The Rational Homotopy of the $K(2)$-Local Sphere and the Chromatic Splitting Conjecture for the Prime 3 and Level 2

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Abstract. We calculate the rational homotopy of the $K(2)$-local sphere $L_{K(2)}S^0$ at the prime 3 and confirm Hopkins' chromatic splitting conjecture for $p = 3$ and $n = 2$.

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1. Introduction

Let $K(n)$ be the n-th Morava K-theory at a fixed prime p. The Adams-Novikov Spectral Sequence for computing the homotopy groups of the $K(n)$-local sphere $L_{K(n)}S^0$ can be identified by [2] with a descent spectral sequence

$$E_2^{s,t} \cong H^s(\mathbb{G}_n, (E_n)_t) \Longrightarrow \pi_{t-s}(L_{K(n)}S^0) \ . \tag{1}$$

Here $\mathbb{G}_n$ denotes the automorphism group of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$, where Γ_n is the Honda formal group law; the group $\mathbb{G}_n$ is a profinite group and cohomology is continuous cohomology. The spectrum E_n is the 2-periodic Landweber exact ring spectrum so that the complete local ring $(E_n)_0$ classifies deformations of Γ_n.

In this paper we focus on the case $p = 3$ and $n = 2$. In [4], we constructed a resolution of the $K(2)$-local sphere at the prime 3 using homotopy fixed point spectra of the form E_2^{hF} where $F \subseteq \mathbb{G}_2$ is a finite subgroup. These fixed point spectra are well-understood. In particular, their homotopy groups have all been calculated (see [4]) and they are closely related to the Hopkins-Miller spectrum of topological modular forms. The resolution was used in [6] to redo and refine the earlier calculation of the homotopy of the $K(2)$-localization of the mod-3

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Moore spectrum [14]. In this paper we show how the results of [6] imply the calculation of the rational homotopy of the $K(2)$-local sphere. Let $\mathbb{Q}_p$ be the field of fractions of the p-adic integers and Λ the exterior algebra functor.

Theorem 1.1. *There are classes $\zeta \in \pi_{-1}(L_{K(2)}S^0)$ and $e \in \pi_{-3}(L_{K(2)}S^0) \otimes \mathbb{Q}$ that induce an isomorphism of algebras*

$$\Lambda_{\mathbb{Q}_3}(\zeta, e) \cong \pi_*(L_{K(2)}S^0) \otimes \mathbb{Q} \ .$$

Our result is in agreement with the result predicted by Hopkins' chromatic splitting conjecture [8], and in fact, we will establish this splitting conjecture for $n = 2$ and $p = 3$.

We will prove a more general result which will be useful for calculations with the Picard group of Hopkins [17]. Before stating that, let us give some notation. If X is a spectrum, then we define

$$(E_n)_*X \stackrel{\text{def}}{=} \pi_*L_{K(n)}(E_n \wedge X) \ .$$

Despite the notation, $(E_n)_*(-)$ is not quite a homology theory, because it doesn't take wedges to sums; however, it is a sensitive and tested algebraic invariant for the $K(n)$-local category. The $(E_n)_*$-module $(E_n)_*X$ is equipped with the $\mathfrak{m}$-adic topology where $\mathfrak{m}$ is the maximal ideal in $(E_n)_0$. With respect to this topology, the group $\mathbb{G}_n$ acts through continuous maps and the action is twisted because it is compatible with the action of $\mathbb{G}_n$ on the coefficient ring $(E_n)_*$. This topology is always topologically complete but need not be separated. See [4] §2 for some precise assumptions which guarantee that $(E_n)_*X$ is complete and separated. All modules which will be used in this paper will in fact satisfy these assumptions.

Let $E(n)$ denote the nth Johnson-Wilson spectrum and L_n localization with respect to $E(n)$. Note that $E(0)_*$ is rational homology and $E(1)$ is the Adams summand of p-local complex K-theory. Let S^n_p denote the p-adic completion of the sphere.

Theorem 1.2. *Let $p = 3$ and let X be any $K(2)$-local spectrum so that $(E_2)_*X \cong (E_2)_* \cong (E_2)_*S^0$ as a twisted $\mathbb{G}_2$-module. Then there is a weak equivalence of $E(1)$-local spectra*

$$L_1X \cong L_1(S^0_3 \vee S^{-1}_3) \vee L_0(S^{-3}_3 \vee S^{-4}_3) \ .$$

We will use Theorem 1.1 to prove Theorem 1.2, but we note that Theorem 1.1 is subsumed into Theorem 1.2. Indeed, $\pi_*X \otimes \mathbb{Q} \cong \pi_*L_1X \otimes \mathbb{Q}$ and

$$\pi_*L_1S^0_3 \otimes \mathbb{Q} \cong \pi_*L_0S^0_3 \cong \mathbb{Q}_3$$

all concentrated in degree zero. The generality of the statement of Theorem 1.2 is not vacuous; there are such X which are not weakly equivalent to $L_{K(2)}S^0$ – "exotic" elements in the $K(2)$-local Picard group. See [5] and [10].

We remark that Theorem 1.1 disagrees with the calculation by Shimomura and Wang in [15]. In particular, Shimomura and Wang find the exterior algebra on ζ only.

An interesting feature of our proof of Theorem 1.1 is that it does not require a preliminary calculation of all of $\pi_*(L_{K(2)}S^0)$. In fact, we get away with much less, namely with only a (partial) understanding of the E_2-term of the Adams-Novikov Spectral Sequence converging to $\pi_*L_{K(2)}(S/3)$ where $S/3$ denotes the mod-3 Moore spectrum (see Corollary 3.4). Our method of proof can also be used to recover the rational homotopy of $L_{K(2)}S^0$ as well as the chromatic splitting conjecture at primes $p > 3$ [16]; we only need to use the analog of Corollary 3.4 for the E_2-term of the Adams-Novikov spectral sequence of the $K(2)$-localization of the mod-p Moore spectrum for $p > 3$.
In section 2 we give some general background on the automorphism group $\mathbb{G}_2$ and we review the main results of [4]. In section 3 we recall those results of [6] which are relevant for the purpose of this paper. Section 4 gives the calculation of the rational homotopy groups of $L_{K(2)}S^0$ and in the final section 5 we prove Theorem 1.2 and the chromatic splitting conjecture for $n = 2$ and $p = 3$. See Corollary 5.11.

