Algebraic Kasparov K-theory. I

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Abstract. This paper is to construct unstable, Morita stable and stable bivariant algebraic Kasparov K-theory spectra of k-algebras. These are shown to be homotopy invariant, excisive in each variable K-theories. We prove that the spectra represent universal unstable, Morita stable and stable bivariant homology theories respectively.

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1. Introduction

K-theory was originally discovered by Grothendieck in the late 50-s. Thanks to works by Atiyah, Hirzebruch, Adams K-theory was firmly entrenched in topology in the 60-s. Along with topological K-theory mathematicians developed algebraic K-theory. After Atiyah-Singer's index theorem for elliptic operators K-theory penetrated into analysis and gave rise to operator K-theory.

The development of operator K-theory in the 70-s took place in a close contact with the theory of extensions of C^*-algebras and prompted the creation of a new technical apparatus, the Kasparov K-theory [24]. The Kasparov bifunctor $KK_*(A,B)$ combines Grothendieck's K-theory $K_*(B)$ and its dual (contravariant) theory $K^*(A)$. The existence of the product $KK_*(A,D)\otimes KK_*(D,B)\to KK_*(A,B)$ makes the bifunctor into a very strong and flexible tool.

One way of constructing an algebraic counterpart of the bifunctor $KK_*(A,B)$ with a similar biproduct and similar universal properties is to define a triangulated category whose objects are algebras. In [8] the author constructed various bivariant K-theories of algebras, but he did not study their universal properties. Motivated by ideas and work of J. Cuntz on bivariant K-theory of locally convex algebras [4, 5, 6], *universal* algebraic bivariant K-theories were constructed by Cortiñas–Thom in [3].

Developing ideas of [8] further, the author introduces and studies in [9] universal bivariant homology theories of algebras associated with various classes $\mathfrak{F}$ of fibrations on an "admissible category of k-algebras" $\Re$. The methods used by the author to construct $D(\Re, \mathfrak{F})$ are very different from those used by Cortiñas–Thom [3] to construct kk. However they coincide in the appropriate stable case. In a certain sense [9] uses the same approach as in constructing E-theory of C^*-algebras [18]. We start with a datum of an admissible category of algebras $\Re$ and a class $\mathfrak{F}$ of fibrations on it and then construct a *universal* algebraic bivariant K-theory $j : \Re \to D(\Re, \mathfrak{F})$ out of the datum $(\Re, \mathfrak{F})$ by inverting certain arrows which we call weak equivalences. The category $D(\Re, \mathfrak{F})$ is naturally triangulated. The most important cases in practice are the class of k-linear split surjections $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$ or the class $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$ of all surjective homomorphisms.

If $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$ (respectively $\mathfrak{F} = \mathfrak{F}_{\mathrm{surj}}$) then $j : \Re \to D(\Re, \mathfrak{F})$ is called the unstable algebraic Kasparov K-theory (respectively unstable algebraic E-theory) of $\Re$. It should be emphasized that [9] does not consider any matrix-invariance in general. This is caused by the fact that many interesting admissible categories of algebras deserving to be considered separately like that of all commutative ones are not closed under matrices.

If we want to have matrix invariance, then [9] introduces matrices into the game and gets universal algebraic, excisive, homotopy invariant *and* "Morita invariant" (respectively "M_∞-invariant") K-theories $j : \Re \to D_{mor}(\Re, \mathfrak{F})$ (respectively $j : \Re \to D_{st}(\Re, \mathfrak{F})$). The triangulated category $D_{mor}(\Re, \mathfrak{F})$ (respectively $D_{st}(\Re, \mathfrak{F})$) is constructed out of $D(\Re, \mathfrak{F})$ just by "inverting matrices" M_nA, $n > 0$, $A \in \Re$ (respectively by inverting the natural arrows $A \to M_\infty A$ with $M_\infty A = \cup_n M_n A$). We call $D_{mor}(\Re, \mathfrak{F}_{\mathrm{spl}})$ and $D_{mor}(\Re, \mathfrak{F}_{\mathrm{surj}})$ (respectively $D_{st}(\Re, \mathfrak{F}_{\mathrm{spl}})$ and $D_{st}(\Re, \mathfrak{F}_{\mathrm{surj}})$) the Morita stable algebraic KK- and E-theories (respectively the stable algebraic KK- and E-theories). A version of the Cortiñas–Thom theorem [3] says that there is a natural isomorphism of $\mathbb{Z}$-graded abelian groups (see [9])

$$D_{st}(\Re, \mathfrak{F})_*(k, A) \cong KH_*(A),$$

where $KH_*(A)$ is the $\mathbb{Z}$-graded abelian group consisting of the homotopy K-theory groups in the sense of Weibel [31].

One of the aims of this paper is to represent unstable, Morita stable and stable algebraic Kasparov K-theories. Here we deal only with the class of k-linear split surjections $\mathfrak{F} = \mathfrak{F}_{\mathrm{spl}}$. We introduce the "unstable, Morita stable and stable algebraic Kasparov K-theory spectra" $\mathbb{K}^\star(A, B)$ of k-algebras $A, B \in \Re$ where $\star \in \{unst, mor, st\}$ and $\Re$ is an appropriate admissible category of algebras. It should be emphasized that the spectra do not use any realizations of categories and are defined by means of algebra homomorphisms only. This makes our constructions rather combinatorial.

Theorem (Excision Theorem A for spectra). *Let $\star \in \{unst, mor, st\}$. The assignment $B \mapsto \mathbb{K}^\star(A, B)$ determines a functor*

$$\mathbb{K}^\star(A, ?) : \Re \to (Spectra)$$

which is homotopy invariant and excisive in the sense that for every $\mathfrak{F}$-extension $F \to B \to C$ the sequence

$$\mathbb{K}^\star(A,F) \to \mathbb{K}^\star(A,B) \to \mathbb{K}^\star(A,C)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots \to \mathbb{K}^\star_{i+1}(A,C) \to \mathbb{K}^\star_i(A,F) \to \mathbb{K}^\star_i(A,B) \to \mathbb{K}^\star_i(A,C) \to \cdots$$

for any $i \in \mathbb{Z}$.

We also have the following

Theorem (Excision Theorem B for spectra). *Let $\star \in \{unst, mor, st\}$. The assignment $B \mapsto \mathbb{K}^\star(B,D)$ determines a functor*

$$\mathbb{K}^\star(?,D) : \Re^{\mathrm{op}} \to (Spectra),$$

which is excisive in the sense that for every $\mathfrak{F}$-extension $F \to B \to C$ the sequence

$$\mathbb{K}^\star(C,D) \to \mathbb{K}^\star(B,D) \to \mathbb{K}^\star(F,D)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots \to \mathbb{K}^\star_{i+1}(F,D) \to \mathbb{K}^\star_i(C,D) \to \mathbb{K}^\star_i(B,D) \to \mathbb{K}^\star_i(F,D) \to \cdots$$

for any $i \in \mathbb{Z}$.

The following result gives the desired representability.

Theorem (Comparison). *Let $\star \in \{unst, mor, st\}$. Then for any algebras $A, B \in \Re$ there is an isomorphism of $\mathbb{Z}$-graded abelian groups*

$$\mathbb{K}^\star_*(A,B) \cong D_\star(\Re,\mathfrak{F})_*(A,B) = \bigoplus_{n\in\mathbb{Z}} D_\star(\Re,\mathfrak{F})(A,\Omega^n B),$$

functorial both in A and in B.

The Cortiñas–Thom theorem [3, 8.2.1] and the Comparison Theorem above imply that the spectrum $\mathbb{K}^{st}(k,-)$ represents KH. Namely, we have the following

Theorem. *For any $A \in \Re$ there is a natural isomorphism of $\mathbb{Z}$-graded abelian groups*

$$\mathbb{K}^{st}_*(k,A) \cong KH_*(A).$$

The preceding theorem is an analog of the same result of KK-theory saying that there is a natural isomorphism $KK_*(\mathbb{C},A) \cong K_*(A)$ for any C^*-algebra A.

It is important to mention that another aim of the present paper together with [10] is to develop the theory of "K-motives of algebras", for which Excision Theorems A-B as well as the Comparison Theorem are of great utility. This theory shares lots of common properties with K-motives and bivariant K-theory of algebraic varieties, introduced and studied by the author and Panin

in [11, 12] in order to solve some problems in [13] for motivic spectral sequence. In fact, K-motives of algebras are a kind of a connecting language between Kasparov K-theory of C^*-algebras and K-motives of algebraic varieties (hence the choice of the title of this paper and that of [10]).
Throughout the paper k is a fixed commutative ring with unit and Alg_k is the category of non-unital k-algebras and non-unital k-homomorphisms.

Organization of the paper. In Section 2 we fix some notation and terminology. We study simplicial algebras and simplicial sets of algebra homomorphisms associated with simplicial algebras there. In Section 3 we discuss extensions of algebras and classifying maps. It is thanks to simplicial algebras and some elementary facts of their extensions that Excision Theorem A is possible to prove. Then comes Section 4 in which Excision Theorem A is proved. We also formulate Excision Theorem B in this section but its proof requires an additional material. The spectrum $\mathbb{K}^{unst}(A,B)$ is introduced and studied in Section 5. In Section 6 we present necessary facts about model categories and Bousfield localization. This material is needed to prove Excision Theorem B. In Section 7 we study relations between simplicial and polynomial homotopies. As an application Comparison Theorem A is proved in the section. Comparison Theorem B is proved in Section 8. It says that the Hom-sets of $D(\Re,\mathfrak{F})$ are represented by stable homotopy groups of spectra $\mathbb{K}^{unst}(A,B)$-s. The spectra $\mathbb{K}^{st},\mathbb{K}^{mor}$ are introduced and studied in Section 9. We also prove there Comparison Theorems for $D^{st}(\Re,\mathfrak{F})$, $D^{mor}(\Re,\mathfrak{F})$ and construct an isomorphism between stable groups $\mathbb{K}^{st}_*(k,A)$ of an algebra A and its homotopy K-theory groups.

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2. Preliminaries

2.1. Algebraic homotopy. Following Gersten [14] a category of k-algebras without unit $\Re$ is *admissible* if it is a full subcategory of Alg_k and

(1) R in $\Re$, I a (two-sided) ideal of R then I and R/I are in $\Re$;
(2) if R is in $\Re$, then so is $R[x]$, the polynomial algebra in one variable;
(3) given a cartesian square

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "\sigma"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

in Alg_k with A,B,C in $\Re$, then D is in $\Re$.

One may abbreviate 1, 2, and 3 by saying that $\Re$ is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number

of variables, and fibre products. For instance, the *category of commutative k-algebras* CAlg_k is admissible.
Observe that every k-module M can be regarded as a non-unital k-algebra with trivial multiplication: $m_1 \cdot m_2 = 0$ for all $m_1, m_2 \in M$. Then $\mathrm{Mod}\, k$ is an admissible category of commutative k-algebras.
If R is an algebra then the polynomial algebra $R[x]$ admits two homomorphisms onto R

$$R[x] \overset{\partial_x^0}{\underset{\partial_x^1}{\rightrightarrows}} R$$

where $\partial_x^i|_R = 1_R$, $\quad \partial_x^i(x) = i$, $\quad i = 0, 1$. Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Definition. Two homomorphisms $f_0, f_1 : S \to R$ are *elementary homotopic*, written $f_0 \sim f_1$, if there exists a homomorphism

$$f : S \to R[x]$$

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. A map $f : S \to R$ is called an *elementary homotopy equivalence* if there is a map $g : R \to S$ such that fg and gf are elementary homotopic to id_R and id_S respectively.

For example, let A be a $\mathbb{N}$-graded algebra, then the inclusion $A_0 \to A$ is an elementary homotopy equivalence. The homotopy inverse is given by the projection $A \to A_0$. Indeed, the map $A \to A[x]$ sending a homogeneous element $a_n \in A_n$ to $a_n t^n$ is a homotopy between the composite $A \to A_0 \to A$ and the identity id_A.
The relation "elementary homotopic" is reflexive and symmetric [14, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). The set of equivalence classes of morphisms $R \to S$ is written $[R, S]$. This equivalence relation will also be called *polynomial or algebraic homotopy.*

Lemma 2.1 (Gersten [15]). *Given morphisms in* Alg_k

$$\begin{tikzcd} R \arrow[r, "f"] & S \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] & T \arrow[r, "h"] & U \end{tikzcd}$$

such that $g \simeq g'$, then $gf \simeq g'f$ and $hg \simeq hg'$.

Thus homotopy behaves well with respect to composition and we have category $Hotalg$, the *homotopy category of k-algebras*, whose objects are k-algebras and such that $Hotalg(R, S) = [R, S]$. The homotopy category of an admissible category of algebras $\Re$ will be denoted by $\mathcal{H}(\Re)$. Call a homomorphism $s : A \to B$ an *I-weak equivalence* if its image in $\mathcal{H}(\Re)$ is an isomorphism. Observe that I-weak equivalences are those homomorphisms which have homotopy inverses.
The diagram in Alg_k

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence if f is injective ($\equiv \operatorname{Ker} f = 0$), g is surjective, and the image of f is equal to the kernel of g. Thus f is a monomorphism and $f = \ker g$.

DEFINITION. An algebra R is *contractible* if $0 \sim 1$; that is, if there is a homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

For example, every square zero algebra $A \in \mathrm{Alg}_k$ is contractible by means of the homotopy $A \to A[x]$, $a \in A \mapsto ax \in A[x]$. In other words, every k-module, regarded as a k-algebra with trivial multiplication, is contractible.
Following Karoubi and Villamayor [23] we define ER, the *path algebra* on R, as the kernel of $\partial_x^0 : R[x] \to R$, so $ER \to R[x] \xrightarrow{\partial_x^0} R$ is a short exact sequence in Alg_k. Also $\partial_x^1 : R[x] \to R$ induces a surjection

$$\partial_x^1 : ER \to R$$

and we define the *loop algebra* ΩR of R to be its kernel, so we have a short exact sequence in Alg_k

$$\Omega R \to ER \xrightarrow{\partial_x^1} R.$$

We call it the *loop extension* of R. Clearly, ΩR is the intersection of the kernels of ∂_x^0 and ∂_x^1. By [14, 3.3] ER is contractible for any algebra R.

2.2. SIMPLICIAL ALGEBRAS. Let *Ord* denote the category of finite nonempty ordered sets and order-preserving maps, and for each $n \geqslant 0$ we introduce the object $[n] = \{0 < 1 < \cdots < n\}$ of *Ord*. We let $\Delta^n = \mathrm{Hom}_{Ord}(-,[n])$, so that $|\Delta^n|$ is the standard n-simplex. In what follows the category of non-unital simplicial k-algebras will be denoted by SimAlg_k.
Given a simplicial set X and a simplicial algebra $A_\bullet$, we denote by $A_\bullet(X)$ the simplicial algebra $\mathrm{Map}(X, A_\bullet) : [n] \mapsto \mathrm{Hom}_{\mathbb{S}}(X \times \Delta^n, A_\bullet)$. We note that all simplicial algebras are fibrant simplicial sets. If $A_\bullet$ is contractible as a simplicial set, then the fact that $\mathbb{S}$ is a simplicial category and thus satisfies axiom M7 for simplicial model categories (see [19, section 9.1.5]) implies that $A_\bullet(X)$ is contractible.
In what follows a unital simplicial k-algebra $A_\bullet$ is an object of SimAlg_k such that all structure maps are unital algebra homomorphisms.

PROPOSITION 2.2. *Suppose $A_\bullet$ is a unital simplicial k-algebra. Then the following statements are equivalent:*

1. *$A_\bullet$ is contractible as a simplicial set;*
2. *$A_\bullet$ is connected;*
3. *there is an element $t \in A_1$ such that $\partial_0(t) = 0$ and $\partial_1(t) = 1$.*

Furthermore, if one of the equivalent assumptions is satisfied then every simplicial ideal $I_\bullet \subset A_\bullet$ is contractible.

Proof. $(1) \Rightarrow (2), (2) \Rightarrow (3)$ are obvious.
$(3) \Rightarrow (1)$. One can construct a homotopy $f : \Delta^1 \times A_\bullet \to A_\bullet$ from 0 to 1 by defining, for $n \geqslant 0$, the map $f_n : \Delta^1_n \times A_n \to A_n$ with the formula $f_n(\alpha, a) = (\alpha^*(t)) \cdot a$. The same contraction applies to $I_\bullet$. $\square$

The main example of a simplicial algebra we shall work with is defined as

$$A^{\Delta} : [n] \mapsto A^{\Delta^n} := A[t_0, \ldots, t_n]/\langle 1 - \sum_i t_i \rangle \quad (\cong A[t_1, \ldots, t_n]),$$

where $A \in \mathrm{Alg}_k$. Given a map $\alpha : [m] \to [n]$ in Ord, the map $\alpha^* : A^{\Delta^n} \to A^{\Delta^m}$ is defined by $\alpha^*(t_j) = \sum_{\alpha(i)=j} t_i$. Observe that $A^{\Delta} \cong A \otimes k^{\Delta}$.
Note that the face maps $\partial_{0;1} : A^{\Delta^1} \to A^{\Delta^0}$ are isomorphic to $\partial_t^{0;1} : A[t] \to A$ in the sense that the diagram

$$\begin{tikzcd} A[t] \arrow[r, "\partial_t^{\varepsilon}"] \arrow[d, "t \mapsto t_0"'] & A \arrow[d] \\ A^{\Delta^1} \arrow[r, "\partial_{\varepsilon}"] & A^{\Delta^0} \end{tikzcd}$$

is commutative and the vertical maps are isomorpisms. Let $A^+ := A \oplus k$ as a group and

$$(a, n)(b, m) = (ab + ma + nb, nm).$$

Then A^+ is a unital k-algebra containing A as an ideal. The simplicial algebra $(A^+)^{\Delta}$ has the element $t = t_0$ in dimension 1, which satisfies $\partial_0(t) = 0$ and $\partial_1(t) = 1$. Thus, t is an edge which connects 1 to 0, making $(A^+)^{\Delta}$ a unital connected simplicial algebra. By Proposition 2.2 $(A^+)^{\Delta}$ is contractible as a simplicial set. It follows that the same is true for A^{Δ}.
There is a mapping space functor $\mathrm{Hom}^{\bullet}_{\mathrm{Alg}_k} : (\mathrm{Alg}_k)^{\mathrm{op}} \times \mathrm{Alg}_k \to \mathbb{S}$, given by

$$(A, B) \mapsto ([n] \mapsto \mathrm{Hom}_{\mathrm{Alg}_k}(A, B^{\Delta^n})).$$

For every $A \in \mathrm{Alg}_k$, the functor $\mathrm{Hom}^{\bullet}_{\mathrm{Alg}_k}(?, A) : (\mathrm{Alg}_k)^{\mathrm{op}} \to \mathbb{S}$ has a left adjoint $A^? : \mathbb{S} \to (\mathrm{Alg}_k)^{\mathrm{op}}$. If $X \in \mathbb{S}$,

$$A^X = \lim_{\Delta^n \to X} A^{\Delta^n}.$$

Observe that

$$A^X = \mathrm{Hom}_{\mathbb{S}}(X, A^{\Delta}).$$

We have

$$\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^X) = \mathrm{Hom}_{\mathbb{S}}(X, \mathrm{Hom}^{\bullet}_{\mathrm{Alg}_k}(A, B)).$$

As above, for any simplicial algebra $A_{\bullet}$ the functor $\mathrm{Hom}_{\mathrm{Alg}_k}(?, A_{\bullet}) : (\mathrm{Alg}_k)^{\mathrm{op}} \to \mathbb{S}$ has a left adjoint $A_{\bullet}\langle ? \rangle = \mathrm{Hom}_{\mathbb{S}}(?, A_{\bullet}) : \mathbb{S} \to (\mathrm{Alg}_k)^{\mathrm{op}}$. We have

$$\mathrm{Hom}_{\mathrm{Alg}_k}(B, A_{\bullet}\langle X \rangle) = \mathrm{Hom}_{\mathbb{S}}(X, \mathrm{Hom}_{\mathrm{Alg}_k}(B, A_{\bullet})).$$

Note that if $A_{\bullet} = A^{\Delta}$ then $A^{\Delta}\langle X \rangle = A^X$.
Let $\mathbb{S}_{\bullet}$ be the category of pointed simplicial sets. For $(K, \star) \in \mathbb{S}_{\bullet}$, put

$$\begin{aligned} A_{\bullet}\langle K, \star \rangle := \mathrm{Hom}_{\mathbb{S}_{\bullet}}((K, \star), A_{\bullet}) &= \ker(\mathrm{Hom}_{\mathbb{S}}(K, A_{\bullet}) \to \mathrm{Hom}_{\mathbb{S}}(\star, A_{\bullet})) \\ &= \ker(A_{\bullet}\langle K \rangle \to A_{\bullet}). \end{aligned}$$

Proposition 2.3 (Cortiñas–Thom [3]). *Let K be a finite simplicial set, $\star$ a vertex of K, and A a k-algebra. Then k^K and $k^{(K,\star)}$ are free k-modules, and there are natural isomorphisms*

$$A \otimes_k k^K \overset{\cong}{\to} A^K \qquad A \otimes_k k^{(K,\star)} \overset{\cong}{\to} A^{(K,\star)}.$$

Proof. This is a consequence of [3, 3.1.3]: $A \otimes_k k^K \cong A \otimes_k (k \otimes_{\mathbb{Z}} \mathbb{Z}^K) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}^K \cong A^K$. $\square$

2.3. Subdivision. To give an explicit fibrant replacement of the simplicial set $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B_\bullet)$ with $B_\bullet$ a simplicial algebra, we should first define ind-algebras. In this paragraph we shall adhere to [3].

If $\mathcal{C}$ is a category, we write $\mathrm{ind}-\mathcal{C}$ for the category of ind-objects of $\mathcal{C}$. It has as objects the directed diagrams in $\mathcal{C}$. An object in $\mathrm{ind}-\mathcal{C}$ is described by a filtering partially ordered set $(I, \leqslant)$ and a functor $X : I \to \mathcal{C}$. The set of homomorphisms from (X, I) to (Y, J) is

$$\lim_{i\in I} \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

We shall identify objects of $\mathcal{C}$ with constant ind-objects, so that we shall view $\mathcal{C}$ as a subcategory of $\mathrm{ind}-\mathcal{C}$. The category of ind-algebras over k will be denoted by $\mathrm{Alg}_k^{\mathrm{ind}}$.

If $A = (A, I), B = (B, J) \in \mathrm{Alg}_k^{\mathrm{ind}}$ we put

$$[A, B] = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathcal{H}(\mathrm{Alg}_k)}(A_i, B_j).$$

Note that there is a natural map $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B) \to [A, B]$. Two homomorphisms $f, g : A \to B$ in $\mathrm{Alg}_k^{\mathrm{ind}}$ are called homotopic if they have the same image in $[A, B]$.

Write $\mathrm{sd} : \mathbb{S} \to \mathbb{S}$ for the simplicial subdivision functor (see [16, Ch. III.§4]). It comes with a natural transformation $h : \mathrm{sd} \to \mathrm{id}_{\mathbb{S}}$, which is usually called the last vertex map. We have an inverse system

$$\mathrm{sd}^\bullet K : \mathrm{sd}^0 K \xleftarrow{h_K} \mathrm{sd}^1 K \xleftarrow{h_{\mathrm{sd}\, K}} \mathrm{sd}^2 K \xleftarrow{h_{\mathrm{sd}^2 K}} \cdots.$$

We may regard $\mathrm{sd}^\bullet K$ as a pro-simplicial set, that is, as an ind-object in $\mathbb{S}^{\mathrm{op}}$. The ind-extension of the functor $A_\bullet\langle ?\rangle : \mathbb{S}^{\mathrm{op}} \to \mathrm{Alg}_k$ with $A_\bullet$ a simplicial algebra maps $\mathrm{sd}^\bullet K$ to

$$A_\bullet\langle \mathrm{sd}^\bullet K\rangle = \{A_\bullet\langle \mathrm{sd}^n K\rangle \mid n \in \mathbb{Z}_{\geqslant 0}\}.$$

If we fix K, we obtain a functor $(?)\langle \mathrm{sd}^\bullet K\rangle : \mathrm{SimAlg}_k \to \mathrm{Alg}_k^{\mathrm{ind}}$, which extends to $(?)\langle \mathrm{sd}^\bullet K\rangle : \mathrm{SimAlg}_k^{\mathrm{ind}} \to \mathrm{Alg}_k^{\mathrm{ind}}$ in the usual manner explained above. In the special case when $A_\bullet = A^\Delta, A \in \mathrm{Alg}_k$, the ind-algebra $A^\Delta\langle \mathrm{sd}^\bullet K\rangle$ is denoted by $A^{\mathrm{sd}^\bullet K}$.

Let $A \in \mathrm{Alg}_k, B_\bullet \in \mathrm{SimAlg}_k^{\mathrm{ind}}$. The space of the preceding paragraph extends to ind-algebras by

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_\bullet) := ([n] \mapsto \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_n)).$$

Let K be a finite simplicial set and $B_\bullet \in \mathrm{SimAlg}_k^{\mathrm{ind}}$. Denote by $\mathbb{B}_\bullet(K)$ the simplicial ind-algebra $([n,\ell] \mapsto B_\bullet\langle \mathrm{sd}^n(K \times \Delta^\ell)\rangle) = \{(Ex^n B_\bullet)^K \mid n \geqslant 0\}$. If $K = *$ we write $\mathbb{B}_\bullet$ for $\mathbb{B}_\bullet(*)$.
Similar to [3, 3.2.2] one can prove that there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{S}}(K, \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet)) \cong \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_\bullet\langle \mathrm{sd}^\bullet K\rangle),$$

where $A \in \mathrm{Alg}_k, B_\bullet \in \mathrm{SimAlg}_k^{\mathrm{ind}}$ and K is a finite simplicial set. Notice that the formula holds even at the ind-level, before taking colimit.

