equivariant then we have an equivariant inclusion $C_0(X/S) \subseteq \mathrm{Cor}_k(S, X)$, induced by $\langle f, id_X \rangle : X \hookrightarrow S \times X$. Let $F : G\,\mathrm{Cor}_k^{op} \to \mathrm{Ab}$ be a presheaf with equivariant transfers. Define the pairing

$$Tr : C_0(X/S)^G \otimes F(X) \to F(S) \tag{6.1}$$

to be the composite

$$\begin{tikzcd} C_0(X/S)^G \otimes F(X) \arrow[r, "Tr"] \arrow[d, hook] & F(S) \\ G\,\mathrm{Cor}_k(S, X) \otimes F(X). \arrow[ur, "\text{evaluate}"'] & \end{tikzcd}$$

Define

$$C_n(X/S) = C_0(X \times \Delta^n / S \times \Delta^n).$$

The assignment $n \mapsto C_n(X/S)^G$ is a simplicial abelian group and hence gives rise to an associated chain complex.

Definition 6.2. The nth *equivariant Suslin homology* of X/S is defined to be

$$H^{Sus}_n(G; X/S) = H_n C_\bullet(X/S)^G.$$

Lemma 6.3. *Let F be a homotopy invariant presheaf with equivariant transfers. The map (6.1) factors through the zeroth Suslin homology group to yield the pairing*

$$Tr : H^{Sus}_0(G; X/S) \otimes F(X) \to F(S).$$

Proof. Because $F(X \times \mathbb{A}^1) = F(X)$ we have the commutative diagram

$$\begin{tikzcd} C_0(X \times \mathbb{A}^1/S \times \mathbb{A}^1)^G \otimes F(X) \arrow[r, "Tr"] \arrow[d, "\partial_0 - \partial_1"'] & F(S \times \mathbb{A}^1) \arrow[d, "i_0 - i_1"] \\ C_0(X/S)^G \otimes F(X) \arrow[r, "Tr"] & F(S), \end{tikzcd}$$

which implies the lemma. □

Our next goal is to compute the equivariant Suslin homology of equivariant relative curves. Recall that an equivariant Cartier divisor on X is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)^G$, see Section 2.3 for a recollection.

Definition 6.4. (1) Let X be a G-scheme and $Y \subseteq X$ an invariant closed subscheme. A *relative equivariant Cartier divisor* on X is an equivariant Cartier divisor on X (see Definition 2.10) such that $\mathrm{Supp}(D) \cap Y = \emptyset$. Write $\mathrm{Div}^G(X, Y)$ for the group of relative equivariant Cartier divisors, where the group operation is induced by that on $\mathrm{Div}^G(X)$.

(2) A *principal relative equivariant Cartier divisor* is an invariant rational function $f \in \Gamma(X, \mathcal{K}^*)^G$ on X such that f is defined and equal to 1 on Y.

(3) Write $\mathrm{Div}^G_{rat}(X, Y)$ for the group of relative equivariant Cartier divisors modulo the principal relative equivariant Cartier divisors.

Let $i : Y \hookrightarrow X$ be an equivariant closed embedding of G-schemes. Define an étale sheaf on X by

$$\mathbb{G}_{X,Y} = \ker(\mathcal{O}_X^* \to i_* \mathcal{O}_Y^*).$$

Since $\mathcal{O}_X^*$ and $\mathcal{O}_Y^*$ are étale G-sheaves, so is $\mathbb{G}_{X,Y}$. See Section 2.2 for a recollection on G-sheaves and their cohomology.

Definition 6.5. Define the *relative equivariant Picard group* by

$$\mathrm{Pic}^G(X, Y) = H^1_{et}(G; X, \mathbb{G}_{X,Y}).$$

From the definition of $\mathbb{G}_{X,Y}$ we have a natural exact sequence

$$\Gamma(X, \mathcal{O}_X^*)^G \to \Gamma(Y, \mathcal{O}_Y^*)^G \to \mathrm{Pic}^G(X, Y) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y). \tag{6.6}$$

Theorem 2.7 and the above exact sequence imply that $\mathrm{Pic}^G(X, Y) = H^1_{Zar}(G; X, \mathbb{G}_{X,Y})$. The following lemma is straightforward to verify.

Lemma 6.7. *The group $\mathrm{Pic}^G(X, Y)$ is isomorphic to the group consisting of isomorphism classes of pairs $(\mathcal{L}, \phi)$ where $\mathcal{L}$ is a G-line bundle on X and ϕ is an equivariant isomorphism $\phi : \mathcal{L}|_Y \cong \mathcal{O}_Y$ of G-line bundles on Y and group operation induced by tensor product of G-line bundles.*

Proposition 6.8. *There is a natural injection $\iota : \mathrm{Div}^G_{rat}(X, Y) \hookrightarrow \mathrm{Pic}^G(X, Y)$. If in addition X has faithful action and Y has an invariant affine open neighborhood, then ι is an isomorphism.*

Proof. Let $D \in \mathrm{Div}^G(X,Y)$. Since $Y \cap \mathrm{Supp}(D) = \emptyset$, there is a canonical equivariant trivialization $s_D : \mathcal{O}_X(D)|_Y \cong \mathcal{O}_Y$. In particular we have a natural homomorphism $\mathrm{Div}^G(X,Y) \to \mathrm{Pic}^G(X,Y)$. If $(\mathcal{O}_X(D), s_D) = (\mathcal{O}_X, \mathrm{id})$ then there is an equivariant isomorphism $\psi : \mathcal{O}_X \cong \mathcal{O}_X(D)$ such that $\psi|_Y = (s_D)^{-1}$. We have an induced isomorphism $\psi : \Gamma(X, \mathcal{O}_X)^G \cong \Gamma(X, \mathcal{O}_X(D))^G$ and letting $f = \psi(1) \in \Gamma(X, \mathcal{O}_X(D))^G \subseteq \Gamma(X, \mathcal{K}_X)^G$ we have that $D = \mathrm{div}(f^{-1})$ and $D|_Y = 1$ which implies that ι is injective.
The image of ι consists of pairs $(\mathcal{L}, \phi)$ such that $\mathcal{L}$ admits an equivariant injection into $\mathcal{K}_X$ and ϕ extends to an equivariant trivialization on an invariant open neighborhood of Y. When the action on X is faithful, every G-line bundle on X admits an equivariant injection into $\mathcal{K}_X$ by Proposition 2.12. When Y has an invariant open affine neighborhood, every equivariant trivialization ϕ extends to an invariant open neighborhood of Y. $\square$

LEMMA 6.9. *If X is normal and Y is reduced then*

$$\mathrm{Pic}^G(X \times \mathbb{A}^1, Y \times \mathbb{A}^1) \cong \mathrm{Pic}^G(X,Y).$$

If additionally, Y has an invariant affine open neighborhood then

$$\mathrm{Div}^G_{rat}(X \times \mathbb{A}^1, Y \times \mathbb{A}^1) \cong \mathrm{Div}^G_{rat}(X,Y).$$

Proof. The first statement follows from the exact sequence (6.6), Lemma 2.9, and the five lemma. For the second statement, observe that if $K = \ker(G \to \mathrm{Aut}(X))$, then G/K acts faithfully on X and $\mathrm{Div}^G_{rat}(X,Y) = \mathrm{Div}^{G/K}_{rat}(X,Y)$. The result then follows from the first part together with Proposition 6.8. $\square$

DEFINITION 6.10. Let $X \to S$ be a smooth relative curve in $G\mathrm{Sm}/k$ (i.e.,an equivariant smooth map of relative dimension one). An *equivariant good compactification* of X over S is an equivariant open embedding $X \subseteq \overline{X}$ of G-schemes over S where $\overline{X} \to S$ is a proper normal (not necessarily smooth) curve with G-action and $X_\infty = (\overline{X} - X)_{red}$ has an invariant open affine neighborhood in $\overline{X}$.

If $X \to S$ is an equivariant smooth relative curve with equivariant good compactification then we have an isomorphism $\mathrm{cyc} : \mathrm{Div}(\overline{X}, X_\infty) \cong C_0(X/S)$. Indeed, if $D \in \mathrm{Div}(\overline{X}, X_\infty)$ then $\mathrm{cyc}(D)$ is supported on X and the assumptions above guarantee that it is finite and surjective over a component of S. It is straightforward to check this is an equivariant isomorphism (see Lemma 2.11) and so we immediately conclude the following.

LEMMA 6.11. *Let $X \to S$ be an equivariant smooth curve with good equivariant compactification $\overline{X} \to S$. Then* cyc *induces a natural isomorphism*

$$\mathrm{cyc} : \mathrm{Div}^G(\overline{X}, X_\infty) \xrightarrow{\cong} C_0(X/S)^G.$$

With these definitions, Suslin-Voevodsky's fundamental computation of the Suslin homology of relative curves extends to the equivariant setting.

Theorem 6.12. *Let $p : X \to S$ be an equivariant smooth quasi-affine curve with equivariant good compactification $\overline{X} \to S$. Then*

$$H_n^{Sus}(G; X/S) \cong \begin{cases} \mathrm{Div}^G_{rat}(\overline{X}, X_\infty) & n = 0 \\ 0 & n > 0. \end{cases}$$

Proof. The argument is similar to [SV96, Theorem 3.1]. Define

$$\mathcal{M}_n(X/S) = \{f \in \Gamma(\overline{X} \times \Delta^n, \mathcal{K}^*) \mid f \text{ is defined and equal to 1 on } S \times \Delta^n\}.$$

As shown in [SV96, proof of Theorem 3.1], the natural map $\mathcal{M}_n(X/S) \to C_n(X/S)$ is an injection. We thus have an exact sequence of complexes

$$0 \to \mathcal{M}_\bullet(X/S)^G \to C_\bullet(X/S)^G \to \mathrm{Div}^G_{rat}(\overline{X} \times \Delta^\bullet, Y \times \Delta^\bullet) \to 0.$$

By Lemma 6.9 the result follows once we show that $\mathcal{M}_\bullet(X/S)^G$ is acyclic. We work with the normalized chain complex. Let $f \in \mathcal{M}_n(X/S)^G$ and suppose that $\partial_i(f) = 1$ for $i = 0, \ldots, n$. We need to show that there is $g \in \mathcal{M}_{n+1}(X/S)^G$ such that $\partial_i(g) = 1$ for $i = 0, \ldots, n$ and $\partial_{n+1}(g) = f$. Following [SV96, Theorem 3.1], we consider

$$g_i = (t_{i+1} + \cdots + t_{n+1}) + (t_0 + \cdots + t_i)s_i(f).$$

Since f is equivariant it follows that g_i is equivariant. The function

$$g = g_n g_{n-1}^{-1} \cdots g_0^{(-1)^n}$$

is then equivariant and by loc. cit. it has the required properties. □

We finish with a discussion of the compatibility of the isomorphism in the previous theorem with respect to push-forwards along finite morphisms.

Lemma 6.13. *Let $W \to X$ be an equivariant finite surjection between normal G-schemes over k. Then the norm map $N : \mathcal{K}^*(W) \to \mathcal{K}^*(X)$ is equivariant.*

Proof. If $g : Y' \to Y$ is an isomorphism then the norm $N : \mathcal{K}^*(Y') \to \mathcal{K}^*(Y)$ is just the inverse of the induced isomorphism $\tilde{g} : \mathcal{K}^*(Y) \to \mathcal{K}^*(Y')$. Thus the lemma follows from functoriality of the norm map. □

Suppose that $f : \overline{X} \to \overline{Y}$ is a finite surjective equivariant map between normal G-schemes which restricts to a finite surjective equivariant map $X_\infty \to Y_\infty$, where $X_\infty \subseteq \overline{X}$, $Y_\infty \subseteq Y$ are invariant closed, reduced subschemes. The norm map induces a map $\mathrm{Div}^G(\overline{X}, X_\infty) \to \mathrm{Div}^G(\overline{Y}, Y_\infty)$ which factors through rational equivalence to give

$$f_* : \mathrm{Div}^G_{rat}(\overline{X}, X_\infty) \to \mathrm{Div}^G_{rat}(\overline{Y}, Y_\infty).$$

If $X_\infty \subseteq \overline{X}$ has an invariant affine neighborhood, every invertible invariant regular function α on X_∞ extends to an invariant rational function $\tilde{\alpha}$ on $\overline{X}$. The difference of two different extensions is a principal relative equivariant divisor and so we have a well-defined homomorphism

$$\mathcal{O}^*(X_\infty)^G \to \mathrm{Div}^G_{rat}(\overline{X}, X_\infty).$$

Additionally we can define $f_* : \mathcal{O}^*(X_\infty)^G \to \mathcal{O}^*(Y_\infty)^G$ by extending α to $\tilde{\alpha}$ as above and then define $f_*(\alpha) = N(\tilde{\alpha})|_{Y_\infty}$. It is easily checked that $N(\tilde{\alpha})|_{Y_\infty}$ lies in $\mathcal{O}^*(Y_\infty)$ and that this value does not depend on the choice of extension.

Lemma 6.14. *Let $(\overline{Y}, Y_\infty)$ and $(\overline{X}, X_\infty)$ be good equivariant compactifications of Y and X. Let $f : \overline{Y} \to \overline{X}$ be a finite map which restricts to a map $f : Y \to X$. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{O}^*(X_\infty)^G \arrow[r] \arrow[d, "f_*"] & \mathrm{Div}^G_{rat}(\overline{X}, X_\infty) \arrow[r, "\cong"] \arrow[d, "f_*"] & H_0^{Sus}(G; X/S) \arrow[d, "f_*"] \\
\mathcal{O}^*(Y_\infty)^G \arrow[r] & \mathrm{Div}^G_{rat}(\overline{Y}, Y_\infty) \arrow[r, "\cong"] & H_0^{Sus}(G; Y/S),
\end{tikzcd}$$

where the left hand and middle vertical maps are induced by the norm map and the right hand vertical map is push forward of cycles.

Proof. The commutativity of the left hand square is immediate from the definitions. If D is a Cartier divisor on $\overline{X}$ then $f_*\mathrm{cyc}(D) = \mathrm{cyc}(f_*D)$ by [Gro67, Proposition 21.10.17]. This implies that the right hand square commutes. □

7. Equivariant triples

In this section we introduce and study an equivariant generalization of Voevodsky's standard triples and establish equivariant analogues of their basic properties. From now on k is assumed to be perfect. Additionally we will usually assume that G satisfies Condition 5.1, i.e., all irreducible representations are assumed to be one dimensional.

Definition 7.1. An *equivariant standard triple* $(\overline{X} \xrightarrow{\overline{p}} S, X_\infty, Z)$ consists of a proper equivariant map $\overline{p}$ of relative dimension one between G-schemes and closed invariant subschemes Z, X_∞ of $\overline{X}$ such that

(1) S is smooth and $\overline{X}$ is normal
(2) $X = \overline{X} - X_\infty$ is quasi-affine and smooth over S
(3) $Z \cap X_\infty = \emptyset$
(4) $X_\infty \cup Z$ has an invariant affine neighborhood in $\overline{X}$.

Note that $\overline{X}$ is an equivariant good compactification of both X and $X - Z$.

Remark 7.2. By [MVW06, Remark 11.6], these conditions imply that S is affine and that Z and X_∞ are finite over S.

Nonequivariantly any smooth quasi-projective scheme fits into a triple, locally around any finite set of points. Equivariantly this is more delicate. If $f : X \to S$ is an equivariant curve which is smooth at a point $x \in X$ then the induced map $\Omega^1_{S/k,f(x)} \otimes_{\mathcal{O}_{S,f(x)}} k(x) \to \Omega^1_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k(x)$ is an injection of I_x-representations over $k(x)$. However it could happen that $\Omega^1_{X/k,x} \otimes k(x)$ has no codimension 1 summand, in which case there can be no such equivariant curve $X \to S$ which is smooth at x. Under the assumption of Condition 5.1 we can construct enough equivariant triples around an orbit in order to establish Theorem 7.13 below.

Write $T_xX := \mathrm{Hom}_{k(x)}(\Omega^1_{X/k,x} \otimes k(x), k(x))$ for the tangent space at $x \in X$.