2. Background

Let Γ_2 be the Honda formal group law of height 2; this is the unique 3-typical formal group law over $\mathbb{F}_9$ with 3-series $[3](x) = x^9$. We begin with a short analysis of the Morava stabilizer group $\mathbb{G}_2$, the group of automorphisms of the pair $(\mathbb{F}_9, \Gamma_2)$. Let $\mathbb{W} = \mathbb{W}(\mathbb{F}_9)$ denote the Witt vectors of $\mathbb{F}_9$ and let $(-)^\sigma : \mathbb{W} \to \mathbb{W}$ be the lift of the Frobenius. Define

$$\mathcal{O}_2 = \mathbb{W}\langle S\rangle/(S^2 = 3, wS = Sw^\sigma)\ .$$

Then $\mathcal{O}_2$ is isomorphic to the ring of endomorphisms of Γ_2 over $\mathbb{F}_9$; hence $\mathcal{O}_2^\times$ is isomorphic to the group $\mathbb{S}_2$ of automorphisms of Γ_2 over $\mathbb{F}_9$. This is a subgroup of the group $\mathbb{G}_2$ of automorphisms of the pair $(\mathbb{F}_9, \Gamma_2)$, which is the group of pairs (f, ϕ) with $\phi : \mathbb{F}_9 \to \mathbb{F}_9$ a field isomorphism and $f : \phi_*\Gamma_2 \to \Gamma_2$ an isomorphism of formal group laws over $\mathbb{F}_9$. Since Γ_2 is defined over $\mathbb{F}_3$, there is a splitting

$$\mathbb{G}_2 \cong \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$$

with Galois action given by $\phi(x + yS) = x^\sigma + y^\sigma S$.
The 3-adic analytic group $\mathbb{S}_2 \subseteq \mathbb{G}_2$ contains elements of order 3; indeed, an explicit such element is given by

$$a = -\frac{1}{2}(1 + \omega S)$$

where ω is a fixed primitive 8-th root of unity in $\mathbb{W}$. If C_3 is the cyclic group of order 3, the map $H^*(\mathbb{S}_2, \mathbb{F}_3) \to H^*(C_3, \mathbb{F}_3)$ defined by a is surjective and, hence, $\mathbb{S}_2$ and $\mathbb{G}_2$ cannot have finite cohomological dimension. As a consequence, the trivial module $\mathbb{Z}_3$ cannot admit a projective resolution of finite length. Nonetheless, $\mathbb{G}_2$ has virtual finite cohomological dimension, and admits a finite length resolution by permutation modules obtained from finite subgroups. Such a resolution was constructed in [4] using the following two finite subgroups of $\mathbb{G}_2$. The notation $\langle - \rangle$ indicates the subgroup generated by the listed elements.

(1) Let $G_{24} = \langle a, \omega^2, \omega\phi \rangle \cong C_3 \rtimes Q_8$. Here Q_8 is the quaternion group of order 8. Note ω^2 acts non-trivially and $\omega\phi$ acts trivially on C_3.
(2) $SD_{16} = \langle \omega, \phi \rangle$. This subgroup is isomorphic to the semidihedral group of order 16.

Remark 2.1. The group $\mathbb{G}_2$ splits as a product $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$. To be specific, the center of $\mathbb{G}_2$ is isomorphic to $\mathbb{Z}_3^\times$ and there is an isomorphism from the additive group $\mathbb{Z}_3$ onto the multiplicative subgroup $1 + 3\mathbb{Z}_3 \subseteq \mathbb{Z}_3^\times$ sending 1 to 4. There is also a reduced determinant map $\mathbb{G}_2 \to \mathbb{Z}_3$. (See [4].) The composition $\mathbb{Z}_3 \to \mathbb{G}_2 \to \mathbb{Z}_3$ is multiplication by 2, giving the splitting. All finite subgroups of $\mathbb{G}_2$ are automatically finite subgroups of $\mathbb{G}_2^1$.

Because of this splitting, any resolution of the trivial $\mathbb{G}_2^1$-module $\mathbb{Z}_3$ can be promoted to a resolution of the trivial $\mathbb{G}_2$-module. See Remark 2.4 below. Thus we begin with $\mathbb{G}_2^1$.
If $X = \lim X_\alpha$ is a profinite set, define $\mathbb{Z}_3[[X]] = \lim \mathbb{Z}/3^n[X_\alpha]$. The following is the main algebraic result of [4].

Theorem 2.2. *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules of the form*

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_3$$

with

$$C_0 = C_3 \cong \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$$

and

$$C_1 = C_2 \cong \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \mathbb{Z}_3(\chi)$$

where $\mathbb{Z}_3(\chi)$ is the SD_{16} module which is free of rank 1 over $\mathbb{Z}_3$ and with ω and ϕ both acting by multiplication by -1.

We recall that a continuous $\mathbb{Z}_3[[\mathbb{G}_2]]$-module M is *profinite* if there is an isomorphism $M \cong \lim_\alpha M_\alpha$ where each M_α is a finite $\mathbb{Z}_3[[\mathbb{G}_2]]$ module.

Corollary 2.3. *Let M be a profinite $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module. Then there is a first quadrant cohomology spectral sequence*

$$E_1^{p,q}(M) \cong \mathrm{Ext}^q_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_p, M) \Longrightarrow H^{p+q}(\mathbb{G}_2^1, M)$$

with

$$E_1^{0,q}(M) = E_1^{3,q}(M) \cong H^q(G_{24}, M)$$

and

$$E_1^{1,q}(M) = E_1^{2,q}(M) \cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3(\chi), M) & q = 0 \\ 0 & q > 0 \ . \end{cases}$$

Remark 2.4. These ideas and techniques can easily be extended to the full group $\mathbb{G}_2$ using the splitting $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$. Let $\psi \in \mathbb{Z}_3$ be a topological generator; then there is a resolution

$$0 \longrightarrow \mathbb{Z}_3[[\mathbb{Z}_3]] \xrightarrow{\psi - 1} \mathbb{Z}[[\mathbb{Z}_3]] \longrightarrow \mathbb{Z}_3 \longrightarrow 0 \ .$$

Write $C_\bullet \to \mathbb{Z}_3$ for the resolution of Theorem 2.2. Then the total complex of the double complex

$$C_\bullet \hat{\otimes} \{ \mathbb{Z}_3[[\mathbb{Z}_3]] \xrightarrow{\psi - 1} \mathbb{Z}[[\mathbb{Z}_3]] \}$$

defines an exact complex $D_\bullet \to \mathbb{Z}_3$ of $\mathbb{Z}_3[[\mathbb{G}_2]]$-modules. The symbol $\hat{\otimes}$ indicates the completion of the tensor product. From this we get a spectral sequence analogous to that of Corollary 2.3.

REMARK 2.5. In our arguments below, we will use the functors on profinite $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules to profinite abelian groups given by

$$M \mapsto E_2^{p,0}(M) = H^p(\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M)) \ .$$

Here $C_\bullet$ is the resolution of Theorem 2.2; thus, we are using the $q = 0$ line of the E_2-page of the spectral sequence of Corollary 2.3. We would like some information on the exactness of these functors; for this we need a hypothesis. If M is a profinite $\mathbb{Z}_3[[\mathbb{G}_2]]$-module then

$$\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M) = \lim_\alpha \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M_\alpha)$$

is also necessarily profinite as a $\mathbb{Z}_3$-module. Since profinite $\mathbb{Z}_3$-modules are closed under kernels and cokernels, the groups $E_2^{p,0}(M)$ are also profinite. We will use later that if M is a finitely generated profinite $\mathbb{Z}_3$-module and $M/3M = 0$, then $M = 0$.