THEOREM 2.4 (Cortiñas–Thom). *Let $A \in \mathrm{Alg}_k$, $B_\bullet \in \mathrm{SimAlg}_k^{\mathrm{ind}}$. Then*

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet) = Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_\bullet).$$

In particular, $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet)$ *is fibrant.*

Proof. The proof is like that of [3, 3.2.3]. □

PROPOSITION 2.5. *Let K be a finite simplicial set, $A \in \mathrm{Alg}_k$ and $(B_\bullet, J) \in \mathrm{SimAlg}_k^{\mathrm{ind}}$. Then*

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet(K)) = (Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_\bullet))^K.$$

In particular, the left hand side is fibrant.

Proof. The proof is like that of [3, 3.2.3].

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{S}}(\Delta^\ell, &\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet(K))) \\
&= \mathrm{colim}_{(j,n)\in J\times\mathbb{Z}_{\geqslant 0}} \mathrm{Hom}_{\mathrm{Alg}_k}(A, B_{\bullet,j}\langle \mathrm{sd}^n(K\times\Delta^\ell)\rangle) \\
&= \mathrm{colim}_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^n(K\times\Delta^\ell), \mathrm{Hom}_{\mathrm{Alg}_k}(A, B_{\bullet,j})) \\
&= \mathrm{colim}_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^n(K\times\Delta^\ell), \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathrm{Alg}_k}(A, B_{\bullet,j})) \\
&= \mathrm{colim}_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{Hom}_{\mathbb{S}}(K\times\Delta^\ell, Ex^n \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathrm{Alg}_k}(A, B_{\bullet,j})) \\
&= \mathrm{Hom}_{\mathbb{S}}(K\times\Delta^\ell, Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B_\bullet)).
\end{aligned}$$

□

COROLLARY 2.6. *Let $A \in \mathrm{Alg}_k$ and let K, L be finite simplicial sets, then*

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet(K))^L = \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet(K\times L)).$$

Denote by $\mathbb{B}_\bullet(I)$ and $\mathbb{B}_\bullet(\Omega)$ the simplicial ind-algebras $\mathbb{B}_\bullet(\Delta^1)$ and $\ker(\mathbb{B}_\bullet(I) \xrightarrow{(d^0,d^1)} \mathbb{B}_\bullet)$ respectively. We define inductively $\mathbb{B}_\bullet(I^n) := (\mathbb{B}_\bullet(I^{n-1}))^I$, $\mathbb{B}_\bullet(\Omega^n) := (\mathbb{B}_\bullet(\Omega^{n-1}))(\Omega)$. Clearly, $\mathbb{B}_\bullet(I^n) = \mathbb{B}_\bullet(\Delta^1\times \overset{n}{\cdots} \times\Delta^1)$ and $\mathbb{B}_\bullet(\Omega^n)$ is a simplicial ideal of $\mathbb{B}_\bullet(I^n)$ that consists in each degree ℓ of simplicial maps $F : \Delta^1\times \overset{n}{\cdots} \times\Delta^1\times\Delta^\ell \to \mathbb{B}_\bullet$ such that $F|_{\partial(\Delta^1\times\overset{n}{\cdots}\times\Delta^1)\times\Delta^\ell} = 0$.

COROLLARY 2.7. *Let $A \in \mathrm{Alg}_k$, then*

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet(\Omega^n)) = \Omega^n(\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet)),$$

where $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}_\bullet)$ *is based at zero.*

Proof. This is a consequence of Theorem 2.4, Proposition 2.5 and Corollary 2.6. □

3. Extensions and Classifying Maps

Throughout, we assume fixed an underlying category $\mathcal{U}$, which is a full subcategory of $\operatorname{Mod} k$. In what follows we denote by $\mathfrak{F}$ the class of k-split surjective algebra homomorphisms. We shall also refer to $\mathfrak{F}$ as *fibrations*.

Definition. An admissible category of algebras $\Re$ is said to be T*-closed* if we have a faithful forgetful functor $F : \Re \to \mathcal{U}$ (i.e. F is the restriction to $\Re$ of the forgetful functor from Alg_k to k-modules) and a left adjoint functor $\widetilde{T} : \mathcal{U} \to \Re$. Notice that the counit map $\eta_A : T(A) := \widetilde{T}F(A) \to A$, $A \in \Re$, is a fibration.
We denote by $\Re^{\mathrm{ind}}$ the category of ind-objects for an admissible category of algebras $\Re$. If $\Re$ is T-closed then TA, $A \in \Re^{\mathrm{ind}}$, is defined in a natural way.

Throughout this section $\Re$ is supposed to be T-closed.

Lemma 3.1. *For every $A \in \Re$ the algebra TA is contractible, i.e. there is a polynomial contraction $\tau : TA \to TA[x]$ such that $\partial_x^0\tau = 0, \partial_x^1\tau = 1$. Moreover, the contraction is functorial in A.*

Proof. Consider a map $u : FTA \to FTA[x]$ sending an element $b \in FTA$ to $bx \in FTA[x]$. If $X \in \operatorname{Ob}\mathcal{U}$ then we denote the unit map $X \to F\widetilde{T}X$ by i_X. The desired contraction τ is uniquely determined by the map $u \circ i_{FA} : FA \to FTA[x]$. By using elementary properties of adjoint functors, one can show that $\partial_x^0\tau = 0, \partial_x^1\tau = 1$. □

Examples. (1) Let $\Re = \mathrm{Alg}_k$. Given an algebra A, consider the algebraic tensor algebra

$$TA = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \cdots$$

with the usual product given by concatenation of tensors. In Cuntz's treatment of bivariant K-theory [4, 5, 6], tensor algebras play a prominent role.
There is a canonical k-linear map $A \to TA$ mapping A into the first direct summand. Every k-linear map $s : A \to B$ into an algebra B induces a homomorphism $\gamma_s : TA \to B$ defined by

$$\gamma_s(x_1 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\cdots s(x_n).$$

$\Re$ is plainly T-closed.
(2) If $\Re = \mathrm{CAlg}_k$ then

$$T(A) = Sym(A) = \oplus_{n\geqslant 1} S^nA, \quad S^nA = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}\rangle, \quad \sigma \in \Sigma_n,$$

the symmetric algebra of A, and $\Re$ is T-closed.

We shall say that a sequence in $\Re$

$$0 \to C \to B \to A \to 0$$

is an F-*split extension* or just an *($\mathfrak{F}$-)extension* if it is split exact in the category of k-modules. We have the natural extension of algebras

$$0 \longrightarrow JA \xrightarrow{\iota_A} TA \xrightarrow{\eta_A} A \longrightarrow 0.$$

Here JA is defined as $\operatorname{Ker}\eta_A$. Clearly, JA is functorial in A. This extension is universal in the sense that given any extension

$$0 \to C \to B \xrightarrow{\alpha} A \to 0$$

with α in $\mathfrak{F}$, there exists a commutative diagram of extensions as follows.

$$\begin{tikzcd} C \arrow[r] & B \arrow[r, "\alpha"] & A \\ J(A) \arrow[u, "\xi"] \arrow[r, "\iota_A"] & T(A) \arrow[u] \arrow[r, "\eta_A"] & A \arrow[u, "\mathrm{id}_A"'] \end{tikzcd}$$

Furthermore, ξ is unique up to elementary homotopy [3, 4.4.1] in the sense that if $\beta, \gamma : A \to B$ are two splittings to α then ξ_β corresponding to β is elementary homotopic to ξ_γ corresponding to γ. Because of this, we shall abuse notation and refer to any such morphism ξ as *the* classifying map of the extension whenever we work with maps up to homotopy. The elementary homotopy $H(\beta,\gamma) : J(A) \to C[x]$ is explicitly constructed as follows. Let $\widetilde{\alpha} : B[x] \to A[x], \sum b_i x^i \mapsto \alpha(b_i)x^i$, be the natural lift of α. Consider a k-linear map

$$u : A \to B[x], \quad a \mapsto \beta(a)(1-x) + \gamma(a)x.$$

It is extended to an algebra homomorphism $\bar{u} : T(A) \to B[x]$. One has a commutative diagram of algebras

$$\begin{tikzcd} C[x] \arrow[r] & B[x] \arrow[r, "\widetilde{\alpha}"] & A[x] \\ J(A) \arrow[u, "H(\beta,\gamma)"] \arrow[r, "\iota_A"] & T(A) \arrow[u, "\bar{u}"] \arrow[r, "\eta_A"] & A \arrow[u, "\iota"'] \end{tikzcd},$$

where ι is the natural inclusion. It follows that $H(\beta,\gamma)$ is an elementary homotopy between ξ_β and ξ_γ.

If we want to specify a particular choice of ξ corresponding to a splitting β then we sometimes denote ξ by ξ_β indicating the splitting.

Also, if

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & B \arrow[r, "\alpha"] \arrow[d, "h"] & A \arrow[d, "g"] \\ C' \arrow[r] & B' \arrow[r, "\alpha'"] & A' \end{tikzcd}$$

is a commutative diagram of extensions, then there is a diagram

$$\begin{tikzcd} J(A) \arrow[r, "\xi_\beta"] \arrow[d, "J(g)"'] & C \arrow[d, "f"] \\ J(A') \arrow[r, "\xi_{\beta'}"] & C' \end{tikzcd}$$

of classifying maps, which is commutative up to elementary homotopy (see [3, 4.4.2]).

The elementary homotopy can be constructed as follows. Let $\widetilde{\alpha}' : B'[x] \to A'[x], \sum b_i' x^i \mapsto \alpha'(b_i')x^i$, be the natural lift of α'. Consider a k-linear map

$$v : A \to B'[x], \quad a \mapsto h\beta(a)(1-x) + \beta' g(a)x.$$

It is extended to a ring homomorphism $\bar{v} : T(A) \to B[x]$. One has a commutative diagram of algebras

$$\begin{tikzcd} C'[x] \arrow[r] & B'[x] \arrow[r, "\widetilde{\alpha}'"] & A'[x] \\ J(A) \arrow[r] \arrow[u, "G(\beta,\beta')"] & T(A) \arrow[r, "\eta_A"] \arrow[u, "\bar{v}"] & A \arrow[u, "\iota' g"'] \end{tikzcd},$$

where $\iota' : A' \to A'[x]$ is the natural inclusion. It follows that $G(\beta, \beta')$ is an elementary homotopy between $f\xi_\beta$ and $\xi_{\beta'} J(g)$.

Let $\mathcal{C}$ be a small category and let $\Re^{\mathcal{C}}$ (respectively $\mathcal{U}^{\mathcal{C}}$) denote the category of $\mathcal{C}$-diagrams in $\Re$ (respectively in $\mathcal{U}$). Then we can lift the functors $F : \Re \to \mathcal{U}$ and $\widetilde{T} : \mathcal{U} \to \Re$ to $\mathcal{C}$-diagrams. We shall denote the functors by the same letters. So we have a faithful forgetful functor $F : \Re^{\mathcal{C}} \to \mathcal{U}^{\mathcal{C}}$ and a functor $\widetilde{T} : \mathcal{U}^{\mathcal{C}} \to \Re^{\mathcal{C}}$, which is left adjoint to F. The counit map $\eta_A : T(A) := \widetilde{T}F(A) \to A$, $A \in \Re^{\mathcal{C}}$, is a levelwise fibration.

Definition. We shall say that a sequence of $\mathcal{C}$-diagrams in $\Re$

$$0 \to C \to B \xrightarrow{\alpha} A \to 0$$

is a *F-split extension* or just an *($\mathfrak{F}$-)extension* if it is split exact in the abelian category $(\operatorname{Mod} k)^{\mathcal{C}}$ of $\mathcal{C}$-diagrams of k-modules.

We have a natural extension of $\mathcal{C}$-diagrams in $\Re$

$$0 \longrightarrow JA \xrightarrow{\iota_A} TA \xrightarrow{\eta_A} A \longrightarrow 0.$$

Here JA is defined as $\operatorname{Ker} \eta_A$. Clearly, JA is functorial in A.

Lemma 3.2. *Given any extension $0 \to C \to B \to A \to 0$ of $\mathcal{C}$-diagrams in $\Re$, there exists a commutative diagram of extensions as follows.*

$$\begin{tikzcd} C \arrow[r] & B \arrow[r, "\alpha"] & A \\ J(A) \arrow[r, "\iota_A"] \arrow[u, "\xi"] & T(A) \arrow[r, "\eta_A"] \arrow[u] & A \arrow[u, "\mathrm{id}_A"'] \end{tikzcd}$$

Furthermore, ξ is unique up to a natural elementary homotopy $H(\beta,\gamma): JA \to C[x]$, where β,γ are two splittings of α.

Proof. The proof is like that for algebras (see above). $\square$

Lemma 3.3. *Let*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & B \arrow[r, "\alpha"] \arrow[d, "h"'] & A \arrow[d, "g"] \\ C' \arrow[r] & B' \arrow[r, "\alpha'"] & A' \end{tikzcd}$$

be a commutative diagram of F-split extensions of $\mathcal{C}$-diagrams with splittings $\beta: A \to B, \beta': A' \to B'$. Then there is a diagram of classifying maps

$$\begin{tikzcd} J(A) \arrow[r, "\xi_\beta"] \arrow[d, "J(g)"'] & C \arrow[d, "f"] \\ J(A') \arrow[r, "\xi_{\beta'}"] & C' \end{tikzcd}$$

which is commutative up to a natural elementary homotopy $G(\beta,\beta'): JA \to C'[x]$.

Proof. The proof is like that for algebras (see above). $\square$

Lemma 3.4. *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r, "u"] \arrow[d, "h"'] & C \arrow[d, "g"] \\ A' \arrow[r] & B' \arrow[r, "u'"] & C' \end{tikzcd}$$

be a commutative diagram of F-split extensions of $\mathcal{C}$-diagrams with splittings $(v,v'):(C,C') \to (B,B')$ being such that (v,v') is a splitting to (u,u') in the category of arrows $\mathrm{Ar}(\mathcal{U}^{\mathcal{C}})$, i.e. $hv = v'g$. Then the diagram of classifying maps

$$\begin{tikzcd} J(C) \arrow[r, "\xi_v"] \arrow[d, "J(g)"'] & A \arrow[d, "f"] \\ J(C') \arrow[r, "\xi_{v'}"] & A' \end{tikzcd}$$

is commutative.

Proof. If we regard h and g as $\{0 \to 1\} \times \mathcal{C}$-diagrams and (u,u') as a map from h to g, then the commutative diagram of lemma is the classifying map corresponding to the splitting (v,v') of $\{0 \to 1\} \times \mathcal{C}$-diagrams. $\square$

4. The Excision Theorems

Throughout this section $\Re$ is assumed to be T-closed. Recall that k^Δ is a contractible unital simplicial object in $\Re$ and $t := t_0 \in k^{\Delta^1}$ is a 1-simplex with $\partial_0(t) = 0, \partial_1(t) = 1$. Given an algebra B, the ind-algebra $\mathbb{B}^\Delta$ is defined as

$$[m,\ell] \mapsto \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^m \Delta^\ell, B^\Delta) = B^{\mathrm{sd}^m \Delta^\ell}.$$

If $B = k$ then $\mathbb{B}^\Delta$ will be denoted by $\Bbbk^\Delta$. B^Δ can be regarded as a k^Δ-module, i.e. there is a simplicial map, induced by multiplication,

$$B^\Delta \times k^\Delta \to B^\Delta.$$

Similarly, $\mathbb{B}^\Delta$ can be regarded as a $\Bbbk^\Delta$-ind-module.
Given two algebras $A, B \in \Re$ and $n \geqslant 0$, consider the simplicial set

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n)) \cong \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, B \otimes_k \Bbbk^\Delta(\Omega^n)).$$

It follows from Proposition 2.5 and Corollary 2.7 that it is fibrant. $\mathbb{B}^\Delta(\Omega^n)$ is a simplicial ideal of the simplicial ind-algebra

$$\mathbb{B}^\Delta(I^n) = ([m,\ell] \mapsto \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^m(\Delta^1 \times \overset{n}{\cdots} \times \Delta^1 \times \Delta^\ell) \to B^\Delta)).$$

There is a commutative diagram of simplicial ind-algebras

$$\begin{tikzcd}
P\mathbb{B}^\Delta(\Omega^n) \arrow[r, tail] \arrow[d, tail] & (\mathbb{B}^\Delta(\Omega^n))^I \arrow[r, two heads, "d_0"] \arrow[d, tail] & \mathbb{B}^\Delta(\Omega^n) \arrow[d, tail] \\
P\mathbb{B}^\Delta(I^n) \arrow[r, tail] & \mathbb{B}^\Delta(I^{n+1}) \arrow[r, two heads, "d_0"] & \mathbb{B}^\Delta(I^n)
\end{tikzcd}$$

with vertical arrows inclusions and the right lower map d_0 applies to the last coordinate.
We claim that the natural simplicial map $d_1 : P\mathbb{B}^\Delta(\Omega^n) \to \mathbb{B}^\Delta(\Omega^n)$ has a natural k-linear splitting. In fact, the splitting is induced by a natural k-linear splitting υ for $d_1 : P\mathbb{B}^\Delta(I^n) \to \mathbb{B}^\Delta(I^n)$. Let $\mathbf{t} \in P\Bbbk^\Delta(I^n)_0$ stand for the composite map

$$\mathrm{sd}^m(\Delta^1 \times \overset{n+1}{\cdots} \times \Delta^1) \xrightarrow{pr} \mathrm{sd}^m \Delta^1 \to \Delta^1 \xrightarrow{t} k^\Delta,$$

where pr is the projection onto the $(n+1)$th direct factor Δ^1. The element $\mathbf{t}$ can be regarded as a 1-simplex of the unital ind-algebra $\Bbbk^\Delta(I^n)$ such that $\partial_0(\mathbf{t}) = 0$ and $\partial_1(\mathbf{t}) = 1$. Let $\imath : \mathbb{B}^\Delta(I^n) \to (\mathbb{B}^\Delta(I^n))^{\Delta^1}$ be the natural inclusion. Multiplication with $\mathbf{t}$ determines a k-linear map $\mathbb{B}^\Delta(I^{n+1}) \xrightarrow{\mathbf{t}\cdot} P\mathbb{B}^\Delta(I^n)$. Now the desired k-linear splitting is defined as

$$\upsilon := \mathbf{t} \cdot \imath.$$

Consider a sequence of simplicial sets
(1)
$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^\Delta) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^\Delta(\Omega)) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^2 A, \mathbb{B}^\Delta(\Omega^2)) \xrightarrow{\varsigma} \cdots$$

Each map ς is defined by means of the classifying map ξ_υ corresponding to the k-linear splitting υ. More precisely, if we consider $\mathbb{B}^\Delta(\Omega^n)$ as a $(\mathbb{Z}_{\geqslant 0}\times\Delta)$-diagram in $\Re$, then there is a commutative diagram of extensions for $(\mathbb{Z}_{\geqslant 0}\times\Delta)$-diagrams

$$\begin{tikzcd} J\mathbb{B}^\Delta(\Omega^n) \arrow[r] \arrow[d, "\xi_\upsilon"'] & T\mathbb{B}^\Delta(\Omega^n) \arrow[r] \arrow[d] & \mathbb{B}^\Delta(\Omega^n) \arrow[d, equal] \\ \mathbb{B}^\Delta(\Omega^{n+1}) \arrow[r] & P\mathbb{B}^\Delta(\Omega^n) \arrow[r, "d_1"] & \mathbb{B}^\Delta(\Omega^n) \end{tikzcd}$$

For every element $f \in \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, \mathbb{B}^\Delta(\Omega^n))$ one sets:

$$\varsigma(f) := \xi_\upsilon \circ J(f) \in \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^\Delta(\Omega^{n+1})).$$

Now consider an $\mathfrak{F}$-extension in $\Re$

$$F \xrightarrow{i} B \xrightarrow{f} C.$$

For any $n \geqslant 0$ one constructs a cartesian square of simplicial ind-algebras

$$\begin{tikzcd} P_f(\Omega^n) \arrow[r, "pr"] \arrow[d, "pr"'] & P(\mathbb{C}^\Delta(\Omega^n)) \arrow[d, "d_1"] \\ \mathbb{B}^\Delta(\Omega^n) \arrow[r, "f"] & \mathbb{C}^\Delta(\Omega^n). \end{tikzcd}$$

We observe that the path space $P(P_f(\Omega^n))$ of $P_f(\Omega^n)$ is the fibre product of the diagram

$$P\mathbb{B}^\Delta(\Omega^n) \xrightarrow{P(f)} P\mathbb{C}^\Delta(\Omega^n) \xleftarrow{Pd_1} P(P\mathbb{C}^\Delta(\Omega^n)).$$

Denote by $\widetilde{P}(P_f(\Omega^n))$ the fibre product of the diagram

$$P\mathbb{B}^\Delta(\Omega^n) \xrightarrow{P(f)} P\mathbb{C}^\Delta(\Omega^n) \xleftarrow{d_1^{P\mathbb{C}^\Delta(\Omega^n)}} P(P\mathbb{C}^\Delta(\Omega^n)).$$

Given a simplicial set X, let

$$\mathsf{sw} : X^{\Delta^1\times\Delta^1} \to X^{\Delta^1\times\Delta^1}$$

be the automorphism swapping the two coordinates of $\Delta^1\times\Delta^1$. If $X = \mathbb{C}^\Delta(\Omega^n)$ then sw induces an automorphism

$$\mathsf{sw} : P(P\mathbb{C}^\Delta(\Omega^n)) \to P(P\mathbb{C}^\Delta(\Omega^n)),$$

denoted by the same letter. Notice that

$$Pd_1 = d_1 \circ \mathsf{sw}.$$

Moreover, the commutative diagram

$$\begin{tikzcd} P\mathbb{B}^\Delta(\Omega^n) \arrow[r, "P(f)"] \arrow[d, equal] & P\mathbb{C}^\Delta(\Omega^n) \arrow[d, equal] & P(P\mathbb{C}^\Delta(\Omega^n)) \arrow[l, "Pd_1"'] \arrow[d, "\mathsf{sw}"] \\ P\mathbb{B}^\Delta(\Omega^n) \arrow[r, "P(f)"] & P\mathbb{C}^\Delta(\Omega^n) & P(P\mathbb{C}^\Delta(\Omega^n)) \arrow[l, "d_1"'] \end{tikzcd}$$

yields an isomorphism of simplicial sets

$$P(P_f(\Omega^n)) \cong \widetilde{P}(P_f(\Omega^n)).$$

The natural simplicial map

$$\partial := (d_1, Pd_1) : \widetilde{P}(P_f(\Omega^n)) \to P_f(\Omega^n)$$

has a natural k-linear splitting $\tau : P_f(\Omega^n) \to \widetilde{P}(P_f(\Omega^n))$ defined as $\tau = (\upsilon, P\upsilon)$. So one can define a sequence of simplicial sets

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, P_f) \xrightarrow{\vartheta} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, P_f(\Omega)) \xrightarrow{\vartheta} \cdots$$

with each map ϑ defined by means of the classifying map ξ_τ corresponding to the k-linear splitting τ.
There is a natural map of simplicial ind-algebras for any $n \geqslant 0$

$$\iota : \mathbb{F}^\Delta(\Omega^n) \to P_f(\Omega^n).$$

PROPOSITION 4.1. *For any $n \geqslant 0$ there is a homomorphism of simplicial ind-algebras $\alpha : J(P_f(\Omega^n)) \to \mathbb{F}^\Delta(\Omega^{n+1})$ such that in the diagram*

$$\begin{tikzcd}
J(\mathbb{F}^\Delta(\Omega^n)) \arrow[r, "\xi_\upsilon"] \arrow[d, "J(\iota)"'] & \mathbb{F}^\Delta(\Omega^{n+1}) \arrow[d, "\iota"] \\
J(P_f(\Omega^n)) \arrow[r, "\xi_\tau"] \arrow[ur, "\alpha"] & P_f(\Omega^{n+1})
\end{tikzcd}$$

$\alpha J(\iota) = \xi_\upsilon$, $\xi_\tau J(\iota) = \iota\xi_\upsilon$, and $\iota\alpha$ is elementary homotopic to ξ_τ.