Proposition 7.3. *Let V be a finite dimensional representation, $Y \subseteq X \subseteq \mathbb{A}(V)$ equivariant closed embeddings of smooth G-schemes, and $x \in Y$ a closed point. Suppose that there are G-representations $W_2 \subseteq W_1$ such that there is an I_x-equivariant isomorphism $f : T_xX \cong (W_1)_{k(x)}$ which restricts to an I_x-equivariant isomorphism $T_xY \cong (W_2)_{k(x)}$. Fix an equivariant splitting $W_1 \to W_2$ of the inclusion $W_2 \subseteq W_1$. Then there is a G-equivariant linear projection $V \to W_1$ such that the composition $X \subseteq \mathbb{A}(V) \to \mathbb{A}(W_1)$ is étale and the induced map $Y \to \mathbb{A}(W_2)$ is also étale.*

Proof. Equivariant linear projections $V \to W_1$, which satisfy the above conditions, are parameterized by an open subset $U \subseteq \mathbb{A}(\mathrm{Hom}_{k[G]}(V, W_1))$ of the affine space associated to the k-vector space of G-equivariant linear maps. More precisely, a point $p \in U$ corresponds to an equivariant $k(p)$-linear map $\mathbb{A}(V)_{k(p)} \to \mathbb{A}(W_1)_{k(p)}$ such that the induced maps $X_{k(p)} \to \mathbb{A}(W_1)_{k(p)}$ and $Y \to \mathbb{A}(W_2)_{k(p)}$ are étale at any point $x' \in Y_{k(p)}$ which lies over $x \in Y$. We need to check that U is nonempty, which implies the result as any nonempty open subset of an affine space has a rational point.
We first treat the case when x is a rational point. Consider the diagram

$$\begin{tikzcd}
 & W_2 & & W_1 \arrow[ll] & \\
 & T_xY \arrow[u, "f'"] \arrow[rr, hook] \arrow[dl] & & T_xX \arrow[u, "f"'] \arrow[r, hook] \arrow[dl, "i"'] & V, \arrow[dll, dashed, bend left, "\rho"] \\
\mathrm{Ind}(T_xY) \arrow[rr] \arrow[uur, bend left] & & \mathrm{Ind}(T_xX) \arrow[uur, bend left] \arrow[urr] & &
\end{tikzcd}$$

where the inclusion $T_xX \subseteq V$ (resp. $T_xY \subseteq V$) is the one induced by $X \subseteq \mathbb{A}(V)$ (resp. $Y \subseteq \mathbb{A}(Y)$) and $\mathrm{Ind}(M) := k[G] \otimes_{k[I_x]} M$ is the G-representation which is induced by the I_x-representation M. The G-equivariant maps $\mathrm{Ind}(T_xX) \to W_1$ and $\mathrm{Ind}(T_xY) \to W_2$ are induced respectively by the I_x-equivariant maps $f : T_xX \to W_1$ and $f' : T_xY \to W_2$. Similarly the I_x-equivariant inclusion $T_xX \subseteq V$ induced by $X \subseteq \mathbb{A}(V)$ induces the G-equivariant linear map $\mathrm{Ind}(T_xX) \to V$. Choose a G-equivariant linear map $\rho : V \to \mathrm{Ind}(T_xX)$ so that the composition $T_xX \to V \xrightarrow{\rho} \mathrm{Ind}(T_xX)$ agrees with the canonical I_x-equivariant linear map i. Then the composition $V \to \mathrm{Ind}(T_xX) \to W_1$ has the required properties and thus U is nonempty in this case.
Now suppose that $x \in Y$ is a nonrational closed point. Consider the G-equivariant embeddings $Y_{k(x)} \subseteq X_{k(x)} \subseteq \mathbb{A}(V)_{k(x)}$ and the points $y_i \in Y_{k(x)}$ which lie over $x \in Y$. Consider the open subset $U' \subseteq \mathbb{A}(\mathrm{Hom}_{k(x)[G]}(V_{k(x)}, (W_1)_{k(x)}))$ consisting of p' such that the corresponding equivariant linear projections $V_{k(p')} \to (W_1)_{k(p')}$ induces maps $X_{k(p')} \to \mathbb{A}(W_1)_{k(p')}$ and $Y_{k(p')} \to \mathbb{A}(W_2)_{k(p')}$ which are étale at any point $y' \in Y_{k(p')}$ lying over a $y_i \in Y_{k(x)}$. Note that $I_{y_i} = I_x$, $T_{y_i}(X_{k(x)}) = T_xX$, and $T_{y_i}(Y_{k(x)}) = T_xY$ and so the hypothesis of the proposition apply to $y_i \in Y_{k(x)}$. Since these are rational points, the previous paragraph shows that U' is

nonempty. Consider the image $p \in \mathbb{A}(\mathrm{Hom}_{k[G]}(V, W_1))$ of a $p' \in U'$. The squares

$$\begin{tikzcd} X_{k(p')} \arrow[r, "p'"] \arrow[d] & \mathbb{A}(W_1)_{k(p')} \arrow[d] \\ X_{k(p)} \arrow[r, "p"] & \mathbb{A}(W_1)_{k(p)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_{k(p')} \arrow[r, "p'"] \arrow[d] & \mathbb{A}(W_2)_{k(p')} \arrow[d] \\ Y_{k(p)} \arrow[r, "p"] & \mathbb{A}(W_2)_{k(p)} \end{tikzcd}$$

are cartesian and so by faithfully flat descent we conclude that the lower horizontal arrows are étale at any point $x' \in Y_{k(p)}$ lying over $x \in Y$. In other words U' maps to U under the projection $\mathbb{A}(\mathrm{Hom}_{k[G]}(V, W_1))_{k(x)} \to \mathbb{A}(\mathrm{Hom}_{k[G]}(V, W_1))$ and so U is nonempty as well. $\square$

THEOREM 7.4. *Assume that G satisfies Condition 5.1. Let X be a smooth quasi-projective G-scheme over k of pure dimension d. Let $Y \subseteq X$ be a smooth invariant closed subscheme containing no component of X and $x \in Y$ a closed point. Then there exists an invariant open affine neighborhood U in Y of y and an equivariant standard triple $(\overline{U} \to S, U_\infty, Z)$ such that $(U, U \cap Y) \cong (\overline{U} - U_\infty, Z)$.*

Proof. First we claim that there are G-representations $W_2 \subseteq W_1$ defined over k, such that there is an isomorphism $(W_1)_{k(x)} \cong T_xX$ of I_x-representations over $k(x)$, which restricts to an I_x-equivariant isomorphism $T_xY \cong (W_2)_{k(x)}$. Indeed, if G satisfies Condition 5.1 then so does I_x. From the fact that $k(x)[I_x] = k[I_x] \otimes_k k(x)$ is the sum of the irreducible representations (over $k(x)$), which are one-dimensional, we see that for any representation M' of I_x defined over $k(x)$ there is a representation M defined over k such that $M' = M_{k(x)}$. Similarly, Condition 5.1 implies that for every I_x-representation N there is a G-representation N' such that $N' = N$ as I_x-representations. These observations easily imply the claim.

Since X is quasi-projective, there is an open invariant affine neighborhood of x and so we may shrink X equivariantly around x and assume that it is affine. Embed X in some representation $\mathbb{A}(V)$. Fixing a choice of equivariant projection $W_1 \to W_2$ and applying Proposition 7.3 we obtain an equivariant linear projection $V \to W_1$ inducing maps $X \to \mathbb{A}(W_1)$ and $Y \to \mathbb{A}(W_2)$ which are étale at x (and hence at all points of $G{\cdot}x$). Let $W \subseteq W_1$ be a codimension-one G-representation containing W_2 and choose an equivariant projections $W_1 \to W$ and $W \to W_2$ factoring $W_1 \to W_2$. The induced equivariant map $p : X \to \mathbb{A}(W)$ is smooth at every point of $G{\cdot}x$. Shrinking X equivariantly around $G{\cdot}x$, we may assume that $p : X \to \mathbb{A}(W)$ is smooth and $Y \to \mathbb{A}(W_2)$ is étale. The map $Y \to \mathbb{A}(W)$ is then quasi-finite.

Let $V' \subseteq V$ be a complementary representation to W so that $V = V' \oplus W$. Let $\overline{X} \subseteq \mathbb{P}(V' \oplus 1) \times \mathbb{A}(W)$ be the closure of X and write $p : \overline{X} \to \mathbb{A}(W)$ for the induced equivariant map. The fiber of $X \to \mathbb{A}(W)$ over any point of its image is one-dimensional. It follows that $\overline{X} - X$ is finite over $\mathbb{A}(W)$. Write $\Sigma \subseteq \overline{X}$ for the set of singular points of $p : \overline{X} \to \mathbb{A}(W)$. Then $\Sigma \subseteq \overline{X}$ is closed,

invariant and is finite over each point of $p(y)$, for any $y \in X$. Therefore there is an invariant affine open neighborhood S of $p(G{\cdot}x)$ in $\mathbb{A}(W)$ over which Σ is finite and over which Y has finite fibers. Note that Σ and Y are disjoint. Define U to be $p^{-1}(S) \cap (X - \Sigma)$. By construction $p : U \to S$ is smooth and equivariant. Define $\overline{U} \subseteq \overline{X}$ to be the preimage of S and set $U_\infty = \overline{U} - U$.
It remains to see that we may arrange that $U_\infty \coprod (U \cap Y)$ has an invariant affine neighborhood in $\overline{U}$. Since $\overline{U}$ is projective over S there is a global section of some very ample line bundle $\mathcal{L}$ whose divisor D misses the finite set of points of U_∞ and $U \cap Y$ over $G{\cdot}y$. As S is affine and $\mathcal{L}$ is very ample, $\overline{U} - D$ is affine. Intersecting all of the translates of this affine neighborhood, we obtain an invariant open affine neighborhood of all of the points of U_∞ and $U \cap Y$ over $G{\cdot}x$. Replacing S by a smaller invariant open affine neighborhood of $p(G{\cdot}x)$ we may assume that D misses all of U_∞ and $U \cap Y$. We thus obtain an invariant affine neighborhood of U_∞ and $U \cap Y$.

□

Let $(\overline{X} \to S, X_\infty, Z)$ be an equivariant triple. Write ΔX for the equivariant Cartier divisor associated to the diagonal $X \subseteq X \times_S X$.

Definition 7.5. An equivariant standard triple is *equivariantly split* over an invariant open $U \subseteq X$ if $\Delta X|_{U \times_S Z}$ is an equivariant principal divisor.

The proof of the following is straightforward.

Lemma 7.6. *Let $f : S' \to S$ be an equivariant map between smooth affine G-schemes over k and $T = (\overline{X} \to S, X_\infty, Z)$ an equivariant triple over S. Then $f^*T = (\overline{X} \times_S S' \to S', X_\infty \times_S S', Z \times_S S')$ is an equivariant triple over S'. If T is equivariantly split over U then f^*T is equivariantly split over $U \times_S S'$.*

In the equivariant case, the question of a triple being locally split is more delicate than its nonequivariant analog. Nonequivariantly, all divisors on $X \times X$ are locally principal when X is smooth. The nonequivariant argument requires more work as an equivariant Weil divisor (equivalently by Lemma 2.11, an equivariant Cartier divisor) on a smooth G-scheme might not be locally equivariantly principal. This can be seen for example from Proposition 2.12 together with the fact that $\mathrm{Pic}^G(S)$ can be nonzero for local rings S.
If $\pi : A \to A/G$ is a quotient and $B \subseteq A$ is an invariant closed subscheme then since $|G|$ is coprime to $\mathrm{char}(k)$ the canonical map $B/G \to \pi(B)$ is an isomorphism. In particular we have a Cartesian square

$$\begin{tikzcd} X \arrow[r, hook, "\Delta"] \arrow[d] & X \times_S X \arrow[d, "\pi"] \\ X/G \arrow[r, hook] & (X \times_S X)/G, \end{tikzcd}$$

whenever the right hand vertical quotient exists.

PROPOSITION 7.7. *Let J be a smooth G-scheme which is finite over k and let $C \to J$ be a smooth equivariant curve. Then the Weyl divisor $(\Delta C)/G \hookrightarrow (C \times_J C)/G$ is locally principal.*

Proof. Consider the coherent sheaf $\mathcal{O}(\Delta C/G)$ on $(C \times_J C)/G$ associated to the Weyl divisor $(\Delta C)/G$. The condition that the divisor $(\Delta C)/G$ is locally principal is equivalent to the condition that the coherent sheaf $\mathcal{O}(\Delta C/G)$ is locally free. Let $\overline{k}$ be an algebraic closure of k. The sheaf $\mathcal{O}(\Delta C/G)$ is locally free if it is so after base change to $\overline{k}$. The base change of $\mathcal{O}(\Delta C/G)$ to $\overline{k}$ is the sheaf associated to $(\Delta C/G)_{\overline{k}}$. Since $(\Delta C/G)_{\overline{k}} = (\Delta C_{\overline{k}})/G$ it is enough to consider the case when k is algebraically closed. We may also assume that J is equivariantly irreducible. Since k is algebraically closed, $J = G/H$ for some subgroup H and so $C = G \times^H C'$ for some smooth H-curve $C' \to \operatorname{Spec}(k)$. Since $(C \times_J C)/G = (C' \times C')/H$ we may replace C by C' and G by H. In other words, we may assume that $J = \operatorname{Spec}(k)$ and it suffices to show that for a smooth G-curve over the algebraically closed field k, the coherent sheaf associated to $\Delta C/G \hookrightarrow (C \times C)/G$ is locally free.
Let $c \in C$ be a closed point. Write $\mathcal{O}(\Delta C/G)_{[c]}$ for the restriction of $\mathcal{O}(\Delta C/G)$ to $(C \times Gc)/G$. The sheaf $\mathcal{O}(\Delta C/G)_{[c]}$ is the coherent sheaf associated to the divisor $(\Delta(G{\cdot}c))/G \hookrightarrow (C \times Gc)/G$. We have that $G{\cdot}c \cong G/I_c$ and so the divisor $(\Delta Gc)/G \hookrightarrow (C \times Gc)/G$ is identified with $[c] \hookrightarrow C/I_c$ under the equivariant isomorphisms $(C \times G{\cdot}c)/G \cong (C \times G/I_c)/G \cong (C \times^{I_c} G)/G \cong C/I_c$, the second isomorphism arising from $(c, [g]) \mapsto (g^{-1}c, g)$. Normality is preserved under taking quotients and so C/I_c is a normal curve and therefore it is also smooth and so $\mathcal{O}(\Delta C/G)_{[c]}$ is locally free of rank one. Every closed point of $(C \times C)/G$ is in some $(C \times G{\cdot}c)/G$ and so $\operatorname{rank}_x[\mathcal{O}(\Delta C/G)] = 1$ (where $\operatorname{rank}_x \mathcal{F} = \dim_{k(x)} \mathcal{F}_x \otimes k(x)$) for every closed point $x \in (C \times C)/G$. But the collection of points where the rank of a coherent sheaf takes on a fixed value is constructible and so $\operatorname{rank}_x[\mathcal{O}(\Delta C/G)] = 1$ for every $x \in (C \times C)/G$. A coherent sheaf $\mathcal{F}$ on a reduced scheme X is locally free exactly when the function $x \mapsto \operatorname{rank}_x \mathcal{F}$ on X is locally constant. We conclude that $\mathcal{O}(\Delta C/G)$ is locally free of rank one. □

COROLLARY 7.8. *Let $X \to S$ be a smooth equivariant curve, with X and S quasi-projective G-schemes. Then the equivariant Cartier divisor $\Delta X \hookrightarrow X \times_S X$ is equivariantly locally principal.*

Proof. We also write ΔX for the associated equivariant Weyl divisor on $X \times_S X$. We have the Cartesian square of normal schemes

$$\begin{tikzcd} \Delta X \arrow[r] \arrow[d] & X \times_S X \arrow[d, "\pi"] \\ \Delta X/G \arrow[r] & (X \times_S X)/G. \end{tikzcd}$$

The equivariant Weyl divisor $\Delta X \hookrightarrow X \times_S X$ is equivariantly locally principal if the Weyl divisor $(\Delta X)/G \hookrightarrow (X \times_S X)/G$ is locally principal. This

Weyl divisor is locally principal exactly when the associated coherent sheaf $\mathcal{O}((\Delta X)/G)$ is locally free. Consider the map $p : (X \times_S X)/G \to S/G$. The fibers over a point $[s] \in S/G$ are $p^{-1}([s]) = (X_{Gs} \times_{Gs} X_{Gs})/G$. The restriction $\mathcal{O}((\Delta X)/G)_{[s]}$ of $\mathcal{O}((\Delta X)/G)$ to the fiber $p^{-1}([s])$ is the coherent sheaf associated to $(\Delta X_{Gs})/G \hookrightarrow (X_{Gs} \times_{Gs} X_{Gs})/G$. By the previous proposition $\mathcal{O}((\Delta X)/G)_{[s]}$ is locally free of rank one for all closed points $[s] \in S/G$. Every closed point of $\mathrm{Supp}(\Delta X/G)$ is in some $p^{-1}([s])$. The sheaf $\mathcal{O}((\Delta X)/G)$ is isomorphic to the trivial line bundle at all points not in $\mathrm{Supp}(\Delta X/G)$. We conclude that the coherent sheaf $\mathcal{O}((\Delta X)/G)$ has rank one at all closed points and hence has rank one at all points. It follows that it is locally free of rank one. □

The following is an important example.