LEMMA 2.6. *Suppose* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence of profinite* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$*-modules such that* $H^1(G_{24}, M_1) = 0$*. Then there is a long exact sequence of profinite* $\mathbb{Z}_3$*-modules*

$$0 \to E_2^{0,0}(M_1) \to E_2^{0,0}(M_2) \to E_2^{0,0}(M_3) \to E_2^{1,0}(M_1) \to \dots$$
$$\dots \to E_2^{3,0}(M_2) \to E_2^{3,0}(M_3) \to 0 \ .$$

Proof. In general the sequence of complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M_1) \to \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M_2) \to \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M_3) \to 0$$

of profinite $\mathbb{Z}_3$-modules need not be exact; however, by Corollary 2.3, the failure of exactness is exactly measured by $H^1(G_{24}, M_1)$. Therefore, if that group is zero, then we do get an exact sequence of complexes, and the result follows. □

REMARK 2.7. By [4], the resolution $C_\bullet \to \mathbb{Z}_3$ of Theorem 2.2 can be promoted to a resolution of $(E_2)_* E_2^{h\mathbb{G}_2^1}$ by twisted $\mathbb{G}_2$-modules

$$(2) \qquad \begin{aligned} (E_2)_* E_2^{h\mathbb{G}_2^1} \to (E_2)_* E_2^{hG_{24}} \to \ & (E_2)_* \Sigma^8 E_2^{hSD_{16}} \\ & \to (E_2)_* \Sigma^8 E_2^{hSD_{16}} \to (E_2)_* E_2^{hG_{24}} \to 0. \end{aligned}$$

We have $\Sigma^8 E_2^{hSD_{16}}$ because C_1 is twisted by a character. From §5 of [4] we get the following topological refinement: there is a sequence of maps between spectra

$$(3) \qquad E_2^{h\mathbb{G}_2^1} \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \to \Sigma^{40} E_2^{hSD_{16}} \to \Sigma^{48} E_2^{hG_{24}}$$

realizing the resolution (2) and with the property that any two successive maps are null-homotopic and all possible Toda brackets are zero modulo indeterminacy. Note that there is an equivalence $\Sigma^8 E_2^{hSD_{16}} \simeq \Sigma^{40} E_2^{hSD_{16}}$, so that suspension is for symmetry only; however,

$$\Sigma^{48} E_2^{hG_{24}} \not\simeq E_2^{hG_{24}}$$

even though

$$(E_2)_* \Sigma^{48} E_2^{hG_{24}} \cong (E_2)_* E_2^{hG_{24}}.$$

This suspension is needed to make the Toda brackets vanish. Because these Toda brackets vanish, the sequence of maps in the topological complex (3) further refines to a tower of fibrations

$$\begin{tikzcd} E_2^{h\mathbb{G}_2^1} \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & E^{hG_{24}} \\ \Sigma^{45} E_2^{hG_{24}} \arrow[u] & \Sigma^{38} E_2^{hSD_{16}} \arrow[u] & \Sigma^{7} E_2^{hSD_{16}} \arrow[u] & \end{tikzcd} \tag{4}$$

There is a similar tower for the sphere itself, using the resolution of Remark 2.4.

Remark 2.8. Let $\Sigma^{-p} F_p$ denote the successive fibers in the tower (4); thus, for example, $F_3 = \Sigma^{48} E_2^{hG_{24}}$. Then combining the descent spectral sequences for the groups G_{24}, SD_{16} and $\mathbb{G}_2^1$ with Corollary 2.3 and the spectral sequence of the tower, we get a square of spectral sequences

$$\begin{tikzcd} E_1^{p,q}((E_2)_t X) \arrow[r, Rightarrow, "2"] \arrow[d, Rightarrow, "1"'] & H^{p+q}(\mathbb{G}_2^1, (E_2)_t X) \arrow[d, Rightarrow, "3"] \\ \pi_{t-q} L_{K(2)}(F_p \wedge X) \arrow[r, Rightarrow, "4"'] & \pi_{t-(p+q)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) . \end{tikzcd} \tag{5}$$

We will use information about spectral sequences (1) and (2) to deduce information about spectral sequences (3) and (4). See Lemmas 4.4 and 5.3.
There is a similar square of spectral sequences where the lower right corner becomes $\pi_* L_{K(2)} S^0$. This uses the resolution of Remark 2.4 and the subsequent tower for the sphere.

3. The algebraic spectral sequences in the case of $(E_2)_*/3$

Let $S/3$ denote the mod-3 Moore spectrum. Then, in the case of $(E_2)_*/3 = (E_2)_*(S/3)$ the spectral sequence of Corollary 2.3 was completely worked out in [6]. We begin with some of the details.
First note that this is a spectral sequence of modules over $H^*(\mathbb{G}_2; (E_2)_*/3)$. We will describe the E_1-term as a module over the subalgebra

$$\mathbb{F}_3[\beta, v_1] \subseteq H^*(\mathbb{G}_2; (E_2)_*/3)$$

where $\beta \in H^2(\mathbb{G}_2, (E_2)_{12}/3)$ detects the image of the homotopy element $\beta_1 \in \pi_{10}S^0$ in $\pi_{10}(L_{K(2)}(S/3))$ and $v_1 := u_1u^{-2}$ detects the image of the homotopy element in $\pi_4(S/3)$

$$S^4 \longrightarrow \Sigma^4(S/3) \xrightarrow{A} S/3$$

of the inclusion of the bottom cell composed with the v_1-periodic map constructed by Adams.

In the next result, the element α of bidegree $(1,4)$ detects the image of the homotopy element $\alpha_1 \in \pi_3S^0$ and the element $\widetilde{\alpha}$ of bidegree $(1,12)$ detects an element in $\pi_{11}(L_{K(2)}(S/3))$ which maps to the image of β_1 in $\pi_{10}(L_{K(2)}S^0)$ under the pinch map $S/3 \to S^1$ to the top cell. For more details on these elements, as well as for the proof of the following theorem we refer to [6]. We write

$$E_r^{p,q,t} = E_r^{p,q}((E_2)_t/3)$$

for the E_r-term of the spectral sequence of Corollary 2.3. For example, if $p = 0$ or $p = 3$, then

$$E_1^{p,*,t} = H^*(G_{24}, (E_2)_t/3)\ .$$

By the calculations of [4] §3, there is an invertible class $\Delta \in H^0(G_{24}, (E_2)_{24})$. We also write Δ for its image in $H^0(G_{24}, (E_2)_{24}/3)$.

THEOREM 3.1. *There are isomorphisms of $\mathbb{F}_3[\beta, v_1]$-modules, with β acting trivially on $E_1^{p,*,*}$ if $p = 1, 2$:*

$$E_1^{p,*,*} \cong \begin{cases} \mathbb{F}_3[[v_1^6\Delta^{-1}]][\Delta^{\pm1}, v_1, \beta, \alpha, \widetilde{\alpha}]/(\alpha^2, \widetilde{\alpha}^2, v_1\alpha, v_1\widetilde{\alpha}, \alpha\widetilde{\alpha} + v_1\beta)e_p & p = 0, 3 \\ \\ \omega^2u^4\mathbb{F}_3[[u_1^4]][v_1, u^{\pm8}]e_p & p = 1, 2\ . \end{cases}$$

REMARK 3.2. The module generators e_p are of tridegree $(p, 0, 0)$. If $p = 0$ or $p = 3$, then $E_1^{p,0,*}$ is isomorphic to a completion of the ring of mod-3 modular forms for smooth elliptic curves. Indeed, by Deligne's calculations [1] §6, the ring of modular forms is $\mathbb{F}_3[b_2, \Delta^{\pm1}]$ where b_2 is the Hasse invariant and Δ is the discriminant. The Hasse invariant of an elliptic curve can be computed as v_1 of the associated formal group, so we can write $b_2 = v_1$.