Proof. We want to construct a commutative diagram of extensions as follows.

$$\begin{tikzcd}
\mathbb{F}^\Delta(\Omega^{n+1}) \arrow[r] \arrow[d, "\mathrm{id}"] & P(\mathbb{F}^\Delta(\Omega^n)) \arrow[r, "d_1^F"] \arrow[d, "\chi"'] & \mathbb{F}^\Delta(\Omega^n) \arrow[d, "\iota"] \\
\mathbb{F}^\Delta(\Omega^{n+1}) \arrow[r] \arrow[d, "\iota"] & P(\mathbb{B}^\Delta(\Omega^n)) \arrow[r, "\pi"] \arrow[d, "\theta"'] & P_f(\Omega^n) \arrow[d, "\mathrm{id}"] \\
P_f(\Omega^{n+1}) \arrow[r] & \widetilde{P}P_f(\Omega^n) \arrow[r, "\partial"] & P_f(\Omega^n)
\end{tikzcd} \tag{2}$$

Here π is a natural map induced by $(d_1 : P(\mathbb{B}^\Delta(\Omega^n)) \to \mathbb{B}^\Delta(\Omega^n), P(f))$. A splitting ν to π is constructed as follows.
Let $g : C \to B, j : B \to F$ be k-linear splittings to $f : B \to C$ and $i : F \to B$ respectively. So $fg = 1_C$, $ji = 1_F$ and $ij + gf = 1_B$. Then the simplicial map

$$ij : \mathbb{B}^\Delta(\Omega^n) \to \mathbb{B}^\Delta(\Omega^n)$$

is k-linear. We define ν as the composite map

$$\begin{tikzcd} P_f(\Omega^n) \arrow[d, tail] & \\ \mathbb{B}^\Delta(\Omega^n) \times P(\mathbb{C}^\Delta(\Omega^n)) \arrow[r, "{(vij,P(g))}"] & P(\mathbb{B}^\Delta(\Omega^n)) \times P(\mathbb{B}^\Delta(\Omega^n)) \arrow[d, "+"] \\ & P(\mathbb{B}^\Delta(\Omega^n)). \end{tikzcd}$$

We have to define the map θ. For this we construct a map of simplicial sets

$$\lambda : \Delta^1 \times \Delta^1 \to \Delta^1.$$

We regard the simplicial set Δ^1 as the nerve of the category $\{0 \to 1\}$. Then λ is obtained from the functor between categories

$$\{0 \to 1\} \times \{0 \to 1\} \to \{0 \to 1\}, \quad (0,1),(1,0),(1,1) \mapsto 1, (0,0) \mapsto 0.$$

The induced map $\lambda^* : \mathbb{B}^\Delta(\Omega^n)^{\Delta^1} \to \mathbb{B}^\Delta(\Omega^n)^{\Delta^1 \times \Delta^1}$ induces a map of path spaces $\lambda^* : P\mathbb{B}^\Delta(\Omega^n) \to P(P\mathbb{B}^\Delta(\Omega^n))$. The desired map θ is defined by the map $(1_{P(\mathbb{B}^\Delta(\Omega^n))}, f\lambda^*)$. Our commutative diagram is constructed.
Consider the following diagrams of classifying maps

$$\begin{tikzcd} J(\mathbb{F}^\Delta(\Omega^n)) \arrow[r, "\xi_\upsilon"] \arrow[d, "J(\iota)"] & \mathbb{F}^\Delta(\Omega^{n+1}) \arrow[d, "\mathrm{id}"] \\ J(P_f(\Omega^n)) \arrow[r, "\alpha"] & \mathbb{F}^\Delta(\Omega^{n+1}) \end{tikzcd} \qquad \begin{tikzcd} J(P_f(\Omega^n)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & \mathbb{F}^\Delta(\Omega^{n+1}) \arrow[d, "\iota"] \\ J(P_f(\Omega^n)) \arrow[r, "\xi_\tau"] & P_f(\Omega^{n+1}) \end{tikzcd}$$

Since $\chi\upsilon = \nu\iota$ then the left square is commutative by Lemma 3.4, because (d_1^F, π) yield a map of $\{0 \to 1\} \times \mathcal{C}$-diagrams split by (υ, ν). Also $\xi_\tau J(\iota) = \iota\xi_\upsilon$, because (d_1^F, ∂) yield a map of $\{0 \to 1\} \times \mathcal{C}$-diagrams split by (υ, τ). The right square is commutative up to elementary homotopy by Lemma 3.3. $\square$

Definition. Given two k-algebras $A, B \in \Re$, the *unstable algebraic Kasparov K-theory space* of (A,B) is the space $\mathcal{K}(\Re)(A,B)$ defined as the (fibrant) space

$$\operatorname{colim}_n \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n)).$$

Its homotopy groups will be denoted by $\mathcal{K}_n(\Re)(A,B)$, $n \geqslant 0$. In what follows we shall often write $\mathcal{K}(A,B)$ to denote the same space omitting $\Re$ from notation.

Remark. The space $\mathcal{K}(\Re)(A,B)$ only depends on the endofunctor $T : \Re \to \Re$. If A, B belong to another admissible category of algebras $\Re'$ with the same endofunctor T, then $\mathcal{K}(\Re)(A,B)$ equals $\mathcal{K}(\Re')(A,B)$.

We call a functor $\mathcal{F}$ from $\Re$ to simplicial sets or spectra *homotopy invariant* if for every $B \in \Re$ the natural map $B \to B[x]$ induces a weak equivalence $\mathcal{F}(B) \simeq \mathcal{F}(B[x])$.

LEMMA 4.2. (1) *For any $n \geqslant 0$ the simplicial functor $B \mapsto \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^{\Delta}(\Omega^n))$ is homotopy invariant. In particular, the simplicial functor $\mathcal{K}(A, ?)$ is homotopy invariant.*
(2) *Given a $\mathfrak{F}$-fibration $f : B \to C$, let $f[x] : B[x] \to C[x]$ be the fibration $\sum b_i x^i \mapsto \sum f(b_i) x^i$. Then $P_{f[x]}(\Omega^n) = P_f(\Omega^n)[x]$ and the natural map of simplicial sets*

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, P_f(\Omega^n)) \to \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, P_{f[x]}(\Omega^n)) \tag{3}$$

is a homotopy equivalence for any $n \geqslant 0$ and $A \in \Re$.

Proof. (1). By Theorem 2.4 $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^{\Delta}) = Ex^{\infty}(\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^{\Delta}))$. It is homotopy invariant by [8, 3.1]. For any $n \geqslant 0$ and $A \in \Re$ there is a commutative diagram of fibre sequences

$$\begin{array}{ccccc}
(A, \mathbb{B}^{\Delta}(\Omega^{n+1})) & \longrightarrow & (A, P\mathbb{B}^{\Delta}(\Omega^n)) & \longrightarrow & (A, \mathbb{B}^{\Delta}(\Omega^n)) \\
\downarrow & & \downarrow & & \downarrow \\
(A, \mathbb{B}[x]^{\Delta}(\Omega^{n+1})) & \longrightarrow & (A, P\mathbb{B}[x]^{\Delta}(\Omega^n)) & \longrightarrow & (A, \mathbb{B}[x]^{\Delta}(\Omega^n)).
\end{array}$$

By induction, if the right arrow is a weak equivalence, then so is the left one because the spaces in the middle are contractible.
(2). The fact that $P_{f[x]}(\Omega^n) = P_f(\Omega^n)[x]$ is straightforward. The map (3) is the fibre product map corresponding to the commutative diagram

$$\begin{array}{ccccc}
(A, \mathbb{B}^{\Delta}(\Omega^n)) & \longrightarrow & (A, \mathbb{C}^{\Delta}(\Omega^n)) & \longleftarrow & (A, P\mathbb{C}^{\Delta}(\Omega^n)) \\
\downarrow & & \downarrow & & \downarrow \\
(A, \mathbb{B}[x]^{\Delta}(\Omega^n)) & \longrightarrow & (A, \mathbb{C}[x]^{\Delta}(\Omega^n)) & \longleftarrow & (A, P\mathbb{C}[x]^{\Delta}(\Omega^n)).
\end{array}$$

The left and the middle vertical arrows are weak equivalences by the first assertion. The right vertical arrow is a weak equivalence, because it is a map between contractible spaces. Since the right horizontal maps are fibrations, we conclude that the desired map is a weak equivalence. $\square$

We are now in a position to prove the following result.

EXCISION THEOREM A. *For any algebra $A \in \Re$ and any $\mathfrak{F}$-extension in $\Re$*

$$F \xrightarrow{i} B \xrightarrow{f} C$$

the induced sequence of spaces

$$\mathcal{K}(A, F) \longrightarrow \mathcal{K}(A, B) \longrightarrow \mathcal{K}(A, C)$$

is a homotopy fibre sequence.

Proof. We have constructed above a sequence of simplicial sets

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, P_f) \xrightarrow{\vartheta} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, P_f(\Omega)) \xrightarrow{\vartheta} \cdots$$

with each map ϑ defined by means of the classifying map ξ_τ corresponding to the k-linear splitting τ. Let $\mathcal{X}$ denote its colimit. One has a homotopy cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "pr"'] & P\mathcal{K}(A,C) \simeq * \arrow[d, "d_1"] \\ \mathcal{K}(A,B) \arrow[r, "f"] & \mathcal{K}(A,C). \end{tikzcd}$$

By Proposition 4.1 for any $n \geqslant 0$ there is a diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, \mathbb{F}^\Delta(\Omega^n)) \arrow[r, "\varsigma"] \arrow[d, "\iota"'] & \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{F}^\Delta(\Omega^{n+1})) \arrow[d, "\iota"] \\ \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, P_f(\Omega^n)) \arrow[r, "\vartheta"] \arrow[ur, "a"] & \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, P_f(\Omega^{n+1})) \end{tikzcd}$$

with $\varsigma(u) = \xi_\upsilon \circ J(u)$, $\vartheta(v) = \xi_\tau \circ J(v)$, $a(v) = \alpha \circ J(v)$. Proposition 4.1 also implies that $a\iota = \varsigma$, $\iota\varsigma = \vartheta\iota$ and that there exists a map

$$H : \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, P_f(\Omega^n)) \to \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, P_f(\Omega^{n+1})[x])$$

such that $\partial_x^0 H = \iota a$ and $\partial_x^1 H = \vartheta$.
One has a commutative diagram

$$\begin{tikzcd} (J^{n+1}A, P_f(\Omega^{n+1})) \arrow[r, "diag"] \arrow[d, "i"'] & (J^{n+1}A, P_f(\Omega^{n+1})) \times (J^{n+1}A, P_f(\Omega^{n+1})) \\ (J^{n+1}A, P_f(\Omega^{n+1})[x]). \arrow[ur, "(\partial_x^0, \partial_x^1)"'] & \end{tikzcd}$$

By Lemma 4.2(2) i is a weak equivalence. So $\operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, P_f(\Omega^{n+1})[x])$ is a path object of $\operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, P_f(\Omega^{n+1}))$ in $\mathbb{S}$. Since all spaces in question are fibrant, we conclude that ιa is simplicially homotopic to ϑ, and hence $\pi_s(\iota a) = \pi_s(\vartheta)$, $s \geqslant 0$. Therefore the induced homomorphisms

$$\pi_s(\iota) : \mathcal{K}_s(A,F) \to \pi_s(\mathcal{X}), \quad s \geqslant 0,$$

are isomorphisms, and hence $\iota : \mathcal{K}(A,F) \to \mathcal{X}$ is a weak equivalence.
Since the vertical arrows in the commutative diagram

$$\begin{tikzcd} P\mathcal{K}(A,F) \arrow[rr] & & \mathcal{K}(A,C) \arrow[dd, equal] & \\ & \mathcal{X} \arrow[ul] \arrow[rr, "pr"] & & \mathcal{K}(A,B) \arrow[ul] \arrow[dd, equal] \\ * \arrow[uu] \arrow[rr] & & \mathcal{K}(A,C) & \\ & \mathcal{K}(A,F) \arrow[ul] \arrow[uu, "\iota"] \arrow[rr, "i"'] & & \mathcal{K}(A,B) \arrow[ul] \end{tikzcd}$$

are weak equivalences and the upper square is homotopy cartesian, then so is the lower one (see [19, 13.3.13]). Thus,

$$\mathcal{K}(A,F) \longrightarrow \mathcal{K}(A,B) \longrightarrow \mathcal{K}(A,C)$$

is a homotopy fibre sequence. The theorem is proved. $\square$

COROLLARY 4.3. *For any algebras $A, B \in \Re$ the space $\Omega\mathcal{K}(A, B)$ is naturally homotopy equivalent to $\mathcal{K}(A, \Omega B)$.*

Proof. Consider the extension

$$\Omega B \longrightarrow EB \xrightarrow{\partial_x^1} B$$

which gives rise to a homotopy fibre sequence

$$\mathcal{K}(A, \Omega B) \to \mathcal{K}(A, EB) \to \mathcal{K}(A, B)$$

by Excision Theorem A. Our assertion will follow once we prove that $\mathcal{K}(A, EB)$ is contractible.
Since EB is contractible, then there is an algebraic homotopy $h : EB \to EB[x]$ contracting EB. There is also a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n)) \arrow[r, "diag"] \arrow[d, "i"'] & \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n)) \times \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n)) \\
\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}[x](\Omega^n)). \arrow[ur, "(\partial_x^0, \partial_x^1)"'] &
\end{tikzcd}$$

By Lemma 4.2(1) i is a weak equivalence. We see that $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}[x](\Omega^n))$ is a path object of $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n))$ in $\mathbb{S}$, and hence the induced map

$$h_* : \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n)) \to \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}[x](\Omega^n))$$

is such that $\partial_x^1 h_* = \mathrm{id}$ is homotopic to $\partial_x^0 h_* = const$. Thus $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{EB}(\Omega^n))$ is contractible, and hence so is $\mathcal{K}(A, EB)$. $\square$

We have proved that the simplicial functor $\mathcal{K}(A, B)$ is excisive in the second argument. It turns out that it is also excisive in the first argument.

EXCISION THEOREM B. *For any algebra $D \in \Re$ and any $\mathfrak{F}$-extension in $\Re$*

$$F \xrightarrow{i} B \xrightarrow{f} C$$

the induced sequence of spaces

$$\mathcal{K}(C, D) \longrightarrow \mathcal{K}(B, D) \longrightarrow \mathcal{K}(F, D)$$

is a homotopy fibre sequence.

The proof of this theorem requires some machinery. We shall use recent techniques and results from homotopical algebra (both stable and unstable). The proof is on page 1242.

5. The spectrum $\mathbb{K}^{unst}(A,B)$

Throughout this section $\Re$ is assumed to be T-closed.

Theorem 5.1. *Let $A,B\in\Re$; then there is a natural isomorphism of simplicial sets*

$$\mathcal{K}(A,B)\cong\Omega\mathcal{K}(JA,B).$$

In particular, $\mathcal{K}(A,B)$ is an infinite loop space with $\mathcal{K}(A,B)$ simplicially isomorphic to $\Omega^n\mathcal{K}(J^nA,B)$.

Proof. For any $n\in\mathbb{N}$ there is a commutative diagram

$$\begin{tikzcd}
P\mathbb{B}^\Delta(\Omega^n) \arrow[r, tail] \arrow[d, "d_{1,\mathbb{B}^\Delta(\Omega^n)}"'] & PP\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, two heads, "Pd_1"] \arrow[d, "d_{1,P\mathbb{B}^\Delta(\Omega^{n-1})}"] & P\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[d, "d_{1,\mathbb{B}^\Delta(\Omega^{n-1})}"] \\
\mathbb{B}^\Delta(\Omega^n) \arrow[r, tail] & P\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, two heads, "d_1"'] & \mathbb{B}^\Delta(\Omega^{n-1}).
\end{tikzcd}$$

The definition of the natural splitting υ to the lower right arrow is naturally lifted to a natural splitting $\nu:=P\upsilon$ for the upper right arrow in such a way that $d_1\circ\nu=\upsilon\circ d_1$. It follows from Lemma 3.4 that the corresponding diagram of the classifying maps

$$\begin{tikzcd}
JP\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, "\xi_\nu"] \arrow[d, "J(d_1)"'] & P\mathbb{B}^\Delta(\Omega^n) \arrow[d, "d_1"] \\
J\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, "\xi_\upsilon"] & \mathbb{B}^\Delta(\Omega^n)
\end{tikzcd}$$

is commutative. There is also a commutative diagram with exact rows for every $n\geqslant 1$

$$\begin{tikzcd}
\mathbb{B}^\Delta(\Omega^{n+1}) \arrow[r, tail, "j"] \arrow[d, tail, "\mathsf{sw}"'] & P\mathbb{B}^\Delta(\Omega^n) \arrow[r, two heads, "d_1"] \arrow[d, equal] & \mathbb{B}^\Delta(\Omega^n) \arrow[d, equal] \\
\mathbb{B}^\Delta(\Omega^{n+1}) \arrow[r, tail, "\mathsf{sw}\circ j"] \arrow[d, tail, "j"'] & P\mathbb{B}^\Delta(\Omega^n) \arrow[r, two heads, "d_1"] \arrow[d, "\mathsf{sw}\circ Pi"] & \mathbb{B}^\Delta(\Omega^n) \arrow[d, tail, "i"] \\
P\mathbb{B}^\Delta(\Omega^n) \arrow[r, tail] & PP\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, two heads, "Pd_1"'] & P\mathbb{B}^\Delta(\Omega^{n-1}).
\end{tikzcd}$$

with i, j natural inclusions and sw permuting the last two coordinates. One has a commutative diagram of classifying maps

$$\begin{tikzcd}
J\mathbb{B}^\Delta(\Omega^n) \arrow[r, "\xi_\upsilon"] \arrow[d, equal] & \mathbb{B}^\Delta(\Omega^{n+1}) \arrow[d, "\mathsf{sw}"] \\
J\mathbb{B}^\Delta(\Omega^n) \arrow[r, "\mathsf{sw}\circ\xi_\upsilon"] \arrow[d, "J(i)"'] & \mathbb{B}^\Delta(\Omega^{n+1}) \arrow[d, "j"] \\
JP\mathbb{B}^\Delta(\Omega^{n-1}) \arrow[r, "\xi_\nu"] & P\mathbb{B}^\Delta(\Omega^n)
\end{tikzcd}$$

Observe that each simplicial map

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, P\mathbb{B}^\Delta(\Omega^{n-1})) \xrightarrow{P\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, P\mathbb{B}^\Delta(\Omega^n))$$

agrees with the map defined like ς but using ξ_ν. To see this, it is enough to consider the following commutative diagram with exact rows and obvious splittings:

$$\begin{tikzcd}
J(P\mathbb{B}^\Delta) \arrow[r, tail] \arrow[d] & T(P\mathbb{B}^\Delta) \arrow[r, two heads, "\eta_{P\mathbb{B}^\Delta}"] \arrow[d] & P\mathbb{B}^\Delta \arrow[d, equal] \\
P(J\mathbb{B}^\Delta) \arrow[r, tail] \arrow[d] & P(T\mathbb{B}^\Delta) \arrow[r, two heads, "P(\eta_{\mathbb{B}^\Delta})"] \arrow[d] & P\mathbb{B}^\Delta \arrow[d, equal] \\
P(\mathbb{B}^\Delta(\Omega)) \arrow[r, tail] & PP\mathbb{B}^\Delta \arrow[r, two heads, "Pd_1"'] & P\mathbb{B}^\Delta
\end{tikzcd}$$

Therefore all squares of the diagram

$$\begin{tikzcd}
\Omega\mathcal{K}(JA,B): & \cdots \arrow[r, "\mathsf{sw}\circ\varsigma"] & (J^nA, \mathbb{B}^\Delta(\Omega^n)) \arrow[r, "\mathsf{sw}\circ\varsigma"] \arrow[d] & (J^{n+1}A, \mathbb{B}^\Delta(\Omega^{n+1})) \arrow[r, "\mathsf{sw}\circ\varsigma"] \arrow[d] & \cdots \\
P\mathcal{K}(JA,B): & \cdots \arrow[r, "P\varsigma"] & (J^nA, P\mathbb{B}^\Delta(\Omega^{n-1})) \arrow[r, "P\varsigma"] \arrow[d, "d_1"'] & (J^{n+1}A, P\mathbb{B}^\Delta(\Omega^n)) \arrow[r, "P\varsigma"] \arrow[d, "d_1"] & \cdots \\
\mathcal{K}(JA,B): & \cdots \arrow[r, "\varsigma"] & (J^nA, \mathbb{B}^\Delta(\Omega^{n-1})) \arrow[r, "\varsigma"] & (J^{n+1}A, \mathbb{B}^\Delta(\Omega^n)) \arrow[r, "\varsigma"] & \cdots
\end{tikzcd}$$

are commutative. The desired isomorphism $\mathcal{K}(A,B) \cong \Omega\mathcal{K}(JA,B)$ is encoded by the following commutative diagram:

$$\begin{tikzcd}
(A, \mathbb{B}^\Delta) \arrow[r, "\varsigma"] \arrow[d, "\varsigma"'] & (JA, \mathbb{B}^\Delta(\Omega)) \arrow[r, "\varsigma"] \arrow[d, "\varsigma"'] & (J^2A, \mathbb{B}^\Delta(\Omega^2)) \arrow[r, "\varsigma"] \arrow[d, "\varsigma"'] & \cdots \\
(JA, \mathbb{B}^\Delta(\Omega)) \arrow[r, "\varsigma"] \arrow[d, equal] & (J^2A, \mathbb{B}^\Delta(\Omega^2)) \arrow[r, "\varsigma"] \arrow[d, "(21)"'] & (J^3A, \mathbb{B}^\Delta(\Omega^3)) \arrow[r, "\varsigma"] \arrow[d, "(321)"'] & \cdots \\
(JA, \mathbb{B}^\Delta(\Omega)) \arrow[r, "\mathsf{sw}\circ\varsigma"] & (J^2A, \mathbb{B}^\Delta(\Omega^2)) \arrow[r, "\mathsf{sw}\circ\varsigma"] & (J^3A, \mathbb{B}^\Delta(\Omega^3)) \arrow[r, "\mathsf{sw}\circ\varsigma"] & \cdots
\end{tikzcd}$$

The colimit of the upper sequence is $\mathcal{K}(A,B)$ and the colimit of the lower one is $\Omega\mathcal{K}(JA,B)$. The cycle $(n\cdots 21)\in\Sigma_n$ permutes coordinates of $\mathbb{B}^\Delta(\Omega^n)$. $\square$

Corollary 5.2. *For any algebras $A,B\in\Re$ the space $\mathcal{K}(A,B)$ is naturally homotopy equivalent to $\mathcal{K}(JA,\Omega B)$.*

Proof. This follows from the preceding theorem and Corollary 4.3. $\square$

Definition. Given two k-algebras $A,B\in\Re$, the *unstable algebraic Kasparov KK-theory spectrum* of (A,B) consists of the sequence of spaces

$$\mathcal{K}(A,B),\mathcal{K}(JA,B),\mathcal{K}(J^2A,B),\ldots$$

together with isomorphisms $\mathcal{K}(J^nA,B)\cong\Omega\mathcal{K}(J^{n+1}A,B)$ constructed in Theorem 5.1. It forms an Ω-spectrum which we also denote by $\mathbb{K}^{unst}(A,B)$. Its homotopy groups will be denoted by $\mathbb{K}_n^{unst}(A,B)$, $n\in\mathbb{Z}$. We sometimes write $\mathbb{K}(A,B)$ instead of $\mathbb{K}^{unst}(A,B)$, dropping “unst” from notation.
Observe that $\mathbb{K}_n(A,B)\cong\mathcal{K}_n(A,B)$ for any $n\geqslant 0$ and $\mathbb{K}_n(A,B)\cong\mathcal{K}_0(J^{-n}A,B)$ for any $n<0$.

Theorem 5.3. *The assignment $B\mapsto\mathbb{K}(A,B)$ determines a functor*

$$\mathbb{K}(A,?):\Re\to(\mathit{Spectra})$$

which is homotopy invariant and excisive in the sense that for every $\mathfrak{F}$-extension $F\to B\to C$ the sequence

$$\mathbb{K}(A,F)\to\mathbb{K}(A,B)\to\mathbb{K}(A,C)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots\to\mathbb{K}_{i+1}(A,C)\to\mathbb{K}_i(A,F)\to\mathbb{K}_i(A,B)\to\mathbb{K}_i(A,C)\to\cdots$$

for any $i\in\mathbb{Z}$.

Proof. This follows from Excision Theorem A. $\square$

We also have the following

Theorem 5.4. *The assignment $B\mapsto\mathbb{K}(B,D)$ determines a functor*

$$\mathbb{K}(?,D):\Re^{\mathrm{op}}\to(\mathit{Spectra}),$$

which is excisive in the sense that for every $\mathfrak{F}$-extension $F\to B\to C$ the sequence

$$\mathbb{K}(C,D)\to\mathbb{K}(B,D)\to\mathbb{K}(F,D)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots\to\mathbb{K}_{i+1}(F,D)\to\mathbb{K}_i(C,D)\to\mathbb{K}_i(B,D)\to\mathbb{K}_i(F,D)\to\cdots$$

for any $i\in\mathbb{Z}$.

Proof. We postpone the proof till subsection 6.6. $\square$

The reader may have observed that we do not involve any matrices in the definition of $\mathcal{K}(A,B)$ as *any* sort of algebraic K-theory does. This is one of *important* differences with usual views on algebraic K-theory. The author is motivated by the fact that many interesting admissible categories of algebras deserving to be considered like that of all commutative ones are not closed under matrices.

6. Homotopy theory of algebras

Let $\Re$ be a *small* admissible category of rings. In order to prove Excision Theorem B, we have to use results from homotopy theory of rings. First, we introduce a model category $U\Re_\bullet^{I,J}$ of pointed simplicial functors from $\Re$ to $\mathbb{S}_\bullet$. This model category is a reminiscence of Morel–Voevodsky [25] motivic model category of pointed motivic spaces. Second, we define a model category of S^1-spectra $Sp(\Re)$ associated with $U\Re_\bullet^{I,J}$. A typical fibrant spectrum of $Sp(\Re)$ is $\mathbb{K}^{unst}(A,-)$, $A \in \Re$. The strategy of proving Excision Theorem B is first to prove a kind of Excision Theorem B for the spectra $\mathbb{K}^{unst}(A,-)$ (see Theorem 6.6) and then use standard facts from homotopical algebra to show the original Excision Theorem B on the level of spaces. We mostly adhere to [8].