Lemma 7.9. *Let J be an equivariantly irreducible, smooth zero-dimensional G-scheme over k and W a G-representation and $L := J \times \mathbb{A}(W)$. Let X_∞ and Z be disjoint, invariant nonempty finite subsets of $\mathbb{P}(L \oplus 1)$. Then*

$$T := (\mathbb{P}(L \oplus 1) \to J, X_\infty, Z)$$

is an equivariant standard triple which is equivariantly split over any invariant open $U \subseteq L$.

Proof. That T is an equivariant standard triple is clear.
Let $x \in J$ be a point. Then $J = G \times^{G_x} \{x\}$, $\mathbb{P}(L \oplus 1) \cong G \times^{G_x} (\{x\} \times \mathbb{P}(W \oplus 1))$, $U = G \times^{G_x} (\{x\} \times U')$ for a G_x-invariant open $U' \subseteq \mathbb{A}(W)$, and $X_\infty = G \times^{G_x} (\{x\} \times X'_\infty)$ and $Z = G \times^{G_x} (\{x\} \times Z')$ for G_x-invariant disjoint subsets X'_∞ and Z of $\mathbb{P}(W \oplus 1)$. It is thus enough to show that the G_x-equivariant triple $(\mathbb{P}(W \oplus 1), X'_\infty, Z')$ is split over any G_x-invariant open $U' \subseteq \mathbb{A}(W)$.
We show that $\Delta\mathbb{A}(W)$ is equivariantly principal on $\mathbb{A}(W) \times \mathbb{A}(W)$. To show this it suffices to show that $\mathcal{O}(\Delta\mathbb{A}(W))$ is the trivial G_x-line bundle. By Corollary 7.8 the equivariant Cartier divisor $\Delta\mathbb{A}(W) \subseteq \mathbb{A}(W) \times \mathbb{A}(W)$ is equivariantly locally principal which implies in turn that $\Delta\mathbb{A}(W)/G_x \subseteq (\mathbb{A}(W) \times \mathbb{A}(W))$ is locally prinicipal. Therefore $\mathcal{O}(\Delta\mathbb{A}(W)/G_x)$, the coherent sheaf on $(\mathbb{A}(W) \times \mathbb{A}(W))/G_x$ associated to the Weil divisor $(\Delta\mathbb{A}(W))/G_x$, is a line bundle. By [Kan79, Theorem 2.4], $\mathrm{Pic}((\mathbb{A}(W) \times \mathbb{A}(W))/G_x) = 0$ and so the G_x-line bundle $\mathcal{O}(\Delta\mathbb{A}(W)) = \pi^*\mathcal{O}(\Delta\mathbb{A}(W)/G_x)$ is trivial as needed. □

Theorem 7.10. *Let $(\overline{X} \to S, X_\infty, Z)$ be an equivariant standard triple. Then any finite set of points in X has an invariant open neighborhood U over which this triple splits.*

Proof. Let $\mathcal{P} \subseteq X$ be a finite set of points in X. Replacing $\mathcal{P}$ by $G{\cdot}\mathcal{P}$ we may assume that $\mathcal{P}$ is invariant. The equivariant map $\pi : X \times_S Z \to X$ is finite and so $\pi^{-1}\mathcal{P} \subseteq X \times_S Z$ is also an invariant finite set of points. It follows from Corollary 7.8 that D/G is locally principal on $(X \times_S Z)/G$. Let $W \subseteq (X \times_S Z/G)$ be a neighborhood of $\mathcal{P}/G$ on which D/G is principal and let $V \subseteq X \times_S Z$ be its preimage. There is some equivariant neighborhood U of $\mathcal{P}$ such that $U \times_S Z \subseteq V$. The equivariant triple is split over this U. □

Two equivariant finite correspondences $\lambda_0, \lambda_1 \in G\,\mathrm{Cor}_k(X,Y)$ are said to be *equivariantly* $\mathbb{A}^1$*-homotopic* provided there is an $H \in G\,\mathrm{Cor}_k(X \times \mathbb{A}^1, Y)$ such that $H|_{X\times\{i\}} = \lambda_i$, $i = 0, 1$.

PROPOSITION 7.11. *Let* $(\overline{X} \xrightarrow{\overline{p}} S, X_\infty, Z)$ *be an equivariant standard triple which is split over an open affine* $U \subseteq X$*. Then there is an equivariant finite correspondence*

$$\lambda : U \to X - Z$$

such that λ *composed with* $j : X - Z \subseteq X$ *is equivariantly* $\mathbb{A}^1$*-homotopic to the inclusion* $i : U \subseteq X$*. In particular for any homotopy invariant presheaf with equivariant transfers* F*, we have the commutative diagram*

$$\begin{tikzcd} F(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] & F(X-Z) \arrow[dl, "\lambda^*"] \\ F(U). & \end{tikzcd}$$

Proof. We write $\overline{X}_U = U \times_S \overline{X}$. Pulling back to U gives the equivariant triple $(\overline{p}' : \overline{X}_U \to U, (X_\infty)_U, Z_U)$. The diagonal $\Delta : U \to X_U$ is an equivariant section of $\overline{p}'$, so is an element of $C_0(X_U/U)^G$. By Theorem 6.12 it thus determines the class $\Delta U \in \mathrm{Div}^G_{rat}(\overline{X}_U, (X_\infty)_U)$. By assumption, ΔU restricted to Z_U is equivariantly principal, say $\Delta U|_{Z_U} = \mathrm{div}(r_U)$, where r_U is an invariant regular function. Since $Z_U \coprod (X_\infty)_U$ has an invariant affine neighborhood in $\overline{X}_U$, we can use the Chinese remainder theorem to find an invariant rational function ϕ on $\overline{X}_U$ which is defined in an invariant neighborhood of $Z_U \coprod (X_\infty)_U$ and is equal to 1 on $(X_\infty)_U$ and equal to r_U on Z_U. Note that $\mathrm{div}(\phi)$ is zero in $\mathrm{Div}^G_{rat}(\overline{X}_U, (X_\infty)_U)$. We lift the class ΔU to a class $[\lambda'] \in \mathrm{Div}^G_{rat}(\overline{X}_U, (X_\infty)_U \coprod Z_U)$ by setting $[\lambda'] = \Delta U - \mathrm{div}(\phi)$.
Let F be any homotopy invariant presheaf with transfers. The diagram

$$\begin{tikzcd} F(X_U) \arrow[r] \arrow[d, "Tr({[\Delta]})"'] & F((X-Z)_U) \arrow[dl, "Tr({[\lambda]})"] \\ F(U) & \end{tikzcd}$$

is commutative, where the vertical and diagonal maps are those obtained from Lemma 6.3. Let $\lambda' \in C_0((X-Z)_U/U)^G \subseteq G\,\mathrm{Cor}_k(U, (X-Z)_U)$ be any representative of $[\lambda']$ and $\lambda : U \to X - Z$ be the composition of λ' together with the projection to $X - Z$. It is easily verified that $j\lambda$ and i are equivariantly $\mathbb{A}^1$-homotopic. □

COROLLARY 7.12. *Assume that* G *satisfies Condition 5.1. Let* F *be a homotopy invariant presheaf with equivariant transfers,* $Z \subseteq X$ *a closed embedding of smooth quasi-projective* G*-schemes over* k*, and* $x \in X$ *a closed point. Then there exists an open invariant neighborhood* U *of* x*, and a map* $\phi : F(X - Z) \to$

$F(U)$ such that the following triangle commutes,

$$\begin{tikzcd} F(X) \arrow[d] \arrow[dr] & \\ F(X-Z) \arrow[r, "\phi"] & F(U). \end{tikzcd}$$

Proof. If $x \notin Z$ there is nothing to prove. If $x \in Z$ then by Theorem 7.4 there is an invariant open neighborhood X' of x and an equivariant triple $(\overline{X'}, X'_\infty, Z')$ such that $(X', X' - Z) = (\overline{X'} - X'_\infty, Z')$. By Theorem 7.10 there is an invariant open neighborhood U of x such that this triple splits over U. Applying Proposition 7.11 to U yields the corollary. □

Theorem 7.13. *Assume that G satisfies Condition 5.1. Let F be a homotopy invariant presheaf with equivariant transfers, S a smooth semilocal affine G-scheme over k with a single closed orbit and $S_0 \subseteq S$ a dense invariant open subscheme. Then the restriction map $F(S) \to F(S_0)$ is injective.*

Proof. Write the G-scheme S as the intersection $\cap X_i$ and $S_0 = \cap V_i$ where X_i are invariant open neighborhoods of a point x of a smooth affine G-scheme X, $V \subseteq X$ an invariant open, and $V_i = V \cap X_i$.
Write $Z = (X - V)_{red}$. First observe that we may assume that Z is smooth. Indeed, since k is perfect, there is a filtration $\emptyset = Z(n+1) \subseteq Z(n) \subseteq \cdots \subseteq Z(1) \subseteq Z(0) = (X - V)_{red}$ by closed invariant subschemes such that $Z(r) - Z(r-1)$ is smooth (take $Z(r) \subseteq Z(r+1)$ to be the set of singular points). Write $Z(r)_i = X_i \cap Z(r)$. Each $X_i - Z(r-1)_i \subseteq X_i - Z(r)_i$ is the complement of an invariant smooth closed subscheme. If the morphism $F(\cap(X_i - Z(r)_i)) \to F(\cap(X_i - Z(r-1)_i))$ is injective for all r, then $F(\cap X_i) \to F(\cap V_i)$ is injective. Thus we may assume that Z is smooth. Consequently $Z_i := X_i - V_i$ is also smooth.
Now the U_i given by Corollary 7.12 is contained in some X_j and so the kernel of $F(X_i) \to F(V_i)$ vanishes in $F(X_j)$. Thus the map $F(S) \to F(S_0)$ is injective. □

Recall that a G-scheme W is called *equivariantly irreducible* if there is an irreducible component W_0 of W such that $G \cdot W_0 = W$. The underlying scheme of an essentially smooth, zero dimensional G-scheme J over k is a disjoint union of the Zariski spectra of finitely generated field extenstions of k.

Corollary 7.14. *Assume that G satisfies Condition 5.1. Suppose that F is a homotopy invariant presheaf with equivariant transfers and that $F(J) = 0$ for any essentially smooth, zero dimensional G-scheme J over k. Then $F_{GNis} = 0$.*

Definition 7.15. An *equivariant covering morphism* $f : T_Y \to T_X$, of two equivariant standard triples $T_Y = (\overline{Y} \to S, Y_\infty, Z_Y)$ and $T_X = (\overline{X} \to S, X_\infty, Z_X)$, is an equivariant finite map $f : \overline{Y} \to \overline{X}$ such that

(1) $f(Y) \subseteq X$,
(2) $f|_Y : Y \to X$ is étale,

(3) f induces and isomorphism $Z_Y \xrightarrow{\cong} Z_X$, and $Z_Y = f^{-1}Z_X \cap Y$.

Write $Q(X, Y, A)$ for the equivariant distinguished square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ A \arrow[r, "i"] & X, \end{tikzcd}$$

where i is an equivariant open embedding and $f : Y \to X$ is an equivariant étale morphism.

Definition 7.16. Let $f : T_Y \to T_X$ be an equivariant covering morphism of equivariant standard triples as above. The *associated equivariant distinguished square* to this morphism is $Q = Q(X, Y, X - Z_X)$ and we say that the square Q *comes from* this covering morphism.

The following is an important class of examples.

Example 7.17. Suppose that X is affine, has an equivariant good compactification $\overline{X}$ over some smooth S (see Definition 6.10), and $X = U \cup V$ is an open cover by invariant open subschemes such that $X - (U \cap V)$ has an invariant open affine neighborhood. Then

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

comes from the morphism of triples $(\overline{X}, \overline{X} - U, X - V) \to (\overline{X}, X_\infty, X - V)$ defined by the identity on $\overline{X}$.

The proof of the following theorem (and the lemmas below on which it depends) are similar to the arguments in the nonequivariant case. We include complete details for the reader's convenience.

Theorem 7.18. *Let X be a smooth equivariantly irreducible G-scheme over k. Let $Q' = Q(X', Y', A')$ and $Q = Q(X, Y, A)$ be equivariant distinguished squares such that Q' is the restriction of Q along an invariant open subscheme $X' \subseteq X$. Write $j : Q' \hookrightarrow Q$ for the inclusion. Assume that X' and Y' are affine and that Q comes from an equivariant covering map*

$$T_Y = (\overline{Y}, Y_\infty, Z_Y) \to T_X = (\overline{X}, X_\infty, Z_X)$$

of equivariant standard triples and that T_X splits over X'.

Let $F : G\,\mathrm{Cor}_k^{op} \to \mathrm{Ab}$ be a homotopy invariant presheaf with equivariant transfers. Then the map of complexes

$$\begin{tikzcd} 0 \arrow[r] & F(X) \arrow[r, "{(i,f)}"] \arrow[d, "j_X"] & F(A) \oplus F(Y) \arrow[r, "{(-f,i)}"] \arrow[d, "\binom{j_A}{j_Y}"] & F(B) \arrow[r] \arrow[d, "j_B"] & 0 \\ 0 \arrow[r] & F(X') \arrow[r, "{(i',f')}"] & F(A') \oplus F(Y') \arrow[r, "{(-f',i')}"] & F(B') \arrow[r] & 0 \end{tikzcd}$$

is chain homotopic to zero.

In particular if $Q' = Q$ then the Mayer-Vietoris sequence

$$0 \to F(X) \to F(A) \oplus F(Y) \to F(B) \to 0$$

is split-exact.