If $p = 1$ or $p = 2$, we have written $E_1^{p,0,*}$ as a submodule of $(E_2)_*/3 = \mathbb{F}_9[[u_1]][u^{\pm1}]$. Recall that there is a 3-typical choice for the universal deformation of the Honda formal group Γ_2 with $v_1 = u_1u^{\pm2}$ and $v_2 = u^{-8}$.

All differentials in the spectral sequence of Corollary 2.3 with $M = (E_2)_*/3$ are v_1-linear. This follows from the fact that v_1 is an element in the homotopy groups of the spectrum $S/3$. In particular, d_1 is determined by continuity and the following formulae established in [6].

THEOREM 3.3. *There are elements*

$$\Delta_k \in E_1^{0,0,24k}, \quad b_{2k+1} \in E_1^{1,0,16k+8}, \quad \overline{b}_{2k+1} \in E_1^{2,0,16k+8}, \quad \overline{\Delta}_k \in E_1^{3,0,24k}$$

for each $k \in \mathbb{Z}$ satisfying

$$\Delta_k \equiv \Delta^k e_0, \quad b_{2k+1} \equiv \omega^2u^{-4(2k+1)}e_1, \quad \overline{b}_{2k+1} \equiv \omega^2u^{-4(2k+1)}e_2, \quad \overline{\Delta}_k \equiv \Delta^k e_3$$

(where the congruences are modulo the ideal $(v_1^6\Delta^{-1})$ resp. $(v_1^4u^8)$ and in the case of Δ_0 we even have equality $\Delta_0 = \Delta^0 e_0 = e_0$) such that

$$d_1(\Delta_k) = \begin{cases} (-1)^{m+1} b_{2.(3m+1)+1} & k = 2m+1, m \in \mathbb{Z} \\ (-1)^{m+1} m v_1^{4.3^n-2} b_{2.3^n(3m-1)+1} & k = 2m.3^n, m \in \mathbb{Z}, \\ & \quad m \not\equiv 0 \bmod (3), n \geq 0 \\ 0 & k = 0 \end{cases}$$

$$d_1(b_{2k+1}) = \begin{cases} (-1)^n v_1^{6.3^n+2} \overline{b}_{3^{n+1}(6m+1)} & k = 3^{n+1}(3m+1), \\ & \quad m \in \mathbb{Z}, n \geq 0 \\ (-1)^n v_1^{10.3^n+2} \overline{b}_{3^n(18m+11)} & k = 3^n(9m+8), \\ & \quad m \in \mathbb{Z}, n \geq 0 \\ 0 & \textit{else} \end{cases}$$

$$d_1(\overline{b}_{2k+1}) = \begin{cases} (-1)^{m+1} v_1^2 \overline{\Delta}_{2m} & 2k+1 = 6m+1, m \in \mathbb{Z} \\ (-1)^{m+n} v_1^{4.3^n} \overline{\Delta}_{3^n(6m+5)} & 2k+1 = 3^n(18m+17), \\ & \quad m \in \mathbb{Z}, n \geq 0 \\ (-1)^{m+n+1} v_1^{4.3^n} \overline{\Delta}_{3^n(6m+1)} & 2k+1 = 3^n(18m+5), \\ & \quad m \in \mathbb{Z}, n \geq 0 \\ 0 & \textit{else}\ . \end{cases}$$

We will actually only need the following consequence of these results, which follows after a little bookkeeping.

Corollary 3.4. *There is an isomorphism*

$$E_2^{p,0}((E_2)_0/3) \cong \begin{cases} \mathbb{F}_3 & p = 0, 3 \\ 0 & p = 1, 2\ . \end{cases}$$

Remark 3.5. We also record here the integral calculation $H^*(G_{24}, (E_2)_*)$ from [4]; we will use this in Proposition 5.5. There are elements c_4, c_6 and Δ in $H^0(G_{24}, (E_2)_*)$ of internal degrees 8, 12 and 24 respectively. The element Δ is invertible and there is a relation

$$c_4^3 - c_6^2 = (12)^3\Delta\ .$$

Define $j = c_4^3/\Delta$ and let M_* be the graded ring

$$M_* = \mathbb{Z}_3[[j]][c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3\Delta, \Delta j = c_4^3).$$

There are also elements $\alpha \in H^1(G_{24}, (E_2)_4)$ and $\beta \in H^2(G_{24}, (E_2)_{12})$ which reduce to the restriction (from $\mathbb{G}_2$ to G_{24}) of the elements of the same name in Theorem 3.1. There are relations

$$\begin{aligned} 3\alpha = 3\beta = \alpha^2 &= 0 \\ c_4\alpha = c_4\beta &= 0 \\ c_6\alpha = c_6\beta &= 0. \end{aligned} \tag{6}$$

Finally

$$H^*(G_{24},(E_2)_*) = M_*[\alpha,\beta]/R$$

where R is the ideal of relations given by (6). The element Δ has already appeared in Theorem 3.1. Modulo 3, $c_4 \equiv v_1^2$ and $c_6 \equiv v_1^3$ up to a unit in $H^0(G_{24},(E_2)_0/3) = \mathbb{F}_3[[j]]$. Compare [3], Proposition 7.

4. The rational calculation

The purpose of this section is to give enough qualitative information about the integral calculation of $H^*(\mathbb{G}_2,(E_2)_*)$ in order to prove Theorem 1.1. Much of this is more refined than we actually need, but of interest in its own right.
The following result implies that the rational homotopy will all arise from $H^*(\mathbb{G}_2,(E_2)_0)$.

Proposition 4.1. *a) Suppose $t = 4.3^k m$ with $m \not\equiv 0$ mod (3). Then $3^{k+1}H^*(\mathbb{G}_2,(E_2)_t) = 0$.*
b) Suppose t is not divisible by 4. Then $H^*(\mathbb{G}_2,(E_2)_t) = 0$.

Proof. Part (b) is the usual sparseness for the Adams-Novikov Spectral Sequence. We can prove this here by considering the spectral sequence

$$H^p(\mathbb{G}_2/\{\pm 1\}, H^q(\{\pm 1\},(E_2)_t) \Longrightarrow H^{p+q}(\mathbb{G}_2,(E_2)_t)$$

given by the inclusion of the central subgroup $\{\pm 1\} \subset \mathbb{Z}_3^\times \subset \mathbb{G}_2$. The central subgroup $\mathbb{Z}_3^\times$ acts trivially on $(E_2)_0$ and by multiplication on u; that is, if $g \in \mathbb{Z}_3^\times$ then $g_*(u) = gu$. In particular we find

$$H^q(\{\pm 1\},(E_2)_t) = 0$$

unless t is a non-zero multiple of 4 and $q = 0$. From this (b) follows.
For (a) we use the spectral sequence

$$H^p(\mathbb{G}_2^1, H^q(\mathbb{Z}_3,(E_2)_t)) \Longrightarrow H^{p+q}(\mathbb{G}_2,(E_2)_t)$$

If $\psi \in \mathbb{Z}_3$ is a topological generator, then $\psi \equiv 4$ modulo 9. In particular,

$$\psi(u^{t/2}) = (1 + 2.3^{k+1}m)u^{t/2} \qquad \mathrm{mod}\ (3^{k+2})$$

and we have that $H^q(\mathbb{Z}_3,(E_2)_t) = 0$ unless $q = 1$ and

$$3^{k+1}H^1(\mathbb{Z}_3^\times,(E_2)_t) = 0\ .$$

Then (a) follows. □

It's not possible to be quite so precise in the case of $\mathbb{G}_2^1$. However, we do have the following result.