6.1. The category of simplicial functors $U\Re$. We shall use the model category $U\Re$ of covariant functors from $\Re$ to simplicial sets (and not contravariant functors as usual). We do not worry about set theoretic issues here, because we assume $\Re$ to be small. We shall consider both the injective and projective model structures on $U\Re$ and refer to Dugger [7] for further details. Both model structures are Quillen equivalent. These are proper, simplicial, cellular model category structures with weak equivalences and cofibrations (respectively fibrations) being defined objectwise, and fibrations (respectively cofibrations) being those maps having the right (respectively left) lifting property with respect to trivial cofibrations (respectively trivial fibrations). The fully faithful contravariant functor

$$r : \Re \to U\Re, \quad A \longmapsto \mathrm{Hom}_\Re(A,-),$$

where $rA(B) = \mathrm{Hom}_\Re(A,B)$ is to be thought of as the constant simplicial set for any $B \in \Re$.

In the injective model structure on $U\Re$, cofibrations are the injective maps. This model structure enjoys the following properties (see Dugger [7, p. 21]):

- ⋄ every object is cofibrant;
- ⋄ being fibrant implies being objectwise fibrant, but is stronger (there are additional diagramatic conditions involving maps being fibrations, etc.);
- ⋄ any object which is constant in the simplicial direction is fibrant.

If $F \in U\Re$ then $U\Re(rA \times \Delta^n, F) = F_n(A)$ (isomorphism of sets). Hence, if we look at simplicial mapping spaces we find

$$\mathrm{Map}(rA, F) = F(A)$$

(isomorphism of simplicial sets). This is a kind of "simplicial Yoneda Lemma". In the projective model structure on $U\Re$, fibrations are defined sectionwise. The class of projective cofibrations is generated by the set

$$I_{U\Re} \equiv \{rA \times (\partial\Delta^n \subset \Delta^n)\}^{n\geqslant 0}$$

indexed by $A \in \Re$. Likewise, the class of acyclic projective cofibrations is generated by

$$J_{U\Re} \equiv \{rA \times (\Lambda^k_n \subset \Delta^n)\}^{n>0}_{0\leqslant k\leqslant n}.$$

The projective model structure on $U\Re$ enjoys the following properties:

- ⋄ every projective cofibration is an injective map (but not vice versa);
- ⋄ if $A \in \Re$ and K is a simplicial set, then $rA\times K$ is a projective cofibrant simplicial functor. In particular, rA is projective cofibrant for every algebra $A \in \Re$;
- ⋄ rA is projective fibrant for every algebra $A \in \Re$.

6.2. Bousfield localization. Recall from [19] that if $\mathcal{M}$ is a model category and S a set of maps between cofibrant objects, one can produce a new model structure on $\mathcal{M}$ in which the maps S are weak equivalences. The new model structure is called the *Bousfield localization* or just localization of the old one. Since all model categories we shall consider are simplicial one can use the simplicial mapping object instead of the homotopy function complex for the localization theory of $\mathcal{M}$.

Definition. Let $\mathcal{M}$ be a simplicial model category and let S be a set of maps between cofibrant objects.

(1) An *S-local object* of $\mathcal{M}$ is a fibrant object X such that for every map $A \to B$ in S, the induced map of $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence of simplicial sets.
(2) An *S-local equivalence* is a map $A \to B$ such that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence for every S-local object X.

In words, the S-local objects are the ones which see every map in S as if it were a weak equivalence. The S-local equivalences are those maps which are seen as weak equivalences by every S-local object.

Theorem 6.1 (Hirschhorn [19]). *Let $\mathcal{M}$ be a cellular, simplicial model category and let S be a set of maps between cofibrant objects. Then there exists a model category $\mathcal{M}/S$ whose underlying category is that of $\mathcal{M}$ in which*

(1) *the weak equivalences are the S-local equivalences;*
(2) *the cofibrations in $\mathcal{M}/S$ are the same as those in $\mathcal{M}$;*
(3) *the fibrations are the maps having the right-lifting-property with respect to cofibrations which are also S-local equivalences.*

Left Quillen functors from $\mathcal{M}/S$ to $\mathcal{D}$ are in one to one correspondence with left Quillen functors $\Phi : \mathcal{M} \to \mathcal{D}$ such that $\Phi(f)$ is a weak equivalence for all $f \in S$. In addition, the fibrant objects of $\mathcal{M}$ are precisely the S-local objects, and this new model structure is again cellular and simplicial.

The model category $\mathcal{M}/S$ whose existence is guaranteed by the above theorem is called the *S-localization* of $\mathcal{M}$. The underlying category is the same as that of $\mathcal{M}$, but there are more trivial cofibrations (and hence fewer fibrations).
Note that the identity maps yield a Quillen pair $\mathcal{M} \rightleftarrows \mathcal{M}/S$, where the left Quillen functor is the map $\mathrm{id} : \mathcal{M} \to \mathcal{M}/S$.

6.3. The model category $U\Re_I$. Let $I = \{i = i_A : r(A[t]) \to r(A) \mid A \in \Re\}$, where each i_A is induced by the natural homomorphism $i : A \to A[t]$. Consider the injective model structure on $U\Re$. We shall refer to the I-local equivalences as (injective) I-weak equivalences. The resulting model category $U\Re/I$ will be denoted by $U\Re_I$ and its homotopy category is denoted by $\mathrm{Ho}_I(\Re)$. Notice that any homotopy invariant functor $F : \Re \to Sets$ is an I-local object in $U\Re$ (hence fibrant in $U\Re_I$).
Let F be a functor from $\Re$ to simplicial sets. There is a *singular functor* $Sing_*(F)$ which is defined at each algebra R as the diagonal of the bisimplicial set $F(R^\Delta)$. Thus $Sing_*(F)$ is also a functor from $\Re$ to simplicial sets. If we consider R as a constant simplicial algebra, then the natural map $R \to R^\Delta$ yields a natural transformation $F \to Sing_*(F)$. It is an I-trivial cofibration by [8, 3.8].
If we consider the projective model structure on $U\Re$, then we shall refer to the I-local equivalences (respectively fibrations in the I-localized model structure) as projective I-weak equivalences (respectively I-projective fibrations). The resulting model category $U\Re/I$ will be denoted by $U\Re^I$. It is shown similar to [26, 3.49] that the classes of injective and projective I-weak equivalences coincide. Hence the identity functor on $U\Re$ is a Quillen equivalence between $U\Re^I$ and $U\Re_I$.
The model category $U\Re^I$ satisfies some finiteness conditions.

Definition ([21]). An object A of a model category $\mathcal{M}$ is *finitely presentable* if the set-valued Hom-functor $\mathrm{Hom}_{\mathcal{M}}(A, -)$ commutes with all colimits of sequences $X_0 \to X_1 \to X_2 \to \cdots$. A cofibrantly generated model category with generating sets of cofibrations $\mathcal{I}$ and trivial cofibrations $\mathcal{J}$ is called *finitely generated* if the domains and codomains of $\mathcal{I}$ and $\mathcal{J}$ are finitely presentable, and *almost finitely generated* if the domains and codomains of $\mathcal{I}$ are finitely presentable and there exists a set of trivial cofibrations $\mathcal{J}'$ with finitely presentable domains and codomains such that a map with fibrant codomain is a fibration if and only if it has the right lifting property with respect to $\mathcal{J}'$.

Using the simplicial mapping cylinder in $U\Re$ (it is the usual one from simplicial sets applied objectwise), we may factor the morphism

$$r(A[t]) \longrightarrow rA$$

into a projective cofibration composed with a simplicial homotopy equivalence in $U\Re$

$$r(A[t]) \longrightarrow \mathrm{cyl}\big(r(A[t]) \to rA\big) \longrightarrow rA. \tag{4}$$

Observe that the maps in (4) are I-weak equivalences.
Let $J_{U\Re^I}$ denote the set of maps

$$r(A[t]) \times \Delta^n \coprod_{r(A[t])\times\partial\Delta^n} \operatorname{cyl}\big(r(A[t]) \to rA\big) \times \partial\Delta^n \to \operatorname{cyl}\big(r(A[t]) \to rA\big) \times \Delta^n$$

indexed by $n \geqslant 0$ and $A \in \Re$.
Let Λ be a set of generating trivial cofibrations for the injective model structure on $U\Re$. Using [19, 4.2.4] a simplicial functor $\mathcal{X}$ is I-local in the injective (respectively projective) model structure if and only if it has the right lifting property with respect to $\Lambda \cup J_{U\Re^I}$ (respectively $J_{U\Re} \cup J_{U\Re^I}$). It follows from [21, 4.2] that $U\Re^I$ is almost finitely generated, because domains and codomains of $J_{U\Re} \cup J_{U\Re^I}$ are finitely presentable.

6.4. The model category $U\Re_J$. Let us introduce the class of excisive functors on $\Re$. They look like flasque presheaves on a site defined by a cd-structure in the sense of Voevodsky [30, section 3].

Definition. Let $\Re$ be an admissible category of algebras. A simplicial functor $\mathcal{X} \in U\Re$ is called *excisive* with respect to $\mathfrak{F}$ if $\mathcal{X}(0)$ is contractible and for any cartesian square in $\Re$

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

with f a fibration (call such squares *distinguished*) the square of simplicial sets

$$\begin{tikzcd} \mathcal{X}(D) \arrow[r] \arrow[d] & \mathcal{X}(A) \arrow[d] \\ \mathcal{X}(B) \arrow[r] & \mathcal{X}(C) \end{tikzcd}$$

is a homotopy pullback square. It immediately follows from the definition that every pointed excisive object takes $\mathfrak{F}$-extensions in $\Re$ to homotopy fibre sequences of simplicial sets.

Consider the injective model structure on $U\Re$. Let α denote a distinguished square in $\Re$

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

and denote the pushout of the diagram

$$\begin{tikzcd} rC \arrow[r] \arrow[d] & rA \\ rB & \end{tikzcd}$$

by $P(\alpha)$. Notice that the diagram obtained is homotopy pushout. There is a natural map $P(\alpha) \to rD$, and both objects are cofibrant. In the case of the degenerate square this map has to be understood as the map from the initial object $\emptyset$ to $r0$.

We can localize $U\Re$ at the family of maps

$$J = \{P(\alpha) \to rD \mid \ \alpha \text{ is a distinguished square}\}.$$

The corresponding J-localization will be denoted by $U\Re_J$. The weak equivalences (trivial cofibrations) of $U\Re_J$ will be referred to as (injective) J-weak equivalences ((injective) J-trivial cofibrations).

It follows that the square "$r(\alpha)$"

$$\begin{tikzcd} rC \arrow[r] \arrow[d] & rA \arrow[d] \\ rB \arrow[r] & rD \end{tikzcd}$$

with α a distinguished square is a homotopy pushout square in $U\Re_J$. A simplicial functor $\mathcal{X}$ in $U\Re$ is J-local if and only if it is fibrant and excisive [8, 4.3].

We are also interested in constructing sets of generating acyclic cofibrations for model structures. Let us apply the simplicial mapping cylinder construction cyl to distinguished squares and form the pushouts:

$$\begin{tikzcd} rC \arrow[r] \arrow[d] & \operatorname{cyl}(rC \to rA) \arrow[r] \arrow[d] & rA \arrow[d] \\ rB \arrow[r] & \operatorname{cyl}(rC \to rA) \coprod_{rC} rB \arrow[r] & rD \end{tikzcd}$$

Note that $rC \to \operatorname{cyl}(rC \to rA)$ is both an injective and a projective cofibration between (projective) cofibrant simplicial functors. Thus $s(\alpha) \equiv \operatorname{cyl}(rC \to rA) \coprod_{rC} rB$ is (projective) cofibrant [20, 1.11.1]. For the same reasons, applying the simplicial mapping cylinder to $s(\alpha) \to rD$ and setting $t(\alpha) \equiv \operatorname{cyl}\big(s(\alpha) \to rD\big)$ we get a projective cofibration

$$\operatorname{cyl}(\alpha)\colon s(\alpha) \longrightarrow t(\alpha).$$

Let $J_{U\Re}^{\operatorname{cyl}(\alpha)}$ consists of maps

$$s(\alpha) \times \Delta^n \coprod_{s(\alpha)\times\partial\Delta^n} t(\alpha) \times \partial\Delta^n \longrightarrow t(\alpha) \times \Delta^n.$$

It is directly verified that a simplicial functor $\mathcal{X}$ is J-local if and only if it has the right lifting property with respect to $\Lambda \cup J_{U\Re}^{\operatorname{cyl}(\alpha)}$, where Λ is a set of generating trivial cofibrations for the injective model structure on $U\Re$.

If one localizes the projective model structure on $U\Re$ with respect to the set of projective cofibrations $\{\operatorname{cyl}(\alpha)\}_\alpha$, the resulting model category shall be denoted by $U\Re^J$. The weak equivalences (trivial cofibrations) of $U\Re^J$ will be referred to as projective J-weak equivalences (projective J-trivial cofibrations). As above,

$\mathcal{X}$ is fibrant in $U\mathfrak{R}^J$ if and only if it has the right lifting property with respect to $J_{U\mathfrak{R}} \cup J_{U\mathfrak{R}}^{\mathrm{cyl}(\alpha)}$. Since both domains and codomains in $J_{U\mathfrak{R}} \cup J_{U\mathfrak{R}}^{\mathrm{cyl}(\alpha)}$ are finitely presentable then $U\mathfrak{R}^J$ is almost finitely generated by [21, 4.2].
It can be shown similar to [26, 3.49] that the classes of injective and projective J-weak equivalences coincide. Hence the identity functor on $U\mathfrak{R}$ is a Quillen equivalence between $U\mathfrak{R}_J$ and $U\mathfrak{R}^J$.

6.5. THE MODEL CATEGORY $U\mathfrak{R}_{I,J}$.

DEFINITION. A simplicial functor $\mathcal{X} \in U\mathfrak{R}$ is called *quasi-fibrant* with respect to $\mathfrak{F}$ if it is homotopy invariant and excisive. For instance, if $\mathfrak{R}$ is T-closed and $A \in \mathfrak{R}$ then the simplicial functor $\mathcal{K}(A, ?)$ is quasi-fibrant by Lemma 4.2 and Excision Theorem A.
Consider the injective model structure on $U\mathfrak{R}$. The model category $U\mathfrak{R}_{I,J}$ is, by definition, the Bousfield localization of $U\mathfrak{R}$ with respect to $I \cup J$. Equivalently, $U\mathfrak{R}_{I,J}$ is the Bousfield localization of $U\mathfrak{R}$ with respect to $\{\mathrm{cyl}(r(A[t]) \to rA)\} \cup \{\mathrm{cyl}(\alpha)\}$, where A runs over the objects from $\mathfrak{R}$ and α runs over the distinguished squares. The weak equivalences (trivial cofibrations) of $U\mathfrak{R}_{I,J}$ will be referred to as (injective) (I, J)-weak equivalences ((injective) (I, J)-trivial cofibrations). By [8, 4.5] a simplicial functor $\mathcal{X} \in U\mathfrak{R}$ is (I, J)-local if and only if it is fibrant, homotopy invariant and excisive.

DEFINITION. Following [8] a homomorphism $A \to B$ in $\mathfrak{R}$ is said to be an $\mathfrak{F}$*-quasi-isomorphism* or just a *quasi-isomorphism* if the map $rB \to rA$ is an (I, J)-weak equivalence.

Consider now the projective model structure on $U\mathfrak{R}$. The model category $U\mathfrak{R}^{I,J}$ is, by definition, the Bousfield localization of $U\mathfrak{R}$ with respect to $\{\mathrm{cyl}(r(A[t]) \to rA)\} \cup \{\mathrm{cyl}(\alpha)\}$, where A runs over the objects from $\mathfrak{R}$ and α runs over the distinguished squares. The weak equivalences (trivial cofibrations) of $U\mathfrak{R}^{I,J}$ will be referred to as projective (I, J)-weak equivalences (projective (I, J)-trivial cofibrations). Similar to [8, 4.5] a simplicial functor $\mathcal{X} \in U\mathfrak{R}$ is fibrant in $U\mathfrak{R}^{I,J}$ if and only if it is projective fibrant, homotopy invariant and excisive or, equivalently, it has the right lifting property with respect to $J_{U\mathfrak{R}} \cup J_{U\mathfrak{R}_I} \cup J_{U\mathfrak{R}}^{\mathrm{cyl}(\alpha)}$. Since both domains and codomains in $J_{U\mathfrak{R}} \cup J_{U\mathfrak{R}_I} \cup J_{U\mathfrak{R}}^{\mathrm{cyl}(\alpha)}$ are finitely presentable then $U\mathfrak{R}^{I,J}$ is almost finitely generated by [21, 4.2].
It can be shown similar to [26, 3.49] that the classes of injective and projective (I, J)-weak equivalences coincide. Hence the identity functor on $U\mathfrak{R}$ is a Quillen equivalence between $U\mathfrak{R}_{I,J}$ and $U\mathfrak{R}^{I,J}$.
It is straightforward to show that the results for the model structures on $U\mathfrak{R}$ have analogs for the category $U\mathfrak{R}_\bullet$ of pointed simplicial functors (see [8]). In order to prove Excision Theorem B, we have to consider a model category of spectra for $U\mathfrak{R}_\bullet^{I,J}$.

6.6. THE CATEGORY OF SPECTRA. In this section we assume $\mathfrak{R}$ to be small and T-closed. We use here ideas and work of Hovey [21], Jardine [22] and Schwede [28].

DEFINITION. The category $Sp(\Re)$ of spectra consists of sequences $\mathcal{E} \equiv (\mathcal{E}_n)_{n\geqslant 0}$ of pointed simplicial functors equipped with structure maps $\sigma_n^{\mathcal{E}} : \Sigma\mathcal{E}_n \to \mathcal{E}_{n+1}$ where $\Sigma = S^1 \wedge -$ is the suspension functor. A map $f : \mathcal{E} \to \mathcal{F}$ of spectra consists of compatible maps $f_n : \mathcal{E}_n \to \mathcal{F}_n$ of pointed simplicial functors in the sense that the diagrams

$$\begin{tikzcd} \Sigma\mathcal{E}_n \arrow[r, "\sigma_n^{\mathcal{E}}"] \arrow[d, "\Sigma f_n"'] & \mathcal{E}_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma\mathcal{F}_n \arrow[r, "\sigma_n^{\mathcal{F}}"] & \mathcal{F}_{n+1} \end{tikzcd}$$

commute for all $n \geqslant 0$.

EXAMPLE. The main spectrum we shall work with is as follows. Let $A \in \Re$ and let $\mathcal{R}(A)$ be the spectrum which is defined at every $B \in \Re$ as the sequence of spaces pointed at zero

$$\operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^\Delta), \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^\Delta), \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^2A, \mathbb{B}^\Delta), \ldots$$

By Theorem 2.4 each $\mathcal{R}(A)_n(B)$ is a fibrant simplicial set and by Corollary 2.7

$$\Omega^k\mathcal{R}(A)_n(B) = \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, \mathbb{B}^\Delta(\Omega^k)).$$

Each structure map $\sigma_n : \Sigma\mathcal{R}(A)_n \to \mathcal{R}(A)_{n+1}$ is defined at B as adjoint to the map $\varsigma : \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^nA, \mathbb{B}^\Delta) \to \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^\Delta(\Omega))$ constructed in (1).

A map $f : \mathcal{E} \to \mathcal{F}$ is a level weak equivalence (respectively fibration) if $f_n : \mathcal{E}_n \to \mathcal{F}_n$ is a (I, J)-weak equivalence (respectively projective (I, J)-fibration). And f is a projective cofibration if f_0 and the maps

$$\mathcal{E}_{n+1} \bigsqcup\nolimits_{\Sigma\mathcal{E}_n} \Sigma\mathcal{F}_n \longrightarrow \mathcal{F}_{n+1}$$

are cofibrations in $U\Re_\bullet^{I,J}$ for all $n \geqslant 0$. By [21, 22, 28] we have:

PROPOSITION 6.2. *The level weak equivalences, projective cofibrations and level fibrations furnish a simplicial and left proper model structure on $Sp(\Re)$. We call this the projective model structure.*

The Bousfield–Friedlander category of spectra [1] will be denoted by Sp. There is a functor

$$Sp \to Sp(\Re)$$

that takes a spectrum of pointed simplicial sets $\mathcal{E}$ to the constant spectrum $A \in \Re \mapsto \mathcal{E}(A) = \mathcal{E}$. For any algebra $D \in \Re$ there is also a functor

$$U_D : Sp(\Re) \to Sp, \quad \mathcal{X} \mapsto \mathcal{X}(D).$$

Given a spectrum $\mathcal{E} \in Sp$ and a pointed simplicial functor K, there is a spectrum $\mathcal{E} \wedge K$ with $(\mathcal{E} \wedge K)_n = \mathcal{E}_n \wedge K$ and having structure maps of the form

$$\Sigma(\mathcal{E}_n \wedge K) \cong (\Sigma\mathcal{E}_n) \wedge K \xrightarrow{\sigma_n \wedge K} \mathcal{E}_{n+1} \wedge K.$$

Given $D \in \Re$, the functor $F_D : Sp \to Sp(\Re)$, $\mathcal{E} \mapsto \mathcal{E} \wedge rD_+$, is left adjoint to $U_D : Sp(\Re) \to Sp$. So there is an isomorphism

$$\operatorname{Hom}_{Sp(\Re)}(\mathcal{E} \wedge rD_+, \mathcal{X}) \cong \operatorname{Hom}_{Sp}(\mathcal{E}, \mathcal{X}(D)). \tag{5}$$

Our next objective is to define the stable model structure. We define the fake suspension functor $\Sigma : Sp(\Re) \to Sp(\Re)$ by $(\Sigma\mathcal{Z})_n = \Sigma\mathcal{Z}_n$ and structure maps

$$\Sigma(\Sigma\mathcal{Z}_n) \xrightarrow{\Sigma\sigma_n} \Sigma\mathcal{Z}_{n+1},$$

where σ_n is a structure map of $\mathcal{Z}$. Note that the fake suspension functor is left adjoint to the fake loops functor $\Omega^\ell : Sp(\Re) \to Sp(\Re)$ defined by $(\Omega^\ell\mathcal{Z})_n = \Omega\mathcal{Z}_n$ and structure maps adjoint to

$$\Omega\mathcal{Z}_n \xrightarrow{\Omega\widetilde{\sigma}_n} \Omega(\Omega\mathcal{Z}_{n+1}),$$

where $\widetilde{\sigma}_n$ is adjoint to the structure map σ_n of $\mathcal{Z}$.

Definition. A spectrum $\mathcal{Z}$ is *stably fibrant* if it is level fibrant and all the adjoints $\widetilde{\sigma}_n^{\mathcal{Z}} : \mathcal{Z}_n \to \Omega\mathcal{Z}_{n+1}$ of its structure maps are (I, J)-weak equivalences.

Example. Given $A \in \Re$, the spectrum $\mathbb{K}(A, -)$ consists of the sequence of simplicial functors

$$\mathcal{K}(A, -), \mathcal{K}(JA, -), \mathcal{K}(J^2A, -), \ldots$$

together with isomorphisms $\mathcal{K}(J^nA, -) \cong \Omega\mathcal{K}(J^{n+1}A, -)$ constructed in Theorem 5.1. Lemma 4.2 and Excision Theorem A imply $\mathbb{K}(A, -)$ is a stably fibrant spectrum. Note that $\mathbb{K}(A, B)$ is stably fibrant in Sp for every $B \in \Re$.

The stably fibrant spectra determine the stable weak equivalences of spectra. Stable fibrations are maps having the right lifting property with respect to all maps which are projective cofibrations and stable weak equivalences.

Definition. A map $f : \mathcal{E} \to \mathcal{F}$ of spectra is a *stable weak equivalence* if for every stably fibrant $\mathcal{Z}$ taking a cofibrant replacement $Qf : Q\mathcal{E} \to Q\mathcal{F}$ of f in the level projective model structure on $Sp(\Re)$ yields a weak equivalence of pointed simplicial sets

$$\operatorname{Map}_{Sp(\Re)}(Qf, \mathcal{Z}) : \operatorname{Map}_{Sp(\Re)}(Q\mathcal{F}, \mathcal{Z}) \longrightarrow \operatorname{Map}_{Sp(\Re)}(Q\mathcal{E}, \mathcal{Z}).$$

By specializing the collection of results in [21, 28] to our setting we have:

Theorem 6.3. *The classes of stable weak equivalences and projective cofibrations define a simplicial and left proper model structure on $Sp(\Re)$.*

If we define the stable model category structure on ordinary spectra Sp similar to $Sp(\Re)$, then by [21, 3.5] it coincides with the stable model structure of Bousfield–Friedlander [1].

Define the *shift functors* $t : Sp(\Re) \to Sp(\Re)$ and $s : Sp(\Re) \to Sp(\Re)$ by $(s\mathcal{X})_n = \mathcal{X}_{n+1}$ and $(t\mathcal{X})_n = \mathcal{X}_{n-1}$, $(t\mathcal{X})_0 = pt$, with the evident structure maps. Note that t is left adjoint to s.