Proof. By Lemmas 7.20 and 7.21 we have maps $s_1 = (\lambda_A, 0) : F(A) \oplus F(Y) \to F(X')$ and $s_2 = (\psi, \lambda_B) : F(B) \to F(A') \oplus F(Y')$. For these maps to form a chain homotopy from j to zero we need that $sd + ds = j$. This boils down to six equations. Three come from the commutativity of the trapezoid in Lemma 7.20. The remaining three which involve ψ are $\psi i \simeq 0$, $j_A \simeq i'\lambda_A - \psi f$ and $j_B \simeq i'\lambda_B - f'\psi$. These follow from Lemma 7.21. □

Lemma 7.19. *Let $f : T_Y \to T_X$ be an equivariant covering morphism of equivariant standard triples. If T_X is equivariantly split over V then T_Y is equivariantly split over $f^{-1}(V) \cap Y$.*

Proof. By assumption the equivariant Cartier divisor $\Delta X|_{V \times_S Z_X}$ is an equivariant principal divisor, say $\Delta X|_{V \times_S Z_X} = \operatorname{div}(\phi)$. Then $(f \times f)^*(\Delta X) = \Delta Y + Q$, where the support of Q is disjoint from that of ΔY. Since $Z_Y \cong Z_X$, $\operatorname{Supp}(Q)$ is also disjoint from $Y \times_S Z_Y$ and therefore $Q|_{Y \times_S Z_Y} = 0$. Since $(\Delta Y + Q)|_{(f^{-1}V \cap Y) \times_S Z_Y} = \operatorname{div}(\phi \circ (f \times f))$, it follows that $\Delta Y|_{(f^{-1}V \cap Y) \times_S Z_Y} = \operatorname{div}(\phi \circ (f \times f))$ as well. □

Lemma 7.20. *Let $j : Q' \hookrightarrow Q$ be as above. Then there are finite equivariant correspondences $\lambda_A : X' \to A$ and $\lambda_B : Y' \to B$ such that the following diagram in $G\operatorname{Cor}_k$ is commutative up to equivariant $\mathbb{A}^1$-homotopy,*

$$\begin{tikzcd} & Y' \arrow[r, "f'"] \arrow[dl, hook, "j_Y"'] \arrow[d, "\lambda_B"] & X' \arrow[d, "\lambda_A"'] \arrow[dr, hook, "j_X"] & \\ Y & B \arrow[l, hook', "i"] \arrow[r, "f"] & A \arrow[r, hook, "i"] & X. \end{tikzcd}$$

Proof. The equivariant triple T_X is split over X'. By Lemma 7.19, T_Y splits over Y'. Proposition 7.11 gives the existence of λ_A and λ_B making the triangles commute up to $\mathbb{A}^1$-homotopy. The square is easily seen to commute up to $\mathbb{A}^1$-homotopy by the construction used in the proof of Proposition 7.11. □

Lemma 7.21. *Let $j : Q' \hookrightarrow Q$ be as above. There is an equivariant correspondence $\psi \in G\operatorname{Cor}_k(A', B)$ such that the square*

$$\begin{tikzcd} B' \arrow[r, "\lambda_B \circ i' - j_B"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ A' \arrow[r, "\lambda_A \circ i' - j_A"'] \arrow[ur, dashed, "\psi"] & A \end{tikzcd}$$

is homotopy commutative in $G\operatorname{Cor}_k$*, where* $\lambda_A \in G\operatorname{Cor}_k(X', A)$*,* $\lambda_B \in G\operatorname{Cor}_k(Y', B)$ *are the equivariant correspondences from Lemma 7.20. Moreover the composite* $i\psi : A' \to Y$ *is equivariantly* $\mathbb{A}^1$*-homotopic to zero.*

Proof. First we define the equivariant correspondence $\psi \in G\operatorname{Cor}_k(A', B)$. Write $\Delta X'$ for the equivariant Cartier divisor on $X' \times_S \overline{X}$ corresponding to the graph of $X' \hookrightarrow \overline{X}$. Similarly, write $\Delta Y'$ for the equivariant Cartier divisor corresponding to the graph of $Y' \hookrightarrow \overline{Y}$. Write $\mathcal{M}$ for the pullback of $\Delta X'$ to $X' \times_S \overline{Y}$.

The support of $\Delta X'$ is disjoint from $A' \times_S Z_X$ so $\Delta X'|_{A' \times_S Z_X} = 0$. Similarly $\mathcal{M}|_{A' \times_S Z_X} = 0$ and $\Delta Y'|_{B' \times_S Z_Y} = 0$.

By assumption the equivariant Cartier divisor $\Delta X'|_{X' \times_S Z_X}$ is equivariantly principal. By Lemma 7.19, $\Delta Y'|_{Y' \times_S Z_Y}$ is equivariantly principal as well. Write $\Delta X'|_{X' \times_S Z_X} = \operatorname{div}(r_X)$, $\mathcal{M}|_{X' \times_S Z_Y} = \operatorname{div}(r_M)$, and $\Delta Y'|_{Y' \times_S Z_Y} = \operatorname{div}(r_Y)$, where r_X, r_M, and r_Y are invariant regular functions. Furthermore we have that r_X is invertible on $A' \times_S Z_X$. Similarly r_M is invertible on $A' \times_S Z_Y$ and r_Y is invertible on $Y' \times_S Z_Y$. Under the isomorphism $Z_Y \cong Z_X$, r_X becomes identified with r_M.

Let U be an invariant affine neighborhood of $Y_\infty \coprod Z_Y$ in $\overline{Y}$. Then $X' \times_S U$ is an invariant affine neighborhood of $X' \times_S (Y_\infty \coprod Z_Y)$. Since points of $(X' \times_S U)/G$ are orbits, $X' \times_S Y_\infty$ and $X' \times_S Z_Y$ remain disjoint in $(X' \times_S U)/G$. Since $\operatorname{char}(k)$ doesn't divide $|G|$, we have $(X' \times_S Y_\infty)/G = \pi(X' \times_S Y_\infty)$ and similarly for $X' \times_S Z_Y$ where $\pi : X' \times_S U \to (X' \times_S U)/G$ is the quotient map. The invariant regular functions 1 and r_M on $X' \times_S Z_Y$ and $X' \times_S Y_\infty$ define invariant regular functions on their quotients. Since $(X' \times_S U)/G$ is affine, we may apply the Chinese remainder theorem, see e.g.,[GW10, Proposition B.1]) to obtain a regular function $\overline{h}$ on $(X' \times_S U)/G$ that equals $\overline{r_M}$ on $(X' \times_S Z_Y)/G$ and 1 on $(X' \times_S Y_\infty)/G$. We thus have an invariant regular function h on $X' \times_S U$ which equals r_M on $X' \times_S Z_Y$ and 1 on $X' \times_S Y_\infty$.

View h as an invariant rational function on $A' \times_S \overline{Y}$. The support of its associated divisor $\operatorname{div}(h)$ is disjoint from $A' \times_S (Z_Y \coprod Y_\infty)$ and so is an element of $\operatorname{Div}^G(A' \times_S \overline{Y}, A' \times_S (Z_X \coprod Y_\infty)) = C_0(A' \times_S B/A')^G$. Since $C_0(A' \times_S B/A')^G \subseteq G\operatorname{Cor}_k(A', B)$, the divisor $-\operatorname{div}(h)$ determines the equivariant correspondence $\psi : A' \to B$. It remains to verify its properties.

First $i\psi \in G\operatorname{Cor}_k(A', Y)$ corresponds to $-\operatorname{div}(h)$ in $\operatorname{Div}^G(A' \times_S \overline{Y} A' \times_S Y_\infty)$. But since $h|_{A' \times_S Y_\infty} = 1$, $-\operatorname{div}(h)$ is a principal relative equivariant Cartier divisor and so represents 0 in $H_0^{Sus}(G; A' \times_S Y/A')$. Thus $i\psi$ is equivariantly $\mathbb{A}^1$-homotopic to zero.

It remains to see that the diagram of the lemma is homotopy commutative. By the construction of λ_A and λ_B the composition $\lambda_A \circ i' \in G\operatorname{Cor}_k(A', A)$ and $\lambda_B \circ i' \in G\operatorname{Cor}_k(B', B)$ are represented by the classes $\Delta A' - \operatorname{div}(\phi_X)$ and $\Delta B' - \operatorname{div}(\phi_Y)$ in $\operatorname{Div}^G_{rat}(A' \times_S \overline{X}, A' \times_S (X_\infty \coprod Z_X))$ and $\operatorname{Div}^G_{rat}(B' \times_S \overline{Y}, B' \times_S (Y_\infty \coprod Z_Y))$, where ϕ_X is an invariant rational function which is 1 on $A' \times_S X_\infty$.

On the other hand the inclusions j_A and j_B are represented by the classes $\Delta A'$ and $\Delta B'$. It follows that the differences $\lambda_A \circ i' - j_A \in G\operatorname{Cor}_k(A', A)$ and $\lambda_B \circ i' - j_B \in G\operatorname{Cor}_k(B', B)$ are represented by the classes $\operatorname{div}(\phi_X)$ and $\operatorname{div}(\phi_Y)$ respectively.

The composition $\psi f' \in G\operatorname{Cor}_k(B', B)$ is represented by the divisor of the rational function hf' which is 1 on $B' \times_S Y_\infty$ and $r_M f' = r_Y$ on $B' \times_S Z_Y$. We thus have $\psi f' = \lambda_B \circ i' - j_B$ in $\operatorname{Div}^G_{rat}(B' \times_S \overline{Y},\, B' \times (Y_\infty \coprod Z_Y))$.

Now the composition $f\psi \in G\operatorname{Cor}_k(A', A)$ represents the push forward of ψ along $H_0^{Sus}(G; A' \times_S B/A') \to H_0^{Sus}(G; A' \times_S A/A')$. By Lemma 6.14 this is represented by the norm $N(h^{-1})$. Since h^{-1} is 1 on $f^{-1}(X_\infty) \subseteq Y_\infty$, $N(h) = 1$ on $A' \times_S X$. By the following lemma we have that $N(h) = r_X$ on $A' \times_S Z_X$ which yields the desired equality $f\psi = \lambda_A \circ i' - j_A \in G\operatorname{Cor}_k(A',\, A)$. □

Lemma 7.22. *Let $f : U \to V$ be a finite equivariant map with U and V normal. Suppose that $Z \subseteq V$ and $Z' \subseteq U$ are reduced closed subschemes such that the induced map $Z' \to Z$ is an isomorphism and $U \to V$ is étale in a neighborhood of Z'. If $h \in \mathcal{O}^*(U)^G$ is 1 on $f^{-1}(Z) - Z'$ then $N(h)|_Z$ and $h|_{Z'}$ are identified by $Z' \cong Z$.*

Proof. This follows immediately from the nonequivariant statement [MVW, Lemma 21.10]. (i.e., forget the G-action then [MVW, Lemma 21.10] tells us that $N(h)|_Z$ and $h|_{Z'}$ are identified by $Z' \cong Z$. □

We finish this section with the following useful application of Theorem 7.18.

Theorem 7.23. *Let F be a homotopy invariant presheaf, J a smooth equivariantly irreducible zero-dimensional G-scheme and W a G-representation. Then for any open invariant $U \subseteq L := J \times W$ we have*

$$H^i_{GNis}(U, F_{GNis}) = \begin{cases} F(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. Corollary 3.9 implies that $H^i_{GNis}(U, F) = 0$ for $i > 1$. Consider an equivariant distinguished square $Q = Q(U, V, A)$,

$$\begin{array}{ccc} B & \longrightarrow & V \\ \downarrow & & \downarrow \\ A & \longrightarrow & U. \end{array}$$

There is an equivariant embedding of V into a smooth projective curve $\overline{V}$ with G-action which is finite over $\mathbb{P}(L \oplus 1)$. Indeed, ignoring the group action on V there is an embedding into a smooth projective curve $\overline{V}$. Rational maps between smooth projective curves extend uniquely to morphisms which implies that $\overline{V}$ inherits a G-action from V and maps equivariantly and finitely to $\mathbb{P}(L \oplus 1)$.

The square Q comes from the equivariant covering morphism of equivariant standard triples, $(\overline{V}, V_\infty, Z) \to (\mathbb{P}(L \oplus 1), U_\infty, Z)$ where $V_\infty = \overline{V} - V$, $U_\infty = \mathbb{P}(L \oplus 1) - U$, and $Z = -U - A$. The triple $(\mathbb{P}(L \oplus 1), U_\infty, Z)$ is split over U

by Lemma 7.9. Applying Theorem 7.18 with $Q = Q'$ we see that the Mayer-Vietoris sequence

$$0 \to F(U) \to F(A) \oplus F(V) \to F(B) \to 0$$

is split exact. This implies that F is a sheaf in the equivariant Nisnevich topology on U and that $\check{H}^1(\mathcal{U}/U, F) = 0$ for any cover $\mathcal{U}$ coming from a distinguished Nisnevich square.

We claim that any equivariant Nisnevich cover of U can be refined by one coming from an equivariant distinguished square, and consequently $\check{H}^1(\mathcal{U}, F) = 0$. This will finish the proof since $H^1_{GNis}(U, F) = \check{H}^1(U, F)$. First, since F takes disjoint unions to sums we can replace a cover $\{V_i \to U\}$ by a single cover $f : V' \to U$. Indeed there is a dense invariant open $A \subseteq U$ over which f has a splitting. The complement $Z = U - A$ is a finite set of closed points and we choose a splitting $Z \subseteq V_Z$. Let $V = V' - (V_Z - Z)$ then $Q(U, V, A)$ is a distinguished square and the associated cover refines $f : V' \to U$. □

8. Homotopy invariance of cohomology

In this section we show that under the assumption of Condition 5.1 equivariant Nisnevich cohomology with coefficients in a homotopy invariant presheaf with transfers is again a homotopy invariant presheaf with equivariant transfers. This result is the equivariant analogue of the fundamental technical result in Voevodsky's machinery of presheaves with transfers.

Unless specified otherwise, G is assumed to satisfy Condition 5.1 throughout this section.

Proposition 8.1. *Let F be a homotopy invariant presheaf with equivariant transfers. Then F_{GNis} is also a homotopy invariant presheaf with equivariant transfers.*

Proof. By Theorem 4.13, F_{GNis} is a presheaf with equivariant transfers. To show homotopy invariance it suffices to show that $i^* : F_{GNis}(X \times \mathbb{A}^1) \to F_{GNis}(X)$ is injective for any equivariantly irreducible X, where $i : X \to X \times \mathbb{A}^1$ is the inclusion at $0 \in \mathbb{A}^1$. It suffices to do this locally in the equivariant Nisnevich topology, so we may assume that X is affine semilocal with a single orbit. Let $\mathcal{Z} \subseteq X$ be the set of generic points with induced G-action. We have a commutative square

$$\begin{tikzcd} F_{GNis}(X \times \mathbb{A}^1) \arrow[r] \arrow[d] & F_{GNis}(X) \arrow[d] \\ F_{GNis}(\mathcal{Z} \times \mathbb{A}^1) \arrow[r, "\cong"] & F_{GNis}(\mathcal{Z}). \end{tikzcd}$$

We may view F as a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/K$, where $K = k(X)^G$. Theorem 7.23 implies that F is an equivariant Nisnevich sheaf on $\mathcal{Z} \times \mathbb{A}^1$ and therefore the bottom horizontal arrow is an isomorphism. The vertical arrows are injective by Theorem 7.13 and so i^* is injective and thus F_{GNis} is homotopy invariant. □

8.1. Equivariant contractions. If F is a presheaf with transfers on Sm/k the contraction $F_{(-1)}(X) := F(X \times \mathbb{A}^1 - \{0\})/F(X \times \mathbb{A}^1)$ plays an important role in the study of presheaves with transfers. We introduce an equivariant analogue and establish a few basic results concerning equivariant contractions.

Definition 8.2. Let F be a presheaf on $G\mathrm{Sm}/k$ and W a representation of G. Define the presheaf $F_{(-W)}$ by

$$F_{(-W)}(X) = \mathrm{coker}(F(X \times \mathbb{A}(W)) \to F(X \times \mathbb{A}(W) - \{0\})).$$

When F is a presheaf with equivariant transfers then so is $F_{(-W)}$ since it is the quotient of such presheaves. Similarly if F is homotopy invariant then $F_{(-W)}$ is as well.
Nonequivariantly the projection $X \times \mathbb{A}^1 \to X$ is split by including at $1 \in \mathbb{A}^1$, inducing a decomposition $F(X \times \mathbb{A}^1 - 0) = F(X) \oplus F_{(-1)}(X)$ whenever F is homotopy invariant. When W is a representation with $W^G = 0$ then there is no such equivariant splitting. Nonetheless when F is a presheaf with equivariant transfers we still obtain this decomposition, at least for affine X.

Proposition 8.3. *Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Let S be a smooth affine G-scheme over k and W be a one-dimensional representation. Then there is an equivariant finite correspondence $\lambda : S \times \mathbb{A}(W) \to S \times \mathbb{A}(W) - 0$ inducing a decomposition*

$$F(S \times \mathbb{A}(W) - \{0\}) = F(S) \oplus F_{(-W)}(S).$$

Moreover, this decomposition is natural for equivariant maps $S' \to S$, where S' is affine.

Proof. We have an equivariant standard triple $(S \times \mathbb{P}(W \oplus 1) \to S, S \times \infty, S \times 0)$. By Lemmas 7.9 and 7.6 this equivariant triple is equivariantly split over $X = S \times \mathbb{A}(W)$. Applying Proposition 7.11 yields the correspondence λ which induces the splitting $F(S \times \mathbb{A}(W) - 0) \to F(S \times \mathbb{A}(W))$. □

Proposition 8.4. *Let F be a homotopy invariant presheaf with equivariant transfers and W a one-dimensional representation. Then*

$$(F_{GNis})_{(-W)}(S) = (F_{(-W)})_{GNis}(S)$$

for any smooth affine Henselian G-scheme over k with a single closed orbit.