Proposition 4.2. *Suppose $s > 3$ or t is not divisible by 4. Then*

$$H^s(\mathbb{G}_2^1,(E_2)_t) \otimes \mathbb{Q} = 0\ .$$

Proof. This follows from tensoring the spectral sequence of Corollary 2.3 with $\mathbb{Q}$ and noting that

$$H^s(G_{24},(E_2)_t) \otimes \mathbb{Q} = H^s(SD_{16},(E_2)_t) \otimes \mathbb{Q} = 0$$

if $s > 0$ or t is not divisible by 4. □

To isolate the torsion-free part of the cohomology of either $\mathbb{G}_2$ or $\mathbb{G}_2^1$ we use the spectral sequences of Corollary 2.3. From Remark 3.5 we have an inclusion which is an isomorphism in positive cohomological degrees

$$\mathbb{Z}_3[\beta^2\Delta^{-1}]/(3\beta^2\Delta^{-1}) \subseteq H^*(G_{24}, (E_2)_0).$$

In the notation of the spectral sequences of Corollary 2.3 and Remark 2.4 we then have inclusions

$$\mathbb{Z}_3[\beta^2\Delta^{-1}]/(3\beta^2\Delta^{-1})e_p \subseteq E_1^{p,*}(\mathbb{G}_2^1, (E_2)_0), \quad p = 0, 3.$$

Here is the main algebraic result.

Theorem 4.3. *a) There is an element $e \in H^3(\mathbb{G}_2^1, (E_2)_0)$ of infinite order so that*

$$H^*(\mathbb{G}_2^1, (E_2)_0) \cong \Lambda(e) \otimes \mathbb{Z}_3[\beta^2\Delta^{-1}]/(3\beta^2\Delta^{-1}) \ .$$

b) There is an element $\zeta \in H^1(\mathbb{G}_2, (E_2)_0)$ of infinite order so that

$$H^*(\mathbb{G}_2, (E_2)_0) \cong \Lambda(\zeta) \otimes H^*(\mathbb{G}_2^1, (E_2)_0) \ .$$

Proof. For the proof of part (a) we consider the functors from the category of profinite $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules to 3-profinite abelian groups introduced in Remark 2.5 and given by

$$M \mapsto E_2^{p,0}(M) = H^p(\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_\bullet, M)) \ .$$

Here $C_\bullet$ is the resolution of Theorem 2.2.
From Remark 3.5 we know that the hypothesis of Lemma 2.6 is satisfied for the short exact sequence

$$0 \to (E_2)_0 \xrightarrow{\times 3} (E_2)_0 \to (E_2)_0/3 \to 0 \ .$$

Then Corollary 3.4, the long exact sequence of Lemma 2.6, and the fact that the groups $E_2^{p,0}(\mathbb{G}_2^1, (E_2)_0)$ are profinite $\mathbb{Z}_3$- modules give

$$E_2^{p,0}(\mathbb{G}_2^1, (E_2)_0) \cong \begin{cases} \mathbb{Z}_3, & p = 0, 3; \\ 0, & p = 1, 2 \ . \end{cases}$$

See Remark 2.5. This implies that the E_2-term of the spectral sequence of Corollary 2.3 is isomorphic to

$$\Lambda(e_3) \otimes \ \mathbb{Z}_3[\beta^2\Delta^{-1}]/(3\beta^2\Delta^{-1}) \ .$$

Since there can be no further differentials, part (a) follows.
Since the central $\mathbb{Z}_3$ acts trivially on $(E_2)_0$, we have a Künneth isomorphism

$$H^*(\mathbb{Z}_3, \mathbb{Z}_3) \otimes H^*(\mathbb{G}_2^1, (E_2)_0) \cong H^*(\mathbb{G}_2, (E_2)_0) \ .$$

Part (b) follows. □

We are now ready to state and prove the main result on rational homotopy. Note that Theorem 1.1 of the introduction is an immediate consequence of Proposition 4.1, of Theorem 4.3, of the spectral sequence

$$H^s(\mathbb{G}_2, (E_2)_t) \otimes \mathbb{Q} \Longrightarrow \pi_{t-s} L_{K(2)} S^0 \otimes \mathbb{Q} \ .$$

and part (b) of the following Lemma.

Let κ_2 be the set of isomorphism classes of $K(2)$-local spectra X so that $(E_2)_*X \cong (E_2)_* = (E_2)_*S^0$ as twisted $\mathbb{G}_2$-modules. This is a subgroup of the $K(2)$-local Picard group; the group operation is given by smash product. In [5] we show that $\kappa_2 \cong (\mathbb{Z}/3)^2$.

For the next result, the spectra F_p were defined in Remark 2.8.

Lemma 4.4. *(a) Let $X \in \kappa_2$. Then for $p = 0, 1, 2, 3$, the edge homomorphism of the localized descent spectral sequence*

$$E_2^{p,q,t} = \mathrm{Ext}^q_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_p, (E_2)_tX) \otimes \mathbb{Q} \Longrightarrow \pi_{t-q}L_{K(2)}(F_p \wedge X) \otimes \mathbb{Q}$$

induces an isomorphism

$$\pi_*L_{K(2)}(F_p \wedge X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_p, (E_2)_*X) \otimes \mathbb{Q}\ .$$

(b) Let $F = \mathbb{G}_2^1$ or $\mathbb{G}_2$. Then the localized spectral sequence

$$H^s(F, (E_2)_tX) \otimes \mathbb{Q} \Longrightarrow \pi_{t-s}L_{K(2)}(E_2^{hF} \wedge X) \otimes \mathbb{Q}$$

converges and collapses.

Proof. For (a), the spectral sequence

$$H^s(F, (E_2)_tX) \Longrightarrow \pi_{t-s}L_{K(2)}(E_2^{hF} \wedge X)$$

has a horizontal vanishing line at E_∞ by the calculations of §3 of [4]. Thus the rationalized spectral sequence

$$H^s(F, (E_2)_tX) \otimes \mathbb{Q} \Longrightarrow \pi_{t-s}L_{K(2)}(E_2^{hF} \wedge X) \otimes \mathbb{Q}$$

converges. The result follows in this case.

For (b) we first do the case of $\mathbb{G}_2^1$. We localize the square of spectral sequences of (5) to get a new square of spectral sequences

$$\begin{tikzcd} E_1^{p,q}((E_2)_tX) \otimes \mathbb{Q} \arrow[r, Rightarrow, "②"] \arrow[d, Rightarrow, "①"'] & H^{p+q}(\mathbb{G}_2^1, (E_2)_tX) \otimes \mathbb{Q} \arrow[d, Rightarrow, "③"] \\ \pi_{t-q}L_{K(2)}(F_p \wedge X) \otimes \mathbb{Q} \arrow[r, Rightarrow, "④"'] & \pi_{t-(p+q)}L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \otimes \mathbb{Q}\ . \end{tikzcd} \tag{7}$$

We will show that spectral sequence (3) converges and the result will follow. First note that spectral sequences (2) and (4) are the localizations of finite and convergent spectral sequences, so must converge. We have noted in the proof of part (a) that the spectral sequences of (1) converge. Now we note that spectral sequence (2) with $q = 0$ and the spectral sequence of (4) have the same d_1, by the construction of the tower.