DEFINITION. Define $\Theta : Sp(\Re) \to Sp(\Re)$ to be the functor $s \circ \Omega^\ell$, where s is the shift functor. Then we have a natural map $\iota_{\mathcal{X}} : \mathcal{X} \to \Theta\mathcal{X}$, and we define

$$\Theta^\infty\mathcal{X} = \operatorname{colim}(\mathcal{X} \xrightarrow{\iota_{\mathcal{X}}} \Theta\mathcal{X} \xrightarrow{\Theta\iota_{\mathcal{X}}} \Theta^2\mathcal{X} \xrightarrow{\Theta^2\iota_{\mathcal{X}}} \cdots \xrightarrow{\Theta^{n-1}\iota_{\mathcal{X}}} \Theta^n\mathcal{X} \xrightarrow{\Theta^n\iota_{\mathcal{X}}} \cdots).$$

Let $j_{\mathcal{X}} : \mathcal{X} \to \Theta^\infty\mathcal{X}$ denote the obvious natural transformation. It is a stable equivalence by [21, 4.11]. We call $\Theta^\infty\mathcal{X}$ the *stabilization* of $\mathcal{X}$.

EXAMPLE. Given $A \in \Re$, there is a natural map of spectra

$$\varkappa : \mathcal{R}(A) \to \mathbb{K}(A, -).$$

One has a commutative diagram

$$\begin{tikzcd} \mathcal{R}(A) \arrow[r, "j"] \arrow[d, "\varkappa"'] & \Theta^\infty\mathcal{R}(A) \arrow[d, "\Theta^\infty\varkappa"] \\ \mathbb{K}(A,-) \arrow[r, "j"] & \Theta^\infty\mathbb{K}(A,-). \end{tikzcd}$$

The upper horizontal map is a stable equivalence, the lower and right arrows are isomorphisms. Therefore the natural map of spectra $\varkappa : \mathcal{R}(A) \to \mathbb{K}(A, -)$ is a stable equivalence. In fact for any algebra $B \in \Re$ the map

$$\varkappa_B : \mathcal{R}(A)(B) \to \mathbb{K}(A, B)$$

is a stable equivalence of ordinary spectra.

By [21, 4.6] we get the following result because $\Omega(-)$ preserves sequential colimits and the model category $U\Re_\bullet^{I,J}$ is almost finitely generated.

LEMMA 6.4. *The stabilization of every level fibrant spectrum is stably fibrant.*

LEMMA 6.5. *For any $D \in \Re$ the adjoint functors $F_D : Sp \rightleftarrows Sp(\Re) : U_D$ form a Quillen adjunction between the stable model category of Bousfield–Friedlander spectra Sp and the stable model category $Sp(\Re)$.*

Proof. Clearly, F_D preserves stable cofibrations. To show that F_D preserves stable trivial cofibrations, it is enough to observe that U_D preserves stable fibrant spectra (see the proof of [21, 3.5]) and use (5). $\square$

We are now in a position to prove the main result of this section.

THEOREM 6.6. *Suppose $F \rightarrowtail B \twoheadrightarrow C$ is an $\mathfrak{F}$-extension in $\Re$. Then the commutative square of spectra*

$$\begin{tikzcd} \mathbb{K}(C,-) \arrow[r] \arrow[d] & \mathbb{K}(B,-) \arrow[d] \\ pt \arrow[r] & \mathbb{K}(F,-) \end{tikzcd}$$

is homotopy pushout and homotopy pullback in $Sp(\Re)$. Moreover, if $D \in \Re$ then the square of ordinary spectra

$$\begin{array}{ccc} \mathbb{K}(C,D) & \longrightarrow & \mathbb{K}(B,D) \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathbb{K}(F,D) \end{array}$$

is homotopy pushout and homotopy pullback.

Proof. Given a distinguished square α

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

in $\Re$, the square $r\alpha_+$

$$\begin{array}{ccc} rC_+ & \longrightarrow & rA_+ \\ \downarrow & & \downarrow \\ rB_+ & \longrightarrow & rD_+ \end{array}$$

is homotopy pushout in $U\Re_\bullet^{I,J}$.

We claim that there is a J-weak equivalence of pointed simplicial functors $rA_+ \to rA$ for any algebra $A \in \Re$. The object rA is a cofibre product of the diagram

$$pt \leftarrow r0_+ \to rA_+,$$

in which the right arrow is an injective cofibration. It follows that for every pointed fibrant object $\mathcal{X}$ in $U\Re_{J,\bullet}$ the sequence of simplicial sets

$$\mathrm{Map}_{U\Re_\bullet}(rA,\mathcal{X}) \to \mathcal{X}(A) \to \mathcal{X}(0)$$

is a homotopy fibre sequence with $\mathcal{X}(0)$ contractible. Hence the left arrow is a weak equivalence of simplicial sets, and so the map of pointed simplicial functors $rA_+ \to rA$ is a J-weak equivalence. Using [19, 13.5.9] the square $r\alpha$ with α as above

$$\begin{array}{ccc} rC & \longrightarrow & rA \\ \downarrow & & \downarrow \\ rB & \longrightarrow & rD \end{array}$$

is homotopy pushout in $U\Re_\bullet^{I,J}$.

Given an algebra $A \in \Re$ and $n \geqslant 0$, there is an I-weak equivalence of simplicial functors pointed at zero $i_{J^nA} : r(J^nA) \to Sing(r(J^nA))$. By Theorem 2.4

$$\mathcal{R}(A)_n = Ex^\infty \circ Sing(r(J^nA)).$$

Since the map

$$\xi_v : JA \to \Omega A,$$

which is functorial in A, is a quasi-isomorphism, then the square

$$\begin{tikzcd} r(J^nC) \arrow[r] \arrow[d] & r(J^nB) \arrow[d] \\ pt \arrow[r] & r(J^nF) \end{tikzcd} \tag{6}$$

is weakly equivalent to the homotopy pushout square in $U\Re_\bullet^{I,J}$

$$\begin{tikzcd} r(\Omega^nC) \arrow[r] \arrow[d] & r(\Omega^nB) \arrow[d] \\ pt \arrow[r] & r(\Omega^nF) \end{tikzcd}$$

By [19, 13.5.9] square (6) is then homotopy pushout in $U\Re_\bullet^{I,J}$. Also, [19, 13.5.9] implies that

$$\begin{tikzcd} Sing(r(J^nC)) \arrow[r] \arrow[d] & Sing(r(J^nB)) \arrow[d] \\ pt \arrow[r] & Sing(r(J^nF)) \end{tikzcd}$$

is homotopy pushout in $U\Re_\bullet^{I,J}$, and hence so is

$$\begin{tikzcd} \mathcal{R}(C)_n \arrow[r] \arrow[d] & \mathcal{R}(B)_n \arrow[d] \\ pt \arrow[r] & \mathcal{R}(F)_n. \end{tikzcd}$$

We see that the square of spectra

$$\begin{tikzcd} \mathcal{R}(C) \arrow[r, "u"] \arrow[d] & \mathcal{R}(B) \arrow[d] \\ pt \arrow[r] & \mathcal{R}(F) \end{tikzcd} \tag{7}$$

is level pushout. We can find a projective cofibration of spectra $\iota : \mathcal{R}(C) \to \mathcal{X}$ and a level weak equivalence $s : \mathcal{X} \to \mathcal{R}(B)$ such that $u = s\iota$. Consider a pushout square

$$\begin{tikzcd} \mathcal{R}(C) \arrow[r, "\iota"] \arrow[d] & \mathcal{X} \arrow[d] \\ pt \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

It is homotopy pushout in the projective model structure of spectra, and hence it is levelwise homotopy pushout in $U\Re_\bullet^{I,J}$. Therefore the induced map $\mathcal{Y} \to \mathcal{R}(F)$ is a level weak equivalence, and so (7) is homotopy pushout in the projective model structure of spectra by [19, 13.5.9].

Since the vertical arrows in the commutative diagram

$$\begin{tikzcd}
pt \arrow[rr] & & \mathbb{K}(F,-) & \\
& \mathbb{K}(C,-) \arrow[ul] \arrow[rr] & & \mathbb{K}(B,-) \arrow[ul] \\
pt \arrow[uu] \arrow[rr] & & \mathcal{R}(F) \arrow[uu] & \\
& \mathcal{R}(C) \arrow[ul] \arrow[uu] \arrow[rr] & & \mathcal{R}(B) \arrow[ul] \arrow[uu, "\sim"']
\end{tikzcd}$$

are stable weak equivalences and the lower square is homotopy pushout in the stable model structure of spectra, then so is the upper square by [19, 13.5.9]. By [21, 3.9; 10.3] $Sp(\Re)$ is a stable model category with respect to the stable model structure, and therefore the square of the theorem is also homotopy pullback by [20, 7.1.12].

It follows from Lemma 6.5 that the square of simplicial spectra

$$\begin{tikzcd}
\mathbb{K}(C,D) \arrow[r] \arrow[d] & \mathbb{K}(B,D) \arrow[d] \\
pt \arrow[r] & \mathbb{K}(F,D)
\end{tikzcd}$$

is homotopy pullback for all $D \in \Re$. It is also homotopy pushout in the stable model category of Bousfield–Friedlander spectra by [20, 7.1.12], because this model structure is stable. □

It is also useful to have the following

Theorem 6.7. *Suppose $u : A \to B$ is a quasi-isomorphism in $\Re$. Then the induced map of spectra*

$$u^* : \mathbb{K}(B,-) \to \mathbb{K}(A,-)$$

is a stable equivalence in $Sp(\Re)$. In particular, the map of spaces

$$u^* : \mathcal{K}(B,C) \to \mathcal{K}(A,C)$$

is a weak equivalence for all $C \in \Re$.

Proof. Consider the square in $U\Re_\bullet$

$$\begin{tikzcd}
rB \arrow[r, "u^*"] \arrow[d] & rA \arrow[d] \\
\mathcal{R}(B)_0 \arrow[r, "u^*"] & \mathcal{R}(A)_0.
\end{tikzcd}$$

The upper arrow is an (I,J)-weak equivalence, the vertical maps are I-weak equivalences. Therefore the lower arrow is an (I,J)-weak equivalence.

Since the endofunctor $J : \Re \to \Re$ respects quasi-isomorphisms, then

$$u^* : \mathcal{R}(B) \to \mathcal{R}(A)$$

is a level weak equivalence of spectra. Consider the square in $Sp(\Re)$

$$\begin{tikzcd} \mathcal{R}(B) \arrow[r, "u^*"] \arrow[d, "\wr"'] & \mathcal{R}(A) \arrow[d, "\wr"] \\ \mathbb{K}(B,-) \arrow[r, "u^*"] & \mathbb{K}(A,-). \end{tikzcd}$$

The upper arrow is a level weak equivalence, the vertical maps are stable weak equivalences. Therefore the lower arrow is a stable weak equivalence.
The map $\mathbb{K}(B,-) \xrightarrow{u^*} \mathbb{K}(A,-)$ is a weak equivalence in the projective model structure on $Sp(\Re)$, because both spectra are stably fibrant and levelwise fibrant in $U\Re_\bullet^{I,J}$. It follows that the map of spaces

$$u^* : \mathcal{K}(B,C) \to \mathcal{K}(A,C)$$

is a weak equivalence for all $C \in \Re$. □

We can now prove Excision Theorem B.

Proof of Excision Theorem B. Let $\Re$ be an arbitrary admissible T-closed category of k-algebras. We have to prove that the square of spaces

$$\begin{tikzcd} \mathcal{K}(C,D) \arrow[r] \arrow[d] & \mathcal{K}(B,D) \arrow[d] \\ pt \arrow[r] & \mathcal{K}(F,D) \end{tikzcd}$$

is homotopy pullback for any extension $F \rightarrowtail B \twoheadrightarrow C$ in $\Re$ and any algebra $D \in \Re$.
A subtle difference with what we have defined for spectra is that we do not assume $\Re$ to be small. So to apply Theorem 6.6 one has to find a small admissible T-closed category of k-algebras $\Re'$ containing F, B, C, D.
We can inductively construct such a category as follows. Let $\Re'_0$ be the full subcategory of $\Re$ such that $\operatorname{Ob}\Re'_0 = \{F, B, C, D\}$. If the full subcategory $\Re'_n$ of $\Re$, $n \geqslant 0$, is constructed we define $\Re'_{n+1}$ by adding the following algebras to $\Re'_n$:

- ▷ all ideals and quotient algebras of algebras from $\Re'_n$;
- ▷ all algebras which are pullbacks for diagrams

$$A \to E \leftarrow L$$

  with $A, E, L \in \Re'_n$;
- ▷ all polynomial algebras in one variable $A[x]$ with $A \in \Re'_n$;
- ▷ all algebras TA with $A \in \Re'_n$.

Then we set $\Re' = \bigcup_n \Re'_n$. Clearly $\Re'$ is a small admissible T-closed category of algebras containing F, B, C, D. It remains to apply Theorem 6.6. □

We can now also prove Theorem 5.4.

Proof of Theorem 5.4. Let $\Re$ be an arbitrary admissible T-closed category of k-algebras. We have to prove that the square of spectra

$$\begin{tikzcd} \mathbb{K}(C,D) \arrow[r] \arrow[d] & \mathbb{K}(B,D) \arrow[d] \\ pt \arrow[r] & \mathbb{K}(F,D) \end{tikzcd}$$

is homotopy pullback for any extension $F \rightarrowtail B \twoheadrightarrow C$ in $\Re$ and any algebra $D \in \Re$.
To apply Theorem 6.6 one has to find a small admissible T-closed category of k-algebras $\Re'$ containing F, B, C, D. Such a category is constructed in the proof of Excision Theorem B. □

Corollary 6.8. *Let $\Re$ be an admissible T-closed category of k-algebras. Then for every $A, B \in \Re$ the spectrum $\mathbb{K}(JA, B)$ has homotopy type of $\Sigma\mathbb{K}(A, B)$.*

Proof. We have an extension $JA \rightarrowtail TA \twoheadrightarrow A$ in which TA is contractible by Lemma 3.1. Hence $\mathbb{K}(TA, B) \simeq *$ by Theorem 6.7 (as above one can choose a small admissible T-closed category of algebras such that all considered algebras belong to it). Now our assertion follows from Excision Theorem B. □

7. Comparison Theorem A

In this section we prove a couple of technical (but important!) results giving a relation between simplicial and polynomial homotopy for algebra homomorphisms. As an application, we prove Comparison Theorem A. Throughout $\Re$ is supposed to be T-closed.

7.1. Categories of fibrant objects.

Definition. Let $\mathcal{A}$ be a category with finite products and a final object e. Assume that $\mathcal{A}$ has two distinguished classes of maps, called *weak equivalences* and *fibrations*. A map is called a *trivial fibration* if it is both a weak equivalence and a fibration. We define a *path space* for an object B to be an object B^I together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal map.
Following Brown [2], we call $\mathcal{A}$ a *category of fibrant objects* or a *Brown category* if the following axioms are satisfied.
(A) Let f and g be maps such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.
(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.
(C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with v a fibration (respectively a trivial fibration), the pullback $A \times_C B$ exists and the map $A \times_C B \to A$ is a fibration (respectively a trivial fibration).
(D) For any object B in $\mathcal{A}$ there exists at least one path space B^I (not necessarily functorial in B).
(E) For any object B the map $B \to e$ is a fibration.

7.2. THE HAUPTLEMMA. Every map u in $\Re$ can be factored $u = pi$, where $p \in \mathfrak{F}$ is a fibration and i is an I-weak equivalence. Indeed, let A' be the the fibre product of the diagram

$$A \xrightarrow{u} B \xleftarrow{\partial_x^0} B[x].$$

Then the map $i : A \to A'$, $a \longmapsto (a, u(a))$, is split and obviously an elementary homotopy equivalence. Hence it is an I-weak equivalence. We define $p : A' \to B$ as composition of the projection $A' \to B[x]$ and ∂_x^1. We call a homomorphism an *I-trivial fibration* if it is both a fibration and an I-weak equivalence. We denote by I^n, $n \geqslant 0$, the simplicial set $\Delta^1 \times \overset{n}{\cdots} \times \Delta^1$ and by $\delta^0, \delta^1 : I^n \to I^{n+1}$ the maps $1_{I^n} \times d^0, 1_{I^n} \times d^1$ whose images are $I^n \times \{1\}, I^n \times \{0\}$ respectively. Let $\mathfrak{W}_{\min}$ be the minimal class of weak equivalences containing the homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathfrak{W}_{\min})$ is a Brown category. We should mention that every excisive, homotopy invariant simplicial functor $\mathcal{X} : \Re \to SSets$ gives rise to a class of weak equivalences $\mathfrak{W}$ containing the homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathfrak{W})$ is a Brown category (see [8]). Precisely, $\mathfrak{W}$ consists of those homomorphisms f for which $\mathcal{X}(f)$ is a weak equivalence of simplicial sets.
We shall denote by $B^{\mathfrak{S}^n}$, $n \geqslant 0$, the ind-algebra

$$B_0^{\mathfrak{S}^n} \to B_1^{\mathfrak{S}^n} \to B_2^{\mathfrak{S}^n} \to \cdots$$

consisting of the 0-simplices of the simplicial ind-algebra $\mathbb{B}(\Omega^n)$. So we have for any $k \geqslant 0$:

$$B_k^{\mathfrak{S}^n} = \operatorname{Ker}(B^{\operatorname{sd}^k I^n} \to B^{\operatorname{sd}^k(\partial I^n)}).$$

One also sets

$$\widetilde{B}_k^{\mathfrak{S}^n} = \operatorname{Ker}(B^{\operatorname{sd}^k I^{n+1}} \to B^{\operatorname{sd}^k(\partial I^n \times I)}).$$

HAUPTLEMMA. *Let $A, B \in \Re$ then for any $m, n \geqslant 0$ we have:*

(1) *If $f : A \to B^{\operatorname{sd}^m \Delta^{n+1}}$ is a homomorphism, then the homomorphism $\partial_i f$ is algebraically homotopic to $\partial_j f$ with $i, j \leqslant n+1$.*
(2) *If $f : A \to B^{\operatorname{sd}^m I^{n+1}}$ (respectively $f : A \to \widetilde{B}_m^{\mathfrak{S}^n}$) is a homomorphism, then $d_0 f, d_1 f : A \to B^{\operatorname{sd}^m I^n}$ (respectively $d_0 f, d_1 f : A \to B_m^{\mathfrak{S}^n}$) are algebraically homotopic, where $d_0, d_1 : B^{\operatorname{sd}^m I^{n+1}} \to B^{\operatorname{sd}^m I^n}$ are induced by δ^0, δ^1.*
(3) *If $f_0, f_1 : A \to B^{\operatorname{sd}^m I^n}$ (respectively $f_0, f_1 : A \to B_m^{\mathfrak{S}^n}$) are two algebraically homotopic homomorphisms by means of a map $h : A \to (B^{\operatorname{sd}^m I^n})^{\operatorname{sd}^k \Delta^1}$ (respectively $h : A \to (B_m^{\mathfrak{S}^n})^{\operatorname{sd}^k \Delta^1}$), then there are a homomorphism $g : A' \to A$, which is obtained by pulling back an I-trivial fibration along h, and hence $g \in \mathfrak{W}_{\min}$, and a homomorphism*

$H : A' \to B^{\mathrm{sd}^m I^{n+1}}$ (respectively $H : A' \to \widetilde{B}^{\mathfrak{S}^n}_m$) such that $d_0 H = f_0 g$ and $d_1 H = f_1 g$.

The Hauptlemma essentially says that the condition for homomorphisms of being simplicially homotopic implies that of being polynomially homotopic. The converse is true up to multiplication with some maps from $\mathfrak{W}_{\min}$.

Proof. (1). Given $i < j$ define a homomorphism $\varphi_{i,j} : B[t_0, \dots, t_{n+1}] \to B^{\Delta^n}[x]$ as

$$\varphi_{i,j}(t_k) = \begin{cases} t_k, & k < i \\ xt_i, & k = i \\ xt_k + (1-x)t_{k-1}, & i < k < j \\ (1-x)t_{j-1}, & k = j \\ t_{k-1}, & k > j \end{cases} \tag{8}$$

It takes $1 - \sum_{i=0}^{n+1} t_i$ to zero, and hence one obtains a homomorphism $\varphi_{i,j} : B^{\Delta^{n+1}} \to B^{\Delta^n}[x]$. We define $\varphi_{j,i}(f)(t_0, \dots, t_n, x) = \varphi_{i,j}(f)(t_0, \dots, t_n, 1-x)$ if $j > i$. It follows that for any $h \in B^{\Delta^{n+1}}$

$$\varphi_{i,j}(h)(t_0, \dots, t_n, x) = \begin{cases} \partial_i h, & x = 0, \\ \partial_j h, & x = 1. \end{cases}$$

We see that $\partial_i \alpha$ is elementary homotopic to $\partial_j \alpha$ for any $\alpha : A \to B^{\Delta^{n+1}}$. If there is no likelihood of confusion we shall denote this homotopy by $\varphi_{i,j}$ omitting α.

Now consider the algebra $B^{\mathrm{sd}^k \Delta^{n+1}}$, $k \geqslant 1$. By definition, it is the fiber product over B^{Δ^n} of $((n+2)!)^k$ copies of $B^{\Delta^{n+1}}$. Let $\alpha : A \to B^{\mathrm{sd}^k \Delta^{n+1}}$ be a homomorphism of algebras. A polynomial homotopy from $\partial_j \alpha$ to $\partial_i \alpha$ can be arranged as follows. We pick up the barycenter of $\partial_j \alpha$ and pull it towards the barycenter of α. This operation consists of finitely many polynomial homotopies. Next we pull the vertex i towards the vertex j. Again we have finitely many elementary polynomial homotopies. Finally, we pull the barycenter of α towards the barycenter of $\partial_i \alpha$, resulting the desired polynomial homotopy. Each step of the polynomial homotopy is determined by homotopies of the form $\varphi_{i,i+1}$ or $\varphi_{i+1,i}$.

In order to give a formal description of the algorithm, it is enough to do this for the identity homomorphism of $B^{\mathrm{sd}^k \Delta^{n+1}}$ without loss of generality. Recall that $\mathrm{sd}^k \Delta^{n+1}$ is the nerve $|\mathfrak{sd}^k \Delta^{n+1}|$ of a poset $\mathfrak{sd}^k \Delta^{n+1}$. We shall also regard the poset as a category. By the length of a path in $\mathfrak{sd}^k \Delta^{n+1}$ between two vertices we mean the maximal number of (non-identity) arrows connecting them. Note that the largest (respectively smallest) possible length equals $n+1$ (respectively 0). We can think about $B^{\mathrm{sd}^k \Delta^{n+1}}$ in the following terms:

- ◇ the poset $\mathfrak{sd}^k \Delta^{n+1}$;
- ◇ to every path π of length $k \geqslant 0$ in $\mathfrak{sd}^k \Delta^{n+1}$, a polynomial in $f_\pi \in B^{\Delta^k}$ is attached;

◇ if $f_\pi, f_{\pi'}$ are two polynomials attached to paths π, π' in $\mathfrak{sd}^k\Delta^{n+1}$ having a non-trivial common subpath π'', then restriction of $f_\pi, f_{\pi'}$ to π'' by means of face operators of B^Δ equals $f_{\pi''}$.

The desired algorithm of an algebraic homotopy from $\partial_j\alpha$ to $\partial_i\alpha$ consists of the following steps (we shall sometimes regard $\partial_j\alpha, \partial_i\alpha$ as posets uniquely associated to them):

Step 0: The poset $A_0 := \partial_j\alpha$, the vertex $w_0 := i$, and the poset $(\mathfrak{sd}^k\Delta^{n+1})_0 := \mathfrak{sd}^k\Delta^{n+1}$.

Step 1: Suppose we have found a subposet A_ℓ, $\ell \geqslant 0$, such that the vertices of the subposet $\partial_j\alpha \cap \partial_i\alpha$ are in A_ℓ, A_ℓ is isomorphic to A_0 by an isomorphism of posets leaving the vertices of $\partial_j\alpha \cap \partial_i\alpha$ unchanged, and A_ℓ has a unique vertex w_ℓ on the edge between vertices i and j. If $A_\ell = (\mathfrak{sd}^k\Delta^{n+1})_\ell = \partial_i\alpha$, in which case $w_\ell = j$, we stop the process. Otherwise, we can find a vertex $v \in A_\ell \setminus (\partial_j\alpha \cap \partial_i\alpha)$, a subposet $A_{\ell+1}$ having the same vertices as A_ℓ except a unique vertex $v' \in A_{\ell+1} \setminus (\partial_j\alpha \cap \partial_i\alpha)$, and a path $v \to v'$ or a path $v' \to v$ of length 1. Having chosen $A_{\ell+1}$, we set $(\mathfrak{sd}^k\Delta^{n+1})_{\ell+1} = (\mathfrak{sd}^k\Delta^{n+1})_\ell \setminus \{v\}$. Notice that $w_{\ell+1} \in A_{\ell+1}$ is either v' or w_ℓ.

