Corrigendum To: “Andreotti–Mayer Loci and the Schottky Problem” cf. Documenta Math. 13 (2008) 398–440

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Abstract. The main result of [2] (C. Ciliberto, G. van der Geer: Andreotti-Mayer Loci and the Schottky problem. *Documenta Math.* 13 (2008), 389–440) is based on a proposition whose proof is incorrect. We therefore give an alternative proof of the main result of [2].

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1. Introduction

In this note we make some corrections to [2]. It was pointed out to us by Grushevsky and Hulek that the proof of Proposition 12.1 of loc. cit. is not correct; see also [4]. Since this proposition is used in the proof of the main result of [2] (the proof of Conjecture 9.1 in the case $N_{g,1}$) we will give an alternative proof of this result that does not use Proposition 12.1. In this proof we make essential use of the analysis in [4]. We refer to [2] for the notation we use. We thank Grushevsky and Hulek for spotting the error in the proof and for subsequent correspondence.

2. The Conjecture for $N_{g,1}$

Recall from [2] that $N_{g,k} \subset \mathcal{A}_g$ is the *Andreotti-Mayer locus* consisting of isomorphism classes of principally polarised abelian varieties (B, Ξ) such that the singular locus of Ξ has dimension $\geq k$.
The main result [2, Thm 20.3] of [2] is:

Theorem 2.1. *Let N be an irreducible component of $N_{g,1}$ with $g \geq 4$, of codimension 3 in $\mathcal{A}_g$. If $g = 4$, assume also that the general point of N is not a product of an elliptic curve with a 3-dimensional abelian variety. Then either $N = \mathcal{J}_5$, the jacobian locus in $\mathcal{A}_5$, or $N = \mathcal{H}_4$, the hyperelliptic locus in $\mathcal{A}_4$.*

The proof of Theorem 2.1 given in [2] is based on [2, Proposition 20.1] and [2, Lemma 20.2], in which we use [2, Proposition 12.1]. The proof of [2, Proposition 12.1] is wrong. The proposition is a generalization of [1, Prop. 3]. Since the proof of [2, Thm. 20.3] depends on it, this proof is therefore faulty. We give here a slightly different proof of [2, Proposition 20.1] and [2, Lemma 20.2], though the basic idea is the same as in [2], but we have to consider other types of degenerations of abelian varieties.

The statement of the theorem is well-established for $g = 4$ and $g = 5$, see the references in [2]. We therefore assume in the following that $g \geq 6$.

We begin by recalling the notation. For a principally polarized abelian variety (B, Ξ) we define $N_{k,1}(B, \Xi)$ to be the locus of $u \in B$ such that singular locus of the scheme-theoretic intersection $\Xi \cdot \Xi_u$ has dimension at least 1.

By $\tilde{\mathcal{A}}_g$ we mean the perfect cone compactification of $\mathcal{A}_g$. This is a stratified space with strata $\mathcal{A}_g^{(r)}$, with $\mathcal{A}_g^{(r)}$ the stratum of quasi–abelian varieties of torus rank r, with closure $\mathcal{A}_g^{(\geq r)}$. Note that $\partial\tilde{\mathcal{A}}_g := \tilde{\mathcal{A}}_g - \mathcal{A}_g = \mathcal{A}_g^{(\geq 1)}$ is the 'boundary' of $\tilde{\mathcal{A}}_g$. The stratum $\mathcal{A}_g^{(1)}$ is common to all toroidal compactifications of $\mathcal{A}_g$, so also to the second Voronoi compactification. Finally, by $\tilde{N}_{g,k}$ we mean the closure of the Andreotti–Mayer locus $N_{g,k}$ in $\tilde{\mathcal{A}}_g$.