From this we conclude that the E_2-term of the spectral sequence (4) is

$$E_2^{p,t} \cong H^p(\mathbb{G}_2^1, (E_2)_tX) \otimes \mathbb{Q}\ .$$

Proposition 4.2 implies that the spectral sequence (4) collapses and that, in fact, if

$$\pi_n L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \otimes \mathbb{Q} \neq 0$$

there are unique integers p and t with $t-p=n$ and

$$\pi_n L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \otimes \mathbb{Q} \cong H^p(\mathbb{G}_2^1, (E_2)_t X) \otimes \mathbb{Q}.$$

It follows immediately that spectral sequence (3) converges and collapses.
There is an analogous argument for $\mathbb{G}_2$, using the expanded square of spectral sequences for this group. See Remark 2.8. The needed properties of $H^p(\mathbb{G}_2, (E_2)_t X) \otimes \mathbb{Q}$ are obtained by combining Proposition 4.1 with Theorem 4.3.b. □

Theorem 4.3 and Lemma 4.4 immediately imply the following results. Let S_p^n denote the p-complete sphere.

Theorem 4.5. *Let $X \in \kappa_2$. Then the rational Hurewicz homomorphism*

$$\pi_0 L_0 X \longrightarrow \pi_0 L_0 L_{K(2)}(E_2 \wedge X) \cong (E_2)_0 X \otimes \mathbb{Q}$$

is injective. Given a choice of isomorphism $f : (E_2)_* \to (E_2)_* X$ of twisted $\mathbb{G}_2$-modules the image of the multiplicative unit 1 under the isomorphism

$$\mathbb{Q}_3 \cong \mathbb{Q} \otimes H^0(\mathbb{G}_2, (E_2)_0) \cong \pi_0 L_0 X$$

extends to a weak equivalence of $L_0 L_{K(2)} S^0$-modules

$$L_0 L_{K(2)} S^0 \simeq L_0 X\ .$$

Theorem 4.6. *The localized spectral sequence of Lemma 4.4*

$$\mathbb{Q} \otimes H^s(\mathbb{G}_2, (E_2)_t) \Longrightarrow \pi_{t-s} L_0 L_{K(2)} S^0$$

determines an isomorphism

$$\Lambda_{\mathbb{Q}_3}(\zeta, e) \cong \pi_* L_0 L_{K(2)} S^0.$$

Furthermore, there is a weak equivalence

$$L_0(S_3^0 \vee S_3^{-1} \vee S_3^{-3} \vee S_3^{-4}) \simeq L_0 L_{K(2)} S^0.$$

5. The chromatic splitting conjecture

In this section we prove a refinement of Theorem 1.2 of the introduction.
Our main result, Theorem 5.10, analyzes $L_1 X$ for $X \in \kappa_2$. For this we will use the chromatic fracture square

(8)
$$\begin{tikzcd} L_1 X \arrow[r] \arrow[d] & L_{K(1)} X \arrow[d] \\ L_0 X \arrow[r] & L_0 L_{K(1)} X\ . \end{tikzcd}$$

We made an analysis of L_0X in Theorem 4.5. The calculation of $L_{K(1)}X$ has a number of interesting features, so we dwell on it. In particular, we will produce weak equivalences

$$L_{K(1)}S^0 \to L_{K(1)}L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X)$$

which will be the key to the entire calculation.

We begin with the following general result. Let S/p^n denote the Moore spectrum.

Lemma 5.1. *Let X be a spectrum. Then*

$$L_{K(1)}X = \mathrm{holim}_n\ v_1^{-1}S/p^n \wedge X$$

where $v_1^t : \Sigma^{2t(p-1)}S/p^n \to S/p^n$ is any choice of v_1-self map.

Proof. By Proposition 7.10(e) of [9] we know that

$$L_{K(1)}X = \mathrm{holim}_n\ S/p^n \wedge L_1X\ . \tag{9}$$

Since L_1 is smashing, we may rewrite (9) as

$$L_{K(1)}X = \mathrm{holim}_n\ L_1(S/p^n) \wedge X\ .$$

Thus it is sufficient to know $L_1(S/p^n) = v_1^{-1}S/p^n$. This follows from the Telescope Conjecture for $n = 1$; see [11] and [12]. □

If R is a discrete ring, then the Laurent series over R is the ring $R((x)) = R[[x]][x^{-1}]$.

Proposition 5.2. *(a) There are isomorphisms*

$$\mathbb{F}_3((v_1^6\Delta^{-1}))[v_1^{\pm 1}] \cong v_1^{-1}H_*(G_{24}, (E_2)_*/3) \tag{10}$$

and

$$\mathbb{F}_3((v_1^4v_2^{-1}))[v_1^{\pm 1}] \cong v_1^{-1}H^*(SD_{16}, (E_2)_*/3)\ . \tag{11}$$

(b) There are isomorphisms

$$\mathbb{F}_3[v_1^{\pm 1}] \otimes \Lambda(v_1^{-1}b_1) \cong v_1^{-1}H^*(\mathbb{G}_2^1, (E_2)_*/3) \tag{12}$$

and

$$\mathbb{F}_3[v_1^{\pm 1}] \otimes \Lambda(v_1^{-1}b_1, \zeta) \cong v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*/3)\ . \tag{13}$$

The element b_1 has bidegree $(1, 8)$ and the element $v^{-1}b_1$ detects the image of the homotopy class $\alpha_1 \in \pi_3S^0/3$. The element ζ has bidegree $(1, 0)$ and is the image of the class of the same name in $H^1(\mathbb{G}_2, (E_2)_0)$ from Theorem 4.3.b.

Proof. The results in (a) are immediate consequences of Theorem 3.1. See also [4] §3. For (b), the two isomorphisms both follow from Theorem 3.3 and the algebraic spectral sequences of Corollary 2.3. That $v_1^{-1}b_1$ detects the image of α_1 is proved in Proposition 1.5 of [6]. □

Here is our key lemma. Compare Lemma 4.4 in the rational case.

Lemma 5.3. *Let $X \in \kappa_2$ and let $X/3 = S/3 \wedge X$.*
(a) Suppose that $F = G_{24}$ or SD_{16}. Then the edge homomorphism induces an isomorphism

$$\pi_* L_{K(1)} L_{K(2)}(E_2^{hF} \wedge X/3) \xrightarrow{\cong} v_1^{-1} H^0(F, (E_2)_* X/3) \ .$$

(b) Let $F = \mathbb{G}_2^1$ or $\mathbb{G}_2$. Then the localized spectral sequence

$$(v_1^{-1} H^s(F, (E_* X)/3))_t \Longrightarrow \pi_{t-s} L_{K(1)}(E_2^{hF} \wedge X/3)$$

converges and collapses.

Proof. The proof of Lemma 4.4 goes through *mutatis mutandis*. We need only replace the localization $H^*(F,M) \mapsto H^*(F,M) \otimes \mathbb{Q}$ with the localization

$$H^*(F,M) \longmapsto v_1^{-1} H^*(F, M/3)$$

throughout, and use Theorem 3.3 in place of Proposition 4.1 and Theorem 4.3. $\square$

We now have the following remarkable calculation.