Proof. By Proposition 8.1, F_{GNis} is a homotopy invariant equivariant Nisnevich sheaf with transfers. Observe that $(F_{(-W)})_{GNis} \to (F_{GNis})_{(-W)}$ is a morphism of presheaves with equivariant transfers. Applying Corollary 7.14 to the kernel and cokernel of this map, it suffices to show that $(F_{(-W)})_{GNis}(J) = (F_{GNis})_{(-W)}(J)$ for any essentially smooth zero dimensional G-scheme over k. The left-hand side is by definition $F(J \times \mathbb{A}(W) - \{0\})/F(J \times \mathbb{A}(W))$. The right-hand side is $F_{GNis}(J \times \mathbb{A}(W) - \{0\})/F_{GNis}(J \times \mathbb{A}(W))$. Applying Proposition 8.1 and Theorem 7.23 shows the two sides are equal. □

DEFINITION 8.5. Let $i : Z \hookrightarrow Y$ be an invariant closed embedding with open complement $j : V \subseteq Y$ and F a presheaf. Define the equivariant Nisnevich sheaf $F_{(Y,Z)}$ on Z as in the nonequivariant case. That is let $K_{(Y,Z)} = K$ be the cokernel of $F \to j_*j^*F$ and define $F_{(Y,Z)} = (i^*K)_{GNis}$.

Since sheafification is exact we have an exact sequence

$$F_{GNis} \to (j_*j^*F)_{GNis} \to i_*F_{(Y,Z)} \to 0. \tag{8.6}$$

LEMMA 8.7. *For $n \geq 0$ we have $H^n_{GNis}(-,F)_{(Y,Z)} = i^*R^nj_*F$.*

Proof. The same argument as in [MVW06, Example 2.3.8] works here. Namely, we have $H^n_{GNis}(-,F)_{GNis} = 0$ and therefore $i_*H^n(F)_{(Y,Z)} \cong (j_*j^*H^n(F))_{GNis} = R^nj_*F$. Since $i^*i_* = \mathrm{id}$ the result follows. $\square$

Let $i : S \hookrightarrow S \times \mathbb{A}(W)$ be the invariant closed embedding determined by $0 \in \mathbb{A}(W)$. Then $F_{(-W)}(U) = K(U \times \mathbb{A}(W))$. We obtain by adjunction the map $K(U \times \mathbb{A}(W)) \to i_*i^*K(U \times \mathbb{A}(W)) = i^*K(U)$. Therefore we have the map of sheaves on S

$$(F_{(-W)})_{GNis} \to F_{(S\times\mathbb{A}(W),S\times 0)}.$$

PROPOSITION 8.8. *Let F be a homotopy invariant presheaf with equivariant transfers, W a one-dimensional G-representation, and S a smooth G-scheme. Then we have an isomorphism*

$$(F_{(-W)})_{GNis}|_S \xrightarrow{\cong} F_{(S\times\mathbb{A}(W),S\times 0)}.$$

Proof. We use the argument of [MVW06, Proposition 23.10]. We need to compare $F_{(-W)}$ and j_*j^*F/F in an invariant neighborhood of an orbit Gs of a point s in a smooth affine G-scheme S. The equivariant standard triple $T = (\mathbb{P}(W \oplus 1)_S, S \times \infty, S \times 0)$ is split over $S \times \mathbb{A}(W)$ by Lemma 7.9. Let U be an affine invariant neighborhood of $S \times 0$ in $S \times \mathbb{A}(W)$ and let $T_U = (\mathbb{P}(W \oplus 1)_S, \mathbb{P}(W \oplus 1)_S - U, S \times 0)$. We need to show that by shrinking S there is an invariant open affine neighborhood of $(\mathbb{P}(W \oplus 1)_S - U) \cup S \times 0$. It follows that T_U is an equivariant standard triple.

There is an invariant open $V \subseteq \mathbb{P}(W \oplus 1)$ so that $Gs \times V$ contains $Gs \times 0$ and the finite invariant set $\mathbb{P}(W \oplus 1)_{Gs} - U_{Gs}$. The complements of U and $S \times V$ intersect in a closed subset, disjoint from the fiber $\mathbb{P}(W \oplus 1)_{Gs}$. Since $\mathbb{P}(W \oplus 1)_S$ is proper over S we may shrink S around Gs to assume that the complements are disjoint. Then $S \times V$ contains both $\mathbb{P}(W \oplus 1)_S - U$ and $S \times 0$ as needed.

The identity on $\mathbb{P}(W \oplus 1)$ is an equivariant covering morphism of triples $T_U \to T$. Let $U_0 = U - S \times 0$. Consider the distinguished square Q

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d] & U \arrow[d] \\ S \times \mathbb{A}(W) - 0 \arrow[r, "j"] & S \times W. \end{tikzcd}$$

Write Q' for the same square. The identity $Q' = Q$ comes from the map of triples $T_U \to T$, see Example 7.17. Applying Theorem 7.18 we have a split exact Mayer-Vietoris sequence

$$0 \to F(S \times \mathbb{A}(W)) \to F(S \times W - 0) \oplus F(U) \to F(U_0) \to 0.$$

This together with the homotopy invariance of F implies we have a pushout square

$$\begin{tikzcd} F(S \times \mathbb{A}(W)) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(S \times \mathbb{A}(W) - 0) \arrow[r] & F(U_0). \end{tikzcd}$$

In particular $F(U) \to F(U_0)$ is injective and $F_{(-W)}(S) = F(U_0)/F(U)$.
Note that $j : S \times \mathbb{A}(W) - 0 \to S \times W$ has $j_* j^* F(U) = F(U_0)$ and thus $j_* j^* F/F(U) = F(U_0)/F(U)$ and the result follows by passing to the limit over U and S. $\square$

Lemma 8.9. *Let $f : Y \to X$ be an equivariant étale morphism and $Z \subseteq X$ an invariant closed subscheme such that $f^{-1}(Z) \to Z$ is an isomorphism. Then for any presheaf F we have*

$$F_{(X,Z)} \xrightarrow{\cong} F_{(Y,f^{-1}(Z))}.$$

Proof. It is enough to check the isomorphism on stalks. We may thus assume that Y, X are semilocal Henselian G-schemes, X has a single closed orbit Gx, and Z is nonempty. Since $f^{-1}(Z) \cong Z$ and $Gx \subseteq Z$, it follows that Y also has a single closed orbit and that $Y \cong X$. $\square$

Recall if G acts on the ring R, we write $R^{\#}[G]$ for the twisted group ring (see Remark 2.4). If H acts on the field L and W is a $k[H]$-module then W_L is a $L^{\#}[H]$-module via $r[g](w \otimes x) = gw \otimes r(gx)$.

Lemma 8.10. *Let $Z \subseteq X$ be an equivariant closed embedding of smooth affine G-schemes over k and $x \in Z$ a closed point. Suppose that there are G-representations $W_2 \subseteq W_1$ and isomorphism $f : T_x X \cong (W_1)_{k(x)}$ of $k(x)^{\#}[G_x]$-modules which restricts to a $k(x)^{\#}[G_x]$-module isomorphism $T_x Y \cong (W_2)_{k(x)}$. Then there is an invariant open neighborhood U of x and an equivariant Cartesian diagram*

$$\begin{tikzcd} U \cap Z \arrow[r] \arrow[d] & U \arrow[d] \\ \mathbb{A}(W_2) \arrow[r] & \mathbb{A}(W_1) \end{tikzcd}$$

with étale vertical arrows.

Proof. Let $L/k(x)$ be a finite extension such that the composite L/k is Galois with Galois group Γ. The schemes X_L and Z_L will be considered as $G \times \Gamma$-schemes over k via the diagonal action. We will construct a $G \times \Gamma$-equivariant maps ϕ, ϕ' which fit into a commutative square

$$\begin{tikzcd} Z_L \arrow[r, hook] \arrow[d, "\phi'"'] & X_L \arrow[d, "\phi"] \\ \mathbb{A}(W_2)_L \arrow[r, hook] & \mathbb{A}(W_1)_L \end{tikzcd}$$

and are equivariant at each point of the $G \times \Gamma$-orbit of y, where $y \in X_L$ lies over x. The set of points at which ϕ is étale is an open (and invariant) subset of X_L which contains the orbit of y. Further shrinking this set equivariantly if necessary, we find an invariant open subset $\tilde{U} \subseteq X_L$ such ϕ is étale on $\tilde{U}$ and ϕ' is étale on $\tilde{U} \cap Z_L$. Now $\tilde{U} \times_{\mathbb{A}(W_1)_L} \mathbb{A}(W_2)_L$ is a disjoint union $\tilde{U} \cap Z_L \coprod C$. Replacing $\tilde{U}$ by $\tilde{U} - C$ we may assume that $\tilde{U}$ satisfies $\tilde{U} \cap Z \subseteq \tilde{U} \times_{\mathbb{A}(W_1)_L} \mathbb{A}(W_2)_L$. Galois descent then yields the desired G-equivariant square of the lemma.

It remains to construct the desired $G \times \Gamma$-equivariant square above. Let $y_1 \in Z_L$ be a point lying over x and let $\{y_1, y_2, \ldots, y_n\}$ be the $G \times \Gamma$-orbit of y_1. Let R be the coordinate ring of X and $I \subseteq R$ the defining ideal of Z. Write m_i and $\overline{m}_i$ respectively for the maximal ideals y_i in $R_L = R \otimes_k L$ and $(R/I)_L = R/I \otimes_k L$. The ideals $\cap m_i \subseteq R_L$ and $\cap \overline{m}_i \subseteq (R/I)_L$ are $G \times \Gamma$-invariant. Consider the morphism $\cap m_i \to m_1/m_1^2 \times \cdots \times m_n/m_n^2$, induced by the quotients $m_i \to m_i/m_i^2$. Using the Chinese Remainder Theorem, we see that it is surjective.

Let $S \subseteq G \times \Gamma$ be the set-theoretic stabilizer of y_1. It is the subgroup $S \subseteq G_x \times \Gamma$ consisting of pairs (g, γ) such that the two maps $k(x) \to L$ given by ιg and $\gamma \iota$ are equal (where $\iota : k(x) \subseteq L$ is the embedding chosen at the beginning of the proof). Let $\alpha_1, \ldots, \alpha_n$ be left coset representatives for $(G \times \Gamma)/S$. For an element $\beta = (g, \gamma)$ of $G \times \Gamma$ write $\beta\alpha_i = \alpha_{j(i)} s_i$, for appropriate indices $j(i)$ and $s_i \in S$. If M is an $L^{\#}[S]$-module we obtain an induced $L^{\#}[G \times \Gamma]$-module (here the action of $G \times \Gamma$ on L is via the projection to Γ). As in the case of an ordinary group ring, we may describe the induced module $\mathrm{Ind}(M) := L^{\#}[G \times \Gamma] \otimes_{L^{\#}[S]} M$ as the direct sum $\oplus([\alpha_i] \otimes M)$ of copies of M with basis $\{[\alpha_i]\}$. The $L^{\#}[G \times \Gamma]$ module structure on $\mathrm{Ind}(M)$ is determined by the equations $[\beta]([\alpha_i] \otimes m_i) = ([\alpha_{j(i)}] \otimes s_i m_i)$, for $\beta \in G \times \Gamma$ and $r([\alpha_i] \otimes m_i) = ([\alpha_i] \otimes (\alpha_i^{-1} r) m_i)$ for $r \in L$. We have an isomorphism $m_1/m_1^2 \times \cdots \times m_n/m_n^2 \cong \mathrm{Ind}(m_1/m_1^2)$ given by sending $\overline{r}_i \in m_i/m_i^2$ to $[\alpha_i] \otimes \overline{\alpha_i^{-1} r_i}$. We thus obtain a surjection $\cap m_i \to \mathrm{Ind}(m_1/m_1^2)$ which is a surjection of $L^{\#}[G \times \Gamma]$-modules. In a similar fashion we obtain a surjection $\cap \overline{m}_i \to \mathrm{Ind}(\overline{m}_1/\overline{m}_1^2)$ of $L^{\#}[G \times \Gamma]$-modules.

Now the isomorphism $T_x X \cong (W_1)_{k(x)}$ of $k(x)^{\#}[G_x]$-modules yields the isomorphism $T_{y_1}(X_L) = T_x(X) \otimes_{k(x)} L \cong (W_1)_L$ of $L^{\#}[S]$-modules, which restricts to an isomorphism $T_{y_1} Z_L \cong (W_2)_L$. Since $m_1/m_1^2 \cong (T_{y_1} X_L)^{\vee}$ and

$\overline{m}_1/\overline{m}_1^2 \cong (T_{y_1}Z_L)^\vee$ we obtain a commutative diagram of $L^\#[G\times\Gamma]$-modules

$$\begin{tikzcd} R_L \arrow[d] & \cap m_i \arrow[l, hook'] \arrow[r, two heads] \arrow[d] & \mathrm{Ind}(m_1/m_1^2) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ind}((W_1)^\vee_L) \arrow[d] \\ (R/I)_L & \cap \overline{m}_i \arrow[l, hook'] \arrow[r, two heads] & \mathrm{Ind}(\overline{m}_1/\overline{m}_1^2) \arrow[r, "\cong"] & \mathrm{Ind}((W_2)^\vee_L). \end{tikzcd}$$

The kernel of the action of $G\times\Gamma$ on L is equal to G. Since $|G|$ is invertible in L, the ring $L^\#[G\times\Gamma]$ is semi-simple, see e.g.,[Kün04, Lemma 1.3]. We may therefore choose compatible splittings $\mathrm{Ind}((W_1)^\vee_L)\to \cap m_i$ and $\mathrm{Ind}((W_2)^\vee_L)\to \cap\overline{m}_i$ to the horizontal arrows. We have as well a $L^\#[G\times\Gamma]$-module map $(W_1)^\vee_L \to \mathrm{Ind}((W_1)^\vee_L)$ given by $\omega\mapsto ([\alpha_i]\otimes\alpha_i^{-1}\omega)_i$ and similarly for $(W_2)^\vee_L\to \mathrm{Ind}((W_2)^\vee_L)$. We thus obtain a commutative square

$$\begin{tikzcd} (R)_L \arrow[d] & \mathrm{Sym}_L((W_1)^\vee_L) \arrow[l] \arrow[d] \\ (R/I)_L & \mathrm{Sym}_L((W_2)^\vee_L) \arrow[l] \end{tikzcd}$$

of k-algebras with $G\times\Gamma$-action. Tracing through the construction of these maps, we see that the compositions $(W_1)^\vee_L \to \cap m_i \to m_i/m_i^2$ and $(W_2)^\vee_L \to \overline{m}_i/\overline{m}_i^2$ are $L^\#[G\times\Gamma]$-module isomorphisms. This implies that the horizontal arrows are étale at the m_i and therefore applying $\mathrm{Spec}(-)$ to this square we obtain the desired square of $G\times\Gamma$-schemes over k.

□

Theorem 8.11. *Let X be a smooth affine G-scheme and $Z\subseteq X$ a closed invariant smooth G-scheme of codimension one. Let $x\in Z$ be a closed point. Let W be a G-representation defined over k such that there is an I_x-equivariant isomorphism $W_{k(x)}\cong T_xX/T_xZ$. Then there is an invariant open neighborhood $U\subseteq X$ of x such that for any smooth G-scheme T we have isomorphisms of sheaves on $(U\cap Z)\times T$*

$$F_{(U\times T,(U\cap Z)\times T)} \cong (F_{(-W)})_{GNis}.$$

Proof. We need to see that Condition 5.1 implies that the hypothesis of the previous lemma are satisfied. It suffices to see that every irreducible $k(x)^\#[G_x]$-module is isomorphic to $M\otimes_k k(x)$ for some $k[G]$-module M. Since $k(x)^\#[G_x]$ is semi-simple, we may write it as a direct sum involving all of the irreducible modules. Recall that if R is semi-simple, then Wedderburn's theorem says that $R=\prod \mathrm{End}_{D_i}(S_i)$ where the S_i are the distinct irreducible modules and $D_i=\mathrm{End}_R(S_i)$. Moreover, D_i is a skew-field and $n_i=\dim_{D_i}(S_i)$ is the multiplicity of S_i in the module decomposition $R=\oplus S_i^{n_i}$. Condition 5.1 implies that there are irreducible $k[G]$-modules $M_1,\ldots,M_d$ which form a complete set of irreducible $k[I_x]$-representations. Each M_i is one-dimensional and $d=|I_x|$. Each $M_i' := M_i\otimes_k k(x)$ is an irreducible $k(x)^\#[G_x]$-module. Any $k(x)^\#[G_x]$-module isomorphism $M_i'\cong M_j'$ is also a $k(x)[I_x]$-module isomorphism. Since the $\{M_i'\}$ form d-distinct irreducible $k(x)[I_x]$-modules, they are also d-distinct irreducible $k(x)^{\#}[G_x]$-modules. We claim that this is a complete list of irreducible $k(x)^{\#}[G_x]$-modules. First note that we have $\mathrm{End}_{k(x)^{\#}[G_x]}(M_i') = F$, where $F = k(x)^{G_x}$ is the fixed field. Write $n = [k(x) : F] = |G_x/I_x|$. Since $\mathrm{End}_F(M_i') = (M_i')^n$, each M_i' appears with multiplicity n in this decomposition. Comparing dimensions (as $k(x)$-vectorspaces) we see that the M_i' form a complete list of irreducible $k(x)^{\#}[G_x]$-modules.