The statement of [2, Proposition 20.1], for which we will give a new proof, is the following:

Proposition 2.2. *Let $g \geq 6$ and let N be an irreducible component of $\tilde{N}_{g,1}$ of codimension 3 in $\tilde{\mathcal{A}}_g$. Then $N \cap \mathcal{A}_g^{(1)} \neq \emptyset$.*

Proof. By [3], N is not complete in $\mathcal{A}_g$. Therefore $N \cap \partial\tilde{\mathcal{A}}_g \neq \emptyset$. We shall show that the closure of $N \cap \mathcal{A}_g$ in $\tilde{\mathcal{A}}_g$ intersects Mumford's partial toroidal compactification ${\mathcal{A}_g}^{(\leq 1)}$ inside $\tilde{\mathcal{A}}_g$. The reason for using the perfect cone compactification is the fact that the locus ${\mathcal{A}_g}^{(\geq r)}$ has codimension r in it.

Since $\partial\tilde{\mathcal{A}}_g$ is a divisor in $\tilde{\mathcal{A}}_g$, it intersects N in codimension one. Let M be any irreducible component of $N \cap \partial\tilde{\mathcal{A}}_g$. We make a case distinction.

Step 1. Suppose first $M \subseteq \mathcal{A}_g^{(\geq 4)}$. For dimension reasons we then have $M = \mathcal{A}_g^{(\geq 4)}$. So the general point of M corresponds to a general standard compactification $(\overline{X}, \overline{\Xi})$ of torus rank 4 (see [2, §16]) with abelian part (B, Ξ) general in $\mathcal{A}_{g-4}$. Recall that $(\overline{X}, \overline{\Xi})$ corresponds to the datum of a point $b = (b_1, \ldots, b_4) \in B^4$ and a 4×4 matrix $T = (t_{ij})$ with entries in $\mathbb{C}^*$ such that $t_{ii} = 1$ and $t_{ij} = t_{ji}^{-1}$. By letting some t_{ij} with $i \neq j$ tend to 0 (or equivalently to ∞) we obtain special points of $M = \mathcal{A}_g^{(4)}$, see [2, §17] for a similar discussion in the case of torus rank 2. We now write down a local equation for the generalized theta divisor. We let $z = (z_1, \ldots, z_{g-4}) \in \mathbb{C}^{g-4}$ be coordinates on the universal cover of B, and $\xi(z) = 0$ the equation of Ξ. Given $\zeta = (\zeta_1, \ldots, \zeta_4) \in \mathbb{C}^4$, we set $u_i = \exp(2\pi\zeta_i)$ for $1 \leqslant i \leqslant 4$, $u = (u_1, \ldots, u_4)$, and $u_I = \prod_{i\in I} u_i$. Finally, we set $t_I = \prod_{i,j\in I, i\leq j} t_{ij}$ and $b_I = \sum_{i\in I} b_i$ and let $\omega_I \in \mathbb{C}^{g-r}$ represent $b_I \in B$. We use the convention $t_\emptyset = 1, b_\emptyset = 0, \omega_\emptyset = 0$. The generalized theta divisor $\overline{\Xi}$

of $\overline{X}$ as in [2, §16] is given by the equation

$$\mathfrak{x}(z,u,b,T)=\sum_{I\subseteq\{1,\dots,4\}} u_I\, t_I\, \xi(z-\omega_I)=0\,. \tag{1}$$

By the hypothesis, the system

$$\begin{aligned} \mathfrak{x} &=0 \\ u_h\partial_{u_h}\mathfrak{x} &=0 \qquad \text{for } 1\leqslant h\leqslant 4 \\ \partial_{z_i}\ \mathfrak{x} &=0 \quad \text{for } 1\leqslant i\leqslant g-4 \end{aligned} \tag{2}$$

has an at least 1–dimensional set of solutions in z,u for all b and T and $(B,\Xi)\in\mathcal{A}_{g-4}$ general. As pointed out above, in (1) we may even take $t_{ij}=0$ for all pairs (i,j) with $i<j$ and $b_i=b\in B$, and accordingly $\omega_i=\omega$, for all i with $1\le i\le 4$. Then (2) reads