Step 2: If we have a path $v' \to v$ from Step 1, then there are a unique $i \in \{0, 1, \ldots, n\}$ and a polynomial homotopy $H_\ell : B^{sd^k\Delta^{n+1}} \to B^{sd^k\Delta^n}[x]$ such that:

- $\partial_x^0 H_\ell$ equals the homomorphism $B^{sd^k\Delta^{n+1}} \to B^{|A_\ell|}$, induced by the inclusion $A_\ell \hookrightarrow \mathfrak{sd}^k\Delta^{n+1}$, and $\partial_x^1 H_\ell$ equals the homomorphism $B^{sd^k\Delta^{n+1}} \to B^{|A_{\ell+1}|}$;
- H_ℓ factors as

$$B^{sd^k\Delta^{n+1}} \to B^{|E_\ell|} \xrightarrow{h_\ell} B^{sd^k\Delta^n}[x],$$

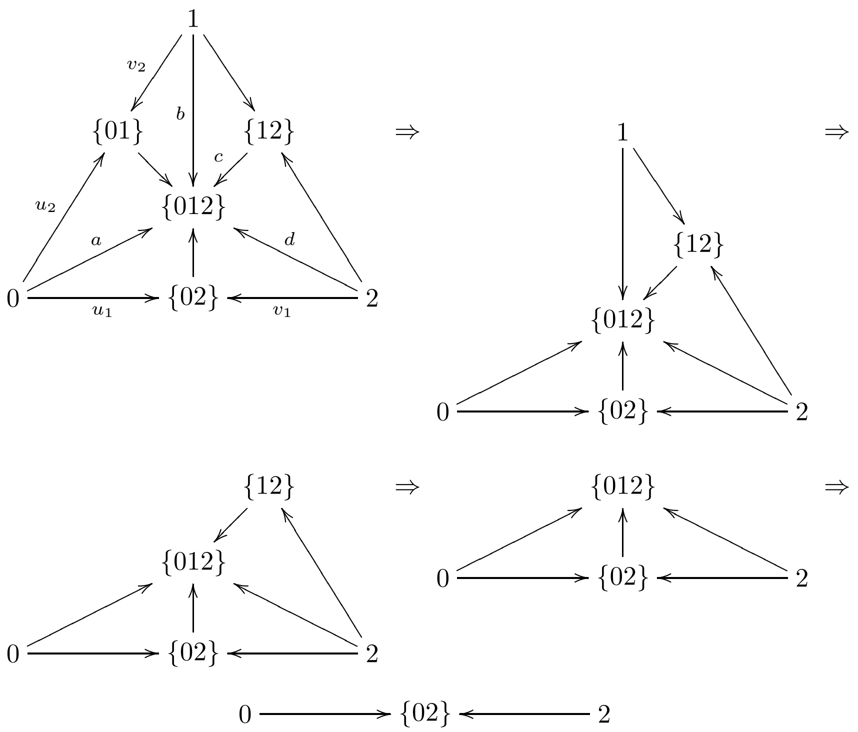
where E_ℓ is the subposet of $(\mathfrak{sd}^k\Delta^{n+1})_\ell$ whose vertices are those of $A_\ell \cup A_{\ell+1}$ and the left homomorphism is induced by the inclusion $E_\ell \hookrightarrow \mathfrak{sd}^k\Delta^{n+1}$;
- h_ℓ is a fibre product of homomorphisms of the form $i : B^{\Delta^n} \hookrightarrow B^{\Delta^n}[x]$ or $\varphi_{i,i+1} : B^{\Delta^{n+1}} \to B^{\Delta^n}[x]$.

If we have a path $v \to v'$ of Step 1, then H_ℓ has the same properties as above except that h_ℓ is a fibre product of homomorphisms of the form $i : B^{\Delta^n} \hookrightarrow B^{\Delta^n}[x]$ or $\varphi_{i+1,i} : B^{\Delta^{n+1}} \to B^{\Delta^n}[x]$.

Step 3: Apply Step 1 to the pair $(A_{\ell+1}, (\mathfrak{sd}^k\Delta^{n+1})_{\ell+1})$.

The algorithm stops in finitely many steps. The desired polynomial homotopy is given by the collection $\{H_\ell\}_\ell$. The homotopy pulls the barycenter of $\partial_j\alpha$ towards the barycenter of α. Also it pulls the vertex i towards j by means of $\{w_\ell\}_\ell$. Let us illustrate the algorithm by considering for simplicity the algebra

$B^{\mathrm{sd}^1\Delta^2}$, which is glued out of six polynomials in B^{Δ^2}.

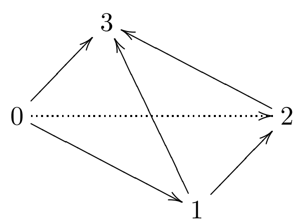


The picture depicts each poset $(\mathfrak{sd}^1\Delta^2)_\ell$. It also says that

$$\begin{aligned}\partial_2\alpha = (0 \xrightarrow{u_2} \{01\} \xleftarrow{v_2} 1) &\overset{H_0}{\sim} (0 \xrightarrow{a} \{012\} \xleftarrow{b} 1, w_1 = 1, v' = \{012\})\\ &\overset{H_1}{\sim} (0 \xrightarrow{a} \{012\} \xleftarrow{c} \{12\}, w_2 = v' = \{12\})\\ &\overset{H_2}{\sim} (0 \xrightarrow{a} \{012\} \xleftarrow{d} 2, w_3 = v' = 2)\\ &\overset{H_3}{\sim} (0 \xrightarrow{u_1} \{02\} \xleftarrow{v_1} 2, w_4 = 2, v' = \{02\}) = \partial_1\alpha\end{aligned}$$

and $\{h_0 = (\varphi_{2,1}, \varphi_{2,1}), E_0 = \langle 0, \{01\}, 1, \{012\}\rangle\}$, $\{h_1 = (i, \varphi_{1,0}), E_1 = \langle 0, \{012\}, 1, \{12\}\rangle\}$, $\{h_2 = (i, \varphi_{1,2}), E_2 = \langle 0, \{012\}, \{12\}, 2\rangle\}$, $\{h_3 = (\varphi_{1,2}, \varphi_{1,2}), E_2 = \langle 0, \{012\}, \{02\}, 2\rangle\}$.

Another example is for the algebra $B^{\mathrm{sd}^1\Delta^3}$, which is glued out of 24 polynomials in B^{Δ^3}. Consider a tetrahedron labeled with $0, 1, 2, 3$

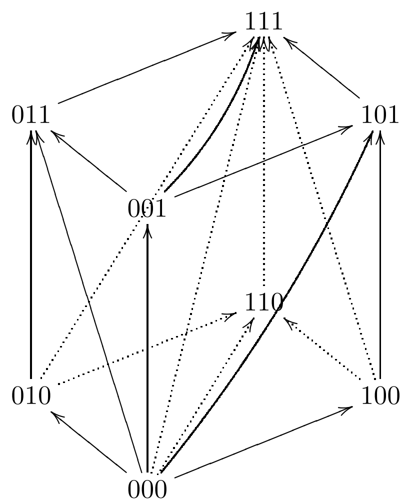


The polynomial homotopy from $\partial_2\alpha$ to $\partial_3\alpha$ is encoded by the following data of the algorithm:

$\diamond$ $v = \{013\}$ and $v' = \{0123\}$;
$\diamond$ $v = \{13\}$ and $v' = \{123\}$;
$\diamond$ $v = \{03\}$ and $v' = \{023\}$;
$\diamond$ $v = 3$ and $v' = \{23\}$;
$\diamond$ $v = \{23\}$ and $v' = 2$;
$\diamond$ $v = \{123\}$ and $v' = \{12\}$;
$\diamond$ $v = \{023\}$ and $v' = \{02\}$
$\diamond$ finally, $v = \{0123\}$ and $v' = \{012\}$.

(2). Note that this assertion follows from the particular case when f is the identity map of its codomain. Nevertheless we shall prove the original statement. The cube I^{n+1} is glued out of $(n+1)!$ simplices of dimension $n+1$. Its vertices can be labeled with $(n+1)$-tuples of numbers which equal either zero or one. The number of vertices equals 2^{n+1}. A homomorphism $\alpha : A \to B^{I^{n+1}}$ is glued out of $(n+1)!$ homomorphisms $\alpha_i : A \to B^{\Delta^{n+1}}$. The set of vertices $V_{d_0\alpha}$ of the face $d_0\alpha$ consists of those $(n+1)$-tuples whose last coordinate equals 1, and the set of vertices $V_{d_1\alpha}$ of the face $d_1\alpha$ consists of those $(n+1)$-tuples whose last coordinate equals 0. The desired algebraic homotopy from $d_0\alpha$ to $d_1\alpha$ is constructed in the following way. We first construct an algebraic homotopy H_0 from $f_0 := d_0\alpha$ to a homomorphism $f_1 : A \to B^{I^n}$. The set of vertices V_1 for f_1 equals $(V_{d_0\alpha} \setminus \{00\ldots01\}) \cup \{00\ldots0\}$. In other words, we pull $\{00\ldots01\}$ towards $\{00\ldots0\}$. The number of $(n+1)$-simplices having vertices from $V_{d_0\alpha} \cup \{00\ldots0\}$ equals $n!$. Let S be the set of such simplices. If $\alpha_i : A \to B^{\Delta^{n+1}}$ is in S, then the result is an algebraic homotopy $\varphi_{0,1}$ defined in (1) from $\partial_0\alpha_i$ to $\partial_1\alpha_i$. The homotopy H_0 at each α_i, $i \leqslant n!$, is $\varphi_{0,1}$. Next one constructs an algebraic homotopy H_1 from f_1 to $f_2 : A \to B^{I^n}$. The set of vertices V_2 of f_2 equals $(V_1 \setminus \{10\ldots01\}) \cup \{10\ldots0\}$. In other word, we pull $\{10\ldots01\}$ towards $\{10\ldots0\}$. The homotopy H_1 at each simplex is either $\varphi_{1,2}$ or id. One repeats this procedure 2^n times. The last step is to pull $(11\ldots11)$ towards $(11\ldots10)$ resulting a polynomial homotopy H_{2^n-1} which is $\varphi_{n,n+1}$ at each simplex. Clearly, if there are boundary conditions as in (2) then the algebraic homotopy behaves on the boundary in a consistent way.
In the case $\alpha : A \to B^{\mathrm{sd}^m I^{n+1}}$, $m > 0$, the desired polynomial homotopy is constructed in a similar way (we should also use the algorithm of (1)).

Let us illustrate the algorithm by considering for simplicity the case $\alpha : A \to B^{I^3}$. Such a map is glued out of six homomorphisms $\alpha_i : A \to B^{\Delta^3}$, $i = 1, \dots, 6$.



The desired algebraic homotopy from $d_0\alpha$ to $d_1\alpha$ is arranged as follows. We first pull (001) towards (000) resulting a polynomial homotopy H_0 from $d_0\alpha$, which is labeled by $\{(001), (101), (011), (111)\}$, to the square labeled by $\{(000), (101), (011), (111)\}$. This step is a result of the algebraic homotopy $\varphi_{0,1}$ described in (1) corresponding to two glued tetrahedra having vertices $\{(000), (001), (011), (111)\}$ and $\{(000), (001), (101), (111)\}$ respectively. So $H_0 = (\varphi_{0,1}, \varphi_{0,1})$. Next we pull (101) towards (100) resulting a polynomial homotopy H_1 from the square labeled by $\{(000), (101), (011), (111)\}$ to the square labeled by $\{(000), (100), (011), (111)\}$. So $H_1 = (\varphi_{1,2}, \mathrm{id})$. The next step is to pull (011) towards (010) resulting a polynomial homotopy H_2 from the square labeled by $\{(000), (100), (011), (111)\}$ to the square labeled by $\{(000), (100), (010), (111)\}$. So $H_2 = (\mathrm{id}, \varphi_{1,2})$. And finally one pulls (111) towards (110) resulting a polynomial homotopy H_3 from the square labeled by $\{(000), (100), (010), (111)\}$ to the square labeled by $\{(000), (100), (010), (110)\}$. In this case $H_3 = (\varphi_{2,3}, \varphi_{2,3})$.
(3) We first want to prove the following statement.

Hauptsublemma. *$B^{\mathrm{sd}^m I^{n+1}}$ (respectively $\widetilde{B}_m^{\mathfrak{S}^n}$) is a path space for $B^{\mathrm{sd}^m I^n}$ (respectively $B_m^{\mathfrak{S}^n}$) in the Brown category $(\Re, \mathfrak{F}, \mathfrak{W}_{\min})$.*

Proof. By (2) the maps

$$d_0, d_1 : B^{\mathrm{sd}^m I^{n+1}} \to B^{\mathrm{sd}^m I^n}$$

are algebraically homotopic, hence equal in the category $\mathcal{H}(\Re)$.
The map

$$(d_0, d_1) : B^{\mathrm{sd}^m I^{n+1}} \to B^{\mathrm{sd}^m I^n} \times B^{\mathrm{sd}^m I^n}$$

is a k-linear split homomorphism, hence a fibration. A splitting is defined as

$$(b_1,b_2)\in B^{\mathrm{sd}^m I^n}\times B^{\mathrm{sd}^m I^n}\mapsto \imath(b_1)\cdot(1-\mathbf{t})+\imath(b_2)\cdot\mathbf{t}\in B^{\mathrm{sd}^m I^{n+1}},$$

where $\mathbf{t}\in k^{\mathrm{sd}^m I^{n+1}}$ is defined on page 1220 and $\imath:\mathbb{B}^{\Delta}(I^n)\to(\mathbb{B}^{\Delta}(I^n))^{\Delta^1}$ is the natural inclusion. There is a commutative diagram

$$\begin{tikzcd} B^{\mathrm{sd}^m I^n} \arrow[rr, "diag"] \arrow[dr, "s"'] & & B^{\mathrm{sd}^m I^n}\times B^{\mathrm{sd}^m I^n} \\ & B^{\mathrm{sd}^m I^{n+1}}, \arrow[ur, "{(d_0,d_1)}"'] & \end{tikzcd}$$

where s is induced by projection of I^{n+1} onto I^n which forgets the last coordinate. To show that $B^{\mathrm{sd}^m I^{n+1}}$ is a path space, we shall check that s is an I-weak equivalence. We have that $d_0 s=\mathrm{id}$. We want to check that sd_0 is algebraically homotopic to id.
In the proof of Proposition 4.1 we have constructed a simplicial map

$$\lambda:I^2\to I.$$

It induces a simplicial homotopy between sd_0 and id

$$\lambda^*:B^{\mathrm{sd}^m I^{n+1}}\to B^{\mathrm{sd}^m I^{n+2}}.$$

By (2) these are algebraically homotopic. We conclude that s,d_0 are I-weak equivalences, and hence so is d_1. The statement for $\widetilde{B}_m^{\mathfrak{S}^n}$ is verified in a similar way. $\square$

The algebra $B':=(B^{\mathrm{sd}^m I^n})^{\mathrm{sd}^k\Delta^1}$ is another path object of $B^{\mathrm{sd}^m I^n}$, and so there is a commutative diagram

$$\begin{tikzcd} B^{\mathrm{sd}^m I^n} \arrow[rr, "diag"] \arrow[dr, "s'"'] & & B^{\mathrm{sd}^m I^n}\times B^{\mathrm{sd}^m I^n} \\ & B', \arrow[ur, "{(d_0',d_1')}"'] & \end{tikzcd}$$

where s is an I-weak equivalence and (d_0',d_1') is a fibration. Let X be the fibre product for

$$B^{\mathrm{sd}^m I^{n+1}}\xrightarrow{(d_0,d_1)}B^{\mathrm{sd}^m I^n}\times B^{\mathrm{sd}^m I^n}\xleftarrow{(d_0',d_1')}B'.$$

Then (s,s') induces a unique map $q:B^{\mathrm{sd}^m I^n}\to X$ such that $pr_1\circ q=s$ and $pr_2\circ q=s'$. We can factor q as

$$B^{\mathrm{sd}^m I^n}\xrightarrow{s''}B''\xrightarrow{p}X,$$

where s'' is an I-weak equivalence and p is a fibration. It follows that $u:=pr_2\circ p$ and $v:=pr_1\circ p$ are I-trivial fibrations, because $vs''=s, us''=s'$. It follows that there is a commutative diagram

$$\begin{tikzcd} B^{\mathrm{sd}^m I^n} \arrow[rr, "diag"] \arrow[dr, "s''"'] & & B^{\mathrm{sd}^m I^n} \times B^{\mathrm{sd}^m I^n} \\ & B'' \arrow[ur, "(d_0'', d_1'')"'] & \end{tikzcd}$$

with $(d_0'', d_1'') := (d_0, d_1) \circ v = (d_0', d_1') \circ u$, and so the algebra B'' is a path object of $B^{\mathrm{sd}^m I^n}$.

Now let us consider a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "h'"] \arrow[d, "g"'] & B'' \arrow[r, "v"] \arrow[d, "u"'] & B^{\mathrm{sd}^m I^{n+1}} \arrow[d, "(d_0,d_1)"] \\ A \arrow[r, "h"'] & B' \arrow[r, "(d_0',d_1')"'] & B^{\mathrm{sd}^m I^n} \times B^{\mathrm{sd}^m I^n} \end{tikzcd}$$

with the left square cartesian. The desired homomorphism $H : A' \to B^{\mathrm{sd}^m I^{n+1}}$ is then defined as vh'. The homomorphism $H : A' \to \widetilde{B}_m^{\mathfrak{S}^n}$ is constructed in a similar way. $\square$

The proof of the Hauptlemma also applies to showing that for any homomorphism $h : A \to B^{\mathrm{sd}^m \Delta^1 \times \Delta^n}$ the induced maps $d_0 h, d_1 h : A \to B^{\mathrm{sd}^m \Delta^n}$ are algebraically homotopic. If $m = 0$ then the homotopy is constructed in n steps similar to that described above for cubes I^n (each step is obtained by applying the polynomial homotopy $\varphi_{i,j}$).

We can use the homotopy to describe explicitly a polynomial contraction of an algebra B^{Δ^n} to B. Precisely, consider the maps $s : B \to B^{\Delta^n}$, $\delta : B^{\Delta^n} \to B$ induced by the unique map $[n] \to [0]$ and the map $[0] \to [n]$ taking 0 to n. Then $\delta s = 1_B$ and $s\delta$ is polynomially homotopic to the identity map of B^{Δ^n}. The homotopy is constructed by lifting the simplicial homotopy that contracts Δ^n to its last vertex. This simplicial homotopy is given by a simplicial map

$$\Delta^1 \times \Delta^n \xrightarrow{h} \Delta^n$$

that takes $(v : [m] \to [1], u : [m] \to [n])$ to $\bar{u} : [m] \to [n]$, where $\bar{u}$ is defined as the composite

$$[m] \xrightarrow{(u,v)} [n] \times [1] \xrightarrow{w} [n]$$

and where $w(j, 0) = j$ and $w(j, 1) = n$.

We have a homomorphism

$$h^* : B^{\Delta^n} \to B^{\Delta^1 \times \Delta^n}$$

which is induced by h. Then $d_0 h^* = 1$ is polynomially homotopic to $d_1 h^* = s\delta$.

If a homomorphism $f : A' \to A$ is homotopic to $g : A' \to A$ by means of a homomorphism $h : A' \to A[x]$ then $J(f)$ is homotopic to $J(g)$. Indeed, consider a commutative diagram of algebras

$$\begin{tikzcd}
JA' \arrow[r] \arrow[d, "J(h)"'] & TA' \arrow[r] \arrow[d, "T(h)"'] & A' \arrow[d, "h"] \\
J(A[x]) \arrow[r] \arrow[d, "\gamma"'] & T(A[x]) \arrow[r] \arrow[d] & A[x] \arrow[d, equal] \\
(JA)[x] \arrow[r] \arrow[d, "\partial_x^{0;1}"'] & (TA)[x] \arrow[r] \arrow[d, "\partial_x^{0;1}"'] & A[x] \arrow[d, "\partial_x^{0;1}"] \\
JA \arrow[r] & TA \arrow[r] & A.
\end{tikzcd}$$

Then $\gamma \circ J(h)$ yields the required homotopy between $J(f)$ and $J(g)$.
Let $A, B \in \Re$ and $n \geqslant 0$. Part (i) of the Hauptlemma implies that there is a map

$$\pi_0(\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}(\Omega^n))) \to [J^n A, B^{\mathfrak{S}^n}]$$

which is consistent with the colimit maps

$$\varsigma : \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}(\Omega^n)) \to \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1} A, \mathbb{B}(\Omega^{n+1}))$$

defined by (1) and $\sigma : [J^n A, B^{\mathfrak{S}^n}] \to [J^{n+1} A, B^{\mathfrak{S}^{n+1}}]$ which is defined like ς. So we get a map

$$\Gamma : \mathcal{K}_0(A, B) \to \mathrm{colim}_n [J^n A, B^{\mathfrak{S}^n}].$$

COMPARISON THEOREM A. *The map* $\Gamma : \mathcal{K}_0(A, B) \to \mathrm{colim}_n [J^n A, B^{\mathfrak{S}^n}]$ *is an isomorphism.*

Proof. It is obvious that

$$\pi_0(\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}(\Omega^n))) \to [J^n A, B^{\mathfrak{S}^n}]$$

is surjective for each $n \geqslant 0$, and hence so is Γ. Suppose $f_0, f_1 : J^n A \to B^{\mathfrak{S}^n}$ are polynomially homotopic by means of h. By the Hauptlemma there are a homomorphism $g : A' \to J^n A$, which is a fibre product of an I-trivial fibration along h, and hence $g \in \mathfrak{W}_{\min}$, and a homomorphism $H : A' \to \widetilde{B}^{\mathfrak{S}^n}$ such that $d_0 H = f_0 g$ and $d_1 H = f_1 g$. Similar to the proof of Excision Theorem B one can construct a small admissible category of algebras $\Re'$ such that it contains all algebras $\{A', J^n A, B^{\mathrm{sd}^m I^n}\}_{m,n}$ we work with and such that g is a quasi-isomorphism of $\Re'$.
By Theorem 6.7 the induced map of graded abelian groups

$$g^* : \mathcal{K}_*(\Re')(J^n A, B) \to \mathcal{K}_*(\Re')(A', B)$$

is an isomorphism. We have that g^* takes $f_0, f_1 \in \mathcal{K}_n(\Re')(J^n A, B)$ to the same element in $\mathcal{K}_n(\Re')(A', B)$, and so $f_0 = f_1$. We see that Γ is also injective, hence it is an isomorphism. $\square$

Corollary 7.1. *The homotopy groups of* $\mathbb{K}(A,B)$ *are computed as follows:*

$$\mathbb{K}_m(A,B) \cong \begin{cases} \operatorname{colim}_n[J^n A, (\Omega^m B)^{\mathfrak{S}^n}], & m \geqslant 0 \\ \operatorname{colim}_n[J^{-m+n} A, B^{\mathfrak{S}^n}], & m < 0 \end{cases}$$

Proof. This follows from Corollary 4.3 and the preceding theorem. □

8. Comparison Theorem B

In this section $\Re$ is supposed to be T-closed. Let $\mathfrak{W}$ be a class of weak equivalences containing homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathfrak{W})$ is a Brown category.

Definition. The *left derived category* $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ of $\Re$ with respect to $(\mathfrak{F}, \mathfrak{W})$ is the category obtained from $\Re$ by inverting the weak equivalences.

By [9] the family of weak equivalences in the category $\mathcal{H}\Re$ admits a calculus of right fractions. The left derived category $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ (possibly "large") is obtained from $\mathcal{H}\Re$ by inverting the weak equivalences. The left derived category $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ is left triangulated (see [8, 9] for details) with Ω a loop functor on it.

There is a general method of stabilizing Ω (see Heller [17]) and producing a triangulated (possibly "large") category $D(\Re, \mathfrak{F}, \mathfrak{W})$ from the left triangulated structure on $D^-(\Re, \mathfrak{F}, \mathfrak{W})$.

An object of $D(\Re, \mathfrak{F}, \mathfrak{W})$ is a pair (A, m) with $A \in D^-(\Re, \mathfrak{F}, \mathfrak{W})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leqslant k\}$. The morphisms between (A, m) and $(B, n) \in D(\Re, \mathfrak{F}, \mathfrak{W})$ are defined by

$$D(\Re, \mathfrak{F}, \mathfrak{W})[(A,m),(B,n)] := \operatorname{colim}_{k \in I_{m,n}} D^-(\Re, \mathfrak{F}, \mathfrak{W})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$

Morphisms of $D(\Re, \mathfrak{F}, \mathfrak{W})$ are composed in the obvious fashion. We define the *loop* automorphism on $D(\Re, \mathfrak{F}, \mathfrak{W})$ by $\Omega(A, m) := (A, m-1)$. There is a natural functor $S : D^-(\Re, \mathfrak{F}, \mathfrak{W}) \to D(\Re, \mathfrak{F}, \mathfrak{W})$ defined by $A \longmapsto (A, 0)$.

$D(\Re, \mathfrak{F}, \mathfrak{W})$ is an additive category [8, 9]. We define a triangulation $\mathcal{T}r(\Re, \mathfrak{F}, \mathfrak{W})$ of the pair $(D(\Re, \mathfrak{F}, \mathfrak{W}), \Omega)$ as follows. A sequence

$$\Omega(A, l) \to (C, n) \to (B, m) \to (A, l)$$

belongs to $\mathcal{T}r(\Re, \mathfrak{F}, \mathfrak{W})$ if there is an even integer k and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \to \Omega^{k-n}(C) \to \Omega^{k-m}(B) \to \Omega^{k-l}(A)$ in $D^-(\Re, \mathfrak{F}, \mathfrak{W})$. Then the functor S takes left triangles in $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ to triangles in $D(\Re, \mathfrak{F}, \mathfrak{W})$. By [8, 9] $\mathcal{T}r(\Re, \mathfrak{F}, \mathfrak{W})$ is a triangulation of $D(\Re, \mathfrak{F}, \mathfrak{W})$ in the classical sense of Verdier [29].