Let W_1 and W_2 be G-representations satisfying the hypothesis of the previous lemma. Set $W = W_1/W_2$. By the previous lemma, after shrinking X around x, there is an equivariant Cartesian square,

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{A}(W_2) \arrow[r] & \mathbb{A}(W_1) \end{tikzcd}$$

where the vertical maps are étale. Proceeding as in [MVW06, Theorem 23.12] yields the result. □

8.2. Proof of homotopy invariance. We remind the reader that G is assumed to satisfy Condition 5.1.

Theorem 8.12 (Homotopy Invariance). *Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Then $H^n_{GNis}(-, F_{GNis})$ is also a homotopy invariant presheaf with equivariant transfers.*

Proof. By Theorem 4.15, $H^n_{GNis}(-, F_{GNis})$ is a presheaf with equivariant transfers and it remains to verify that it is homotopy invariant. The case $n = 0$ is Proposition 8.1. We may thus assume that $F = F_{GNis}$ and we proceed by induction on n. Let X be a smooth G-scheme and consider the map $\pi : X \times \mathbb{A}^1 \to X$ and the Leray spectral sequence

$$H^p_{GNis}(X, R^q\pi_*F) \Rightarrow H^{p+q}_{GNis}(X \times \mathbb{A}^1, F).$$

We have that $\pi_*F = F$ since $\pi_*F(U) = F(U \times \mathbb{A}^1) \cong F(U)$. By induction we have that $R^q\pi_*F = 0$ for $0 < q < n$. The spectral sequence collapses by Theorem 8.14, yielding the desired isomorphism $H^n_{GNis}(X; F_{GNis}) = H^n_{GNis}(X \times \mathbb{A}^1; F_{GNis})$. □

Lemma 8.13. *Let X be a smooth G-scheme over k, $Z \subseteq X$ a closed invariant subset such that* $\mathrm{codim}(Z) \geq 1$, *and x a point of X. Then there is an open invariant neighborhood $U \subseteq X$ of x and a sequence of invariant reduced closed subschemes $\emptyset = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_k$ in U satisfying the following two properties.*

1. *The G-schemes $Y_i - Y_{i-1}$ are smooth invariant divisors on $U - Y_{i-1}$.*
2. *$U \cap Z \subseteq Y_k$.*

Proof. The argument is similar to [Voe00, Lemma 4.31]. The key point is that under our assumptions, there is a smooth equivariant curve $p : U \to V$ and so the induction argument of loc. cit. applies here.

□

Now we are ready to prove the vanishing of $R^n\pi_*F$.

Theorem 8.14. *Let X be a smooth G-scheme over k and F a homotopy invariant equivariant Nisnevich sheaf with transfers. Assume that that $R^q\pi_*F = 0$ for $0 < q < n$, and that $H^p_{GNis}(-,F)$ is homotopy invariant for $p < n$. Then $R^n\pi_*F = 0$ as well.*

Proof. We may assume that X is equivariantly irreducible. We need to show that given an $\alpha \in H^n_{GNis}(X \times \mathbb{A}^1, F)$ it becomes zero on an equivariant Nisnevich cover of X. Let J denote the set of generic points of X. By Theorem 7.23, $H^n_{GNis}(J \times \mathbb{A}^1, F) = 0$. This implies that there is an open dense $V \subseteq X$ such that $\alpha|_V$ vanishes. Let $Z = X - V$ with its reduced structure. It now suffices to show that

$$H^n_{GNis}(X \times \mathbb{A}^1, F) \to H^n_{GNis}((X - Z) \times \mathbb{A}^1, F)$$

is injective locally in the equivariant Nisnevich topology on X. Using Lemma 8.13 we may assume that Z is a smooth invariant divisor. We are thus reduced to showing that

$$H^n_{GNis}(X' \times \mathbb{A}^1, F) \to H^n_{GNis}((X' - Z') \times \mathbb{A}^1, F) \tag{8.15}$$

is injective where X' is a smooth affine Heneselian semilocal G-scheme over k with a single closed orbit and $Z' \subseteq X'$ is a smooth invariant divisor.
Write $i : Z' \to X'$ and $j : U' = X' - Z' \to X'$. The map (8.15) factors as

$$H^n_{GNis}(X' \times \mathbb{A}^1, F) \xrightarrow{\tau} H^n_{GNis}(X' \times \mathbb{A}^1, j_*j^*F) \xrightarrow{\eta} H^n_{GNis}(X' - Z' \times \mathbb{A}^1, j^*F)$$

(where we view F as a sheaf on $X' \times \mathbb{A}^1$). We show that each of these maps is injective.
First we show that η is injective. We begin by showing that $R^qj_*F = 0$ for $0 < q < n$. By the inductive hypothesis we have that $H^q_{GNis}(-,F)$ is a homotopy invariant presheaf with equivariant transfers. Since $q > 0$ we have $H^q_{GNis}(-,F)_{GNis} = 0$. By Theorem 8.11 there is a G-representation W such that $(H^q(F)_{(-W)})_{GNis} \cong H^q(F)_{(X'\times\mathbb{A}^1, Z'\times\mathbb{A}^1)}$. By Proposition 8.4 we have that $(H^q(F)_{(-W)})_{GNis} = ((H^q(F)_{GNis})_{(-W)}))_{GNis} = 0$. Finally, by Lemma 8.7, we have

$$R^qj_*F \cong i_*H^q(F)_{(X'\times\mathbb{A}^1, Z'\times\mathbb{A}^1)} \cong i_*(H^q(F)_{GNis})_{GNis} = 0$$

and so $R^qj_*F = 0$ as claimed. Now consider the Leray spectral sequence

$$H^p_{GNis}(X' \times \mathbb{A}^1, R^qj_*(j^*F)) \Rightarrow H^{p+q}_{GNis}(X' - Z', j^*F).$$

Since $R^qj_*F = 0$ for $0 < q < n$ we obtain an exact sequence

$$0 \to H^n_{GNis}(X'\times\mathbb{A}^1, j_*j^*F) \xrightarrow{\eta} H^n_{GNis}(U'\times\mathbb{A}^1, j^*F) \to H^0(X'\times\mathbb{A}^1, R^nj_*j^*F).$$

In particular η is injective as required.
It remains to show that τ is injective as well. By Theorem 7.13 we have an injection $F \to j_*j^*F$. Combining this with (8.6) we have an exact sequence

$$0 \to F \to j_*j^*F \to i_*F_{(X'\times\mathbb{A}^1, Z'\times\mathbb{A}^1)} \to 0.$$

As noted in the previous paragraph we have $((F)_{(-W)})_{GNis} \cong (F)_{(X'\times\mathbb{A}^1, Z'\times\mathbb{A}^1)}$ as sheaves on $Z'\times\mathbb{A}^1$. Consider the long exact sequence associated to the above short exact sequence,

$$H^{n-1}(X'\times\mathbb{A}^1, j_*j^*F) \to H^{n-1}(Z'\times\mathbb{A}^1, F_{(-W)}) \to H^n(X'\times\mathbb{A}^1, F)$$
$$\to H^n(X'\times\mathbb{A}^1, j_*j^*F) \to H^n(Z'\times\mathbb{A}^1, F_{(-W)}).$$

It suffices to show that $H^{n-1}(X'\times\mathbb{A}^1, j_*j^*F) \to H^{n-1}(Z'\times\mathbb{A}^1, F_{(-W)})$ is onto. For $n > 1$ we have that $H^{n-1}(Z'\times\mathbb{A}^1, F_{(-W)}) = H^{n-1}(Z', F_{(-W)}) = 0$, by homotopy invariance of $F_{(-W)}$, the induction hypothesis and that Z' is a semilocal Hensel G-scheme with one orbit. It remains to consider $n = 1$ and show that

$$F(U'\times\mathbb{A}^1) = H^0(X'\times\mathbb{A}^1, j_*j^*F) \to H^0(Z'\times\mathbb{A}^1, F_{(-W)})$$

is surjective. By homotopy invariance of F and $F_{(-W)}$ this map is identified with the map $F(U') \to F_{(-W)}(Z')$. By Theorem 8.11 and (8.6) we have a surjection

$$j_*j^*F \to i_*(F_{(-W)})_{GNis} \to 0$$

which shows that $F(U') \to F_{(-W)}(Z')$ is surjective because X' is a Henselian semilocal G-scheme with a single orbit. We conclude that τ is injective and the proof of the theorem is complete. □

8.3. Applications of homotopy invariance. As before, we assume that G satisfies Condition 5.1.

Theorem 8.16. *Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Let S be a smooth affine semilocal G-scheme over k with a single closed orbit and W a one-dimensional representation. Then for any invariant open $U \subseteq \mathbb{A}(W) \subseteq \mathbb{P}(W\oplus 1)$ and any $n > 0$,*

$$H^n_{GNis}(S\times U, F) = 0.$$

In particular, $H^n_{GNis}(S, F) = 0$ for $n > 0$.

Proof. By Theorem 8.12, $H^n_{GNis}(-, F)$ is a homotopy invariant presheaf with equivariant transfers, in particular the second statement follows from the first. By Corollary 7.14 it suffices to show that $H^n_{GNis}(J\times U, F) = 0$ for any equivariantly irreducible zero dimensional G-scheme J over k. This follows from Theorem 7.23.

□

Theorem 8.17. *Let F be a homotopy invariant equivariant Nisnevich sheaf with transfers on $G\mathrm{Sm}/k$. Let W be a one-dimensional representation and X a smooth G-scheme over k. Then*

$$H^n_{GNis}(X\times(\mathbb{A}(W)-0), F) \cong H^n_{GNis}(X, F) \oplus H^n_{GNis}(X, F_{(-W)}).$$

Proof. Write $\pi : X\times(\mathbb{A}(W)-0) \to X$ for the projection. Consider the Leray spectral sequence $H^p_{GNis}(X, R^q\pi_*F) \Rightarrow H^{p+q}_{GNis}(X\times(\mathbb{A}(W)-0), F)$. We have by Theorem 8.16 that $H^q_{GNis}(S\times(\mathbb{A}(W)-0), F) = 0$ for any smooth affine semilocal G-scheme S with a single closed orbit and any $q > 0$ and therefore this spectral sequence collapses. We therefore have that $H^n_{GNis}(X \times (\mathbb{A}(W) - 0), F) \cong H^n_{GNis}(X, \pi_* F)$. Since F is a sheaf, Proposition 8.3 is seen to imply that there is a decomposition $\pi_* F = F \oplus F_{(-W)}$ of sheaves on X and we are done. □

Our final application in this section is to show that the equivariant Nisnevich and equivariant Zariski cohomology with coefficients in a homotopy invariant presheaf with transfers agree.

Theorem 8.18. *Let F be a homotopy invariant, equivariant Nisnevich sheaf with transfers on $G\mathrm{Sm}/k$. Then for any smooth quasi-projective G-scheme X we have an isomorphism*

$$H^n_{GZar}(X, F) \cong H^n_{GNis}(X, F).$$

Proof. Consider the Leray spectral sequence $H^p_{GZar}(X, \mathcal{H}^q) \Rightarrow H^{p+q}_{GNis}(X, F)$ where $\mathcal{H}^q$ is the equivariant Zariski sheafification of the presheaf $U \mapsto H^q_{GNis}(U, F)$ on X_{GZar}. The result will follow if we see that $\mathcal{H}^q = 0$ for $q > 0$. The points of X_{GZar} are the semilocal rings $\mathcal{O}_{X,Gx}$ of an orbit $Gx \subseteq X$. By Theorem 8.12, the presheaf $H^q_{GNis}(-, F)$ is a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$ and so the vanishing of $\mathcal{H}^q$ follows from Theorem 8.16. □

8.4. The group $(\mathbb{Z}/2)^n$. In this subsection we let $G = (\mathbb{Z}/2)^n$ and write ϵ_i for the generator of the ith factor.

Definition 8.19. Let $I \subseteq \{1, \dots n\}$. Define the $(\mathbb{Z}/2)^n$-scheme $\mathbb{G}_m^{\sigma_I}$ to be $\mathbb{G}_m$ equipped with the $(\mathbb{Z}/2)^n$-action specified by letting the generator ϵ_i of the ith factor act by

$$\epsilon_i(x) = \begin{cases} 1/x & i \in I \\ x & \text{else.} \end{cases}$$

When $G = \mathbb{Z}/2$ we simply write $\mathbb{G}_m^{\sigma} := \mathbb{G}_m^{\sigma_{\{1\}}}$.

If $I = \emptyset$, then $\mathbb{G}_m^{\sigma_I}$ is simply $\mathbb{G}_m$ equipped with trivial action. Note that if $I \neq \emptyset$ then $\mathbb{G}_m^{\sigma_I}$ is not an invariant open in any representation and so the considerations in the previous subsections do not immediately apply to $\mathbb{G}_m^{\sigma_I}$. These schemes will be important for the cancellation theorem in the next section. We first record a few useful analogues of the previous results for these G-scheme. Define $F_{(-\sigma_I)}$ to be the presheaf given by

$$F_{(-\sigma_I)}(X) := \mathrm{coker}(F(X) \xrightarrow{\pi^*} F(X \times \mathbb{G}_m^{\sigma_I}))$$

where $\pi : \mathbb{G}_m^{\sigma_I} \to \mathrm{Spec}(k)$ is the structure map. Note that inclusion at $\{1\}$ yields an equivariant section $i_1 : \mathrm{Spec}(k) \to \mathbb{G}_m^{\sigma_I}$. We thus have

$$F(X \times \mathbb{G}_m^{\sigma_I}) = F(X) \oplus F_{(-\sigma_I)}(X).$$

In particular if F is a presheaf with equivariant transfers, homotopy invariant, or a sheaf then so if F.

Recall that σ_I is the representation specified by letting ϵ_i act on k by -1 if $i \in I$ (and by the identity otherwise). The scheme underlying $\mathbb{P}(\sigma_I \oplus 1)$ is $\mathbb{P}^1$ and ϵ_i acts by $[a : b] \mapsto [-a : b]$ if $i \in I$ (and by the identity otherwise) and $\mathbb{G}_m^{\sigma_I}$ embeds into $\mathbb{P}(\sigma_I \oplus 1)$ as an open invariant subscheme (but is not contained in $\mathbb{A}(\sigma_I)$).

LEMMA 8.20. *Let X_∞ be the complement of $\mathbb{G}_m^{\sigma_I} \subseteq \mathbb{P}(\sigma_I \oplus 1)$ and Z a finite, invariant set of closed points, disjoint from X_∞. The triple $(\mathbb{P}(\sigma_I \oplus 1), X_\infty, Z)$ is split over any invariant open subscheme $U \subseteq \mathbb{G}_m^{\sigma_I}$.*

Proof. The argument is a simpler version of the argument given in Lemma 7.9. The key point is that $\mathrm{Pic}((\mathbb{G}_m^{\sigma_I} \times \mathbb{G}_m^{\sigma_I})/(\mathbb{Z}/2)^n) = 0$ by [Kan79, Lemma 2.1] or [Mag80, Corollary 12]. □

THEOREM 8.21. *Let F be a homotopy invariant presheaf, S a smooth semilocal G-scheme over k with a single closed orbit, and $U \subseteq \mathbb{G}^{\sigma_I}$ an invariant open subscheme. Then*

$$H^i_{GNis}(S \times U, F_{GNis}) = \begin{cases} F(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. For each i, $H^i_{GNis}(-, F_{GNis})$ is a homotopy invariant presheaf with equivariant transfers. It thus suffices by Corollary 7.14 to treat the case when S is a zero dimensional smooth $(\mathbb{Z}/2)^n$-scheme. This case follows exactly as in the argument for Theorem 7.23, replacing the use of Lemma 7.9 with Lemma 8.20. □

PROPOSITION 8.22. *Let F be a homotopy invariant presheaf with equivariant transfers on $G\mathrm{Sm}/k$. Then*

$$(F_{GNis})_{(-\sigma_I)} = (F_{(-\sigma_I)})_{GNis}$$

for any smooth semilocal G-scheme S with a single closed orbit.