$$\begin{aligned} \xi(z)+(\sum_{i=1}^4 u_i)\xi(z-\omega) &=0 \\ u_h\xi(z-\omega) &=0 \qquad \text{for } 1\leqslant h\leqslant 4 \\ \partial_{z_i}\xi(z)+(\sum_{i=1}^4 u_i)\partial_{z_i}\xi(z-\omega) &=0 \quad \text{for } 1\leqslant i\leqslant g-4 \end{aligned} \tag{3}$$

which still has an at least 1-dimensional set of solutions in z,u for all $b\in B$ and general (B,Ξ) in $\mathcal{A}_{g-4}$. If for all these solutions one has $\sum_{i=1}^4 u_i=0$, then (3) implies

$$\xi(z)=\partial_{z_i}\xi(z)=0 \text{ for } 1\leqslant i\leqslant g-4$$

implying $(B,\Xi)\in N_{g-4,1}$, a contradiction, because $(B,\Xi)\in\mathcal{A}_{g-4}$ is taken general. So we may assume $\sum_{i=1}^4 u_i\neq 0$; in particular, there is an h with $1\le h\le 4$ such that $u_h\neq 0$. Then (3) reads

$$\xi(z)=0,\ \xi(z-\omega)=0,\ \partial_{z_i}\xi(z)+(\sum_{i=1}^4 u_i)\partial_{z_i}\xi(z-\omega)=0 \text{ for } 1\leqslant i\leqslant g-4\,.$$

This implies that $N_{1,1}(B,\Xi)=B$, which is impossible by [2, Proposition 11.6] since $(B,\Xi)\in\mathcal{A}_{g-4}$ is general. Or, alternatively, [1, Proposition 3] implies $(B,\Xi)\in N_{g-4,0}$, again a contradiction.

Remark 2.3. In the argument here we used specialization. This may bring us into a deeper stratum. But the local equation for the generalized theta divisor in the deeper stratum is a limit of the local equation of the generalized theta divisor. Similarly the limits of the equations (2) that describe the singularities gives the singular locus in the limit. Since the flat limit of the solution space is contained in the set of solutions of the limit set of equations, we see that the singular locus has a dimension that is not less than the dimension of the singular locus in the generic fibre of our local family. This argument will be used repeatedly in what follows.

Step 2. Assume next $M \cap \mathcal{A}_g^{(3)} \neq \emptyset$, so that M has codimension 1 in $\mathcal{A}_g^{(\geq 3)}$. There are three possibilities. Either the general point of M corresponds to a standard compactification, or M coincides with one of the two strata of codimension 4 and torus rank 3 of the perfect cone decomposition. See the table on p. 1321 of [4].

Step 2a. We first deal with the first case. We have a map $q : \mathcal{A}_g^{(\geq 3)} \dashrightarrow \tilde{\mathcal{A}}_{g-3}$. derived from the map to the Satake compactification. We denote by $F_{(B,\Xi)}$ the fibre of q over $(B,\Xi) \in \mathcal{A}_{g-3}$. By Step 1, the rational map q is defined at the general point of M and $q(M) \cap \mathcal{A}_{g-3} \neq \emptyset$. Then,

(1) either $q_{|M}$ is dominant onto $\tilde{\mathcal{A}}_{g-3}$ and, for $(B,\Xi) \in \mathcal{A}_{g-3}$ general, M intersects $F_{(B,\Xi)}$ in codimension 1;
(2) or $q(M)$ has codimension 1 in $\tilde{\mathcal{A}}_{g-3}$ with full fibres contained in M.