Let $\mathcal{E}$ be the class of all $\mathfrak{F}$-extensions of k-algebras

$$(E) : A \to B \to C. \tag{9}$$

Definition. Following Cortiñas–Thom [3] a *($\mathfrak{F}$-)excisive homology theory* on $\Re$ with values in a triangulated category $(\mathcal{T}, \Omega)$ consists of a functor $X : \Re \to \mathcal{T}$, together with a collection $\{\partial_E : E \in \mathcal{E}\}$ of maps $\partial_E^X = \partial_E \in \mathcal{T}(\Omega X(C), X(A))$. The maps ∂_E are to satisfy the following requirements.

(1) For all $E \in \mathcal{E}$ as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathcal{T}$.

(2) If

$$\begin{tikzcd} (E): & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ (E'): & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

is a map of extensions, then the following diagram commutes

$$\begin{tikzcd} \Omega X(C) \arrow[r, "\partial_E"] \arrow[d, "\Omega X(\gamma)"'] & X(A) \arrow[d, "X(\alpha)"] \\ \Omega X(C') \arrow[r, "\partial_{E'}"'] & X(A). \end{tikzcd}$$

We say that the functor $X : \Re \to \mathcal{T}$ is *homotopy invariant* if it maps homotopic homomomorphisms to equal maps, or equivalently, if for every $A \in \mathrm{Alg}_k$, X maps the inclusion $A \subset A[t]$ to an isomorphism.

Denote by $\mathfrak{W}_\triangle$ the class of homomorphisms f such that $X(f)$ is an isomorphism for any excisive, homotopy invariant homology theory $X : \Re \to \mathcal{T}$. We shall refer to the maps from $\mathfrak{W}_\triangle$ as *stable weak equivalences*. The triple $(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ is a Brown category. In what follows we shall write $D^-(\Re, \mathfrak{F})$ and $D(\Re, \mathfrak{F})$ to denote $D^-(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ and $D(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ respectively, dropping $\mathfrak{W}_\triangle$ from notation.

In this section we prove the following theorem.

COMPARISON THEOREM B. *For any algebras $A, B \in \Re$ there is an isomorphism of $\mathbb{Z}$-graded abelian groups*

$$\mathbb{K}_*(A, B) \cong D(\Re, \mathfrak{F})_*(A, B) = \bigoplus_{n \in \mathbb{Z}} D(\Re, \mathfrak{F})(A, \Omega^n B),$$

functorial both in A and in B.

The graded isomorphism consists of a zig-zag of isomorphisms each of which is constructed below.

COROLLARY 8.1. *$D(\Re, \mathfrak{F})$ is a category with small Hom-sets.*

DEFINITION. Let $\Re$ be a small T-closed admissible category of algebras. A homomorphism $A \to B$ in $\Re$ is said to be a *stable $\mathfrak{F}$-quasi-isomorphism* or just a *stable quasi-isomorphism* if the map $\Omega^n A \to \Omega^n B$ is a quasi-isomorphism for some $n \geqslant 0$. The class of quasi-isomorphisms will be denoted by $\mathfrak{W}_{qis}$. By [8] the triple $(\Re, \mathfrak{F}, \mathfrak{W}_{qis})$ is a Brown category.

Consider the ind-algebra $(B^{\mathfrak{S}^n},\mathbb{Z}_{\geqslant 0})$ with each $B_k^{\mathfrak{S}^n}$, $k \in \mathbb{Z}_{\geqslant 0}$, being $\ker(B^{\mathrm{sd}^k I^n} \to B^{\partial(\mathrm{sd}^k I^n)})$, that is $B^{\mathfrak{S}^n}$ is the underlying ind-algebra of 0-simplices of $\mathbb{B}(\Omega^n)$. We shall denote by $B^{\mathcal{S}^n}$ the algebra $B_0^{\mathfrak{S}^n}$. Notice that $B^{\mathcal{S}^1} = \Omega B$. There is a sequence of maps

$$\mathrm{Hom}_{\mathrm{Alg}_k}(B,B) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k}(JB,B_k^{\mathfrak{S}^1}) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k}(J^2B,B_k^{\mathfrak{S}^2}) \xrightarrow{\varsigma} \cdots$$

One sets $1_B^{n,k} := \varsigma^n(1_B)$.
Recall that $\mathfrak{W}_{\min}$ is the least collection of weak equivalences containing $A \to A[x]$, $A \in \Re$, such that the triple $(\Re,\mathfrak{F},\mathfrak{W}_{\min})$ is a Brown category.

Lemma 8.2. *Let $\Re$ be a T-closed admissible category of algebras and $B \in \Re$. Then for any $n \geqslant 0$ all morphisms of the sequence*

$$B_0^{\mathfrak{S}^n} \to B_1^{\mathfrak{S}^n} \to B_2^{\mathfrak{S}^n} \to \cdots$$

belong to $\mathfrak{W}_{\min}$.

Proof. Recall that the simplicial ind-algebra $P\mathbb{B}^{\Delta}(\Omega^n)$ is indexed over $\mathbb{Z}_{\geqslant 0}$ and defined as $\ker((\mathbb{B}^{\Delta}(\Omega^n))^I \xrightarrow{d_0} \mathbb{B}^{\Delta}(\Omega^n))$. The proof of the Hauptsublemma shows that on the level of 0-simplices d_0 is an I-trivial fibration. Its kernel consists of 0-simplices of $P\mathbb{B}^{\Delta}(\Omega^n)$ and whose underlying sequence of algebras is denoted by

$$PB_0^{\mathfrak{S}^n} \to PB_1^{\mathfrak{S}^n} \to PB_2^{\mathfrak{S}^n} \to \cdots$$

For each algebra of the sequence $PB_k^{\mathfrak{S}^n}$ one has $(0 \to PB_k^{\mathfrak{S}^n}) \in \mathfrak{W}_{\min}$, because it is the kernel of an I-trivial fibration.
The assertion is obvious for $n = 0$. We have a commutative diagram of extensions for all $n \geqslant 1, k \geqslant 0$

$$\begin{tikzcd}
B_k^{\mathfrak{S}^n} \arrow[r] \arrow[d] & PB_k^{\mathfrak{S}^{n-1}} \arrow[r] \arrow[d] & B_k^{\mathfrak{S}^{n-1}} \arrow[d] \\
B_{k+1}^{\mathfrak{S}^n} \arrow[r] & PB_{k+1}^{\mathfrak{S}^{n-1}} \arrow[r] & B_{k+1}^{\mathfrak{S}^{n-1}}
\end{tikzcd}$$

with the right and the middle arrows belonging to $\mathfrak{W}_{\min}$ by induction, hence so is the left one. □

Lemma 8.3. *Let $\Re$ be a T-closed admissible category of algebras and $B \in \Re$. Then each $1_B^{n,k}$, $n,k \geqslant 0$, belongs to $\mathfrak{W}_{\min}$.*

Proof. We fix k. The identity map $1_B = 1_B^{0,k}$ belongs to $\mathfrak{W}_{\min}$. The map $1_B^{1,k}$ is the classifying map $\xi_v : JB \to B_k^{\mathfrak{S}^1}$, which is in $\mathfrak{W}_{\min}$. Suppose $1_B^{n-1,k}$, $n > 1$, belongs to $\mathfrak{W}_{\min}$. Then $1_B^{n,k} = \xi_v J(1_B^{n-1,k})$, where $\xi_v : J(B_k^{\mathfrak{S}^{n-1}}) \to B_k^{\mathfrak{S}^n}$ is in $\mathfrak{W}_{\min}$. Since J respects maps from $\mathfrak{W}_{\min}$, then $1_B^{n,k}$ is in $\mathfrak{W}_{\min}$. □

Lemma 8.4. *The following conditions are equivalent for a homomorphism $f : A \to B$ in $\Re$:*

(1) *f is a stable quasi-isomorphism;*

(2) $J^n(f) : J^n A \to J^n B$ *is a quasi-isomorphism for some* $n \geqslant 0$*;*
(3) *for any* $k \geqslant 0$ *there is a* $n \geqslant 0$ *such that* $f^{\mathfrak{S}^n} : A_k^{\mathfrak{S}^n} \to B_k^{\mathfrak{S}^n}$ *is a quasi-isomorphism.*

Proof. (1) $\Leftrightarrow$ (2). Consider a commutative diagram of extensions

$$\begin{tikzcd} JA \arrow[r] \arrow[d, "\rho_A"'] & TA \arrow[r] \arrow[d] & A \arrow[d, equal] \\ \Omega A \arrow[r] & EA \arrow[r] & A, \end{tikzcd}$$

where TA, EA are contractible. It follows that ρ_A is a quasi-isomorphism. It is plainly functorial in A. Since J respects quasi-isomorphisms, it follows that there is a commutative diagram for any $n \geqslant 1$

$$\begin{tikzcd} J^n A \arrow[r] \arrow[d, "J^n(f)"'] & \Omega^n A \arrow[d, "\Omega^n(f)"] \\ J^n B \arrow[r] & \Omega^n B, \end{tikzcd}$$

in which the horizontal maps are quasi-isomorphisms. We see that $\Omega^n(f)$ is a quasi-isomorphism if and only if $J^n(f)$ is.
(2) $\Leftrightarrow$ (3). There is a commutative diagram of extensions for all $n \geqslant 1, k \geqslant 0$

$$\begin{tikzcd} J(A_k^{\mathfrak{S}^{n-1}}) \arrow[r] \arrow[d] & T(A_k^{\mathfrak{S}^{n-1}}) \arrow[r] \arrow[d] & A_k^{\mathfrak{S}^{n-1}} \arrow[d, equal] \\ A_k^{\mathfrak{S}^n} \arrow[r] & PA_k^{\mathfrak{S}^{n-1}} \arrow[r] & A_k^{\mathfrak{S}^{n-1}} \end{tikzcd}$$

in which the right and the middle arrows are quasi-isomorphisms, hence so is the left one. The middle arrow is actually quasi-isomorphic to zero. Since J respects quasi-isomorphisms, we get a chain of quasi-isomorphisms

$$J^n A \to J^{n-1}(A_k^{\mathfrak{S}^1}) \to \cdots \to J(A_k^{\mathfrak{S}^{n-1}}) \to A_k^{\mathfrak{S}^n},$$

functorial in A. It follows that there is a commutative diagram for any $n \geqslant 1$

$$\begin{tikzcd} J^n A \arrow[r] \arrow[d, "J^n(f)"'] & A_k^{\mathfrak{S}^n} \arrow[d, "f^{\mathfrak{S}^n}"] \\ J^n B \arrow[r] & B_k^{\mathfrak{S}^n}, \end{tikzcd}$$

in which the horizontal maps are quasi-isomorphisms. We see that $f_k^{\mathfrak{S}^n}$ is a quasi-isomorphism if and only if $J^n(f)$ is. $\square$

We call a homomorphism $f : A \to B$ in a T-closed category $\Re$ a *$\mathcal{K}$-equivalence* if the induced map $\mathcal{K}(C, A) \to \mathcal{K}(C, B)$ is a weak equivalence of spaces.

PROPOSITION 8.5. *Let $\Re$ be a small T-closed admissible category of algebras. A homomorphism $t : A \to B$ in $\Re$ is a stable quasi-isomorphism if and only if it is a $\mathcal{K}$-equivalence.*

Proof. Suppose $t : A \to B$ is a stable quasi-isomorphism. Then $\Omega^n(t)$ is a quasi-isomorphism for some $n \geqslant 0$, and hence a $\mathcal{K}$-equivalence. For any algebra $C \in \Re$ the induced map

$$\mathcal{K}(J^nC, \Omega^nA) \to \mathcal{K}(J^nC, \Omega^nB)$$

is a weak equivalence of spaces. By Corollaries 4.3 and 5.2 the map

$$\Omega^n\mathcal{K}(J^nC, A) \to \Omega^n\mathcal{K}(J^nC, B)$$

is a weak equivalence, hence so is the map

$$t_* : \mathcal{K}(C, A) \to \mathcal{K}(C, B).$$

Thus t is a $\mathcal{K}$-equivalence.

Suppose now $t : A \to B$ is a $\mathcal{K}$-equivalence. Then the induced map

$$\mathcal{K}(B, A) \to \mathcal{K}(B, B)$$

is a homotopy equivalence of spaces. There are $k, n \geqslant 0$, a map $e : J^nB \to A_k^{\mathfrak{S}^n}$, and a sequence of maps

$$J^nB \xrightarrow{e} A_k^{\mathfrak{S}^n} \xrightarrow{t^{\mathfrak{S}^n}} B_k^{\mathfrak{S}^n}$$

such that $t^{\mathfrak{S}^n}e$ is simplicially homotopic to $1_B^{n,k}$. By the Hauptlemma $t^{\mathfrak{S}^n}e$ is polynomially homotopic to $1_B^{n,k}$. By Lemma 8.3 $1_B^{n,k}$ is a quasi-isomorphism. It follows that e is a right unit in the category $D^-(\Re, \mathfrak{F}, \mathfrak{W}_{qis})$. For every $m \geqslant 0$ one has

$$\varsigma^m(t^{\mathfrak{S}^n}e) = p \circ J^m(t^{\mathfrak{S}^n}) \circ J^m(e) \simeq 1_B^{n+m,k}, \tag{10}$$

where p is a quasi-isomorphism. By Lemma 8.3 $1_B^{n+m,k}$ is a quasi-isomorphism. It follows that $J^m(e)$ is a right unit in $D^-(\Re, \mathfrak{F}, \mathfrak{W}_{qis})$.

We claim that $t^{\mathfrak{S}^n}$ is a $\mathcal{K}$-equivalence. By assumption $t^{\mathfrak{S}^0} = t$ is a $\mathcal{K}$-equivalence. Suppose $t^{\mathfrak{S}^{n-1}}$ is a $\mathcal{K}$-equivalence for $n \geqslant 1$. There is a commutative diagram of extensions

$$\begin{tikzcd}
A_k^{\mathfrak{S}^n} \arrow[r] \arrow[d, "t^{\mathfrak{S}^n}"'] & PA_k^{\mathfrak{S}^{n-1}} \arrow[r] \arrow[d] & A_k^{\mathfrak{S}^{n-1}} \arrow[d, "t^{\mathfrak{S}^{n-1}}"] \\
B_k^{\mathfrak{S}^n} \arrow[r] & PB_k^{\mathfrak{S}^{n-1}} \arrow[r] & B_k^{\mathfrak{S}^{n-1}},
\end{tikzcd}$$

in which the right and the middle arrows are $\mathcal{K}$-equivalences by induction, hence so is the left one. The middle arrow is actually quasi-isomorphic to zero. We see that $t^{\mathfrak{S}^n}e$ is a $\mathcal{K}$-equivalence. The two out of three property implies e is a $\mathcal{K}$-equivalence. Therefore the induced map

$$e_* : \mathcal{K}(J^nA, J^nB) \to \mathcal{K}(J^nA, A_k^{\mathfrak{S}^n})$$

is a homotopy equivalence of spaces. Let $q = e_*^{-1}(1_A^{n,k}) : J^{n+m}A \to (J^nB)_l^{\mathfrak{S}^m}$; then $e^{\mathfrak{S}^m}q$ is simplicially homotopic to $\varsigma^m(1_A^{n,k})$. By the Hauptlemma $e^{\mathfrak{S}^m}q$ is polynomially homotopic to $\varsigma^m(1_A^{n,k})$. It follows from Lemma 8.3 that $\varsigma^m(1_A^{n,k})$ is a quasi-isomorphism. We see that $e^{\mathfrak{S}^m}$ is a left unit in $D^-(\Re, \mathfrak{F}, \mathfrak{W}_{qis})$. The

proof of Lemma 8.4 shows that $J^m(e)$ is quasi-isomorphic to $e^{\mathfrak{S}^m}$. Thus $J^m(e)$ is a left unit in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{qis})$.
By above $J^m(e)$ is also a right unit in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{qis})$, and so is an isomorphism in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{qis})$. Since the canonical functor $\Re\to D^-(\Re,\mathfrak{F},\mathfrak{W}_{qis})$ has the property that a homomorphism of algebras is a quasi-isomorphism if and only if its image in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{qis})$ is an isomorphism, we see that $J^m(e)$ is a quasi-isomorphism.
By (10) $J^m(t^{\mathfrak{S}^n})$ is a quasi-isomorphism, because so are p, $1_B^{n+m,k}$ and $J^m(e)$. Since J preserves quasi-isomorphisms, the proof of Lemma 8.4 shows that there is a commutative diagram

$$\begin{tikzcd} J^{n+m}A \arrow[r] \arrow[d, "J^{n+m}(t)"'] & J^m(A_k^{\mathfrak{S}^n}) \arrow[d, "J^m(t^{\mathfrak{S}^n})"] \\ J^{n+m}B \arrow[r] & J^m(B_k^{\mathfrak{S}^n}), \end{tikzcd}$$

in which the horizontal maps are quasi-isomorphisms. We see that $J^{n+m}(t)$ is a quasi-isomorphism, because so is $J^m(t^{\mathfrak{S}^n})$. So t is a stable quasi-isomorphism by Lemma 8.4 as required. □

The next result is an improvement of Theorem 6.7. It will also be useful when proving Comparison Theorem B.

Theorem 8.6. *Suppose $\Re$ is an admissible T-closed category of algebras and $u:A\to B$ is a $\mathcal{K}$-equivalence in $\Re$. Then the induced map*

$$u^*:\mathbb{K}(B,D)\to\mathbb{K}(A,D)$$

is a weak equivalence of spectra for any $D\in\Re$.

Proof. Similar to the proof of Excision Theorem B one can construct a small admissible T-closed full subcategory of algebras $\Re'$ such that it contains A,B,D. By assumption u is a $\mathcal{K}$-equivalence in $\Re'$, hence $J^n(u)$ is a quasi-isomorphism of $\Re'$ for some $n\geqslant 0$ by the preceding proposition and Lemma 8.4.
By Theorem 6.7 the induced map of spectra

$$(J^n(u))^*:\mathbb{K}(\Re')(J^nB,D)\to\mathbb{K}(\Re')(J^nA,D)$$

is a weak equivalence. Corollary 6.8 now implies the claim. □

Lemma 8.7. *Suppose $\Re$ is an admissible T-closed category of algebras. Then every stable weak equivalence in $\Re$ is a $\mathcal{K}$-equivalence.*

Proof. Using Theorem 5.3 for every $A\in\Re$ the map

$$\mathbb{K}(A,-):\Re\to\mathrm{Ho}(Sp)$$

with $\mathrm{Ho}(Sp)$ the homotopy category of spectra yields an excisive, homotopy invariant homology theory. Therefore it takes stable weak equivalence to isomorphisms in $\mathrm{Ho}(Sp)$. □

Given an ind-algebra $(B, J) \in \mathfrak{R}^{\mathrm{ind}}$ and $A \in \mathfrak{R}$, we set

$$D^-(\mathfrak{R}, \mathfrak{F})(A, B) = \operatorname{colim}_{j \in J} D^-(\mathfrak{R}, \mathfrak{F})(A, B_j).$$

Using the fact that J respects polynomial homotopy and stable weak equivalences, we can extend the map $\varsigma : \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, B^{\mathfrak{S}^n}) \to \operatorname{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, B^{\mathfrak{S}^{n+1}})$ to a functor

$$\sigma : D^-(\mathfrak{R}, \mathfrak{F})(A, B^{\mathfrak{S}^n}) \to D^-(\mathfrak{R}, \mathfrak{F})(JA, B^{\mathfrak{S}^{n+1}}).$$

The functor σ takes a map

$$\begin{tikzcd} & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & & B^{\mathfrak{S}^n} \end{tikzcd}$$

in $D^-(\mathfrak{R}, \mathfrak{F})(A, B^{\mathfrak{S}^n})$, where $s \in \mathfrak{W}_\triangle$, to the map

$$\begin{tikzcd} & JA' \arrow[dl, "J(s)"'] \arrow[dr, "\varsigma(f)"] & \\ JA & & B^{\mathfrak{S}^{n+1}} \end{tikzcd}.$$

Since J respects weak equivalences and homotopy, it follows that σ is well-defined.
The map $\Gamma : \mathcal{K}_0(A, B) \to \operatorname{colim}_n [J^n A, B^{\mathfrak{S}^n}]$ is an isomorphism by Comparison Theorem A. There is a natural map

$$\Gamma_1 : \operatorname{colim}_n [J^n A, B^{\mathfrak{S}^n}] \to \operatorname{colim}_n D^-(\mathfrak{R}, \mathfrak{F})(J^n A, B^{\mathfrak{S}^n}).$$

Lemma 8.8. Γ_1 *is an isomorphism, functorial in A and B.*

Proof. Suppose maps $f_0, f_1 : J^n A \to B^{\mathfrak{S}^n}$ are such that $\Gamma_1(f_0) = \Gamma_1(f_1)$. Using the Hauptlemma, we may choose n big enough to find a stable weak equivalence $t : A' \to J^n A$ such that $f_0 t$ is simplicially homotopic to $f_1 t$. By Lemma 8.7 t is a $\mathcal{K}$-equivalence of $\mathfrak{R}$. By Theorem 8.6 the induced map of graded abelian groups

$$t^* : \mathcal{K}_*(J^n A, B) \to \mathcal{K}_*(A', B)$$

is an isomorphism. We have that t^* takes $f_0, f_1 \in \mathcal{K}_n(J^n A, B)$ to the same element in $\mathcal{K}_n(A', B)$, and so $f_0 = f_1$. We see that Γ_1 is injective.
Consider a map

$$\begin{tikzcd} & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ J^n A & & B^{\mathfrak{S}^n} \end{tikzcd}$$

with $s \in \mathfrak{W}_\triangle$. By Lemma 8.7 s is a $\mathcal{K}$-equivalence of $\mathfrak{R}$. By Theorem 8.6 the induced map of abelian groups

$$s^* : \mathcal{K}_n(J^n A, B) \to \mathcal{K}_n(A', B)$$

is an isomorphism. Then there are a $m \geqslant 0$, a morphism $g : J^{n+m}A \to B^{\mathfrak{S}^{n+m}}$ such that $\varsigma^m(f)$ is simplicially homotopic to $g \circ J^m(s) : J^m A' \to B^{\mathfrak{S}^{n+m}}$. By the Hauptlemma these are polynomially homotopic. It follows that $\Gamma_1(g) = fs^{-1}$, and so Γ_1 is also surjective. $\square$

LEMMA 8.9. *The natural map*

$$\Gamma_2 : \operatorname{colim}_n D^-(\Re, \mathfrak{F})(J^n A, B^{\mathcal{S}^n}) \to \operatorname{colim}_n D^-(\Re, \mathfrak{F})(J^n A, B^{\mathfrak{S}^n})$$

is an isomorphism, functorial in A and B.