Proposition 5.4. *Let $X \in \kappa_2$. Then the $K(1)$-localized Hurewicz homomorphism*

$$\pi_0 L_{K(1)} X/3 \longrightarrow \pi_0 L_{K(1)} L_{K(2)}(E_2 \wedge X/3)$$

is injective. Any choice of isomorphism $(E_2)_* \cong (E_2)_* X$ of twisted $\mathbb{G}_2$-modules uniquely defines a generator of

$$\pi_0(L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X/3)) \cong (v_1^{-1} H^0(\mathbb{G}_2^1, (E_2)_*/3)_0 \cong \mathbb{F}_3 \ .$$

This generator extends uniquely to a weak equivalence

$$L_{K(1)} S^0/3 \simeq L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X/3) \ .$$

Proof. We use the localized spectral sequence

$$(v_1^{-1} H^s(\mathbb{G}_2^1, (E_2)_*/3))_t \Longrightarrow \pi_{t-s} L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X/3).$$

This converges by Lemma 5.3.b. The choice of isomorphism $(E_2)_* \cong (E_2)_* X$ is used to identify the E_2-term. By the isomorphism of (12) this spectral sequence collapses. By [11], we know that there is an isomorphism

$$\mathbb{F}_3[v_1^{\pm 1}] \otimes \Lambda(\alpha) \cong \pi_* L_{K(1)} S/3$$

where α is the image of $\alpha_1 \in \pi_3 S^0/3$. The result now follows from Proposition 5.2. $\square$

This result will be extended to an integral calculation in Proposition 5.7.
For a complete local ring A with maximal ideal $\mathfrak{m}$ define

$$A((x)) = \lim_k \left\{ A/\mathfrak{m}^k((x)) \right\} \ .$$

This a completion of the ring of Laurent series. Recall that $v_1 = u_1 u^{-2}$ and $v_2 = u^{-8}$ for the standard p-typical deformation of the Honda formal group over $(E_2)_*$. As a first example, Lemma 5.1 immediately gives

$$\pi_* L_{K(1)} E_2 = \mathbb{W}((u_1))[u^{\pm 1}] \ . \tag{14}$$

We now give a calculation of $\pi_* L_{K(1)} E_2^{hF}$ for our two important finite subgroups. The elements c_4, c_6, Δ were all introduced in Remark 3.5.

PROPOSITION 5.5. *Let $X \in \kappa_2$ and fix an isomorphism $(E_2)_* X \cong (E_2)_*$ of twisted $\mathbb{G}_2$-modules.*
(a) The edge homomorphism of the homotopy fixed point spectral sequence induces an isomorphism

$$\pi_* L_{K(1)} L_{K(2)}(E_2^{hG_{24}} \wedge X) \cong \lim\, v_1^{-1} H^0(G_{24}, (E_2)_*/3^n) \ .$$

Define $b_2 = c_6/c_4$ and $j = c_4^3/\Delta$. Then these choices define an isomorphism

$$\mathbb{Z}_3((j))[b_2^{\pm 1}] \cong \lim\, v_1^{-1} H^0(G_{24}, (E_2)_*/3^n) \ .$$

(b) The edge homomorphism of the homotopy fixed point spectral sequence induces an isomorphism

$$\pi_* L_{K(1)} L_{K(2)}(E_2^{hSD_{16}} \wedge X) \cong \lim\, v_1^{-1} H^0(SD_{16}, (E_2)_*/3^n) \ .$$

Define $w = v_1^4/v_2$. Then this determines an isomorphism

$$\mathbb{Z}_3((w))[v_1^{\pm 1}] \cong \lim\, v_1^{-1} H^0(SD_{16}, (E_2)_*/3^n) \ .$$

Proof. For (a), the first isomorphism follows from Proposition 5.2, Lemma 5.1, and a five lemma argument. For the second isomorphism, we know by Remark 3.5 that $c_4 \equiv v_1^2$ and $c_6 \equiv v_1^3$ modulo 3 and up to a unit. It follows that c_4 is invertible in the inverse limit and that we have a map

$$\mathbb{Z}_3((j))[b_2^{\pm 1}] \to \lim\, v_1^{-1} H^0(G_{24}, (E_2)_*/3^n) \ .$$

By Proposition 5.2, this map induces an isomorphism modulo 3 and the result follows.
Part (b) follows directly from [4] §3. □

LEMMA 5.6. *Let $X \in \kappa_2$ and let $F = G_{24}$ or SD_{16}. Given a choice of isomorphism $(E_2)_* \cong (E_2)_* X$ of twisted $\mathbb{G}_2$-modules the image of the multiplicative unit 1 under the isomorphisms*

$$\lim(v_1^{-1} H^0(F, E_*/3^n))_0 \cong \lim(v_1^{-1} H^0(F, E_* X/3^n))_0 \cong \pi_0 L_{K(1)} L_{K(2)}(E_2^{hF} \wedge X)$$

extends to a weak equivalence of $L_{K(1)} E_2^{hF}$-modules

$$L_{K(1)} E_2^{hF} \simeq L_{K(1)} L_{K(2)}(E_2^{hF} \wedge X) \ .$$

Proof. Let $Z = L_{K(2)}(E_2^{hF} \wedge X)$. By Proposition 5.5, the given isomorphism of Morava modules determines a map $g : S^0 \to L_{K(1)} Z$. By induction and a five lemma argument, the induced map $S^0 \to L_{K(1)} Z \wedge S/3^n$ extends to a weak equivalence of $L_{K(1)} E_2^{hF}$-modules

$$L_{K(1)} E_2^{hF}/3^n \simeq L_{K(1)} Z \wedge S/3^n$$

and the result follows from Proposition 5.1. □

Theorem 5.7. *Let $X \in \kappa_2$. Then the localized Hurewicz homomorphism*

$$\pi_0 L_{K(1)} X \longrightarrow \pi_0 L_{K(1)} L_{K(2)}(E_2 \wedge X)$$

is injective. Given a choice of isomorphism $(E_2)_* \cong (E_2)_* X$ of twisted $\mathbb{G}_2$-modules the image of the multiplicative unit 1 under the isomorphisms

$$\mathbb{Z}_3 \cong \lim(v_1^{-1} H^0(\mathbb{G}_2^1, (E_2)_*/3^n))_0 \cong \pi_0(L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X))$$

extends to a weak equivalence of $L_{K(1)}S^0$-modules

$$L_{K(1)} S^0 \simeq L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \ .$$

Proof. Let $Y = L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X)$. Take the tower of 2.7 and apply the localization functor $L_{K(1)} L_{K(2)}(- \wedge X)$ to produce a tower with homotopy inverse limit $L_{K(1)} Y$. By Lemma 5.6, the fibers are all of the form $\Sigma^{8k} L_{K(1)} E_2^{hF}$ with $F = G_{24}$ or $F = SD_{16}$. Using Proposition 5.5, we then see that the map

$$S^0 \to L_{K(1)} L_{K(2)}(E_2^{hG_{24}} \wedge X) \simeq L_{K(1)} E_2^{hG_{24}}$$

induced by the given isomorphism of Morava modules lifts uniquely to a map

$$\iota : L_{K(1)} S^0 \to L_{K(1)} Y \ .$$

By Proposition 5.4 this induces a weak equivalence

$$L_{K(1)} S/3 \simeq L_{K(1)} Y \wedge S/3 \ .$$

Then, using the natural fiber sequence

$$L_{K(1)} S/3 \wedge Y \to L_{K(1)} S/3^n \wedge Y \to L_{K(1)} S/3^{n-1} \wedge Y,$$

induction, and Lemma 5.1, we obtain the desired weak equivalence. □

We record the following result for later use. It is an immediate consequence of Theorem 5.7.