As in Step 1, $F_{(B,\Xi)}$ contains a dense open subset consisting of pairs (b,T), with $b = (b_1, b_2, b_3) \in B^3$ and T a 3×3 matrix with entries in $\mathbb{C}^*$ such that $t_{ii} = 1$ and $t_{ij} = t_{ji}^{-1}$. We can obtain special points of $F_{(B,\Xi)}$ as limits by letting some t_{ij}, with $i \neq j$, tend to 0 (or, which is the same, to ∞), cf. Remark 2.3. Case (2) can be excluded by repeating the same argument as in Step 1 and taking into account that, by [2, Theorem 8.6], $q(M)$ is not contained in $N_{g-3,1}$. (Note that the general point of $q(M)$ corresponds to a simple abelian variety, so that [2, Proposition 11.6] can be applied; see [2, §7].)

In case (1), we take $(B,\Xi) \in \mathcal{A}_{g-3}$ general. The fibre $F_{(B,\Xi)}$ is fibered over B^3 with fibres torus embeddings of $\mathbb{G}_m^3$. The image L of $M \cap F_{(B,\Xi)}$ in B^3 under the projection map to B^3 is of codimension ≤ 1.

First, suppose the codimension is 1. Since the fibre of $M \cap F_{(B,\Xi)} \to B^3$ contains the full fibre of $p : F_{(B,\Xi)} \to B^3$, we can let the t_{ij} for $i < j$ tend to zero and find the set of equations

$$\begin{aligned}
&\xi(z) + u_1\xi(z-\omega_1) + u_2\xi(z-\omega_2) + u_3\xi(z-\omega_3) = 0\\
&u_1\xi(z-\omega_1) = 0,\ u_2\xi(z-\omega_2) = 0,\ u_3\xi(z-\omega_3) = 0\\
&\partial_{z_i}\xi(z) + u_1\partial_{z_i}\xi(z-\omega_1) + u_2\partial_{z_i}\xi(z-\omega_2) + u_3\partial_{z_i}\xi(z-\omega_3) = 0
\end{aligned} \tag{4}$$

which for every triple (b_1, b_2, b_3) in the image has an at least 1-dimensional set of solutions in z and u_i. If for general $(b_1, b_2, b_3) \in L$ we have $u_1 = u_2 = u_3 = 0$, then $B \in N_{g-3,1}$, against our assumptions. If for general (b_1, b_2, b_3) in L exactly s (with $1 \leq s \leq 3$) u_i are not identically zero then we see that the projection of L to B^s lies in $N_{1,s}(B,\Xi)$. But this contradicts [2, Proposition 11.6], which says that the codimension in B^s is at least 2. (Alternatively, use [1, Proposition 3].)

Second, we treat the case $L = B^3$. Then $M \cap F_{(B,\Xi)}$ intersects the general fibre of p, which is a torus embedding of $\mathbb{G}_m^3$, in a surface. Then we may let two t_{ij}, e.g., t_{13} and t_{23}, tend to 0, cf. Remark 2.3. Set $t = t_{12}$. Then we find the system

(5)

$$\begin{aligned}
&\xi(z)+u_1\xi(z-\omega_1)+u_2\xi(z-\omega_2)+u_3\xi(z-\omega_3)+tu_1u_2\xi(z-\omega_1-\omega_2)=0\\
&u_i(\xi(z-\omega_i)+tu_{3-i}\xi(z-\omega_1-\omega_2))=0,\ 1\leqslant i\leqslant 2\\
&u_3\xi(z-\omega_3)=0\\
&\partial_{z_i}\xi(z)+u_1\partial_{z_i}\xi(z-\omega_1)+u_2\partial_{z_i}\xi(z-\omega_2)+u_3\partial_{z_i}\xi(z-\omega_3)\\
&\qquad\qquad\qquad\qquad+tu_1u_2\partial_{z_i}\xi(z-\omega_1-\omega_2)=0,\ 1\leqslant i\leqslant g-3
\end{aligned}$$