Proof. The argument is the same as in Proposition 8.4. □

THEOREM 8.23. *Let F be a homotopy invariant equivariant Nisnevich sheaf with transfers on $G\mathrm{Sm}/k$. Then*

$$H^n(X \times \mathbb{G}_m^{\sigma_I}, F) \cong H^n(X, F) \oplus H^n(X, F_{(-\sigma_I)}).$$

Proof. Write $\pi : X \times \mathbb{G}_m^{\sigma_I} \to X$ for the projection. By Theorem 8.21 we have that $H^q(S \times \mathbb{G}_m^{\sigma_I}, F) = 0$ for any smooth semilocal G-scheme S over k and $q > 0$. Therefore $R^q\pi_*F = 0$ for $q > 0$ and so the Leray spectral sequence degenerates yielding $H^n(X \times \mathbb{G}_m^{\sigma_I}, F) \cong H^n(X, \pi_*F)$. Since $\pi_*F = F \oplus F_{(-\sigma_I)}$ we are done. □

9. Cancellation Theorem

We apply the machinery developed in the previous sections in order to establish an equivariant version of Voevodsky's Cancellation Theorem for $(\mathbb{Z}/2)^n$. The argument given here is an equivariant modification of Voevodsky's argument in [Voe10a]. In this section we write $G = (\mathbb{Z}/2)^n$ and ϵ_i denotes the generator of the ith copy of $\mathbb{Z}/2$. Note that $G = (\mathbb{Z}/2)^n$ satisfies Condition 5.1 over any field (of characteristic different from two).

Remark 9.1. Voevodsky's Cancellation Theorem involves Tate spheres and the equivariant version involves Tate spheres equipped with a group action. If V is a one-dimensional representation, then the scheme underlying $\mathbb{A}(V) - \{0\}$ is $\mathbb{G}_m$ and so these would be a natural candidate for an equivariant cancellation theorem. Our arguments do not work for these equivariant Tate spheres. The argument we give requires that the divisor $D(g_n)$ used in [Voe10a] (see below) is invariant, when equipped with the action coming from that on $\mathbb{G}_m$. This only happens if $\mathbb{G}_m$ is equipped with an action which is compatible with the group structure on $\mathbb{G}_m$. The action on $\mathbb{A}(V) - \{0\}$ is not compatible with the group structure on $\mathbb{G}_m$ and so the argument below does not apply to these equivariant Tate spheres. Since $\mathrm{Aut}(\mathbb{G}_m) = \mathbb{Z}/2$, the possible actions on the Tate sphere, which are suitable for the argument below, are limited. Furthermore, we need that *each* one dimensional representation corresponds to an action on $\mathbb{G}_m$, compatible with the group structure of $\mathbb{G}_m$. These two requirements limit the groups G for which the argument below can work to $G = (\mathbb{Z}/2)^n$. An equivariant cancellation theorem for other groups satisfying Condition 5.1 would likely involve the equivariant Tate spheres $\mathbb{A}(V) - \{0\}$ and would have to find a way around the fact that $D(g_n)$ is not invariant.

Let Z be a smooth G-scheme and $z \in Z$ an invariant rational point. Write $e : Z \to Z$ for the equivariant idempotent defined as the composition $Z \to z \to Z$. Now for G-schemes X, Y define

$$\begin{aligned}\mathrm{Cor}_k(X \wedge (Z, z), &Y \wedge (Z, z)) := \\ &\{\mathcal{V} \in \mathrm{Cor}_k(X \times Z, Y \times Z) \,|\, \mathcal{V} \circ (id_X \times e) = 0 = (id_Y \circ e) \circ \mathcal{V}\}.\end{aligned}$$

We usually omit the basepoint $z \in Z$ from the notation when it is understood that Z is a pointed scheme and simply write $\mathrm{Cor}_k(X \wedge Z, Y \wedge Z)$ for this group. Note that this group inherits a natural G-action from that on $\mathrm{Cor}_k(X \times Z, Y \times Z)$. We write as usual

$$G\,\mathrm{Cor}_k(X \wedge Z, Y \wedge Z) := \mathrm{Cor}_k(X \wedge Z, Y \wedge Z)^G.$$

This construction applies in particular to the G-varieties $(\mathbb{G}_m^{\sigma_I}, 1)$ introduced in Definition 8.19. Write f_i, $i = 1, 2$ for the projection $f_i : X \times \mathbb{G}_m \times Y \times \mathbb{G}_m \to \mathbb{G}_m$ to the ith copy of $\mathbb{G}_m$. More generally write $f_i^{\sigma_I} : X \times \mathbb{G}_m^{\sigma_I} \times Y \times \mathbb{G}_m^{\sigma_I} \to \mathbb{G}_m$ to the ith copy of $\mathbb{G}_m$ (considered with trivial action). Define rational functions $g_n^{\sigma_I}$ by

$$g_n^{\sigma_I} := \frac{(f_1^{\sigma_I})^{n+1} - 1}{(f_1^{\sigma_I})^{n+1} - (f_2^{\sigma_I})}.$$

We consider the associated divisors $D(g_n^{\sigma_I})$. Observe that this is an invariant divisor. Of course the various $D(g_n^{\sigma_I})$ are all exactly the same divisor nonequivariantly, only the G-actions differ. The underlying divisor is the divisor of the rational function $g_n = (f_1^{n+1}-1)/(f_1^{n+1}-f_2)$. This is the divisor considered in [Voe10a]. We simply write D_n for the divisor $D(g_n^{\sigma_I})$ when context makes clear the G-action.

Lemma 9.2 ([Voe10a, Lemma 4.1]). *For any $\mathcal{Z} \in \mathrm{Cor}_k(X\times\mathbb{G}_m^{\sigma_I}, Y\times\mathbb{G}_m^{\sigma_I})$ there exists an N such that for all $n\geq N$ the divisor D_n intersects $\mathcal{Z}$ properly over X.*

For $\mathcal{Z} \in \mathrm{Cor}_k(X\times\mathbb{G}_m^{\sigma_I}, Y\times\mathbb{G}_m^{\sigma_I})$ the intersection $\mathcal{Z}\cdot D_n$ is an equidimensional relative cycle once n is large enough. Define $\rho_n(\mathcal{Z}) \in \mathrm{Cor}_k(X,Y)$ to be the projection to $X\times Y$ of this intersection. Observe that $g\cdot\rho_n(\mathcal{Z}) = \rho_n(g\cdot\mathcal{Z})$ for $g\in G$. Therefore if $\mathcal{Z}\in G\,\mathrm{Cor}_k(X\times\mathbb{G}_m^{\sigma_I}, Y\times\mathbb{G}_m^{\sigma_I})$ then $\rho_n(\mathcal{Z})\in G\,\mathrm{Cor}_k(X,Y)$. If both $\rho_n(\mathcal{Z})$ and $\rho_m(\mathcal{Z})$ are defined, they differ only up to equivariant $\mathbb{A}^1$-homotopy, see [Voe10a].

Lemma 9.3 ([Voe10a, Lemmas 4.3, 4.4, 4.5]).

(1) *For $\mathcal{W}\in G\,\mathrm{Cor}_k(X,Y)$ and $n\geq 1$ we have $\rho_n(\mathcal{W}\times \mathrm{id}_{\mathbb{G}_m^{\sigma_I}}) = \mathcal{W}$.*
(2) *Let e denote the composition $\mathbb{G}_m^{\sigma_I}\to\{1\}\to\mathbb{G}_m^{\sigma_I}$. Then $\rho_n(\mathrm{id}_X\times e)=0$ for all $n\geq 0$ and all $g\in G$.*
(3) *Let $\mathcal{Z}\in G\,\mathrm{Cor}_k(X\times\mathbb{G}_m^{\sigma_I}, Y\times\mathbb{G}_m^{\sigma_I})$ such that $\rho_n\mathcal{Z}$ is defined. Consider an arbitrary $\mathcal{W}\in G\,\mathrm{Cor}_k(X',X)$. Then $\rho_n(\mathcal{Z}\circ(\mathcal{W}\times\mathrm{id}_{\mathbb{G}_m^{\sigma_I}}))$ is defined and*
$$\rho_n(\mathcal{Z}\circ(\mathcal{W}\times\mathrm{id}_{\mathbb{G}_m^{\sigma_I}})) = \rho_n(\mathcal{Z})\circ\mathcal{W},$$
where $\circ$ denotes composition of correspondences.
(4) *Let $\mathcal{Z}\in G\,\mathrm{Cor}_k(X\times\mathbb{G}_m^{\sigma_I}, Y\times\mathbb{G}_m^{\sigma_I})$ be such that $\rho_n\mathcal{Z}$ is defined and $f: X'\to Y'$ a morphism of schemes. Then $\rho_n(\mathcal{Z}\times f)$ is defined and*
$$\rho_n(\mathcal{Z}\times f) = \rho_n(\mathcal{Z})\times f.$$

Write $\mathcal{I}_{\sigma_I}\in G\,\mathrm{Cor}_k(\mathbb{G}_m^{\sigma_I},\mathbb{G}_m^{\sigma_I})$ for the finite correspondence given by $\mathcal{I}_{\sigma_I} := \mathrm{id}-e$. As usual, when context makes the action clear, we simply write $\mathcal{I}$.

Proposition 9.4. *There is an equivariant homotopy*
$$H\in G\,\mathrm{Cor}_k(\mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I}\wedge\mathbb{A}^1, \mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I})$$
such that $H_0-H_1 = \tau - \mathrm{id}_{\mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I}}$, where τ is the endomorphism of $\mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I}$ which switches the factors.

Proof. The case of $\mathbb{G}_m$ with trivial action (i.e. $I=\emptyset$) is [BV08, Proposition 3.2].
There is a canonical map $p:\mathbb{G}_m^{\sigma_I}\times\mathbb{G}_m^{\sigma_I}\to\mathrm{Sym}^2(\mathbb{G}_m^{\sigma_I})$ with transpose p^t. Then $p^tp\in\mathrm{Cor}_k(\mathbb{G}_m^{\sigma_I}\times\mathbb{G}_m^{\sigma_I},\mathbb{G}_m^{\sigma_I}\times\mathbb{G}_m^{\sigma_I})$ is equal to $\mathrm{id}+\tau$. Write $\alpha:\mathbb{G}_m^{\sigma_I}\times\mathbb{G}_m^{\sigma_I}\to\mathbb{G}_m^{\sigma_I}\times\mathbb{G}_m^{\sigma_I}$ for the map defined by $(x,y)\mapsto(xy,1)$.
Define the G-scheme M_I to have underlying scheme $M_I=\mathbb{G}_m\times\mathbb{A}^1$ and the action is specified by letting ϵ_i act by $(x,y)\mapsto(x^{-1},y/x)$ if $i\in I$ (and the

identity otherwise). The map $\mathbb{G}_m^{\sigma_I} \times \mathbb{G}_m^{\sigma_I} \to M_I$, $(x,y) \mapsto (xy, x+y)$ is an equivariant isomorphism. Therefore we have an equivariant $\mathbb{A}^1$-homotopy between the correspondences p and $p\alpha$. Explicitly, we have an equivariant homotopy $H : M_I \times \mathbb{A}^1 \to M_I$, given by $(x,y,t) \mapsto (x, t(1+x) + (1-t)y)$ which induces the desired homotopy.
We therefore have that $\mathrm{id}+\tau = p^t p \simeq p^t p\alpha = \alpha+\tau\alpha$. Now $\alpha+\tau\alpha$ takes values in $1\times\mathbb{G}_m^{\sigma_I}\cup\mathbb{G}_m^{\sigma_I}\times 1$ and therefore $\mathrm{id} = \tau$ in $G\,\mathrm{Cor}_k(\mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I}, \mathbb{G}_m^{\sigma_I}\wedge\mathbb{G}_m^{\sigma_I})/\sim_{\mathbb{A}^1}$ □

For $\mathcal{W} \in G\,\mathrm{Cor}_k(X \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$ define

$$\mathcal{W} \times^{(\tau)} \mathcal{I} \in G\,\mathrm{Cor}_k(X \wedge \mathbb{G}_m^{\sigma_I} \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I} \wedge \mathbb{G}_m^{\sigma_I})$$

by $\mathcal{W} \times^{(\tau)} \mathcal{I} = (\mathrm{id}_Y \times \tau) \circ (\mathcal{W} \times \mathcal{I}) \circ (\mathrm{id}_X \times \tau)$.

Lemma 9.5. *Let $\mathcal{W} \in G\,\mathrm{Cor}_k(X \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$. There is an equivariant homotopy*

$$\phi = \phi_{\mathcal{W}} \in G\,\mathrm{Cor}_k(X \times \mathbb{A}^1 \wedge \mathbb{G}_m^{\sigma_I} \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I} \wedge \mathbb{G}_m^{\sigma_I})$$

such that $\phi_0 - \phi_1 = \mathcal{W} \times^{(\tau)} \mathcal{I} - \mathcal{W} \times \mathcal{I}$.

Proof. Let $H \in G\,\mathrm{Cor}_k(\mathbb{G}_m^{\sigma_I} \times \mathbb{G}_m^{\sigma_I} \times \mathbb{A}^1, \mathbb{G}_m^{\sigma_I} \times \mathbb{G}_m^{\sigma_I})$ be the homotopy as in the previous proposition. We proceed as in [Sus03, Lemma 4.70]. Let $\phi = \phi_{\mathcal{W}}$ be defined by

$$\begin{aligned}\phi = (\mathrm{id}_Y \times H) \circ [(\pm(\mathcal{W} \times \mathcal{I}) \times \mathrm{id}_{\mathbb{A}^1}]+\\ +(\mathrm{id}_Y \times \tau) \circ (\mathcal{W} \times \mathcal{I}) \circ (\mathrm{id}_X \times H).\end{aligned}$$

If $\mathcal{W}$ is invariant then ϕ is also invariant. □

Recall that if F is a presheaf we write $C_n F$ for the presheaf $X \mapsto F(X \times \Delta^n_k)$.

Theorem 9.6. *Let X, Y be smooth $(\mathbb{Z}/2)^n$-schemes over k. The homomorphism of simplicial abelian groups*

$$G\,\mathrm{Cor}_k(X \times \Delta^\bullet_k, Y) \to G\,\mathrm{Cor}_k(X \times \Delta^\bullet_k \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$$

given by $\mathcal{Z} \mapsto \mathcal{Z} \times \mathcal{I}$ is a weak equivalence.

Proof. We follow the nonequivariant argument, [Sus03, Theorem 4.7]. We work with the associated normalized chain complexes to the displayed simplicial abelian groups.
First we show that this map is injective on homology groups. Suppose that $\mathcal{W} \in G\,\mathrm{Cor}_k(X \times \Delta^n, Y)$ is a cycle such that $\mathcal{W} \times \mathcal{I}$ is a boundary. Then there is $\mathcal{V} \in G\,\mathrm{Cor}_k((X \times \Delta^{n+1}) \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$ and $\partial_{n+1}(\mathcal{V}) = \mathcal{W} \times \mathcal{I}$ and $\partial_i(\mathcal{W}) = 0$ for $0 \le i \le n$. By Lemma 9.2 there is N such that $\rho_N(\mathcal{V})$ is defined. By Lemma 9.3 we have that $\rho_N(\partial_i \mathcal{W})$ is defined as well. Moreover by Lemma 9.3 we have

$$\begin{aligned}&\partial_i(\rho_N(\mathcal{V})) = \rho_N(\partial_i(\mathcal{V})) = 0, \quad 0 \le i \le n\\ &\partial_{n+1}(\rho_N(\mathcal{V})) = \rho_N(\partial_{n+1}(\mathcal{V})) = \rho_N(\mathcal{W} \times \mathcal{I}) = \mathcal{W}.\end{aligned}$$

Therefore $\mathcal{W}$ is itself a boundary and so the map on homology is an injection.