Corollary 5.8. *Let $X \in \kappa_2$. Given a choice of isomorphism $(E_2)_* \cong (E_2)_* X$ of twisted $\mathbb{G}_2$-modules the image of the multiplicative unit 1 under the isomorphisms*

$$\mathbb{Z}_3 \cong \lim(v_1^{-1} H^0(\mathbb{G}_2^1, (E_2)_*/3^n))_0 \cong \pi_0(L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X))$$

extends to a weak equivalence of $L_{K(1)} E_2^{h\mathbb{G}_2^1}$-modules

$$L_{K(1)} E_2^{h\mathbb{G}_2^1} \simeq L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \ .$$

We now want to extend Theorem 5.7 to the sphere itself. Recall that there is a fiber sequence

$$L_{K(2)} S^0 \longrightarrow E_2^{h\mathbb{G}_2^1} \xrightarrow{\psi - 1} E_2^{h\mathbb{G}_2^1}$$

where ψ is a topological generator of the central $\mathbb{Z}_3 \subseteq \mathbb{G}_2$. For any $K(2)$-local X, we may apply the functor $L_{K(2)}((-) \wedge X)$ to get a fiber sequence

$$X \longrightarrow L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \xrightarrow{\psi-1} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X)\ . \tag{15}$$

THEOREM 5.9. *a) Let $X \in \kappa_2$. Given a choice of isomorphism $(E_2)_* \cong (E_2)_* X$ of twisted $\mathbb{G}_2$-modules the image of the multiplicative unit 1 under the isomorphisms*

$$\mathbb{Z}_3 \cong \lim(v_1^{-1} H^0(\mathbb{G}_2, (E_2)_*/3^n)_*)_0 \cong \pi_0 L_{K(1)} X$$

extends to a weak equivalence of $L_{K(1)} L_{K(2)} S^0$-modules

$$L_{K(1)} L_{K(2)} S^0 \simeq L_{K(1)} X\ .$$

b) The weak equivalence $L_{K(1)} S^0 \simeq L_{K(1)} L_{K(2)} E_2^{h\mathbb{G}_2^1}$ of Proposition 5.7 factors uniquely though $L_{K(1)} L_{K(2)} S^0$ and extends to a weak equivalence

$$L_{K(1)} S^0 \vee L_{K(1)} S^{-1} \simeq L_{K(1)} L_{K(2)} S^0$$

where $L_{K(1)} S^{-1} \to L_{K(1)} L_{K(2)} S^0$ is induced by $\zeta \in \pi_{-1} L_{K(2)} S^0$.

Proof. Let $f : L_{K(1)} E_2^{h\mathbb{G}_2^1} \to L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X)$ be the equivalence of Corollary 5.8. Since $\psi : E_2^{h\mathbb{G}_2^1} \to E_2^{h\mathbb{G}_2^1}$ is a morphism of ring spectra, we get a diagram of $L_{K(1)} E_2^{h\mathbb{G}_2^1}$-module maps

$$\begin{tikzcd} L_{K(1)} E_2^{h\mathbb{G}_2^1} \arrow[r, "f"] \arrow[d, "\psi-1"'] & L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X) \arrow[d, "(\psi-1)\wedge X"] \\ L_{K(1)} E_2^{h\mathbb{G}_2^1} \arrow[r, "f"] & L_{K(1)} L_{K(2)}(E_2^{h\mathbb{G}_2^1} \wedge X)\ . \end{tikzcd}$$

By Theorem 5.7, there is an equivalence $L_{K(1)} S^0 \simeq L_{K(1)} E_2^{h\mathbb{G}_2^1}$. Hence, to check that the diagram commutes, we need only verify that it commutes after applying π_0, and this is obvious. Part (a) follows.

We now prove part (b). Let $f_0 : L_{K(1)} S^0 \longrightarrow L_{K(1)} E_2^{h\mathbb{G}_2^1}$ be the equivalence, as in Theorem 5.7. The composition $(\psi - 1) f_0$ is zero, as ψ induces a ring map on $(E_2)_0$. Because $\pi_1 L_{K(1)} S^0 = 0$, f_0 lifts uniquely to a map $f : L_{K(1)} S^0 \to L_{K(1)} L_{K(2)} S^0$ and we get a weak equivalence

$$f \vee g : L_{K(1)} S^0 \vee L_{K(1)} S^{-1} \longrightarrow L_{K(1)} L_{K(2)} S^0$$

where g is the desuspension of the composition

$$L_{K(1)} S^0 \xrightarrow[\simeq]{f_0} L_{K(1)} E_2^{h\mathbb{G}_2^1} \longrightarrow \Sigma L_{K(1)} L_{K(2)} S^0\ .$$

As ζ is defined to be the image of unit in $\pi_0 E_2^{h\mathbb{G}_2^1}$ in $\pi_{-1} L_{K(2)} S^0$, the result follows. □

We now come to our main theorems.

Theorem 5.10. *Let $X \in \kappa_2$. Then the localized Hurewicz homomorphism*

$$\pi_0 L_1 X \longrightarrow \pi_0 L_1 L_{K(2)}(E_2 \wedge X)$$

is injective. A choice of isomorphism $f : (E_2)_* \to (E_2)_* X$ determines a generator of $\pi_0 L_1 X \cong \mathbb{Z}_3$. This generator extends uniquely to a weak equivalence of $L_1 L_{K(2)} S^0$-modules

$$L_1 L_{K(2)} S^0 \simeq L_1 X \ .$$

Proof. From Theorem 5.9 we have that $\pi_1 L_{K(1)} X = 0$ for all $X \in \kappa_2$. The result then follows by the chromatic fracture square (8), Theorem 4.5 and Theorem 5.9. □

Theorem 5.11 (Chromatic Splitting). *If $n = 2$ and $p = 3$, then*

$$L_1 L_{K(2)} S^0 \simeq L_1(S^0_3 \vee S^{-1}_3) \vee L_0(S^{-3}_3 \vee S^{-4}_3)$$

where S^n_p denotes the p-complete sphere.

Proof. We use the chromatic square of (8). Let $X = L_{K(2)} S^0$. Theorem 5.9 implies

$$L_0 L_{K(1)} X \simeq L_0 L_{K(1)}(S^0 \vee S^{-1}) \ .$$

From Theorem 4.6 we have that

$$L_0 X \simeq L_0(S^0_3 \vee S^{-1}_3 \vee S^{-3}_3 \vee S^{-4}_3) \ .$$

Thus we need only show that the map

$$L_0 X \longrightarrow L_0 L_{K(1)} X$$

is equivalent to the composition

$$L_0(S^0_3 \vee S^{-1}_3 \vee S^{-3}_3 \vee S^{-4}_3) \longrightarrow L_0(S^0_3 \vee S^{-1}_3) \longrightarrow L_0 L_{K(1)}(S^0 \vee S^{-1})$$

where the first map is projection and the second map is the L_0 localization of the canonical map $S^0_3 \vee S^{-1}_3 \to L_{K(1)}(S^0 \vee S^{-1})$. This follows from Theorem 4.6 and Theorem 5.9.b. □

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