which for every triple $(b_1, b_2, b_3) \in L = B^3$ and a suitable t has an at least 1-dimensional set Z of solutions in z and u_i. If t is not constant, we may let t tend to 0 (or ∞ which amounts to the same), in which case we find a contradiction as before. Similarly if $t = 0$. So we may assume that t is a non–zero constant. If $u_1 = u_2 = 0$, then either $(B, \Xi) \in N_{g-3,1}$ (if $u_3 = 0$) or $N_{1,1}(B, \Xi) = B$ (if $u_3 \neq 0$); the former case is not possible because $q(M) = \tilde{\mathcal{A}}_{g-3}$, the latter is not possible by [2, Proposition 11.6]. So we may assume that one of the variables u_1, u_2, e.g., u_1, is not identically zero. Suppose $u_2 = 0$. Since we are assuming $(b_1, b_2, b_3) \in B^3$ general, we can specialize to $b_1 = b_3$. Then (5) becomes

$$\begin{aligned}
&\xi(z)+(u_1+u_3)\xi(z-\omega_1)=0\\
&u_1\xi(z-\omega_1)=u_3\xi(z-\omega_3)=0\\
&\partial_{z_i}\xi(z)+(u_1+u_3)\partial_{z_i}\xi(z-\omega_1)=0,\ 1\leqslant i\leqslant g-3
\end{aligned}$$

which implies that either $(B, \Xi) \in N_{g-3,1}$ (if $u_1 + u_3 = 0$) or $N_{1,1}(B, \Xi) = B$ (if $u_1 + u_3 \neq 0$), both leading to contradictions. Thus we may assume that u_1 and u_2 are both not identically 0. Suppose that one of u_1 and u_2 is constant on the solution set Z of (5) for general $(b_1, b_2) \in B^2$. Let C be an irreducible curve in the projection of Z to B. By either the second or the third equation in (5) we have $\mathcal{O}_C(\Xi_{b_i} - \Xi_{b_1+b_2}) \cong \mathcal{O}_C$. Since B is general, the map $B \to \mathrm{Pic}(C)$ sending b to $\mathcal{O}_C(\Xi_b - \Xi_{b_1+b_2})$ has finite kernel. Hence b_i has finite order, a contradiction, since $b_i \in B$ is general. So u_1 and u_2 are non–constant. Then by taking the limit case where $u_2 = 0$, the system (5) becomes like (4) with $u_2 = 0$. If u_1 and u_3 are both non–zero, then $N_{0,2}(B, \Xi) = B^2$, contradicting [2, Proposition 11.6]. If one of u_1 and u_3 is zero, a similar argument works.

Step 2b. Here we assume that M coincides with the component of the perfect cone compactification that parametrizes compactifications of semi-abelian varieties that are a union of two $\mathbb{P}^1 \times \mathbb{P}^2$-bundles over an abelian variety B of dimension $g - 3$. This type has been described in [4, Section 7]. Here we find a theta divisor given on one component by an equation

$$\begin{aligned}
\xi(z)+&u_1\,\xi(z-\omega_1)+u_2\,\xi(z-\omega_2)+u\,\xi(z-\omega_3)+\\
&u_1u\,t_{13}\xi(z-\omega_1-\omega_3)+u_2ut_{23}\,\xi(z-\omega_2-\omega_3)=0\,.
\end{aligned}$$

Again we have cases (1) and (2) as in Step 2a. Case (2) can be eliminated as above. In case (1) we can assume that t_{13} and t_{23} tend to 0. Then we find a system of equations

$$\begin{aligned}
&\xi(z)+u_1\xi(z-\omega_1)+u_2\xi(z-\omega_2)+u\xi(z-\omega_3)=0\\
&u_1\xi(z-\omega_1)=0,\quad u_2\xi(z-\omega_2)=0,\quad u\xi(z-\omega_3)=0\\
&\partial_{z_i}\xi(z)+u_1\partial_{z_i}\xi(z-\omega_1)+u_2\partial_{z_i}\xi(z-\omega_2)+x\partial_{z_i}\xi(z-\omega_3)=0
\end{aligned}$$

which has a 1-dimensional set of solutions in z and u_1, u_2, u (the affine coordinates on $\mathbb{P}^2$ and $\mathbb{P}^1$) for all $b_i \in B$ and $(B, \Xi) \in \mathcal{A}_{g-3}$ general. Note that the equations coincide with those of (4) and this case can be eliminated by a variant of the arguments used above.