Now we show that the map on homology is surjective. Let $\mathcal{V} \in G\,\mathrm{Cor}_k(X \times \Delta^n \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$ be a cycle (i.e., $\mathcal{V} \in G\,\mathrm{Cor}_k((X \times \Delta^n) \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$ satisfies $\partial_i(\mathcal{V}) = 0$ for $0 \leq i \leq n$). Consider the homotopy $\phi = \phi_{\mathcal{V}}$ from Lemma 9.5 which satisfies

$$\phi_0 - \phi_1 = \mathcal{V} \times^{(\tau)} \mathcal{I} - \mathcal{V} \times \mathcal{I}.$$

Applying ρ_N (with respect to the second factor of $\mathbb{G}_m^{\sigma_I}$) and using Lemma 9.3 we have

$$\rho_N(\mathcal{V} \times \mathcal{I}) = \mathcal{V}$$
$$\rho_N(\mathcal{V} \times^{(\tau)} \mathcal{I}) = \rho_N(\mathcal{V}) \times \mathcal{I}.$$

Let $\psi_N = \rho_N(\phi)$. We have

$$(\psi_N)_0 - (\psi_N)_1 = \rho_N(\mathcal{V}) \times \mathcal{I} - \mathcal{V}$$

and so $\partial_i(\psi_N) = \rho_N(\partial_i \phi_{\mathcal{V}}) = 0$ (because $\partial_i \phi_{\mathcal{V}} = \phi_{\partial_i \mathcal{V}} = 0$). We therefore have that $\psi_N \in G\,\mathrm{Cor}_k(X \times \Delta_k^\bullet \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$ is a cycle. The two restrictions

$$G\,\mathrm{Cor}_k(X \times \mathbb{A}^1 \times \Delta_k^\bullet \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I}) \to G\,\mathrm{Cor}_k(X \times \Delta_k^\bullet \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$$

induced by $0 \in \mathbb{A}^1$ and $1 \in \mathbb{A}^1$ induce the same map in homology. Therefore $(\psi_N)_0 - (\psi_N)_1 = \rho_N(\mathcal{V}) \times \mathcal{I} - \mathcal{V}$ is a boundary in $G\,\mathrm{Cor}_k(X \times \Delta_k^\bullet \wedge \mathbb{G}_m^{\sigma_I}, Y \wedge \mathbb{G}_m^{\sigma_I})$. □

We are now ready to prove the equivariant cancellation theorem. Let F be a homotopy invariant equivariant Nisnevich sheaf with transfers.

THEOREM 9.7 (Equivariant Cancellation). *Let X be a smooth $(\mathbb{Z}/2)^n$-scheme. Then*

$$H^n_{GNis}(X, C_*\mathbb{Z}_{tr,G}(Y)) = H^n_{GNis}(X \wedge \mathbb{G}_m^{\sigma_I}, C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})).$$

Proof. The argument is formally the same as in the nonequivariant case given all of the machinery developed in the previous sections. For convenience we give some details. Consider the projection $\pi : X \times \mathbb{G}_m^{\sigma_I} \to X$. We first consider the Leray spectral sequence (which is convergent as X has bounded cohomological dimension)

$$E_2^{p,q} = H^p(X, R^q\pi_* C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})) \Longrightarrow H^{p+q}(X \times \mathbb{G}_m^{\sigma_I}, C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})).$$

Write H^q for the qth cohomology sheaf of the complex $C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})$. To compute the complex $\mathbb{R}\pi_* C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})$ we use the hypercohomology spectral sequence,

$$E_2^{p,q} = R^p\pi_* H^q \Longrightarrow H^{p+q}(\mathbb{R}\pi_* C_*\mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})).$$

The stalks of $R^p\pi_* H^q$ are $H^p(S \times \mathbb{G}_m^{\sigma_I}, H^q)$ where S is a smooth affine semilocal Henselian G-scheme over k with a single closed orbit. By Theorem 8.21 we have $R^p\pi_* H^q = H^p_{GNis}(S \times \mathbb{G}_m^{\sigma_I}, H^q) = 0$ for $p > 0$. The spectral sequence thus degenerates and we have

$$H^q(\mathbb{R}\pi_* \mathbb{Z}_{tr,G}(Y \wedge \mathbb{G}_m^{\sigma_I})) = \pi_* H^q.$$

The stalks of $\pi_* H^q$ are $H^0_{GNis}(S\times \mathbb{G}_m^{\sigma_I}, H^q)$ which, by Theorem 8.23, split into the direct sum

$$H^0_{GNis}(S\times \mathbb{G}_m^{\sigma_I}, H^q) = H^q(S)\oplus H^q_{(-\sigma_I)}(S).$$

By Proposition 8.22 we have that $(\mathcal{H}^q_{GNis})_{(-\sigma_I)} = (\mathcal{H}^q_{(-\sigma_I)})_{GNis}$. Therefore we have

$$H^q(\mathbb{R}\pi_* C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I})) = H^q(C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I}))\oplus H^q(C_*\mathbb{Z}_{tr,G}(Y)).$$

Thus $C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I})\oplus C_*\mathbb{Z}_{tr,G}(Y)\to \mathbb{R}\pi_* C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I})$ is a quasi-isomorphism and therefore

$$H^*(X\times \mathbb{G}_m^{\sigma_I}, C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I})) = H^*(X, C_*\mathbb{Z}_{tr,G}(Y\wedge \mathbb{G}_m^{\sigma_I}))\oplus H^*(X, C_*\mathbb{Z}_{tr,G}(Y)),$$

as required. □

We finish by relating the complexes $C_*\mathbb{Z}_{tr,G}(\mathbb{G}_m^{\sigma_I})$ to the ones introduced in Section 5. After a change of coordinates $\mathbb{P}(\sigma_I\oplus 1)$ can be viewed as $\mathbb{P}^1$ where ϵ_i acts by $[x:y]\mapsto [y:x]$ if $i\in I$ (and is the identity otherwise) and $\mathbb{G}_m^{\sigma_I}$ becomes identified with $\mathbb{P}^1 - \{[0:1],[1:0]\}$. Write $\mathcal{Z}_I$ for $\mathbb{Z}/2$ considered with the action where ϵ_i acts nontrivially if $i\in I$ (and trivially otherwise). Consider the Cartesian square in $G\mathrm{Sm}/k$

$$\begin{tikzcd}
\mathbb{G}_m^{\sigma_I}\times \mathcal{Z}_I \arrow[r] \arrow[d] & \mathbb{A}^1\times \mathcal{Z}_I \arrow[d, "\phi"] \\
\mathbb{G}_m^{\sigma_I} \arrow[r] & \mathbb{P}^1.
\end{tikzcd}$$

The action on $\mathbb{A}^1\times \mathcal{Z}_I = \mathbb{A}^1\coprod \mathbb{A}^1$ is specified by letting ϵ_i switching the factors if $i\in I$ (and is the identity otherwise). The map ϕ sends (x,e) to $[x:1]$ and (x,σ) to $[1:x]$. Note that $\phi^{-1}(\{[0:1],[1:0]\})\cong \{[0:1],[1:0]\}$ is an equivariant isomorphism. In particular, the above square is an equivariant distinguished square.

Recall that we write S^{σ_I} for the topological representation sphere associated to the $(\mathbb{Z}/2)^n$-representation σ_I and $\mathbb{Z}_{top}(\sigma_I) = \mathrm{cone}(\mathbb{Z}_{tr,G}(\mathcal{Z}_I)\to \mathbb{Z})$, see Example 5.11. Using the square above, we obtain a quasi-isomorphism

$$(C_*\big(\mathbb{Z}_{tr,G}(\mathbb{G}_m^{\sigma_I})/\mathbb{Z})\otimes^{\mathbb{L}}_{tr}\mathbb{Z}_{top}(\sigma_I)\big)\simeq C_*\big(\mathbb{Z}_{tr,G}(\mathbb{P}(\sigma_I\oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(\sigma_I))\big).$$

For a representation V, define the equivariant Nisnevich sheaf with transfers $\mathbb{Z}_{tr,G}(T^V)$ by

$$\mathbb{Z}_{tr,G}(T^V) := \mathbb{Z}_{tr,G}(\mathbb{P}(V\oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(V))$$

and similarly for expressions such as $\mathbb{Z}_{tr,G}(X\wedge T^V)$.

Write $i_X : \mathbb{Z}(X\times \mathbb{P}(V))\to \mathbb{Z}(X\times \mathbb{P}(V\oplus 1))$ for the inclusion. If F is a complex of equivariant Nisnevich sheaves, write $H^n_{GNis}(X\wedge T^V, F) := \mathrm{Ext}^n_{GNis}(\mathrm{cone}(i_X), F)$.

Theorem 9.8. *Let X be a smooth $(\mathbb{Z}/2)^n$-scheme and V a finite dimensional representation. Then*

$$H^n_{GNis}(X, C_*\mathbb{Z}_{tr,G}(Y)) \cong H^n_{GNis}(X \wedge T^V, C_*\mathbb{Z}_{tr,G}(Y \wedge T^V)).$$

Proof. It is enough to treat the case of a one dimensional representation. Every one dimensional representation of $(\mathbb{Z}/2)^n$ is of the form σ_I. Using Theorem 4.15 and a standard spectral sequence argument, one sees that the displayed map of hypercohomology groups can be computed as

$$\begin{aligned}&\mathrm{Ext}^n(C_*\mathbb{Z}_{tr,G}(X), C_*\mathbb{Z}_{tr,G}(Y))\\ &\qquad \to \mathrm{Ext}^n(C_*\mathbb{Z}_{tr,G}(X) \otimes^{\mathbb{L}}_{tr} C_*\mathbb{Z}_{tr,G}(T^V), C_*\mathbb{Z}_{tr,G}(Y) \otimes^{\mathbb{L}}_{tr} C_*\mathbb{Z}_{tr,G}(T^V)),\end{aligned}$$

where Ext is computed in $D^-(G\,\mathrm{Cor}_k)$, the derived category of equivariant Nisnevich sheaves with transfers. If V is a trivial representation (i.e., $I = \emptyset$) then we have $C_*\mathbb{Z}_{tr,G}(T^V) \simeq C_*(\mathbb{Z}_{tr,G}(\mathbb{G}_m)/\mathbb{Z})[1]$. More generally if $V = \sigma_I$, we have $C_*\mathbb{Z}_{tr,G}(T^V) \simeq C_*(\mathbb{Z}_{tr,G}(\mathbb{G}_m^{\sigma_I})/\mathbb{Z}) \otimes^{\mathbb{L}}_{tr} \mathbb{Z}_{top}(\sigma_I)$. Both shift and $\mathbb{Z}_{top}(\sigma_I)$ are invertible (see Lemma 5.12), and so the theorem follows from Theorem 9.7. $\square$

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References

[Bre67] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34, Springer-Verlag, Berlin-New York, 1967. MR 0214062 (35 #4914)

[BV08] A. Beilinson and V. Vologodsky, *A DG guide to Voevodsky's motives*, Geom. Funct. Anal. 17 (2008), no. 6, 1709–1787. MR 2399083 (2009d:14018)

[CR62] C. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962. MR 0144979 (26 #2519)

[Del09] P. Deligne, *Voevodsky's lectures on motivic cohomology 2000/2001*, Algebraic topology, Abel Symp., vol. 4, Springer, Berlin, 2009, pp. 355–409. MR 2597743 (2011e:18002)

[dS03] P. F. dos Santos, *A note on the equivariant Dold-Thom theorem*, J. Pure Appl. Algebra 183 (2003), no. 1-3, 299–312. MR 1992051 (2004b:55021)

[Dug05] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, K-Theory 35 (2005), no. 3-4, 213–256 (2006). MR MR2240234 (2007g:19004)

[EG98] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), no. 3, 595–634. MR 1614555 (99j:14003a)

[EG00] ______, *Riemann-Roch for equivariant Chow groups*, Duke Math. J. 102 (2000), no. 3, 567–594. MR 1756110 (2001f:14018)

[EG08] ______, *Algebraic cycles and completions of equivariant K-theory*, Duke Math. J. 144 (2008), no. 3, 489–524. MR 2444304 (2009i:14009)

[Gir71] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-New York, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179. MR 0344253 (49 #8992)

[Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 0238860 (39 #220)

[GW10] U. Görtz and T. Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises. MR 2675155 (2011f:14001)

[Her13] P. Herrmann, *Equivariant Motivic Homotopy Theory*, ArXiv e-prints (2013).

[HHR] M. Hill, M. Hopkins, and D. Ravenel, *On the non-existence of elements of kervair invariant one*, ArXiv e-prints, 0908.3724v2.

[HKO11] P. Hu, I. Kriz, and K. Ormsby, *The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism*, Adv. Math. 228 (2011), no. 1, 434–480. MR 2822236

[HVØ] J. Heller, M. Voineagu, and P.A. Østvær, *An equivariant Beilinson-Lichtenbaum comparison theorem*.

[Jos07] R. Joshua, *Bredon-style homology, cohomology and Riemann-Roch for algebraic stacks*, Adv. Math. 209 (2007), no. 1, 1–68. MR 2294217 (2008i:14016)

[Kan79] M. C. Kang, *Picard groups of some rings of invariants*, J. Algebra 58 (1979), no. 2, 455–461. MR 540650 (82m:14004)

[KØ12] A. Krishna and P. A. Østvær, *Nisnevich descent for K-theory of Deligne-Mumford stacks*, J. K-Theory 9 (2012), no. 2, 291–331. MR 2922391

[Kün04] M. Künzer, *On representations of twisted group rings*, J. Group Theory 7 (2004), no. 2, 197–229. MR 2049017 (2005h:20004)

[LLFM96] H. B. Lawson, Jr., P. Lima-Filho, and M.-L. Michelsohn, *Algebraic cycles and equivariant cohomology theories*, Proc. London Math. Soc. (3) 73 (1996), no. 3, 679–720. MR 1407465 (97i:55022)

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000. MR 1771927 (2001f:14006)

[LMM81] G. Lewis, J. P. May, and J. McClure, *Ordinary $RO(G)$-graded cohomology*, Bull. Amer. Math. Soc. (N.S.) 4 (1981), no. 2, 208–212. MR 598689 (82e:55008)

[LS08] M. Levine and C. Serpé, *On a spectral sequence for equivariant K-theory*, K-Theory 38 (2008), no. 2, 177–222. MR 2366561 (2009f:19006)

[Mag80] A. R. Magid, *Picard groups of rings of invariants*, J. Pure Appl. Algebra 17 (1980), no. 3, 305–311. MR 579090 (81g:14004)

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR MR1304906 (95m:14012)

[MV99] F. Morel and V. Voevodsky, *$\mathbf{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 45–143 (2001). MR 1813224 (2002f:14029)

[MVW06] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006. MR 2242284 (2007e:14035)

[Nie08] Z. Nie, *Karoubi's construction for motivic cohomology operations*, Amer. J. Math. 130 (2008), no. 3, 713–762. MR 2418926 (2009d:14019)

[Ray70] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Mathematics, Vol. 169, Springer-Verlag, Berlin, 1970. MR 0277519 (43 #3252)

[SGA03] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR 2017446 (2004g:14017)

[Sus03] A. Suslin, *On the Grayson spectral sequence*, Tr. Mat. Inst. Steklova 241 (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 218–253. MR 2024054 (2005g:14043)

[SV96] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties*, Invent. Math. 123 (1996), no. 1, 61–94. MR 1376246 (97e:14030)

[Tot99] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281. MR 1743244 (2001f:14011)

[Voe00] V. Voevodsky, *Cohomological theory of presheaves with transfers*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 87–137. MR 1764200

[Voe03] ______, *Motivic cohomology with* $\mathbf{Z}/2$*-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104. MR 2031199 (2005b:14038b)

[Voe10a] ______, *Cancellation theorem*, Doc. Math. (2010), no. Extra volume: Andrei A. Suslin sixtieth birthday, 671–685. MR 2804268 (2012d:14035)

[Voe10b] ______, *Homotopy theory of simplicial sheaves in completely decomposable topologies*, J. Pure Appl. Algebra 214 (2010), no. 8, 1384–1398. MR 2593670 (2011a:55022)

[Voe10c] ______, *Unstable motivic homotopy categories in Nisnevich and cdh-topologies*, J. Pure Appl. Algebra 214 (2010), no. 8, 1399–1406. MR 2593671 (2011e:14041)

[VSF00] V. Voevodsky, A. Suslin, and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143, Princeton University Press, Princeton, NJ, 2000. MR 1764197 (2001d:14026)

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

[Yos83] T. Yoshida, *On G-functors. II. Hecke operators and G-functors*, J. Math. Soc. Japan 35 (1983), no. 1, 179–190. MR 679083 (84b:20010)

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