Proof. It follows from Lemma 8.2 that

$$D^-(\Re, \mathfrak{F})(J^n A, B^{\mathcal{S}^n}) \to D^-(\Re, \mathfrak{F})(J^n A, B^{\mathfrak{S}^n})$$

is bijective for all $n \geqslant 0$. Therefore Γ_2 is an isomorphism. $\square$

Consider a commutative diagram of algebras

$$\begin{tikzcd}
B^{\mathcal{S}^n} \arrow[r] & PB^{\mathcal{S}^{n-1}} \arrow[r] & B^{\mathcal{S}^{n-1}} \\
J(B^{\mathcal{S}^{n-1}}) \arrow[u, "\xi^{n-1}"] \arrow[r] \arrow[d, "\rho^{n-1}"'] & T(B^{\mathcal{S}^{n-1}}) \arrow[u] \arrow[r] \arrow[d] & B^{\mathcal{S}^{n-1}} \arrow[u, equal] \arrow[d, equal] \\
\Omega B^{\mathcal{S}^{n-1}} \arrow[r] & E(B^{\mathcal{S}^{n-1}}) \arrow[r] & B^{\mathcal{S}^{n-1}}
\end{tikzcd}$$

The middle arrows are stably weak equivalent to zero and ρ^{n-1}, ξ^{n-1} are stable weak equivalences, functorial in B. Since Ω respects stable weak equivalences, one obtains a functorial zig-zag of stable weak equivalences of length $2n$

$$B^{\mathcal{S}^n} \xleftarrow{\xi^{n-1}} J(B^{\mathcal{S}^{n-1}}) \xrightarrow{\rho^{n-1}} \Omega B^{\mathcal{S}^{n-1}} \xleftarrow{\Omega\xi^{n-2}} \cdots \xleftarrow{\Omega^{n-1}\xi^0} \Omega^{n-1}JB \xrightarrow{\Omega^{n-1}\rho^0} \Omega^n B.$$

The zig-zag yields an isomorphism $\delta^n : B^{\mathcal{S}^n} \to \Omega^n B$ in $D^-(\Re, \mathfrak{F})$.
Let us define a map

$$\Gamma_3 : \operatorname{colim}_n D^-(\Re, \mathfrak{F})(J^n A, B^{\mathcal{S}^n}) \to \operatorname{colim}_n D^-(\Re, \mathfrak{F})(J^n A, \Omega^n B)$$

by taking

$$\begin{tikzcd}
 & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
J^n A & & B^{\mathcal{S}^n}
\end{tikzcd}$$

to $\delta^n f s^{-1}$. We have to verify that Γ_3 is consistent with colimit maps, where a colimit map on the right hand side $u_n : D^-(\Re, \mathfrak{F})(J^n A, \Omega^n B) \to D^-(\Re, \mathfrak{F})(J^{n+1}A, \Omega^{n+1}B)$ takes

$$\begin{tikzcd}
 & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
J^n A & & \Omega^n B
\end{tikzcd}$$

to $\rho^0_{\Omega^n B} J(f)(J(s))^{-1}$. Let

$$v_n : D^-(\Re,\mathfrak{F})(J^n A, B^{\mathcal{S}^n}) \to D^-(\Re,\mathfrak{F})(J^{n+1}A, B^{\mathcal{S}^{n+1}})$$

be a colimit map on the left. So we have to check that $\Gamma_3(v_n(fs^{-1})) = u_n(\Gamma_3(fs^{-1}))$.
The map $u_n(\Gamma_3(fs^{-1}))$ is a zig-zag

$$J^{n+1}A \xleftarrow{Js} JA' \xrightarrow{Jf} JB^{\mathcal{S}^n} \xleftarrow{J\xi^{n-1}} J^2(B^{\mathcal{S}^{n-1}}) \xrightarrow{J\rho^{n-1}} J\Omega B^{\mathcal{S}^{n-1}} \xleftarrow{J\Omega\xi^{n-2}} \cdots$$
$$\xleftarrow{J\Omega^{n-1}\xi^0} J\Omega^{n-1}JB \xrightarrow{J\Omega^{n-1}\rho^0} J\Omega^n B \xrightarrow{\rho^0_{\Omega^n B}} \Omega^{n+1}B.$$

The map $\Gamma_3(v_n(fs^{-1}))$ is a zig-zag

$$J^{n+1}A \xleftarrow{Js} JA' \xrightarrow{Jf} JB^{\mathcal{S}^n} \xrightarrow{\xi^n} B^{\mathcal{S}^{n+1}} \xleftarrow{\xi^n} JB^{\mathcal{S}^n} \xrightarrow{\rho^n} \Omega B^{\mathcal{S}^n} \xleftarrow{\Omega\xi^{n-1}} \cdots$$
$$\xrightarrow{\Omega^{n-1}\rho^1} \Omega^n B^{\mathcal{S}^1} \xleftarrow{\Omega^n\xi^0} \Omega^n JB \xrightarrow{\Omega^n\rho^0} \Omega^{n+1}B.$$

We can cancel two ξ^n-s. One has therefore to check that the zig-zag

$$JB^{\mathcal{S}^n} \xleftarrow{J\xi^{n-1}} J^2(B^{\mathcal{S}^{n-1}}) \xrightarrow{J\rho^{n-1}} J\Omega B^{\mathcal{S}^{n-1}} \xleftarrow{J\Omega\xi^{n-2}} \cdots$$
$$\xleftarrow{J\Omega^{n-1}\xi^0} J\Omega^{n-1}JB \xrightarrow{J\Omega^{n-1}\rho^0} J\Omega^n B \xrightarrow{\rho^0_{\Omega^n B}} \Omega^{n+1}B$$

equals the zig-zag

$$JB^{\mathcal{S}^n} \xrightarrow{\rho^n} \Omega B^{\mathcal{S}^n} \xleftarrow{\Omega\xi^{n-1}} \cdots \xrightarrow{\Omega^{n-1}\rho^1} \Omega^n B^{\mathcal{S}^1} \xleftarrow{\Omega^n\xi^0} \Omega^n JB \xrightarrow{\Omega^n\rho^0} \Omega^{n+1}B.$$

For this one should use the property that if $g : A \to B$ is a homomorphism then there is a commutative diagram

(11)
$$\begin{tikzcd}
J(A) \arrow[r, "\rho_A"] \arrow[d, "J(g)"'] & \Omega A \arrow[r] \arrow[d, "\Omega(g)"] & EA \arrow[r] \arrow[d] & A \arrow[d, "g"] \\
J(B) \arrow[r, "\rho_B"] & \Omega B \arrow[r] & EB \arrow[r] & B.
\end{tikzcd}$$

So the desired compatibility with colimit maps determines a map of colimits.

LEMMA 8.10. *The map Γ_3 is an isomorphism, functorial in A and B.*

Proof. This follows from the fact that all δ^n-s are isomorphisms in $D^-(\Re,\mathfrak{F})$. □

Consider a sequence of stable weak equivalences

$$J^n A \xrightarrow{\rho} \Omega J^{n-1}A \xrightarrow{\Omega\rho} \Omega^2 J^{n-2}A \xrightarrow{\Omega^2\rho} \cdots \xrightarrow{\Omega^{n-1}\rho} \Omega^n A,$$

which is functorial in A. Denote its composition by γ_n.
Let us define a map

$$\Gamma_4 : \operatorname{colim}_n D^-(\Re,\mathfrak{F})(J^n A, \Omega^n B) \to \operatorname{colim}_n D^-(\Re,\mathfrak{F})(\Omega^n A, \Omega^n B)$$

by taking

$$\begin{tikzcd} & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ J^nA & & \Omega^nB \end{tikzcd}$$

to $fs^{-1}\gamma_n^{-1}$. We have to verify that Γ_4 is consistent with colimit maps, where a colimit map on the right hand side $w_n : D^-(\Re,\mathfrak{F})(\Omega^nA,\Omega^nB) \to D^-(\Re,\mathfrak{F})(\Omega^{n+1}A,\Omega^{n+1}B)$ takes

$$\begin{tikzcd} & A' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ \Omega^nA & & \Omega^nB \end{tikzcd}$$

to $\Omega(f)(\Omega(s))^{-1}$. So we have to check that $\Gamma_4(u_n(fs^{-1})) = w_n(\Gamma_4(fs^{-1}))$. The map $\Gamma_4(u_n(fs^{-1}))$ equals the zig-zag from $\Omega^{n+1}A$ to $\Omega^{n+1}B$

$$\Omega^{n+1}B \xleftarrow{\rho} J\Omega^nB \xleftarrow{Jf} JA' \xrightarrow{Js} J^{n+1}A \xrightarrow{\rho} \Omega J^nA \xrightarrow{\Omega\rho}$$

$$\Omega^2J^{n-1}A \xrightarrow{\Omega^2\rho} \cdots \xrightarrow{\Omega^n\rho} \Omega^{n+1}A.$$

In turn, the map $w_n(\Gamma_4(fs^{-1}))$ equals the zig-zag from $\Omega^{n+1}A$ to $\Omega^{n+1}B$

$$\Omega^{n+1}B \xleftarrow{\Omega f} \Omega A' \xrightarrow{\Omega s} \Omega J^nA \xrightarrow{\Omega\rho} \Omega^2J^{n-1}A \xrightarrow{\Omega^2\rho}$$

$$\Omega^3J^{n-2}A \xrightarrow{\Omega^3\rho} \cdots \xrightarrow{\Omega^n\rho} \Omega^{n+1}A.$$

The desired compatibility would be checked if we showed that the zig-zag

$$\Omega J^nA \xleftarrow{\rho} J^{n+1}A \xleftarrow{Js} JA' \xrightarrow{Jf} J\Omega^nB \xrightarrow{\rho} \Omega^{n+1}B \tag{12}$$

equals the zig-zag

$$\Omega J^nA \xleftarrow{\Omega s} \Omega A' \xrightarrow{\Omega f} \Omega^{n+1}B.$$

For this we use commutative diagram (11) to show that $\rho_{J^nA} \circ Js = \Omega s \circ \rho_{A'}$ and $\rho_{\Omega^nB} \circ Jf = \Omega f \circ \rho_{A'}$. We see that (12) equals $\Omega f \circ \rho_{A'} \circ \rho_{A'}^{-1} \circ (\Omega s)^{-1} = \Omega f \circ (\Omega s)^{-1}$ in $D^-(\Re,\mathfrak{F})$ and the desired compatibility follows.

Lemma 8.11. *The map Γ_4 is an isomorphism, functorial in A and B.*

Proof. This follows from the fact that all γ_n-s are isomorphisms in $D^-(\Re,\mathfrak{F})$. □

Proof of Comparison Theorem B. Using Comparison Theorem A, Lemmas 8.8, 8.9, 8.10, 8.11, the isomorphism of abelian groups

$$\mathbb{K}_0(A,B) \cong D(\Re,\mathfrak{F})(A,B)$$

is defined as $\Gamma_4\Gamma_3\Gamma_2^{-1}\Gamma_1$. Using Corollary 7.1, we get that

$$\mathbb{K}_{n>0}(A,B) \cong D(\Re,\mathfrak{F})(A,\Omega^{n>0}B)$$

and

$$\mathbb{K}_{n<0}(A,B) \cong D(\Re,\mathfrak{F})(J^{-n}A,B).$$

It remains to observe that $D(\Re,\mathfrak{F})(J^{-n}A,B)\cong D(\Re,\mathfrak{F})(A,\Omega^n B)$ for all negative n. $\square$

Corollary 8.12. *Let $\Re$ be T-closed. Then the classes of stable weak equivalences and $\mathcal{K}$-equivalences coincide.*

Corollary 8.13. *Let $\Re'$ be a full admissible T-closed subcategory of an admissible T-closed category of algebras $\Re$. Then the natural functor*

$$D(\Re',\mathfrak{F})\to D(\Re,\mathfrak{F})$$

is full and faithful.

Proof. This follows from Comparison Theorem B. $\square$

We want to introduce the class of unstable weak equivalences on $\Re$. Recall that $\mathfrak{W}_{\min}$ is the minimal class of weak equivalences containing the homomorphisms $A\to A[t]$, $A\in\Re$, such that the triple $(\Re,\mathfrak{F},\mathfrak{W}_{\min})$ is a Brown category. We do not know whether the canonical functor $\Re\to D^-(\Re,\mathfrak{F},\mathfrak{W}_{\min})$ has the property that $f\in\mathfrak{W}_{\min}$ if and only if its image in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{\min})$ is an isomorphism. For this reason we give the following

Definition. Let $\Re$ be T-closed. A homomorphism of algebras $f:A\to B$, $A,B\in\Re$, is called an *unstable weak equivalence* if its image in $D^-(\Re,\mathfrak{F},\mathfrak{W}_{\min})$ is an isomorphism. The class of unstable weak equivalences will be denoted by $\mathfrak{W}_{unst}$.

By construction, $D^-(\Re,\mathfrak{F},\mathfrak{W}_{\min})=\Re[\mathfrak{W}_{\min}^{-1}]$ and $\mathfrak{W}_{\min}\subseteq\mathfrak{W}_{unst}$. Using universal properties of localization, one obtains that $D^-(\Re,\mathfrak{F},\mathfrak{W}_{\min})=\Re[\mathfrak{W}_{unst}^{-1}]$. We can now apply results of the section to prove the following

Theorem 8.14. *Let $\Re$ be T-closed. Then*

$$\mathfrak{W}_{\triangle}=\{f\in\operatorname{Mor}(\Re)\mid\Omega^n(f)\in\mathfrak{W}_{unst}\text{ for some }n\geqslant 0\}.\tag{13}$$

Proof. By minimality of $\mathfrak{W}_{\min}$ the functor

$$\mathbb{K}(A,-):\Re\to \mathit{Spectra}$$

takes the maps from $\mathfrak{W}_{\min}$ (hence the maps from $\mathfrak{W}_{unst}$) to weak equivalences for all $A\in\Re$. Corollary 8.12 implies the right hand side of (13) is contained in $\mathfrak{W}_{\triangle}$.
The proof of Proposition 8.5 can literally be repeated for $\mathfrak{W}_{unst}$ to show that for every stable weak equivalence f there is $n\geqslant 0$ such that $\Omega^n(f)$ is in $\mathfrak{W}_{unst}$. $\square$

We can now make a table showing similarity of spaces and non-unital algebras. It is a sort of a dictionary for both categories. Precisely, one has:

Spaces Top	Algebras $\Re$
Fibrations	Fibrations $\mathfrak{F}$
Loop spaces ΩX	Algebras $\Omega A = (x^2 - x)A[x]$
Homotopies $X \to Y^I$	Polynomial homotopies $A \to B[x]$
Unstable weak equivalences	Unstable weak equivalences $\mathfrak{W}_{unst}$
Unstable homotopy category	The category $D^-(\Re, \mathfrak{F}, \mathfrak{W}_{unst})$
Stable weak equivalences	Stable weak equivalences $\mathfrak{W}_{\triangle}$
Stable homotopy category of spectra	The category $D(\Re, \mathfrak{F})$

To conclude the section, we should mention that Comparison Theorem B implies representability of the Hom-set $D(\Re, \mathfrak{F})(A, B)$, $A, B \in \Re$, by the spectrum $\mathbb{K}(A, B)$. By [9] the natural functor $j : \Re \to D(\Re, \mathfrak{F})$ is the universal excisive, homotopy invariant homology theory in the sense that any other such a theory $X : \Re \to \mathcal{T}$ uniquely factors through j.

9. Morita stable and stable bivariant K-theories

In this section we introduce matrices into the game. We start with preparations. If A is an algebra and $n \leqslant m$ are positive integers, then there is a natural inclusion $\iota_{n,m} : M_nA \to M_mA$ of rings, sending M_nA into the upper left corner of M_mA. We write $M_\infty A = \cup_n M_nA$. Let ΓA, $A \in \mathrm{Alg}_k$, be the algebra of $\mathbb{N} \times \mathbb{N}$-matrices which satisfy the following two properties.

(i) The set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is finite.

(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most N nonzero entries.

$M_\infty A \subset \Gamma A$ is an ideal. We put

$$\Sigma A = \Gamma A / M_\infty A.$$

We note that ΓA, ΣA are the cone and suspension rings of A considered by Karoubi and Villamayor in [23, p. 269], where a different but equivalent definition is given. By [3] there are natural ring isomorphisms

$$\Gamma A \cong \Gamma k \otimes A, \quad \Sigma A \cong \Sigma k \otimes A.$$

We call the short exact sequence

$$M_\infty A \rightarrowtail \Gamma A \twoheadrightarrow \Sigma A$$

the *cone extension*. By [3] $\Gamma A \twoheadrightarrow \Sigma A \in \mathfrak{F}_{\mathrm{spl}}$.

Throughout this section we assume that $\Re$ is a T-closed admissible category of k-algebras with $k, M_nA, \Gamma A \in \Re$, $n \geqslant 1$, for all $A \in \Re$. Then $M_\infty A, \Sigma A \in \Re$ for any $A \in \Re$ and $M_\infty(f) \in \mathfrak{F}$ for any $f \in \mathfrak{F}$. Note that $M_\infty A \cong A \otimes M_\infty(k) \in \Re$ for any $A \in \Re$. It follows from Proposition 2.3 that for any finite simplicial set L, there are natural isomorphisms

$$M_\infty A \otimes k^L \cong (M_\infty A)^L \cong A \otimes (M_\infty k)^L.$$

Given an algebra A, one has a natural homomorphism $\iota : A \to M_\infty(k) \otimes A \cong M_\infty(A)$ and an infinite sequence of maps

$$A \xrightarrow{\iota} M_\infty(k) \otimes A \xrightarrow{\iota} M_\infty(k) \otimes M_\infty(k) \otimes A \longrightarrow \cdots \longrightarrow M_\infty^{\otimes n}(k) \otimes A \longrightarrow \cdots$$

Definition. (1) The *stable algebraic Kasparov K-theory space* of two algebras $A, B \in \Re$ is the space

$$\mathcal{K}^{st}(A,B) = \operatorname{colim}_n \mathcal{K}(A, M_\infty k^{\otimes n} \otimes B).$$

Its homotopy groups will be denoted by $\mathcal{K}_n^{st}(A,B)$, $n \geqslant 0$.
(2) The *Morita stable algebraic Kasparov K-theory space* of two algebras $A, B \in \Re$ is the space

$$\mathcal{K}^{mor}(A,B) = \operatorname{colim}(\mathcal{K}(A,B) \to \mathcal{K}(A, M_2 k \otimes B) \to \mathcal{K}(A, M_3 k \otimes B) \to \cdots).$$

Its homotopy groups will be denoted by $\mathcal{K}_n^{mor}(A,B)$, $n \geqslant 0$.
(3) A functor $X : \Re \to \mathbb{S}/(Spectra)$ is *M_∞-invariant* (respectively *Morita invariant*) if $X(A) \to X(M_\infty A)$ (respectively each $X(A) \to X(M_n A), n > 0$) is a weak equivalence.
(4) An excisive, homotopy invariant homology theory $X : \Re \to \mathcal{T}$ is *M_∞-invariant* (respectively *Morita invariant*) if $X(A) \to X(M_\infty A)$ (respectively each $X(A) \to X(M_n A)$, $n > 0$) is an isomorphism.

Lemma 9.1. *The functor $\mathcal{K}^{st}(A,-)$ (respectively $\mathcal{K}^{mor}(A,-)$) is M_∞-invariant (respectively Morita invariant) for all $A \in \Re$.*

Proof. Straightforward. □

Theorem 9.2 (Excision). *For any algebra $A \in \Re$ and any $\mathfrak{F}$-extension in $\Re$*

$$F \xrightarrow{i} B \xrightarrow{f} C$$

the induced sequences of spaces

$$\mathcal{K}^\star(A,F) \longrightarrow \mathcal{K}^\star(A,B) \longrightarrow \mathcal{K}^\star(A,C)$$

and

$$\mathcal{K}^\star(C,A) \longrightarrow \mathcal{K}^\star(B,A) \longrightarrow \mathcal{K}^\star(F,A)$$

are homotopy fibre sequences, where $\star \in \{st, mor\}$.

Proof. This follows from Excision Theorems A, B and some elementary properties of simplicial sets. □

Definition. (1) Given two k-algebras $A, B \in \Re$ and $\star \in \{st, mor\}$, the sequence of spaces

$$\mathcal{K}^\star(A,B), \mathcal{K}^\star(JA,B), \mathcal{K}^\star(J^2A,B), \ldots$$

together with isomorphisms $\mathcal{K}^\star(J^nA,B) \cong \Omega\mathcal{K}^\star(J^{n+1}A,B)$ constructed in Theorem 5.1 forms an Ω-spectrum which we also denote by $\mathbb{K}^\star(A,B)$. Its homotopy groups will be denoted by $\mathbb{K}_n^\star(A,B)$, $n \in \mathbb{Z}$. Observe that $\mathbb{K}_n^\star(A,B) \cong \mathcal{K}_n^\star(A,B)$ for any $n \geqslant 0$ and $\mathbb{K}_n^\star(A,B) \cong \mathcal{K}_0^\star(J^{-n}A,B)$ for any $n < 0$.

(2) The *stable algebraic Kasparov K-theory spectrum* of (A, B) (respectively *Morita stable algebraic Kasparov K-theory spectrum*) is the Ω-spectrum $\mathbb{K}^{st}(A, B)$ (respectively $\mathbb{K}^{mor}(A, B)$).

THEOREM 9.3. *Let $\star \in \{st, mor\}$. The assignment $B \mapsto \mathbb{K}^\star(A, B)$ determines a functor*

$$\mathbb{K}^\star(A, ?) : \Re \to (Spectra)$$

which is homotopy invariant and excisive in the sense that for every $\mathfrak{F}$-extension $F \to B \to C$ the sequence

$$\mathbb{K}^\star(A, F) \to \mathbb{K}^\star(A, B) \to \mathbb{K}^\star(A, C)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots \to \mathbb{K}^\star_{i+1}(A, C) \to \mathbb{K}^\star_i(A, F) \to \mathbb{K}^\star_i(A, B) \to \mathbb{K}^\star_i(A, C) \to \cdots$$

for any $i \in \mathbb{Z}$.

Proof. This follows from Theorem 9.2. □

We also have the following

THEOREM 9.4. *Let $\star \in \{st, mor\}$. The assignment $B \mapsto \mathbb{K}^\star(B, D)$ determines a functor*

$$\mathbb{K}^\star(?, D) : \Re^{\mathrm{op}} \to (Spectra),$$

which is excisive in the sense that for every $\mathfrak{F}$-extension $F \to B \to C$ the sequence

$$\mathbb{K}^\star(C, D) \to \mathbb{K}^\star(B, D) \to \mathbb{K}^\star(F, D)$$

is a homotopy fibration of spectra. In particular, there is a long exact sequence of abelian groups

$$\cdots \to \mathbb{K}^\star_{i+1}(F, D) \to \mathbb{K}^\star_i(C, D) \to \mathbb{K}^\star_i(B, D) \to \mathbb{K}^\star_i(F, D) \to \cdots$$

for any $i \in \mathbb{Z}$.

Proof. This follows from Theorem 9.2. □

THEOREM 9.5 (Comparison). *There are natural isomorphisms*

$$\mathcal{K}_0^{st}(A, B) \to \operatorname{colim}_{m,n}[J^n A, M_\infty(k)^{\otimes m} \otimes B^{\mathfrak{S}^n}]$$

and

$$\mathcal{K}_0^{mor}(A, B) \to \operatorname{colim}_{m,n}[J^n A, M_m(k) \otimes B^{\mathfrak{S}^n}],$$

functorial in A and B.

Proof. This follows from Comparison Theorem A. □

COROLLARY 9.6. (1) *The homotopy groups of* $\mathbb{K}^{st}(A,B)$ *are computed as follows:*

$$\mathbb{K}_i^{st}(A,B) \cong \begin{cases} \operatorname{colim}_{m,n}[J^nA, (\Omega^i M_\infty(k)^{\otimes m} \otimes B)^{\mathfrak{S}^n}], & i \geqslant 0 \\ \operatorname{colim}_{m,n}[J^{-i+n}A, M_\infty(k)^{\otimes m} \otimes B^{\mathfrak{S}^n}], & i < 0 \end{cases}$$

(2) *The homotopy groups of* $\mathbb{K}^{mor}(A,B)$ *are computed as follows:*

$$\mathbb{K}_i^{mor}(A,B) \cong \begin{cases} \operatorname{colim}_{m,n}[J^nA, (\Omega^i M_m(B))^{\mathfrak{S}^n}], & i \geqslant 0 \\ \operatorname{colim}_{m,n}[J^{-i+n}A, M_m(B)^{\mathfrak{S}^n}], & i < 0 \end{cases}$$

Proof. This follows from Corollary 4.3 and the preceding theorem. □

We denote by $D^-_{st}(\Re,\mathfrak{F})$ the category whose objects are those of $\Re$ and whose maps between $A, B \in \Re$ are defined as

$$\operatorname{colim}_n D^-(\Re,\mathfrak{F})(A, M_\infty(k)^{\otimes n}(B)).$$

Similarly, denote by $D^-_{mor}(\Re,\mathfrak{F})$ the category whose objects are those of $\Re$ and whose maps between $A, B \in \Re$ are defined as

$$\operatorname{colim}_n D^-(\Re,\mathfrak{F})(A, M_n(B)).$$

It follows from [9] that $D^-_{st}(\Re,\mathfrak{F})$ and $D^-_{mor}(\Re,\mathfrak{F})$ are naturally left triangulated. Similar to the definition of $D(\Re,\mathfrak{F})$ we can stabilize the loop endofunctor Ω to get new categories $D_{mor}(\Re,\mathfrak{F})$ and $D_{st}(\Re,\mathfrak{F})$ which are in fact triangulated.

THEOREM 9.7 ([9]). *The functor* $\Re \to D_{st}(\Re,\mathfrak{F})$ *(respectively* $\Re \to D_{mor}(\Re,\mathfrak{F})$*) is the universal* $\mathfrak{F}$*-excisive, homotopy invariant,* M_∞*-invariant (respectively Morita invariant) homology theory on* $\Re$.

The next result says that the Hom-sets $D_{st}(\Re,\mathfrak{F})(A,B)$ $(D_{mor}(\Re,\mathfrak{F})(A,B))$, $A, B \in \Re$, can be represented as homotopy groups of spectra $\mathbb{K}^{st}(\Re,\mathfrak{F})(A,B)$ $(\mathbb{K}^{mor}(A,B))$.

THEOREM 9.8 (Comparison). *Let* $\star \in \{st, mor\}$. *Then for any algebras* $A, B \in \Re$ *there is an isomorphism of* $\mathbb{Z}$*-graded abelian groups*

$$\mathbb{K}^\star_*(A,B) \cong D_\star(\Re,\mathfrak{F})_*(A,B) = \bigoplus_{n\in\mathbb{Z}} D_\star(\Re,\mathfrak{F})(A,\Omega^n B),$$

functorial both in A *and in* B.

Proof. This follows from Comparison Theorem B. □

THEOREM 9.9 (Cortiñas–Thom). *There is a natural isomorphism of* $\mathbb{Z}$*-graded abelian groups*

$$D_{st}(\Re,\mathfrak{F})_*(k,A) \cong KH_*(A),$$

where $KH_*(A)$ *is the* $\mathbb{Z}$*-graded abelian group consisting of the homotopy* K*-theory groups in the sense of Weibel* [31].

Proof. See [9]. □

We end up the paper by proving the following

THEOREM 9.10. *For any $A \in \Re$ there is a natural isomorphism of $\mathbb{Z}$-graded abelian groups*

$$\mathbb{K}_*^{st}(k, A) \cong KH_*(A).$$

Proof. This follows from Theorems 9.8 and 9.9. □

The preceding theorem is an analog of the same result of KK-theory saying that there is a natural isomorphism $KK_*(\mathbb{C}, A) \cong K(A)$ for any C^*-algebra A.

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