Step 2c. The final case left to be dealt with is the case where M coincides with the codimension 4 stratum of torus rank 3 which parametrizes compactifications whose toric part corresponds to a union of two $\mathbb{P}^3$ and an intersection of two quadrics in $\mathbb{P}^5$, see [4, Section 7]. The case of a 1-dimensional singular locus on one of the two $\mathbb{P}^3$ follows along the same lines as the cases above. For the intersection of quadrics, the theta divisor is described by an equation of the form (see [4])

$$\begin{aligned}
&\xi(z-\omega_1)+u_1\xi(z-\omega_2-\omega_3)+tu_2\xi(z-\omega_2)+\\
&tu_3\xi(z-\omega_1-\omega_3)+su_4\xi(z-\omega_3)+su_5\xi(z-\omega_1-\omega_2)=0
\end{aligned}\tag{6}$$

where the $\omega_1, \omega_2, \omega_3$ can vary freely in $\mathbb{C}^{g-2}$. Again we can let s and t tend to 0 and then the arguments of Step 1 can be applied.

Step 3. Next we may assume $M \cap \mathcal{A}_g^{(2)} \neq \emptyset$, and then M has codimension 2 in $\mathcal{A}_g^{(\geq 2)}$. Consider the dominant map $q : \mathcal{A}_g^{(\geq 2)} \dashrightarrow \tilde{\mathcal{A}}_{g-2}$, which, by Steps 1 and 2, is defined at the general point of M and we have $q(M) \cap \mathcal{A}_{g-2} \neq \emptyset$. The fibre $F_{(B,\Xi)}$ of q is now a compactified $\mathbb{G}_m$-bundle over B^2 with $t := t_{12}$ the coordinate on $\mathbb{G}_m$. There are three possibilities:

(1) either $q_{|M}$ is dominant onto $\tilde{\mathcal{A}}_{g-2}$ and, for $(B, \Xi) \in \mathcal{A}_{g-2}$ general, M intersects $F_{(B,\Xi)}$ in codimension 2;
(2) or $q(M)$ has codimension 1 in $\tilde{\mathcal{A}}_{g-2}$ and, for $(B, \Xi) \in q(M)$ general, M intersects $F_{(B,\Xi)}$ in codimension 1;
(3) or $q(M)$ has codimension 2 in $\tilde{\mathcal{A}}_{g-2}$ with full fibres contained in M.

Case (3) can be excluded with the same argument in Step 1. Case (2) can be excluded with an argument similar to the one we gave in Step 2. Indeed, we take $(B, \Xi) \in q(M)$ general. The image L of the projection of an irreducible component of $M \cap F_{(B,\Xi)}$ to B^2 has either codimension 1 in B^2 or $L = B^2$. The relevant system is now

(7)
$$\begin{aligned}
&\xi(z)+u_1\,\xi(z-\omega_1)+u_2\,\xi(z-\omega_2)+tu_1u_2\,\xi(z-\omega_1-\omega_2)=0\\
&u_1\,\xi(z-\omega_1)+u_1u_2t\,\xi(z-\omega_1-\omega_2)=0,\\
&u_2\,\xi(z-\omega_2)+u_1u_2t\,\xi(z-\omega_1-\omega_2)=0,\\
&\partial_{z_i}\xi(z)+u_1\,\partial_{z_i}\xi(z-\omega_1)+u_2\,\partial_{z_i}\xi(z-\omega_2)+u_1u_2t\,\partial_{z_i}\xi(z-\omega_1-\omega_2)=0\,.
\end{aligned}$$

with an at least 1–dimensional set of solutions Z.

If $\mathrm{codim}_{B^2}(L) = 1$, then the full fibre of $F_{(B,\Xi)} \to B^2$ is contained in $M \cap F_{(B,\Xi)}$. Therefore we can let $t = 0$ in (7) and we can argue as in Step 2 (Case (1), $\mathrm{codim}_{B^3}(L) = 1$).

If $L = B^2$, we may assume, as above, that both u_1, u_2 do not vanish identically on Z. By taking $u_i = 0$ in (7), we see that $N_{0,1}(B, \Xi) = B$, contradicting [2, Proposition 11.6].

Also case (1) can be treated in a similar way. We take $(B, \Xi) \in \mathcal{A}_{g-2}$ general. The image L of the projection of an irreducible component of $M \cap F_{(B,\Xi)}$ onto B^2 has either codimension 1 or 2 in B^2. We let L_1 (resp. L_2) be the projection of L on the first (resp. second) factor of B^2.

If $\mathrm{codim}_{B^2}(L) = 2$, then the full fibre of $F_{(B,\Xi)} \to B^2$ is contained in $M \cap F_{(B,\Xi)}$. Therefore we can let $t = 0$ in (7). As usual we may get rid of the case in which $u_1 = u_2 = 0$ on the set Z of solutions of (7). Assume next that only one among u_1, u_2, e.g. u_1 vanishes identically. Then $L_2 \subseteq N_{1,1}(B, \Xi)$ and [2, Proposition 11.6] tells us that $L_2 = N_{1,1}(B, \Xi)$ has codimension 2 in B. But then $N_{1,1}(B, \Xi)$ is positive-dimensional and Proposition [1, Proposition 3] yields $(B, \Xi) \in N_{g-2,0}$, again a contradiction.

So we may assume that u_1, u_2 both do not vanish identically on Z. By taking $u_i = 0$, we see that $L_i \subseteq N_{0,1}(B, \Xi)$, for $i \in \{1, 2\}$. By [2, Proposition 11.6], we have $L_i = N_{0,1}(B, \Xi)$, for $1 \leqslant i \leqslant 2$, and $L = L_1 \times L_2$, hence L intersects the diagonal of B^2. Therefore we can take $\omega_1 = \omega_2 = \omega$ (corresponding to $b \in N_{0,1}(B, \Xi)$ general) and $t = 0$ in (7), so that this system becomes

$$\begin{aligned} &\xi(z) + (u_1 + u_2)\,\xi(z - \omega) = 0 \\ &u_1\,\xi(z - \omega) = 0,\ u_2\,\xi(z - \omega) = 0 \\ &\partial_{z_i}\xi(z) + (u_1 + u_2)\,\partial_{z_i}\xi(z - \omega) = 0,\ 1 \leqslant i \leqslant g - 2. \end{aligned} \tag{8}$$

If $u_1 + u_2 = 0$ identically on Z, we have $(B, \Xi) \in N_{g,0}$, a contradiction. If $u_1 + u_2$ is not identically 0, then (8) shows that $N_{1,1}(B, \Xi) = N_{0,1}(B, \Xi)$ which is a divisor on B, contradicting [2, Proposition 11.6].

Finally, assume $\mathrm{codim}_{B^2}(L) = 1$. As before we get rid of the case in which one of u_1 and u_2 vanishes identically on the solution set Z for some triple (b_1, b_2, t). If for some pair $(b_1, b_2) \in L$ we can take the limiting case $t = 0$, we are also done. So we may assume that u_1, u_2, t are non–zero and t is constant. If u_1 and u_2 are constant on Z we find a contradiction as in Step 2a. So we may assume that u_1 and u_2 are both non–constant and again we can finish as in Step 2a.

In conclusion $M \not\subseteq \mathcal{A}_g^{(\geq 2)}$, hence $M \cap \mathcal{A}_g^{(1)} \neq \emptyset$, proving the assertion.

□

Keeping the above notation, by Proposition 2.2 the map $q : \mathcal{A}_g^{(1)} \dashrightarrow \tilde{\mathcal{A}}_{g-1}$ is defined at the general point of M. We finish by giving a new proof of [2, Lemma 20.2].

Lemma 2.4. *In the above setting, the Zariski closure of $q(M)$ in $\tilde{\mathcal{A}}_{g-1}$ is:*

(i) *either an irreducible component N_1 of $\tilde{N}_{g-1,1}$ of codimension 3 in $\tilde{\mathcal{A}}_{g-1}$; or*

(ii) *an irreducible component N_0 of $\tilde{N}_{g-1,0}$ and if $(B,\Xi) \in N_0$ is general, then $M \cap F_{(B,\Xi)}$ is an irreducible component of $N_{1,1}(B,\Xi)$ of codimension 2 in B. Moreover, if $\xi = (\overline{X}, \overline{\Xi}) \in M$ is a general point, then $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ meets the singular locus D of $\overline{X}$ in one or two points, whose associated quadric has corank 1.*

Proof. If $q(M) \subseteq N_{g-1,1}$, then [2, Theorem 8.6] implies $3 \leq \mathrm{codim}_{\mathcal{A}_{g-1}}(q(M)) \leq \mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(M) = 3$ and we are in case (i). If $q(M) \not\subseteq N_{g-1,1}$, the argument in the proof of Proposition 2.2 shows that $q(M) \subseteq N_{g-1,0}$ and $M \cap F_{(B,\Xi)} \subseteq N_{1,1}(B,\Xi)$. By [2, Proposition 11.6] we have $\mathrm{codim}_B(N_{1,1}(B,\Xi)) \geq 2$ and since $N_{g-1,0}$ is a divisor in $\mathcal{A}_{g-1}$ we are in case (ii). The final assertion follows just as in the proof of [2, Lemma 20.2, (ii)] that does not depend on [2, Proposition 12.1], and goes through without change. □

3. Further Corrections

In the proof of [2, Corollary 12.2] we erroneously refer to (a non–existing) Corollary 11.1. The reference should instead have been to Proposition 12.1, whose proof is incorrect. Hence the proof of Corollary 12.2 is faulty. However, this Corollary 12.2 has never been used in [2]. Proposition 12.1 of [2] is used in the proof of [2, Lemma 16.1], which is therefore incorrect. Lemma 16.1 is used to justify the final assertion of [2, Remark 18.1], which is however not used later. Remark [2, 12.3] is also based on the incorrect analysis of the proof of [2, Prop. 21.1].

We point out one more (inconsequential) error in [2, §17, p. 485, l. 22–23]. It is stated there that, for a general quasi–abelian variety $(\overline{X}, \overline{\Xi})$ of torus rank 2 with abelian part (B,Ξ), the point $x \in \overline{X} - \mathrm{Sing}(\overline{X})$, corresponding to coordinates (u,z), is a vertical singularity of $\overline{\Xi}$ if and only if z is singular for the divisor H of B defined by

$$\xi(\tau, z-\omega_1)\xi(\tau, z-\omega_2) = t\,\xi(\tau,z)\xi(\tau, z-\omega_1-\omega_2).$$

Four lines below, we state that the existence of a vertical singularity of $\overline{\Xi}$ implies that Ξ, Ξ_{b_1}, Ξ_{b_2} and $\Xi_{b_1+b_2}$ are tangentially degenerate at some point of B. Both assertions are incorrect. If x as above is a vertical singularity, then z is singular for H, but in general the converse does not hold. Moreover, if the theta divisors Ξ, Ξ_{b_1}, Ξ_{b_2} and $\Xi_{b_1+b_2}$ are tangentially degenerate at some point of B, this may give rise to a vertical singularity, but does not do so necessarily.

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